EDUCATIONAL ASSESSMENT AND EXAMINATIONS SERVICE
ETHIOPIAN SECONDARY SCHOOL LEAVING CIRTIFICATE EXAMINATION (ESSLCE)

***MATHEMATICS* FOR *NATURAL SCIENCE* SOLUTIONS**

Hamlie 2015 E.C/ July 2023 G.C

SUBJECT CODE: 02 **BOOKLET CODE: 340**

Number of Items: 65 **Time Allowed: 3 hours**

1. Givens sets A and B where A = $\{x\in \mathbb{N}: x < 3\} = \{1, 2\}$, and B is the set of all possible factors of 13 so that B = {-13, -1, 1, 13}. Then

$B\times A = \{(x, y): x \in B \text{ and } y \in A\}$, By definition

$= \{(-13, 1), (-13, 2), (-1, 1), (-1, 2), (1, 1), (1, 2), (13, 1), (13, 2)\}$

Answer: No Answer is given ■

2. Given quadratic function $f(x) = x^2 - 6x + 10$ so that for the general form $f(x) = ax^2 + bx + c$; a =1, b = -6 and c =10. Then,

Axis of symmetry of graph of f is the line $x = -\frac{b}{2a}$, i.e., $x = -(\frac{-6}{2(1)}) = 3$.

Vertex of graph of f is at a point $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$ or simply:

$$(-\frac{b}{2a}, f(-\frac{b}{2a})) = (3, 3^2 - 6(3)+10) = (3, 1).$$

And, a =1> 0 implies graph of f opens upward and range of f is

$$\{y: y\geq f(-\frac{b}{2a})\} \text{ or } y: y\geq \frac{4ac-b^2}{4a}\}, \text{ i.e., } f(x) \geq 1$$

Answer: C ■

3. The area A of a regular n-sided polygon with radius r (or inscribed in a circle of radius r) is given by

$$A = \frac{1}{2} nr^2 \sin \frac{360^o}{n}$$

Given hexagon implies n=6; and given r= 6 cm give

$$A = \frac{1}{2}(6)(6 \text{ cm})^2 \sin \frac{360^o}{6}$$

$$= 108 \sin 60^o \text{ cm}^2 = 54\sqrt{3} \text{ cm}^2, \text{ recall that } \sin 60^o = \frac{\sqrt{3}}{2}$$

Answer: A ■

4. A prime number is a natural number that has only two positive factor (i.e., 1 and itself).

Answer: D ■

5. A rational number r is in its standard form if it can be put as:

$$r = a \times 10^n \text{ where } 1 \leq a < 10 \text{ and } n \in \mathbb{Z}$$

Thus, only 5×10^5 is in standard form since $1 \leq 5 < 10$ and $5 \in \mathbb{Z}$.

Answer: C ■

6. For $a > 0$, and integers m, n, recall the exponential rules that

(i) $\sqrt{a} = a^{\frac{1}{2}}$; (iii) $a^n a^m = a^{n+m}$

(ii) $(a^n)^m = a^{nm}$; (iv) $a^n = a^m \Leftrightarrow n = m$

First, by (ii) $125^x = (5^3)^x = 5^{3x}$, and $25^{3x+1} = (5^2)^{3x+1} = 5^{6x+2}$ …(*)

Then, by (iii) and (*): $125^x 5^{2x-1} = 5^{3x} 5^{2x-1} = 5^{5x-1}$ …(**)

And, by (i), (ii) and (**): $\sqrt{125^x 5^{2x-1}} = (5^{5x-1})^{\frac{1}{2}} = 5^{\frac{5x-1}{2}}$ …(***)

Therefore, by (*), and (***):

$$\sqrt{125^x 5^{2x-1}} = 25^{3x+1} \Rightarrow 5^{\frac{5x-1}{2}} = 5^{6x+2}$$

$$\Rightarrow \frac{5x-1}{2} = 6x + 2, \text{ by (iv)}$$

$$\Rightarrow 5x - 1 = 12x + 4$$

$$\Rightarrow x = -\frac{5}{7}$$

Since the given equation is defined for every real number x, $x = -\frac{5}{7}$.

Answer: A ■

7. First, determine the domain of the variable (**Universe**).

By absolute value definition, in the left $3|2x - 4| \geq 0$ so that

$$6 - 2|1 - 5x| \geq 0 \Rightarrow |1 - 5x| \leq 3 \Rightarrow -3 \leq 1\text{-}5x \leq 3$$

$$\Rightarrow -\frac{2}{5} \leq x \leq \frac{4}{5}$$

$$\Rightarrow x \in [-\frac{2}{5}, \frac{4}{5}] \ldots\ldots [I_1]$$

Again $3|2x - 4| = 6 - 2|1 - 5x|$ implies $2|1 - 5x| = 6 - 3|2x - 4|$ so that by absolute value definition, in the left $2|1 - 5x| \geq 0$ so that

$$6 - 3|2x - 4| \geq 0 \Rightarrow |2x - 4| \leq 2 \Rightarrow -2 \leq 2x\text{-}4 \leq 2$$

$$\Rightarrow 1 \leq x \leq 3$$

$$\Rightarrow x \in [1, 3] \ldots\ldots [I_2]$$

Hence, the domain of variable is: $I_1 \cap I_2 = [-\frac{2}{5}, \frac{4}{5}] \cap [1, 3] = \{\ \}$.

Since the universe is empty set, the solution set is also empty set.

