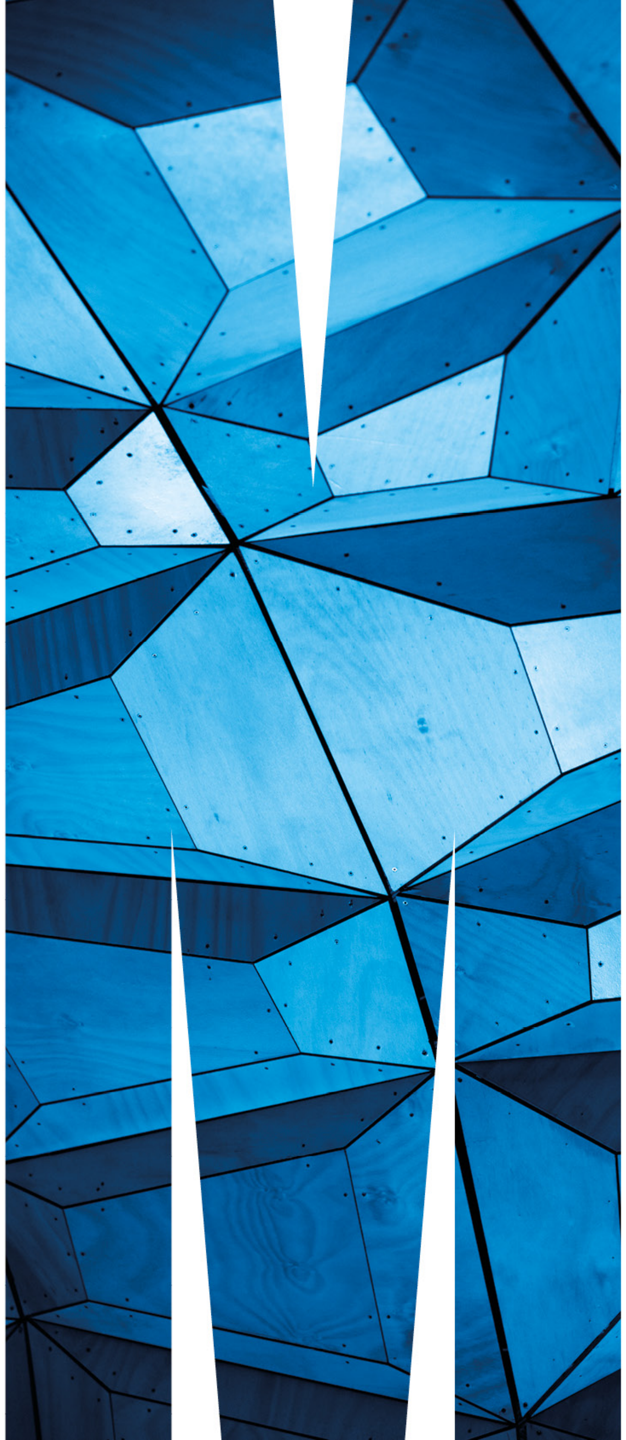
MONASH University

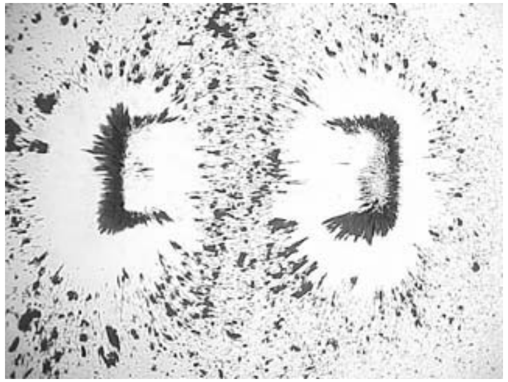
Magnetostatics

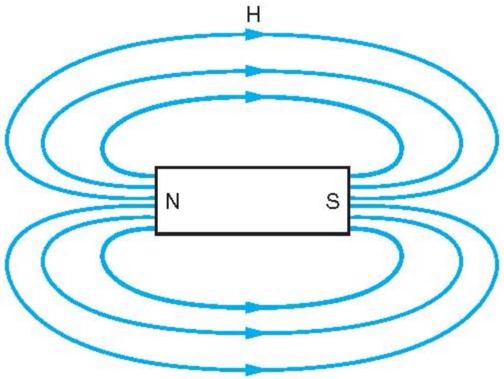
GROUP OF EIGHT AUSTRALIA



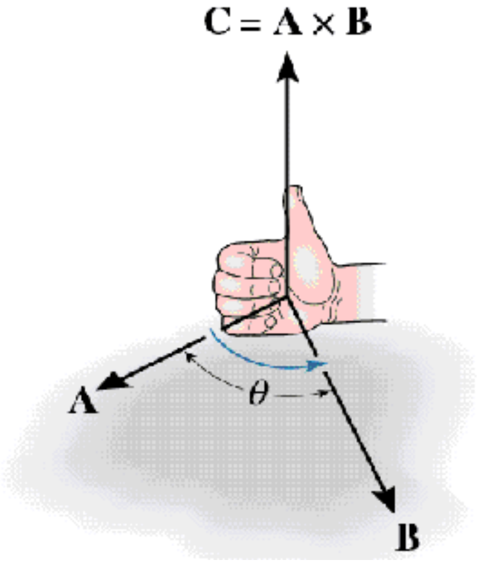
Magnetic field

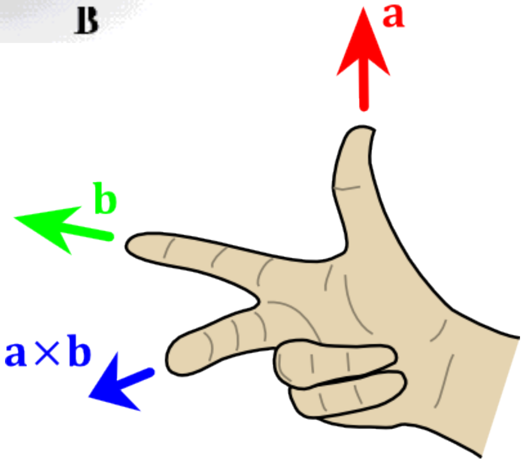
- Magnetic field can be caused by
  - permanent magnet
  - an electric field changing linearly with time
  - a direct current
- Our focus will mostly be on direct current for this section





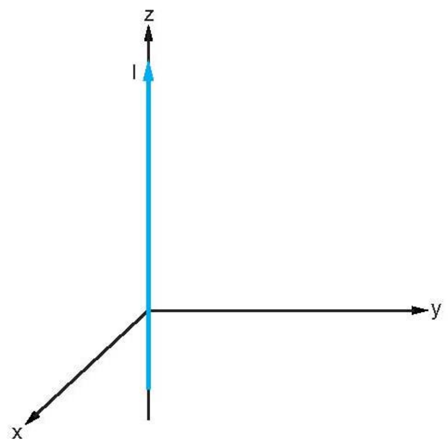
Cross product

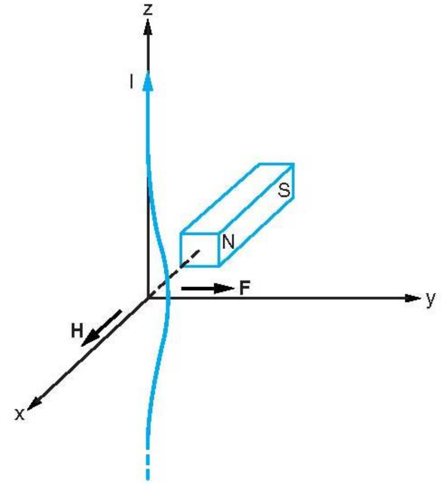




$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}|\,|\mathbf{B}| \sin\theta_{AB}$$

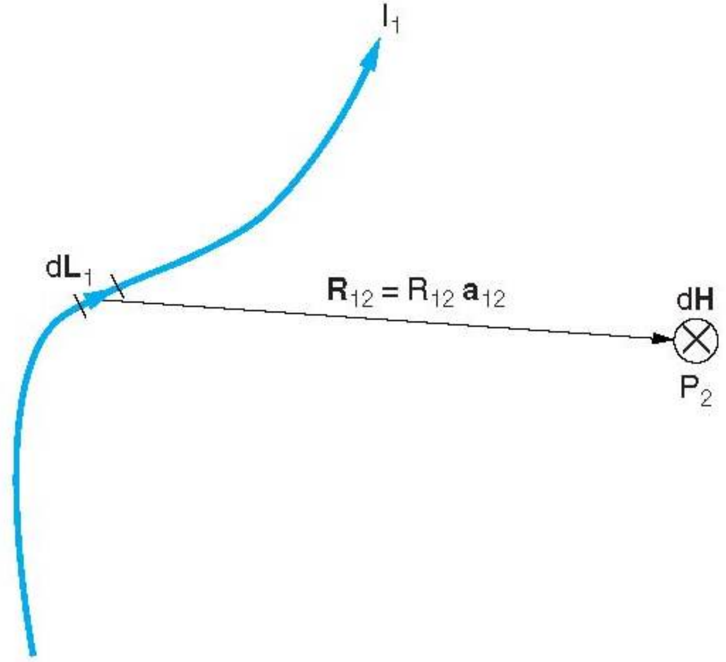
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$





Biot-Savart's Law

- At point P, the **differential magnetic field intensity** (*dH*) produced by the differential current element is **proportional** to the product *Idl* and the **sine of the angle** between the element and the line joining *P* to the element and is **inversely proportional** to the **square of the distance** *R* between *P* and the element.



