Fractal in Action

G. V. V. Sharma
Dept. of Electrical Engg.,
IIT Hyderabad.

June 16, 2024

Question

The centre of a circle is at (2,0). If one end of a diameter is at (6,0), then the other end is at :

1. $(0, 0)$
2. $(4, 0)$
3. $(-2, 0)$
4. $(-6, 0)$

Solution

Let

$$\mathbf{O} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1}$$

If $\mathbf{O}$ divides AB in the ratio $k : 1$,

$$\mathbf{O} = \frac{(\mathbf{A} + k\mathbf{B})}{1 + k} \tag{2}$$

In this case, $\because k = 1$,

$$\mathbf{O} = \frac{(\mathbf{A} + \mathbf{B})}{2} \tag{3}$$

$$\implies \mathbf{B} = 2\mathbf{O} - \mathbf{A} \tag{4}$$

$$= 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{5}$$

Question

$ABCD$ is a rectangle formed by the points $A\left(-1,-1\right)$,$B\left(-1,6\right)$,$C\left(3,6\right)$ and $D\left(3,-1\right)$. P,Q,R and S are mid-points of sides AB,BC,CD and DA respectively. Show that the diagonals of the quadrilateral $PQRS$ bisect each other.

Solution

From (2),

$$\mathbf{P} = \frac{\mathbf{A}+\mathbf{B}}{2}, \ \mathbf{Q} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{6}$$

$$\mathbf{R} = \frac{\mathbf{C}+\mathbf{D}}{2}, \ \mathbf{S} = \frac{\mathbf{D}+\mathbf{A}}{2} \tag{7}$$

Let $\mathbf{O}_1$ and $\mathbf{O}_2$ be the midpoints of PR and QS respectively

$$\mathbf{O}_1 = \frac{\mathbf{P}+\mathbf{R}}{2} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}}{4} \tag{8}$$

$$\mathbf{O}_2 = \frac{\mathbf{Q}+\mathbf{S}}{2} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}}{4} \tag{9}$$

Since

$$\mathbf{O}_1 = \mathbf{O}_2, \tag{10}$$

the diagonals bisect each other.

Question

AD is a median of ΔABC with vertices $A\left(5,-6\right), B\left(6,4\right)$ and $C\left(0,0\right)$. Length AD is equal to:

1. $\sqrt{68}$
2. $2\sqrt{15}$
3. $\sqrt{101}$
4. 10

Solution

The midpoint of **BC** is

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{11}$$

$$= \frac{1}{2}\begin{pmatrix}6\\4\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}, \tag{12}$$

Since

$$\mathbf{A}-\mathbf{D} = \begin{pmatrix}5\\-6\end{pmatrix} - \begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}2\\-8\end{pmatrix} \tag{13}$$

$$\implies ||\mathbf{A}-\mathbf{D}|| \triangleq \sqrt{(\mathbf{A}-\mathbf{D})^\top(\mathbf{A}-\mathbf{D})} \tag{14}$$

$$= \sqrt{\begin{pmatrix}2 & -8\end{pmatrix}\begin{pmatrix}2\\-8\end{pmatrix}} = \sqrt{2^2+8^2} = \sqrt{68} \tag{15}$$

Question

If the distance between the points $(3, -5)$ and $(x, -5)$ is 15 units, then the values of x are

1. $12, -18$
2. $-12, 18$
3. $18, 5$
4. $-9, -12$

Solution

$$\mathbf{A} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} x \\ -5 \end{pmatrix} \tag{16}$$

$$\implies \mathbf{A} - \mathbf{B} = \begin{pmatrix} 3 - x \\ -5 - (-5) \end{pmatrix} = \begin{pmatrix} 3 - x \\ 0 \end{pmatrix} \tag{17}$$

$$\implies ||\mathbf{A} - \mathbf{B}|| = \sqrt{\begin{pmatrix} 3 - x & 0 \end{pmatrix} \begin{pmatrix} 3 - x \\ 0 \end{pmatrix}} = \sqrt{(3 - x)^2} \tag{18}$$

$$\implies 15 = \pm (3 - x) \tag{19}$$

$$\implies x = -12, 18 \tag{20}$$

Question

Solve the following system of linear equations algebraically

$$\begin{aligned} 2x + 5y &= -4 \\ 4x - 3y &= 5 \end{aligned} \tag{21}$$

Solution

(21) can be expressed as

$$\begin{aligned} \mathbf{n}_1^\top \mathbf{x} &= c_1 \\ \mathbf{n}_2^\top \mathbf{x} &= c_2 \end{aligned} \tag{22}$$

where

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \ c_1 = 5 \tag{23}$$

$$\mathbf{n}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \ c_2 = -4 \tag{24}$$

(22) gives the normal forms of the equations given in (21) where

$$\mathbf{n}_1, \mathbf{n}_2 \tag{25}$$

are defined to be the normal vectors of the respective lines.

