

Spin Plaquette Hamiltonian on a square lattice

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1 Hamiltonian

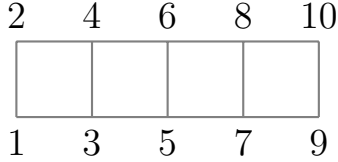
We consider the spin plaquette model

$$H = -J \sum_{\square} \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z + h \sum_{i=1}^N \sigma_i^x \quad (1)$$

where $\sigma_i^{\{x,y,z\}}$ are the Pauli matrices on the lattice site i , \square means the interaction is for a plaquette's on a square lattice, N is the total number of sites.

1.1 2-Ladder

Here we define the Hamiltonian on a geometry as shown below, the spins sits on the vertices and interacts as given in equation 1



Now the Hamiltonian in equation 1 can be written as

$$H = -J \sum_{i=1,3,5,\dots}^{N-3} \sigma_i^z \sigma_{i+1}^z \sigma_{i+2}^z \sigma_{i+3}^z + h \sum_{i=1}^N \sigma_i^x \quad (2)$$

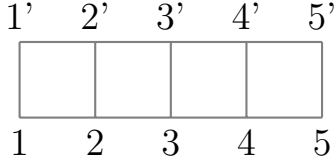
The boundary condition used are free(open). For periodic boundary conditions the sum in the first term will go till $N - 1$.

1.1.1 Analytic Solution

Domain wall variables on short rungs To make any further progress analytically it is useful to relabel the vertices as shown in diagram below. Now we can use new domain wall variables defined on the short rungs as

So the Hamiltonian becomes

$$H = -J \sum_{i=1}^{N/2} \sigma_i^z \sigma_{i'}^z \sigma_{i+1}^z \sigma_{i'+1}^z + h \sum_{i=1}^{N/2} (\sigma_i^x + \sigma_{i'}^x) \quad (3)$$



$$S_i^z = \sigma_i^z \sigma_{i'}^z \quad S_i^x = \sigma_i^x \quad T_i^x = \sigma_i^x \sigma_{i'}^x \quad (4)$$

Now for the S variables to satisfy the pauli algebra we find $S_i^y = \sigma_i^y \sigma_{i'}^z$. Also $\sigma_{i'}^x = S_i^x T_i^x$. So the Hamiltonian in equation 2 can be written as

$$H = -J \sum_{i=1}^{N/2} S_i^z S_{i+1}^z + h \sum_{i=1}^{N/2} S_i^x (1 + T_i^z) \quad (5)$$

Here we found that the Hamiltonian is Transverse Field Ising like in S variables but we have an extra term coupled with S^x with the same coupling constant h . Also it is worth noticing that T_i^x separately commutes with the whole Hamiltonian.

Suggestions :

Refer garrahan's paper for 2-ladder [Strong zero modes in a class of generalised Ising spin ladders with plaquette interactions](#).

Jordan Weigner transformation We can also think of performing a Jordan-Wiegnier transformations on the original Hamiltonian(equation 2) to go to the spinless fermionic variables. It is obvious to expect a plaquette interaction in the fermionic variables as well. But let's check

First we do a canonical transformation

$$\sigma_z \rightarrow -\sigma_x \quad \sigma_x \rightarrow \sigma_z \quad (6)$$

So the Hamiltonian in equation 2 becomes

$$H = -J \sum_{i=1,3,5,\dots}^{N-3} \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x + h \sum_{i=1}^N \sigma_i^z \quad (7)$$

Now because of the labelling used here we can introduce the transformations in the same form as for a 1D chain

$$\sigma_i^z = 2a_i^\dagger a_i - 1 \quad (8)$$

$$a_i = \left(\prod_{j<i} \sigma_j^z \right) \sigma_j^+ \quad (9)$$

$$a_i^\dagger = \left(\prod_{j<i} \sigma_j^z \right) \sigma_j^- \quad (10)$$

here a_i and a_i^\dagger are the annihilation and creation operator for the fermions at site i . Also $\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y)$ and $\sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y)$ are raising and lowering operators at each site. We can invert equation 9 and 10 to get σ_j^+ and σ_j^- and then solve for σ_j^x .

$$\sigma_i^x = \left(\prod_{j<i} \sigma_j^z \right) (a_i + a_i^\dagger) \quad (11)$$

Now we solve for the interaction term using the transformation of σ_z and the commutation relations $\{a_i, a_j^\dagger\} = \delta_{ij}$.

$$\sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x = (a_i + a_i^\dagger) \sigma_i^z (a_{i+1} + a_{i+1}^\dagger) (a_{i+2} + a_{i+2}^\dagger) \sigma_{i+2}^z (a_{i+3} + a_{i+3}^\dagger) \quad (12)$$

$$\sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x = (a_i - a_i^\dagger) (a_{i+1} + a_{i+1}^\dagger) (a_{i+2} - a_{i+2}^\dagger) (a_{i+3} + a_{i+3}^\dagger) \quad (13)$$

Does not seem to be useful as there are quartic interaction in these variables as well. Going to Majorana fermions will not help again.