Biot-Savart's Law

- The definition can also be written as:

$$\boxed{d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}}$$

- Sum of contributions from each *current* element

$$\boxed{\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

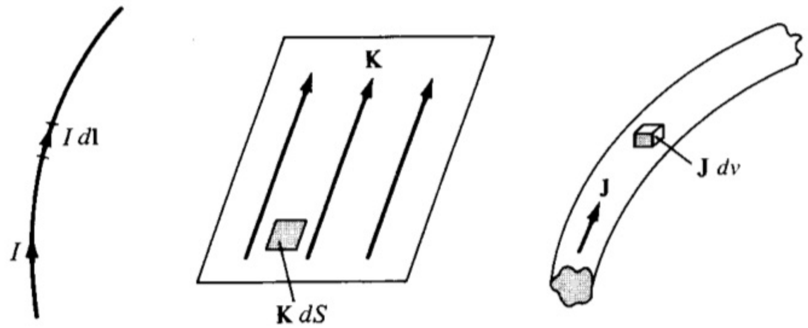
Biot-Savart's Law

- Similar to electric charge distributions, there are different current distributions

$$I\,d\mathbf{L} = \mathbf{K}\,dS = \mathbf{J}\,dv$$

K – surface current density (A/m)

J – volume current density (A/m^2)



$$\mathbf{H} = \int_L \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(line current)}$$

$$\mathbf{H} = \int_S \frac{\mathbf{K}\,dS \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(surface current)}$$

$$\mathbf{H} = \int_v \frac{\mathbf{J}\,dv \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(volume current)}$$

Magnetic field of an infinite length conductor

- Extending along the z axis in a cylindrical coordinate system from $-\infty$ to ∞

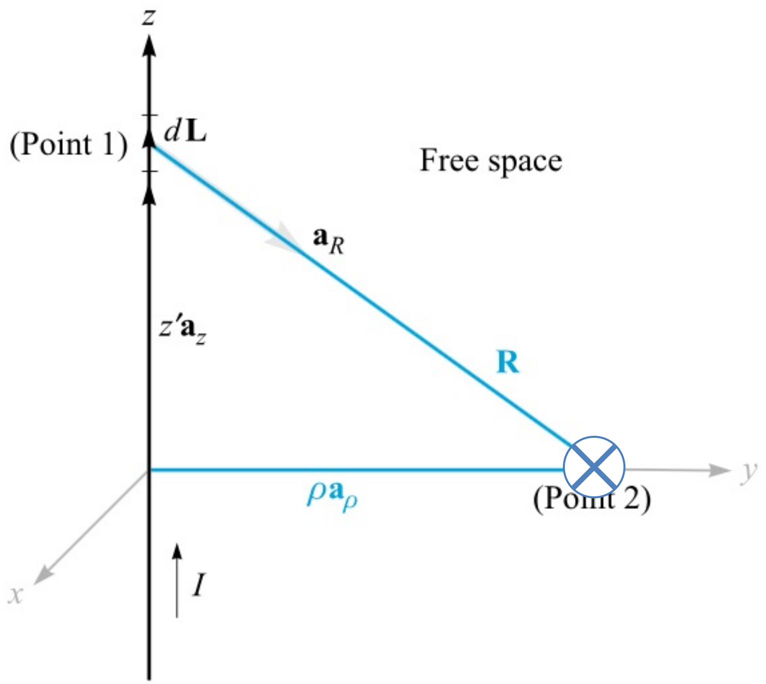
$$\mathbf{R}_{12} = \mathbf{r} - \mathbf{r}' = \rho \mathbf{a}_\rho - z' \mathbf{a}_z$$

$$\mathbf{a}_{R12} = \frac{\rho \mathbf{a}_\rho - z' \mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

$$d\mathbf{H}_2 = \frac{I\, dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$\mathbf{H}_2 = \int_{-\infty}^{\infty} \frac{I\, dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

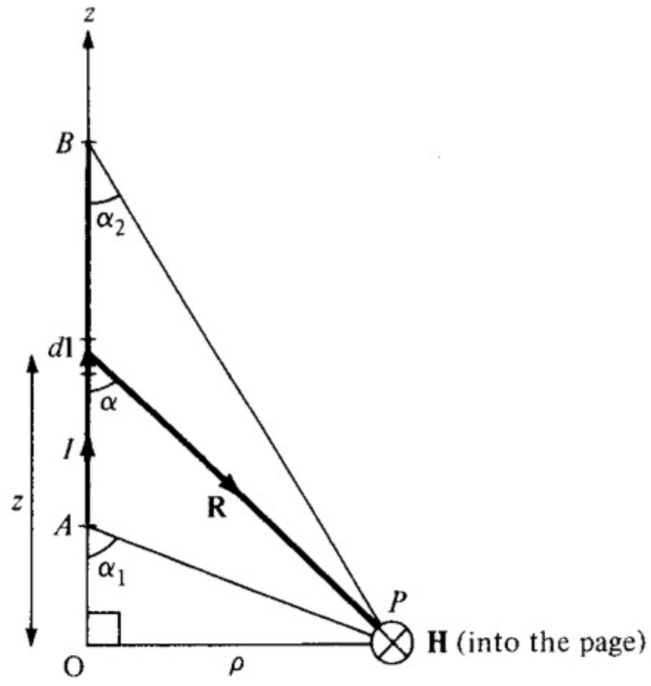
$$\boxed{\mathbf{H}_2 = \frac{I}{2\pi\rho} \mathbf{a}_\phi}$$



Magnetic field of a finite length conductor

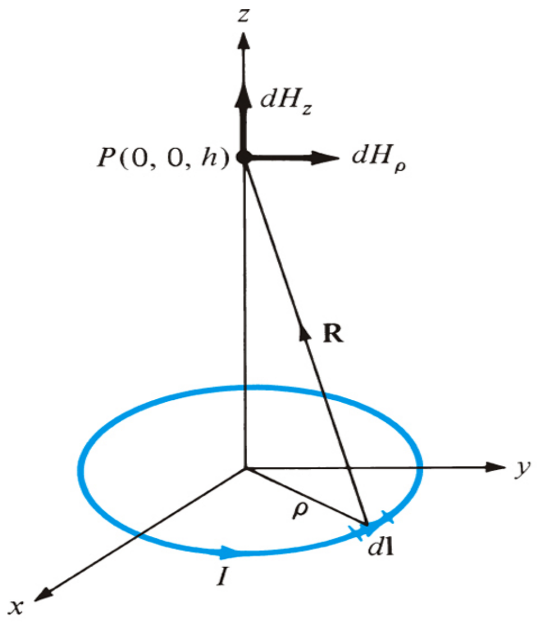
For a finite length conductor,

$$\mathbf{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi$$

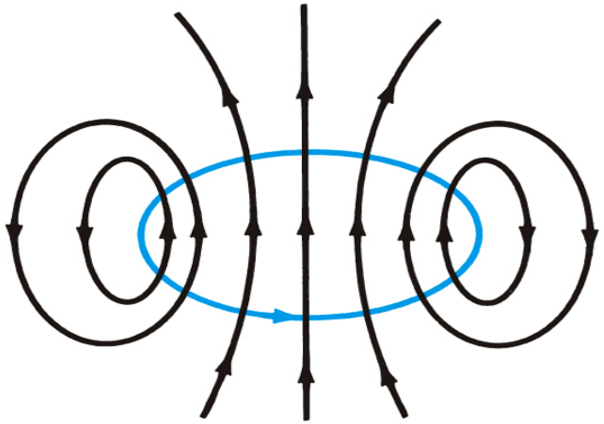


Example

A circular loop located on $x^2 + y^2 = 9$, $z = 0$ carries a direct current of 10 A along $\mathbf{a}_\phi$. Determine $\mathbf{H}$ at (0, 0, 4) and (0, 0, −4).



(a)



(b)

Example

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho\mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}}(\rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z) = dH_\rho\, \mathbf{a}_\rho + dH_z\, \mathbf{a}_z$$

Cancels out by symmetry

a_z component adds up

$$\mathbf{H} = \int dH_z\, \mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2\, d\phi\, \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi\mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2\mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10\,A$, $\rho = 3$, $h = 4$ gives

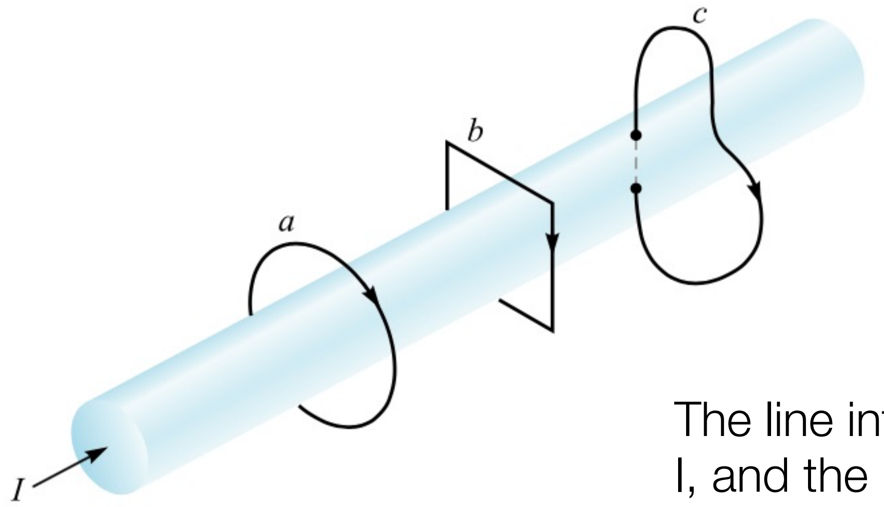
$$\mathbf{H}(0, 0, 4) = \frac{10\,(3)^2\, \mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z \text{ A/m}$$

(b) Notice from $d\mathbf{l} \times \mathbf{R}$ above that if h is replaced by $-h$, the z-component of $d\mathbf{H}$ remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence

$$\mathbf{H}(0, 0, -4) = \mathbf{H}(0, 0, 4) = 0.36\mathbf{a}_z \text{ A/m}$$

Ampere's Circuital Law

- The line integral of **H** around a closed path is the same as the net current I_{enc} enclosed by the path



$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

The line integral of H about the closed paths *a* and *b* is equal to I, and the integral around path *c* is less than I, since the entire current is not enclosed by the path.

Ampere's Circuital Law

- Gauss's law involves finding the total charge enclosed by a closed surface.
- Ampere's circuital law involves finding the total current enclosed by a closed path
- Symmetry is important, as it makes our lives easier
- The secret to calculating magnetic fields is to choose a path that makes the integration simple.

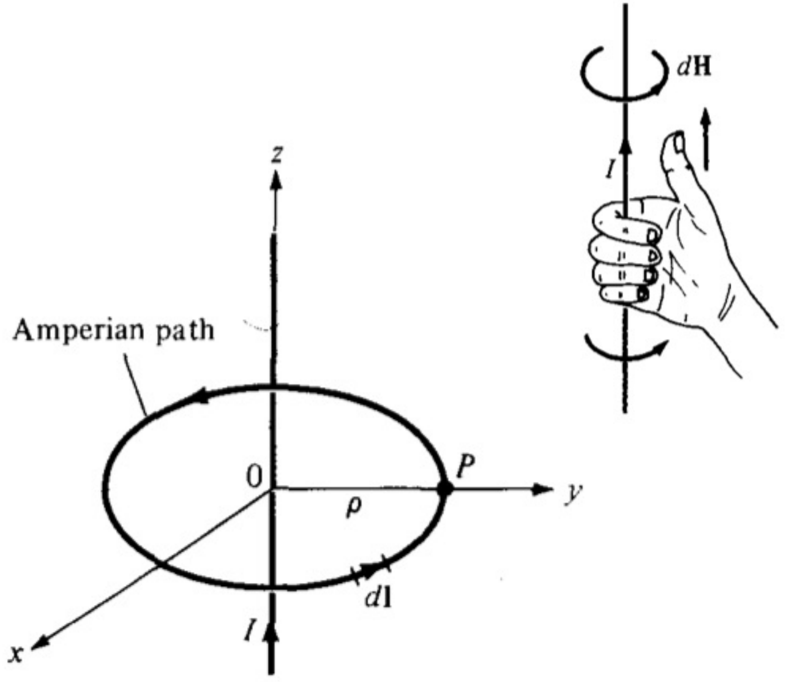
Applications of Ampere's Circuit Law

H in ϕ direction only

Differential element of L

$$I = \int H_\phi \mathbf{a}_\phi \cdot \rho \, d\phi \, \mathbf{a}_\phi = H_\phi \int \rho \, d\phi = H_\phi \cdot 2\pi\rho$$