Solution

(22) can be expressed as

$$\begin{pmatrix}\mathbf{n}_1 & \mathbf{n}_2\end{pmatrix}^\top \mathbf{x} = \begin{pmatrix}c_1\\ c_2\end{pmatrix} \tag{26}$$

yielding the matrix equation

$$\begin{pmatrix}2 & 5\\ 4 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-4\\ 5\end{pmatrix} \tag{27}$$

Writing the augmented matrix for using Gauss elimination

$$\left(\begin{array}{cc|c}2 & 5 & -4\\ 4 & -3 & 5\end{array}\right) \xleftarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix}2 & 5 & -4\\ 0 & -13 & 13\end{pmatrix} \tag{28}$$

$$\begin{pmatrix}2 & 5 & -4\\ 0 & -13 & 13\end{pmatrix} \xleftarrow{R_1 \to \frac{13}{5}R_1 + R_2} \begin{pmatrix}\frac{26}{5} & 0 & \frac{13}{5}\\ 0 & -13 & 13\end{pmatrix} \tag{29}$$

$$\implies \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ -1\end{pmatrix} \tag{30}$$

Question

Find the ratio in which the point $C\left(\frac{8}{5}, y\right)$ divides the line segment joining the points $A(1,2)$ and $B(2,3)$. Also, find the value of y.

Solution

For collinearity,

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{31}$$

Performing row reduction,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 8/5 \\ 2 & 3 & y \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & (2-1) & \left(\frac{8}{5}-1\right) \\ 0 & (3-2) & (y-3) \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 1 & y-3 \end{pmatrix} \tag{32}$$

$$\xleftrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & y-\frac{18}{5} \end{pmatrix} \implies y = \frac{18}{5} \tag{33}$$

in order to get a 0 row.

Question

The sum of the digits of a 2-digit number is 14. The number obtained by interchanging its digits exceeds the given number by 18. Find the number.

Solution

Let the digits of the number be x_1(tens) and x_2(units).Given

$$x_1 + x_2 = 14 \tag{34}$$

$$10x_2 + x_1 = 18 + 10x_1 + x_2 \tag{35}$$

$$\implies x_1 - x_2 = -2 \tag{36}$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \end{pmatrix} \tag{37}$$

$$\tag{38}$$

Solution

If

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{39}$$

$$\mathbf{A}^\top \mathbf{A} = \mathbf{I} \tag{40}$$

A is then defined to be an orthogonal matrix.

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ -2 \end{pmatrix} \tag{41}$$

$$\implies 2\mathbf{I}\mathbf{x} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \tag{42}$$

$$\implies \mathbf{x} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \tag{43}$$

Topics covered so far

1. Vectors
2. Section Formula
3. Norm
4. Gauss Elimination
5. Rank
6. Orthogonal matrix

Question

If $\overrightarrow{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{b} = \hat{i} + \hat{j} - \hat{k}$, then $\overrightarrow{a}$ and $\overrightarrow{b}$ are:

1. Collinear vectors which are not parallel
2. Parallel vectors
3. Perpendicular vectors
4. Unit vectors

Solution

Let

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{44}$$

Applying concept of rank from (31)

$$\text{rank} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} = 2 \neq 1, \text{Not parallel} \tag{45}$$

Applying condition for perpendicularity:

$$\mathbf{a}^\top \mathbf{b} = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \implies \mathbf{a} \perp \mathbf{b} \tag{46}$$

Question

If α,β and γ are the angles which a line makes with positive directions of x,y and z axes respectively, then which of the following are not true?

1. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
2. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$
3. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$
4. $\cos\alpha + \cos\beta + \cos\gamma = 1$

Solution

Let $\mathbf{m}$ represent the unit direction vector of the line. Then,

$$\mathbf{m} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \tag{47}$$

with

$$\|\mathbf{m}\| = 1 \tag{48}$$

Parametric Form

Also,

$$
\begin{aligned}
2x + 5y &= -4 && (49)\\
\implies 2x &= -4 - 5y && (50)\\
\implies \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -2 \\ 0 \end{pmatrix} + y \begin{pmatrix} -\frac{5}{2} \\ 1 \end{pmatrix} && (51)\\
\mathbf{x} &= \begin{pmatrix} -2 \\ 0 \end{pmatrix} - \frac{5y}{2} \begin{pmatrix} 1 \\ -\frac{2}{5} \end{pmatrix} && (52)\\
&= \mathbf{A} + k\mathbf{m} && (53)
\end{aligned}
$$

m is defined to be the direction vector of the line.

Question

$\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$ are three mutually perpendicular unit vectors. If θ is the angle between $\overrightarrow{a}$ and $\left(\overrightarrow{2a} + \overrightarrow{3b} + \overrightarrow{6c}\right)$, find the value of $\cos\theta$.

Solution

Given:

$$\mathbf{a}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{c} = \mathbf{c}^\top \mathbf{a} = 0 \tag{54}$$

$$||\mathbf{a}|| = ||\mathbf{b}|| = ||\mathbf{c}|| = 1 \tag{55}$$

$$\cos\theta = \frac{\mathbf{a}^\top (2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c})}{||\mathbf{a}||||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}||} \tag{56}$$

Now,

$$\mathbf{a}^\top (2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}) = 2\mathbf{a}^\top \mathbf{a} + 3\mathbf{a}^\top \mathbf{b} + 6\mathbf{a}^\top \mathbf{c} = 2 + 0 + 0 = 2 \tag{57}$$

$$||\mathbf{a}||||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| = ||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| \tag{58}$$

Solution

From (14) norm definition:

$$(||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}||)^2 = ||4\mathbf{a}^2|| + ||9\mathbf{b}^2|| + ||36\mathbf{c}^2|| = 49 \tag{59}$$

$$\implies ||2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}|| = +7 \tag{60}$$

$$\implies \cos\theta = \frac{2}{7} \tag{61}$$

Question

Find the position vector of point **C** which divides the line segment joining points **A** and **B** having position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio $4 : 1$ externally. Further, find $|\overrightarrow{AB}| : |\overrightarrow{BC}|$.

Solution

We know that

$$\mathbf{C} = \frac{4\mathbf{B} - \mathbf{A}}{4 - 1} \tag{62}$$

Simplify the above for $\mathbf{C}$.

Question

Two vertices of the parallelogram **ABCD** are given as $\mathbf{A}\,(-1, 2, 1)$ and $\mathbf{B}\,(1, -2, 5)$. If the equation if the line passing through **C** and **D** is $\frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$, then find the distance between the sides AB and CD. Hence, find the area of parallelogram $ABCD$.

Solution

Let the two parallel lines be

$$\mathbf{x} = \mathbf{A} + k_1\mathbf{m} \tag{63}$$

$$\mathbf{x} = \mathbf{C} + k_2\mathbf{m} \tag{64}$$

If $\mathbf{P}$ be a point on the second line,

$$\mathbf{P} = \mathbf{C} + k_2\mathbf{m} \tag{65}$$

$$(\mathbf{A} - \mathbf{P})^\top \mathbf{m} = 0 \tag{66}$$

From the above,

$$(\mathbf{A} - \mathbf{C})^\top \mathbf{m} - k_2\|\mathbf{m}\|^2 = 0 \tag{67}$$

$$\implies k_2 = \frac{(\mathbf{A} - \mathbf{C})^\top \mathbf{m}}{\|\mathbf{m}\|^2} \tag{68}$$

Question

Find the equation of the line passing through the point of intersection of the lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x-1}{0} = \frac{y}{-3} = \frac{z-7}{2}$ and perpendicular to these given lines.