Answer: D ■

8. The lateral surface area of a regular pyramid with length of lateral edge ℓ, and n-sided regular polygon base of side length s is given by:

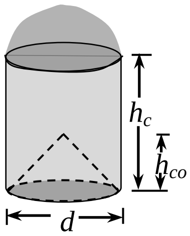
$$A_\ell = \frac{1}{2}ns\ell$$

Given, n = 6 (hexagonal base), s = 12 cm, and ℓ = 10 cm, we have

$$A_\ell = \frac{1}{2}(6)(12\text{ cm})(10\text{ cm}) = 360\text{ cm}^2.$$

Answer: B ■

9. Suppose an object is given as in the figure below. Before the cut off, the volume of the object V_o was the sum of the volumes of the hemisphere V_H and the cylinder V_C. Diameter of the hemisphere and base of the cylinder are equal, i.e., given d = 6 cm; and height of the cylinder is given by h_C = 12 cm so that:



$$\begin{aligned} V_o = V_H + V_C &= \frac{1}{12}\pi d^3 + \frac{1}{4}\pi d^2 h_C \\ &= \frac{1}{12}\pi(6\text{ cm})^3 + \frac{1}{4}\pi(6\text{ cm})^2(12\text{ cm}) \\ &= 126\pi\text{cm}^3 \qquad \text{(volume before cut off)} \end{aligned}$$

The cut off right circular cone has the same base diameter d = 6 cm, and given height $h_{Co} = 8$ cm so that the volume V_{Co} of the cone is:

$$V_{Co} = \frac{1}{12}\pi d^2 h_{Co} = \frac{1}{12}\pi(6\text{ cm})^2(8\text{ cm}) = 24\pi\text{cm}^3$$

Therefore, the volume V of the remaining object after cut off is:

$$V = V_o - V_{Co} = 126\pi\text{cm}^3 - 24\pi\text{cm}^3 = 102\pi\text{cm}^3.$$

Answer: B ■

10. The domain of a power function $f(x) = ax^{\frac{n}{m}}$ where $a \neq 0$, $\frac{n}{m} > 0$ and m is odd integer is $\mathbb{R}$ because f is defined for every real number x so that domain of $f(x) = 2x^{\frac{2}{3}}$ is $\mathbb{R}$.

Answer: D ■

11. The slope of a line that makes an angle of θ with the positive x-axis (recall this angle is called angle of inclination) is given by

$$m = \tan\theta.$$

Given, $\theta = 135^o$ implies:

$$\begin{aligned} m &= \tan 135^o \\ &= -\tan(180^o - 135^o), \text{ quadrant II reference angle formula} \\ &= -\tan 45^o = -1. \end{aligned}$$

Answer: A ■

12. By co-terminal angles relation and periodicity of tan by 180^o:

$$\tan\theta = \tan(\theta + n180^o) \text{ for every integer n.}$$

For $\theta = 509^o$ choose $n = -3$ so that:

$$\begin{aligned} \tan 509^o &= \tan(509^o + (-3)180^o) \\ &= \tan(-31^o) \\ &= -\tan 31^o \text{ , because tangent function is odd} \\ &= -0.81 \text{ , because given } \tan 31^o = 0.81 \end{aligned}$$

Answer: A ■

13. Recall the identities $\sec\theta = \frac{1}{\cos\theta}$ and $\cos\theta = \pm\sqrt{1-\sin^2\theta}$.

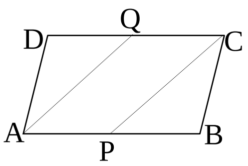
Thus, $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}}$, since $\sec\theta > 0$ in 4th quadrant

$$= \frac{1}{\sqrt{1-(-\frac{3}{5})^2}}, \text{ given } \sin\theta = -\frac{3}{5}$$

$$= \frac{5}{4}.$$

Answer: A ■

14. Suppose in the following figure that ABCD is a parallelogram with P a mid-point of $\overline{AB}$ and Q a mid-point of $\overline{CD}$. Then,



First note that:

i) By opposite sides of parallelogram $\overline{AD} \equiv \overline{CB}$

ii) By opposite angles of parallelogram

$$\angle ADQ \equiv \angle CBP, \text{ and}$$

iii) By opposite sides of parallelogram and given mid points P and Q, $\overline{DQ} \equiv \overline{BP}$

Therefore, by SAS congruency, $\Delta ADQ \equiv \Delta CBP$ so that B is necessarily true; and by corresponding sides of congruent triangles and symmetric property of congruency of a line segment, A is necessarily true. Since we have parallel $\overline{AP}$ and $\overline{DC}$, by definition of trapezium, APCD is a trapezium so that D is necessarily true.

In ΔCBP, $\overline{PB}$ and $\overline{PC}$ may not be congruent since $\angle PBC$ might be obtuse so that $\overline{PC}$ be the longest side. Thus, $PC \neq PB = AP$ (P is mid-point) gives; APCQ isn't a rhombus so that C is **NOT** necessarily true.

Answer: C ■

15. First $5\log_9 x - 2\log_9(x^2) - \log_4 8 = -1$ is defined if $x > 0$.

Recall that for $1 \neq a > 0$:

(i) $\log_{a^k} x = \frac{1}{k}\log_a x$, (ii) $\log_a x^m = m\log_a x$,

(iii) $\log_a a = 1$, (iv) $\log_a x + \log_a y = \log_a xy$ for $x, y > 0$.