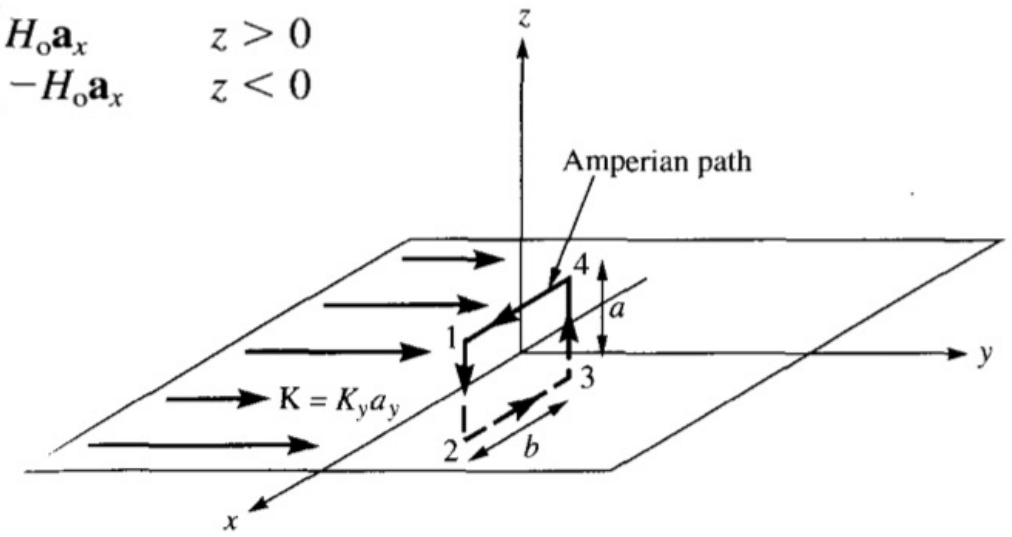
$$\boxed{H_\phi = \frac{I}{2\pi\rho}}$$



Applications of Ampere's Circuit Law

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b$$

$$\mathbf{H} = \begin{cases} H_o \mathbf{a}_x & z > 0 \\ -H_o \mathbf{a}_x & z < 0 \end{cases}$$



$$\oint \mathbf{H} \cdot d\mathbf{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathbf{H} \cdot d\mathbf{l}$$
$$= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b)$$
$$= 2H_o b$$

$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases}$$

a_N - unit vector normal (outward) to the current sheet

$$\boxed{\mathbf{H} = \frac{1}{2} \mathbf{K} \times \mathbf{a}_n}$$

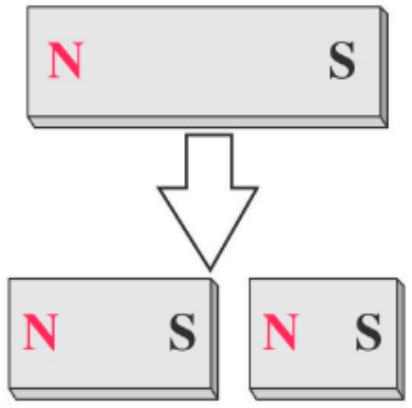
Magnetic Poles Vs Electric Charge

- We observed monopoles in electricity. A (+) or (-) alone was stable, and field lines could be drawn around it.

- Magnets cannot exist as monopoles. If you break a bar magnet between N and S poles, you get two smaller magnets, each with its own N and S pole.

In contrast to electric charges, magnetic poles always come in pairs and can't be isolated.

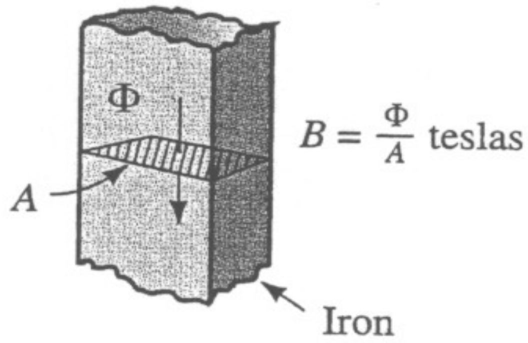
Breaking a magnet in two ...



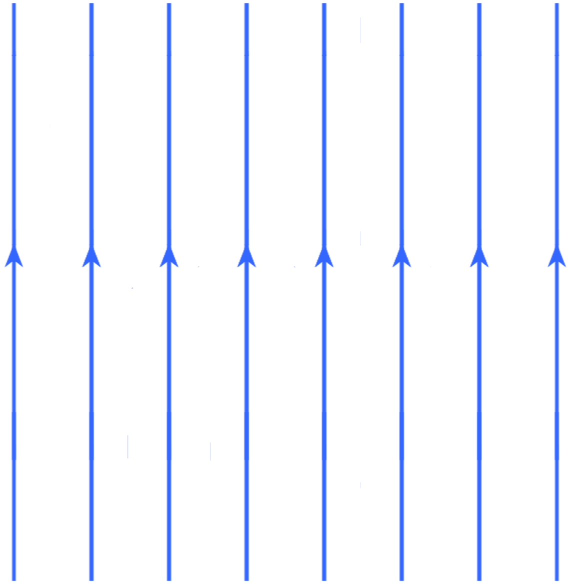
... yields two magnets, not two isolated poles.

Magnetic Flux

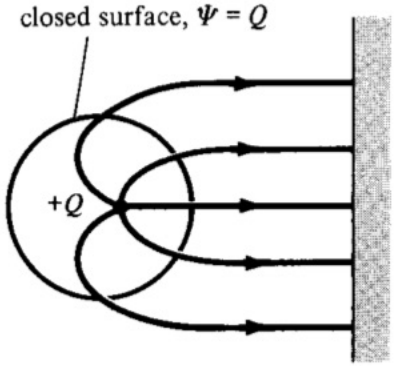
- Density of flux (or field) lines determines forces on magnetic poles
- Direction of flux indicates direction of force on a North pole



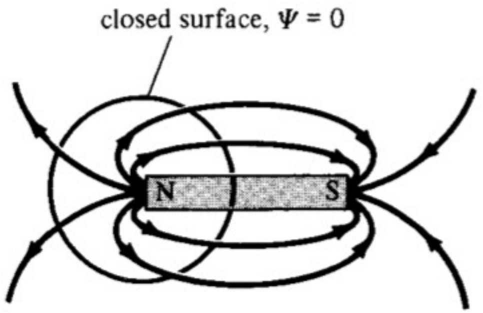
$$B = \phi / A$$



Magnetic Flux Density



Electric	Magnetic
$\mathbf{D} = \epsilon_0 \mathbf{E}$	$\mathbf{B} = \mu_0 \mathbf{H}$
$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$	$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ Wb
$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$



μ_0 – permeability of free space = 4π x 10^{-7} H/m

Maxwell's Equations

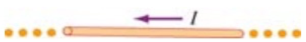
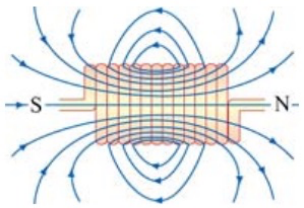
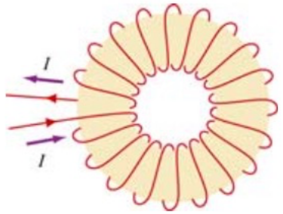
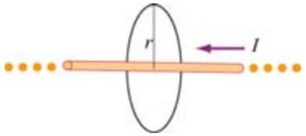
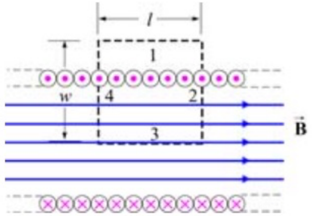
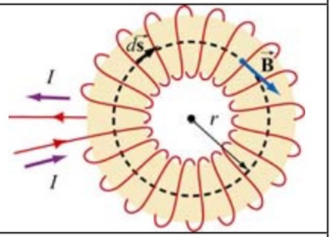
- Maxwell's equations for static fields, in integral and differential (point) forms

Integral		Differential
$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv$	Gauss's Law for elec. field	$\nabla \cdot \mathbf{D} = \rho_v$
$\oint \mathbf{E} \cdot d\mathbf{L} = 0$	Kirchhoff's Voltage Law	$\nabla \times \mathbf{E} = 0$
$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S}$	Ampere's Circuit Law	$\nabla \times \mathbf{H} = \mathbf{J}$
$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Gauss's Law for mag. field	$\nabla \cdot \mathbf{B} = 0$

Forces due to magnetic fields

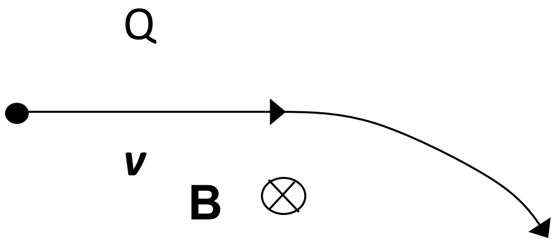
- Three case studies
  - Force on a moving charged particle in B field
  - Force on a current element in B field
  - Force between two current elements

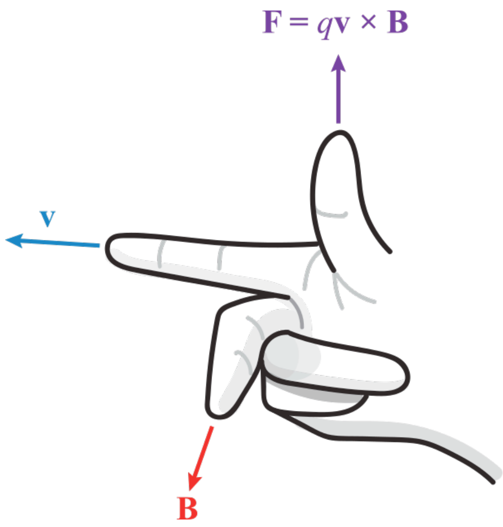
Ampere's Law Summary

System	Infinite wire	Ideal solenoid	Toroid
Figure			
(1) Draw the Amperian loop			
(2) Find the current enclosed by the Amperian loop	$I_{\text{enc}} = I$	$I_{\text{enc}} = NI$	$I_{\text{enc}} = NI$
(3) Calculate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ along the loop	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r)$	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl$	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B(2\pi r)$
(4) Equate $\mu_0 I_{\text{enc}}$ with $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ to obtain $\vec{\mathbf{B}}$	$B = \dfrac{\mu_0 I}{2\pi r}$	$B = \dfrac{\mu_0 NI}{l} = \mu_0 nI$	$B = \dfrac{\mu_0 NI}{2\pi r}$

