Question 49, ME Gate 2023

EE23BTECH11017 - Eachempati Mihir Divyansh*

Question: Consider the second-order linear differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0,\ x \geq 1$$

with the initial conditions

$$y(x = 1) = 6, \quad \left.\frac{dy}{dx}\right|_{x=1} = 2.$$

Then the value of y at $x = 2$ is (GATE ME 2023)

Solution:

Symbol	Value	Description
y(x)	?	Function
y(1)	6	Initial Condition
y'(1)	2	Initial Condition

TABLE 0
GIVEN INFORMATION

By Euler-Cauchy substitution for the given question,

$$x = e^t, \quad t \geq 0 \tag{1}$$

$$\implies \frac{dt}{dx} = e^{-t} \tag{2}$$

$$\implies \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt} \tag{3}$$

$$\implies \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{d^2t}{dx^2}\left(\frac{dy}{dt}\right) \tag{4}$$

$$= e^{-2t}\frac{d^2y}{dt^2} + e^{-2t}\frac{dy}{dt} \tag{5}$$

So the given equation becomes

$$e^{2t}\left(e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right) + e^{-t}\left(e^{-t}\frac{dy}{dt}\right) - y = 0 \tag{6}$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} - y = 0 \tag{7}$$

$$\frac{d^2y}{dt^2} - y = 0, \quad t \geq 0 \tag{8}$$

Taking Laplace Transform,

$$s^2Y(s) - sy(0) - y'(0) - Y(s) \quad = 0 \tag{9}$$

$$Y(s) = \frac{sy(0) + y'(0)}{s^2 - 1} \tag{10}$$

$$= \frac{6s + 2}{s^2 - 1} \tag{11}$$

By partial fractions,

$$Y(s) = \frac{4}{s-1} + \frac{2}{s+1} \tag{12}$$

Taking Inverse Laplace using

$$\frac{1}{s+a} \xleftrightarrow{\mathcal{L}^{-1}} e^{-at} \tag{13}$$

We get,

$$y(t) = 4e^t + 2e^{-t} \tag{14}$$

From (1)

$$y(t) = 4x + \frac{2}{x} \tag{15}$$

$$\implies y(2) = 9 \tag{16}$$

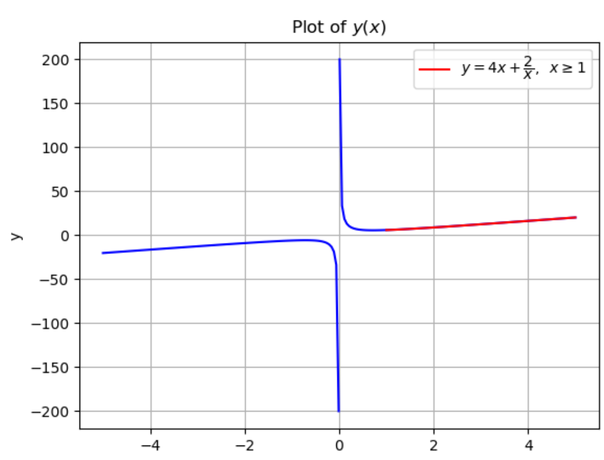

Fig. 0. Plot of $y(x)$ v/s x