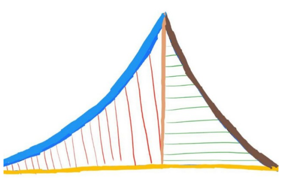
MATRICES
In Geometry



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CONTENTS

1 Vectors

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1}$$

1.1 Sides

1.1.1. The direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Find the direction vectors of AB, BC and CA.
Solution:

a) The Direction vector of AB is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4-1 \\ 6-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.1.2}$$

b) The Direction vector of BC is

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3-(-4) \\ -5-6 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.1.3}$$

c) The Direction vector of CA is

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1-(-3) \\ -1-(-5) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.1.4}$$

1.1.2. The length of side BC is

$$c = \|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{A})} \tag{1.1.2.1}$$

where

$$\mathbf{A}^\top \triangleq \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{1.1.2.2}$$

Similarly,

$$b = \|\mathbf{C} - \mathbf{B}\|, \ a = \|\mathbf{A} - \mathbf{C}\| \tag{1.1.2.3}$$

Find a, b, c.

a) From (1.1.1.2),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \tag{1.1.2.4}$$

$$\implies c = \|\mathbf{B} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \tag{1.1.2.5}$$

$$= \sqrt{\begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix}} = \sqrt{(5)^2 + (7)^2} \tag{1.1.2.6}$$

$$= \sqrt{74} \tag{1.1.2.7}$$

b) Similarly, from (1.1.1.3),

$$a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix}} \tag{1.1.2.8}$$

$$= \sqrt{(1)^2 + (11)^2} = \sqrt{122} \tag{1.1.2.9}$$

and from (1.1.1.4),

c)

$$b = \|\mathbf{A} - \mathbf{C}\| = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}} \tag{1.1.2.10}$$

$$= \sqrt{(4)^2 + (4)^2} = \sqrt{32} \tag{1.1.2.11}$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{1.1.3.1}$$

Are the given points in (1) collinear?
Solution: From (1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \tag{1.1.3.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{5} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & \frac{-48}{5} \end{pmatrix} \tag{1.1.3.3}$$

There are no zero rows. So,

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{1.1.3.4}$$

Hence, the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear. This is visible in Fig. 1.1.3.

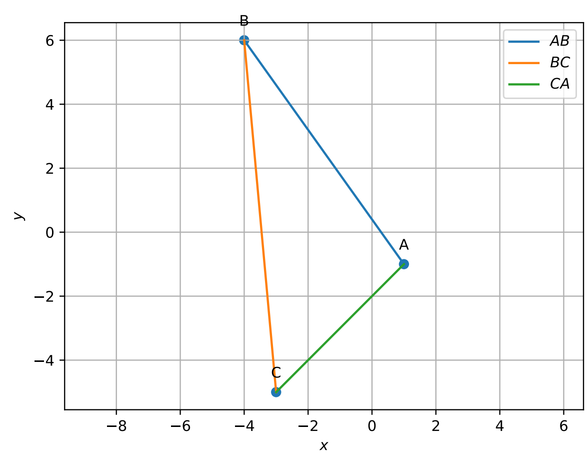

Fig. 1.1.3: $\triangle ABC$

1.1.4. The parameteric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \quad k \neq 0, \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

is the direction vector of AB. Find the parameteric equations of AB, BC and CA.
Solution: From (1.1.4.1) and (1.1.1.2), the parametric

equation for AB is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.4.3}$$

Similarly, from (1.1.1.3) and (1.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.4.4}$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.4.5}$$

1.1.5. The normal form of the equation of AB is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.1}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \tag{1.1.5.2}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.3}$$

Find the normal form of the equations of AB, BC and CA.
Solution:

a) From (1.1.1.3), the direction vector of side **BC** is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.5.4}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} -11 \\ -1 \end{pmatrix} \tag{1.1.5.5}$$

from (1.1.5.3). Hence, from (1.1.5.1), the normal equation of side BC is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.1.5.6}$$

$$\implies \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{1.1.5.7}$$

$$\implies BC : \quad \begin{pmatrix} 11 & 1 \end{pmatrix} \mathbf{x} = -38 \tag{1.1.5.8}$$

b) Similarly, for AB, from (1.1.1.2),

$$\mathbf{m} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.5.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \tag{1.1.5.10}$$

and

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.11}$$

$$\implies AB : \quad \mathbf{n}^\top \mathbf{x} = \begin{pmatrix} 7 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.12}$$

$$\implies \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} = 2 \tag{1.1.5.13}$$

c) For CA, from (1.1.1.4),

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.5.14}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.15}$$

$$\tag{1.1.5.16}$$

$$\implies \mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.1.5.17}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = 2 \tag{1.1.5.18}$$

1.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2} \left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\right\| \tag{1.1.6.1}$$

where

$$\mathbf{A} \times \mathbf{B} \triangleq \begin{vmatrix} 1 & -4 \\ -1 & 6 \end{vmatrix} \tag{1.1.6.2}$$

Find the area of $\triangle ABC$.
Solution: From (1.1.1.2) and (1.1.1.4),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.6.3}$$

$$\implies (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix} 5 & 4 \\ -7 & 4 \end{vmatrix} \tag{1.1.6.4}$$

$$= 5 \times 4 - 4 \times (-7) \tag{1.1.6.5}$$

$$= 48 \tag{1.1.6.6}$$

$$\implies \frac{1}{2} \left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\right\| = \frac{48}{2} = 24 \tag{1.1.6.7}$$

which is the desired area.

1.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top \mathbf{C} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{A}\|} \tag{1.1.7.1}$$

Solution:

a) From (1.1.1.2), (1.1.1.4), (1.1.2.7) and (1.1.2.11)

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & 7 \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{1.1.7.2}$$

$$= -8 \tag{1.1.7.3}$$

$$\implies \cos A = \frac{-8}{\sqrt{74}\sqrt{32}} = \frac{-1}{\sqrt{37}} \tag{1.1.7.4}$$

$$\implies A = \cos^{-1} \frac{-1}{\sqrt{37}} \tag{1.1.7.5}$$

b) From (1.1.1.2), (1.1.1.3), (1.1.2.7) and (1.1.2.9)

$$(\mathbf{C} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 & -11 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix} \tag{1.1.7.6}$$

$$= 82 \tag{1.1.7.7}$$

$$\implies \cos B = \frac{82}{\sqrt{74}\sqrt{122}} = \frac{41}{\sqrt{2257}} \tag{1.1.7.8}$$

$$\implies B = \cos^{-1} \frac{41}{\sqrt{2257}} \tag{1.1.7.9}$$

c) From (1.1.1.3), (1.1.1.4), (1.1.2.9) and (1.1.2.11)

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}4 & 4\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} \tag{1.1.7.10}$$

$$= 40 \tag{1.1.7.11}$$

$$\implies \cos C = \frac{40}{\sqrt{32}\sqrt{122}} = \frac{5}{\sqrt{61}} \tag{1.1.7.12}$$

$$\implies C = \cos^{-1}\frac{5}{\sqrt{61}} \tag{1.1.7.13}$$

All codes for this section are available at

codes/triangle/sides.py

1.2 Median

1.2.1. If $\mathbf{D}$ divides BC in the ratio $k:1$,

$$\mathbf{D} = \frac{k\mathbf{C}+\mathbf{B}}{k+1} \tag{1.2.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.

Solution: Since $\mathbf{D}$ is the midpoint of BC,

$$k = 1, \tag{1.2.1.2}$$

$$\implies \mathbf{D} = \frac{\mathbf{C}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix} \tag{1.2.1.3}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix}-1\\-3\end{pmatrix} \tag{1.2.1.4}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix} \tag{1.2.1.5}$$

1.2.2. Find the equations of AD, BE and CF.

Solution: :

a) The direction vector of AD is

$$\mathbf{m} = \mathbf{D}-\mathbf{A} = \begin{pmatrix}\frac{-7}{2}\\\frac{1}{2}\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-9\\3\end{pmatrix} \equiv \begin{pmatrix}-3\\1\end{pmatrix} \tag{1.2.2.1}$$

$$\implies \mathbf{n} = \begin{pmatrix}1\\3\end{pmatrix} \tag{1.2.2.2}$$

Hence the normal equation of median AD is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{A}\right) = 0 \tag{1.2.2.3}$$

$$\implies \begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = -2 \tag{1.2.2.4}$$

b) For BE,

$$\mathbf{m} = \mathbf{E}-\mathbf{B} = \begin{pmatrix}-1\\-3\end{pmatrix} - \begin{pmatrix}-4\\6\end{pmatrix} = \begin{pmatrix}3\\-9\end{pmatrix} \equiv \begin{pmatrix}1\\-3\end{pmatrix} \tag{1.2.2.5}$$

$$\implies \mathbf{n} = \begin{pmatrix}3\\1\end{pmatrix} \tag{1.2.2.6}$$

Hence the normal equation of median BE is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{B}\right) = 0 \tag{1.2.2.7}$$

$$\implies \begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}3 & 1\end{pmatrix}\begin{pmatrix}-4\\6\end{pmatrix} = -6 \tag{1.2.2.8}$$

c) For median CF,

$$\mathbf{m} = \mathbf{F}-\mathbf{C} = \begin{pmatrix}\frac{-3}{2}\\\frac{5}{2}\end{pmatrix} - \begin{pmatrix}-3\\-5\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\\frac{15}{2}\end{pmatrix} \equiv \begin{pmatrix}1\\5\end{pmatrix} \tag{1.2.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix}5\\-1\end{pmatrix} \tag{1.2.2.10}$$

Hence the normal equation of median CF is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{C}\right) = 0 \tag{1.2.2.11}$$

$$\implies \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}-3\\-5\end{pmatrix} = -10 \tag{1.2.2.12}$$

1.2.3. Find the intersection $\mathbf{G}$ of BE and CF.

Solution: From (1.2.2.8) and (1.2.2.12), the equations of

BE and CF are, respectively,

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}-6\end{pmatrix} \tag{1.2.3.1}$$

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}-10\end{pmatrix} \tag{1.2.3.2}$$

From (1.2.3.1) and (1.2.3.2) the augmented matrix is

$$\begin{pmatrix}3 & 1 & -6\\5 & -1 & -10\end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1+R_2} \begin{pmatrix}8 & 0 & -16\\5 & -1 & -10\end{pmatrix} \tag{1.2.3.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix}1 & 0 & -2\\5 & -1 & -10\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2-5R_1} \begin{pmatrix}1 & 0 & -2\\0 & -1 & 0\end{pmatrix} \tag{1.2.3.4}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix}1 & 0 & -2\\0 & 1 & 0\end{pmatrix} \tag{1.2.3.5}$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.2.3.6}$$

1.2.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.1}$$

Solution:

a) From (1.2.1.4) and (1.2.3.6),

$$\mathbf{G}-\mathbf{B} = \begin{pmatrix}2\\-6\end{pmatrix}, \mathbf{E}-\mathbf{G} = \begin{pmatrix}1\\-3\end{pmatrix} \tag{1.2.4.2}$$

$$\implies \mathbf{G}-\mathbf{B} = 2\left(\mathbf{E}-\mathbf{G}\right) \tag{1.2.4.3}$$

$$\implies \left\|\mathbf{G}-\mathbf{B}\right\| = 2\left\|\mathbf{E}-\mathbf{G}\right\| \tag{1.2.4.4}$$

$$\text{or, } \frac{BG}{GE} = 2 \tag{1.2.4.5}$$

b) From (1.2.1.5) and (1.2.3.6),

$$\mathbf{F}-\mathbf{G} = \frac{1}{2}\begin{pmatrix}1\\5\end{pmatrix}, \mathbf{G}-\mathbf{C} = \begin{pmatrix}1\\5\end{pmatrix} \tag{1.2.4.6}$$

$$\implies \mathbf{G}-\mathbf{C} = 2\left(\mathbf{F}-\mathbf{G}\right) \tag{1.2.4.7}$$

$$\implies \left\|\mathbf{G}-\mathbf{C}\right\| = 2\left\|\mathbf{F}-\mathbf{G}\right\| \tag{1.2.4.8}$$

$$\text{or, } \frac{CG}{GF} = 2 \tag{1.2.4.9}$$

c) From (1.2.1.3) and (1.2.3.6),

$$\mathbf{G}-\mathbf{A} = \begin{pmatrix}-3\\1\end{pmatrix}, \mathbf{D}-\mathbf{G} = \frac{1}{2}\begin{pmatrix}-3\\1\end{pmatrix} \tag{1.2.4.10}$$

$$\mathbf{G}-\mathbf{A} = 2\left(\mathbf{D}-\mathbf{G}\right) \tag{1.2.4.11}$$

$$\implies \left\|\mathbf{G}-\mathbf{A}\right\| = 2\left\|\mathbf{D}-\mathbf{G}\right\| \tag{1.2.4.12}$$

$$\text{or, } \frac{AG}{GD} = 2 \tag{1.2.4.13}$$

From (1.2.4.5), (1.2.4.9), (1.2.4.13)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.14}$$

1.2.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear.

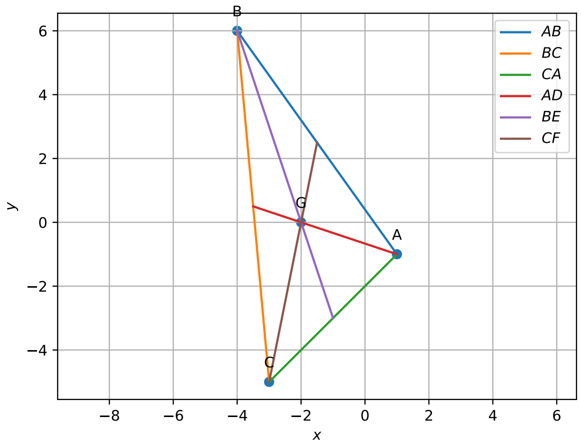

Fig. 1.2.5: Medians of $\triangle ABC$ meet at $\mathbf{G}$.

Solution: Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\operatorname{rank}\begin{pmatrix}1 & 1 & 1\\\mathbf{A} & \mathbf{D} & \mathbf{G}\end{pmatrix} = 2 \tag{1.2.5.1}$$

$$\implies \begin{pmatrix}1 & 1 & 1\\1 & -\frac{7}{2} & -2\\-1 & \frac{1}{2} & 0\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3+R_2} \begin{pmatrix}1 & 1 & 1\\1 & -\frac{7}{2} & -2\\0 & -3 & -2\end{pmatrix} \tag{1.2.5.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2-R_1} \begin{pmatrix}1 & 1 & 1\\0 & -\frac{9}{2} & -3\\0 & -3 & -2\end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3-\frac{2}{3}R_2} \begin{pmatrix}1 & 1 & 1\\0 & -\frac{9}{2} & -3\\0 & 0 & 0\end{pmatrix} \tag{1.2.5.3}$$

Thus, the matrix (1.2.5.1) has rank 2 and the points are collinear. Thus, the medians of a triangle meet at the point $\mathbf{G}$. See Fig. 1.2.5.

1.2.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.2.6.1}$$

$\mathbf{G}$ is known as the *centroid* of $\triangle ABC$.

Solution:

$$\mathbf{G} = \frac{\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}-4\\6\end{pmatrix} + \begin{pmatrix}-3\\-5\end{pmatrix}}{3} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.2.6.2}$$

1.2.7. Verify that

$$\mathbf{A}-\mathbf{F} = \mathbf{E}-\mathbf{D} \tag{1.2.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

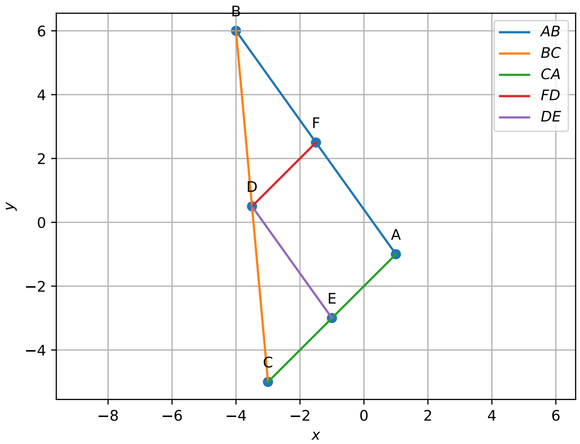

Fig. 1.2.7: $AFDE$ forms a parallelogram in triangle ABC

Solution:

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.2.7.2}$$

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.2.7.3}$$

$$\implies \mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.4}$$

See Fig. 1.2.7,
All codes for this section are available in

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codes/triangle/medians.py
codes/triangle/pgm.py
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1.3 Altitude

1.3.1. $\mathbf{D}_1$ is a point on BC such that

$$AD_1 \perp BC \tag{1.3.1.1}$$

and AD_1 is defined to be the altitude. Find the normal vector of AD_1.
Solution: The normal vector of AD_1 is the direction vector BC and is obtained from (1.1.1.3) as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.3.1.2}$$

1.3.2. Find the equation of AD_1.
Solution: The equation of AD_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.3.2.1}$$

$$\implies \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -12 \tag{1.3.2.2}$$

1.3.3. Find the equations of the altitudes BE_1 and CF_1 to the sides AC and AB respectively.

Solution:

a) From (1.1.1.4), the normal vector of CF_1 is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.3.3.1}$$

and the equation of CF_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.3.3.2}$$

$$\implies \begin{pmatrix} -5 & 7 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} \right) = 0 \tag{1.3.3.3}$$

$$\implies \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = 20, \tag{1.3.3.4}$$

b) Similarly, from (1.1.1.2), the normal vector of BE_1 is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.3.3.5}$$

and the equation of BE_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.3.3.6}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right) = 0 \tag{1.3.3.7}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{1.3.3.8}$$

1.3.4. Find the intersection $\mathbf{H}$ of BE_1 and CF_1.

Solution: The intersection of (1.3.3.8) and (1.3.3.4), is obtained from the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 5 & -7 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} \tag{1.3.4.1}$$

which can be solved as

$$\begin{pmatrix} 1 & 1 & 2 \\ 5 & -7 & 20 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -12 & 10 \end{pmatrix} \tag{1.3.4.2}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-12}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{17}{6} \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \tag{1.3.4.3}$$

yielding

$$\mathbf{H} = \frac{1}{6}\begin{pmatrix}17\\-5\end{pmatrix}, \tag{1.3.4.4}$$

See Fig. 1.3.4.1

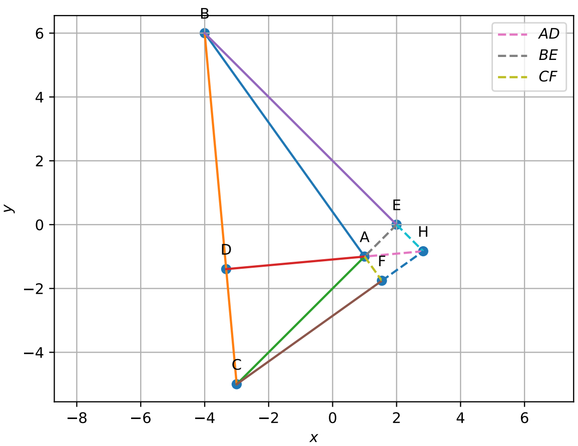

Fig. 1.3.4.1: Altitudes BE_1 and CF_1 intersect at $\mathbf{H}$

1.3.5. Verify that

$$(\mathbf{A}-\mathbf{H})^\top(\mathbf{B}-\mathbf{C}) = 0 \tag{1.3.5.1}$$

Solution: From (1.3.4.4),

$$\mathbf{A}-\mathbf{H} = -\frac{1}{6}\begin{pmatrix}11\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{1.3.5.2}$$

$$\implies (\mathbf{A}-\mathbf{H})^\top(\mathbf{B}-\mathbf{C}) = \frac{1}{6}\begin{pmatrix}11 & 1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} = 0 \tag{1.3.5.3}$$

All codes for this section are available at

codes/triangle/altitude.py

1.4 Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x}-\frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{1.4.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.
Solution: From (1.1.1.2), (1.1.1.3), (1.1.1.4), (1.2.1.3), (1.2.1.4) and (1.2.1.5),

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{1.4.1.2}$$

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix}5\\-7\end{pmatrix} \tag{1.4.1.3}$$

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \begin{pmatrix}-1\\-3\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix} \tag{1.4.1.4}$$

$$\tag{1.4.1.5}$$

yielding

$$(\mathbf{B}-\mathbf{C})^\top\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix}-1 & 11\end{pmatrix}\begin{pmatrix}-\frac{7}{2}\\\frac{1}{2}\end{pmatrix} = 9 \tag{1.4.1.6}$$

$$(\mathbf{A}-\mathbf{B})^\top\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix}5 & -7\end{pmatrix}\begin{pmatrix}-\frac{3}{2}\\\frac{5}{2}\end{pmatrix} = -25 \tag{1.4.1.7}$$

$$(\mathbf{C}-\mathbf{A})^\top\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}-1\\-3\end{pmatrix} = 16 \tag{1.4.1.8}$$

Thus, the perpendicular bisectors are obtained from (1.4.1.1) as

$$BC: \quad \begin{pmatrix}-1 & 11\end{pmatrix}\mathbf{x} = 9 \tag{1.4.1.9}$$

$$CA: \quad \begin{pmatrix}5 & -7\end{pmatrix}\mathbf{x} = -25 \tag{1.4.1.10}$$

$$AB: \quad \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = -4 \tag{1.4.1.11}$$

1.4.2. Find the intersection $\mathbf{O}$ of the perpendicular bisectors of AB and AC.
Solution:
The intersection of (1.4.1.10) and (1.4.1.11), can be obtained as

$$\begin{pmatrix}5 & -7 & -25\\1 & 1 & -4\end{pmatrix} \xleftrightarrow{R_2\leftarrow 5R_2-R_1} \begin{pmatrix}5 & -7 & -25\\0 & 12 & 5\end{pmatrix} \tag{1.4.2.1}$$

$$\xleftrightarrow{R_1\leftarrow\frac{12}{7}R_1+R_2} \begin{pmatrix}\frac{60}{7} & 0 & \frac{-265}{7}\\0 & 12 & 5\end{pmatrix} \xleftrightarrow[R_1\leftarrow\frac{7}{60}R_1]{R_2\leftarrow\frac{1}{12}R_2} \begin{pmatrix}1 & 0 & \frac{-53}{12}\\0 & 1 & \frac{5}{12}\end{pmatrix} \tag{1.4.2.2}$$

$$\implies \mathbf{O} = \begin{pmatrix}\frac{-53}{12}\\\frac{5}{12}\end{pmatrix} \tag{1.4.2.3}$$

1.4.3. Verify that $\mathbf{O}$ satisfies (1.4.1.1). $\mathbf{O}$ is known as the circumcentre.
Solution: Substituing from (1.4.2.3) in (1.4.1.1), when

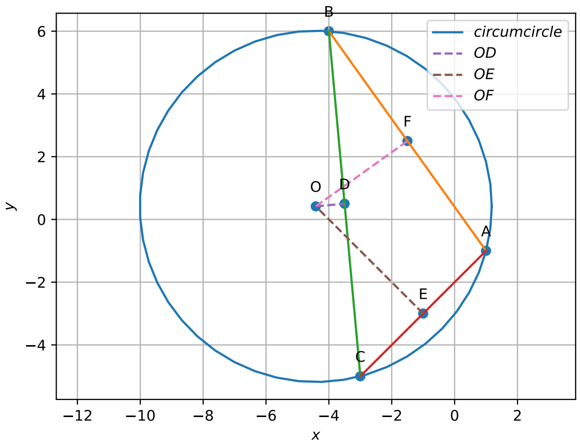

Fig. 1.4.5.1: Circumcircle of $\triangle ABC$ with centre $\mathbf{O}$.

substituted in the above equation,

$$\left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)^\top (\mathbf{B}-\mathbf{C})$$

$$= \left(\frac{1}{12}\begin{pmatrix}-53\\5\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}\right)^\top \begin{pmatrix}-1\\11\end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix}-11 & -1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} = 0 \quad (1.4.3.1)$$

1.4.4. Verify that

$$OA = OB = OC \quad (1.4.4.1)$$

1.4.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$R = OA \quad (1.4.5.1)$$

This is known as the *circumradius*.
Solution: See Fig. 1.4.5.1.

