Filter Design

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I. Introduction

We must design an IIR and FIR filter with a given filter number. This is a bandpass filter whose specifications are shown below.

II. Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz. If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$.

A. Digital Filter

1) *Passband* :
$\{4 + 0.6(j)\}$ kHz to $\{4 + 0.6(j + 2)\}$ kHz, where

$$j = (r - 11000) \mod \sigma \tag{1}$$

where σ is the sum of the digits of my roll number and r is the roll number.

$$r = 11017 \tag{2}$$
$$\sigma = 10 \tag{3}$$
$$j = 7 \tag{4}$$

So, the un-normalized, discrete-time passband frequencies are

$$F_{p_1} = 8.2 \text{ kHz} \tag{5}$$
$$F_{p_2} = 9.4 \text{ kHz} \tag{6}$$

The corresponding normalized digital frequencies are:

$$\omega_{p_1} = 2\pi\frac{F_{p_1}}{F_s} = 0.34\pi \tag{7}$$
$$\omega_{p_2} = 2\pi\frac{F_{p_2}}{F_s} = 0.39\pi \tag{8}$$
$$\omega_c = \frac{\omega_{p_1} + \omega_{p_2}}{2} \tag{9}$$

Where ω_c is the center frequency.

2) *Tolerance:* The deviations in the passband and stopband (δ_p and δ_s) are called the tolerances of a filter. Assuming that they are equal, let

$$\delta_p = \delta_s = 0.15 \tag{10}$$

3) *Stopband*: The transition band on either side of the passband is given to be

$$\Delta F = 0.3 \text{ kHz} \tag{11}$$

Therefore,

$$F_{s_1} = F_{p_1} - \Delta F = 7.9 \text{ kHz} \tag{12}$$
$$F_{s_2} = F_{p_2} + \Delta F = 9.7 \text{ kHz} \tag{13}$$

and the corresponding normalized digital frequencies are:

$$\omega_{s_1} = 0.33\pi \tag{14}$$
$$\omega_{s_2} = 0.40\pi \tag{15}$$

B. Analog Filter

The analog filter frequency is related to the digital frequency by

$$\Omega = \tan\frac{\omega}{2} \tag{16}$$
$$\implies \Omega_{p_1} = 0.59,\ \Omega_{p_2} = 0.70, \tag{17}$$
$$\implies \Omega_{s_1} = 0.57,\ \Omega_{s_2} = 0.74 \tag{18}$$

III. The IIR Filter Design

For the design of our bandpass IIR filter, we require a stopband with a monotonic characteristic and an equiripple passband. Consequently, the Chebyshev approximation is employed.

A. The Analog Filter

1) *Low Pass Filter Specifications:* Let $H_{a,BP}(j\Omega)$ be the desired analog bandpass filter, with the specifications provided in **??**, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter. Define

$$\Omega_0 = \sqrt{\omega_{p_1}\omega_{p_2}} = 0.64 \tag{19}$$
$$B = \omega_{p_2} - \omega_{p_1} = 0.11 \tag{20}$$

Then,

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{21}$$

Substituting Ω_{s_1} and Ω_{s_2} in (**??**) we obtain the stopband edges of lowpass filter

$$\Omega_{Ls1} = \frac{\Omega_{s_1}^2 - \Omega_0^2}{B\Omega_{s_1}} = -1.50 \tag{22}$$

$$\Omega_{Ls2} = \frac{\Omega_{s_2}^2 - \Omega_0^2}{B\Omega_{s_2}} = 1.56 \tag{23}$$

We choose the minimum of these two stopband edges

$$\Omega_{Ls} = \min\left(\left|\Omega_{Ls_1}\right|, |\Omega|_{Ls_2}\right) = 1.50 \tag{24}$$

2) *The Low Pass Chebyschev Filter Parameters:* The magnitude of an nth order Chebyschev low-pass filter transfer function is given by:

$$H_{a,LP}\left(j\Omega_L\right) = \frac{1}{\sqrt{1 + \varepsilon^2 T_N^2\left(\Omega_L/\Omega_{Lp}\right)}} \tag{25}$$

where

$$\varepsilon = \sqrt{10^{\delta_p/10} - 1}$$

is the ripple factor, Ω_{Lp} is the cutoff frequency and

$$T_n\left(x\right) = \cosh\left(N \cosh^{-1} x\right)$$

is a Chebyshev polynomial of the nth order. The recurrence relation for the polynomials is as follows

$$T_{N+2} = 2xT_{N+1} - T_N \tag{26}$$

The filter parameters have the following constraints

$$\frac{\sqrt{D_2}}{c_N\left(\Omega_{Ls}\right)} \le \varepsilon \le \sqrt{D_1} \tag{27}$$

$$N \ge \left\lceil \frac{\cosh^{-1}\sqrt{\frac{D_2}{D_1}}}{\cosh^{-1}\Omega_{Ls}} \right\rceil \tag{28}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$ and $\lceil . \rceil$ is known as the ceiling operator.