By (i), (ii) and (iii), $\log_4 8 = \log_{2^2} 2^3 = \frac{3}{2}\log_2 2 = \frac{3}{2}$.

And, $x > 0$ implies $\log_9(x^2) = 2\log_9 x$ so that if $\log_9 x = y$, then

$$5\log_9 x - 2\log_9(x^2) - \log_4 8 = -1 \Longrightarrow 5y - 2(2y) - \frac{3}{2} = -1 \Longrightarrow y = \frac{1}{2}$$

Thus, $\log_9 x = y \Longrightarrow \log_9 x = \frac{1}{2} \Longrightarrow x = 9^{\frac{1}{2}} = 3$.

Therefore, solution set = {3}.

Answer: A ■

16. See rules under solution of Question 6. Here, $x \in \mathbb{R}$.

$$5\left(\frac{125}{8}\right)^{\frac{1}{3}x^2-\frac{2}{3}x} = 2 \Longrightarrow \left(\left(\frac{5}{2}\right)^3\right)^{\frac{1}{3}x^2-\frac{2}{3}x} = \frac{2}{5}$$

$$\Longrightarrow \left(\frac{5}{2}\right)^{3\left(\frac{1}{3}x^2-\frac{2}{3}x\right)} = \frac{2}{5}$$

$$\Longrightarrow \left(\frac{5}{2}\right)^{x^2-2x} = \frac{2}{5}$$

$$\Longrightarrow \left(\frac{2}{5}\right)^{2x-x^2} = \frac{2}{5}, \text{ by } \left(\frac{a}{b}\right)^n = \left(\frac{b}{a}\right)^{-n}$$

$$\Longrightarrow 2x - x^2 = 1$$

$$\Longrightarrow (x-1)^2 = 0 \Longrightarrow x = 1$$

Answer: B ■

17. You may use Case or Sign Chart or Graphical method.

Using case method for $(2x-3)(x+5) \leq 0$ implies

Case I: $2x-3 \leq 0 \cap x+5 \geq 0 \Longrightarrow x \leq 3/2 \cap x \geq -5 \Longrightarrow -5 \leq x \leq 3/2$

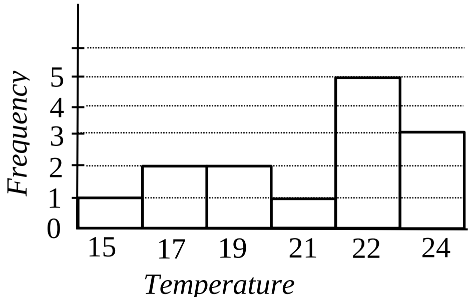
Case II: $2x-3 \geq 0 \cap x+5 \leq 0 \Longrightarrow x \geq 3/2 \cap x \leq -5 \Longrightarrow \emptyset$

Thus, Solution Set = $\emptyset \cup \{x: -5 \leq x \leq 3/2\} = [-5, 3/2]$

Or graph of f(x) = (2x-3)(x+5) is open upward and has x-intercepts 3/2 and -5 so that it is on and below x- axis for $x \in [-5, 3/2]$.

Answer: C ■

18. The histogram below shows the recorded temperature (in ^{o}C) of certain town in Ethiopia for the first 15 days of March 2023.



First prepare the following frequency distribution table.

Frequency, f_i (No. of days)	1	2	2	1	5	3
Temperature, T_5	15	17	19	21	22	24

Then, the highest frequency is $f_5 = 5$ of $T_5 = 22^{o}$C (the **Mode**) so that A is **true** but D is false. The temperature was $T_6 = 24\ ^{o}$C for $f_6 = 3$ (three days) so that B is false.

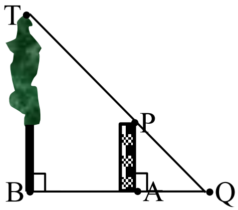
The temperature was more than $T_3 = 19\ ^{o}$C for $f_4 = 1$ (i.e., $T_4 = 21^{o}$C), for $f_5 = 5$ (i.e., $T_5 = 19^{o}$C), and for $f_6 = 3$ (i.e., $T_6 = 24^{o}$C). Thus, the temperature was more than $T_3 = 19\ ^{o}$C totally for:

$$f_4 + f_5 + f_6 = 1 + 5 + 3 = 9 \text{ days.}$$

, i.e., $\frac{f_4+f_5+f_6}{\sum_1^6 f_i} \times 100\% = \frac{9}{1+2+2+1+5+3} \times 100\% = 64.30\%$ of the days was with temperature more than 19 ^{o}C so that C is false.

Answer: A ■

19. Given a 20 m high building $\overline{AP}$ casts a shadow of AQ = 4 m, at the same time a BT = 35 m tree casts a shadow of x = BA + AQ m as shown in the figure below.



