ASSIGNMENT-7
GATE ST-2022 53-65

EE24BTECH11019 - DWARAK A

Q.36 - Q.65 carry TWO marks each

1) Let $X_1, X_2, \ldots, X_7$ be a random sample from a normal population with mean 0 and variance $\theta > 0$. Let
$$K = \frac{X_1^2 + X_2^2}{X_1^2 + X_2^2 + \cdots + X_7^2}$$
Consider the following statements:
   (I) The statistics K and $X_1^2 + X_2^2 + \cdots + X_7^2$ are linearly independent.
   (II) $\frac{7K}{2}$ has an F-distribution with 2 and 7 degrees of freedom.
   (III) $E(K^2) = \frac{8}{63}$
   Then which of the following statements is/are true?
   a) (I) and (II) only
   b) (I) and (III) only
   c) (II) and (III) only
   d) (I) only
2) Consider the following statements:
   (I) Let a random variable X have the probability density function
   $$f_X(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty$$
   Then there exist *i.i.d* random variables X_1 and X_2 such that X and $X_1 - X_2$ have the same distribution.
   (II) Let a random variable Y have the probability density function
   $$f_Y(y) = \begin{cases} \frac{1}{4}, & -2 < y < 2 \\ 0, & \text{elsewhere} \end{cases}$$
   Then there exist *i.i.d* random variables Y_1 and Y_2 such that Y and $Y_1 - Y_2$ have the same distribution.
   Then which of the above statements is/are true?
   a) (I) only
   b) (II) only
   c) Both (I) and (II)
   d) Neither (I) nor (II)
3) Suppose $X_1, X_2, \ldots, X_n, \ldots$ are independent exponential random variables with the mean $\frac{1}{2}$. Let the notation *i.o.* denote 'infinitely often'. Then which of the following

is/are true?

a) $P\left(\left\{X_n > \frac{\epsilon}{2}\log_e n\right\} i.o.\right) = 1$ for $0 < \epsilon \leq 1$
b) $P\left(\left\{X_n < \frac{\epsilon}{2}\log_e n\right\} i.o.\right) = 1$ for $0 < \epsilon \leq 1$
c) $P\left(\left\{X_n > \frac{\epsilon}{2}\log_e n\right\} i.o.\right) = 1$ for $\epsilon > 1$
d) $P\left(\left\{X_n < \frac{\epsilon}{2}\log_e n\right\} i.o.\right) = 1$ for $\epsilon > 1$

4) Let $\{X_n\}, n \geq 1$, be a sequence of random variables with the probability mass functions

$$p_{X_n}(x) = \begin{cases} \frac{n}{n+1}, & x = 0, \\ \frac{1}{n+1}, & x = n, \\ 0, & \text{elsewhere.} \end{cases}$$

Let X be a random variable with $P(X = 0) = 1$. Then which of the following statements is/are true?

a) X_n converges to X in distribution
b) X_n converges to X in probability
c) $E(X_n) \to E(X)$
d) There exists a subsequence $\{X_n\}$ of X_n such that X_{n_k} converges to X almost surely

5) Let M be any 3×3 symmetric matrix with eigenvalues 1, 2 and 3. Let N be any 3×3 matrix with real eigenvalues such that $MN + N^\top M = 3I$, where I is the 3×3 identity matrix. Then which of the following cannot be eigenvalue(s) of the matrix N ?

a) $\frac{1}{4}$
b) $\frac{3}{4}$
c) $\frac{1}{2}$
d) $\frac{7}{4}$

6) Let M be a 3×2 real matrix having a singular value decomposition as $M = USV^\top$ where the matrix $S = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^\top$, U is a 3×3 orthogonal matrix, and V is a 2×2 orthogonal matrix. Then which of the following statements is/are true?

a) The rank of the matrix M is 1
b) The trace of the matrix $M^\top M$ is 4
c) The largest singular value of the $(M^\top M)^{-1} M^\top$ matrix is 1
d) The nullity of the matrix M is 1

7) Let X be a random variable such that

$$P\left(\frac{a}{2\pi} X \in \mathbb{Z}\right) = 1, \quad a > 0,$$

where $\mathbb{Z}$ denotes the set of all integers. If $\phi_X(t), t \in \mathbb{R}$, denotes the characteristic function of X, then which of the following is/are true?

a) $\phi_X(a) = 1$
b) $\phi_X(\cdot)$ is periodic with period a
c) $|\phi_X(t)| < 1$ for all $t \neq a$

d) $\int_{0}^{2\pi} e^{-itn}\phi_X(t)dt = \pi P\left(X = \frac{2\pi n}{a}\right), n \in \mathbb{Z}, i = \sqrt{-1}$

8) Which of the following real valued functions is/are uniformly continuous on $[0, \infty)$?
   a) $\sin^2 x$
   b) $x \sin x$
   c) $\sin(\sin x)$
   d) $\sin(x \sin x)$
9) Two independent random samples, each of size 7, from two populations yield the following values: If Mann-Whitney U test is performed at 5% level of significance

Population 1	18	20	16	20	17	18	14
Population 2	17	18	14	20	14	13	16

   to test the null hypothesis H_0: Distributions of the populations are same, against the alternative hypothesis H_1: Distributions of the populations are not same, then the value of the test statistic U (in integer) for the given data, is ______
10) Consider the multiple regression model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \epsilon,$$

   where ϵ is normally distributed with mean 0 and variance $\sigma^2 > 0$, and $\beta_0, \beta_1, \beta_2, \beta_3$ are unknown parameters. Suppose 52 observations of (Y, X_1, X_2, X_3) yield sum of squares due to regression as 18.6 and total sum of squares as 79.23. Then, for testing the null hypothesis $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ against the alternative hypothesis $H_1 : \beta_i \neq 0$ for some i = 1, 2, 3, the value of the test statistic (rounded off to three decimal places), based on one way analysis of variance, is ______
11) Suppose a random sample of size 3 is taken from a distribution with the probability density function

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   If p is the probability that the largest sample observation is at least twice the smallest sample observation, then the value of p (rounded off to three decimal places) is ______
12) Let a linear model $Y = \beta_0 + \beta_1 X + \epsilon$ be fitted to the following data, where ϵ is normally distributed with mean 0 and unknown variance $\sigma^2 > 0$ Let $\hat{Y}_0$ denote the

x_i	0	1	2	3	4
y_i	3	4	5	6	7

   ordinary least-square estimator of Y at $X = 6$, and the variance of $hatY_0 = c\sigma^2$. Then the value of the real constant c (rounded off to one decimal place) is equal to ______
13) Let $0, 1, 1, 2, 0$ be five observations of a random variable X which follows a Poisson distribution with the parameter $\theta > 0$. Let the minimum variance unbiased estimate of $P(X \leq 1)$, based on this data, be α. Then $5^4\alpha$ (in integer) is equal to ______