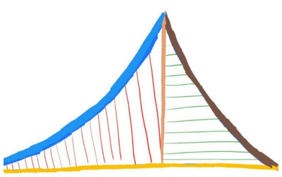
MATRICES
In Geometry



G. V. V. Sharma

Contents

1 Vectors

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1}$$

1.1 Sides

1.1.1. The direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Find the direction vectors of AB, BC and CA.
Solution:

a) The Direction vector of AB is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4-1 \\ 6-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.1.2}$$

b) The Direction vector of BC is

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3-(-4) \\ -5-6 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.1.3}$$

c) The Direction vector of CA is

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1-(-3) \\ -1-(-5) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.1.4}$$

1.1.2. The length of side BC is

$$c = \|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{A})} \tag{1.1.2.1}$$

where

$$\mathbf{A}^\top \triangleq \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{1.1.2.2}$$

Similarly,

$$b = \|\mathbf{C} - \mathbf{B}\|, \, a = \|\mathbf{A} - \mathbf{C}\| \tag{1.1.2.3}$$

Find a, b, c.

a) From (1.1.1.2),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \tag{1.1.2.4}$$

$$\implies c = \|\mathbf{B} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \tag{1.1.2.5}$$

$$= \sqrt{\begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix}} = \sqrt{(5)^2 + (7)^2} \tag{1.1.2.6}$$

$$= \sqrt{74} \tag{1.1.2.7}$$

b) Similarly, from (1.1.1.3),

$$a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix}} \tag{1.1.2.8}$$

$$= \sqrt{(1)^2 + (11)^2} = \sqrt{122} \tag{1.1.2.9}$$

and from (1.1.1.4),

c)

$$b = \|\mathbf{A} - \mathbf{C}\| = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}} \tag{1.1.2.10}$$

$$= \sqrt{(4)^2 + (4)^2} = \sqrt{32} \tag{1.1.2.11}$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{1.1.3.1}$$

Are the given points in (1) collinear?
Solution: From (1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \tag{1.1.3.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{5} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & \frac{-48}{5} \end{pmatrix} \tag{1.1.3.3}$$

There are no zero rows. So,

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{1.1.3.4}$$

Hence, the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear. This is visible in Fig. 1.1.3.

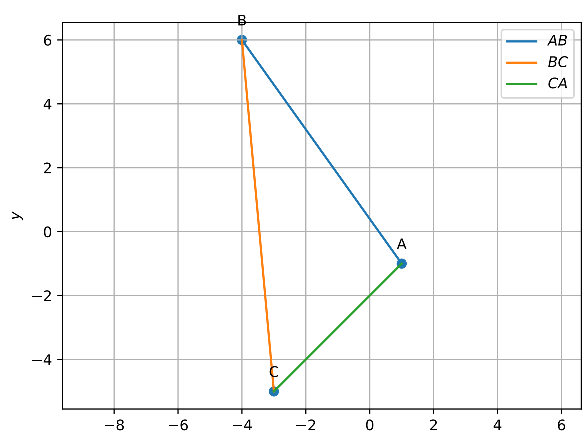


Fig. 1.1.3: $\triangle ABC$

1.1.4. The parameteric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \quad k \neq 0, \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

is the direction vector of AB. Find the parameteric equations of AB, BC and CA.
Solution: From (1.1.4.1) and (1.1.1.2), the parametric

equation for AB is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.4.3}$$

Similarly, from (1.1.1.3) and (1.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.4.4}$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.4.5}$$

1.1.5. The normal form of the equation of AB is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.1}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \tag{1.1.5.2}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.3}$$

Find the normal form of the equations of AB, BC and CA.
Solution:

a) From (1.1.1.3), the direction vector of side **BC** is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.5.4}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} -11 \\ -1 \end{pmatrix} \tag{1.1.5.5}$$

from (1.1.5.3). Hence, from (1.1.5.1), the normal equation of side BC is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.1.5.6}$$

$$\implies \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{1.1.5.7}$$

$$\implies BC : \quad \begin{pmatrix} 11 & 1 \end{pmatrix} \mathbf{x} = -38 \tag{1.1.5.8}$$

b) Similarly, for AB, from (1.1.1.2),

$$\mathbf{m} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.5.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \tag{1.1.5.10}$$

and

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.11}$$

$$\implies AB : \quad \mathbf{n}^\top \mathbf{x} = \begin{pmatrix} 7 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.12}$$

$$\implies \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} = 2 \tag{1.1.5.13}$$

c) For CA, from (1.1.1.4),

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.5.14}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.15}$$

$$\tag{1.1.5.16}$$

$$\implies \mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.1.5.17}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = 2 \tag{1.1.5.18}$$