By AA Similarity, $\Delta TBQ \sim \Delta PAQ$ so that by proportionality of corresponding sides:

$$\frac{BA + AQ}{AQ} = \frac{BT}{AP} \Longrightarrow \frac{x}{4m} = \frac{35m}{20m} \Longrightarrow x = 7\text{ m}$$

Answer: D ■

20. How far means the distance so that the total distance traveled by the car is: 9 km + 4 km + 12 km = 25 km.

Answer: B ■

21. By Remainder Theorem if a polynomial p(x) is divided by x-a, then the remainder is p(a) so that if p(x) = $3x^6+5x^4-7x^3+2kx^2+3$ is divided by x+1=x-(-1), then the remainder is p(-1) so that given a remainder 4 implies:

$$p(-1) = 4 \Longrightarrow 3(-1)^6+5(-1)^4-7(-1)^3+2k(-1)^2+3 = 4$$
$$\Longrightarrow 2k = -14 \Longrightarrow k = -7.$$

Answer: A ■

22. Let V(h, k)= vertex and $F(x_F, y_F)$ = Focus. Then,

Vertex: V(h, k)=V(2, -1), and Focus: $F(x_F, y_F)$ =F(-1, -1) so that

k = -1 = y_F, h = 2, x_F = -1, and y_F = -1.

It is horizontal parabola, because y_F = k = -1.

It is open left parabola $(y - k)^2$= -4p(x - h), because -1 = $x_F < h = 2$.

So F(h-p, k) = F(2-p, -1) = F(-1, -1) ⇒ 2 - p = -1⇒ p = 3

Therefore, the parabola is, $(y + 1)^2$ = -12(x-2)

Answer: A ■

23. We need equation of hyperbola with foci F_1 = (-2, 1), F_2 = (8, 1) and the length of the conjugate axis is 2b = 8 units. Then, distance between the foci F_1 = (-2, 1) and F_2 = (8, 1) is 2c

$$\Rightarrow 2c = 8 - (-2) = 10 \Rightarrow c = 5 \Rightarrow c^2 = 25$$

Length of conjugate axis: 8 ⇒ 2b = 8 ⇒ b = 4 ⇒ $b^2 = 16$

Center = $(\frac{-2+8}{2}, \frac{1+1}{2})$ = (3, 1) = (h, k) ←mid-point between foci , and

$$a^2 = c^2 - b^2 = 25 - 16 = 9.$$

It is a horizontal hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, because $y_{F_1} = 1 = y_{F_2}$

Therefore, the hyperbola is $\frac{(x-3)^2}{9} - \frac{(y-1)^2}{16} = 1$.

Answer: No Answer is given ■

24. Given proposition p with p ≡T and any proposition q

A. $(\neg p \wedge q) \Rightarrow q \equiv (\neg T \wedge q) \Rightarrow q$, given

$\equiv (F \wedge q) \Rightarrow q$, rule of negation

$\equiv F \Rightarrow q$, rule of conjunction

$\equiv T$, rule of implication

Similarly, we can show that the (B), (C), and (D) are F.

Answer: A ■

25. A. Since |x| = x for x∈ (0, ∞) we have f(x) = x – 1 which is a linear function so that f is one-to-one function.

B. Not one-to-one, because for instance f(1) = f(6) = 5 but 1 ≠ 6.

C. Not one-to-one, because y may have two or more children.

D. Not one-to-one, because for instance f(-1) = f(1) = 0 but -1 ≠ 1.

Answer: A ■

26. Given $\frac{x^2+14}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\Rightarrow \frac{x^2+14}{(x+2)(x-1)^2} = \frac{A(x-1)^2+B(x-1)(x+2)+C(x+2)}{(x+2)(x-1)^2}$

$\Rightarrow \frac{x^2+14}{(x+2)(x-1)^2} = \frac{(A+B)x^2+(-2A+B+C)x+(A-2B+2C)}{(x+2)(x^2+1)}$

⇒ x^2 +14 = (A+B)x^2 +(-2A+B+C)x +(A-2B+2C)

$$\Rightarrow \begin{cases} A + B = 1 & \ldots\ldots\ldots (1) \\ -2A + B + C = 0 & \ldots\ldots\ldots (2) \\ A - 2B + 2C = 14 & \ldots\ldots\ldots (3) \end{cases}$$

$\Rightarrow \begin{cases} -3A + C = -1 \\ 3A + 2C = 16 \end{cases}$ ← replacing B =1−A from (1) in to (2) & (3)

⇒ A = 2 and C= 5 so that by B = 1-A, B = -1

Answer: C ■

27. Given a rational function f(x) = $\frac{x^3-3x}{x^2-6}$. Then

A. y-intercept is y = f(0) = $\frac{0^3-3(0)}{0^2-6}$ = 0.

B & C. f(x) = $\frac{x^3-3x}{x^2-6}$ is odd because it has a form $\frac{\text{odd}}{\text{even}}$ so that graph of f is symmetrical with respect to the origin.

D. x-intercept is the value of x for which x^3-3x = 0, i.e., x = 0, $\pm\sqrt{3}$.

Answer: C ■

28. Let θ be angle between the lines ℓ_1:y =2x+3 with slope m_1 = 2 and ℓ_2:y =1-3x with slope m_1 = -3. Then, 0 ≤ θ < π and

$$\tan\theta = \frac{m_2-m_1}{1+m_1m_2} = \frac{-3-2}{1+2(-3)} = 1 \text{ so that } \theta = \tan^{-1}1 = \pi/4.$$

Answer: B ■

29. Given position vectors **u** = (-1, 6) and **v** = (2, 4). Then

4**u**-3**v** = 4(-1, 6)-3(2, 4) = (-4, 24) - (6, 12)

= (-4-6, 24-12) = (-10, 12).

Answer: B ■

30. Given a function f(x) = 3sin(4x) has period p = $\frac{2\pi}{4} = \frac{\pi}{2}$. And, the function y = sin x lies below x-axis on (π, 2π); dividing bounds of this interval by 4, f lies below x-axis on (π/4, π/2) so that A is **true**.