Force on a charged particle

$$\mathbf{F}_{\mathrm{m}} = Q\mathbf{u} \times \mathbf{B}$$





1. The force depends on the direction of the magnetic field (*i.e.* whether it emanates from a north pole or a south pole).
2. The force is perpendicular to both the velocity and magnetic field directions
3. The force is zero if the particle velocity is zero (and depends on the sign of ***v***)
4. The force depends on the sign of the electric charge

Force on a charged particle

- For a stationary charge,

$$\mathbf{F}_e = \mathbf{QE}$$

- For a moving charge

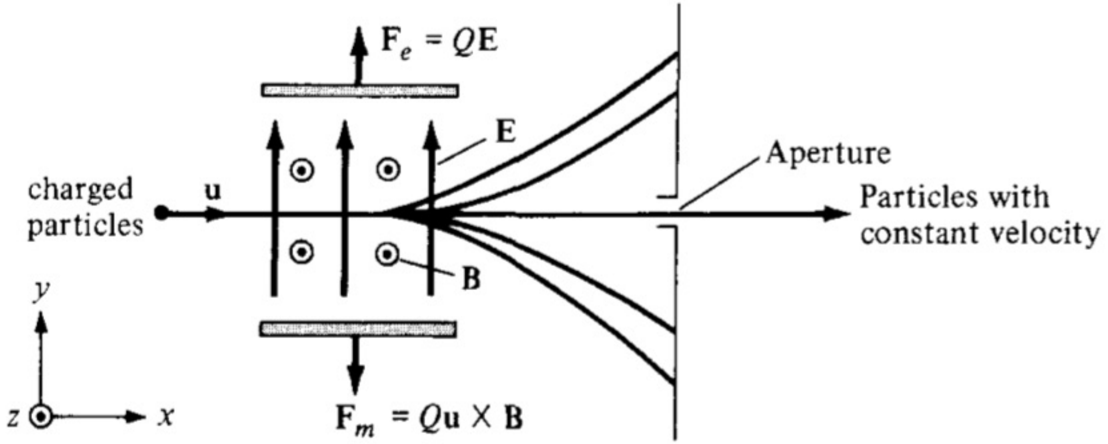
$$\mathbf{F} = \mathbf{QE} + (\mathbf{Qu} \times \mathbf{B})$$

Lorentz force equation

State of Particle	E Field	B Field	Combined E and B Fields
Stationary	$Q\mathbf{E}$	—	$Q\mathbf{E}$
Moving	$Q\mathbf{E}$	$Q\mathbf{u} \times \mathbf{B}$	$Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

Example

A charged particle moves with a uniform velocity $4\mathbf{a}_x$ m/s in a region where $\mathbf{E} = 20\,\mathbf{a}_y$ V/m and $\mathbf{B} = B_o\mathbf{a}_z$ Wb/m^2. Determine B_o such that the velocity of the particle remains constant.



Example

If the particle moves with a constant velocity, it implies that its acceleration is zero. In other words, the particle experiences no net force. Hence,

$$0 = \mathbf{F} = m\mathbf{a} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$0 = Q\,(20\mathbf{a}_y + 4\mathbf{a}_x \times B_o\mathbf{a}_z)$$

or

$$-20\mathbf{a}_y = -4B_o\mathbf{a}_y$$

$$\text{Thus } B_o = 5.$$

Force on a current element

- A small segment of the current element of charge *dQ* and velocity *u*

$$d\mathbf{F} = dQ\mathbf{u} \times \mathbf{B}$$

$$\mathbf{u} = \frac{d\mathbf{L}}{dt}$$

$$d\mathbf{F} = I\, d\mathbf{L} \times \mathbf{B}$$

$$\boxed{\mathbf{F} = \int I\, d\mathbf{L} \times \mathbf{B}}$$

$$\mathbf{J}\, dv = \mathbf{K}\, dS = I\, d\mathbf{L}$$

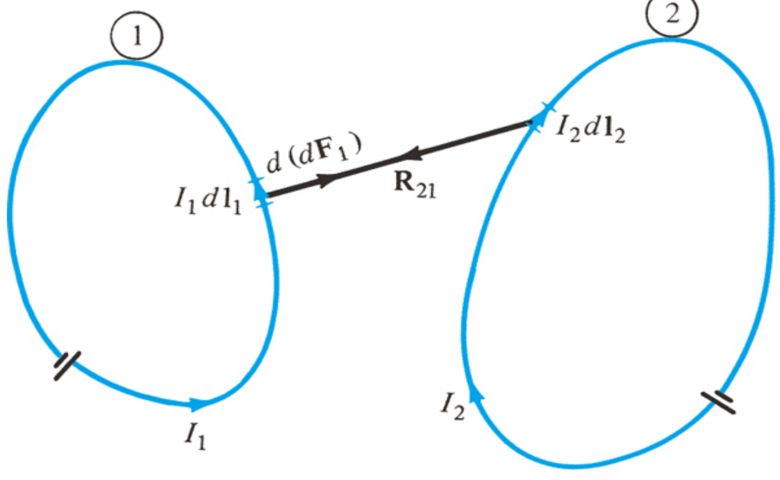
Force between two current elements

Force on 1 from magnetic field of 2

$$\boxed{d(d\mathbf{F}_1) = I_1\, d\mathbf{l}_1 \times d\mathbf{B}_2}$$

Apply Biot-Savart's Law to find $d\mathbf{B}_2$

$$\boxed{d\mathbf{B}_2 = \frac{\mu_0 I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}}}{4\pi R_{21}^2}}$$



$$d(d\mathbf{F}_1) = \frac{\mu_0 I_1\, d\mathbf{l}_1 \times (I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{4\pi R_{21}^2}$$

$$\mathbf{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2}$$

Force between two parallel current elements

Simpler case, where current elements are parallel

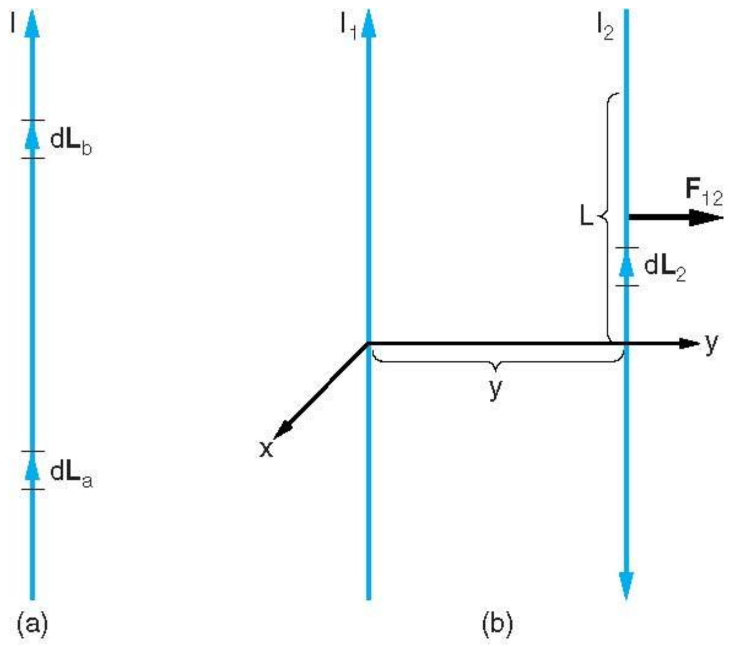
NOTE: ρ = y, $\mathbf{a}_\phi = -\mathbf{a}_x$

$$d\mathbf{F}_{12} = I_2 d\mathbf{L}_2 \times \mathbf{B}_1$$

$$\mathbf{B}_1 = \frac{\mu_o I_1}{2\pi\rho}\mathbf{a}_\phi$$

$$d\mathbf{F}_{12} = I_2 dz\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi y}(-\mathbf{a}_x) = \frac{\mu_o I_1 I_2}{2\pi y}dz(-\mathbf{a}_y)$$

$$\mathbf{F}_{12} = \frac{\mu_o I_1 I_2}{2\pi y}(-\mathbf{a}_y)\int_L^0 dz = \frac{\mu_o I_1 I_2 L}{2\pi y}\mathbf{a}_y$$



Another example

A rectangular loop carrying current I_2 is placed parallel to an infinitely long filamentary wire carrying current I_1 as shown in Figure 8.4(a). Show that the force experienced by the loop is given by

$$\mathbf{F} = -\frac{\mu_o I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right]\mathbf{a}_\rho \text{ N}$$

$$\mathbf{F}_\ell = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = I_2 \oint d\mathbf{l}_2 \times \mathbf{B}_1 \qquad \mathbf{B}_1 = \frac{\mu_o I_1}{2\pi\rho_o}\,\mathbf{a}_\phi$$

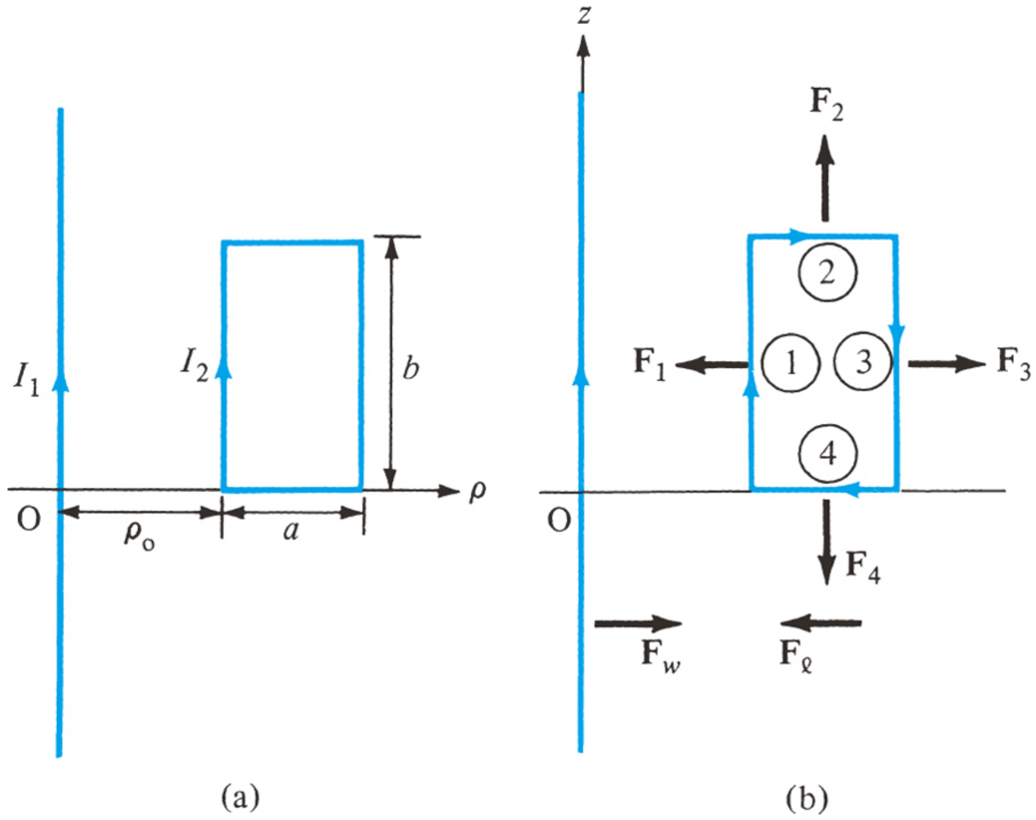
Example

$$\mathbf{F}_\ell = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = I_2 \oint d\mathbf{l}_2 \times \mathbf{B}_1$$