1.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| \tag{1.1.6.1}$$

where

$$\mathbf{A} \times \mathbf{B} \triangleq \begin{vmatrix} 1 & -4 \\ -1 & 6 \end{vmatrix} \tag{1.1.6.2}$$

Find the area of $\triangle ABC$.
Solution: From (1.1.1.2) and (1.1.1.4),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.6.3}$$

$$\implies (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix} 5 & 4 \\ -7 & 4 \end{vmatrix} \tag{1.1.6.4}$$

$$= 5 \times 4 - 4 \times (-7) \tag{1.1.6.5}$$

$$= 48 \tag{1.1.6.6}$$

$$\implies \frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| = \frac{48}{2} = 24 \tag{1.1.6.7}$$

which is the desired area.

1.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top \mathbf{C} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{A}\|} \tag{1.1.7.1}$$

Solution:

a) From (1.1.1.2), (1.1.1.4), (1.1.2.7) and (1.1.2.11)

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & 7 \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{1.1.7.2}$$

$$= -8 \tag{1.1.7.3}$$

$$\implies \cos A = \frac{-8}{\sqrt{74}\sqrt{32}} = \frac{-1}{\sqrt{37}} \tag{1.1.7.4}$$

$$\implies A = \cos^{-1} \frac{-1}{\sqrt{37}} \tag{1.1.7.5}$$

b) From (1.1.1.2), (1.1.1.3), (1.1.2.7) and (1.1.2.9)

$$(\mathbf{C} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 & -11 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix} \tag{1.1.7.6}$$

$$= 82 \tag{1.1.7.7}$$

$$\implies \cos B = \frac{82}{\sqrt{74}\sqrt{122}} = \frac{41}{\sqrt{2257}} \tag{1.1.7.8}$$

$$\implies B = \cos^{-1} \frac{41}{\sqrt{2257}} \tag{1.1.7.9}$$

c) From (1.1.1.3), (1.1.1.4), (1.1.2.9) and (1.1.2.11)

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C}) = \begin{pmatrix}4 & 4\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} \tag{1.1.7.10}$$

$$= 40 \tag{1.1.7.11}$$

$$\implies \cos C = \frac{40}{\sqrt{32}\sqrt{122}} = \frac{5}{\sqrt{61}} \tag{1.1.7.12}$$

$$\implies C = \cos^{-1}\frac{5}{\sqrt{61}} \tag{1.1.7.13}$$

All codes for this section are available at

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codes/triangle/sides.py
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1.2 Median

1.2.1. If $\mathbf{D}$ divides BC in the ratio $k:1$,

$$\mathbf{D} = \frac{k\mathbf{C}+\mathbf{B}}{k+1} \tag{1.2.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.

Solution: Since $\mathbf{D}$ is the midpoint of BC,

$$k = 1, \tag{1.2.1.2}$$

$$\implies \mathbf{D} = \frac{\mathbf{C}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix} \tag{1.2.1.3}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix}-1\\-3\end{pmatrix} \tag{1.2.1.4}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix} \tag{1.2.1.5}$$

1.2.2. Find the equations of AD, BE and CF.

Solution: :

a) The direction vector of AD is

$$\mathbf{m} = \mathbf{D}-\mathbf{A} = \begin{pmatrix}\frac{-7}{2}\\\frac{1}{2}\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-9\\3\end{pmatrix} \equiv \begin{pmatrix}-3\\1\end{pmatrix} \tag{1.2.2.1}$$

$$\implies \mathbf{n} = \begin{pmatrix}1\\3\end{pmatrix} \tag{1.2.2.2}$$

Hence the normal equation of median AD is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{A}\right) = 0 \tag{1.2.2.3}$$

$$\implies \begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = -2 \tag{1.2.2.4}$$

b) For BE,

$$\mathbf{m} = \mathbf{E}-\mathbf{B} = \begin{pmatrix}-1\\-3\end{pmatrix} - \begin{pmatrix}-4\\6\end{pmatrix} = \begin{pmatrix}3\\-9\end{pmatrix} \equiv \begin{pmatrix}1\\-3\end{pmatrix} \tag{1.2.2.5}$$

$$\implies \mathbf{n} = \begin{pmatrix}3\\1\end{pmatrix} \tag{1.2.2.6}$$

Hence the normal equation of median BE is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{B}\right) = 0 \tag{1.2.2.7}$$

$$\implies \begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}3 & 1\end{pmatrix}\begin{pmatrix}-4\\6\end{pmatrix} = -6 \tag{1.2.2.8}$$

c) For median CF,

$$\mathbf{m} = \mathbf{F}-\mathbf{C} = \begin{pmatrix}\frac{-3}{2}\\\frac{5}{2}\end{pmatrix} - \begin{pmatrix}-3\\-5\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\\frac{15}{2}\end{pmatrix} \equiv \begin{pmatrix}1\\5\end{pmatrix} \tag{1.2.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix}5\\-1\end{pmatrix} \tag{1.2.2.10}$$