Period p = $\frac{\pi}{2}$ implies graph of f completes one cycle on interval [0, $\frac{\pi}{2}$] so that B & D are false.

The function y = sin x rises up on the interval [0, π/2]; dividing bounds of this interval by 4, f rises up on [0, π/8] but [0, π/4]$\not\subseteq$ [0, π/8] so that C is false.

Answer: A ■

31. Given a translation T that takes a point (-1, 1) to point (3, 2).

The translation vector is: $\vec{U}_T$ = (3, 2) – (-1, 1) = (4, 1).

Translating a circle means translating the center and use the same radius. x^2+y^2-2x+4y+1=0 has center C = (-(-2)/2, -4/2) = (1, -2).

So, T(1, -2) = $\vec{U}_T$ + (1, -2) = (5, -1); Center of image circle

And, for radius r, $r^2 = \frac{1}{4}((-2)^2+4^2-4(1))$ = 4.

Thus, the image circle is $(x-5)^2+(y+1)^2$ = 4.

Answer: D ■

32. Let A be a square matrix of order n and let B be a matrix obtained by multiplying one row of A by 2 and C be a matrix found by adding one row of A to another row of A.

If a row/a column is multiplied by a scalar r, then the new determinant equals r times the old determinant; and the determinant remains the same if a multiple of a row/column is added to another row/column. Thus,

$$\det(B) = 2\det(A) = 20 \text{ and } \det(C) = \det(A) = 10$$

Answer: B ■

33. You may check that $\Delta = 0$, $\Delta_x = 0$, $\Delta_y = 0$ and $\Delta_z = 0$ so that from Cramer's rule we conclude that the system has infinitely many solution. And it is better to apply row/column operation method.

The augmented matrix is $\left(\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & -1 & 1 & -1 \end{array}\right)$

Pivoting R_2 and R_3 using R_1 to make $a_{21} = 0 = a_{31}$ as:

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & -1 & 1 & -1 \end{array}\right) \xrightarrow[R_3 \to R_3 - \frac{1}{2}R_1]{R_2 \to R_2 - \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 3/2 & 3/2 & 9/2 \\ 0 & -1/2 & -1/2 & -3/2 \end{array}\right)$$

After rescaling as $R_2 \longrightarrow 2R_2$ and $R_3 \longrightarrow -2R_3$: Pivoting R_3 using R_2 to make $a_{32} = 0$ as:

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 3 & 3 & 9 \\ 0 & 1 & 1 & 3 \end{array}\right) \xrightarrow{R_3 \to R_3 + \frac{1}{3}R_2} \left(\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 3 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Rescaling as $R_2 \longrightarrow \frac{1}{3}R_2$: $\left(\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Therefore, the system is **Dependent**.

From R_1 and R_2 we get two equations in three unknowns as

$$\begin{cases} 2x - y + 3z = 1 \ldots\ldots (1) \\ \quad y + z = 3 \ldots\ldots (2) \end{cases}$$

Let $z = k$ for $k \in \mathbb{R}$. Then (2) gives $y = 3 - k$.

Then, substituting $z = k$ and $y = 3 - k$ in to (1) gives $x = 2 - 2k$

Therefore, the solution set is: $\{(2-2k, 3-k, k) \mid k \in \mathbb{R}\}$, and we say the system has infinitely many solution.

Answer: D ■

34. $$\begin{aligned} z_1 - (z_2 - z_3) &= z_1 + (-1)(z_2 - z_3) \\ &= z_1 + (-1)(z_2) - (-1)z_3 \\ &= z_1 - z_2 + z_3 \neq (z_1 - z_2) - z_3 \text{ so that A is False} \end{aligned}$$

B, C, and, D are commutativity, associativity, and distributivity properties which hold in real number system so that they also hold in set of complex numbers.

Answer: A ■

35. In the Argand plane, the complex numbers $a + bi$ corresponds to the point (a, b) so that $-3 - i$ corresponds to the point (-3, -1).

Answer: C ■

36. A group of students, 5 from grade 10, 8 from grade 11 and 2 from grade 12 to be formed. If a student is chosen randomly from this group, the probability that the chosen student is either from grade 10 or from grade 12 is:

$$p(\text{G10 or G12}) = \frac{n(\text{G10 and G12})}{n(\text{all students})} = \frac{7}{15}$$

Answer: B ■

37. Use matrices multiplication rule and definition of equal matrices.

$$\begin{pmatrix} 2 & -1 \\ a & 4 \end{pmatrix}\begin{pmatrix} 1 & b \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 18 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2(1) + (-1)(-1) & 2b + (-1)3 \\ a(1) + 4(-1) & ab + 4(3) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 18 \end{pmatrix}, \text{ multiplication}$$

$$\Rightarrow \begin{pmatrix} 3 & 2b - 3 \\ a - 4 & ab + 12 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 18 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3 = 3 \\ 2b - 3 = 1 \\ a - 4 = -1 \\ ab + 12 = 18 \end{cases}, \quad \text{by definition of equal matrices}$$

$$\Rightarrow \begin{cases} 3 = 3 \\ b = 2 \\ a = 3 \\ ab = 6 \end{cases}, \quad \text{by definition of equal matrices}$$

Since $ab = 6$ is also satisfied, $a = 3$, and $b = 2$.