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \quad (1.4.6.1)$$

Solution:

a) To find the value of $\angle BOC$:

$$\mathbf{B}-\mathbf{O} = \begin{pmatrix}\frac{5}{12}\\\frac{67}{12}\end{pmatrix}, \mathbf{C}-\mathbf{O} = \begin{pmatrix}\frac{17}{12}\\\frac{-65}{12}\end{pmatrix} \quad (1.4.6.2)$$

$$\implies (\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O}) = \frac{-4270}{144} \quad (1.4.6.3)$$

$$\implies \|\mathbf{B}-\mathbf{O}\| = \frac{\sqrt{4514}}{12}, \|\mathbf{C}-\mathbf{O}\| = \frac{\sqrt{4514}}{12} \quad (1.4.6.4)$$

Thus,

$$\cos BOC = \frac{(\mathbf{B}-\mathbf{O})^\top(\mathbf{C}-\mathbf{O})}{\|\mathbf{B}-\mathbf{O}\|\,\|\mathbf{C}-\mathbf{O}\|} = \frac{-4270}{4514} \quad (1.4.6.5)$$

$$\implies \angle BOC = \cos^{-1}\left(\frac{-4270}{4514}\right) \quad (1.4.6.6)$$

$$= 161.07536^\circ \text{ or } 198.92464^\circ \quad (1.4.6.7)$$

b) To find the value of $\angle BAC$:

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix} \quad (1.4.6.8)$$

$$\implies (\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A}) = -8 \quad (1.4.6.9)$$

$$\|\mathbf{B}-\mathbf{A}\| = \sqrt{74}\,\|\mathbf{C}-\mathbf{A}\| = 4\sqrt{2} \quad (1.4.6.10)$$

Thus,

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{A}\|} = \frac{-8}{4\sqrt{148}} \quad (1.4.6.11)$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{-8}{4\sqrt{148}}\right) \quad (1.4.6.12)$$

$$= 99.46232^\circ \quad (1.4.6.13)$$

From (1.4.6.13) and (1.4.6.7),

$$2 \times \angle BAC = \angle BOC \quad (1.4.6.14)$$

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix} \quad (1.4.7.1)$$

where

$$\theta = \angle BOC \quad (1.4.7.2)$$

Verify that

$$\mathbf{B}-\mathbf{O} = \mathbf{P}(\mathbf{C}-\mathbf{O}) \quad (1.4.7.3)$$

All codes for this section are available at

codes/triangle/perp−bisect.py

1.5 Angle Bisector

1.5.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \tag{1.5.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Problem 1.1.2.
Solution: From the given information,

$$a = m + n, \tag{1.5.1.2}$$
$$b = n + p, \tag{1.5.1.3}$$
$$c = m + p \tag{1.5.1.4}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.5}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.6}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{1.5.1.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.5.1.8}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.9}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.10}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.5.1.11}$$

Therefore,

$$\begin{aligned} p &= \frac{c + b - a}{2} = \frac{\sqrt{74} + \sqrt{32} - \sqrt{122}}{2} \\ m &= \frac{a + c - b}{2} = \frac{\sqrt{74} + \sqrt{122} - \sqrt{32}}{2} \\ n &= \frac{a + b - c}{2} = \frac{\sqrt{122} + \sqrt{32} - \sqrt{74}}{2} \end{aligned} \tag{1.5.1.12}$$

upon substituting from (1.1.2.7), (1.1.2.9) and (1.1.2.11).

1.5.2. Using section formula, find

$$\mathbf{D}_3 = \frac{m\mathbf{C} + n\mathbf{B}}{m + n}, \mathbf{E}_3 = \frac{n\mathbf{A} + p\mathbf{C}}{n + p}, \mathbf{F}_3 = \frac{p\mathbf{B} + m\mathbf{A}}{p + m} \tag{1.5.2.1}$$

1.5.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the *incentre* and *inradius* of $\triangle ABC$.

1.5.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the *incircle* of $\triangle ABC$.
Solution: See Fig. 1.5.4.1

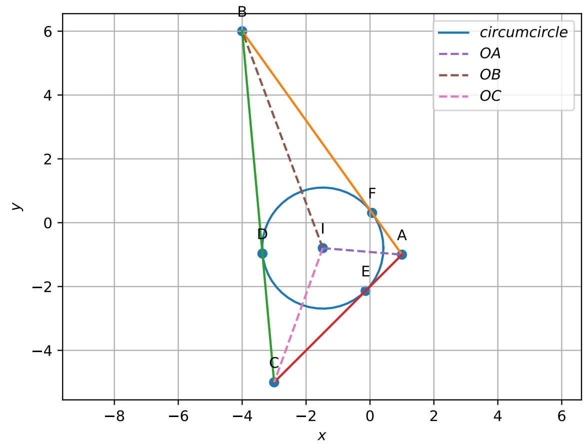

Fig. 1.5.4.1: Incircle of $\triangle ABC$

1.5.5. Using (1.1.7.1) verify that

$$\angle BAI = \angle CAI. \tag{1.5.5.1}$$

AI is the bisector of $\angle A$.

1.5.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.
All codes for this section are available at

codes/triangle/ang−bisect.py

1.6 Eigenvalues and Eigenvectors

The equation of the incircle is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1.6.1}$$

where

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\| - r^2, \tag{1.6.2}$$

$\mathbf{O}$ being the incentre and r the inradius. Here $\mathbf{I}$ is the identity matrix.

1.6.1. Compute

$$\boldsymbol{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\mathbf{V} \tag{1.6.1.1}$$

for $\mathbf{h} = \mathbf{A}$.

1.6.2. Find the roots of the equation

$$\left|\lambda\mathbf{I} - \boldsymbol{\Sigma}\right| = 0 \tag{1.6.2.1}$$

These are known as the eigenvalues of $\boldsymbol{\Sigma}$.

1.6.3. Find $\mathbf{p}$ such that

$$\boldsymbol{\Sigma}\mathbf{p} = \lambda\mathbf{p} \tag{1.6.3.1}$$

using row reduction. These are known as the eigenvectors of Σ.

1.6.4. Define

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{1.6.4.1}$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} \end{pmatrix} \tag{1.6.4.2}$$

1.6.5. Verify that

$$\mathbf{P}^\top = \mathbf{P}^{-1}. \tag{1.6.5.1}$$

$\mathbf{P}$ is defined to be an orthogonal matrix.

1.6.6. Verify that

$$\mathbf{P}^\top \boldsymbol{\Sigma}\mathbf{P} = \mathbf{D}, \tag{1.6.6.1}$$

This is known as the spectral (eigenvalue) decomposition of a symmetric matrix

1.6.7. The direction vectors of the tangents from a point $\mathbf{h}$ to the circle in (1.6.1) are given by

$$\mathbf{m} = \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{1.6.7.1}$$

1.6.8. The points of contact of the pair of tangents to the circle in (1.6.1) from a point $\mathbf{h}$ are given by

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{1.6.8.1}$$

where

$$\mu = -\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{m}} \tag{1.6.8.2}$$

for $\mathbf{m}$ in (1.6.7.1). Compute the points of contact. You should get the same points that you obtained in the previous section.

All codes for this section are available at

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codes/triangle/tangpair.py
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1.7 Addition and Subtraction

1.7.1 Find the sum of the vectors $\mathbf{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\mathbf{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\mathbf{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

1.7.2 In triangle ABC (Fig. 1.7.2.1), which of the following is not true:

a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$
b) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \mathbf{0}$
c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \mathbf{0}$
d) $\overrightarrow{AB} - \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$

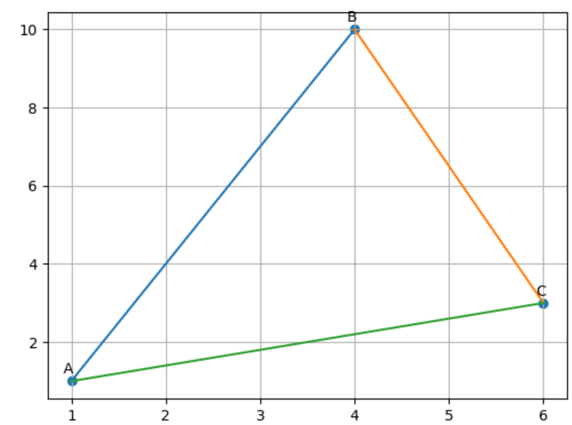

Fig. 1.7.2.1

Solution:

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{B} - \mathbf{A} + \mathbf{C} - \mathbf{B} + \mathbf{A} - \mathbf{C} = 0 \tag{1.7.2.1}$$

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \mathbf{B} - \mathbf{A} + \mathbf{C} - \mathbf{B} - (\mathbf{C} - \mathbf{A}) = 0 \tag{1.7.2.2}$$

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC} = \mathbf{B} - \mathbf{A} + \mathbf{C} - \mathbf{B} + \mathbf{C} - \mathbf{A} = 2(\mathbf{C} - \mathbf{A}) \tag{1.7.2.3}$$

$$\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \mathbf{B} - \mathbf{A} - (\mathbf{B} - \mathbf{C}) + \mathbf{A} - \mathbf{C} = 0 \tag{1.7.2.4}$$

1.7.3 A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution: Let

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \mathbf{C} - \mathbf{B} = 3\begin{pmatrix} \cos 60° \\ \sin 60° \end{pmatrix} \tag{1.7.3.1}$$

$$\implies \mathbf{C} = \begin{pmatrix} \frac{-5}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} \tag{1.7.3.2}$$

which is the displacement. See Fig. 1.7.3.1.

1.7.4 Without using distance formula, show that points A(– 2, – 1), B(4, 0), C(3, 3) and D(–3, 2) are the vertices of a parallelogram.

Solution:

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} = \begin{pmatrix} -6 \\ -1 \end{pmatrix} \tag{1.7.4.1}$$

Hence, $ABCD$ is a parallelogram. See Fig. 1.7.4.1.

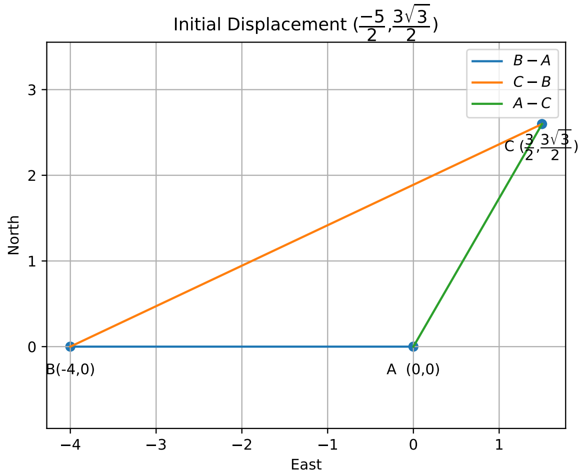

Fig. 1.7.3.1

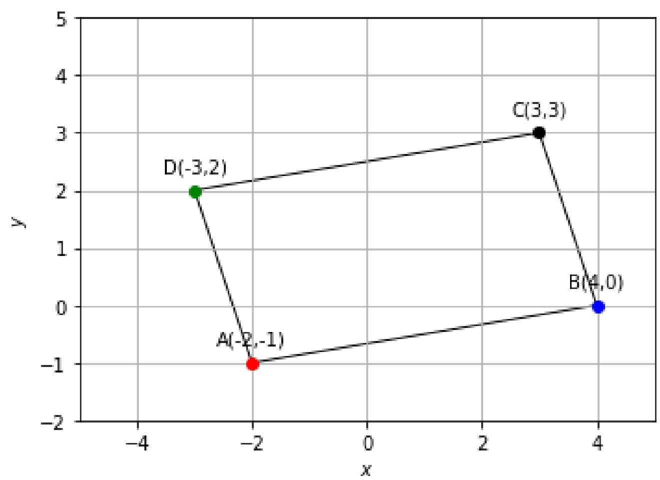

Fig. 1.7.4.1

1.7.5 The fourth vertex **D** of a parallelogram **ABCD** whose three vertices are $\mathbf{A}(-2,3), \mathbf{B}(6,7)$ and $\mathbf{C}(8,3)$ is

a) $(0,1)$
b) $(0,-1)$
c) $(-1,0)$
d) $(1,0)$

1.7.6 Points $\mathbf{A}(4,3), \mathbf{B}(6,4), \mathbf{C}(5,-6)$ and $\mathbf{D}(-3,5)$ are the vertices of a parallelogram.

1.8 *Section Formula*

1.8.1 Find the coordinates of the point which divides the join of $(-1,7)$ and $(4,-3)$ in the ratio 2:3.

Solution: Using section formula (1.2.1.1), the desired point is

$$\frac{1}{1+\frac{3}{2}}\left(\begin{pmatrix}4\\-3\end{pmatrix}+\frac{3}{2}\begin{pmatrix}-1\\7\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \tag{1.8.1.1}$$

See Fig. 1.8.1.1

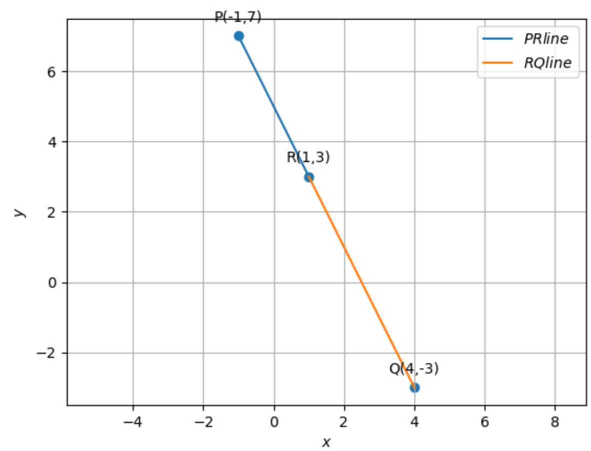

Fig. 1.8.1.1

1.8.2 Find the coordinates of the points of trisection of the line segment joining $(4,-1)$ and $(-2,3)$.

Solution: Using section formula,

$$\mathbf{R}=\frac{1}{1+\frac{1}{2}}\left(\begin{pmatrix}4\\-1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-2\\-3\end{pmatrix}\right)=\begin{pmatrix}2\\\frac{-5}{3}\end{pmatrix} \tag{1.8.2.1}$$

$$\mathbf{S}=\frac{1}{1+\frac{2}{1}}\left(\begin{pmatrix}4\\-1\end{pmatrix}+\frac{2}{1}\begin{pmatrix}-2\\-3\end{pmatrix}\right)=\begin{pmatrix}0\\\frac{-7}{3}\end{pmatrix} \tag{1.8.2.2}$$

which are the desired points of trisection. See Fig. 1.8.2.1

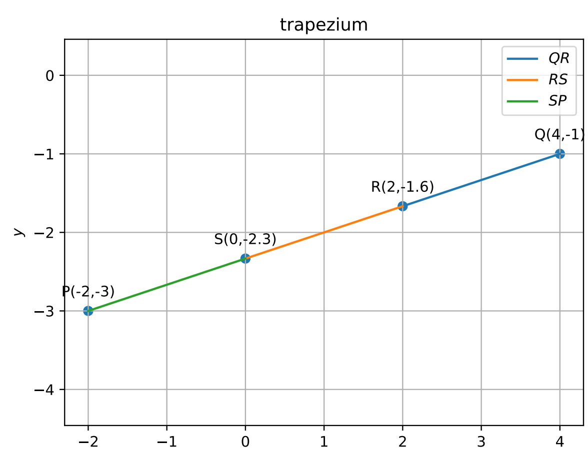

Fig. 1.8.2.1

1.8.3 Find the ratio in which the line segment joining the points $(-3,10)$ and $(6,-8)$ is divided by $(-1,6)$.
Solution: Using section formula,

$$\begin{pmatrix}-1\\6\end{pmatrix} = \frac{\begin{pmatrix}-3\\10\end{pmatrix} + k\begin{pmatrix}6\\-8\end{pmatrix}}{1+n} \tag{1.8.3.1}$$

$$\implies 7k\begin{pmatrix}1\\-2\end{pmatrix} = 2\begin{pmatrix}1\\-2\end{pmatrix} \tag{1.8.3.2}$$

$$\text{or, } k = \frac{2}{7} \tag{1.8.3.3}$$

See Fig. 1.8.3.1.

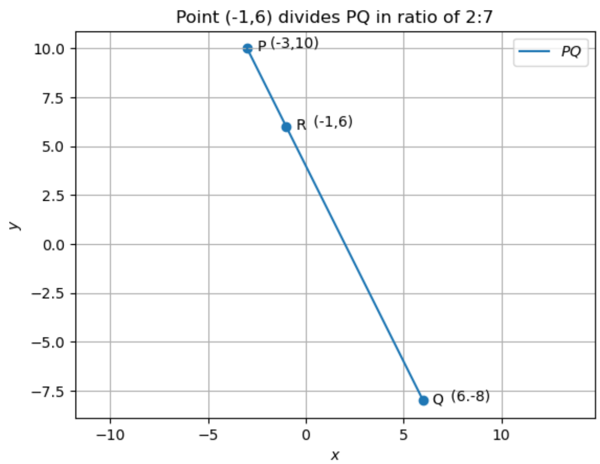

Fig. 1.8.3.1

1.8.4 If $(1,2),(4,y),(x,6),(3,5)$ are the vertices of a parallelogram taken in order, find x and y.
Solution: Since $ABCD$ is a parallellogram,

$$\begin{pmatrix}4\\y\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}x\\6\end{pmatrix} - \begin{pmatrix}3\\5\end{pmatrix} \tag{1.8.4.1}$$

$$\implies \begin{pmatrix}3\\y-2\end{pmatrix} = \begin{pmatrix}x-3\\1\end{pmatrix} \tag{1.8.4.2}$$

$$\text{or, } x = 6, y = 3. \tag{1.8.4.3}$$

See Fig. 1.8.4.1.

1.8.5 Find the coordinates of a point A, where AB is the diameter of a circle whose centre is C$(2,-3)$ and B is $(1,4)$.
Solution:

$$\mathbf{C} = \frac{\mathbf{A}+\mathbf{B}}{2} \implies \mathbf{A} = 2\mathbf{C} - \mathbf{B} = \begin{pmatrix}3\\-10\end{pmatrix} \tag{1.8.5.1}$$

See Fig. 1.8.5.1.

1.8.6 If A and B are $(-2,-2)$ and $(2,-4)$, respectively, find the coordinates of P such that AP= $\frac{3}{7}$AB and P lies on the line segment AB.
Solution: Using section formula,

$$\mathbf{P} = \frac{1}{1+\frac{3}{4}}\left(\begin{pmatrix}-2\\-2\end{pmatrix} + \frac{3}{4}\begin{pmatrix}2\\-4\end{pmatrix}\right) = \begin{pmatrix}\frac{-2}{7}\\\frac{-20}{7}\end{pmatrix} \tag{1.8.6.1}$$

See Fig. 1.8.6.1.

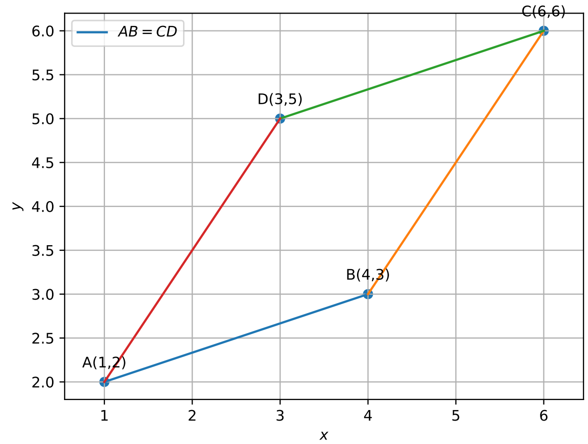

Fig. 1.8.4.1

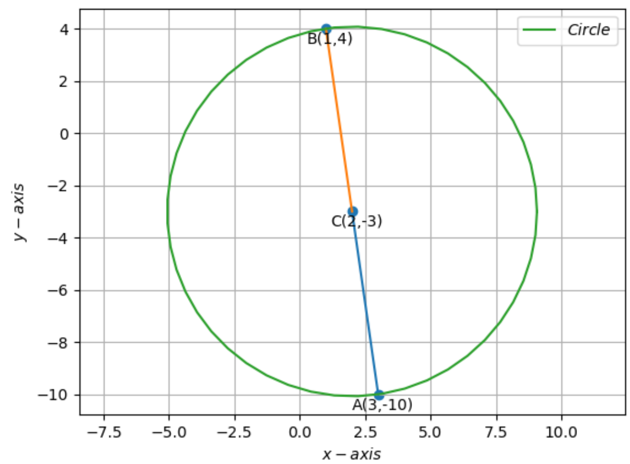

Fig. 1.8.5.1

1.8.7 Find the coordinates of the points which divide the line segment joining $A(-2,2)$ and $B(2,8)$ into four equal parts.
Solution: Using section formula,

$$\mathbf{R}_k = \frac{\mathbf{B}+k\mathbf{A}}{1+k} \tag{1.8.7.1}$$

See Table 1.8.7 and Fig. 1.8.7.1

TABLE 1.8.7

k	$\mathbf{R}_k$
3	$\begin{pmatrix}-1\\\frac{7}{2}\end{pmatrix}$
1	$\begin{pmatrix}0\\5\end{pmatrix}$
$\frac{1}{3}$	$\begin{pmatrix}1\\\frac{13}{2}\end{pmatrix}$

1.8.8 Find the position vector of a point $\mathbf{R}$ which divides the

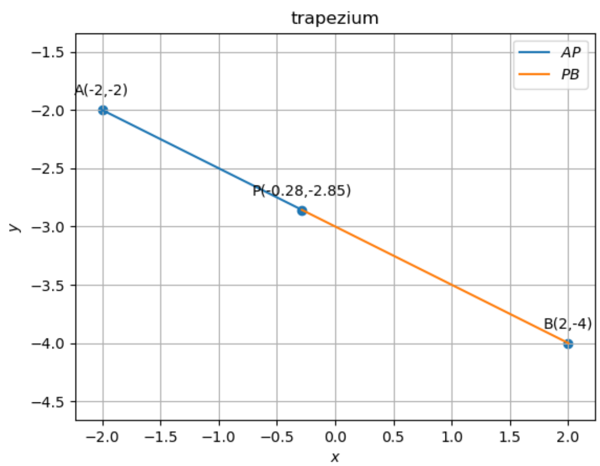

Fig. 1.8.6.1

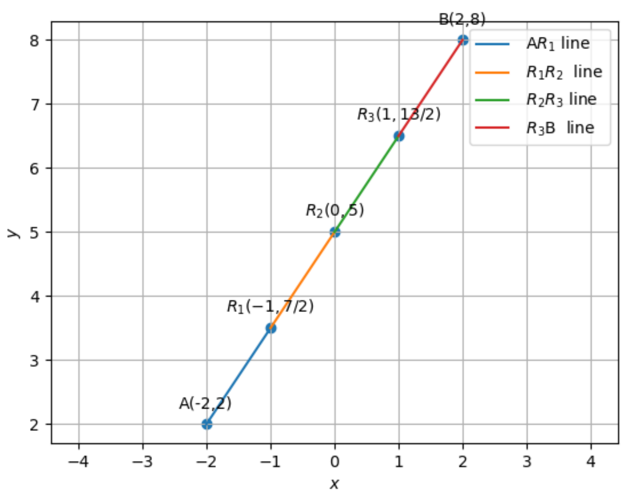

Fig. 1.8.7.1

line joining two points $\mathbf{P}$ and $\mathbf{Q}$ whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively, in the ratio 2 : 1

a) internally
b) externally

Solution: See Table 1.8.8.

TABLE 1.8.8

k	R_k
2	$\frac{1}{3}\begin{pmatrix}-1\\4\\1\end{pmatrix}$
-2	$\begin{pmatrix}-3\\0\\3\end{pmatrix}$

1.8.9 Find the position vector of the mid point of the vector joining the points $\mathbf{P}$(2, 3, 4) and $\mathbf{Q}$(4, 1, –2).

Solution: The desired vector is

$$\frac{1}{2}\begin{pmatrix}2\\3\\4\end{pmatrix}+\frac{1}{2}\begin{pmatrix}4\\1\\-2\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix} \tag{1.8.9.1}$$

1.8.10 Determine the ratio in which the line $2x+y-4=0$ divides the line segment joining the points $\mathbf{A}(2,-2)$ and $\mathbf{B}(3,7)$.