Solution

Let the given lines be denoted by $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. From (53):

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \mathbf{A} + k_1 \mathbf{m}_1 \tag{69}$$

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = \mathbf{B} + k_2 \mathbf{m}_2 \tag{70}$$

Solution

Let the unknown line in its parametric form be denoted as follows from (53).

$$\mathbf{x}_3 = \mathbf{C} + k_3\mathbf{m} \tag{71}$$

The two equations required to solve for the direction of line are

$$\mathbf{m}^\top \mathbf{m}_1 = 0 \tag{72}$$

$$\mathbf{m}^\top \mathbf{m}_2 = 0 \tag{73}$$

$$\implies \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix}^\top \mathbf{m} = 0 \tag{74}$$

yielding

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & 2 \end{pmatrix} \xleftarrow{R_1 \to 2R_2 + 3R_1} \begin{pmatrix} 3 & 0 & 13 \\ 0 & -3 & 2 \end{pmatrix} = 0 \tag{75}$$

$$\implies \begin{pmatrix} 3 & 0 & 13 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = 0 \implies \mathbf{m} = \begin{pmatrix} -\frac{13}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} \tag{76}$$

Solution

Equating (69) and (70),

$$\mathbf{A} + k_1\mathbf{m}_1 = \mathbf{B} + k_2\mathbf{m}_2 \tag{77}$$

$$\begin{pmatrix}\mathbf{m}_1 & \mathbf{m}_2\end{pmatrix}\begin{pmatrix}k_1 \\ -k_2\end{pmatrix} = \mathbf{B} - \mathbf{A} \tag{78}$$

From the above, k_1 and k_2 can be found by gauss elimination given in (29) and thus **C**.

Question

Find the shortest distance between the lines whose vector equations are

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix}1\\2\\3\end{pmatrix} + \kappa_1 \begin{pmatrix}1\\-3\\2\end{pmatrix}\\ \mathbf{x} &= \begin{pmatrix}4\\5\\6\end{pmatrix} + \kappa_2 \begin{pmatrix}2\\3\\1\end{pmatrix}\end{aligned} \tag{79}$$

Solution

From (78) the lines will intersect if

$$\operatorname{rank}\begin{pmatrix}\mathbf{M} & \mathbf{B}-\mathbf{A}\end{pmatrix} = 2 \tag{80}$$

where

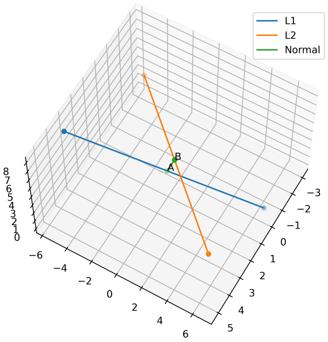
$$\mathbf{M} = \begin{pmatrix}\mathbf{m_1} & \mathbf{m_2}\end{pmatrix} \tag{81}$$

Solution

If L_1, L_2, do not intersect, let

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{A} + \kappa_1 \mathbf{m_1} \\ \mathbf{x}_2 &= \mathbf{B} + \kappa_2 \mathbf{m_2}\end{aligned} \tag{82}$$

be points on L_1, L_2 respectively, that are closest to each other.



Figure

Solution

Then, from (82)

$$\mathbf{x_1} - \mathbf{x_2} = \mathbf{A} - \mathbf{B} + \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} \tag{83}$$

Also,

$$(\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_1 = (\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_2 = 0 \tag{84}$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^\top \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} = \mathbf{0} \tag{85}$$

$$\text{or, } \mathbf{M}^\top (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0} \tag{86}$$

$$\implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) + \mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{0} \tag{87}$$

from (83), yielding

$$\mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{M}^\top (\mathbf{B} - \mathbf{A}) \tag{88}$$

This is known as the *least squares solution*.

Question

8) The sum of first and eight terms of an A.P is 32 and their product is 60. Find the first term and common difference of the A.P. Hence, also find the sum of its first 20 terms.