Answer: B ■

38. A^T is the transpose of A that is found by making the i^{th} row in to i^{th} column, i.e., $A^T = (a_{ji})$ for A$= (a_{ij})$ so that

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ -3 & 6 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 0 & -3 \\ -1 & 2 & 6 \end{pmatrix}$$

Answer: B ■

39. Cumulative frequency (commonly the less than cumulative frequency) of a class (except the first class whose cumulative frequency and frequency are equal) is the sum of frequencies of all the classes before it and its frequency.

Temperature (in ^oC)	10-14	15-19	20-24	25-29	30-34
Frequency (Number of days)	2	9	11	5	3
Cumulative frequencies	2	11	22	**27**	30

Hence, as shown in bold the cumulative frequency of the fourth class is 27 (i.e., 2+9+11+5).

Answer: A ■

40. Given a universal set ℕ, the proposition $(\exists_x)(\exists_y)(x-y = y-x)$ is True just by taking $x = y$ so that B is True.

To show the falseness of A, C, and D we use disprove by counter example. Take $x = 1$ and $y = 2$. Then,

$x \neq y \Rightarrow x > y \equiv T \Rightarrow F \equiv F$ so that A is False.

No value of $y \in \mathbb{N}$ such that $xy < x$ if $x = 1$ so that C is False.

No value of $y \in \mathbb{N}$ such that $x - y = 5$ if $x = 1$ so that D is False.

Answer: B ■

41. The following is distribution of the weight of 40 students in a certain class.

Weight (in Kg)	40-48	49-57	58-66	67-75	76-84
Frequency, f_i	8	18	7	5	2

Then, to get the median weight in Kg, first calculate class boundaries for correction factor 0.5 & (Less than) cumulative frequencies as follows.

Class boundaries	39.5-48.5	48.5-57.5	57.5-66.5	66.5-75.5	75.5-84.5
Frequency, f_i	8	18	7	5	2
Cumulative frequencies, Cf_i	8	26	33	38	40

Then, $n = \sum_{i=1}^{5} f_i = 40 \Rightarrow Cf_i \geq \frac{n}{2} = 20$ so that $Cf_i \geq 20$ for first time in 2^{nd} class, i.e., the median class is the 2^{nd} class with class boundaries 48.5 - 57.5 and lower bound $B_L = 48.5$, cumulative frequency before the median class $Cf_b = 8$, median class frequency f_c =18, and class size $i =$ 9 so that the median is:

$$\tilde{x} = B_L + \left[\frac{\frac{n}{2} - Cf_b}{f_c}\right] i = 48.5 + \left[\frac{20-8}{18}\right] 9 = 54.5 \text{ Kg}$$

Answer: B ■

42. Recall that: Two events are said to be ***mutually exclusive***, if both cannot occur simultaneously. If a fair die is rolled once, then the sample space is S = {1, 2, 3, 4, 5, 6} and among events:

E_1 = an even number occurs = {2, 4, 6};

E_2 = a prime number occurs = {2, 3, 5};

E_3 = a multiple of 3 occurs = {3, 6};

E_4 = a multiple of 5 occurs = {5}.

, the pairs of mutually exclusive events are: E_1 and E_4; and E_3 and E_4, because their intersection is empty set.

Answer: A ■

43. This question must be specified for a polynomial function; otherwise the question is wrong. For instance: $f(x) = 2^x - 1$ has zero at 0, but x is not factor of f.

Answer: Wrong Item ■

44. Mean Value Theorem: If a function f is continuous on the interval [a, b] and differentiable on (a, b), then there exists (at least one) c $\in$(a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

$f(x)=3x^2-5x+1$ is continuous on [0, 1] and differentiable on (0, 1) so that by Mean Value Theorem, there exists c $\in$(0, 1) such that

$$f'(c)=\frac{f(1)-f(0)}{1-0} \Longrightarrow 6c-5=\frac{-1-1}{1} \Longrightarrow c=1/2.$$

Answer: B ■

45. Volume of a cone with height h and base radius r is $V=\frac{1}{3}\pi r^2 h$. Given h =10 cm and r = 5 cm implies

$$\frac{r}{h}=\frac{5}{10} \Longrightarrow r=\frac{1}{2}h \text{ or } h=2r\,.$$

Then, $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi(\frac{1}{2}h)^2 h=\frac{1}{12}\pi h^3$ so that by chain rule

$$\frac{dV}{dt}=\frac{1}{4}\pi h^2\frac{dh}{dt} \quad \Longrightarrow \frac{dh}{dt}=\frac{4}{\pi h^2}\frac{dV}{dt}$$

By given at the moment $\frac{dV}{dt}=6\pi cm^3/sec$, the rate at which the water level rising when the water reaches h = 2 cm deep is

$$\frac{dh}{dt}=\frac{4}{\pi h^2}\frac{dV}{dt}=\frac{4}{\pi(2cm)^2}(6\pi cm^3/sec)=6\text{ cm/sec}.$$

Answer: A ■

46. The graphs of $f(x)=\frac{1}{\sqrt{x}}$ and $g(x)=2-\frac{1}{6}x^3$ have tangent lines of the same slope at x where $f'(x)=g'(x)$. Thus,

$$f'(x)=g'(x) \Longrightarrow -\frac{1}{2}x^{-\frac{3}{2}}=-\frac{1}{2}x^2 \Longrightarrow x=1 \text{ } (\textbf{WHY?})$$