Solution: The given equation can be expressed as

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x}=4 \tag{1.8.10.1}$$

Using section formula in (1.8.10.1),

$$\mathbf{n}^\top\left(\tfrac{k\mathbf{B}+\mathbf{A}}{k+1}\right)=c \tag{1.8.10.2}$$

$$\implies k=\frac{c-\mathbf{n}^\top\mathbf{A}}{\mathbf{n}^\top\mathbf{B}-c} \tag{1.8.10.3}$$

upon simplification. Substituting numerical values,

$$k=\frac{2}{9} \tag{1.8.10.4}$$

See Fig. 1.8.10.1.

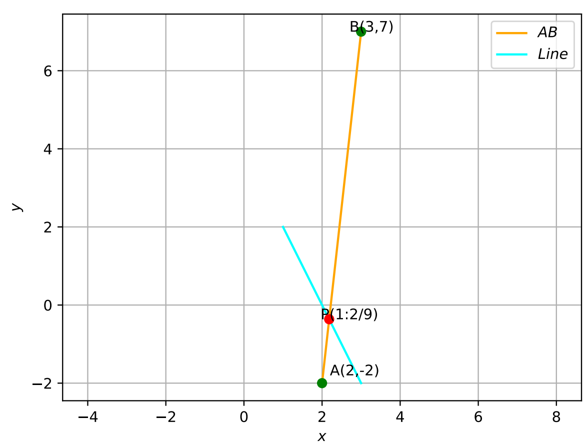

Fig. 1.8.10.1

1.8.11 Let $\mathbf{A}(4,2), \mathbf{B}(6,5)$ and $\mathbf{C}(1,4)$ be the vertices of $\triangle ABC$.

a) The median from $\mathbf{A}$ meets BC at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.
b) Find the coordinates of the point $\mathbf{P}$ on AD such that $AP : PD = 2 : 1$.
c) Find the coordinates of points $\mathbf{Q}$ and $\mathbf{R}$ on medians BE and CF respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.
d) What do you observe?
e) If $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are the vertices of $\triangle ABC$, find the coordinates of the centroid of the triangle.

Solution:

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \tag{1.8.11.1}$$

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix} \frac{5}{2} \\ 3 \end{pmatrix} \tag{1.8.11.2}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix} \tag{1.8.11.3}$$

$$\mathbf{P} = \mathbf{Q} = \mathbf{R} = \frac{1}{3}\begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.8.11.4}$$

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \frac{1}{3}\begin{pmatrix} 11 \\ 11 \end{pmatrix} \tag{1.8.11.5}$$

is the centroid. See Fig. 1.8.11.1.

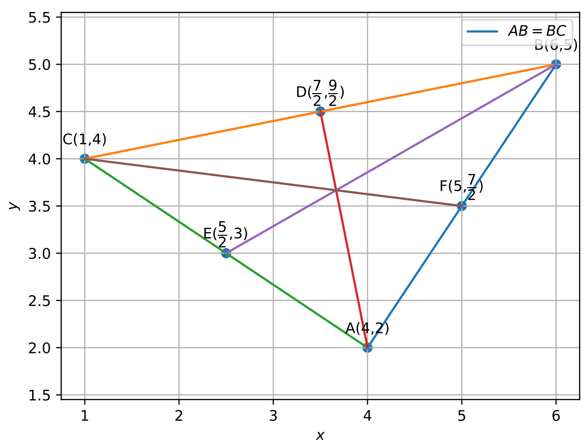

Fig. 1.8.11.1

1.8.12 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\mathbf{a}+\mathbf{b})$ and $(\mathbf{a}-3\mathbf{b})$ externally in the ratio 1 : 2. Also, show that P is the mid point of the line segment RQ.

Solution:

$$\mathbf{R} = \frac{\mathbf{Q}-2\mathbf{P}}{-1} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \tag{1.8.12.1}$$

$$\frac{(\mathbf{R}+\mathbf{Q})}{2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{P}. \tag{1.8.12.2}$$

See Fig. 1.8.12.1.

1.8.13 The point which divides the line segment joining the points $\mathbf{P}(7,-6)$ and $\mathbf{Q}(3,4)$ in the ratio 1 : 2 internally lies in the

a) I quadrant
b) II quadrant
c) III quadrant
d) IV quadrant

1.8.14 If the point $\mathbf{P}(2,1)$ lies on the line segment joining points$\mathbf{A}(4,2)$ and $\mathbf{B}(8,4)$, then

a) $AP = \frac{1}{3}AB$
b) $AP = PE$
c) $PB = \frac{1}{3}AB$

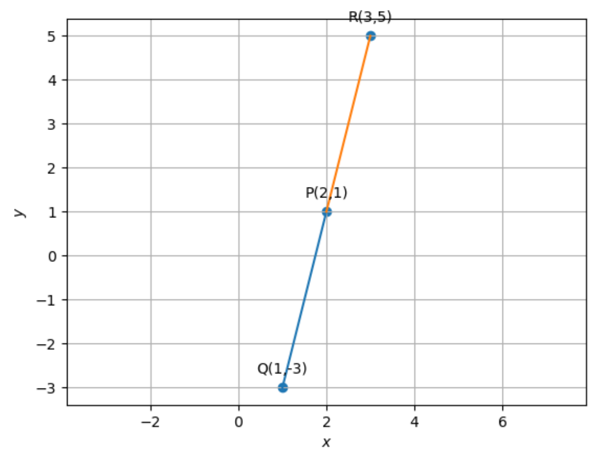

Fig. 1.8.12.1

d) $AP = \frac{1}{2}AB$

1.8.15 If $\mathbf{P}\frac{a}{3}$ is the mid-point of the line segment joining the points $\mathbf{Q}(-6,5)$ and $\mathbf{R}(-2,3)$, then the value of a is

a) – 4
b) – 12
c) 12
d) – 6

1.8.16 A line intersects the y-axis and x-axis of the points $\mathbf{P}$ and $\mathbf{Q}$, respectiveiy. lf $(2,5)$ is the mid-point of $\mathbf{PQ}$, then the coordinates of $\mathbf{P}$ and $\mathbf{Q}$ are, respectively

a) $(0,-5)$ and $(2,0)$
b) $(0,-10)$ and $(-4,0)$
c) $(0,4)$ and $(-10,0)$
d) $(0,-10)$ and $(4,0)$

1.8.17 Point $\mathbf{P}(5,-3)$ is one of the two points of trisection of line segment joining the points $\mathbf{A}(7,-2)$ and $\mathbf{B}(1,-5)$

1.8.18 Points $\mathbf{A}(-6,10), \mathbf{B}(-4,6)$ and $\mathbf{C}(3,-8)$ are collinear such that $\mathbf{AB} = \frac{2}{9}\mathbf{AC}$

1.8.19 In what ratio does the x-axis divide the line segment joining the points $(-4,-6)$ and $(-1,7)$? Find the coordinates of the point of division.

1.8.20 Find the ratio in which the point $\mathbf{P}\left(\frac{3}{4},\frac{5}{12}\right)$ divides the line segment joining the points $\mathbf{A}\left(\frac{1}{2},\frac{3}{2}\right)$ and $\mathbf{B}(2,-5)$.

1.8.21 If $\mathbf{P}(9a-2,-b)$ divides line segment joining $\mathbf{A}(3a+1,-3)$ and $\mathbf{B}(8a,5)$ in the ratio 3:1, find the values of a and b.

1.8.22 The line segment joining the points $\mathbf{A}(3,2)$ and $\mathbf{B}(5,1)$ is divided at the point $\mathbf{P}$ in the ratio 1:2 which lies on $3x-18y+k=0$. Find the value of k.

1.8.23 Find the coordinates of the point $\mathbf{R}$ on the line segment joining the points $\mathbf{P}(-1,3)$ and $\mathbf{Q}(2,5)$ such that $\mathbf{PR} = \frac{3}{5}\mathbf{PQ}$.

1.8.24 Find the ratio in which the line 2x+3y-5=0 divides the line segment joining the points $(8,-9)$ and $(2,1)$. Also find the coordinates of the point of division,

1.8.25 If $\mathbf{a}$ and $\mathbf{b}$ are the postion vectors of A and B, respectively, find the position vector of a point C in BA produced such that BC=1.5BA.

1.8.26 The position vector of the point which divides the join of points 2**a**-3**b** and $\mathbf{a}+\mathbf{b}$ in the ratio 3:1 is

a) $\frac{3\mathbf{a}-2\mathbf{b}}{2}$
b) $\frac{7\mathbf{a}-8\mathbf{b}}{4}$
c) $\frac{3\mathbf{a}}{4}$
d) $\frac{5\mathbf{a}}{4}$

1.8.27 Find the ratio in which the line segment joining $A(1,-5)$ and $B(-4,5)$ is divided by the x-axis. Also find the coordinates of the point of division.

1.8.28 Find the position vector of a point **R** which divides the line joining two points **P** and **Q** whose position vectors are $2\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-3\mathbf{b}$ externally in the ratio 1 : 2.

1.9 Rank

1.9.1 By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.
Solution: The collinearity matrix can be expressed as

$$\begin{pmatrix} -5 & -2 \\ 5 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} -5 & -2 \\ 0 & 0 \end{pmatrix} \tag{1.9.1.1}$$

which is a rank 1 matrix. See Fig. 1.9.1.1.

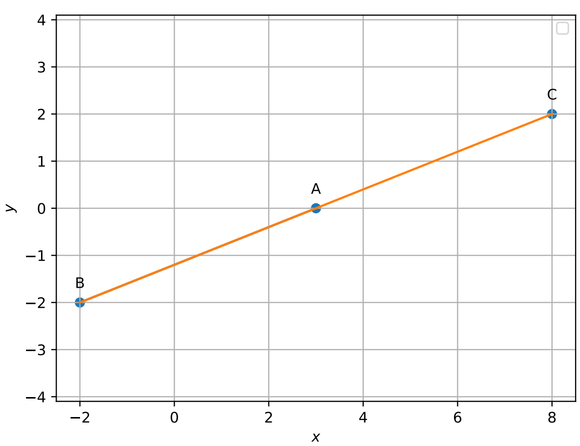

Fig. 1.9.1.1

1.9.2 Determine if the points $(1,5),(2,3)$ and $(-2,-11)$ are collinear.
Solution: Use (A.4.5).

1.9.3 Show that the points $\mathbf{A}(1,2,7),\mathbf{B}(2,6,3)$ and $\mathbf{C}(3,10,-1)$ are collinear.
Solution:

1.9.4 Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear.
Solution:

1.9.5 Show that the points (2, 3, 4), (–1, –2, 1), (5, 8, 7) are collinear.
Solution:

1.9.6 In each of the following, find the value of 'k', for which the points are collinear.

a) $(7,-2),(5,1),(3,k)$
b) $(8,1),(k,-4),(2,-5)$

Solution:

1.9.7 Find a relation between x and y if the points $(x,y),(1,2)$ and $(7,0)$ are collinear.
Solution:

1.9.8 If three points $(x,-1),(2,1)$ and $(4,5)$ are collinear, find the value of x.

1.9.9 If three points $(h,0),(a,b)$ and $(0,k)$ lie on a line, show that

$$\frac{a}{h}+\frac{b}{k}=1 \tag{1.9.9.1}$$

1.9.10 Show that the points A (1, -2, -8), B (5, 0, -2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

1.9.11 If the points $\mathbf{A}(1,2),\mathbf{0}(0,0)$ and $\mathbf{C}(a,b)$ are collinear,then

a) a=b
b) a=2b
c) 2a=b
d) a=-b

True/false

1.12 $\triangle\mathbf{ABC}$ with vertices $\mathbf{A}(-2,0),\mathbf{B}(2,0)$ and $\mathbf{C}(0,2)$ is similar to $\triangle\mathbf{DEF}$ with vertices $\mathbf{D}(-4,0),\mathbf{E}(4,0)$ and $\mathbf{F}(0,4)$

1.13 Point $(-4,2)$ lies on the line segment joining the points $\mathbf{A}(-4,6)$ and $\mathbf{B}(-4,-6)$

1.14 The points $(0,5),(0,-9)$ and $(3,6)$ are collinear

1.15 Points $\mathbf{A}(3,1),\mathbf{B}(12,-2)$ and $\mathbf{C}(0,2)$ cannot be the vertices of a triangle

1.16 Find the value of m if the points $(5,1),(-2,-3)$ and $(8,2m)$ are collinear.

1.17 Find the values of k if the points $\mathbf{A}(k+1,2k),\mathbf{B}(3k,2k+3)$ and $\mathbf{C}(5k-1,5k)$ are collinear

1.18 Using vectors, find the value of k such that the points $(k,-10,3)$, $(1,-1,3)$ and $(3,5,3)$ are collinear.

1.10 Length

1.10.1 Compute the magnitude of the following vectors:

$$\mathbf{a} = \hat{i}+\hat{j}+\hat{k} \tag{1.10.1.1}$$
$$\mathbf{b} = 2\hat{i}-7\hat{j}-3\hat{k} \tag{1.10.1.2}$$
$$\mathbf{c} = \frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{3}\hat{k} \tag{1.10.1.3}$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{1.10.1.4}$$

Then

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^\top\mathbf{a}} = \sqrt{3}, \tag{1.10.1.5}$$
$$\|\mathbf{b}\| = \sqrt{\mathbf{b}^\top\mathbf{b}} = \sqrt{62}, \tag{1.10.1.6}$$
$$\|\mathbf{c}\| = \sqrt{\mathbf{c}^\top\mathbf{c}} = 1 \tag{1.10.1.7}$$

1.10.2 Find the value of x for which $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector
Solution:

$$\because \mathbf{x} = x\begin{pmatrix}1\\1\\1\end{pmatrix}, \|\mathbf{x}\| = 1 \implies x\sqrt{3} = 1 \tag{1.10.2.1}$$

$$\text{or, } x = \frac{1}{\sqrt{3}} \tag{1.10.2.2}$$

1.10.3 If $\mathbf{a} = \mathbf{b}+\mathbf{c}$, then is it true that $|\mathbf{a}| = |\mathbf{b}|+|\mathbf{c}|$? Justify your answer.
Solution: Let

$$\mathbf{b} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{c} = \begin{pmatrix}2\\-1\\-2\end{pmatrix} \tag{1.10.3.1}$$

Then

$$\mathbf{a} = \mathbf{b}+\mathbf{c} = \begin{pmatrix}3\\1\\1\end{pmatrix} \tag{1.10.3.2}$$

$$\implies \|\mathbf{a}\| = \sqrt{11}, \|\mathbf{b}\| = \sqrt{14}, \|\mathbf{c}\| = 3. \tag{1.10.3.3}$$

Thus

$$\|\mathbf{a}\| \neq \|\mathbf{b}\| + \|\mathbf{c}\| \tag{1.10.3.4}$$

1.10.4 If $\overrightarrow{a}$ is a nonzero vector of magnitude ‘a’ and λ a nonzero scalar, then $\lambda\overrightarrow{a}$ is a unit vector if

a) $\lambda = 1$
b) $\lambda = -1$
c) $a = |\lambda|$
d) $a = 1/|\lambda|$

1.10.5 A vector $\mathbf{r}$ is inclined at equal angles to the three axis. If the magnitude of $\mathbf{r}$ is $2\sqrt{3}$ units, find $\mathbf{r}$.
1.10.6 Find the unit vector in the direction of sum of vectors $\mathbf{a}= 2\hat{i}-\hat{j}+\hat{k}$ and $\mathbf{b} = 2\hat{j}+\hat{k}$.
1.10.7 If $\mathbf{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\mathbf{b}=2\hat{i}+\hat{j}-2\hat{k}$, find the unit vector in the direction of

a) $6\mathbf{a}$
b) $2\mathbf{a}$-$\mathbf{b}$

1.10.8 Find a unit vector in the direction of $\overline{PQ}$, where P and Q have co-ordinates(5,0,8) and (3,3,2),respectively.
1.10.9 The vector in the direction of the vector $\hat{i}-2\hat{j}+2\hat{k}$ that has magnitude 9 is

a) $\hat{i}-2\hat{j}+2\hat{k}$
b) $\hat{i}-2\hat{j}$
c) $3(\hat{i}-2\hat{j}+2\hat{k})$
d) $9(\hat{i}-2\hat{j}+2\hat{k})$

1.10.10 If $|\mathbf{a}| = 4$ and $-3 \leq \lambda \leq 2$, then the range of $|\lambda\mathbf{a}|$ is

a) $[0,8]$
b) $[-12,8]$
c) $[0,12]$
d) $[8,12]$

1.10.11 The values of k for which $|\mathbf{ka}| < |\mathbf{a}|$ and $k\mathbf{a}+\frac{1}{2}\mathbf{a}$ is parallel to $\mathbf{a}$ holds true are ______.
1.10.12 If $|\mathbf{a}| = |\mathbf{b}|$, then necessarily it implies $\mathbf{a} = \pm\mathbf{b}$.
1.10.13 The direction cosines of the vector $(2\hat{i}+2\hat{j}-\hat{k})$ are ____________.
1.10.14 Position vector of point P is a vector whose intial point is origin.

1.11 Direction

1.11.1 Find the slope of a line, which passes through the origin and the mid point of the line segment joining the points **P**(0,-4) and **B**(8,0).
Solution: The mid point of PB is

$$\mathbf{M} = \frac{1}{2}(\mathbf{P}+\mathbf{B}) = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.11.1.1}$$

which is equal to the direction vector of OM.

$$\because \mathbf{M} \equiv \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}, m = -\frac{1}{2} \tag{1.11.1.2}$$

which is the desired slope. See Fig. 1.11.1.1.

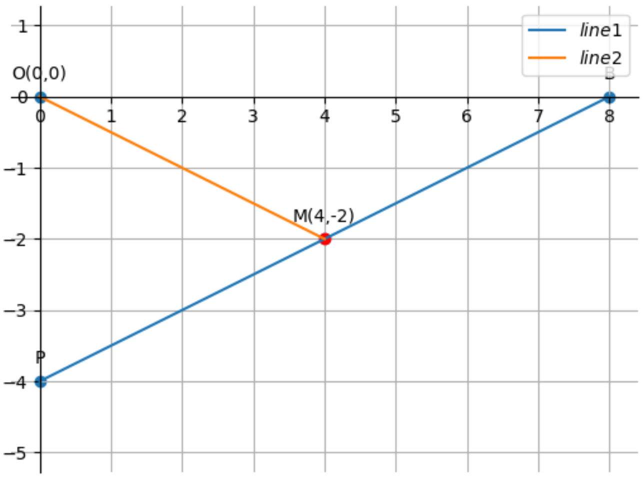

Fig. 1.11.1.1

1.11.2 A line passes through $A(x_1, y_1)$ and $B(h, k)$. If slope of the line is m, show that $(k - y_1) = m(h - x_1)$.
Solution: The direction vector

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} h - x_1 \\ k - y_1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \frac{k-y_1}{h-x_1} \end{pmatrix} \tag{1.11.2.1}$$

1.11.3 For given vectors, $\mathbf{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\mathbf{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\mathbf{a} + \mathbf{b}$.
Solution:

$$\because \mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \tag{1.11.3.1}$$

$$\|\mathbf{a} + \mathbf{b}\| = \sqrt{2} \tag{1.11.3.2}$$

$$\implies \frac{\mathbf{a} + \mathbf{b}}{\|\mathbf{a} + \mathbf{b}\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{1.11.3.3}$$

which is the desired the unit vector.

1.11.4 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + \hat{k}$.

1.11.5 If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}, \mathbf{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\mathbf{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

Solution:

$$2\mathbf{a} - \mathbf{b} + 3\mathbf{c} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \implies \frac{2\mathbf{a} - \mathbf{b} + 3\mathbf{c}}{\|2\mathbf{a} - \mathbf{b} + 3\mathbf{c}\|} = \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \tag{1.11.5.1}$$

1.11.6 Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.
Solution: Let the required vector be

$$c \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}. \tag{1.11.6.1}$$

From the given information,

$$\left\| c \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right\| = 8 \tag{1.11.6.2}$$

$$\implies |c| = \frac{4\sqrt{30}}{15} \tag{1.11.6.3}$$

1.11.7 Find the unit vector in the direction of the vector $\mathbf{a} = \hat{i} + \hat{j} + 2\hat{k}$.

1.11.8 Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where **P** and **Q** are the points (1, 2, 3) and (4, 5, 6), respectively.

1.11.9 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + \hat{k}$.
Solution:

$$\because \mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{1.11.9.1}$$

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \implies \|\mathbf{a} + \mathbf{b}\| = \sqrt{10} \tag{1.11.9.2}$$

From problem 1.11.3, the unit vector in the direction of $\mathbf{a} + \mathbf{b}$ is

$$\frac{\mathbf{a} + \mathbf{b}}{\|\mathbf{a} + \mathbf{b}\|} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{1.11.9.3}$$

The desired vector can then be expressed as

$$\pm \frac{5}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{1.11.9.4}$$

1.11.10 If a line makes angles $90°, 135°, 45°$ with x,y and z-axis respectivly. Find its direction cosines.
Solution: The direction vector is

$$\mathbf{A} = \begin{pmatrix} \cos 90° \\ \cos 135° \\ \cos 45° \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1.11.10.1}$$

1.11.11 Find the direction cosines of the vector joining the points **A** (1, 2, –3) and **B**(–1, –2, 1), directed from **A** to **B**.

Solution: The unit vector in the direction of AB is

$$\frac{\mathbf{B}-\mathbf{A}}{\|\mathbf{B}-\mathbf{A}\|} = \frac{1}{3}\begin{pmatrix}-1\\-2\\2\end{pmatrix} \tag{1.11.11.1}$$

and the direction cosines are the elements of the above vector.

1.11.12 Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axes OX, OY and OZ.

Solution: Since all entries of the given vector

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.11.12.1}$$

are equal, it is equally inclined to the axes.

1.11.13 If a line has the direction ratios –18, 12, –4, then what are its direction cosines?

Solution: Let

$$\mathbf{A} = \begin{pmatrix}-18\\12\\-4\end{pmatrix} \tag{1.11.13.1}$$

Then the unit direction vector of the line is

$$\frac{\mathbf{A}}{\|\mathbf{A}\|} = \begin{pmatrix}\frac{-9}{11}\\\frac{6}{11}\\\frac{-2}{11}\end{pmatrix} \tag{1.11.13.2}$$

1.11.14 Find the direction cosines of the sides of a triangle whose vertices are $\begin{pmatrix}3\\5\\-4\end{pmatrix}$, $\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}-5\\-5\\-2\end{pmatrix}$.