Hence the normal equation of median CF is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{C}\right) = 0 \tag{1.2.2.11}$$

$$\implies \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}-3\\-5\end{pmatrix} = -10 \tag{1.2.2.12}$$

1.2.3. Find the intersection $\mathbf{G}$ of BE and CF.

Solution: From (1.2.2.8) and (1.2.2.12), the equations of

BE and CF are, respectively,

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \end{pmatrix} \tag{1.2.3.1}$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -10 \end{pmatrix} \tag{1.2.3.2}$$

From (1.2.3.1) and (1.2.3.2) the augmented matrix is

$$\begin{pmatrix} 3 & 1 & -6 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 8 & 0 & -16 \\ 5 & -1 & -10 \end{pmatrix} \tag{1.2.3.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix} 1 & 0 & -2 \\ 5 & -1 & -10 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \tag{1.2.3.4}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.2.3.5}$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{1.2.3.6}$$

1.2.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.1}$$

Solution:

a) From (1.2.1.4) and (1.2.3.6),

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \mathbf{E} - \mathbf{G} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.2.4.2}$$

$$\implies \mathbf{G} - \mathbf{B} = 2\left(\mathbf{E} - \mathbf{G}\right) \tag{1.2.4.3}$$

$$\implies \left\|\mathbf{G} - \mathbf{B}\right\| = 2\left\|\mathbf{E} - \mathbf{G}\right\| \tag{1.2.4.4}$$

$$\text{or, } \frac{BG}{GE} = 2 \tag{1.2.4.5}$$

b) From (1.2.1.5) and (1.2.3.6),

$$\mathbf{F} - \mathbf{G} = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \mathbf{G} - \mathbf{C} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{1.2.4.6}$$

$$\implies \mathbf{G} - \mathbf{C} = 2\left(\mathbf{F} - \mathbf{G}\right) \tag{1.2.4.7}$$

$$\implies \left\|\mathbf{G} - \mathbf{C}\right\| = 2\left\|\mathbf{F} - \mathbf{G}\right\| \tag{1.2.4.8}$$

$$\text{or, } \frac{CG}{GF} = 2 \tag{1.2.4.9}$$

c) From (1.2.1.3) and (1.2.3.6),

$$\mathbf{G} - \mathbf{A} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \mathbf{D} - \mathbf{G} = \frac{1}{2}\begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.2.4.10}$$

$$\mathbf{G} - \mathbf{A} = 2\left(\mathbf{D} - \mathbf{G}\right) \tag{1.2.4.11}$$

$$\implies \left\|\mathbf{G} - \mathbf{A}\right\| = 2\left\|\mathbf{D} - \mathbf{G}\right\| \tag{1.2.4.12}$$

$$\text{or, } \frac{AG}{GD} = 2 \tag{1.2.4.13}$$

From (1.2.4.5), (1.2.4.9), (1.2.4.13)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.14}$$

1.2.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear.

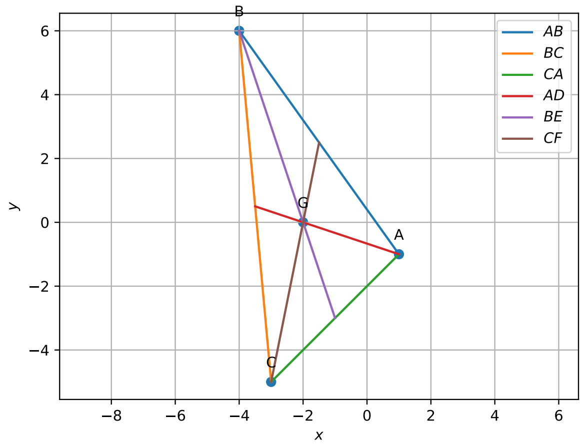

Fig. 1.2.5: Medians of $\triangle ABC$ meet at $\mathbf{G}$.