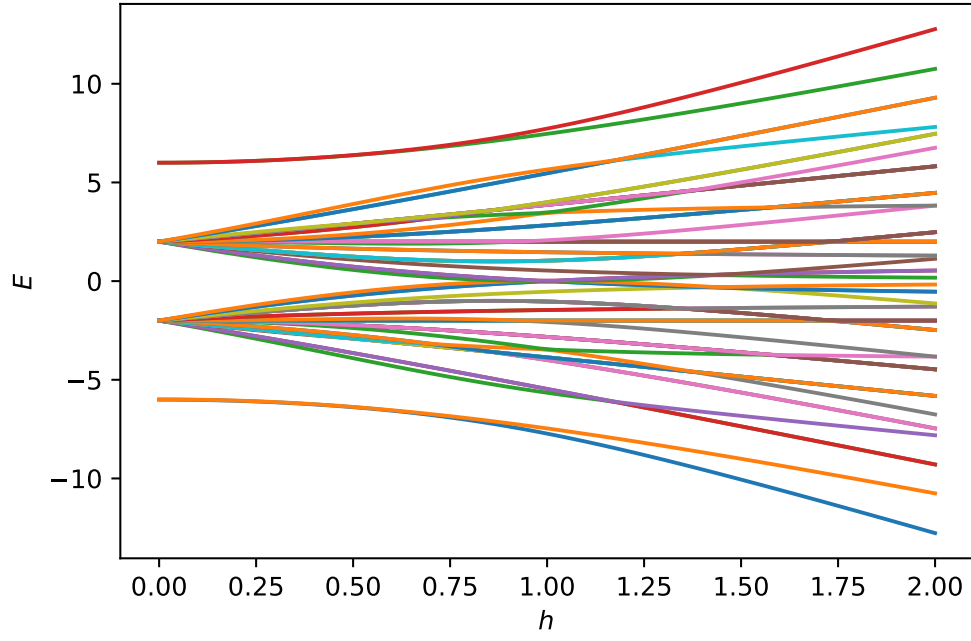
We can try to do this for the Hamiltonian in equation 5 which has only quadratic terms but has an extra term coupled to the S_i^x . Let's see how is that going to affect the form of the transformed Hamiltonian from the Ising case.

1.1.2 Numerics

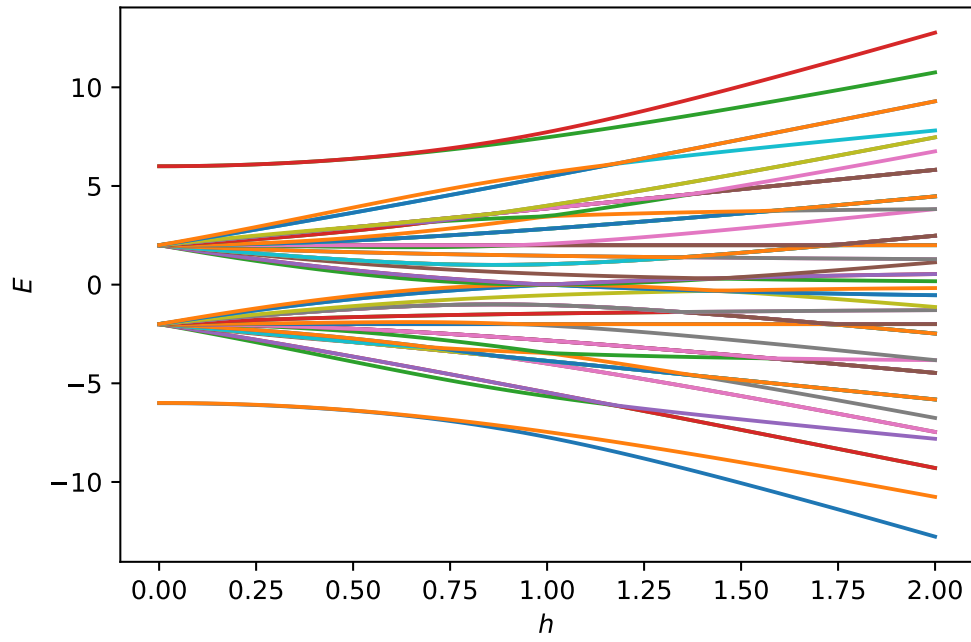
We numerically diagonalize the Hamiltonian and plot the energy spectrum as shown in figure . The gap between the ground state and the lowest excited state is also plotted. The average magnetization $\langle m_z \rangle$ defined as

