Question 37, EE Gate 2022

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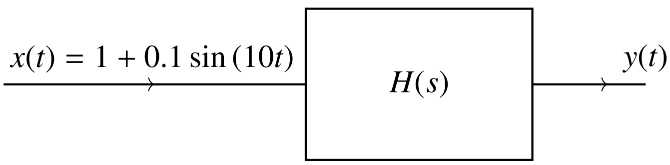
Question: An LTI system is shown in the figure where

$$H(s) = \frac{100}{s^2 + 0.1s + 10}$$

The steady state output of the system for an input $x(t)$ is given by $y(t) = a + b\sin(10t + \theta)$. The values of $'a'$ and $'b'$ are



Solution:

Symbol	Value	Description
$x(t)$	$1 + 0.1\sin(10t)$	Input Signal
$y(t)$	?	Output of the system
$H(s)$	$\frac{100}{s^2+0.1s+10}$	Impulse Response

TABLE 0
GIVEN INFORMATION

1) **Theory:** If a sinusoidal input is given to a system, whose transfer function is known, the output can be calculated as follows

$$y(t) = h(t) * x(t) \tag{1}$$
$$Y(s) = H(s)X(s) \tag{2}$$

Let $s = j\omega$

$$Y(j\omega) = H(j\omega)X(j\omega) \tag{3}$$

If Φ is the phase of $H(j\omega)$,

$$H(j\omega) = |H(j\omega)|\, e^{j\Phi(\omega)} \tag{4}$$

If $x(t) = \cos(\omega_0 t)$,

$$X(j\omega) = \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right) \tag{5}$$

Now,

$$Y(j\omega) = \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)|H(j\omega)|\, e^{j\Phi(\omega)} \tag{6}$$
$$\tag{7}$$

Since $|H(j\omega)|\,\delta(\omega - \omega_0)$ is zero everywhere except at ω_0

$$Y(j\omega) = |H(j\omega_0)|\, e^{j\Phi(\omega_0)}\delta(\omega - \omega_0) \tag{8}$$
$$+ |H(-j\omega_0)|\, e^{j\Phi(-j\omega_0)}\delta(\omega + \omega_0) \tag{9}$$

As $h(t)$ is real,

$$H(\omega) = H^*(-\omega)$$

$$\Phi(-\omega_0) = -\Phi(\omega_0)$$

Hence

$$Y(\omega) = |H(\omega_0)|\left(e^{j\Phi(\omega_0)}\delta(\omega - \omega_0) + e^{-j\Phi(\omega_0)}\delta(\omega + \omega_0)\right) \tag{10}$$

Taking Inverse Fourier Transform,

$$\delta(\omega - \omega_0) \xleftrightarrow{\mathcal{F}} \frac{1}{2}e^{j\omega_0 t} \tag{11}$$

$$\implies y(t) = |H(\omega_0)|\,\frac{1}{2}\left(e^{j(\omega_0 t + \Phi(\omega_0))} + e^{-j(\omega_0 t + \Phi(\omega_0))}\right) \tag{12}$$

$$\implies y(t) = |H(\omega_0)|\cos(\omega_0 t + \Phi(\omega_0)) \tag{13}$$

2) The given input can be assumed to be a superposition of $u(t)$ and $0.1\sin(\omega_0 t)u(t)$.

$$\omega_0 = 0 \text{ and } \omega_0 = 10$$

for the constant input and the sinusoidal input respectively.

$$y(t) = |H(0)| + |H(10)|\sin(10t + \Phi(10)) \tag{14}$$

Here

$$H(\omega) = \frac{100}{(j\omega)^2 + 0.1(j\omega) + 10} \tag{15}$$

$$\implies H(\omega) = \frac{100}{10 - \omega^2 + j(0.1\omega)} \tag{16}$$

$$\implies |H(\omega)| = \frac{100}{\sqrt{(10 - \omega^2)^2 + (0.1\omega)^2}} \tag{17}$$

$$\therefore |H(0)| = 10 \text{ and } |H(10)| \approx 1 \tag{18}$$

The phase $\Phi(\omega)$ is given by

$$\Phi(\omega) = \tan^{-1}\frac{0.1\omega}{\omega^2 - 10} \tag{19}$$

$$\implies \Phi(10) = \tan^{-1}\frac{1}{90} \tag{20}$$

Hence the output of the system

$$y(t) = 10 + \sin\left(10t + \tan^{-1}\frac{1}{90}\right) \tag{21}$$

Hence $a = 10$ and $b = 1$

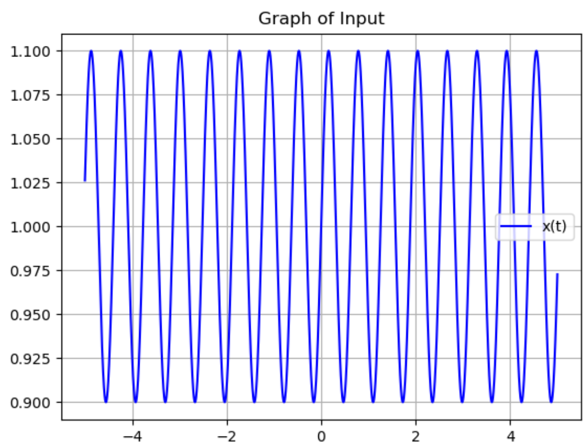


Fig. 2. Input of the system, $x(t)$

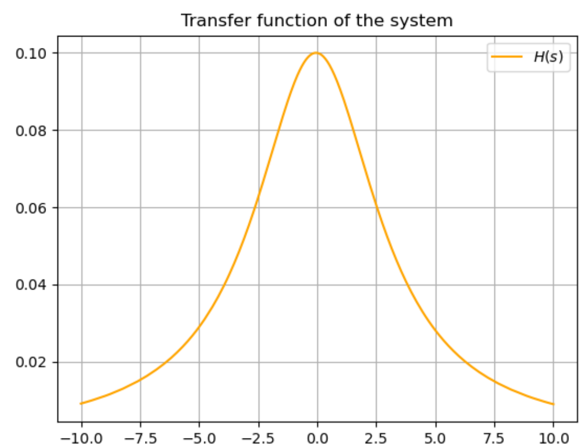


Fig. 2. Transfer function of the system, $H(s)$

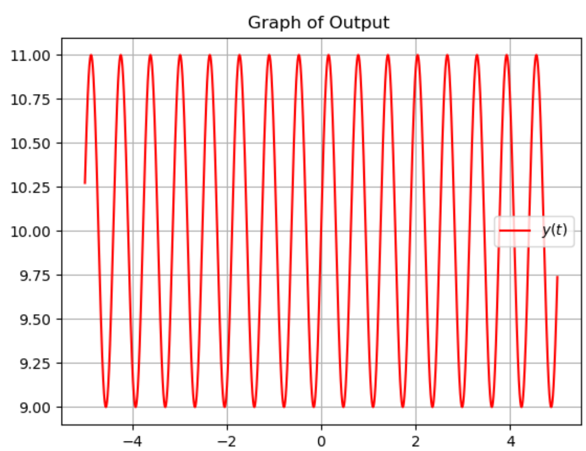


Fig. 2. Output of the system, $y(t)$