Solution: Let the vertices be

$$\mathbf{A} = \begin{pmatrix}3\\5\\-4\end{pmatrix}, \mathbf{B} = \begin{pmatrix}-1\\1\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-5\\-5\\-2\end{pmatrix} \tag{1.11.14.1}$$

The direction vectors of the sides are,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}4\\4\\-6\end{pmatrix} = \mathbf{m_1}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}4\\6\\4\end{pmatrix} = \mathbf{m_2}, \tag{1.11.14.2}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix}-8\\-10\\2\end{pmatrix} = \mathbf{m_3}, \tag{1.11.14.3}$$

The corresponding unit vectors are then obtained as

$$\begin{pmatrix}\frac{2}{\sqrt{17}}\\\frac{2}{\sqrt{17}}\\\frac{-3}{\sqrt{17}}\end{pmatrix}, \begin{pmatrix}\frac{2}{\sqrt{17}}\\\frac{3}{\sqrt{17}}\\\frac{2}{\sqrt{17}}\end{pmatrix}, \begin{pmatrix}\frac{-4}{\sqrt{42}}\\\frac{-5}{\sqrt{42}}\\\frac{1}{\sqrt{42}}\end{pmatrix} \tag{1.11.14.4}$$

1.11.15 Find the direction cosines of the vector $\hat{i}+2\hat{j}+3\hat{k}$.

Solution: The unit vector in the direction of the given vector is

$$\mathbf{A} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} \tag{1.11.15.1}$$

1.11.16 Find the direction cosines of a line which makes equal angles with the coordinate axes.

Solution: Let α be the angle made by the line with the axes. The unit direction vector can be expressed as

$$\mathbf{x} = \begin{pmatrix}\cos\alpha\\\cos\alpha\\\cos\alpha\end{pmatrix} \implies \|\mathbf{x}\| = 1 \tag{1.11.16.1}$$

$$\text{or, } \cos\alpha = \frac{1}{\sqrt{3}} \tag{1.11.16.2}$$

Thus the unit direction vector of the given line is

$$\mathbf{x} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{1.11.16.3}$$

1.11.17 Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

1.12 Scalar Product

1.12.1 Find the angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\overrightarrow{a} \cdot \overrightarrow{b} = \sqrt{6}$.
Solution: From the given information,

$$\|\mathbf{a}\| = \sqrt{3}, \|\mathbf{b}\| = 2, \mathbf{a}^\top \mathbf{b} = \sqrt{6} \quad (1.12.1.1)$$

$$\implies \cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{1}{\sqrt{2}} \quad (1.12.1.2)$$

$$\text{or, } \theta = 45^\circ \quad (1.12.1.3)$$

1.12.2 Find the angle between the the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.
Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \quad (1.12.2.1)$$

From problem 1.12.1,

$$\cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{10}{\sqrt{14} \times \sqrt{14}} = \frac{5}{7} \quad (1.12.2.2)$$

1.12.3 Find $\left|\overrightarrow{a}\right|$ and $\left|\overrightarrow{b}\right|$, if $(\overrightarrow{a} + \overrightarrow{b}).(\overrightarrow{a} - \overrightarrow{b}) = 8$ and $\left|\overrightarrow{a}\right| = 8\left|\overrightarrow{b}\right|$.
Solution:

$$\because (\mathbf{a} + \mathbf{b})^\top (\mathbf{a} - \mathbf{b}) = 8, \|\mathbf{a}\| = 8\|\mathbf{b}\|, \quad (1.12.3.1)$$

$$\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 = 8 \quad (1.12.3.2)$$

$$\implies \|8\mathbf{b}\|^2 - \|\mathbf{b}\|^2 = 8 \quad (1.12.3.3)$$

$$\implies \|\mathbf{b}\| = \frac{2\sqrt{2}}{3\sqrt{7}} \quad (1.12.3.4)$$

Thus,

$$\|\mathbf{a}\| = 8\|\mathbf{b}\| = \frac{16\sqrt{2}}{3\sqrt{7}} \quad (1.12.3.5)$$

1.12.4 Evaluate the product $(3\overrightarrow{a} - 5\overrightarrow{b}) \cdot (2\overrightarrow{a} + 7\overrightarrow{b})$.
Solution:

$$(3\mathbf{a} - 5\mathbf{b})^\top (2\mathbf{a} + 7\mathbf{b}) = 3\mathbf{a}^\top (2\mathbf{a} + 7\mathbf{b}) - 5\mathbf{b}^\top (2\mathbf{a} + 7\mathbf{b})$$

$$= 6\|\mathbf{a}\|^2 - 35\|\mathbf{b}\|^2 + 11\mathbf{a}^\top \mathbf{b} \quad (1.12.4.1)$$

1.12.5 Find the magnitude of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.
Solution: Given

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \cos\theta = \frac{1}{2}, \mathbf{a}^\top \mathbf{b} = \frac{1}{2}, \quad (1.12.5.1)$$

$$\implies \frac{1}{2} = \frac{\frac{1}{2}}{\|\mathbf{a}\|^2} \implies \|\mathbf{a}\| = \|\mathbf{b}\| = 1 \quad (1.12.5.2)$$

by using the definition of the scalar product.

1.12.6 Find $\left|\overrightarrow{x}\right|$, if for a unit vector $\overrightarrow{a}$, $(\overrightarrow{x} - \overrightarrow{a}) \cdot (\overrightarrow{x} + \overrightarrow{a})$=12.
Solution: From the given information,

$$(\mathbf{x} - \mathbf{a})^\top (\mathbf{x} + \mathbf{a}) = 12 \quad (1.12.6.1)$$

$$\implies \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 = 12 \quad (1.12.6.2)$$

$$\implies \|\mathbf{x}\| = \sqrt{13} \quad (1.12.6.3)$$

1.12.7 If the vertices A, B, C of a triangle ABC are (1,2,3), (-1,0,0), (0,1,2), respectively, then find $\angle ABC$.
Solution: From the given information,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} - \mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (1.12.7.1)$$

$$\implies \angle ABC = \cos^{-1} \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{C} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{C} - \mathbf{B}\|} \quad (1.12.7.2)$$

$$= \cos^{-1} \frac{10}{\sqrt{102}} \quad (1.12.7.3)$$

$$(1.12.7.4)$$

1.12.8 Find a unit vector perpendicular to each of the vector $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$, where $\overrightarrow{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\overrightarrow{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.
Solution: Let the desired vector be $\mathbf{x}$. Then,

$$\begin{pmatrix} \mathbf{a} + \mathbf{b} & \mathbf{a} - \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \quad (1.12.8.1)$$

$$(1.12.8.2)$$

$$\because \mathbf{a} + \mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.12.8.3)$$

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (1.12.8.4)$$

(1.12.8.2) can be expressed as

$$\left\{ \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\}^\top \mathbf{x} = 0 \quad (1.12.8.5)$$

$$\implies \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^\top \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \quad (1.12.8.6)$$

$$\implies \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^\top \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \quad (1.12.8.7)$$

$$\text{or, } \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \quad (1.12.8.8)$$

which can be expressed as

$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix} \xleftrightarrow[R_2 = \frac{R_2}{4}]{R_2 = 3R_2 - R_1} \begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & -2 \end{pmatrix} \quad (1.12.8.9)$$

$$\xleftrightarrow[R_1 = \frac{R_1}{3}]{R_1 = R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} \quad (1.12.8.10)$$

yielding

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned} \implies \mathbf{x} = x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad (1.12.8.11)$$

Thus, the desired unit vector is

$$\mathbf{x} = \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad (1.12.8.12)$$

1.12.9 If a unit vector $\overrightarrow{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\overrightarrow{a}$.

Solution: From the given information,

$$\mathbf{a} = \begin{pmatrix} \cos\frac{\pi}{3} \\ \cos\frac{\pi}{4} \\ \cos\theta \end{pmatrix} = \mathbf{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \cos\theta \end{pmatrix} \tag{1.12.9.1}$$

$$\because \|\mathbf{a}\| = 1, \tag{1.12.9.2}$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1 \tag{1.12.9.3}$$

$$\implies \cos\theta = \frac{1}{2} \tag{1.12.9.4}$$

$\because \theta$ is an acute angle. Hence

$$\mathbf{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \tag{1.12.9.5}$$

1.12.10 If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}\cdot\mathbf{b} \geq 0$ only when

a) $0 < \theta < \frac{\pi}{2}$
b) $0 \leq \theta \leq \frac{\pi}{2}$
c) $0 < \theta < \pi$
d) $0 \leq \theta \leq \pi$

Solution:

$$\because \mathbf{a}^\top\mathbf{b} = \cos\theta \|\mathbf{a}\| \|\mathbf{b}\|, \tag{1.12.10.1}$$

$$\mathbf{a}^\top\mathbf{b} \geq 0 \implies \cos\theta \geq 0 \tag{1.12.10.2}$$

$$\therefore 0 \leq \theta \leq \frac{\pi}{2}, \frac{3\pi}{2} \leq \theta \leq 2\pi. \tag{1.12.10.3}$$

1.12.11 Find the angle between x-axis and the line joining points (3,-1) and (4,-2).

Solution: The direction vector of the given line is

$$\mathbf{C} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.12.11.1}$$

Hence, the desired angle is given by

$$\cos\theta = \frac{\mathbf{C}^\top\mathbf{e}_1}{\|\mathbf{C}\| \|\mathbf{e}_1\|} = -\frac{1}{\sqrt{2}} \tag{1.12.11.2}$$

$$\implies \theta = 135° \tag{1.12.11.3}$$

1.12.12 The slope of a line is double of the slope of another line. If tangent of the angle between them is 1/3, find the slopes of the lines.

Solution: The direction vectors of the lines can be expressed as

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ m \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ 2m \end{pmatrix} \tag{1.12.12.1}$$

If the angle between the lines be θ,

$$\tan\theta = \frac{1}{3} \implies \cos\theta = \frac{3}{\sqrt{10}} \tag{1.12.12.2}$$

Thus,

$$\frac{3}{\sqrt{10}} = \frac{\mathbf{m}_1^\top\mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \tag{1.12.12.3}$$

$$= \frac{2m^2+1}{\sqrt{m^2+1}\sqrt{4m^2+1}} \tag{1.12.12.4}$$

$$\implies \frac{9}{10} = \frac{4m^4+4m^2+1}{4m^4+5m^2+1} \tag{1.12.12.5}$$

$$\text{or, } 4m^4 - 5m^2 + 1 = 0 \tag{1.12.12.6}$$

yielding

$$m = \pm\frac{1}{2}, \pm 1 \tag{1.12.12.7}$$

1.12.13 Find angle between the lines, $\sqrt{3}x+y = 1$ and $x+\sqrt{3}y$=1.

Solution: From the given equations, the normal vectors can be expressed as

$$\mathbf{n}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{1.12.13.1}$$

The angle between the lines can then be expressed as

$$\cos\theta = \frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{\sqrt{3}}{2} \tag{1.12.13.2}$$

$$\text{or, } \theta = 30° \tag{1.12.13.3}$$

1.12.14 The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

1.12.15 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ is the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

1.12.16 If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}\times\mathbf{b}|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

1.12.17 A vector $\mathbf{r}$ has a magnitude 14 and direction ratios 2, 3, -6. Find the direction cosines and components of $\mathbf{r}$, given that $\mathbf{r}$ makes an acute angle with x-axis.

1.12.18 Find the angle between the vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$.

1.12.19 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the three vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$ and $|\mathbf{a}| = 2$, $|\mathbf{b}|$=3, $|\mathbf{c}|$=5, the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is

a) 0
b) 1
c) -19
d) 38

1.12.20 If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors such that $\mathbf{a}$+$\mathbf{b}$+$\mathbf{c}$=0, then the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is

a) 1
b) 3
c) $\frac{-3}{2}$
d) None of these

1.12.21 The angles between two vectors $\mathbf{a}, \mathbf{b}$ with magnitude $\sqrt{3}, 4$ respectively, and $\mathbf{a} \cdot \mathbf{b} = 2\sqrt{3}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{3}$
c) $\frac{\pi}{2}$
d) $\frac{5\pi}{2}$

1.12.22 The vector $\mathbf{a} + \mathbf{b}$ bisects the angle between the non-collinear vectors $\mathbf{a}$ and $\mathbf{b}$ if ______.

1.12.23 The vectors $\mathbf{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{k}$ are the adjancent sides of a parallelogram. The acute angle between its diagonals is ______.

1.12.24 If $\mathbf{a}$ is any non-zero vector, then $(\mathbf{a} \cdot \hat{i})\hat{i} + (\mathbf{a} \cdot \hat{j})\hat{j} + (\mathbf{a} \cdot \hat{k})\hat{k}$ equals ______.

1.12.25 If $\mathbf{a}$ and $\mathbf{b}$ are adjacent sides of a rhombus, then $\mathbf{a} \cdot \mathbf{b} = 0$.

1.12.26 Find the angle between the lines

$$\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and} \tag{1.12.26.1}$$
$$\vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k}) \tag{1.12.26.2}$$

1.12.27 Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$.

1.12.28 If a variable line in two adjacent positions has directions cosines l, m, n and $l + \delta l, m + \delta m, n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2 \tag{1.12.28.1}$$

1.12.29 The sine of the angle between the straight line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and the plane $2x - 2y + z = 5$ is

a) $\frac{10}{6\sqrt{5}}$
b) $\frac{4}{5\sqrt{2}}$
c) $\frac{2\sqrt{3}}{5}$
d) $\frac{\sqrt{2}}{10}$

1.12.30 The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

a) $\frac{\sqrt{3}}{2}$
b) $\frac{\sqrt{2}}{3}$
c) $\frac{2}{7}$
d) $\frac{3}{7}$

1.12.31 The angle between the line $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

1.12.32 The angle between the planes $\vec{r} \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} - \hat{j}) = 4$ is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

1.12.33 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ is the angle between them. Then $\mathbf{a} + \mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

1.12.34 The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$ is

a) 0
b) -1
c) 1
d) 3

1.12.35 If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a} \times \mathbf{b}|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

1.12.36 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ the angle between them. Then $\mathbf{a} + \mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

Solution:

$$\because \|\mathbf{a}\| = \|\mathbf{b}\| = 3 \, \|\mathbf{a} + \mathbf{b}\| = 1, \tag{1.12.36.1}$$
$$\|\mathbf{a} + \mathbf{b}\|^2 = 1^2 \tag{1.12.36.2}$$
$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = 1 \tag{1.12.36.3}$$
$$\implies (\|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta) = \frac{-1}{2} \tag{1.12.36.4}$$
$$\implies \cos\theta = \frac{-1}{2}, \text{or, } \theta = \frac{2\pi}{3} \tag{1.12.36.5}$$

1.12.37 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ is the angle between them.Then $\mathbf{a} + \mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

1.12.38 The value of $\hat{i}.(\hat{j} \times \hat{k}) + \hat{j}.(\hat{i} \times \hat{k}) + \hat{k}.(\hat{i} \times \hat{j})$ is

a) 0
b) -1
c) 1
d) 3

1.12.39 If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$,then $|\mathbf{a}.\mathbf{b}| = |\mathbf{a} \times \mathbf{b}|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

1.12.40 A vector $\mathbf{r}$ has a magnitude 14 and direction ratios 2,3,-6. Find the direction cosines and components of $\mathbf{r}$, given that $\mathbf{r}$ makes an acute angle with x-axis.

1.12.41 Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.

1.12.42 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the three vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and $|\mathbf{a}| = 2$, $|\mathbf{b}|=3$, $|\mathbf{c}|=5$, the value of $\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{c} + \mathbf{c}.\mathbf{a}$ is

a) 0
b) 1
c) -19
d) 38

1.12.43 If **a**, **b**, **c** are unit vectors such that **a**+**b**+**c**=0, then the value of **a**.**b** + **b**.**c** + **c**.**a** is

a) 1
b) 3
c) $\frac{-3}{2}$
d) None of these

1.12.44 The angles between two vectors **a** and **b** with magnitude $\sqrt{3}$ and 4, respectively, and **a**, **b**= $2\sqrt{3}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{3}$
c) $\frac{\pi}{2}$
d) $\frac{5\pi}{2}$

1.12.45 The vector **a** + **b** bisects the angle between the non-collinear vectors **a** and **b** if ______.

1.12.46 The vectors $\mathbf{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{k}$ are the adjancent sides of a parallelogram. The acute angle between its diagonals is ______.

1.12.47 If **a** is any non-zero vector, then $(\mathbf{a}.\hat{i})\hat{i}+(\mathbf{a}.\hat{j})\hat{j}+(\mathbf{a}.\hat{k})\ \hat{k}$ equals ______.

1.12.48 If **a** and **b** are adjacent sides of a rhombus, then **a**.**b**.=0.

1.12.49 Find the angle between the lines

$$\overrightarrow{r} = 3\hat{i}-2\hat{j}+6\hat{k}+\lambda(2\hat{i}+\hat{j}+2\hat{k}) \text{ and } \overrightarrow{r} = (2\hat{j}-5\hat{k})+\mu(6\hat{i}+3\hat{j}+2\hat{k})$$

1.12.50 Find the angle between the lines whose direction cosines are given by the equations $l+m+n=0$, $l^2+m^2-n^2=0$.

1.12.51 If a variable line in two adjacent positions has directions cosines l, m, n and $l+\delta l, m+\delta m, n+\delta n$, show that the small angle $\delta\theta$ between the two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$$

1.12.52 The sine of the angle between the straight line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and the plane $2x-2y+z=5$ is

a) $\frac{10}{6\sqrt{5}}$
b) $\frac{4}{5\sqrt{2}}$
c) $\frac{2\sqrt{3}}{5}$
d) $\frac{\sqrt{2}}{10}$

1.12.53 The plane $2x-3y+6z-11=0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

a) $\frac{\sqrt{3}}{2}$
b) $\frac{\sqrt{2}}{3}$
c) $\frac{2}{7}$
d) $\frac{3}{7}$

1.12.54 The angle between the line $\overrightarrow{r} = (5\hat{i}-\hat{j}-4\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ and the plane $\overrightarrow{r}\cdot(3\hat{i}-4\hat{j}-\hat{k})+5=0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

1.12.55 The angle between the planes $\overrightarrow{r}\cdot(2\hat{i}-3\hat{j}+\hat{k})=1$ and $\overrightarrow{r}\cdot(\hat{i}-\hat{j})=4$ is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

1.12.56 Let **a** and **b** be two unit vectors and θ is the angle between them.Then **a** + **b** is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

1.12.57 The value of $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}.(\hat{i}\times\hat{j})$ is

a) 0
b) -1
c) 1
d) 3

1.12.58 If θ is the angle between any two vectors **a** and **b**,then $|\mathbf{a}.\mathbf{b}| = |\mathbf{a}\times\mathbf{b}|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

1.13 Orthogonality

1.13.1 Name the type of quadrilateral formed, if any, by the following points,and give reasons for your answer

a) $A(-1,-2), B(1,0), (C-1,2), D(-3,0)$
b) $A(-3,5), B(-3,1), C(0,3), D(-1,-4)$
c) $A(4,5), B(7,6), C(4,3), D(1,2)$

Solution: See Table 1.13.1, Fig. 1.13.1.1, Fig. 1.13.1.2. and Fig. 1.13.1.3. In b), forming the collinearity matrix

$$\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{B}\end{pmatrix} = \begin{pmatrix}6 & -3\\ -4 & 2\end{pmatrix} \xleftrightarrow{R_2 \to R_2+\frac{2}{3}R_1} = \begin{pmatrix}6 & -3\\ 0 & 0\end{pmatrix} \tag{1.13.1.1}$$

which is a rank 1 matrix. Hence, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear.

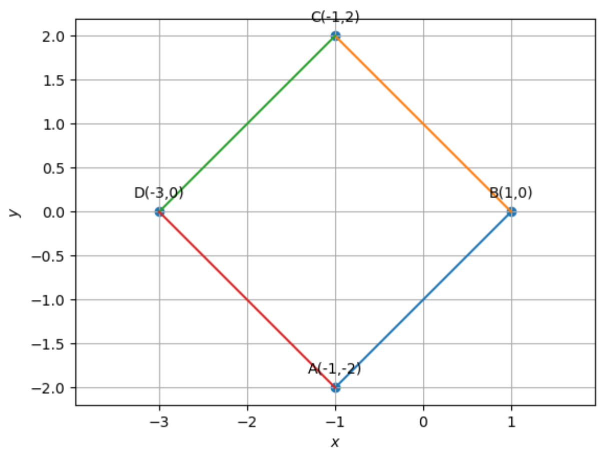

Fig. 1.13.1.1

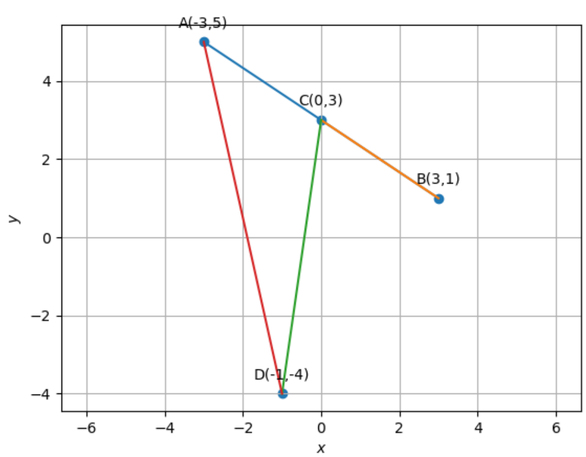

Fig. 1.13.1.2

1.13.2 Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.

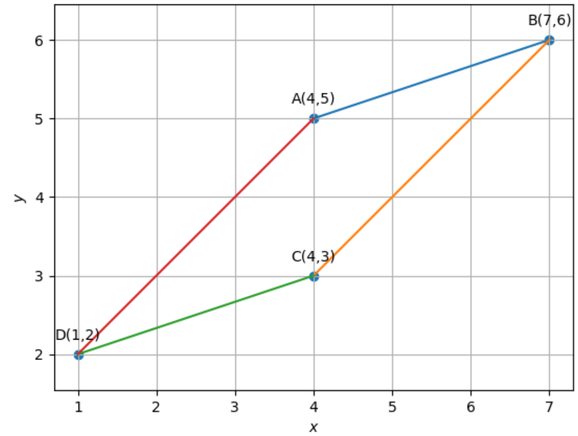

Fig. 1.13.1.3

	$\mathbf{B}-\mathbf{A}=\mathbf{C}-\mathbf{D}$?	$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{B})=0$?	$(\mathbf{C}-\mathbf{A})^\top(\mathbf{D}-\mathbf{B})=0$	**Geometry**
a)	Yes	Yes	Yes	Square
b)	No	-	-	Triangle
c)	Yes	No	No	Parallelogram

TABLE 1.13.1

Solution: Let

$$\mathbf{A} = \begin{pmatrix}1\\3\\7\end{pmatrix}, \mathbf{B} = \begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{1.13.2.1}$$

The projection of $\mathbf{A}$ on $\mathbf{B}$ is defined as the foot of the perpendicular from $\mathbf{A}$ to $\mathbf{B}$ and obtained in (D.1.3). Substituting numerical values,

$$\mathbf{C} = \frac{10}{19}\begin{pmatrix}7\\-1\\8\end{pmatrix} \tag{1.13.2.2}$$

1.13.3 Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.
Solution: The given points are

$$\mathbf{A} = \begin{pmatrix}1\\-1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}1\\1\end{pmatrix} \tag{1.13.3.1}$$

Since

$$\mathbf{A}^\top\mathbf{B} = 0, \tag{1.13.3.2}$$

from (D.1.3), the projection vector is the origin. See Fig. 1.13.3.1.

1.13.4 Show that each of the given three vectors is a unit vector: $\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$, $\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$, $\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$. Also,show that they are mutually perpendicular to each other.
Solution:

$$\mathbf{A} = \begin{pmatrix}\frac{2}{7} & \frac{3}{7} & \frac{6}{7}\\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7}\\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7}\end{pmatrix} \tag{1.13.4.1}$$

is an orthogonal matrix satisfying (D.5.1), which verifies

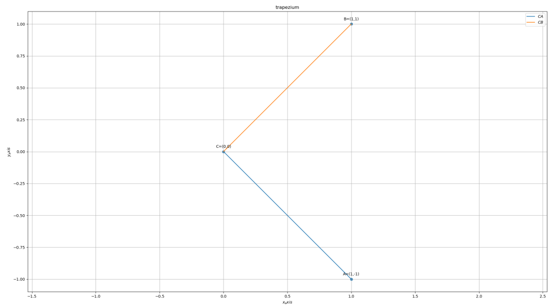

Fig. 1.13.3.1

the given conditions.

1.13.5 If $\overrightarrow{a} = 2\hat{i} + 2\hat{j}3\hat{k}, \overrightarrow{b} = \hat{-i} + 2\hat{j} + \hat{k}$ and $\overrightarrow{c} = 3\hat{i} + \hat{j}$ are such that $\overrightarrow{a} + \lambda\overrightarrow{b}$ is perpendicular to $\overrightarrow{c}$,then find the value of λ.

Solution:

$$\because (\mathbf{a} + \lambda\mathbf{b})^\top \mathbf{c} = 0, \tag{1.13.5.1}$$

$$\lambda = -\frac{\mathbf{a}^\top \mathbf{c}}{\mathbf{b}^\top \mathbf{c}} = 8, \tag{1.13.5.2}$$

upon substituting numerical values.