$$\langle m_z \rangle = \frac{1}{N} \sum_{i=1}^N \langle \sigma_i^z \rangle \quad (14)$$

where $\langle \sigma_i^z \rangle = \langle \psi | \sigma_i^z | \psi \rangle$ is also plotted for the ground state with h .

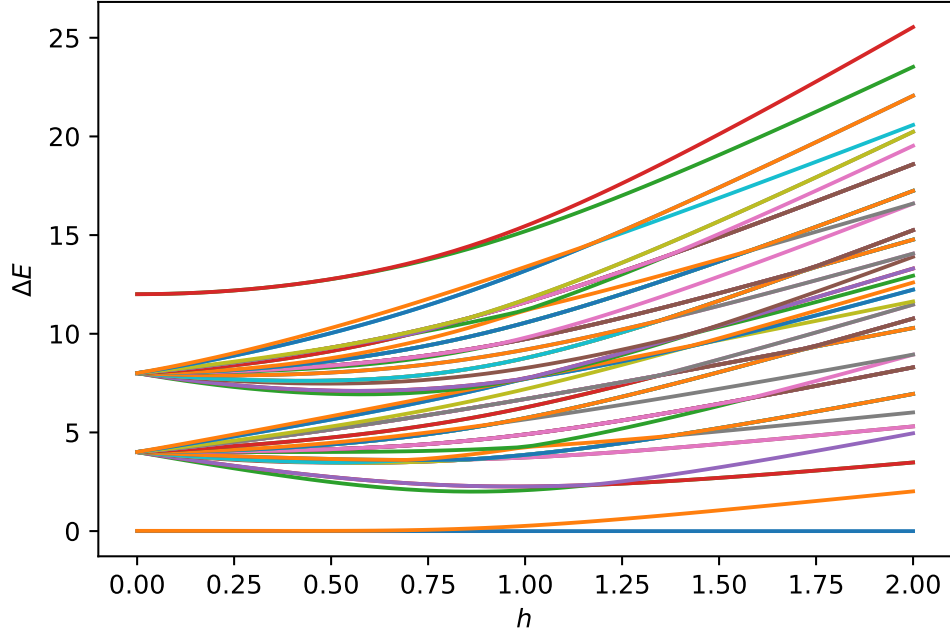


(a) free boundary condition

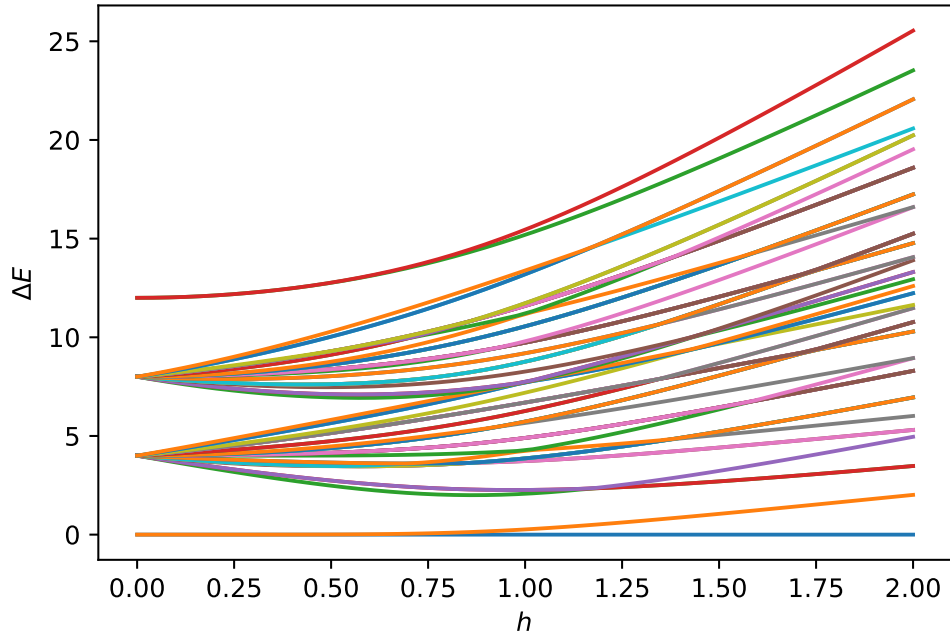


(b) periodic boundary condition

Figure 1: $N = 6$, $j = 1$, Energy of all the levels with h .