After the necessary substitutions, we get N $\ge$ 4 and $0.278 \le \varepsilon \le 0.61$. Plotting the magnitude of the transfer function vs ε, we get

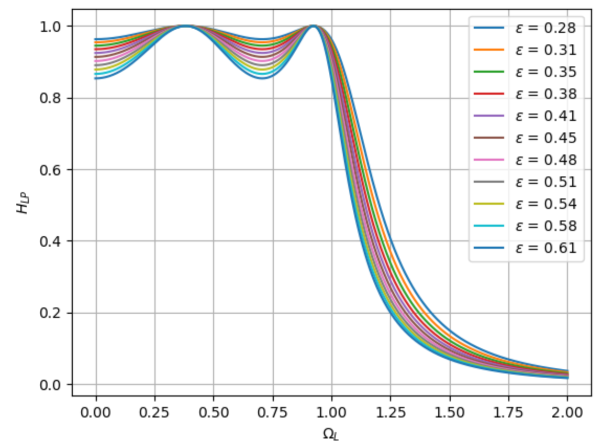

Fig. III-1. Analog Low-Pass Frequency Response for $0.28 \le \varepsilon \le 0.61$

As the value of ε increases, the roll-off becomes steeper. We choose $\varepsilon = 0.29$, as it perfectly transitions from the passband to the stopband.

3) *The Low Pass Chebyschev Filter:* The next step in design is to find an expression for magnitude response in the s domain.
Using $s = j\Omega$ or in this case $s_L = j\Omega_L$ we obtain:

$$\left|H_{a,LP}(j\Omega_L)\right|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\frac{s_L}{j})} \tag{29}$$

To find poles equate the denominator to zero:

$$1 + \epsilon^2 T_N^2\left(\frac{s_L}{j}\right) = 0 \tag{30}$$

where

$$T_N(x) = \cos\left(N\cos^{-1}(x)\right), \ |x| < 1 \tag{31}$$

$$T_N(x) = \cosh\left(N\cosh^{-1}(x)\right), \ |x| \ge 1 \tag{32}$$

On solving (**??**) we obtain poles :

$$s_k = -\Omega_{Lp}\sin\left(A_k\right)\sinh\left(B_k\right) - j\Omega_{Lp}\cos\left(A_k\right)\cosh\left(B_k\right) \tag{33}$$

where k is the index of the pole and

$$A_k = (2k+1)\frac{\pi}{2N} \tag{34}$$

$$B_k = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{35}$$

The s_k values, calculated and plotted by the following Python code are as follows:

https://github.com/Mihir−Divyansh/EE1205/blob/main/Filter_Design/codes/sk.py

Pole	Value
s_1	0.460 + 0.428j
s_2	0.460 - 0.428j
s_3	0.191 - 1.032j
s_4	0.191 + 1.032j
s_5	-0.191 - 1.032j
s_6	-0.460 - 0.428j
s_7	-0.460 + 0.428j
s_8	-0.191 + 1.032j

TABLE III-1
VALUES OF s_k

The poles in the left half of the plane i.e., with -ve real part are taken in the filter design as we intend to design a stable system.

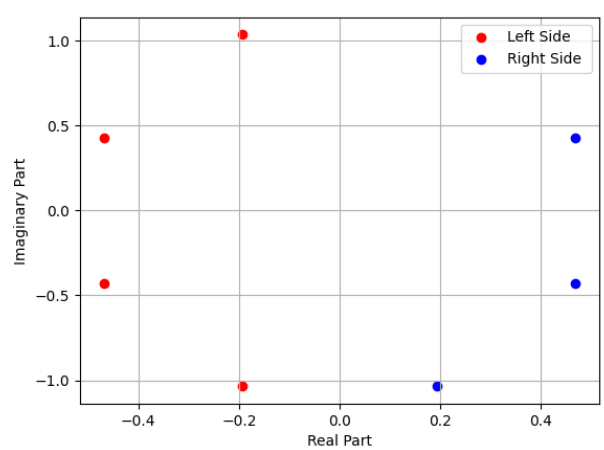

Fig. III-2. Pole-Zero Plot

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{36}$$

where G_{LP} is the gain of the Low pass filter. Refer to Table **??** for s_k values.