$$\mathbf{B}_1 = \frac{\mu_o I_1}{2\pi\rho_o}\mathbf{a}_\phi$$

$$\mathbf{F}_1 = I_2 \int d\mathbf{l}_2 \times \mathbf{B}_1 = I_2 \int_{z=0}^{b} dz\,\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi\rho_o}\mathbf{a}_\phi$$
$$= -\frac{\mu_o I_1 I_2 b}{2\pi\rho_o}\mathbf{a}_\rho \qquad \text{(attractive)}$$

$$\mathbf{F}_3 = I_2 \int d\mathbf{l}_2 \times \mathbf{B}_1 = I_2 \int_{z=b}^{0} dz\,\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi(\rho_o + a)}\mathbf{a}_\phi$$
$$= \frac{\mu_o I_1 I_2 b}{2\pi(\rho_o + a)}\mathbf{a}_\rho \qquad \text{(repulsive)}$$

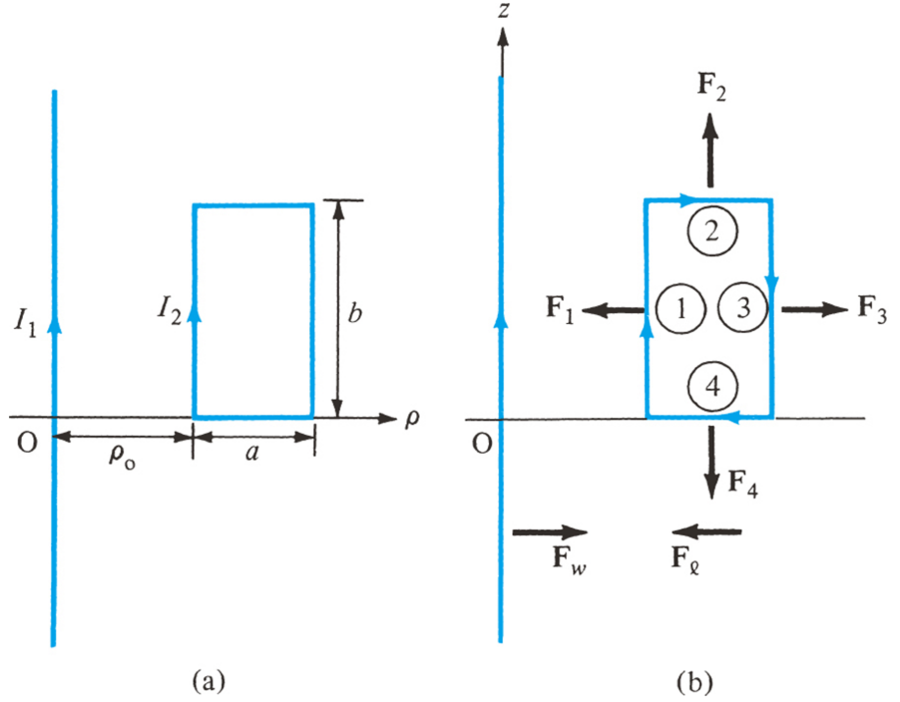


Parallel, same I direction – ATTRACTIVE
Parallel, opposite I direction - REPULSIVE

Example

$$\mathbf{F}_2 = I_2 \int_{\rho=\rho_o}^{\rho_o+a} d\rho\, \mathbf{a}_\rho \times \frac{\mu_o I_1\, \mathbf{a}_\phi}{2\pi\rho}$$
$$= \frac{\mu_o I_1 I_2}{2\pi} \ln \frac{\rho_o + a}{\rho_o}\, \mathbf{a}_z \qquad \text{(parallel)}$$

$$\mathbf{F}_4 = I_2 \int_{\rho=\rho_o+a}^{\rho_o} d\rho\, \mathbf{a}_\rho \times \frac{\mu_o I_1\, \mathbf{a}_\phi}{2\pi\rho}$$
$$= -\frac{\mu_o I_1 I_2}{2\pi} \ln \frac{\rho_o + a}{\rho_o}\, \mathbf{a}_z \qquad \text{(parallel)}$$



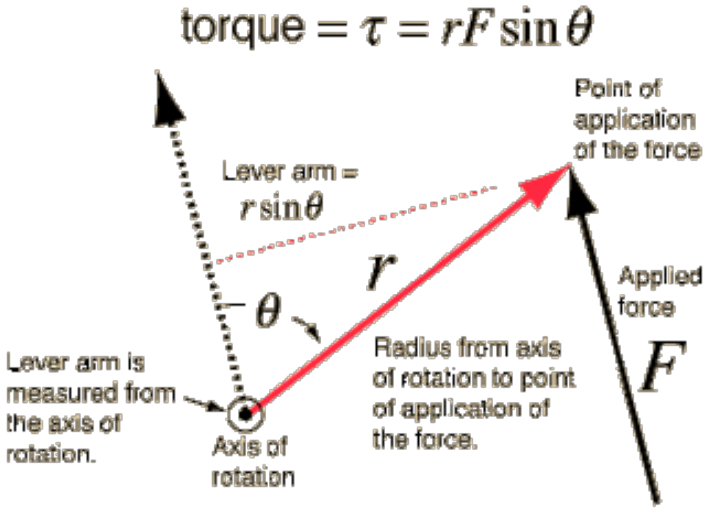
$$\mathbf{F}_\ell = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = I_2 \oint d\mathbf{l}_2 \times \mathbf{B}_1$$

$$\boxed{\mathbf{F}_\ell = \frac{\mu_o I_1 I_2 b}{2\pi}\left[\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right](-\mathbf{a}_\rho)}$$

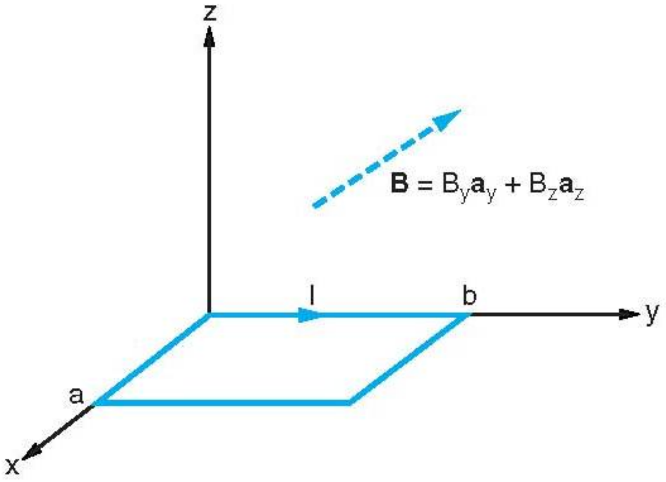
Magnetic Torque

- Torque is the vector product of the moment arm **r** and the force **F**

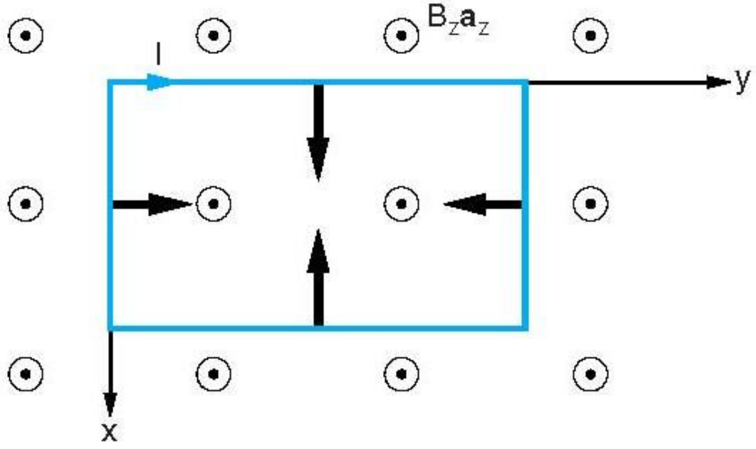
$$\vec{\tau} = \vec{\mathrm{r}} \times \vec{\mathrm{F}}$$



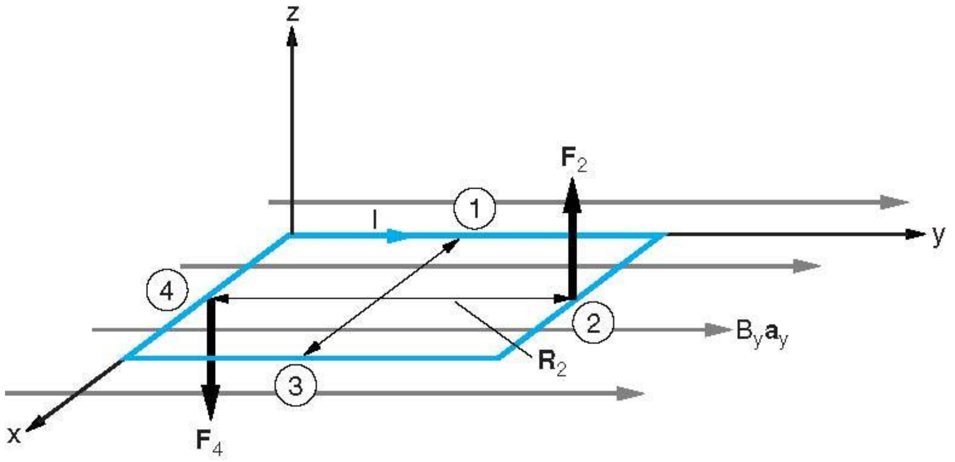
Example



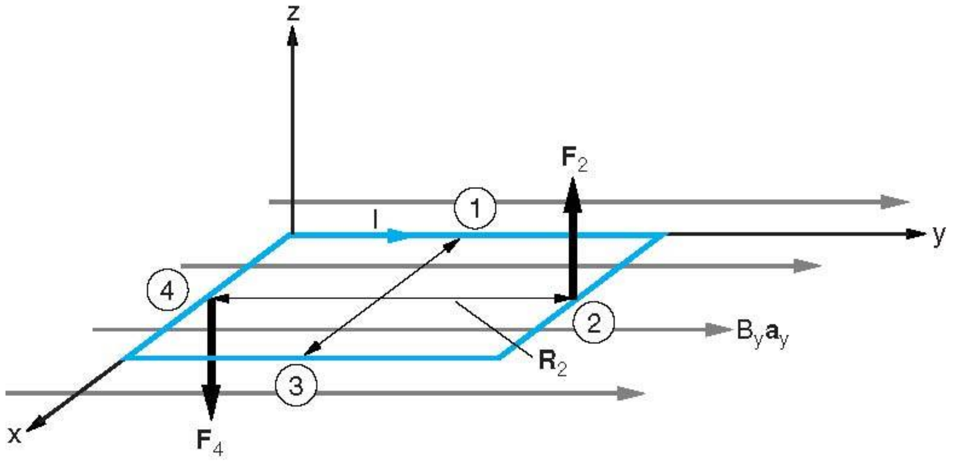
Normal component



Parallel component



Example



The force on side ① is zero.
The force on side ② is $\mathbf{F}_2 = Ia\,\mathbf{a}_x \times B_y\mathbf{a}_y = B_y Ia\,\mathbf{a}_z$
The force on side ③ is zero.
The force on side ④ is $\mathbf{F}_4 = Ia(-\mathbf{a}_x) \times B_y\mathbf{a}_y = -B_y Ia\,\mathbf{a}_z$

$$\mathbf{T} = B_y I ab\,\mathbf{a}_x = \mathbf{m} \times \mathbf{B}$$

$$\mathbf{m} = IS\mathbf{a}_{\mathrm{N}}$$

m = magnetic dipole moment
I = current
S = area of current loop
a_N = normal vector to the loop

