Question 49, ME Gate 2023

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Question: Consider the second-order linear differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0,\ x \geq 1$$

with the initial conditions

$$y(x=1) = 6, \quad \frac{dy}{dx}\Big|_{x=1} = 2.$$

Then the value of y at $x = 2$ is __________. GATE ME 2023

Solution:

Symbol	Value	Description
y(x)	?	Function
y(1)	6	-
y'(1)	2	-

TABLE 0
GIVEN INFORMATION

Consider the Mellin Transform

$$y(x) \overset{\mathcal{M}}{\longleftrightarrow} \int_{-\infty}^{\infty} x^{s-1}y(x)dx \tag{1}$$

Let

$$Y(s) = \int_{-\infty}^{\infty} x^{s-1}y(x)dx$$

Properties of the Mellin transform for a system at initial rest include

$$y'(x) \overset{\mathcal{M}}{\longleftrightarrow} -(s-1)Y(s-1) \tag{2}$$

$$xy'(x) \overset{\mathcal{M}}{\longleftrightarrow} -sY(s) \tag{3}$$

$$(x\frac{d}{dx})^n y \overset{\mathcal{M}}{\longleftrightarrow} (-s)^n Y(s) \tag{4}$$

To modify this, evaluating the Mellin Transform specifically,

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} \int_{-\infty}^{\infty} x^{s-1}(x\frac{dy}{dx})dx, \quad x \geq 1 \tag{5}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} \int_{1}^{\infty} x^{s}(\frac{dy}{dx})dx \tag{6}$$

Integrating by parts,

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} [x^s \int \frac{dy}{dx}dx]\Big|_1^\infty - \int_1^\infty sx^{s-1}y(x)dx \tag{7}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} x^s y(x)\Big|_1^\infty - sY(s) \tag{8}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} \lim_{x\to\infty}(x^s y(x)) - y(1) - sY(s) \tag{9}$$

Let

$$L = \lim_{x\to\infty}(x^s y(x)) \tag{10}$$

Subject to $L = 0$, from (25),

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} -y(1) - sY(s) \tag{11}$$

$$(x\frac{d}{dx})^2 y \overset{\mathcal{M}}{\longleftrightarrow} s^2Y(s) + sy(1) - y'(1) \tag{12}$$

The given differential equation can be written as:

$$x\frac{d}{dx}(x\frac{dy}{dx}) = y, \quad x \geq 1 \tag{13}$$

$$\implies (x\frac{d}{dx})^2 y = y, \quad x \geq 1 \tag{14}$$

Taking Mellin transform on both sides, and from (27)

$$s^2Y(s) + sy(1) - y'(1) = Y(s), \quad s < -1 \tag{15}$$

From Table 0

$$Y(s) = s^2Y(s) + 6s - 2 \tag{16}$$

$$\implies Y(s) = \frac{6s-2}{1-s^2} \tag{17}$$

$$= -\frac{4}{s+1} - \frac{2}{s-1} \tag{18}$$

Property of Laplace Transform

$$e^{at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \quad \Re s > a \tag{19}$$

Taking inverse Mellin transform,

$$Y(s) \overset{\mathcal{M}^{-1}}{\longleftrightarrow} y(x) \equiv Y(s) \overset{\mathcal{L}^{-1}}{\longleftrightarrow} y(e^{-x}) \tag{20}$$

$$-\frac{4}{s+1} - \frac{2}{s-1} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} -4e^{-x} - 2e^{x} \tag{21}$$

$$\implies L^{-1}\{Y(s)\} = -4e^{-x} - 2e^{x} \tag{22}$$

Substituting x by $-\ln x$

$$y(x) = -4x - \frac{2}{x} \tag{23}$$

To find ROC of s, substituting y(x) in (10)

$$\lim_{x\to\infty} x^s \left(-4x - \frac{2}{x}\right) = 0 \tag{24}$$

$$\implies \lim_{x\to\infty} \left(4x^{s+1} + 2x^{s-1}\right) = 0 \tag{25}$$

$$\implies \Re s + 1 < 0, \; \Re s - 1 < 0 \tag{26}$$

$$\implies \Re s < -1 \tag{27}$$

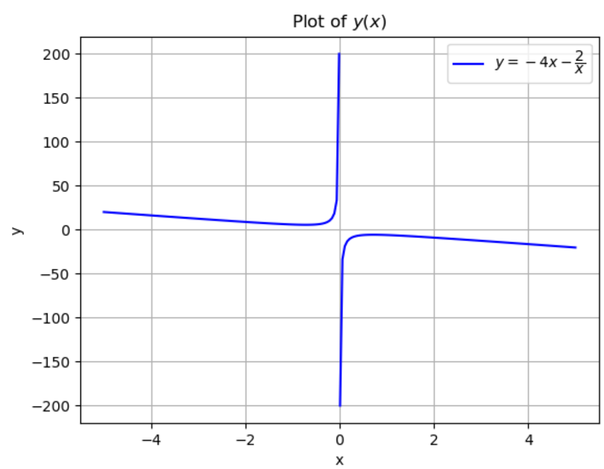

Fig. 0. Plot of $y(x)$ v/s x