Solution

Let the first and eighth terms be $x(0)$ and $x(7)$ respectively,given:

$$x(0)+x(7)=32 \tag{89}$$
$$x(0)=32-x(7) \tag{90}$$
$$x(0)\,x(7)=60 \tag{91}$$

From (90) and (91)

$$x(7)\,(32-x(7))=60 \tag{92}$$

The roots are $(30,2)$, therefore ,if $x(7)=30$ then $x(0)=2$ and if $x(7)=2$ then $x_0=30$
Now

$$x(n)=(x(0)+nd)\,u(n) \tag{93}$$

Where d is the common difference of the A.P and $u(n)$ is the unit step function.
$(u(n)=0\forall n<0, u(n)=1\forall n\geq 0)$

Solution

$$\implies x(7) = (x(0) + 7d) \tag{94}$$

$$\implies 7d = \pm 28 \implies d = \pm 4 \tag{95}$$

Therefore the A.P is $2, 6, 10...$ or $30, 26, 22....$
Considering the former for calculations and taking Z-Transform of (93) for sum.
Since

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{96}$$

Let $y(n)$ denote the sum, let:

$$y(n) = x(n) * h(n) \tag{97}$$

$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k) \tag{98}$$

Solution

Replace $h(n)$ with $u(n)$.

$$y(n) = \sum_{k=0}^{n} x(k)u_{(n-k)} \tag{99}$$

$$= x(0)u(n) + x(1)u(n-1) + ...x(n)u_{(0)} \tag{100}$$

This denotes the sum of terms $x(0), x(1)....x(n)$ i.e. first $n+1$ terms.
From (96)

$$u(n) \xrightarrow{\mathcal{Z}} \frac{1}{(1-z^{-1})} \tag{101}$$

$$nu(n) \xrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^2} \tag{102}$$

$$\implies X(z) = \frac{2}{(1-z^{-1})} + \frac{4z^{-1}}{(1-z^{-1})^2}, \quad |z| > |1| \tag{103}$$

Solution

Now as convolution in the time domain corresponds to multiplication in the frequency domain and (103) and (101).

$$
\begin{aligned}
Y(z) &= X(z) * H(z) && (104) \\
&= \left(\frac{2}{(1 - z^{-1})} + \frac{4z^{-1}}{(1 - z^{-1})^2} \right) \left(\frac{1}{(1 - z^{-1})} \right), \quad |z| > |1| && (105)
\end{aligned}
$$

Using normal inversion for inverse Z-transform:

$$
\begin{aligned}
Y(z) &= \frac{2}{(1 - z^{-1})^2} + \frac{4z^{-1}}{(1 - z^{-1})^3}, \quad |z| > |1| && (106) \\
&= \frac{8z^{-1}}{1 - z^{-1}} + \frac{10z^{-2}}{(1 - z^{-1})^2} + \frac{4z^{-3}}{(1 - z^{-1})^3} + 2 && (107)
\end{aligned}
$$

Solution

For proceeding forwards here are some important generalizations.
Shifting property

$$x(n-k) \leftrightarrow z^{-k}X(z) \tag{108}$$

Differentiation property

$$nx(n) \leftrightarrow -zX'(z) \tag{109}$$

From (101) and (108)

$$u(n-1) \xrightarrow{\mathcal{Z}} \frac{z^{-1}}{1-z^{-1}} \tag{110}$$

From (101) and (109)

$$nu(n) \xrightarrow{\mathcal{Z}} -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right) \tag{111}$$

$$nu(n) \xrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1-z^{-1})^2} \tag{112}$$

Solution

From (108)

$$(n-1)u(n-1) \xrightarrow{\mathcal{Z}} z^{-1}\frac{z^{-1}}{(1-z^{-1})^2} \tag{113}$$

$$(n-1)u(n-1) \xrightarrow{\mathcal{Z}} \frac{z^{-2}}{(1-z^{-1})^2} \tag{114}$$

Now, using (109) and writing the corresponding L.H.S

$$(n)(n-1)u(n-1) \xrightarrow{\mathcal{Z}} \frac{2z^{-2}}{(1-z^{-1})^3} \tag{115}$$

Using (108)

$$\frac{(n-1)(n-2)u(n-2)}{2} \xrightarrow{\mathcal{Z}} \frac{z^{-3}}{(1-z^{-1})^3} \tag{116}$$

Solution

The inverse-Z of a constant will be $\delta(n)$, so it is ruled out. Plugging these values in (107) we get

$$y(n) = 8u\,(n-1) + 10(n-1)u\,(n-1) + 4\frac{(n-1)(n-2)u\,(n-2)}{2} + 2\delta(n) \tag{117}$$

Putting $n = 19$

$$y(19) = 2(19+1)^2 = 800 \tag{118}$$

Solution

Using contour integration for inverse Z-transform

$$y(19) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{18}\, dz \tag{119}$$

$$= \frac{1}{2\pi j} \oint_C \left(2z^{20} (z-1)^{-2} + 4z^{20} (z-1)^{-3}\right)\, dz \tag{120}$$

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z)\right) \tag{121}$$

For R_1 , $m = 2$, where m corresponds to number of repeated poles .