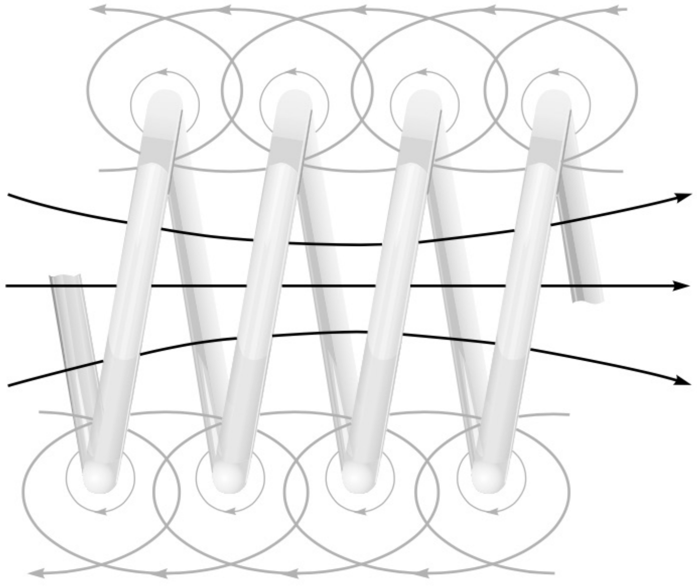
Inductance

- The flux linkage λ is product of the number of turns N and the flux linking each of them

$$\lambda = N\boldsymbol{\Phi}$$

- Inductance (or self-inductance) as the ratio of the total flux link- ages to the current which they link

$$\boxed{L = \frac{N\Phi}{I}}$$



Example

Calculate the self-inductance per unit length of an infinitely long solenoid.

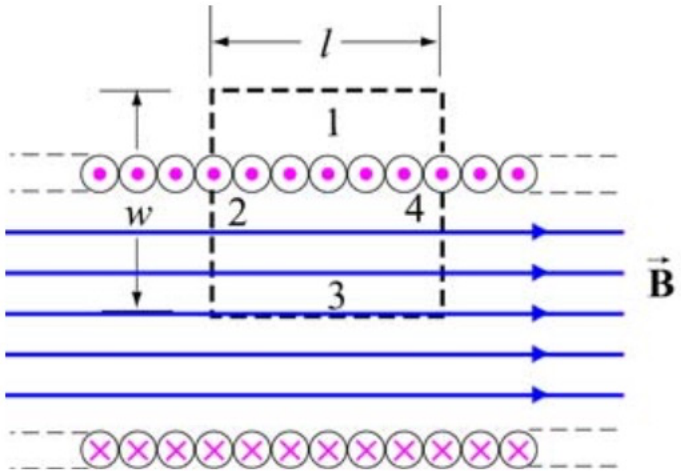
Example

- Steps to find inductance:

1. Choose a suitable coordinate system.
2. Let the inductor carry current I.
3. Determine $\mathbf{B}$ from Biot–Savart's law (or from Ampere's law if symmetry exists) and calculate Ψ from $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$.
4. Finally find L from $L = \dfrac{\lambda}{I} = \dfrac{N\Psi}{I}$.

Flashback – Ampere's circuital law

$$\boxed{\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}}$$



$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \int_1 \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} + \int_2 \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} + \int_3 \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} + \int_4 \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}}$$

$$= 0 + 0 + Bl + 0$$

In our notation,

$$ds = dl$$
$$ds \neq dS$$

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = Bl = \mu_0 NI \qquad B = \frac{\mu_0 NI}{l} = \mu_0 nI$$

Example

- Recall B inside a solenoid

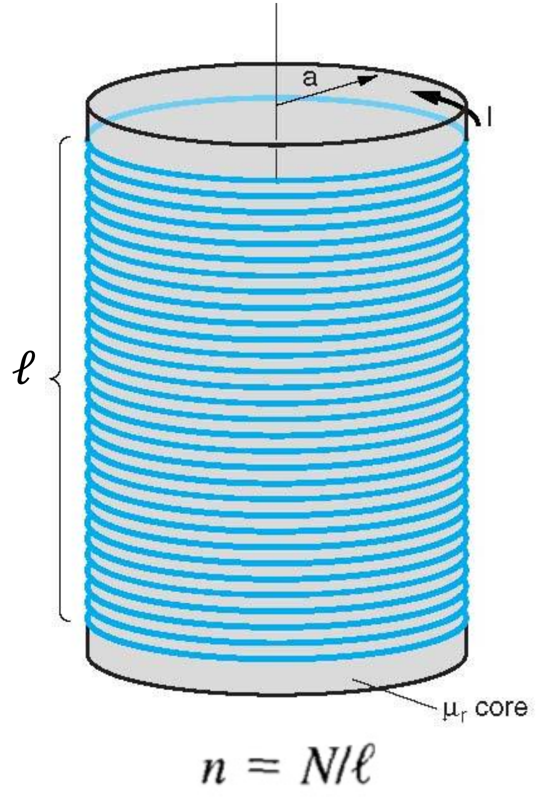
$$B = \mu H = \mu I n$$

- If *S* is the x-sectional area, total flux is

$$\Phi = BS = \frac{\mu INS}{l}$$

- Finally, find L

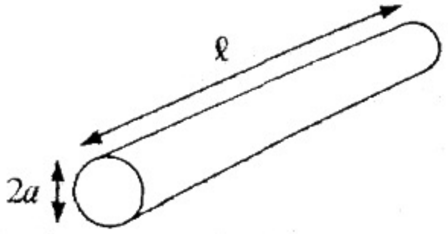
$$L = \frac{N\Phi}{I} = \frac{N^2 \mu S}{l}$$



Inductance

Wire

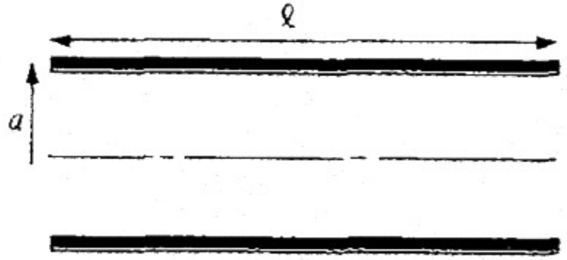
$$L = \frac{\mu_o \ell}{8\pi}$$



Solenoid

$$L = \frac{\mu_o N^2 S}{\ell}$$

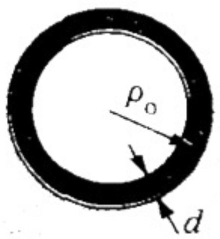
$$\ell \gg a$$



Circular loop

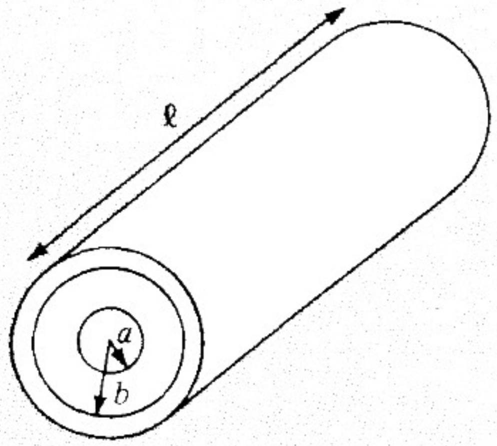
$$L = \frac{\mu_o \ell}{2\pi}\left(\ln\frac{4\ell}{d} - 2.45\right)$$

$$\ell = 2\pi\rho_o,\ \rho_o \gg d$$



Coaxial conductor

$$L = \frac{\mu_o \ell}{\pi}\ln\frac{b}{a}$$



Magnetic Energy

- For electric field, work done was:

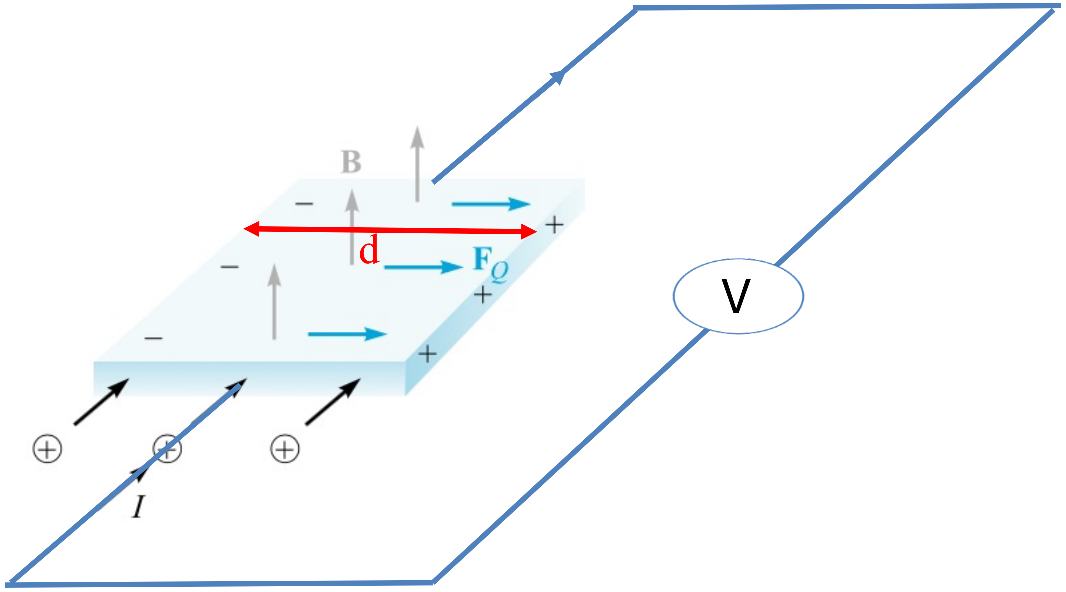
$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D}\cdot\mathbf{E}\, dv$$