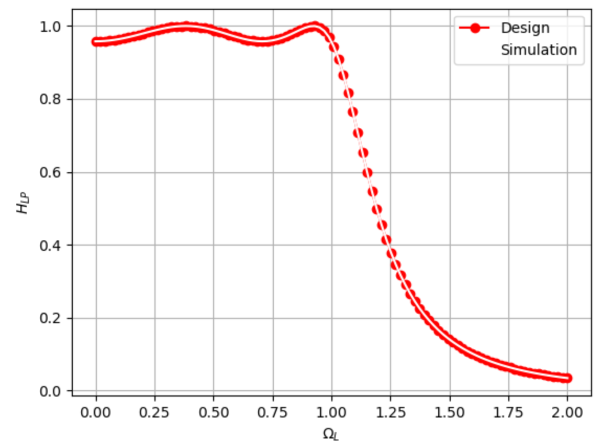

Fig. III-3. Design v/s Simulation for the filter

We know that from (**??**):-

$$\left|H_{a,LP}(s_L)\right| = \frac{1}{\sqrt{1+\epsilon^2}} \text{at } \Omega_L = 1 \implies s_L = j \tag{37}$$

Substituting values from (**??**) in (**??**) we get $G_{LP} = 0.4166$

$$H_{a,LP}(s_L) = \frac{0.42}{s_L^4 + 1.30s_L^3 + 1.85s_L^2 + 1.17s_L + 0.43} \tag{38}$$

4) *The Band Pass Chebyschev Filter:* After verifying the design with the required specifications the next step is to jump to the required type of filter using frequency transformation.

$$s_L = \frac{s^2 + \Omega_0^2}{Bs} \tag{39}$$

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L = \frac{s^2+\Omega_0^2}{Bs}} \tag{40}$$

As there is one-to-one correspondence between the filters so $\Omega = \omega_{p_1}$ should correspond to Ω_{Lp}

$$s = j\omega_{p_1} \tag{41}$$

$$s_L = \frac{(j\omega_{p_1})^2 + \Omega_0^2}{B(j\omega_{p_1})} \tag{42}$$

$$\left|H_{a,BP}(j\omega_{p_1})\right| = 1 \tag{43}$$

$$G_{BP}\left|H_{a,LP}(s_L)\right| = 1 \tag{44}$$

Substituting (**??**) in (**??**) we obtain Gain of required bass pass filter:

$$G_{BP} = 1.04 \tag{45}$$

Thus the response in the s domain

$$H_{a,BP}(s) = \frac{6.49 \times 10^{-5} s^4}{s^8 + 0.14s^7 + 0.17s^6 + 0.18s^5 + 1.05s^4 + 0.75s^3 + 0.29s^2 + 0.01s + 0.03} \tag{46}$$

The expressions in the s-domain and gain factors have been computed using Python code.

In Figure **??**, we present the magnitude of $H_{a,BP}(j\Omega)$ as a function of Ω for both positive and negative frequencies. The pass-band and stop-band frequencies depicted in the figure exhibit close alignment with those determined analytically through the bilinear transformation.

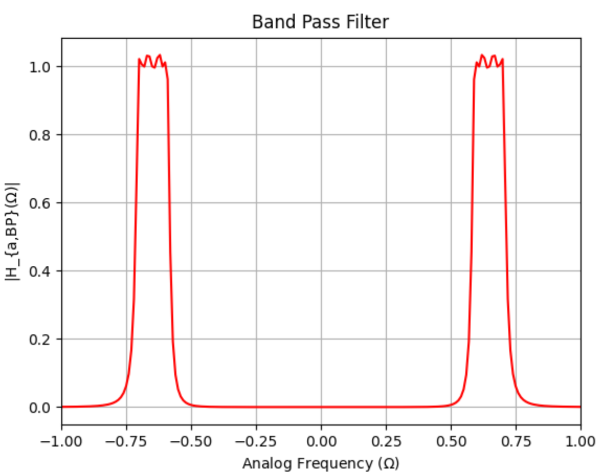

Fig. III-4. Magnitude response of the Analog Band Pass Filter

B. The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{47}$$

Substituting

$$s = \frac{1-z^{-1}}{1+z^{-1}}$$

in (**??**) and calculating expression using a python code we get :