1.13.6 Show that $\left|\overrightarrow{a}\right|\overrightarrow{b} + \left|\overrightarrow{b}\right|\overrightarrow{a}$ is perpendicular to $\left|\overrightarrow{a}\right|\overrightarrow{b} - \left|\overrightarrow{b}\right|\overrightarrow{a}$, for any two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

Solution:

$$\|\mathbf{a}\|\,\mathbf{b} + \|\mathbf{b}\|\,\mathbf{a} = \|\mathbf{a}\|\,\|\mathbf{b}\|\left(\frac{\mathbf{b}}{\|\mathbf{b}\|} + \frac{\mathbf{a}}{\|\mathbf{a}\|}\right) \tag{1.13.6.1}$$

$$\|\mathbf{a}\|\,\mathbf{b} - \|\mathbf{b}\|\,\mathbf{a} = \|\mathbf{a}\|\,\|\mathbf{b}\|\left(\frac{\mathbf{b}}{\|\mathbf{b}\|} - \frac{\mathbf{a}}{\|\mathbf{a}\|}\right) \tag{1.13.6.2}$$

$$\implies (\|\mathbf{a}\|\,\mathbf{b} + \|\mathbf{b}\|\,\mathbf{a})^\top (\|\mathbf{a}\|\,\mathbf{b} - \|\mathbf{b}\|\,\mathbf{a}) = 0 \tag{1.13.6.3}$$

from (D.2.1).

1.13.7 If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are unit vectors such that $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0}$, find the value of $\overrightarrow{a}.\overrightarrow{b} + \overrightarrow{b}.\overrightarrow{c} + \overrightarrow{c}.\overrightarrow{a}$.

Solution:

$$\begin{aligned}
\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 &= 0 \\
\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + 2(\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a}) &= 0 \\
\implies 3 + 2(\mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a}) &= 0 \\
\implies \mathbf{a}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{c} + \mathbf{c}^\top\mathbf{a} &= -\frac{3}{2}
\end{aligned} \tag{1.13.7.1}$$

1.13.8 If either vector $\overrightarrow{a} = 0$ or $\overrightarrow{b} = 0$, then $\overrightarrow{a}.\overrightarrow{b}$=0. But the converse need not be true. Justify your answer with an example.

Solution:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.13.8.1}$$

$$\implies \mathbf{a}^\top\mathbf{b} = 0 \tag{1.13.8.2}$$

1.13.9 Show that the vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ from the vertices of a right angled triangle.

Solution:

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}, \tag{1.13.9.1}$$

$$\implies \mathbf{B} - \mathbf{C} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \tag{1.13.9.2}$$

$$\text{or, } (\mathbf{B} - \mathbf{C})^\top (\mathbf{C} - \mathbf{A}) = 0 \tag{1.13.9.3}$$

1.13.10 Show that the points A, B and C with position vectors, $3\hat{i} - 4\hat{j} - 4\hat{k}, 2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} - 3\hat{j} - 5\hat{k}$, respectively, form the vertices of a right angled triangle.

Solution:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}, \mathbf{C} - \mathbf{B} = \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \tag{1.13.10.1}$$

$$\implies (\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = 0 \tag{1.13.10.2}$$

Hence, $\triangle ABC$ is right angled at $\mathbf{A}$.

1.13.11 Let $\mathbf{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \mathbf{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\mathbf{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\mathbf{d}$ which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{c} \cdot \mathbf{d}$=15.

Solution: From the given information,

$$\mathbf{a}^\top\mathbf{d} = 0 \tag{1.13.11.1}$$

$$\mathbf{b}^\top\mathbf{d} = 0 \tag{1.13.11.2}$$

$$\mathbf{c}^\top\mathbf{d} = 15 \tag{1.13.11.3}$$

yielding

$$\begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \\ \mathbf{c}^\top \end{pmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \tag{1.13.11.4}$$

$$\implies \begin{pmatrix} 1 & 4 & 2 \\ 3 & -2 & 7 \\ 2 & -1 & 4 \end{pmatrix} \mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \tag{1.13.11.5}$$

Forming the augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 3 & -2 & 7 & 0 \\ 2 & -1 & 4 & 15 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & 1 & 0 \\ 0 & -9 & 0 & 15 \end{array}\right)$$
$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{9}{14}R_2} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & 1 & 0 \\ 0 & 0 & -\frac{9}{14} & 15 \end{array}\right) \tag{1.13.11.6}$$

yielding

$$\mathbf{d} = \begin{pmatrix} \frac{160}{3} \\ -\frac{5}{3} \\ -\frac{70}{3} \end{pmatrix} \tag{1.13.11.7}$$

upon back substitution.

1.13.12 Prove that $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2$, if and only if $\mathbf{a}, \mathbf{b}$ are perpendicular, given $\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$.

Solution:

$$\because (\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2, \tag{1.13.12.1}$$
$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \tag{1.13.12.2}$$
$$\implies \mathbf{a}^\top\mathbf{b} = 0 \tag{1.13.12.3}$$

1.13.13 $ABCD$ is a rectangle formed by the points $\mathbf{A}(-1,-1), \mathbf{B}(-1,4), \mathbf{C}(5,4)$ and $\mathbf{D}(5,-1)$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are the mid-points of AB, BC, CD and DA respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer.
Solution: See Fig. 1.13.13.1. From (D.4.3), $PQRS$ is a parallelogram.

$$\mathbf{P} = \frac{3}{2}, \mathbf{Q} = \begin{pmatrix}2\\4\end{pmatrix}, \mathbf{R} = \begin{pmatrix}5\\\frac{3}{2}\end{pmatrix}, \mathbf{S} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{1.13.13.1}$$
$$\implies (\mathbf{Q}-\mathbf{P})^\top(\mathbf{R}-\mathbf{Q}) \neq 0 \tag{1.13.13.2}$$
$$(\mathbf{R}-\mathbf{P})^\top(\mathbf{S}-\mathbf{Q}) = 0 \tag{1.13.13.3}$$

Therefore $PQRS$ is a rhombus.

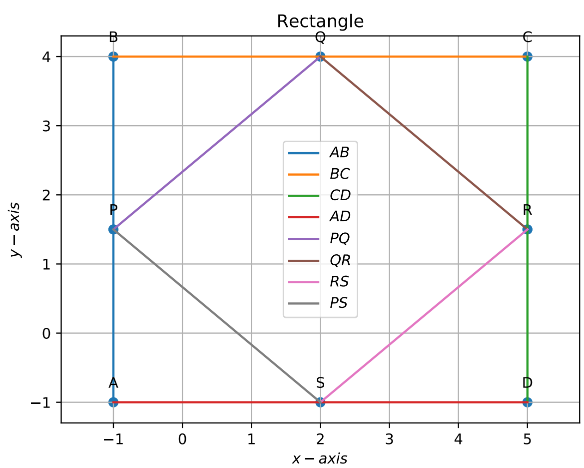

Fig. 1.13.13.1

1.13.14 Without using the Baudhayana theorem, show that the points $A(4,4), B(3,5)$ and $C(-1,-1)$ are the vertices of a right angled triangle. See Fig. 1.13.14.1.

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix}-5\\-5\end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix}1\\-1\end{pmatrix} \tag{1.13.14.1}$$
$$\implies (\mathbf{C}-\mathbf{A})^\top(\mathbf{A}-\mathbf{B}) = 0 \tag{1.13.14.2}$$

Thus, $AB \perp AC$.

1.13.15 The line through the points $(h,3)$ and $(4,1)$ intersects the line $7x-9y-19=0$ at a right angle. Find the value of h.
Solution: The direction vectors of the given lines are

$$\begin{pmatrix}4-h\\-2\end{pmatrix}, \begin{pmatrix}9\\7\end{pmatrix} \tag{1.13.15.1}$$
$$\implies \begin{pmatrix}9 & 7\end{pmatrix}\begin{pmatrix}4-h\\-2\end{pmatrix} = 0 \tag{1.13.15.2}$$
$$\implies h = \frac{22}{9} \tag{1.13.15.3}$$

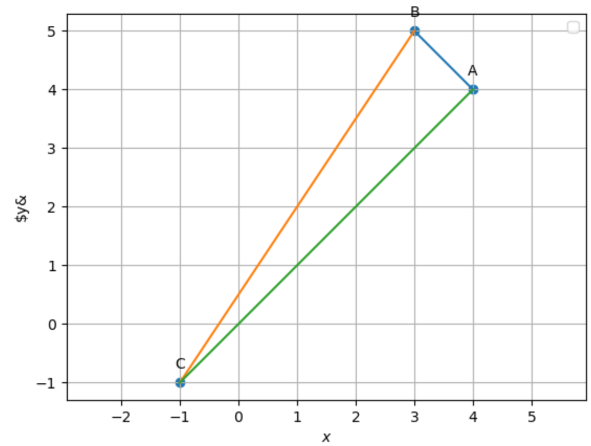

Fig. 1.13.14.1

See Fig. 1.13.15.1.

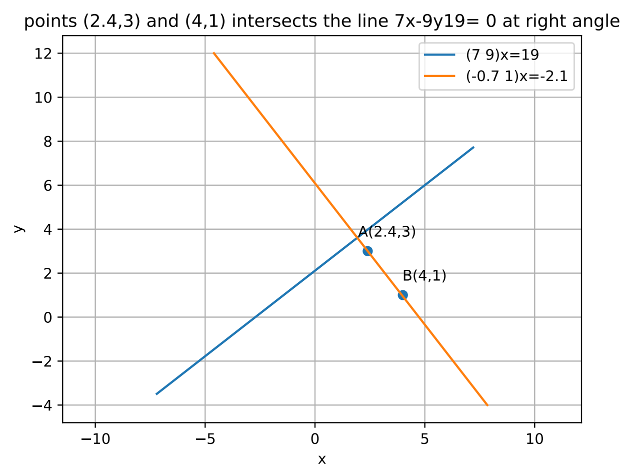

Fig. 1.13.15.1

1.13.16 In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $7x+5y+6z+30=0$ and $3x–y–10z+4=0$
b) $2x+y+3z–2=0$ and $x–2y+5=0$
c) $2x–2y+4z+5=0$ and $3x–3y+6z–1=0$
d) $2x–y+3z–1=0$ and $2x–y+3z+3=0$
e) $4x+8y+z–8=0$ and $y+z–4=0$

Solution: See Table 1.13.16.

1.13.17 Show that the line joining the origin to the point $P(2,1,1)$ is perpendicular to the line determined by the points $A(3,5,-1), B(4,3,-1)$.
Solution:

$$(\mathbf{A}-\mathbf{B})^\top\mathbf{P} = \begin{pmatrix}-1 & 2 & 0\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = 0 \quad \square \tag{1.13.17.1}$$

1.13.18 If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direc-

TABLE 1.13.16

$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1^\top\mathbf{n}_2$	$\|\mathbf{n}_1\|$	$\|\mathbf{n}_2\|$	Angle
$\begin{pmatrix}7\\5\\6\end{pmatrix}$	$\begin{pmatrix}3\\-1\\-10\end{pmatrix}$	-44	$\sqrt{110}$	$\sqrt{110}$	$\cos^{-1}-\frac{2}{5}$
$\begin{pmatrix}2\\1\\3\end{pmatrix}$	$\begin{pmatrix}1\\-2\\0\end{pmatrix}$	0			perpendicular
$\begin{pmatrix}2\\-2\\4\end{pmatrix}$	$\begin{pmatrix}3\\-3\\6\end{pmatrix}$	36	$\sqrt{24}$	$\sqrt{54}$	parallel
$\begin{pmatrix}2\\-1\\3\end{pmatrix}$	$\begin{pmatrix}2\\-1\\3\end{pmatrix}$	14	$\sqrt{14}$	$\sqrt{14}$	parallel
$\begin{pmatrix}4\\8\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	9	9	$\sqrt{2}$	45°

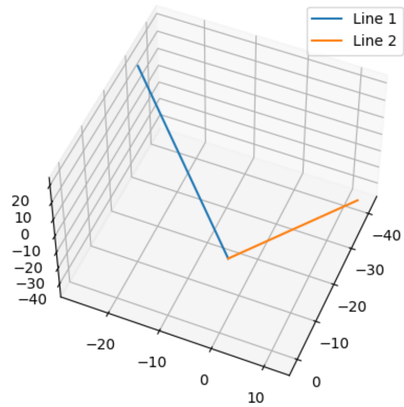

Fig. 1.13.19.1: lines represented for the given points and direction vector with k=$\frac{-10}{7}$

tion cosines of the line perpendicular to both these are $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.

Solution:

$$\mathbf{P} = \begin{pmatrix} l_1 & l_2 & m_1n_2 - m_2n_1 \\ m_1 & m_2 & n_1l_2 - n_2l_1 \\ n_1 & n_2 & l_1m_2 - l_2m_1 \end{pmatrix} \tag{1.13.18.1}$$

satisfies (D.5.1). Hence, the three vectors are mutually perpendicular.

1.13.19 If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Solution: From the given information,

$$\mathbf{m}_1 = \begin{pmatrix}-3\\2k\\2\end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix}3k\\1\\-5\end{pmatrix} \tag{1.13.19.1}$$

$$\implies \begin{pmatrix}-3 & 2k & 2\end{pmatrix}^\top \begin{pmatrix}3k\\1\\-5\end{pmatrix} = 0 \tag{1.13.19.2}$$

$$\implies k = -\frac{10}{7} \tag{1.13.19.3}$$

See Fig. 1.13.19.1

1.13.20 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\mathbf{c} \cdot \mathbf{d}$=15 is equally inclined to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

1.13.21 If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are mutually perpendicular vectors of equal magnitudes,show that the $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is equally inclined to $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

1.13.22 Check whether $(5, -2), (6, 4)$ and $(7, -2)$ are the vertices of an isosceles triangle.

1.13.23 The perpendicular bisector of the line segment joining the points $\mathbf{A}(1, 5)$ and $\mathbf{B}(4, 6)$ cuts the y-axis at

a) $(0, 13)$
b) $(0, -13)$
c) $(0, 12)$
d) $(13, 0)$

1.13.24 The point which lies on the perpendicular bisector of the line segment joining the points $\mathbf{A}(-2, -5)$ and $\mathbf{B}(2, 5)$ is

a) $(0, 0)$
b) $(0, 2)$
c) $(2, 0)$
d) $(-2, 0)$

1.13.25 The points $(-4, 0), (4, 0), (0, 3)$ are the vertices of

a) right triangle
b) isosceles triangle
c) equilateral triangle
d) scalene triangle

1.13.26 The point $\mathbf{A}(2, 7)$ lies on the perpendicular bisector of line segment joining the points $\mathbf{P}(6, 5)$ and $\mathbf{Q}(0, -4)$.

1.13.27 The points $\mathbf{A}(-1, -2), \mathbf{B}(4, 3), \mathbf{C}(2, 5)$ and $\mathbf{D}(-3, 0)$ in that order form a rectangle.

1.13.28 Name the type of triangle formed by the points $\mathbf{A}(-5, 6), \mathbf{B}(-4, -2)$, and $\mathbf{C}(7, 5)$.

1.13.29 What type of a quadrilateral do the points $\mathbf{A}(2, -2), \mathbf{B}(7, 3), \mathbf{C}(11, -1)$, and $\mathbf{D}(6, -6)$ taken in that order, form?

1.13.30 Find the coordinates of the point $\mathbf{Q}$ on the x-axis which lies on the perpendicular bisector of the line segment joining the points $\mathbf{A}(-5, -2)$ and $\mathbf{B}(4, -2)$. Name the type of triangle formed by points $\mathbf{Q}, \mathbf{A}$ and $\mathbf{B}$.

1.13.31 The points $\mathbf{A}(2, 9), \mathbf{B}(a, 5)$ and $\mathbf{C}(5, 5)$ are the vertices of a triangle $\mathbf{ABC}$ right angled at $\mathbf{B}$. Find the values of a and hence the area of $\triangle\mathbf{ABC}$.

1.13.32 Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i} - \hat{j}+2\hat{k}$ and $4\hat{i} - \hat{j} + 3\hat{k}$.

1.13.33 If A,B,C,D are the points with position vectors $\hat{i} + \hat{j} - \hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{k}$, $3\hat{i}$-$2\hat{j}$+$\hat{k}$, respectively, find the projection of $\overline{AB}$ along $\overline{CD}$.

1.13.34 Find the value of λ such that the vectors $\mathbf{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\mathbf{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal.

a) 0
b) 1
c) $\frac{3}{2}$
d) $-\frac{5}{2}$

1.13.35 Projection vector of $\mathbf{a}$ on $\mathbf{b}$ is

a) $\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)$
b) $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$
c) $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$
d) $\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)$

1.13.36 The vectors $\lambda\hat{i}+\lambda\hat{j}+2\hat{k}$, $\hat{i}+\lambda\hat{j}-\hat{k}$ and $2\hat{i}-\hat{j}+\lambda\hat{k}$ are coplanar if
a) $\lambda=-2$
b) $\lambda=0$
c) $\lambda=1$
d) $\lambda=-1$

1.13.37 The number of vectors of unit length perpendicular to the vectors $\mathbf{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\mathbf{b}=\hat{j}+\hat{k}$ is
a) one
b) two
c) three
d) infinite

1.13.38 If $\mathbf{r}\cdot\mathbf{a}=0, \mathbf{r}\cdot\mathbf{b}=0$ and $\mathbf{r}\cdot\mathbf{c}=0$ for some non-zero vector $\mathbf{r}$, then the value of $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is ______.

1.13.39 If $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$, then the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal.

1.13.40 Prove that the lines $x=py+q, z=ry+s$ and $x=p'y+q', z=r'y+s'$ are perpendicular if $pp'+rr'+1=0$.

1.13.41 Find the equation of a plane which bisects perpendicularly the line joining the points A$(2,3,4)$ and B$(4,5,8)$ at right angles.

1.13.42 $\overrightarrow{AB}=3\hat{i}-\hat{j}+\hat{k}$ and $\overrightarrow{CD}=-3\hat{i}+2\hat{j}+4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i}+7\hat{j}+4\hat{k}$ and $-9\hat{j}+2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

1.13.43 Show that the straight lines whose direction cosines are given by $2l+2m-n=0$ and $mn+nl+lm=0$ are at right angles.

1.13.44 If $l_1,m_1,n_1;l_2,m_2,n_2;l_3,m_3,n_3$ are the direction cosines of the three mutually perpendcular lines, prove that the line whose direction cosines are propotional to $l_1+l_2+l_3, m_1+m_2, m_3, n_1+n_2+n_3$ make angles with them.

1.13.45 The intercepts made by the plane $2x-3y+5z+4=0$ on the co-ordinate axis are $\left(-2,\frac{4}{3},-\frac{4}{5}\right)$.

1.13.46 The line $\overrightarrow{r}=2\hat{i}-3\hat{j}-\hat{k}+\lambda(\hat{i}-\hat{j}+2\hat{k})$ lies in the plane $\overrightarrow{r}\cdot(3\hat{i}+\hat{j}-\hat{k})+2=0$.

1.14 Vector Product

1.14.1 Find $\left|\overrightarrow{a}\times\overrightarrow{b}\right|$, if $\overrightarrow{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\overrightarrow{b}=3\hat{i}-2\hat{j}+2\hat{k}$.
Solution: From (D.6.3),

$$\begin{vmatrix}\mathbf{A}_{23} & \mathbf{B}_{23}\end{vmatrix}=\begin{vmatrix}-7 & -2\\ 7 & 2\end{vmatrix}=0 \tag{1.14.1.1}$$

$$\begin{vmatrix}\mathbf{A}_{31} & \mathbf{B}_{31}\end{vmatrix}=\begin{vmatrix}1 & 3\\ 7 & 2\end{vmatrix}=-19 \tag{1.14.1.2}$$

$$\begin{vmatrix}\mathbf{A}_{12} & \mathbf{B}_{12}\end{vmatrix}=\begin{vmatrix}1 & 3\\ -7 & -2\end{vmatrix}=19, \tag{1.14.1.3}$$

$$\|\mathbf{a}\times\mathbf{b}\|=\left\|\begin{pmatrix}\begin{vmatrix}\mathbf{A}_{23} & \mathbf{B}_{23}\end{vmatrix}\\ \begin{vmatrix}\mathbf{A}_{31} & \mathbf{B}_{31}\end{vmatrix}\\ \begin{vmatrix}\mathbf{A}_{12} & \mathbf{B}_{12}\end{vmatrix}\end{pmatrix}\right\|=19\sqrt{2} \tag{1.14.1.4}$$

from (D.7.1).

1.14.2 Find λ and μ if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\overrightarrow{0}$.
Solution: From Appendix D.9, performing row reduction,

$$\begin{pmatrix}2 & 6 & 27\\ 1 & \lambda & \mu\end{pmatrix}\xleftrightarrow{R_2\leftarrow 2R_2-R_1}\begin{pmatrix}2 & 6 & 27\\ 0 & 2\lambda-6 & 2\mu-27\end{pmatrix} \tag{1.14.2.1}$$

$$R_2=0 \implies \mu=\frac{27}{2}, \lambda=3. \tag{1.14.2.2}$$

1.14.3 Find the area of the triangle with vertices $A(1,1,2), B(2,3,5)$ and $C(1,5,5)$.
Solution:

$$\because \mathbf{B}-\mathbf{A}=\begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{C}-\mathbf{A}=\begin{pmatrix}0\\4\\3\end{pmatrix}, \tag{1.14.3.1}$$

$$\frac{1}{2}\left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}0\\4\\3\end{pmatrix}\right\|=\frac{1}{2}\left\|\begin{pmatrix}-6\\3\\4\end{pmatrix}\right\|=\frac{\sqrt{61}}{2} \tag{1.14.3.2}$$

using (1.1.6.1), which is the the desired area.

1.14.4 Find the area of the parallelogram whose adjacent sides are determined by the vectors $\overrightarrow{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\overrightarrow{b}=2\hat{i}-7\hat{j}+\hat{k}$.
Solution: From (1.1.6.1), the desired area is obtained as

$$\left\|\begin{pmatrix}1\\-1\\3\end{pmatrix}\times\begin{pmatrix}2\\-7\\1\end{pmatrix}\right\|=\left\|\begin{pmatrix}20\\5\\-5\end{pmatrix}\right\|=15\sqrt{2} \tag{1.14.4.1}$$

1.14.5 Find the area of a rhombus if its vertices are $A(3,0), B(4,5), C(-1,4)$ and $D(-2,-1)$ taken in order.
Solution: The area of the rhombus is

$$\left\|\left(\mathbf{A}-\mathbf{D}\right)\times\left(\mathbf{B}-\mathbf{A}\right)\right\|=\begin{vmatrix}5 & 1\\ 1 & 5\end{vmatrix}=24 \tag{1.14.5.1}$$

See Fig. 1.14.5.1.

1.14.6 Let the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ be such that $|\overrightarrow{a}|=3$ and $|\overrightarrow{b}|=\frac{\sqrt{2}}{3}$, then $\overrightarrow{a}\times\overrightarrow{b}$ is a unit vector, if the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is
a) $\frac{\pi}{6}$
b) $\frac{\pi}{4}$

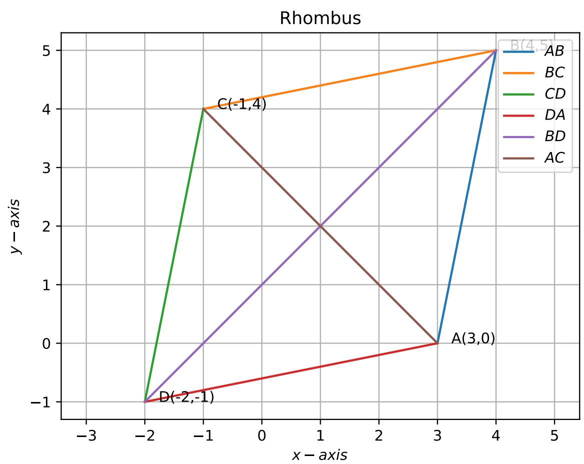

Fig. 1.14.5.1

c) $\frac{\pi}{3}$
d) $\frac{\pi}{2}$

Solution: From the given information and (D.10.1)

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta = 1 \tag{1.14.6.1}$$

$$\implies \sin\theta = \frac{1}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{1}{\sqrt{2}} \tag{1.14.6.2}$$

$$\implies \theta = \frac{\pi}{4} \tag{1.14.6.3}$$

1.14.7 Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is

a) $\frac{1}{2}$
b) 1
c) 2
d) 4

Solution: Since

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \tag{1.14.7.1}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \tag{1.14.7.2}$$

area of the rectangle is

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{C} - \mathbf{D})\| = 2 \tag{1.14.7.3}$$

See Fig. 1.14.7.1

1.14.8 Find the area of the triangle whose vertices are

a) $(2, 3), (-1, 0), (2, -4)$
b) $(-5, -1), (3, -5), (5, 2)$

Solution: See Table 1.14.8.