Answer: A ■

47. Mechanical (also called physical) meaning of derivative is its interpretation as an instantaneous rate of change.

Answer: D ■

48. Using sum rule of derivative:

$$f(x)=5x^3+2x-\sin x+e^x \Longrightarrow f'(x)=15x^2+2-\cos x+e^x$$

$$\text{Thus, } f'(0)=15(0)^2+2-\cos 0+e^0=2.$$

Answer: B ■

49. Given h(x) = g(f(3x+1)), $f'(1)=f(1)=2$, and $g'(2)=1$. Then, using chain rule

$$h'(x) = (g(f(3x+1)))' = g'(f(3x+1))\, f'(3x+1)(3x+1)'$$
$$= 3g'(f(3x+1))\, f'(3x+1)$$

Thus, $h'(0) = 3g'(f(3(0)+1))f'(3(0)+1)$
$= 3g'(f(1))f'(1)$
$= 3g'(2)(2)$, given $f'(1) = f(1) = 2$
$= 3(1)(2)$, given $g'(2) = 1$
$= 6.$

Answer: D ■

50. Validity of an argument can be checked using either of Formal proof, Truth table, Tautology or Rules of Inference.

A. [Using Formal proof]

Given argument: $p, p \Rightarrow q, r \vee \neg q \vdash p \wedge \neg r$

(1) $p \equiv T$ and $p \Rightarrow q \equiv T$ …premise
(2) $q \equiv T$ …(1) and rule of "$\Rightarrow$"
(3) $\neg q \equiv F$ …(2) and rule of "$\neg$"
(4) $r \vee \neg q \equiv T$ …premise
(5) $r \equiv T$ …(3) and rule of "$\vee$" on (4)
(6) $\neg r \equiv F$ …(5) and rule of "$\neg$"
(7) $p \wedge \neg r \equiv F$ …(6) and rule of "$\wedge$"

Therefore, the argument form is invalid.

B. [Using Truth table]

Given argument: $p \vee q, p \Rightarrow q \vdash \neg q$

p	q	¬q	p ∨ q	p ⇒ q
T	T	F	T	T
T	F	T	T	F
F	T	F	T	T
F	F	T	F	T

In first row the premises are T but the conclusion is F so that the argument form is invalid.

C. [Using Formal proof]

Given argument: $p \Rightarrow q, \neg r \wedge q \vdash p \vee r$

(1) $\neg r \wedge q \equiv T$ …premise
(2) $\neg r \equiv T$ and $q \equiv T$ …(1) and rule of "$\wedge$"

(3) r ≡ F …(2) and rule of "¬"
(4) p ⇒ q ≡ T …premise
(5) p ≡ T or p ≡ F …(2) and (4) with rule of "⇒"
(6) p ∨ r ≡ T or p ∨ r ≡ F …by (3) and (5) with rule of "∨"
Therefore, the argument form is invalid.

D. [Using Formal proof]

Given argument: p ⇒ q, q, r ⇒ p ⊢ r
(1) q ≡ T and p ⇒ q …premise
(2) p ≡ T or p ≡ F …(1) and rule of "⇒"
(3) r ⇒ p ≡ T … premise
(4) If p ≡ T, then r ≡ T or r ≡ F…(2) and rule of "⇒"
(5) If p ≡ F, then r ≡ F…(2) and rule of "⇒"
Therefore, the argument form is invalid.

Answer: No Answer is given ■

51. The mid-point of line segment joining (-1, -3, 3) and (-1, 5, -7) in space is $(\frac{-1+(-1)}{2}, \frac{-3+5}{2}, \frac{3+(-7)}{2})$ = (-1, 1, -2).

Answer: D ■

52. Principle of Mathematical Induction (PMI) is applicable on a proposition with variables' value set of Natural Numbers.

A. $n = 0 \Longrightarrow 0! \geq 2^0$ which is false so that no use of PMI.

B. $n = 0 \Longrightarrow 2^{0+1} - 1 = 1$ (True). Hence, first note that

$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for $n \geq 0$ is similar to

$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$ for $n \geq 1$

$n = 1 \Longrightarrow 2^1 - 1 = 1$ (True).

Assume it holds for $n = k > 1$ and $k \in \mathbb{N}$

, i.e., $1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$ is true. (IH)

Then, for $n = k + 1$,

$$1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^k = 2^k - 1 + 2^k, \text{ by IH}$$
$$= 2 \times 2^k - 1$$
$$= 2^{k+1} - 1$$

Thus, the formula holds also for $n = k + 1$ so that this assertion is true applying the Principle of Mathematical Induction.

C. $n = 1 \Longrightarrow 1^2 \neq 3$ so that no use of PMI.

D. $n = 2 \Longrightarrow \frac{2(2+1)}{2} + 1 = 4 \neq 2$ so that no use of PMI.

Answer: B ■

53. Let F be the anti-derivative of f. For constant of integration *c*

A. $F' = f(x)$ so that A is true.

B. $\frac{d}{dx}\int F(x)dx = F(x)$ for every function F so that B is **NOT** true.

C. $\int \frac{d}{dx} f(x)dx = f(x) + c$ so that C is true.

D. $\int f(x)dx = F(x) + c$ so that D is **NOT** true.

Answer: B and D ■

54. $\int 3\cos 3x\, dx = 3\int \cos 3x\, dx$

Let $u = 3x$. Then, $dx = \frac{1}{3}du$ so that

$$3\int \cos 3x\, dx = 3\int \cos u \frac{1}{3} du$$
$$= \int \cos u\, du = \sin u + C$$
$$= \sin 3x + C \text{ by } u = 3x$$

