10.5.2.7

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Question: Consider the second-order linear differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0,\ x \geq 1$$

with the initial conditions

$$y(x=1) = 6, \quad \frac{dy}{dx}\Big|_{x=1} = 2.$$

Then the value of y at $x = 2$ is __________. GATE ME 2023

Solution: Consider the Mellin transform as the

Symbol	Value	Description
y(x)	?	Function
y(1)	6	-
y'(1)	2	-

TABLE 0
GIVEN INFORMATION

combined operation of substituting x by e^{-x} and subsequently taking the laplace transform:

$$y(x) \overset{\mathcal{M}}{\longleftrightarrow} \int_{-\infty}^{\infty} x^{s-1}y(x)dx \tag{1}$$

Let $Y(s) = \int_{-\infty}^{\infty} x^{s-1}y(x)dx$ Properties of the Mellin transform include

$$y'(x) \overset{\mathcal{M}}{\longleftrightarrow} -(s-1)Y(s-1) \tag{2}$$

$$xy'(x) \overset{\mathcal{M}}{\longleftrightarrow} -sY(s) \tag{3}$$

$$(x\frac{d}{dx})^n f \overset{\mathcal{M}}{\longleftrightarrow} (-s)^n Y(s) \tag{4}$$

If the initial conditions are zero. To modify this, consider the definition of Mellin transform.

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} \int_{-\infty}^{\infty} x^{s-1}(x\frac{dy}{dx})dx, \quad x \geq 1 \tag{5}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} \int_{1}^{\infty} x^{s}(\frac{dy}{dx})dx \tag{6}$$

Integrating by parts,

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} [x^s \int \frac{dy}{dx}dx]\Big|_1^{\infty} - \int_1^{\infty} sx^{s-1}y(x)dx \tag{7}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} x^s y(x)\Big|_1^{\infty} - sY(s) \tag{8}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} \lim_{x\to\infty}(x^s y(x)) - y(1) - sY(s) \tag{9}$$

Subject to $\lim_{x\to\infty}(x^s y(x)) = 0$,

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} -y(1) - sY(s) \tag{10}$$

Similarly,

$$(x\frac{d}{dx})^2 y \overset{\mathcal{M}}{\longleftrightarrow} s^2Y(s) + sy(1) - y'(1) \tag{11}$$

The given differential equation can be written as:

$$x\frac{d}{dx}(x\frac{dy}{dx}) = y, \quad x \geq 1 \tag{12}$$

$$\implies (x\frac{d}{dx})^2 y = y \tag{13}$$

Taking Mellin transform on both sides,

$$s^2Y(s) + sy(1) - y'(1) = Y(s) \tag{14}$$

From Table 0

$$Y(s) = s^2Y(s) + 6s - 2 \tag{15}$$

$$\implies Y(s) = \frac{6s-2}{1-s^2} \tag{16}$$

$$= -\frac{4}{s+1} - \frac{2}{s-1} \tag{17}$$

Property of Laplace Transform

$$e^{at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \quad \Re s > a \tag{18}$$

Taking inverse Mellin transform is equivalent to taking an inverse laplace transform and substituting x by $-\ln x$.

$$-\frac{4}{s+1} - \frac{2}{s-1} \overset{\mathcal{L}'}{\longleftrightarrow} -4e^{-x} - 2e^{x} \tag{19}$$

$$\implies L^{-1}\{Y(s)\} = -4e^{-x} - 2e^{x} \tag{20}$$

Substituting x by $-\ln x$

$$y(x) = -4x - \frac{2}{x} \tag{21}$$

$$\implies y(2) = -8 \tag{22}$$

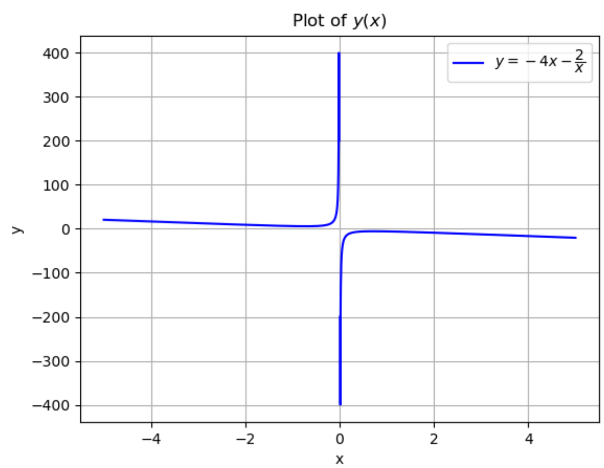

Fig. 0. Plot of $y(x)$ v/s x