- In a steady magnetic field, total energy stored is

$$\boxed{W_H = \frac{1}{2}\int_{\text{vol}} \mathbf{B}\cdot\mathbf{H}\, dv}$$

- Magnetic energy stored in an inductor is: $\boxed{W_m = \frac{1}{2}LI^2}$

Hall effect



When an electrical current passes through a sample placed in a magnetic field, a potential proportional to the current and to the magnetic field is developed across the material in a direction perpendicular to both the current and to the magnetic field

Hall effect

- At equilibrium, the force on charge carrier qE due the developed electric field just balances the force due to the magnetic flux density

$$q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = 0 \qquad \mathbf{E} = \mathbf{B} \times \mathbf{v}$$

- The drift velocity v is related to the current density J:

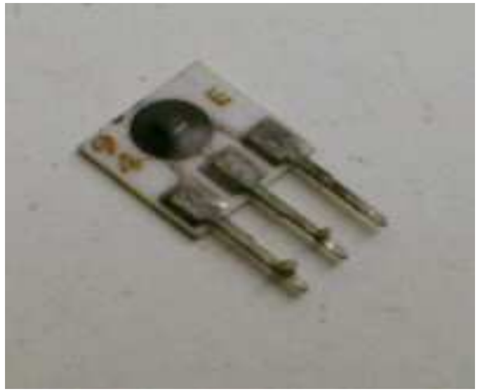
$$\mathbf{J} = Nq\mathbf{v} \qquad \mathbf{E} = R_H \mathbf{B} \times \mathbf{J}$$

$$V_H = |\mathbf{E}| d$$

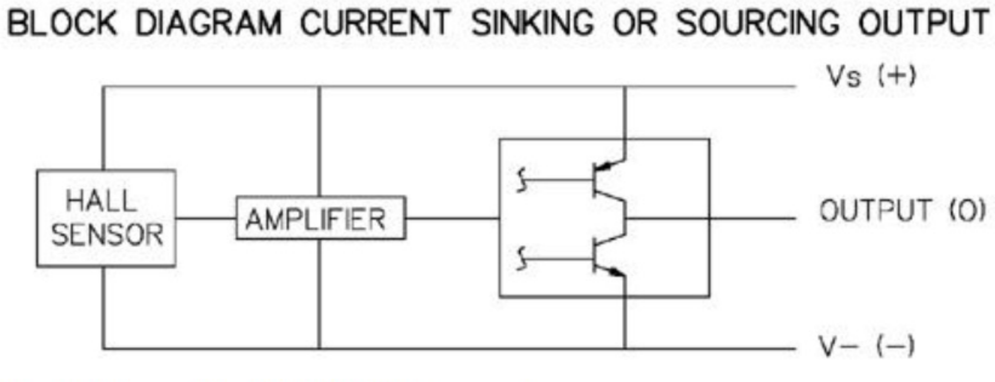
R_H = Hall coefficient ($1/Nq$)

Hall effect

- Applications
  - determination of n-type and p-type semiconductors
  - measuring conductivity σ and mobility μ
  - Hall effect sensor
- Hall effect sensor detects presence of magnetic field



Honeywell SS495A



Faraday's Law

- The story so far…
  - stationary charges ➞ electrostatic field
  - steady currents ➞ magnetostatic fields
  - time-varying currents ➞ ?
- Although attributed to Faraday, Henry discovered the same effect at around the same time

Faraday's Law

- Induced EMF, in a closed circuit, is equal to the rate of change of magnetic flux linkage

$$\boxed{\mathrm{emf} = -N\frac{d\Phi}{dt}}$$

- The negative sign shows that the induced EMF acts to oppose the flux producing it

Interactive simulation: http://phet.colorado.edu/sims/html/faradays-law/latest/faradays-law_en.html

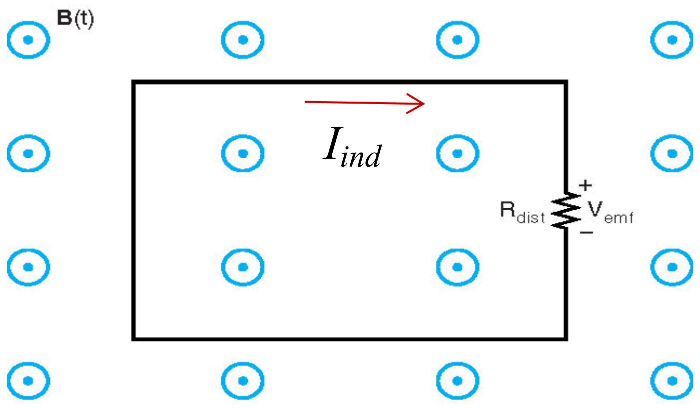
Faraday's Law

- EMF around a closed path in a time-varying field:

$$\text{emf} = -N\frac{d\Phi}{dt}$$

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

- *Transformer emf*: magnetic field changes with time
- *Motional emf*: surface containing the flux changes with time



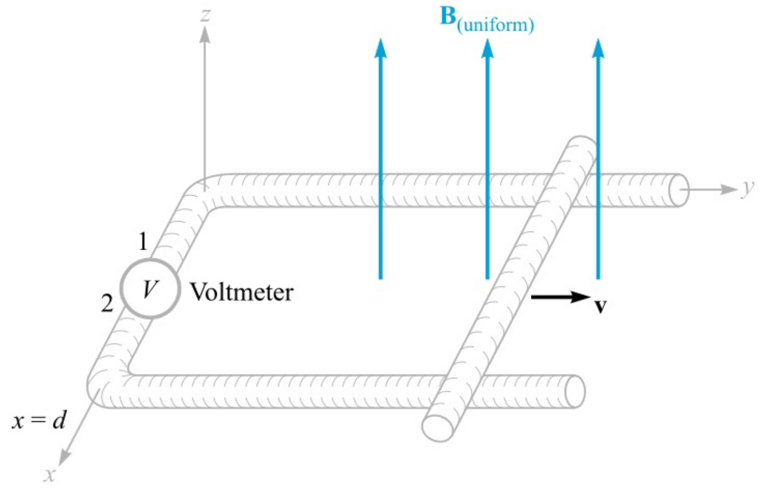
An increasing magnetic field out of the page induces an emf across R_{dist}

Motional emf

- Occurs when a conducting loop is moving in a static B field

$$\boxed{\mathbf{F} = Q\mathbf{v} \times \mathbf{B}}$$

$$\frac{\mathbf{F}}{Q} = \mathbf{v} \times \mathbf{B}$$



- Motional electric field is

$$\boxed{\mathbf{E}_m = \mathbf{v} \times \mathbf{B}}$$

$$\boxed{\text{emf} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}}$$

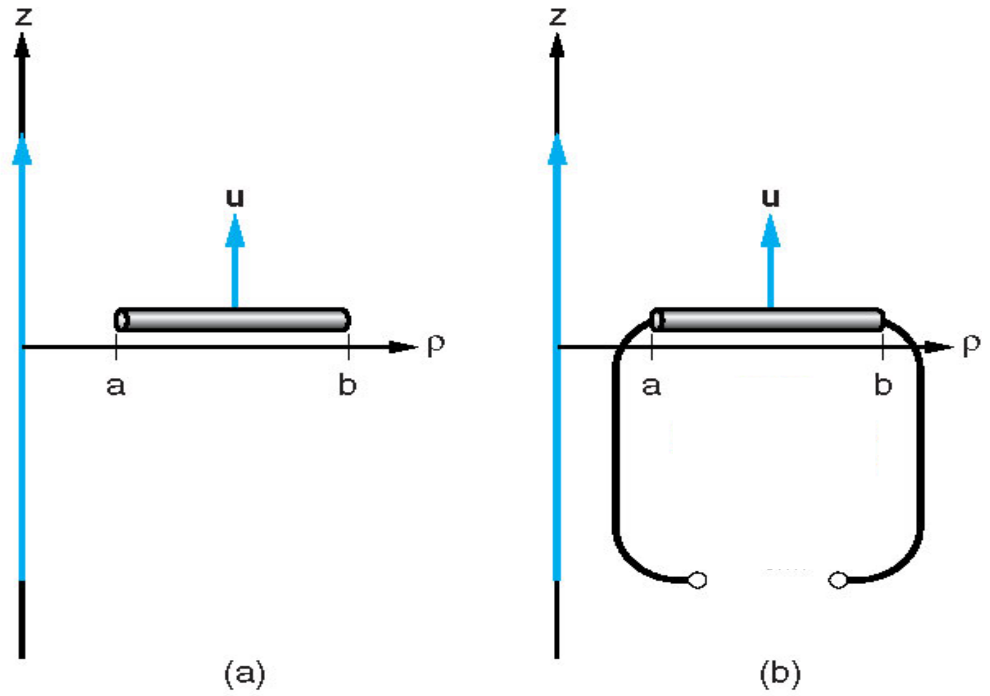
Motional emf

- Now imagine a conducting loop is moving in a time-varying B field
- What happens?

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = \underbrace{\boxed{-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}}}_{\text{transformer}} + \underbrace{\boxed{\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}}}_{\text{motional}}$$

Example

2 A current is flowing along z-direction. A conductor of length 3.5 m is moving along z-axis with a velocity of 2.5 m/s. Position a = 1.5 m from the origin. Find the induced EMF.