$$H_{d,BP}(z) = \frac{G\left(1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8}\right)}{3.62 - 6.95z^{-1} + 26.74z^{-2} - 60.15z^{-3} + 73.76z^{-4} - 49.57z^{-5} + 26.14z^{-6} - 7.62z^{-7} + 1.45z^{-8}} \tag{48}$$

where $G = 6.49 \times 10^{-5}$

The plots for the filter magnitude by approximate calculation (using arrays) and symbolic math in Python are given below. The specifications for the filter are met.

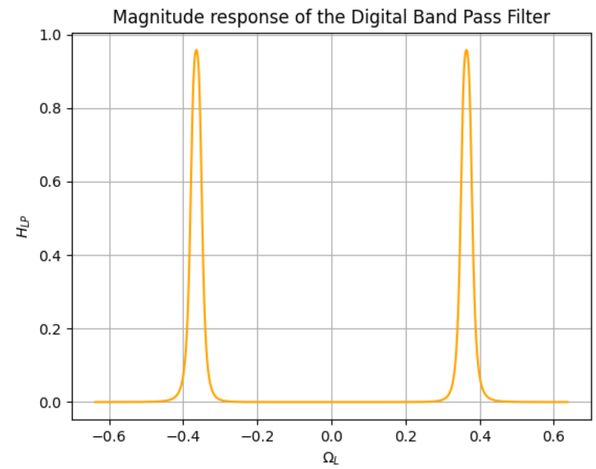

Fig. 4. Approximation of the Filter

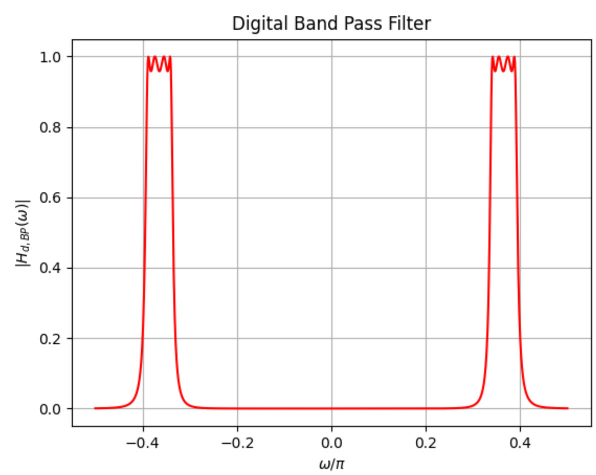

Fig. 4. Magnitude of the Digital Band Pass Filter

IV. The FIR Filter

We design the FIR filter by obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

A. The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is $\delta_s = 0.15$.The cutoff-frequency is given by :

$$\omega_{Lp} = \frac{B}{2} = 0.025\pi \tag{49}$$

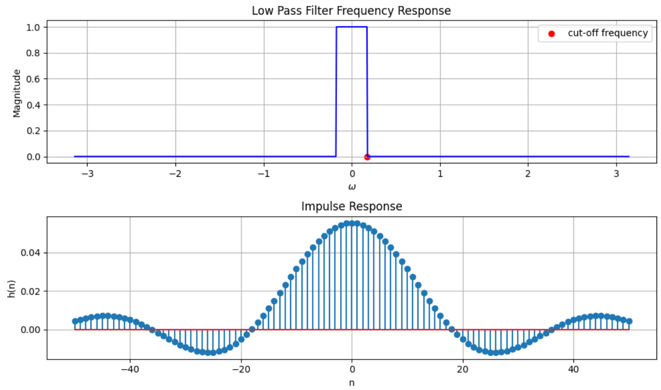

Fig. 4. Impulse Response of an Ideal Low Pass Filter

The impulse response of the ideal Low Pass Filter is given by :

$$h(n) = \begin{cases} \frac{w_l}{\pi}, & \text{if } n = 0 \\ \frac{\sin(w_l n)}{n\pi}, & \text{if } n \neq 0 \end{cases} \tag{50}$$

From (**??**) we conclude that $h(n)$ for an ideal Low Pass Filter is not causal and can neither be made causal by introducing a finite delay. And $h(n)$ do not converge, so the system is unstable. Therefore we move on to windowing the impulse response.