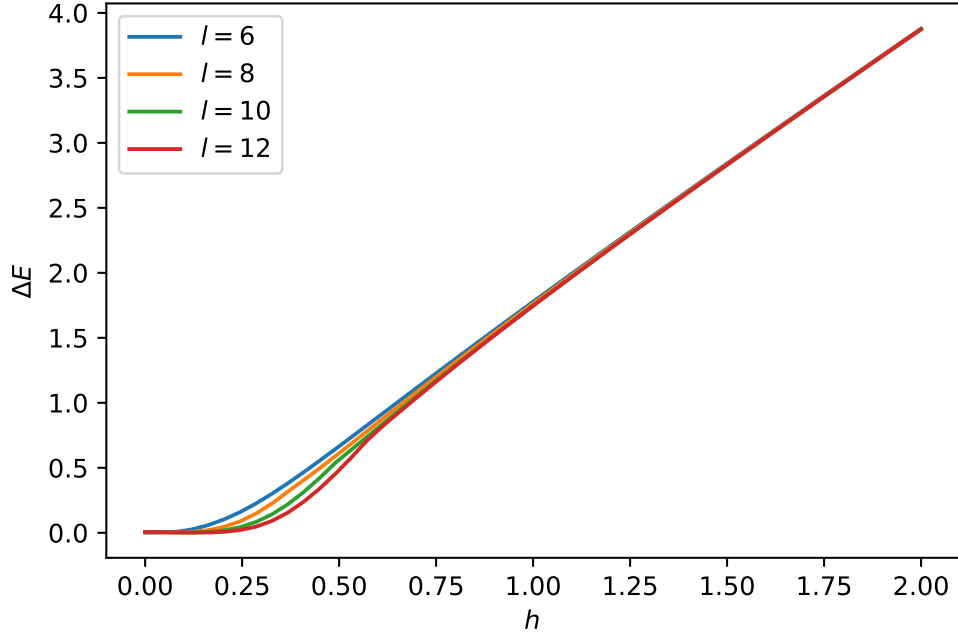


(a) free boundary condition

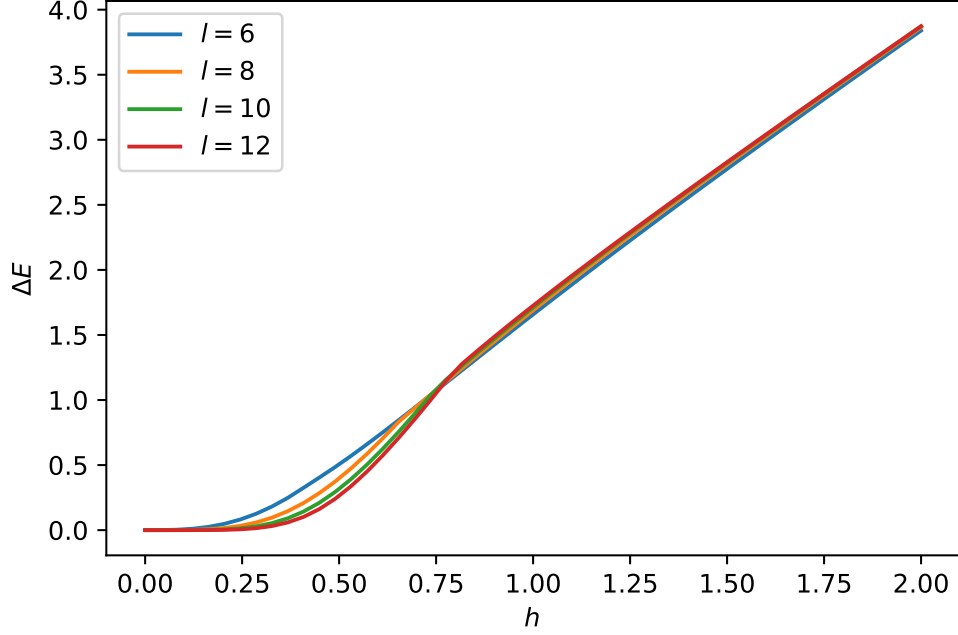


(b) periodic boundary condition

Figure 2: $N = 6$, $j = 1$, Energy gap (with the ground state) of all the levels with h .

