

10.5.2.7

EE23BTECH11017 - Eachempati Mihir Divyansh*

Question: Consider the second-order linear differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0, \quad x \geq 1$$

with the initial conditions

$$y(x=1) = 6, \quad \frac{dy}{dx} \Big|_{x=1} = 2.$$

Then the value of y at $x = 2$ is _____. GATE ME 2023

Solution:

Symbol	Value	Description
$y(x)$?	Function
$y(1)$	6	-
$y'(1)$	2	-

TABLE 0
GIVEN INFORMATION

Consider the Mellin transform as the combined operation of substituting x by e^{-x} and subsequently taking the laplace transform:

$$y(x) \xleftrightarrow{M} \int_{-\infty}^{\infty} x^{s-1} y(x) dx \quad (1)$$

Let $Y(s) = \int_{-\infty}^{\infty} x^{s-1} y(x) dx$ Properties of the Mellin transform include

$$y'(x) \xleftrightarrow{M} -(s-1)Y(s-1) \quad (2)$$

$$xy'(x) \xleftrightarrow{M} -sY(s) \quad (3)$$

$$(x \frac{d}{dx})^n y \xleftrightarrow{M} (-s)^n Y(s) \quad (4)$$

If the initial conditions are zero. To modify this, consider the definition of Mellin transform.

$$(x \frac{dy}{dx}) \xleftrightarrow{M} \int_{-\infty}^{\infty} x^{s-1} (x \frac{dy}{dx}) dx, \quad x \geq 1 \quad (5)$$

$$\xleftrightarrow{M} \int_1^{\infty} x^s (\frac{dy}{dx}) dx \quad (6)$$

Integrating by parts,

$$(x \frac{dy}{dx}) \xleftrightarrow{M} [x^s \int \frac{dy}{dx} dx] \Big|_1^{\infty} - \int_1^{\infty} s x^{s-1} y(x) dx \quad (7)$$

$$\xleftrightarrow{M} x^s y(x) \Big|_1^{\infty} - sY(s) \quad (8)$$

$$\xleftrightarrow{M} \lim_{x \rightarrow \infty} (x^s y(x)) - y(1) - sY(s) \quad (9)$$

Subject to $\lim_{x \rightarrow \infty} (x^s y(x)) = 0$ from (23),

$$(x \frac{dy}{dx}) \xleftrightarrow{M} -y(1) - sY(s) \quad (10)$$

Similarly,

$$(x \frac{d}{dx})^2 y \xleftrightarrow{M} s^2 Y(s) + sy(1) - y'(1) \quad (11)$$

The given differential equation can be written as:

$$x \frac{d}{dx} (x \frac{dy}{dx}) = y, \quad x \geq 1 \quad (12)$$

$$\Rightarrow (x \frac{d}{dx})^2 y = y \quad (13)$$

Taking Mellin transform on both sides, and from (25)

$$s^2 Y(s) + sy(1) - y'(1) = Y(s), \quad s < -1 \quad (14)$$

From Table 0

$$Y(s) = s^2 Y(s) + 6s - 2 \quad (15)$$

$$\Rightarrow Y(s) = \frac{6s - 2}{1 - s^2} \quad (16)$$

$$= -\frac{4}{s+1} - \frac{2}{s-1} \quad (17)$$

Property of Laplace Transform

$$e^{at} \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}, \quad \Re s > a \quad (18)$$

Taking inverse Mellin transform is equivalent to taking an inverse laplace transform and substituting x by $-\ln x$.

$$-\frac{4}{s+1} - \frac{2}{s-1} \xleftrightarrow{\mathcal{L}'} -4e^{-x} - 2e^x \quad (19)$$

$$\Rightarrow L^{-1}\{Y(s)\} = -4e^{-x} - 2e^x \quad (20)$$

Substituting x by $-\ln x$

$$y(x) = -4x - \frac{2}{x} \quad (21)$$

To find ROC of s , substituting $y(x)$ in the limit

$$\lim_{x \rightarrow \infty} x^s \left(-4x - \frac{2}{x} \right) = 0 \quad (22)$$

$$\Rightarrow \lim_{x \rightarrow \infty} (4x^{s+1} + 2x^{s-1}) = 0 \quad (23)$$

$$\Rightarrow \Re s + 1 < 0, \Re s - 1 < 0 \quad (24)$$

$$\Rightarrow \Re s < -1 \quad (25)$$

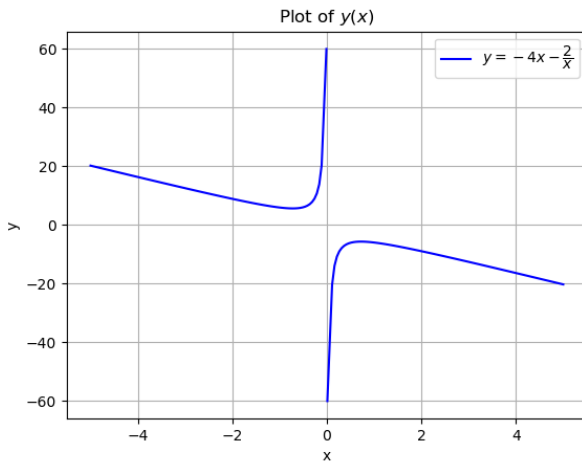


Fig. 0. Plot of $y(x)$ v/s x