1.14.9 Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are $A(0, -1), B(2, 1)$ and $C(0, 3)$. Find the ratio of this area to the area of the given triangle.

Solution: Using (1.2.1.1), the mid point coordinates are

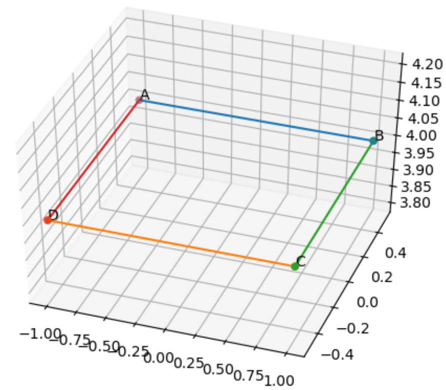

Fig. 1.14.7.1

TABLE 1.14.8

	$\mathbf{A} - \mathbf{B}$	$\mathbf{A} - \mathbf{C}$	$\frac{1}{2}\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\|$
a)	$\begin{pmatrix} 3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 7 \end{pmatrix}$	$\frac{21}{2}$
b)	$\begin{pmatrix} -8 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -10 \\ -3 \end{pmatrix}$	32

given by

$$\mathbf{P} = \frac{1}{2}(\mathbf{A} + \mathbf{B}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.14.9.1}$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{B} + \mathbf{C}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.14.9.2}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{A} + \mathbf{C}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.14.9.3}$$

$$\because \mathbf{P} - \mathbf{Q} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \mathbf{Q} - \mathbf{R} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.14.9.4}$$

$$ar(PQR) = \frac{1}{2}\|(\mathbf{P} - \mathbf{Q}) \times (\mathbf{Q} - \mathbf{R})\| = 1 \tag{1.14.9.5}$$

Similarly,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{1.14.9.6}$$

$$\implies ar(ABC) = \frac{1}{2}\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| = 4 \tag{1.14.9.7}$$

$$\implies \frac{ar(PQR)}{ar(ABC)} = \frac{1}{4} \tag{1.14.9.8}$$

See Fig. 1.14.9.1

1.14.10 Find the area of the quadrilateral whose vertices, taken in order, are $A(-4, -2), B(-3, -5), C(3, -2)$ and $D(2, 3)$.

Solution: See Fig. 1.14.10.1

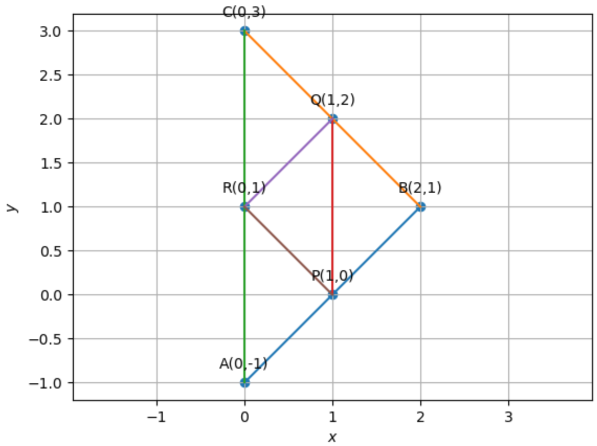

Fig. 1.14.9.1

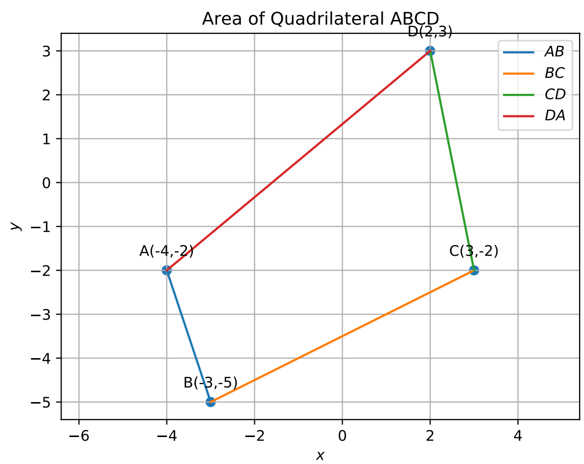

Fig. 1.14.10.1

$$\because \mathbf{A}-\mathbf{B} = \begin{pmatrix}-1\\3\end{pmatrix}, \mathbf{A}-\mathbf{D} = \begin{pmatrix}-6\\-5\end{pmatrix}, \tag{1.14.10.1}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix}-6\\-5\end{pmatrix}, \mathbf{B}-\mathbf{D} = \begin{pmatrix}-3\\-8\end{pmatrix}, \tag{1.14.10.2}$$

$$ar(ABD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| = \frac{23}{2} \tag{1.14.10.3}$$

$$ar(BCD) = \frac{1}{2}\left\|(\mathbf{B}-\mathbf{C})\times(\mathbf{B}-\mathbf{D})\right\| = \frac{33}{2} \tag{1.14.10.4}$$

$$\implies ar(ABCD) = ar(ABD) + ar(BCD) = 28 \tag{1.14.10.5}$$

1.14.11 Verify that a median of a triangle divides it into two triangles of equal areas for $\triangle ABC$ whose vertices are $\mathbf{A}(4,-6), \mathbf{B}(3,2),$ and $\mathbf{C}(5,2)$.

Solution:

$$\mathbf{D} = \frac{\mathbf{B}+\mathbf{C}}{2} = \begin{pmatrix}4\\0\end{pmatrix}, \tag{1.14.11.1}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}1\\-4\end{pmatrix}, \mathbf{A}-\mathbf{D} = \begin{pmatrix}0\\-6\end{pmatrix} \tag{1.14.11.2}$$

$$\implies ar(ABD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\right\| = 3 \tag{1.14.11.3}$$

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix}-1\\-8\end{pmatrix}, \mathbf{A}-\mathbf{D} = \begin{pmatrix}0\\-6\end{pmatrix} \tag{1.14.11.4}$$

$$\implies ar(ACD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{C})\times(\mathbf{A}-\mathbf{D})\right\| \tag{1.14.11.5}$$

$$= 3 = ar(ABD) \tag{1.14.11.6}$$

See Fig. 1.14.11.1.

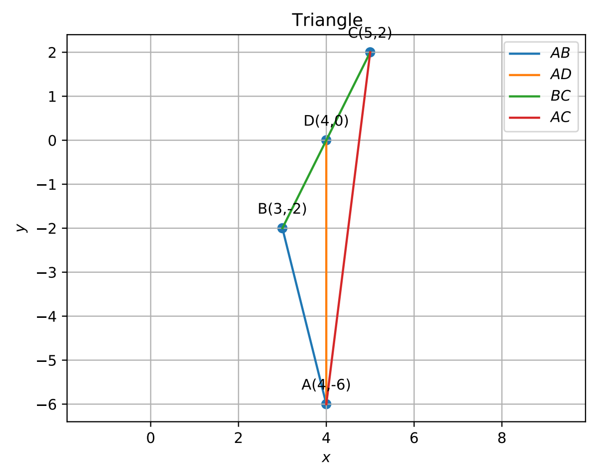

Fig. 1.14.11.1

1.14.12 The two adjacent sides of a parallelogram are $\mathbf{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution: The diagonals of the parallelogram are given by

$$\mathbf{a}+\mathbf{b} = \begin{pmatrix}3\\-6\\2\end{pmatrix}, \mathbf{a}-\mathbf{b} = \begin{pmatrix}1\\-2\\8\end{pmatrix} \tag{1.14.12.1}$$

and the corresponding unit vectors are

$$\frac{\mathbf{a}+\mathbf{b}}{\|\mathbf{a}+\mathbf{b}\|} = \begin{pmatrix}\frac{3}{\sqrt{45}}\\-\frac{6}{\sqrt{45}}\\\frac{2}{\sqrt{45}}\end{pmatrix}, \frac{\mathbf{a}-\mathbf{b}}{\|\mathbf{a}-\mathbf{b}\|} = \begin{pmatrix}\frac{1}{\sqrt{69}}\\-\frac{2}{\sqrt{69}}\\\frac{8}{\sqrt{69}}\end{pmatrix} \tag{1.14.12.2}$$

The area of the parallelogram is given by

$$\|\mathbf{a}\times\mathbf{b}\| = \left\|\begin{pmatrix}22\\-11\\0\end{pmatrix}\right\| = \sqrt{605} \tag{1.14.12.3}$$

1.14.13 The vertices of a $\triangle ABC$ are $\mathbf{A}(4,6), \mathbf{B}(1,5)$ and $\mathbf{C}(7,2)$. A line is drawn to intersect sides AB and AC at $\mathbf{D}$ and $\mathbf{E}$ respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of $\triangle ADE$ and compare it with the area of the $\triangle ABC$.
Solution: See Fig. 1.14.13.1. Using section formula

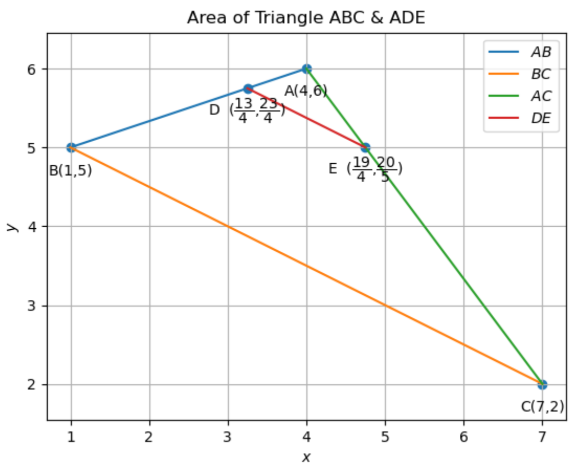

Fig. 1.14.13.1

(1.2.1.1),

$$\mathbf{D} = \frac{3\mathbf{A}+\mathbf{B}}{4} = \frac{1}{4}\begin{pmatrix}13\\23\end{pmatrix} \tag{1.14.13.1}$$

$$\mathbf{E} = \frac{3\mathbf{A}+\mathbf{C}}{4} = \frac{1}{4}\begin{pmatrix}19\\20\end{pmatrix} \tag{1.14.13.2}$$

$$\mathbf{A}-\mathbf{D} = \frac{1}{4}\begin{pmatrix}3\\1\end{pmatrix}, \mathbf{A}-\mathbf{E} = \frac{1}{4}\begin{pmatrix}-3\\1\end{pmatrix} \tag{1.14.13.3}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}3\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-6\\3\end{pmatrix} \tag{1.14.13.4}$$

$$\implies ar(ABD) = \frac{1}{2}\|(\mathbf{A}-\mathbf{D})\times(\mathbf{A}-\mathbf{E})\| = \frac{15}{32} \tag{1.14.13.5}$$

$$ar(ABC) = \frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{B}-\mathbf{C})\| = \frac{15}{2} \tag{1.14.13.6}$$

$$\implies \frac{ar(ADE)}{ar(ABC)} = \frac{1}{16} \tag{1.14.13.7}$$

1.14.14 Draw a quadrilateral in the Cartesian plane, whose vertices are

$$\mathbf{A} = \begin{pmatrix}-4\\5\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\7\end{pmatrix}, \mathbf{C} = \begin{pmatrix}5\\-5\end{pmatrix}, \mathbf{D} = \begin{pmatrix}-4\\-2\end{pmatrix}. \tag{1.14.14.1}$$

Also, find its area.
Solution: See Fig. 1.14.14.1. From (D.11.2),

$$ar(ABCD) = \frac{121}{2} \tag{1.14.14.2}$$

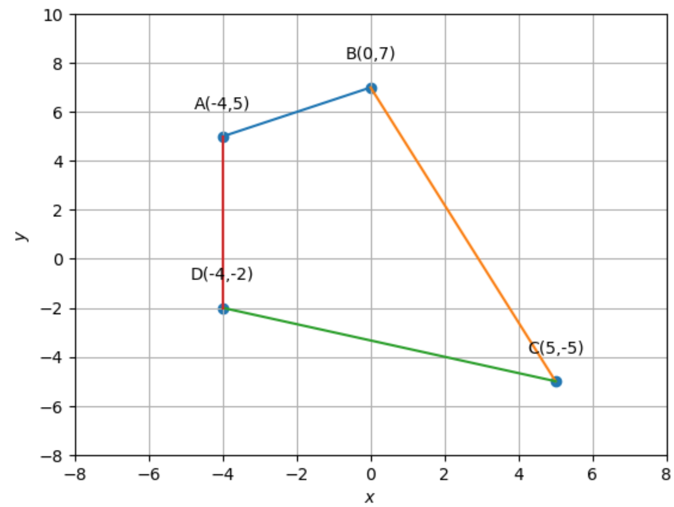

Fig. 1.14.14.1: Plot of quadrilateral $ABCD$

1.14.15 Find the area of region bounded by the triangle whose vertices are $(1,0),(2,2)$ and $(3,1)$.

1.14.16 Find the area of region bounded by the triangle whose vertices are $(-1,0),(1,3)$ and $(3,2)$.

1.14.17 Find the area of the $\triangle ABC$, coordinates of whose vertices are $\mathbf{A}(2,0), \mathbf{B}(4,5)$, and $\mathbf{C}(6,3)$.

1.14.18 Show that

$$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$$

Solution:

$$\begin{aligned}(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) &= \mathbf{a}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}+\mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{a}\\ &= 2\left(\mathbf{a}\times\mathbf{b}\right)\end{aligned} \tag{1.14.18.1}$$

from (D.8.1). and (D.8.2)

1.14.19 If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a}\times\vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.
Solution: For

$$\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\0\\0\end{pmatrix} \tag{1.14.19.1}$$

$$\mathbf{a}\times\mathbf{b} = \mathbf{0}. \tag{1.14.19.2}$$

1.14.20 Given that $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\times\vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

1.14.21 The area of a triangle with vertices $\mathbf{A}(3,0), \mathbf{B}(7,0)$ and $\mathbf{C}(8,4)$ is

a) 14
b) 28
c) 8
d) 6

1.14.22 The area of a triangle with vertices $(a, b+c), (b, c+a)$ and $(c, a+b)$ is

a) $(a+b+c)^2$
b) 0
c) a+b+c
d) abc

1.14.23 Find the area of the triangle whose vertices are $(-8,4), (-6,6)$and $(-3,9)$.

1.14.24 If $\mathbf{D}\left(\frac{-1}{2}, \frac{5}{2}\right), \mathbf{E}(7,3)$ and $\mathbf{F}\left(\frac{7}{2}, \frac{7}{2}\right)$ are the midpoints of sides of $\triangle ABC$, find the area of the $\triangle ABC$.

1.14.25 If $\mathbf{a}+\mathbf{b}+\mathbf{c}$=0, show that $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$. Interpret the result geometrically.

1.14.26 Find the sine of the angle between the vectors $\mathbf{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\mathbf{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

1.14.27 Using vectors, find the area of $\triangle ABC$ with vertices A(1,2,3), B(2,-1,4) and C(4,5,-1).

1.14.28 Using vectors, prove that the parallelograms on the same base and between the same parallels are equal in area.

1.14.29 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$, determine the vertices of a triangle, show that $\frac{1}{2}$ $[\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b}]$ gives the vector area of the triangle. Hence deduce the condition that the three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$, are collinear. Also find the unit vector normal to the plane of the triangle.

1.14.30 Find the area of the parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$.

1.14.31 The vector from origin to the points A and B are $\mathbf{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ and $\mathbf{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, respectively, then the area of $\triangle OAB$ is

a) 340
b) $\sqrt{25}$
c) $\sqrt{229}$
d) $\frac{1}{2}\sqrt{229}$

1.14.32 For any vector $\mathbf{a}$, the value of $(\mathbf{a}\times\hat{i})^2 + (\mathbf{a}\times\hat{j})^2 + (\mathbf{a}\times\hat{k})^2$ is equal to

a) a
b) 3a
c) 4a
d) 2a

1.14.33 If $|\mathbf{a}| = 10, |\mathbf{b}| = 2$ and $\mathbf{a}, \mathbf{b}$=12, then value of $|\mathbf{a}\times\mathbf{b}|$ is

a) 5
b) 10
c) 14
d) 16

1.14.34 If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$ and $\mathbf{b} = \hat{j} - \hat{k}$, find a vector $\mathbf{c}$ such that $\mathbf{a}\times\mathbf{c} = \mathbf{b}$ and $\mathbf{a}\cdot\mathbf{c} = 3$.

1.14.35 The area of the quadrilateral ABCD, where $A(0,4,1)$, $B(2,3,-1)$, $C(4,5,0)$ and $D(2,6,2)$, is equal to

a) 9 sq. units
b) 18 sq. units
c) 27 sq. units
d) 81 sq. units

1.14.36 Find the area of region bounded by the triangle whose vertices are (-1, 1), (0, 5) and (3, 2).

1.15 Miscellaneous

1.15.1 The two opposite vertices of a square are $(-1,2)$ and $(3,2)$. Find the coordinates of the other two vertices.
Solution: Let

$$\mathbf{A} = \begin{pmatrix}-1\\2\end{pmatrix}, \mathbf{C} = \begin{pmatrix}3\\2\end{pmatrix} \tag{1.15.1.1}$$

The given square is available in Fig. 1.15.1.1. Shifting **A**

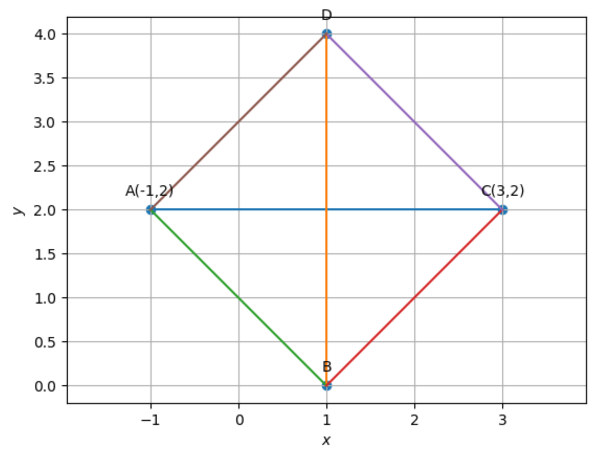

Fig. 1.15.1.1

to origin with reference to Fig. 1.15.1.2,

$$\mathbf{A}_1 = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{C}_1 = \mathbf{C} - \mathbf{A} = \begin{pmatrix}4\\0\end{pmatrix} \tag{1.15.1.2}$$

Since

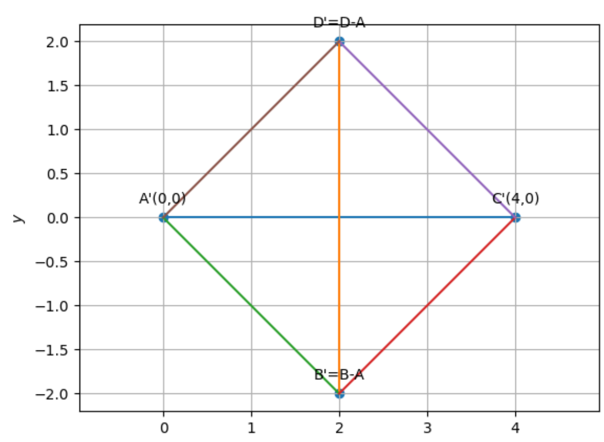

Fig. 1.15.1.2

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix}4\\0\end{pmatrix} \equiv \begin{pmatrix}1\\0\end{pmatrix}, \ \theta = 0^\circ \tag{1.15.1.3}$$

where θ is the angle made by AC with the x-axis. Considering the rotation matrix

$$\mathbf{P} = \begin{pmatrix} \cos\left(\frac{\pi}{4}-\theta\right) & -\sin\left(\frac{\pi}{4}-\theta\right) \\ \sin\left(\frac{\pi}{4}-\theta\right) & \cos\left(\frac{\pi}{4}-\theta\right) \end{pmatrix} \tag{1.15.1.4}$$

From Fig. 1.15.1.3,

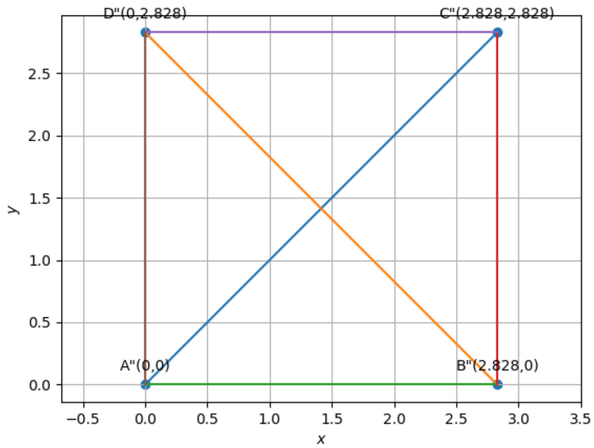

Fig. 1.15.1.3

$$\mathbf{C}_2 = \mathbf{P}(\mathbf{C}-\mathbf{A}) \tag{1.15.1.5}$$

$$\mathbf{B}_2 = \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} \end{pmatrix} \mathbf{C}_2 \tag{1.15.1.6}$$

$$\mathbf{D}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{e}_2 \end{pmatrix} \mathbf{C}_2 \tag{1.15.1.7}$$

Now,

$$\mathbf{B} = \mathbf{P}^\top \mathbf{B}_2 + \mathbf{A} \tag{1.15.1.8}$$

$$\mathbf{D} = \mathbf{P}^\top \mathbf{D}_2 + \mathbf{A} \tag{1.15.1.9}$$

by reversing the process of translation and rotation. Thus, from (1.15.1.8) (1.15.1.6), (1.15.1.9) and (1.15.1.7)

$$\mathbf{B} = \mathbf{P}^\top \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} \end{pmatrix} \mathbf{P}(\mathbf{C}-\mathbf{A}) + \mathbf{A} \tag{1.15.1.10}$$

$$\mathbf{D} = \mathbf{P}^\top \begin{pmatrix} \mathbf{0} & \mathbf{e}_2 \end{pmatrix} \mathbf{P}(\mathbf{C}-\mathbf{A}) + \mathbf{A} \tag{1.15.1.11}$$

yielding

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{D}\begin{pmatrix} 1 \\ 4 \end{pmatrix}. \tag{1.15.1.12}$$

1.15.2 The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

Solution: Let the base be BC. From the given information,

$$\mathbf{B} = a\mathbf{e}_2, \mathbf{C} = -a\mathbf{e}_2 \tag{1.15.2.1}$$

Since $\mathbf{A}$ lies on the x-axis,

$$\mathbf{A} = k\mathbf{e}_1 \tag{1.15.2.2}$$

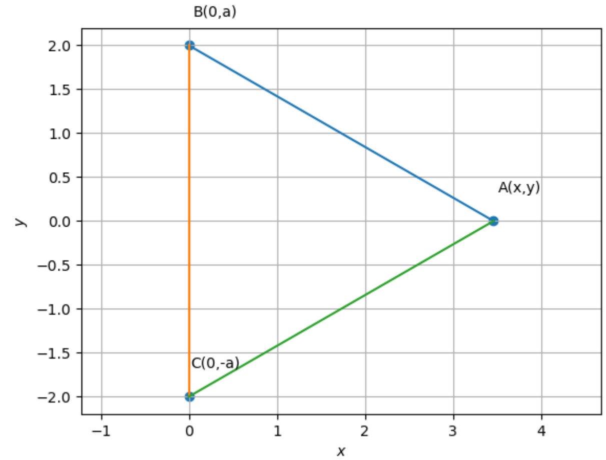

Fig. 1.15.2.1

and

$$\|\mathbf{A}-\mathbf{C}\|^2 = (2a)^2 \tag{1.15.2.3}$$

$$\implies \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^\top\mathbf{C} = 4a^2 \tag{1.15.2.4}$$

$$\implies k^2 + a^2 = 4a^2 \tag{1.15.2.5}$$

$$\text{or, } k = \pm a\sqrt{3} \tag{1.15.2.6}$$

Thus,

$$\mathbf{A} = \pm\sqrt{3}a\mathbf{e}_1 \tag{1.15.2.7}$$

Fig. 1.15.2.1 is plotted for $a = 2$.