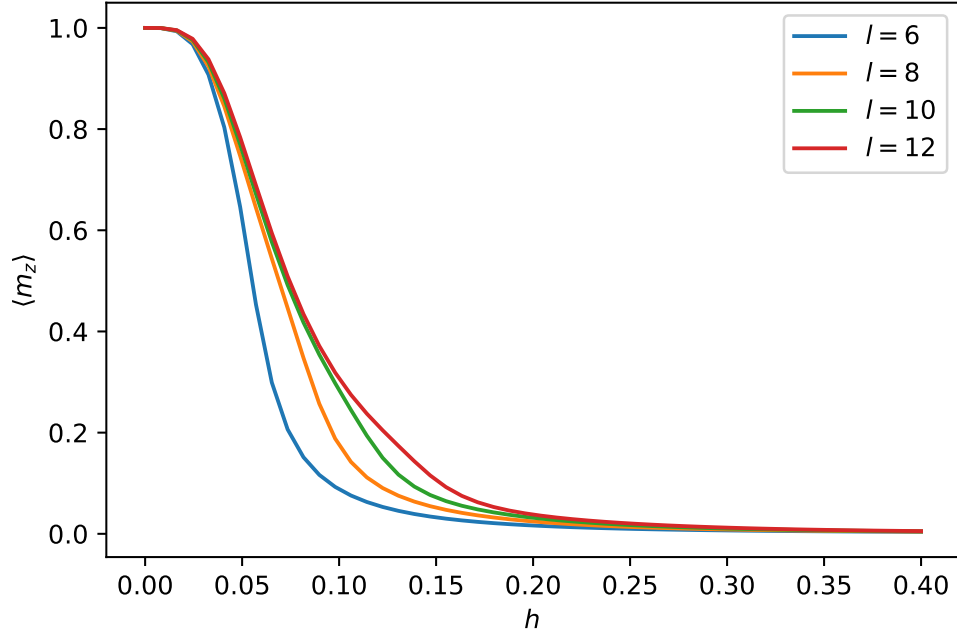


(a) free boundary condition

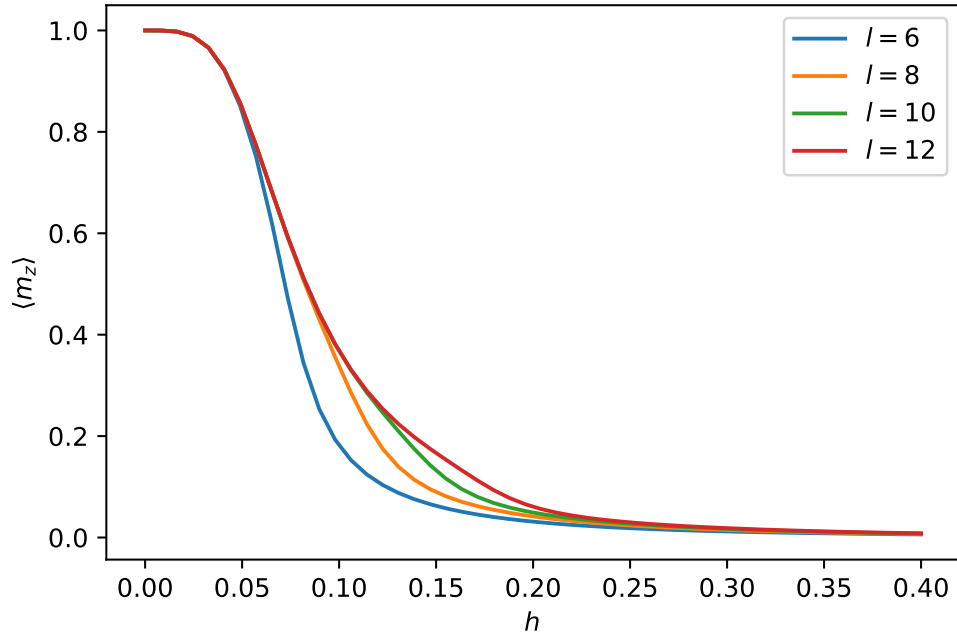


(b) periodic boundary condition

Figure 3: $N = 6$, $j = 1$, Energy gap (between ground state and first excited state) with h and different colours are different system size.



(a) free boundary condition



(b) periodic boundary condition

Figure 4: $N = 6$, $j = 1$, Average magnetization of the ground state with h .

1.2 3-Ladder

1.3 p-Ladder