Solution: Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \tag{1.2.5.1}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ 0 & -3 & -2 \end{pmatrix} \tag{1.2.5.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & -3 & -2 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{3}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.5.3}$$

Thus, the matrix (1.2.5.1) has rank 2 and the points are collinear. Thus, the medians of a triangle meet at the point $\mathbf{G}$. See Fig. 1.2.5.

1.2.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.2.6.1}$$

$\mathbf{G}$ is known as the *centroid* of $\triangle ABC$.

Solution:

$$\mathbf{G} = \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \end{pmatrix}}{3} \tag{1.2.6.2}$$

$$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

1.2.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

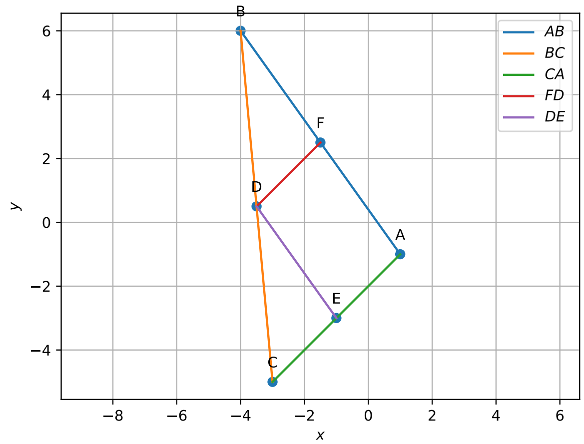

Fig. 1.2.7: $AFDE$ forms a parallelogram in triangle ABC

Solution:

$$\mathbf{A}-\mathbf{F} = \begin{pmatrix}1\\-1\end{pmatrix} - \begin{pmatrix}\frac{-3}{2}\\\frac{5}{2}\end{pmatrix} = \begin{pmatrix}\frac{5}{2}\\\frac{-7}{2}\end{pmatrix} \tag{1.2.7.2}$$

$$\mathbf{E}-\mathbf{D} = \begin{pmatrix}-1\\-3\end{pmatrix} - \begin{pmatrix}\frac{-7}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{5}{2}\\\frac{-7}{2}\end{pmatrix} \tag{1.2.7.3}$$

$$\implies \mathbf{A}-\mathbf{F} = \mathbf{E}-\mathbf{D} \tag{1.2.7.4}$$

See Fig. 1.2.7,
All codes for this section are available in

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codes/triangle/medians.py
codes/triangle/pgm.py
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1.3 Altitude

1.3.1. $\mathbf{D}_1$ is a point on BC such that

$$AD_1 \perp BC \tag{1.3.1.1}$$

and AD_1 is defined to be the altitude. Find the normal vector of AD_1.
Solution: The normal vector of AD_1 is the direction vector BC and is obtained from (1.1.1.3) as

$$\mathbf{n} = \begin{pmatrix}1\\-11\end{pmatrix} \tag{1.3.1.2}$$

1.3.2. Find the equation of AD_1.
Solution: The equation of AD_1 is

$$\mathbf{n}^\top(\mathbf{x}-\mathbf{A}) = 0 \tag{1.3.2.1}$$

$$\implies \begin{pmatrix}-1 & 11\end{pmatrix}\mathbf{x} = \begin{pmatrix}-1 & 11\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = -12 \tag{1.3.2.2}$$

1.3.3. Find the equations of the altitudes BE_1 and CF_1 to the sides AC and AB respectively.

Solution:

a) From (1.1.1.4), the normal vector of CF_1 is

$$\mathbf{n} = \begin{pmatrix}-5\\7\end{pmatrix} \tag{1.3.3.1}$$

and the equation of CF_1 is

$$\mathbf{n}^\top(\mathbf{x}-\mathbf{C}) = 0 \tag{1.3.3.2}$$

$$\implies \begin{pmatrix}-5 & 7\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}-3\\-5\end{pmatrix}\right) = 0 \tag{1.3.3.3}$$

$$\implies \begin{pmatrix}5 & -7\end{pmatrix}\mathbf{x} = 20, \tag{1.3.3.4}$$

b) Similarly, from (1.1.1.2), the normal vector of BE_1 is

$$\mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix} \tag{1.3.3.5}$$

and the equation of BE_1 is

$$\mathbf{n}^\top(\mathbf{x}-\mathbf{B}) = 0 \tag{1.3.3.6}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}-4\\6\end{pmatrix}\right) = 0 \tag{1.3.3.7}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 2, \tag{1.3.3.8}$$