1.16 Triangle

1.16.1 Construct a triangle ABC in which $BC = 7cm, \angle B = 75°$ and $AB + AC = 13cm$.
Solution: From (D.12.3) and (D.12.4), we obtain Fig. 1.16.1.1. See

codes/triangle/const−aBsum.py

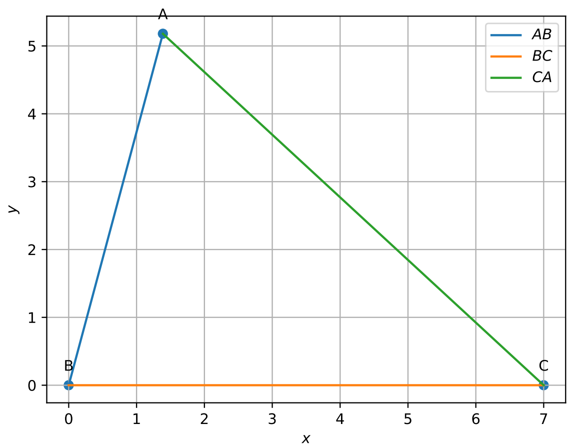

Fig. 1.16.1.1

1.16.2 Construct a triangle ABC in which $BC = 8cm, \angle B = 45°$ and $AB - AC = 3.5cm$.
Solution: See Fig. 1.16.2.1.

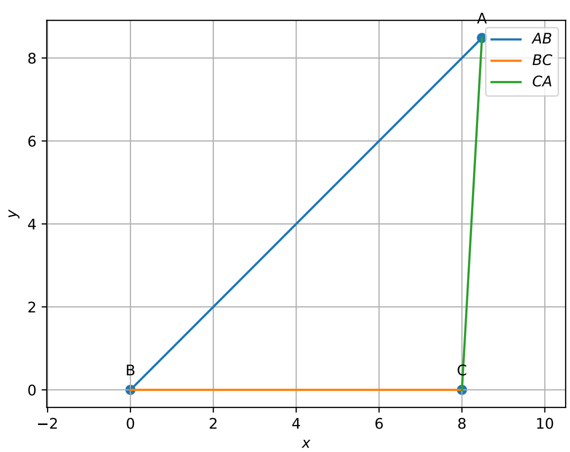

Fig. 1.16.2.1

1.16.3 Construct a triangle ABC in which $BC = 6cm, \angle B = 60°$ and $AC - AB = 2cm$.
Solution: See Fig. 1.16.3.1 obtained by substituting $K = -2$.

1.16.4 Construct a right triangle whose base is $12cm$ and sum of its hypotenuse and other side is $18cm$.
Solution: For $a = 12, \angle B = 90°, b + c = 18$, we obtain Fig. 1.16.4.1.

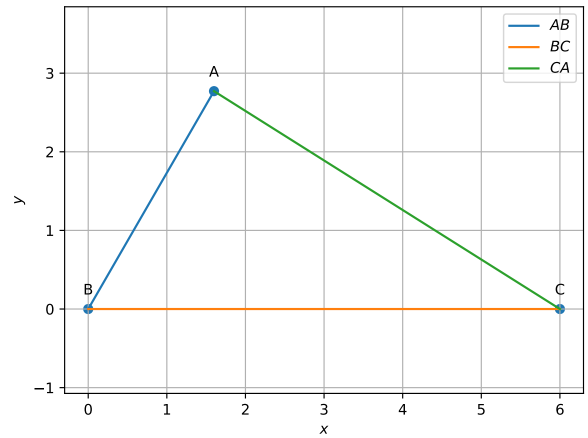

Fig. 1.16.3.1

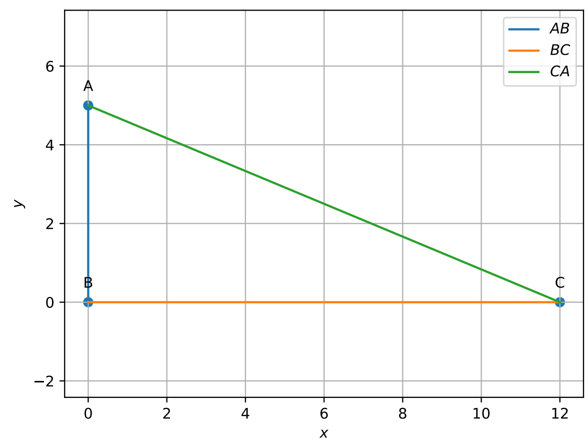

Fig. 1.16.4.1

1.16.5 Construct a triangle ABC in which $\angle B = 30°, \angle C = 90°$ and $AB + BC + CA = 11cm$.
Solution: From (D.13.4) and (D.13.5), Fig. 1.16.5.1 is generated. See

codes/triangle/const−BCsum.py

1.16.6 Draw a right triangle ABC in which $BC = 12cm$, $AB = 5cm$ and $\angle B = 90°$.

1.16.7 Draw an isosceles triangle ABC in which $AB = AC = 6cm$ and $BC = 6cm$.

1.16.8 Draw a triangle ABC in which $AB = 5cm, BC = 6cm$ and $\angle ABC = 60°$.

1.16.9 Draw a triangle ABC in which $AB = 4cm, BC = 6cm$ and $AC = 9cm$.

1.16.10 Draw a triangle ABC in which $BC = 6cm, CA = 5cm$ and $AB = 4cm$.

1.16.11 Is it possible to construct a triangle with lengths of its sides as $4cm$, $3cm$ and $7cm$? Give reason for your answer.

1.16.12 Is it possible to construct a triangle with lengths of its sides as $9cm$, $7cm$ and $17cm$? Give reason for your

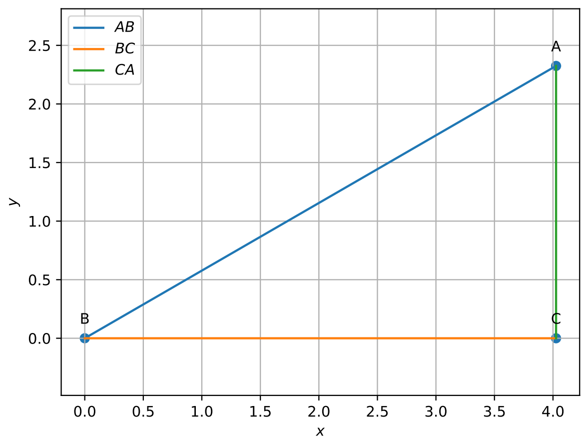

Fig. 1.16.5.1

answer.

1.16.13 Is it possible to construct a triangle with lengths of its sides as $8cm$, $7cm$ and $4cm$? Give reason for your answer.

1.16.14 Two sides of a triangle are of lengths $5cm$ and $1.5cm$. The length of the third side of the triangle cannot be

a) $3.6cm$
b) $4.1cm$
c) $3.8cm$
d) $3.4cm$

1.16.15 The construction of a triangle ABC, given that $BC = 6cm, \angle B = 45°$ is not possible when difference of AB and AC is equal to

a) $6.9cm$
b) $5.2cm$
c) $5.0cm$
d) $4.0cm$

1.16.16 The construction of a triangle ABC, given that $BC = 6cm, \angle C = 60°$ is possible when difference of AB and AC is equal to

a) $3.2cm$
b) $3.1cm$
c) $3cm$
d) $2.8cm$

1.16.17 Construct a triangle whose sides are $3.6cm$, $3.0cm$ and $4.8cm$. Bisect the smallest angle and measure each part.

1.16.18 Construct a triangle ABC in which $BC = 5cm$, $\angle B = 60°$ and $AC + AB = 7.5cm$.

Construct each of the following and give justification :

1.19 A triangle if its perimeter is $10.4cm$ and two angles are $45°$and $120°$.

1.20 A triangle PQR given that $QR = 3cm$, $\angle PQR = 45°$ and $QP - PR = 2cm$.

1.21 A right triangle when one side is $3.5cm$ and sum of other sides and the hypotenuse is $5.5cm$.

1.22 An equilateral triangle if its altitude is $3.2cm$.

Write true or false in each of the following. Give reasons for your answer:

1.23 A triangle ABC can be constructed in which $AB = 5cm, \angle A = 45°$ and $BC + AC = 5cm$.

1.24 A triangle ABC can be constructed in which $BC = 6cm, \angle B = 30°$ and $AC - AB = 4cm$.

1.25 A triangle ABC can be constructed in which $\angle B = 105°, \angle C = 90°$ and $AB + BC + AC = 10cm$.

1.26 A triangle ABC can be constructed in which $\angle B = 60°, \angle C = 45°$ and $AB + BC + AC = 12cm$.

2 Quadrilateral

2.1 Draw a parallelogram $ABCD$ in which $BC = 5cm, AB = 3cm$ and $\angle ABC = 60°$, divide it into triangles ACB and ABD by the diagonal BD.
2.2 Construct a square of side $3cm$.
2.3 Construct a rectangle whose adjacent sides are of lengths $5cm$ and $3.5cm$.
2.4 Construct a rhombus whose side is of length $3.4cm$ and one of its angles is 45°.
2.5 Construct a rhombus whose diagonals are 4 cm and 6 cm in lengths.

3 Matrices

The matrix of the veritices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{3.1}$$

3.1 Vectors

3.1. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{3.1.1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{3.1.1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{3.1.1.3}$$

where the second matrix above is known as a *circulant* matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

3.2. Obtain the normal matrix of the sides of $\triangle ABC$
Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.1.2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{3.1.2.2}$$

3.3. Obtain a, b, c.
Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \tag{3.1.3.1}$$

3.4. Obtain the constant terms in the equations of the sides of the triangle.
Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag}\left\{\left(\mathbf{N}^\top \mathbf{P}\right)\right\} \tag{3.1.4.1}$$

3.2 Median

3.2.1. Obtain the mid point matrix for the sides of the triangle
Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{3.2.1.1}$$

3.2.2. Obtain the median direction matrix.
Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F} \end{pmatrix} \tag{3.2.2.1}$$

$$= \begin{pmatrix} \mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \tag{3.2.2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{3.2.2.3}$$

3.2.3. Obtain the median normal matrix.

3.2.4. Obtian the median equation constants.
3.2.5. Obtain the centroid by finding the intersection of the medians.

3.3 Altitude

3.3.1. Find the normal matrix for the altitudes
Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B} \end{pmatrix} \tag{3.3.1.1}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{3.3.1.2}$$

3.3.2. Find the constants vector for the altitudes.
Solution: The desired vector is

$$\mathbf{c}_2 = \text{diag}\left\{\left(\mathbf{M}^\top \mathbf{P}\right)\right\} \tag{3.3.2.1}$$

3.4 Perpendicular Bisector

3.4.1. Find the normal matrix for the perpendicular bisectors
Solution: The normal matrix is $\mathbf{M}_2$
3.4.2. Find the constants vector for the perpendicular bisectors.
Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right\} \tag{3.4.2.1}$$

3.5 Angle Bisector

3.5.1. Find the points of contact.
Solution: The points of contact are given by

$$\begin{pmatrix} \frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & \frac{n}{b} & \frac{m}{c} \\ \frac{n}{a} & 0 & \frac{p}{c} \\ \frac{m}{a} & \frac{p}{b} & 0 \end{pmatrix} \tag{3.5.1.1}$$

All codes for this section are available at

codes/triangle/mat−alg.py

4 Linear Forms

4.1 Equation of a Line

Find the equation of line

4.1 passing through the point $\mathbf{P}(-4, 3)$ with slope $\frac{1}{2}$.
Solution: Since the normal vector

$$\mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \tag{4.1.1}$$

$$\tag{4.1.2}$$

the desired equation (1.1.5.1) is

$$\mathbf{n}^\top (\mathbf{x}-\mathbf{P}) = 0 \tag{4.1.3}$$

$$\implies \begin{pmatrix} \frac{1}{2} & -1 \end{pmatrix} \mathbf{x} = -5 \tag{4.1.4}$$

See Fig. 4.1.1.

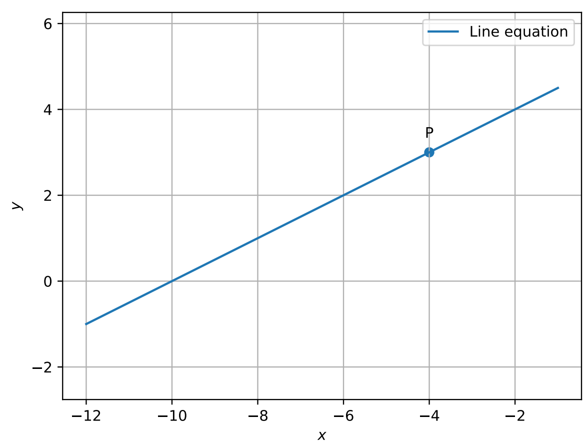

Fig. 4.1.1

4.2 passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m.
Solution:

$$\because \mathbf{n} = \begin{pmatrix} m \\ -1 \end{pmatrix}, \tag{4.2.1}$$

the desired equation is

$$\begin{pmatrix} m & -1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 0 \tag{4.2.2}$$

$$\implies \begin{pmatrix} m & -1 \end{pmatrix} \mathbf{x} = 0 \tag{4.2.3}$$

4.3 passing through $\mathbf{A} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x-axis at an angle of 75°.
Solution:

$$\mathbf{n} = \begin{pmatrix} -1 \\ 2+\sqrt{3} \end{pmatrix} \tag{4.3.1}$$

$$\implies \implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} = 4\left(\sqrt{3}+1\right) \tag{4.3.2}$$

$$\implies \begin{pmatrix} -1 & 2+\sqrt{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 2+\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix} \tag{4.3.3}$$

$$= 4\left(\sqrt{3}+1\right) \tag{4.3.4}$$

is the desired equation. See Fig. 4.3.1.

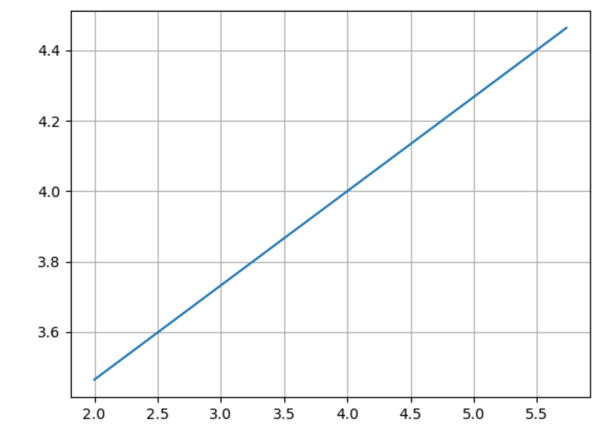

Fig. 4.3.1

4.4 intersecting the x-axis at a distance of 3 units to the left of origin with slope of -2.
Solution: From the given information,

$$\mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{4.4.1}$$

The desired equation of the line is

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \right) = 0 \tag{4.4.2}$$

$$\text{or, } \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = -6 \tag{4.4.3}$$

See Fig. 4.4.1.

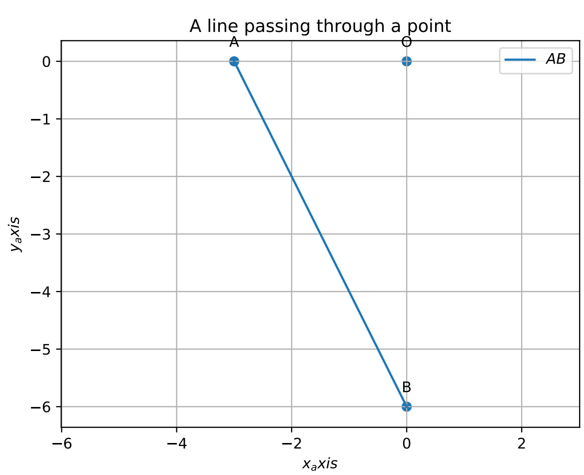

Fig. 4.4.1

4.5 intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.
Solution:

$$\mathbf{n} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \tag{4.5.1}$$

Hence, the equation of the line is given by

$$\begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \tag{4.5.2}$$

$$\text{or, } \begin{pmatrix} -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \mathbf{x} = 2 \tag{4.5.3}$$

See Fig. 4.5.1.

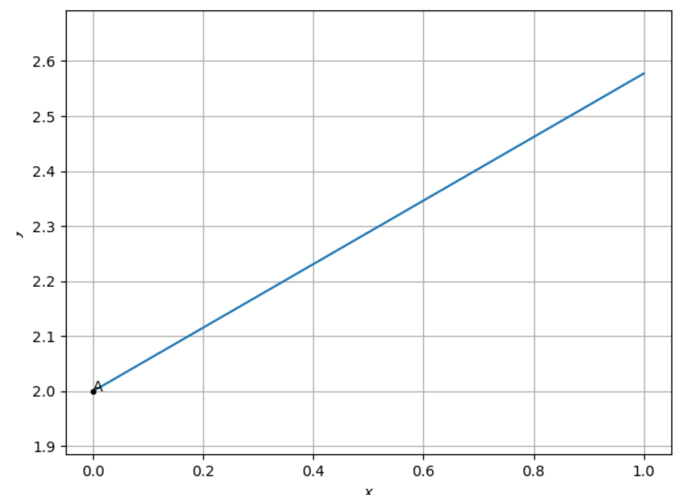

Fig. 4.5.1

4.6 Find the equation of the line passing through the points $\mathbf{A}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{B}\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.
Solution:

$$\mathbf{m} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{4.6.1}$$

Thus, the equation of line is

$$\begin{pmatrix} 5 & 3 \end{pmatrix} \mathbf{x} = -2 \tag{4.6.2}$$

See Fig. 4.6.1.

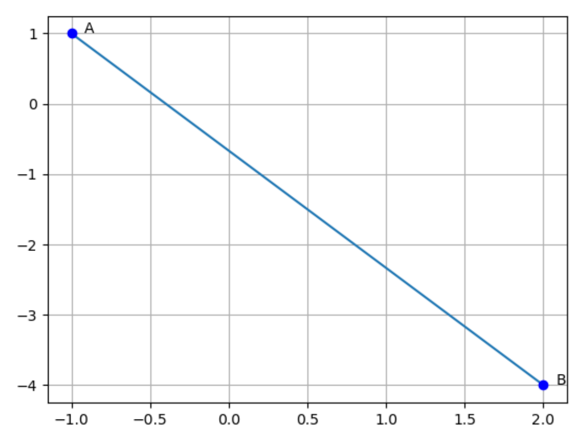

Fig. 4.6.1

4.7 Find the equation of line whose perpendicular distance from the origin is 5 units and the angle made by the

perpendicular with the positive x-axis is 30°.
Solution: From (D.14.3), Thus, the equation of lines are

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \pm 5 \tag{4.7.1}$$

See Fig. 4.7.1.

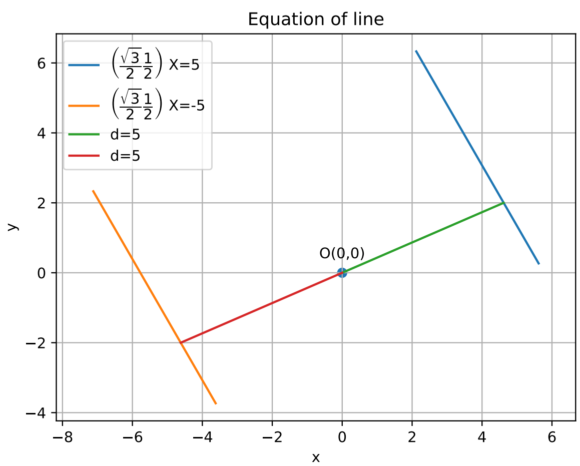

Fig. 4.7.1

4.8 The vertices of triangle PQR are $\mathbf{P}(2,1), \mathbf{Q}(-2,3), \mathbf{R}(4,5)$. Find the equation of the median through $\mathbf{R}$.
Solution: See Fig. 4.8.1. Using section formula, the mid

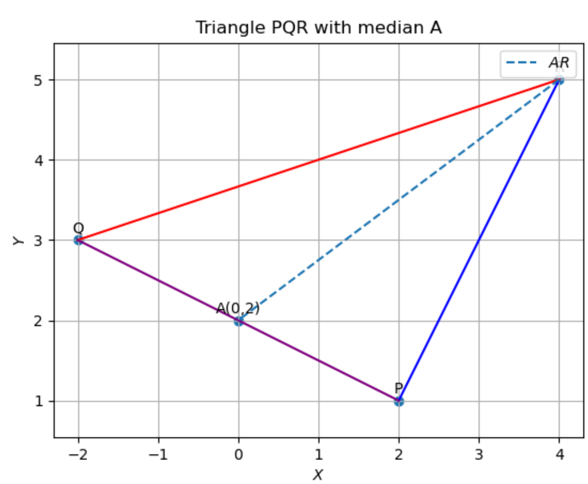

Fig. 4.8.1

point of PQ is

$$\mathbf{A} = \frac{\mathbf{P}+\mathbf{Q}}{2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{4.8.1}$$

So, the direction vector of AR is

$$\mathbf{m} = \mathbf{R} - \mathbf{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{4.8.2}$$

$$\implies \mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{4.8.3}$$

which is the normal vector. Thus, the equation of the line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix} (\mathbf{x} - \mathbf{R}) = 0 \tag{4.8.4}$$

$$\implies \begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 8 \tag{4.8.5}$$

4.9 Find the equation of the line passing through (-3,5) and perpendicular to the line through the points (2,5) and (-3,6).
Solution: The normal vector is

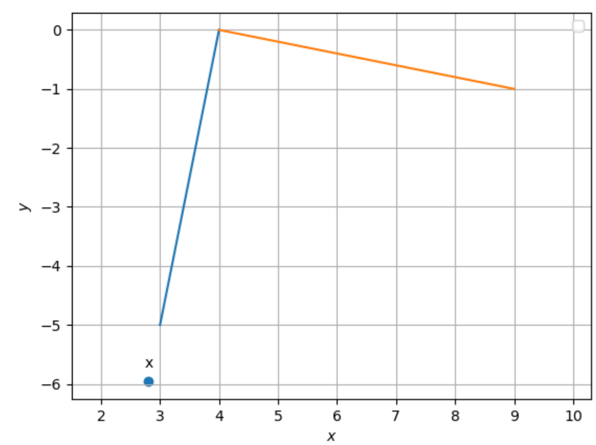

Fig. 4.9.1

$$\mathbf{n} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{4.9.1}$$

Thus, the equation of the line is

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right) = 0 \tag{4.9.2}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = -20 \tag{4.9.3}$$

See Fig. 4.9.1.

4.10 A line perpendicular to the line segment joining the points $\mathbf{P}(1,0)$ and $\mathbf{Q}(2,3)$ divides it in the ratio $1:n$. Find the equation of the line.
Solution: The direction vector of PQ is

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4.10.1}$$

Using section formula,

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1+n} \tag{4.10.2}$$

is the point of intersection. The equation of the desired line is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{R}) = 0 \tag{4.10.3}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2+n}{1+n} \\ \frac{3}{1+n} \end{pmatrix} \tag{4.10.4}$$

$$= \frac{11+n}{1+n} \tag{4.10.5}$$

See Fig. 4.10.1.

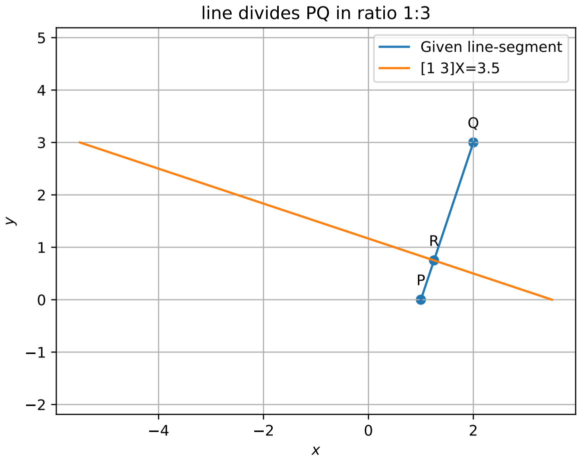

Fig. 4.10.1

4.11 Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point $(2, 3)$.
Solution: Let $(a, 0)$ and $(0, a)$ be the intercept points.

$$\mathbf{m} = \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4.11.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.11.2}$$

and the equation of the line is

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) = 0 \tag{4.11.3}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 5 \tag{4.11.4}$$

See Fig. 4.11.1.