Answer: A ■

55. $f(x) = x^2 - 4$ crosses x-axis at $x = \pm 2$ and $f \leq 0$ on [-2, 2] so that the area of the region bounded by the graph of f and x-axis, using the evenness of f is

$$A = 2\int_0^2 -f(x)dx = 2\int_0^2 (4 - x^2)dx$$
$$= 2(4x - \frac{x^3}{3})_0^2$$
$$= 2[(4(2) - \frac{2^3}{3}) - (4(0) - \frac{0^3}{3})] = \frac{32}{3}$$

Answer: C ■

56. Let f be a continuous function on [0, 1] and let F be an anti-derivative of f with F(1) = -11 and F(0) = 11. Then

$$\int_0^1 f(x)dx = F(x)]_0^1 = F(1) - F(0) = -11 - 11 = -22$$

Answer: B ■

57. On xz-plane $y = 0$ so that the set describing equation of the xz-plane is $\{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } y = 0\}$.

Answer: A ■

58. If the sequence $\{a_n\}$ converges to 3 and the sequence $\{b_n\}$ converges to -1, then by convergence theorem, $\{a_n - b_n\}$ converges to 3 – (-1) = 4.

Answer: A ■

59. $f(x) = \frac{x^2-1}{x^2+1}$ is defined and continuous on $\mathbb{R}$ so that

$$\lim_{x\to 1}\frac{x^2-1}{x^2+1} = \frac{1^2-1}{1^2+1} = 0.$$

Answer: C ■

60. $f(x) = \begin{cases}\alpha x + 1, & x \geq 1\\ x^2 - 4, & x < 1\end{cases}$ is continuous at 1 if

i) $1\in$ domain of f. Of course assume $f(1) = \alpha + 1 \in \mathbb{R}$.

ii) $\lim_{x\to 1} f(x)$ must exist so that side limits exist and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \implies \lim_{x\to 1^-}(x^2 - 4) = \lim_{x\to 1^+}(\alpha x + 1)$$
$$\implies 1^2 - 4 = \alpha(1)+1$$
$$\implies \alpha = -4$$

Answer: A ■

61. First length of the interval $[-\frac{\pi}{4}, \frac{11\pi}{4}]$ is $\ell = \frac{11\pi}{4} - (-\frac{\pi}{4}) = 3\pi$ which is a period of the function which completes one cycle.

Period of $f(x) = 2\sin(\frac{2}{3}x + \frac{\pi}{6})$ is: $p = \frac{2\pi}{2/3} = 3\pi$ (see solution of Q62).

Thus, $\ell = p$ implies $f(x) = 2\sin(\frac{2}{3}x + \frac{\pi}{6})$ has a graph that completes one full cycle on $[-\frac{\pi}{4}, \frac{11\pi}{4}]$.

Answer: D ■

62. Graph of $f(x) = c + a\cos(kx + b)$ has:

$$\text{Period: } p = \frac{2\pi}{k} \quad \text{and Phase shift: } -\frac{b}{k}.$$

Thus, for $f(x) = -\frac{1}{3}\cos(\frac{3}{2}x - \frac{\pi}{3})$ we have $k = \frac{3}{2}$, $b = -\frac{\pi}{3}$ so that

$$\text{Period: } p = \frac{2\pi}{3/2} = \frac{4\pi}{3} \quad \text{and Phase shift: } -\frac{b}{k} = -\frac{-\pi/3}{3/2} = \frac{2\pi}{9}.$$

Answer: B ■

63. Since n starts from 0, the fifth term is at $n = 4$

, i.e., $2^4 + 4(4) - 5 = 27$.

Answer: B ■

64. By long division $\frac{2n+1}{n+1} = 2 - \frac{1}{n+1}$

Then, $n \geq 1 \Longrightarrow n + 1 \geq 2 \Longrightarrow 0 < \frac{1}{n+1} \leq \frac{1}{2}$, reciprocal

$\Longrightarrow -\frac{1}{2} \leq -\frac{1}{n+1} < 0$, multiplying by -1

$\Longrightarrow \frac{3}{2} \leq 2 - \frac{1}{n+1} < 2$, adding 2

$\Longrightarrow \frac{3}{2} \leq \frac{2n+1}{n+1} < 2$

Thus, every real number greater than or equal to 2 is an upper bound.

Or simply by listing some elements of $\frac{2n+1}{n+1} = 2 - \frac{1}{n+1}$ as:

$$2-\frac{1}{2}, 2-\frac{1}{3}, 2-\frac{1}{4}, \ldots, 2-0$$

, we can see it is going to 2 increasingly so that every real number greater than or equal to 2 is an upper bound.

Answer: C ■

65. Given arithmetic sequence $\{a_n\}$ with $a_6 = 35$, and $a_{11} = 75$. Then, the common difference is: $d = \frac{a_{11}-a_6}{11-6} = \frac{75-35}{5} = 8$.

And, using $a_n = a_1 + (n-1)d$ for $n = 6$:

$$a_1 = 35 - (6-1)8 = -5.$$

Thus, by $S_n = \frac{n}{2}(2a_1 + (n-1)d)$ the sum of the first 31 terms is

$$S_{31} = \frac{31}{2}(2(-5)+(31-1)8) = 3565$$

Answer: No Answer is given ■