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 2z^{20} (z-1)^{-2}\right) \tag{122}$$

$$= 2 \lim_{z \to 1} \frac{d}{dz} (z^{20}) \tag{123}$$

$$= 40 \tag{124}$$

Solution

$$
\begin{aligned}
R_2 &= \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 4z^{20} (z-1)^{-3} \right) && (125)\\
&= (2) \lim_{z \to 1} \frac{d^2}{dz^2} (z^{20}) && (126)\\
&= 760 && (127)\\
R_1 + R_2 &= 800 && (128)\\
\implies y(19) &= 800 && (129)
\end{aligned}
$$

Similarly, the sum for the A.P. $30, 26, 22...$ can be found by the same procedure.

Question

9) In an A.P. of 40 terms, the sum of first 9 terms is 153 and the sum of last 6 terms is 687. Determine the first term and the common difference of the A.P. Also find the sum of all the terms of the A.P.

Given:

$$y(8) = 153 \tag{130}$$
$$y(39) - y(34) = 687 \tag{131}$$

Now, let the first term be $x(0)$ and common difference be d. From (96) and (93)

$$X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > |1| \tag{132}$$

For finding the sum (Assuming $h(n) = u(n)$)

$$y(n) = x(n) * h(n) \tag{133}$$
$$Y(z) = X(z) * H(z) \tag{134}$$
$$= \left(\frac{x(0)}{(1 - z^{-1})} + \frac{dz^{-1}}{(1 - z^{-1})^2}\right)\left(\frac{1}{(1 - z^{-1})}\right), \quad |z| > |1| \tag{135}$$

Solution

$$Y\left(z\right)=\frac{\left(2x(0)+d\right)z^{-1}}{1-z^{-1}}+\frac{\left(x(0)+2d\right)z^{-2}}{\left(1-z^{-1}\right)^{2}}+\frac{dz^{-3}}{\left(1-z^{-1}\right)^{3}}+x(0) \quad (136)$$

Using normal inversion for inverse Z-transform:
Using the results (110),(114) and (116)

$$y(n)=(2x_0+d)u\left(n-1\right)+(n-1)u\left(n-1\right)\left(x_0+2d\right)+ \frac{d(n-1)(n-2)u\left(n-2\right)}{2}+x(0)\delta(n) \quad (137)$$

Now use (130) and (131) to solve for $x(0)$ and d and put in (147) for the sum of 40 terms.

Solution

Using contour integration for inverse Z-transform

$$y(n) = \frac{1}{2\pi j} \oint_C Y(z)\, z^{n-1}\, dz \tag{138}$$

$$= \frac{1}{2\pi j} \oint_C \left(x(0) z^{n+1} (z-1)^{-2} + dz^{20} (z-1)^{-3} \right) dz \tag{139}$$

$$R = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z) \right) \tag{140}$$

For R_1 , $m = 2$, where m corresponds to number of repeated poles .

$$R_1 = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 x(0) z^{n+1} (z-1)^{-2} \right) \tag{141}$$

$$= x(0) \lim_{z \to 1} \frac{d}{dz} (z^{n+1}) \tag{142}$$

$$= (n+1)\, x(0) \tag{143}$$

Solution

For R_2 , $m = 3$

$$R_2 = \frac{1}{(2)!} \lim_{z \to 1} \frac{d^2}{dz^2} \left((z-1)^3 dz^{n+1} (z-1)^{-3} \right) \tag{144}$$

$$= \left(\frac{d}{2} \right) \lim_{z \to 1} \frac{d^2}{dz^2} (z^{n+1}) \tag{145}$$

$$= \left(\frac{d}{2} \right) (n) (n+1) \tag{146}$$

$$y(n) = R_1 + R_2 = \left(\frac{n+1}{2} \right) (2x(0) + nd) \tag{147}$$

Now use (130) and (131) to solve for $x(0)$ and d and put in (147) for the sum of 40 terms.