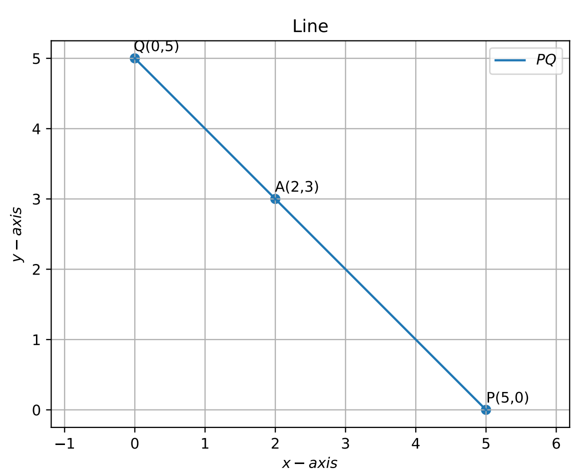

Fig. 4.11.1

4.12 Find the equation of a line passing trough a point (2,2) and cutting off intercepts on the axes whose sum is 9.
Solution: Let the intercept points be

$$\mathbf{P} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ b \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{4.12.1}$$

be the given point. Forming the collinearity matrix,

$$\begin{pmatrix} \mathbf{P} - \mathbf{Q} & \mathbf{P} - \mathbf{R} \end{pmatrix} = \begin{pmatrix} a & a-2 \\ -b & -2 \end{pmatrix} \tag{4.12.2}$$

which is singular if

$$ab - 2(a+b) = 0 \implies ab = 18 \tag{4.12.3}$$

$$\because a + b = 9. \tag{4.12.4}$$

$\therefore a, b$ are the roots of

$$x^2 - 9x + 18 = 0. \tag{4.12.5}$$

yielding

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \tag{4.12.6}$$

Since

$$\mathbf{m} = \begin{pmatrix} a \\ -b \end{pmatrix}, \mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.12.7}$$

Thus, the possible equations of the line are

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \tag{4.12.8}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{4.12.9}$$

See Fig. 4.12.1.

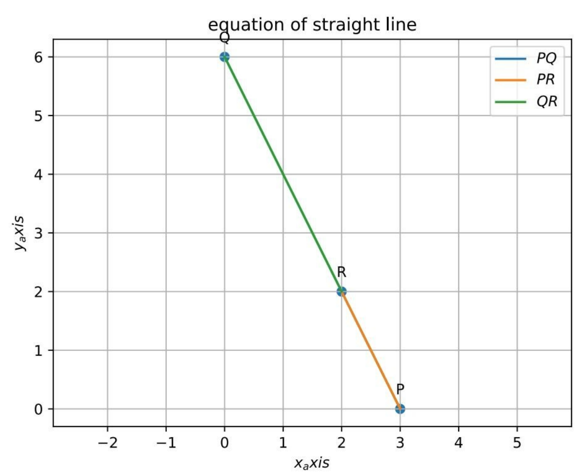

Fig. 4.12.1

4.13 Find the equation of the line through the point (0,2) making an angle $\frac{2\pi}{3}$ with the positive X-axis. Also find the equation of the line parallel to it and crossing the Y-axis at a distance of 2 units below the origin.
Solution: The equation of the first line is

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \tag{4.13.1}$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 2 \tag{4.13.2}$$

The equation of the second line is

$$\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}0\\-2\end{pmatrix}\right)=0 \tag{4.13.3}$$

$$\implies \begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\mathbf{x}=-2 \tag{4.13.4}$$

See Fig. 4.13.1.

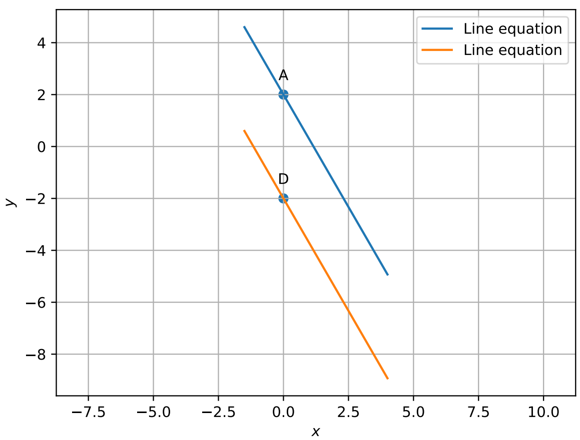

Fig. 4.13.1

4.14 The perpendicular from the origin to a line meets it at the point $(-2, 9)$. Find the equation of the line.
Solution:
The equation of the line is

$$\begin{pmatrix}2 & -9\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}2\\-9\end{pmatrix}\right)=0 \tag{4.14.1}$$

$$\implies \begin{pmatrix}2 & -9\end{pmatrix}\mathbf{x}=85 \tag{4.14.2}$$

See Fig. 4.14.1.

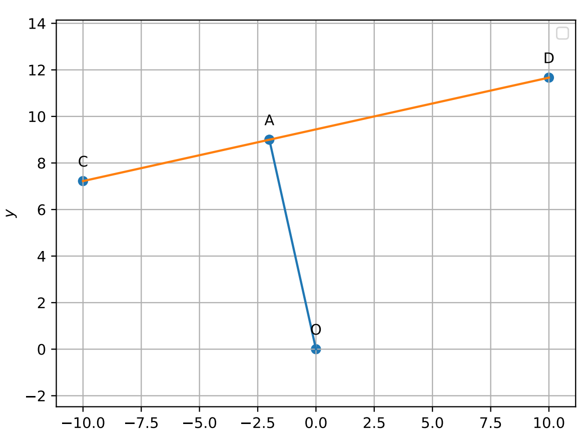

Fig. 4.14.1

4.15 $P(a, b)$ is the mid-point of the line segment between axes. Show that the equation of the line is $\frac{x}{a}+\frac{y}{b}=2$

Solution: From Problem 4.12,

$$\mathbf{n}=\begin{pmatrix}b\\a\end{pmatrix} \tag{4.15.1}$$

$$\implies \begin{pmatrix}b & a\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}a\\b\end{pmatrix}\right)=0 \tag{4.15.2}$$

$$\text{or, } \begin{pmatrix}b & a\end{pmatrix}\mathbf{x}=2ab. \tag{4.15.3}$$

is the desired line equation.

4.16 Point $\mathbf{R}(h, k)$ divides a line segment between the axes in the ratio 1: 2. Find the equation of the line.
Solution: Choosing the intercept points in Problem 4.12,

$$\mathbf{R}=\frac{2\mathbf{A}+\mathbf{B}}{3} \implies \begin{pmatrix}h\\k\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2a\\b\end{pmatrix} \tag{4.16.1}$$

$$\text{or, } \begin{pmatrix}b\\a\end{pmatrix}=\mathbf{n}\equiv\begin{pmatrix}2k\\h\end{pmatrix} \tag{4.16.2}$$

Thus, the equation of the line is given by,

$$\begin{pmatrix}2k & h\end{pmatrix}\mathbf{x}=\begin{pmatrix}2k & h\end{pmatrix}\begin{pmatrix}h\\k\end{pmatrix}=3hk \tag{4.16.3}$$

4.17 Find the equation of the line parallel to the line 3x-4y+2=0 and passing through the point (-2,3).
Solution:

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x}=\begin{pmatrix}3 & -4\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix}=-18 \tag{4.17.1}$$

is the required equation of the line.

4.18 Find the equation of line perpendicular to the line $x-7y+5=0$ and having x intercept 3
Solution: The desired equation is

$$\begin{pmatrix}7 & 1\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}3\\0\end{pmatrix}\right)=0 \tag{4.18.1}$$

$$\implies \begin{pmatrix}7 & 1\end{pmatrix}\mathbf{x}=21 \tag{4.18.2}$$

See Fig. 4.18.1.

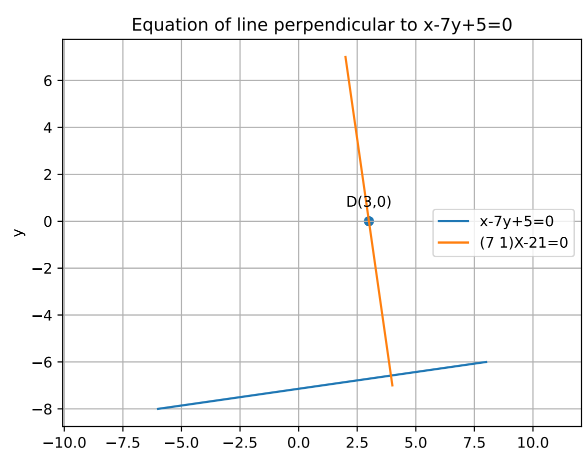

Fig. 4.18.1

4.19 Prove that the line through the point (x_1, y_1) and parallel to the line $Ax+By+C=0$ is $A(x-x_1)+B(y-y_1)=0$.

Solution: The equation of the desired line is

$$\begin{pmatrix} A & B \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = 0 \tag{4.19.1}$$

$$\implies \begin{pmatrix} A & B \end{pmatrix}\mathbf{x} = Ax_1 + By_1 \tag{4.19.2}$$

4.20 Find the equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and} \tag{4.20.1}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \tag{4.20.2}$$

Solution: The direction vector of the desired line is given by

$$\begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix}\mathbf{m} = 0 \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 3 & -16 & 7 \\ 0 & 24 & -12 \end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{2}{3}R_2} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 24 & -12 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2/12} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$

yielding

$$\mathbf{m} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{4.20.3}$$

Hence the vector equation of the line passing through $(1, 2, -4)$ is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + \kappa \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} \tag{4.20.4}$$

4.21 Find the vector equation of the line passing through $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^\top$ and parallel to the planes $\begin{pmatrix} 1 & -1 & 2 \end{pmatrix}\mathbf{x} = 5$ and $\begin{pmatrix} 3 & 1 & 1 \end{pmatrix}\mathbf{x} = 6$.

Solution: The direction vector of the line is given by

$$\begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}\mathbf{m} = 0 \xleftrightarrow{R_2 \to -\frac{3}{4}R_1 + \frac{1}{4}R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{5}{4} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{5}{4} \end{pmatrix} \xleftrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -\frac{5}{4} \end{pmatrix}$$

$$\implies \mathbf{m} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$$

$\therefore$ the equation of the line is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} \tag{4.21.1}$$

4.22 Two lines passing through the point (2,3) intersect each other at an angle of 60°. If slope of one line is 2, find the equation of the other line.

Solution: Using the scalar product

$$\cos 60^\circ = \frac{1}{2} = \frac{\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ m \end{pmatrix}}{\sqrt{5}\sqrt{m^2+1}} \tag{4.22.1}$$

$$\implies 11m^2 + 16m - 1 = 0 \tag{4.22.2}$$

$$or, m = \frac{-8 \pm 5\sqrt{3}}{11} \tag{4.22.3}$$

So, the desired equation of the line is

$$\begin{pmatrix} \frac{-8\pm5\sqrt{3}}{11} & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} \frac{-8\pm5\sqrt{3}}{11} & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{4.22.4}$$

$$= \frac{-49 \pm 16\sqrt{3}}{11} \tag{4.22.5}$$

See Fig. 4.22.1.

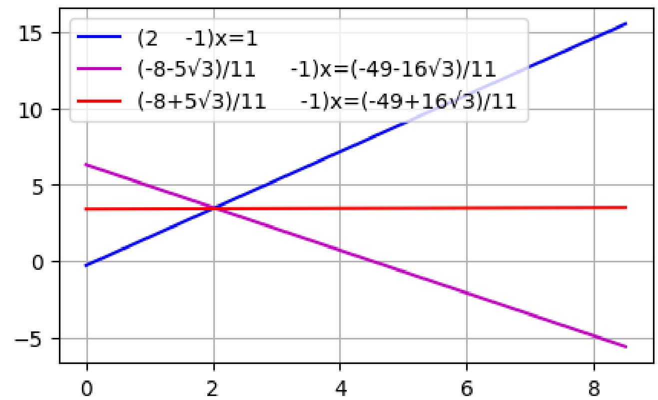

Fig. 4.22.1

4.23

4.24

4.25 The perpendicular from the origin to the line $y = mx + c$ meets it at the point $(-1, 2)$. Find the values of m and c.

Solution:

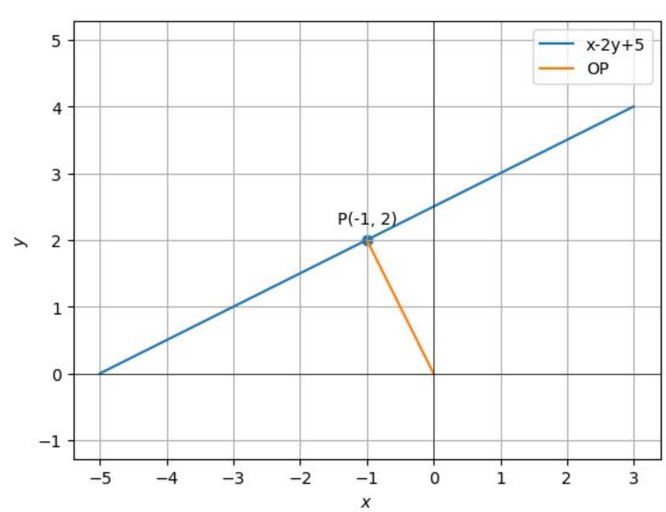

Fig. 4.25.1: Graph

4.26 Find the equation of the lines through the point (3, 2) which make an angle of 45°with the line x–$2y$ = 3.
Solution:

4.27 Consider the following population and year graph, Find the slope of the line AB and using it, find what will be the population in the year 2010?
Solution: The direction vector of the line in Fig. 4.27.1

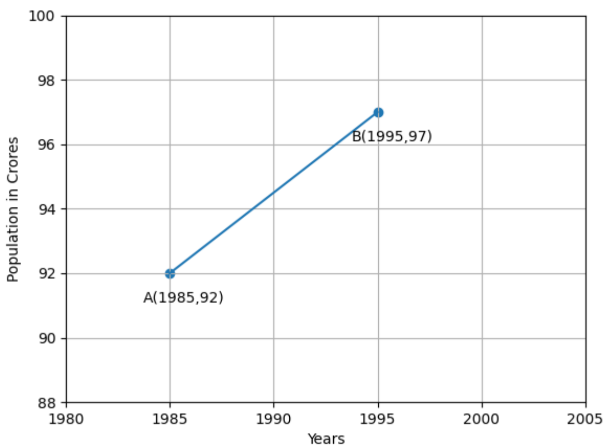

Fig. 4.27.1

is

$$\mathbf{m} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{4.27.1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{4.27.2}$$

The equation of the line is then given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{4.27.3}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = 1801 \tag{4.27.4}$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 2010 \\ y \end{pmatrix} = 1801 \tag{4.27.5}$$

$$\implies y = \frac{209}{2} \tag{4.27.6}$$

5 Perpendicular

6 Plane

7 Miscellaneous

8 Exemplar

9 Singular Value Decomposition

Appendix A
Points on a Line

A.1. The equation of a line is given by

$$y = mx + c \tag{A.1.1}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} + x \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{A.1.2}$$

yielding (1.1.4.1).

A.2. (A.1.1) can also be expressed as

$$y - mx = c \tag{A.2.1}$$

$$\implies \begin{pmatrix} -m & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \tag{A.2.2}$$

yielding (1.1.5.1).

A.3. From (1.1.4.1), if $\mathbf{A}, \mathbf{D}$ and $\mathbf{C}$ are on the same line,

$$\mathbf{D} = \mathbf{A} + q\mathbf{m} \tag{A.3.1}$$

$$\mathbf{C} = \mathbf{D} + p\mathbf{m} \tag{A.3.2}$$

$$\implies p\left(\mathbf{D} - \mathbf{A}\right) + q\left(\mathbf{D} - \mathbf{C}\right) = 0, \quad p, q \neq 0 \tag{A.3.3}$$

$$\implies \mathbf{D} = \frac{p\mathbf{A} + q\mathbf{C}}{p + q} \tag{A.3.4}$$

yielding (1.2.1.1) upon substituting

$$k = \frac{p}{q}. \tag{A.3.5}$$

$\left(\mathbf{D} - \mathbf{A}\right), \left(\mathbf{D} - \mathbf{C}\right)$ are then said to be *linearly dependent*.

A.4. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear, from (1.1.5.1),

$$\mathbf{n}^\top \mathbf{A} = c \tag{A.4.1}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{A.4.2}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{A.4.3}$$

which can be expressed as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{A.4.4}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \begin{pmatrix} \mathbf{n} \\ -c \end{pmatrix} = \mathbf{0} \tag{A.4.5}$$

yielding (1.1.3.1). Rank is defined to be the number of linearly indpendent rows or columns of a matrix.

Appendix B
Tangents to a Circle

The equation of the *incircle* is given by

$$\|\mathbf{x}-\mathbf{O}\|^2 = r^2 \tag{B.1}$$

which can be expressed as (1.6.1) using (1.6.2). In Fig. 1.5.4.1, Let (1.6.8.1) be the equation of AB. Then, the intersection of (1.6.8.1) and (1.6.1) can be expressed as

$$(\mathbf{h}+\mu\mathbf{m})^\top \mathbf{V}(\mathbf{h}+\mu\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h}+\mu\mathbf{m}) + f = 0 \tag{B.2}$$

$$\implies \mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + g(\mathbf{h}) = 0 \tag{B.3}$$

For (B.3) to have exactly one root, the discriminant

$$\left\{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right\}^2 - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{B.4}$$

and (1.6.8.2) is obtained. (B.4) can be expressed as

$$\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\,\mathbf{m} - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{B.5}$$

$$\implies \mathbf{m}^\top\mathbf{\Sigma}\mathbf{m} = 0 \tag{B.6}$$

for $\mathbf{\Sigma}$ defined in (B.6). Substituting (1.6.6.1) in (B.6),

$$\mathbf{m}^\top\mathbf{P}\mathbf{D}\mathbf{P}^\top\mathbf{m} = 0 \tag{B.7}$$

$$\implies \mathbf{v}^\top\mathbf{D}\mathbf{v} = 0 \tag{B.8}$$

where

$$\mathbf{v} = \mathbf{P}^\top\mathbf{m} \tag{B.9}$$

(B.8) can be expressed as

$$\lambda_1 v_1^2 - \lambda_2 v_2^2 = 0 \tag{B.10}$$

$$\implies \mathbf{v} = \begin{pmatrix}\sqrt{|\lambda_2|}\\ \pm\sqrt{|\lambda_1|}\end{pmatrix} \tag{B.11}$$

after some algebra. From (B.11) and (B.9) we obtain (1.6.7.1).

Appendix C
Matrices

Appendix D
2 × 1 vectors

D.1. Mathematically, the projection of $\mathbf{A}$ on $\mathbf{B}$ is defined as

$$\mathbf{C} = k\mathbf{B}, \text{ such that } (\mathbf{A}-\mathbf{C})^\top\mathbf{C} = 0 \tag{D.1.1}$$

yielding

$$(\mathbf{A}-k\mathbf{B})^\top\mathbf{B} = 0 \tag{D.1.2}$$

$$\text{or, } k = \frac{\mathbf{A}^\top\mathbf{B}}{\|\mathbf{B}\|^2} \implies \mathbf{C} = \frac{\mathbf{A}^\top\mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} \tag{D.1.3}$$

D.2. If $\mathbf{A}, \mathbf{B}$ are unit vectors,

$$(\mathbf{A}-\mathbf{B})^\top(\mathbf{A}+\mathbf{B})$$

$$\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 = 0 \tag{D.2.1}$$

D.3. If $ABCD$ be a parallelogram,

$$\mathbf{B}-\mathbf{A} = \mathbf{C}-\mathbf{D} \tag{D.3.1}$$

D.4. If $PQRS$ is formed by joining the mid points of $ABCD$,

$$\mathbf{P} = \frac{1}{2}(\mathbf{A}+\mathbf{B}),\ \mathbf{Q} = \frac{1}{2}(\mathbf{B}+\mathbf{C}) \tag{D.4.1}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{C}+\mathbf{D}),\ \mathbf{S} = \frac{1}{2}(\mathbf{D}+\mathbf{A}) \tag{D.4.2}$$

$$\implies \mathbf{P}-\mathbf{Q} = \mathbf{S}-\mathbf{R}. \tag{D.4.3}$$

Hence, $PQRS$ is a parallelogram from (D.3.1).

D.5. If

$$\mathbf{A}^\top\mathbf{A} = \mathbf{I}, \tag{D.5.1}$$

then $\mathbf{A}$ is an *orthogonal* matrix.

D.6. Let

$$\mathbf{A} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \equiv a_1\vec{i} + a_2\vec{j} + a_3\vec{j}, \tag{D.6.1}$$

$$\mathbf{B} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}, \tag{D.6.2}$$

and

$$\begin{aligned}\mathbf{A}_{ij} &= \begin{pmatrix}a_i\\a_j\end{pmatrix},\\ \mathbf{B}_{ij} &= \begin{pmatrix}b_i\\b_j\end{pmatrix}.\end{aligned} \tag{D.6.3}$$

D.7. The *cross product* or *vector product* of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A}\times\mathbf{B} = \begin{pmatrix}\begin{vmatrix}\mathbf{A}_{23} & \mathbf{B}_{23}\end{vmatrix}\\ \begin{vmatrix}\mathbf{A}_{31} & \mathbf{B}_{31}\end{vmatrix}\\ \begin{vmatrix}\mathbf{A}_{12} & \mathbf{B}_{12}\end{vmatrix}\end{pmatrix} \tag{D.7.1}$$

D.8. Verify that

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{D.8.1}$$

$$\mathbf{A}\times\mathbf{A} = \mathbf{0} \tag{D.8.2}$$

D.9. If

$$\mathbf{A}\times\mathbf{B} = \mathbf{0}, \tag{D.9.1}$$

$\mathbf{A}$ and $\mathbf{B}$ are linearly independent.

D.10.

$$\|\mathbf{A}\times\mathbf{B}\| = \|\mathbf{A}\|\times\|\mathbf{B}\|\sin\theta \tag{D.10.1}$$

where θ is the angle between the vectors.

D.11.

$$ar(ABCD) = \frac{1}{2}((\mathbf{C}-\mathbf{A})\times(\mathbf{D}-\mathbf{B})) \tag{D.11.1}$$

(D.11.2)

D.12. Construct a $\triangle ABC$ given $a, \angle B$ and $K = b + c$.
Solution: Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{D.12.1}$$

$$\implies (K-c)^2 = a^2 + c^2 - 2ac\cos B \tag{D.12.2}$$

$$\implies c = \frac{K^2-a^2}{2(K-a\cos B)} \tag{D.12.3}$$

The coordinates of $\triangle ABC$ can then be expressed as

$$\mathbf{A} = c\begin{pmatrix}\cos B\\ \sin B\end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix}a\\ 0\end{pmatrix}. \tag{D.12.4}$$

D.13. Construct a $\triangle ABC$ given $\angle B, \angle C$ and $K = a + b + c$.
Solution:

$$a + b + c = K \tag{D.13.1}$$
$$b\cos C + c\cos B - a = 0 \tag{D.13.2}$$
$$b\sin C - c\sin B = 0 \tag{D.13.3}$$

resulting in the matrix equation

$$\begin{pmatrix}1 & 1 & 1\\ -1 & \cos C & \cos B\\ 0 & \sin C & -\sin B\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = K\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} \tag{D.13.4}$$

which can be solved to obtain all the sides. $\triangle ABC$ can then be plotted using

$$\mathbf{A} = \begin{pmatrix}a\\ b\end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix}a\\ 0\end{pmatrix} \tag{D.13.5}$$

D.14. Let the perpendicular distance from the origin to a line be p and the angle made by the perpendicular with the positive x-axis be θ. Then

$$p\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \tag{D.14.1}$$

is a point on the line as well as the normal vector. Hence, the equation of the line is

$$p\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\left\{\mathbf{x} - p\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right\} = 0 \tag{D.14.2}$$
$$\implies \begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\mathbf{x} = p \tag{D.14.3}$$