B. The Kaiser Window

The Kaiser window is defined as

$$w(n) = \begin{cases} \dfrac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, & -N \leq n \leq N, \quad \beta > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $I_0(x)$ is a modified Bessel Function of the first kind of zero order in x. β and N are the window-shaping factors.

1) N is chosen according to

$$N \geq \frac{A-8}{4.57\Delta\omega}, \tag{51}$$

where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \geq 48$.

2) β is chosen according to

$$\beta N = \begin{cases} 0.11(A-8.7) & A > 50 \\ 0.58(A-21)^{0.4} + 0.08(A-21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \tag{52}$$

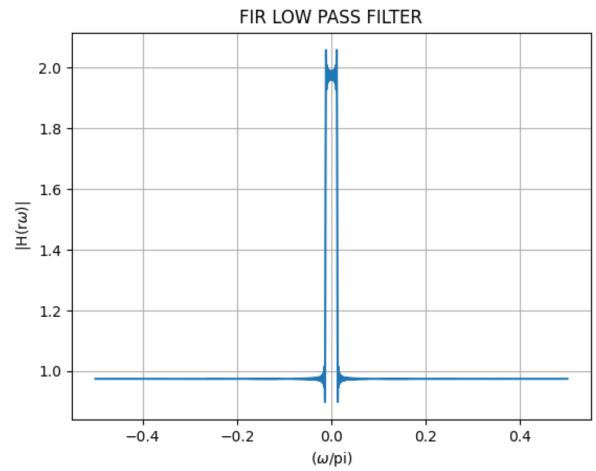

Fig. 2. Magnitude Response of the FIR Low Pass Filter

The window function is defined as :

$$w(n) = \begin{cases} 1, & \text{for } -48 \leq n \leq 48 \\ 0, & \text{otherwise} \end{cases} \tag{53}$$

Therefore the desired impulse response is :

$$h_{lp} = h_n w_n \tag{54}$$

$$h(n) = \begin{cases} \dfrac{\sin(w_l n)}{n\pi}, & \text{for } -48 \leq n \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

C. The Equivalent Band Pass Filter

A Band-Pass Filter (BPF) can be obtained by subtracting the magnitude response of a Low-Pass Filter (LPF) with cutoff frequency ω_{p_1} from another LPF magnitude response with cutoff frequency ω_{p_2}.

$$h_{BP}(n) = \begin{cases} \dfrac{\sin(\omega_{p_2} n)}{n\pi} - \dfrac{\sin\left(\omega_{p_1} n\right)}{n\pi}, & \text{for } n \neq 0 \\ \dfrac{\omega_{p_2} - \omega_{p_1}}{\pi} & \text{for } n = 0 \end{cases} \tag{56}$$

$$\frac{\sin(\omega_{p_2} n)}{n\pi} - \frac{\sin\left(\omega_{p_1} n\right)}{n\pi} = 2\cos\left(\frac{\omega_{p_2} n + \omega_{p_1} n}{2}\right)\sin\left(\frac{\omega_{p_2} n - \omega_{p_1} n}{2}\right) \tag{57}$$

$$= \frac{2\cos(0.37n\pi)\sin(0.02n\pi)}{n\pi} \tag{58}$$

Multiplying by window function we get :

$$h_{BP}(n) = \begin{cases} \frac{2\cos(0.37n\pi)\sin(0.02n\pi)}{n\pi}, & \text{for } |n| \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{59}$$

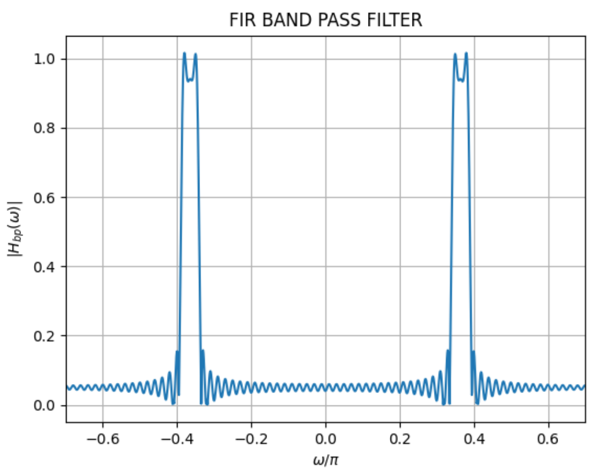

Fig. 2. Magnitude Response of the FIR Band Pass Filter