Example

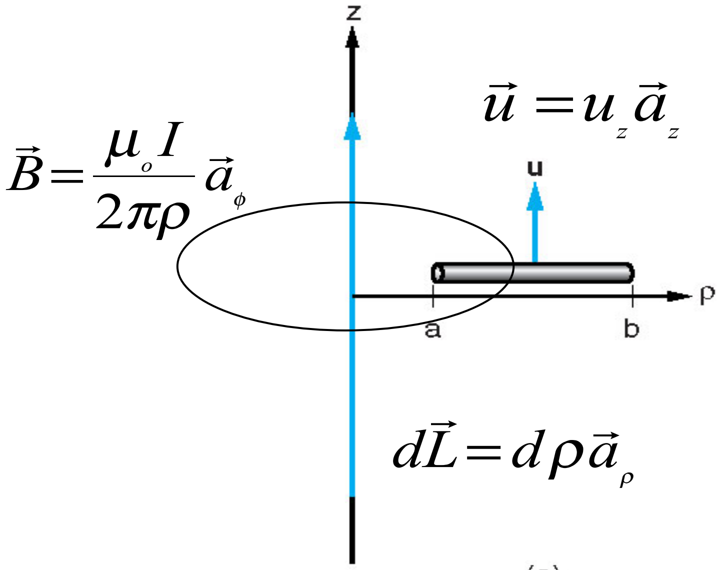
- Steps to solve the problem:
  - Recall for steady B, $\text{emf} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}$
  - Define *d*L
  - Find B for the current-carrying wire
  - Apply the formula

Example

Solution:

$$V_{emf} = \oint \left(u_z \vec{a}_z \times \frac{\mu_o I}{2\pi\rho} \vec{a}_\phi \right) . d\rho \vec{a}_\rho$$

$$= -\frac{\mu_o I u_z}{2\pi} \int_a^b \frac{d\rho}{\rho} = -\frac{\mu_o I u_z}{2\pi} \ln\left(\frac{b}{a}\right)$$



(a)

$$V_{emf} = -\frac{4\pi \times 10^{-7} \left(H/m\right) \times 2(A) \times 2.5(m/s)}{2\pi} \ln\left(\frac{4.5}{1.5}\right)$$

$$= -1.1\ \mu V$$

Transformers

- Electrical device that transfers energy from one circuit to another purely by magnetic coupling
- Transformers are often used to:
  - convert between high and low voltages
  - change impedance, and
  - provide electrical isolation between circuits

Transformer application

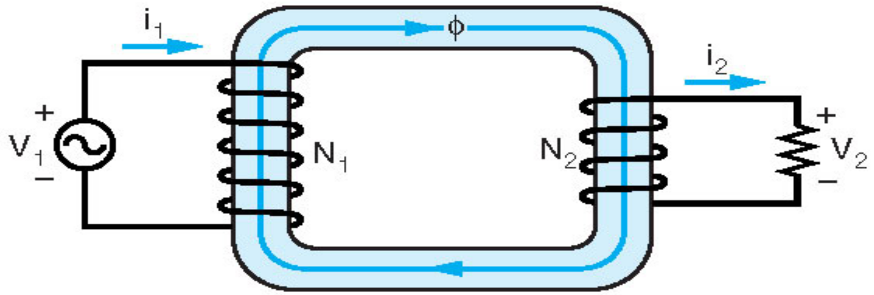
- To start a car, around 40,000 volts is required across the sparkplug
- How do you start a car with only a 12 V battery?

- Answer is here:

  https://nationalmaglab.org/education/magnet-academy/watch-play/interactive/ignition-coil

Transformer

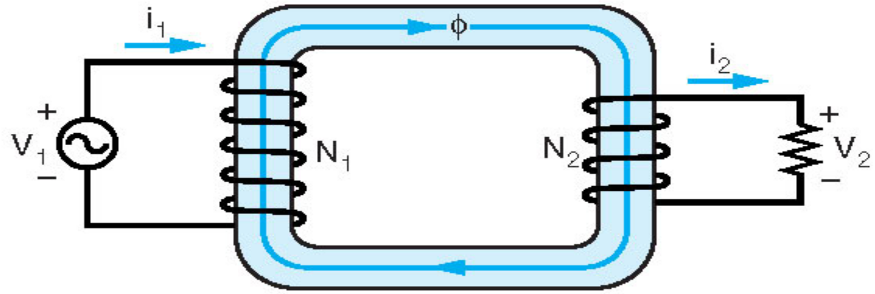
- A **transformer** consists of *primary* (N_1) and *secondary* (N_2) coils wrapped around a magnetic core.
- *Primary* side is called the *driving side* of the transformer with ac voltage v_1 across and current i_1 through the primary coil of N_1 turns.
- *Secondary* side is called the *driven* side of the transformer with ac voltage v_2 across and current i_2 through the secondary coil of N_2 turns.



Transformer

- Primary side of the transformer with AC voltage *v1* across and current *I1* through the primary coil of *N1* turns establishes flux.
- Using Faraday's law:

$$v_1 = -N_1 \frac{d\phi}{dt}$$



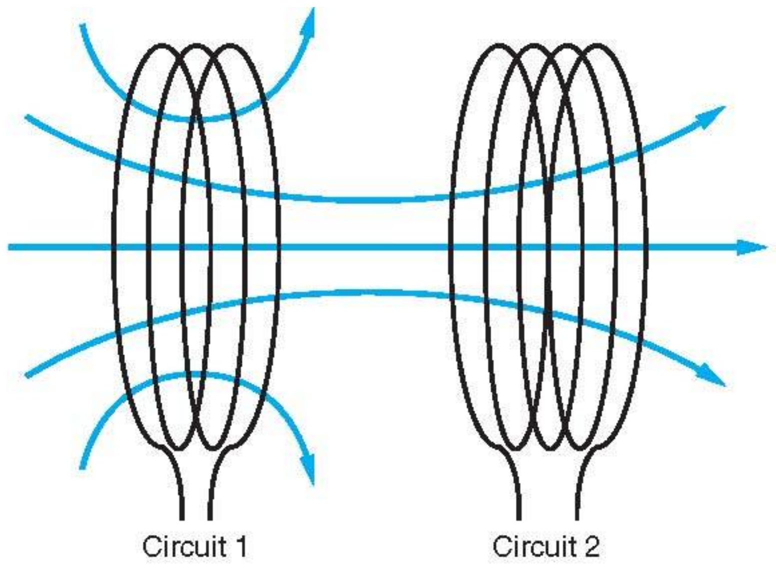
- Secondary side has the same flux

$$v_2 = -N_2 \frac{d\phi}{dt}$$

$$\frac{v_1}{v_2} = \frac{N_1}{N_2}$$

Mutual Inductance

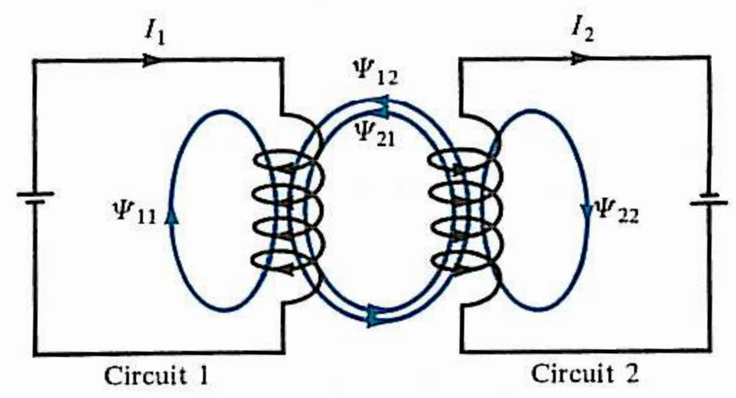
- So far, when we talked about inductance, we referred to self-inductance
- Mutual inductance, *M*, is the flux that is common to both coils



Mutual Inductance

Self inductance of circuit 1:

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \Psi_1}{I_1}$$



Mutual inductance at circuit 1:

$$\boxed{M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1 \Psi_{12}}{I_2}}$$

Self inductance of circuit 2:

$$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \Psi_2}{I_2}$$

Mutual inductance at circuit 2:

$$\boxed{M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2 \Psi_{21}}{I_1}}$$

$$\boxed{\Psi_1 = \Psi_{11} + \Psi_{12} \qquad \Psi_2 = \Psi_{21} + \Psi_{22}}$$

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Maxwell's Equation

- From Faraday's law, we have seen

$$\boxed{\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}}$$

- Applying Stoke's theorem,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- But in electrostatics we learnt

$$\boxed{\nabla \times \mathbf{E} = 0}$$

Our old Maxwell's equations are limited to static fields

Maxwell's Equations

- *Continuity of current*: rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

- From Ampere's circuital law for steady magnetic field, we learnt

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S} \qquad \nabla \times \mathbf{H} = \mathbf{J}$$

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Maxwell's Equation

- When it come's to time varying fields, the equation is inadequate

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

Divergence of a curl is identical to 0

$$\nabla \cdot \nabla \times \mathbf{H} \equiv 0 = \nabla \cdot \mathbf{J}$$

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}}$$

- Solution: add a new term

Maxwell's Equation

- Rewrite Ampere's law as:

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$$

Displacement current

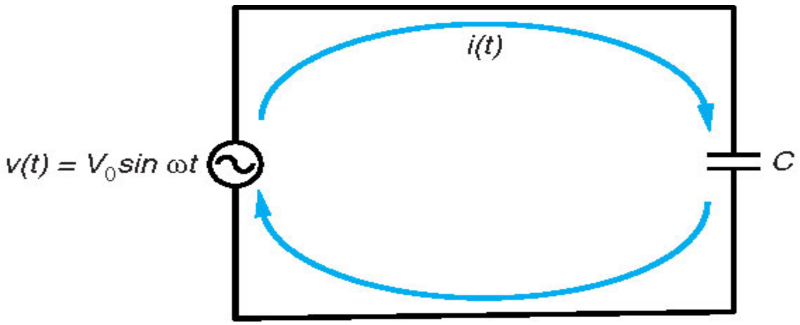
Displacement current

A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage $50 \sin 10^3 t$ V applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\varepsilon_o$.

Solution:

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d} \frac{dV}{dt}$$



Hence,

$$I_d = J_d \cdot S = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

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Maxwell's Equation

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's circuit law

*This is also referred to as Gauss's law for magnetic fields.

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Maxwell's Equations

- Maxwell's fundamental contribution is to fill up the gap between the static and dynamic electromagnetic fields via the current continuity equation
- Maxwell unified all theories of electricity and magnetism into a set of four formulae that are known as Maxwell's equations
- Predominantly used to solved all sorts of static and dynamic electromagnetics problems with appropriate boundary conditions.

Electric and Magnetic fields summary

Electric fields	Magnetic fields
$\mathbf{E}$(V/m)	$\mathbf{H}$(A/m)
$\mathbf{D}$(C/m^2)	$\mathbf{B}$(Wb/m^2)
ψ(C)	ϕ(Wb)
ε(F/m)	μ(H/m)
$\mathbf{D} = \varepsilon \mathrm{E}$	$\mathrm{B} = \mu \mathrm{H}$
$\nabla \cdot \mathbf{D} = \rho_v$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathrm{E} = \mathbf{0}$	$\boldsymbol{\nabla} \times \mathbf{H} = \mathrm{J}$
$\psi = \int \mathrm{D} \cdot \mathbf{dS}$	$\phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S}$
$\mathbf{F}(\mathrm{N}) = Q\mathrm{E}$	$\mathbf{F}(\mathrm{N}) = Q\mathbf{u} \times \mathrm{B}$
$W_{\mathrm{E}}(\mathrm{J}) = \frac{1}{2} \int \mathrm{D} \cdot \mathrm{E}\, dv$	

Lorentz force equation	$\mathrm{F} = q(\mathrm{E} + \mathbf{u} \times \mathrm{B})$
Constitutive relations	$\begin{cases} \mathrm{D} = \varepsilon \mathbf{E} \\ \mathrm{B} = \boldsymbol{\mu} \mathbf{H} \\ \mathrm{J} = \sigma \mathrm{E} \text{ (Ohm's law)} \end{cases}$
Current continuity equation	$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$