Question 49, ME Gate 2023

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Question: Consider the second-order linear differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0,\ x \geq 1$$

with the initial conditions

$$y(x=1) = -6, \quad \left.\frac{dy}{dx}\right|_{x=1} = 2.$$

Then the value of y at $x = 2$ is __________. GATE ME 2023

Solution:

Symbol	Value	Description
y(x)	?	Function
y(1)	-6	-
y'(1)	2	-

TABLE 0
GIVEN INFORMATION

Consider the Mellin Transform

$$y(x) \overset{\mathcal{M}}{\longleftrightarrow} \int_{-\infty}^{\infty} x^{v-1}y(x)dx \tag{1}$$

Let

$$Y(v) = \int_{-\infty}^{\infty} x^{v-1}y(x)dx$$

Properties of the Mellin transform for a system at initial rest include

$$y'(x) \overset{\mathcal{M}}{\longleftrightarrow} -(v-1)Y(v-1) \tag{2}$$

$$xy'(x) \overset{\mathcal{M}}{\longleftrightarrow} -vY(v) \tag{3}$$

$$(x\frac{d}{dx})^n y \overset{\mathcal{M}}{\longleftrightarrow} (-v)^n Y(v) \tag{4}$$

To modify this, evaluating the Mellin Transform specifically,

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} \int_{-\infty}^{\infty} x^{v-1}(x\frac{dy}{dx})dx, \quad x \geq 1 \tag{5}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} \int_{1}^{\infty} x^{v}(\frac{dy}{dx})dx \tag{6}$$

Integrating by parts,

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} [x^v \int \frac{dy}{dx}dx]\Big|_1^{\infty} - \int_1^{\infty} vx^{v-1}y(x)dx \tag{7}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} x^v y(x)\Big|_1^{\infty} - vY(v) \tag{8}$$

$$\overset{\mathcal{M}}{\longleftrightarrow} \lim_{x\to\infty}(x^v y(x)) - y(1) - vY(v) \tag{9}$$

Let

$$L = \lim_{x\to\infty}(x^v y(x)) \tag{10}$$

Subject to $L = 0$, from (25),

$$(x\frac{dy}{dx}) \overset{\mathcal{M}}{\longleftrightarrow} -y(1) - vY(v) \tag{11}$$

$$(x\frac{d}{dx})^2 y \overset{\mathcal{M}}{\longleftrightarrow} v^2Y(v) + vy(1) - y'(1) \tag{12}$$

The given differential equation can be written as:

$$x\frac{d}{dx}(x\frac{dy}{dx}) = y, \quad x \geq 1 \tag{13}$$

$$\implies (x\frac{d}{dx})^2 y = y, \quad x \geq 1 \tag{14}$$

Taking Mellin transform on both sides, and from (27)

$$v^2Y(v) + vy(1) - y'(1) = Y(v), \quad v < -1 \tag{15}$$

From Table 0

$$Y(v) = v^2Y(v) + 6v - 2 \tag{16}$$

$$\implies Y(v) = \frac{6v-2}{1-v^2} \tag{17}$$

$$= -\frac{4}{v+1} - \frac{2}{v-1} \tag{18}$$

Mellin Inversion theorem

$$Y(s) \overset{\mathcal{M}^{-\infty}}{\longleftrightarrow} \frac{1}{2\pi j}\lim_{T\to\infty}\int_{a-jT}^{a+jT} x^v Y(v)dv \tag{19}$$

Here, there are 2 poles corresponding to $v = 1$ and $v = -1$. The limits of integration indicate a contour that can be assumed to cover a plane on eiher side of the verical line $x = a$.

$$y(x) = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{1-jT}^{1+jT} x^v Y(v)dv \tag{20}$$

This can be thought of as a contour encompassing the plane on the left of $x = 1$. Therefore

$$y(x) = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{1-jT}^{1+jT} \left(-\frac{4x^v}{v+1} - \frac{2x^v}{v-1}\right) dv \tag{21}$$

By Cauchy's Reidue Theorem

$$y(x) = -\frac{1}{0!} \lim_{v\to-1} (v+1) \frac{4x^v}{v+1} - \frac{1}{0!} \lim_{v\to1} (v-1) \frac{2x^v}{v-1} \tag{22}$$

$$= -\frac{4}{x} - 2x \tag{23}$$

To find ROC of v, substituting y(x) in (10)

$$\lim_{x\to\infty} x^v \left(-2x - \frac{4}{x}\right) = 0 \tag{24}$$

$$\implies \lim_{x\to\infty} \left(4x^{v-1} + 2x^{v+1}\right) = 0 \tag{25}$$

$$\implies \Re(v+1) < 0,\ \Re(v-1) < 0 \tag{26}$$

$$\implies \Re v < -1 \tag{27}$$

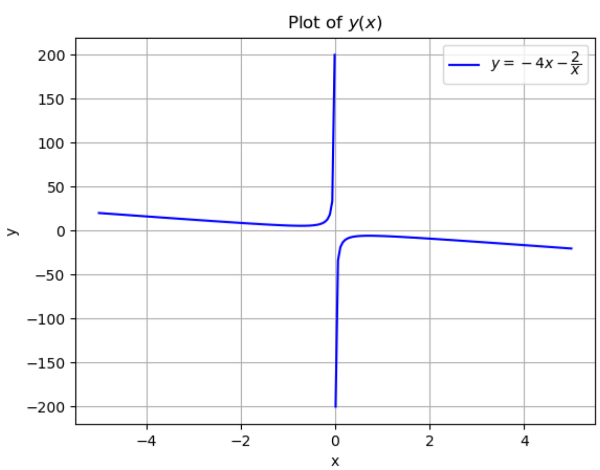

Fig. 0. Plot of $y(x)$ v/s x