1.3.4. Find the intersection $\mathbf{H}$ of BE_1 and CF_1.

Solution: The intersection of (1.3.3.8) and (1.3.3.4), is obtained from the matrix equation

$$\begin{pmatrix}1 & 1\\5 & -7\end{pmatrix}\mathbf{x} = \begin{pmatrix}2\\20\end{pmatrix} \tag{1.3.4.1}$$

which can be solved as

$$\begin{pmatrix}1 & 1 & 2\\5 & -7 & 20\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2-5R_1} \begin{pmatrix}1 & 1 & 2\\0 & -12 & 10\end{pmatrix} \tag{1.3.4.2}$$

$$\xleftrightarrow{R_2\leftarrow \frac{R_2}{-12}} \begin{pmatrix}1 & 1 & 2\\0 & 1 & \frac{-5}{6}\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-R_2} \begin{pmatrix}1 & 0 & \frac{17}{6}\\0 & 1 & \frac{-5}{6}\end{pmatrix} \tag{1.3.4.3}$$

yielding

$$\mathbf{H} = \frac{1}{6}\begin{pmatrix}17\\-5\end{pmatrix}, \tag{1.3.4.4}$$

See Fig. 1.3.4.1

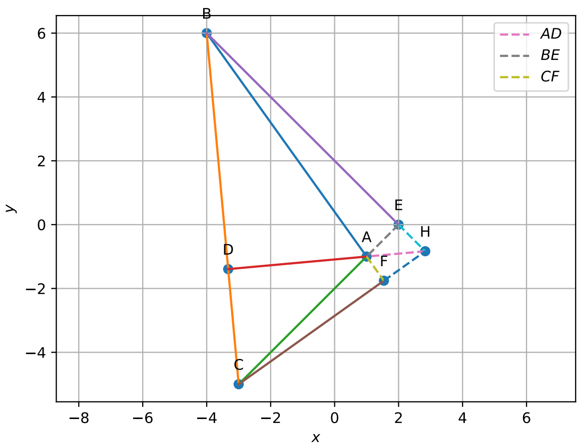

Fig. 1.3.4.1: Altitudes BE_1 and CF_1 intersect at $\mathbf{H}$

1.3.5. Verify that

$$(\mathbf{A}-\mathbf{H})^\top(\mathbf{B}-\mathbf{C}) = 0 \tag{1.3.5.1}$$

Solution: From (1.3.4.4),

$$\mathbf{A}-\mathbf{H} = -\frac{1}{6}\begin{pmatrix}11\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{1.3.5.2}$$

$$\implies (\mathbf{A}-\mathbf{H})^\top(\mathbf{B}-\mathbf{C}) = \frac{1}{6}\begin{pmatrix}11 & 1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} = 0 \tag{1.3.5.3}$$

All codes for this section are available at

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codes/triangle/altitude.py
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1.4 Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x}-\frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{1.4.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.
Solution: From (1.1.1.2), (1.1.1.3), (1.1.1.4), (1.2.1.3), (1.2.1.4) and (1.2.1.5),

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}-1\\11\end{pmatrix} \tag{1.4.1.2}$$

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix}5\\-7\end{pmatrix} \tag{1.4.1.3}$$

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \begin{pmatrix}-1\\-3\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix} \tag{1.4.1.4}$$

$$\tag{1.4.1.5}$$

yielding

$$(\mathbf{B}-\mathbf{C})^\top\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix}-1 & 11\end{pmatrix}\begin{pmatrix}-\frac{7}{2}\\\frac{1}{2}\end{pmatrix} = 9 \tag{1.4.1.6}$$

$$(\mathbf{A}-\mathbf{B})^\top\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix}5 & -7\end{pmatrix}\begin{pmatrix}-\frac{3}{2}\\\frac{5}{2}\end{pmatrix} = -25 \tag{1.4.1.7}$$

$$(\mathbf{C}-\mathbf{A})^\top\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = \begin{pmatrix}-4 & -4\end{pmatrix}\begin{pmatrix}-1\\-3\end{pmatrix} = 16 \tag{1.4.1.8}$$

Thus, the perpendicular bisectors are obtained from (1.4.1.1) as

$$BC: \quad \begin{pmatrix}-1 & 11\end{pmatrix}\mathbf{x} = 9 \tag{1.4.1.9}$$

$$CA: \quad \begin{pmatrix}5 & -7\end{pmatrix}\mathbf{x} = -25 \tag{1.4.1.10}$$

$$AB: \quad \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = -4 \tag{1.4.1.11}$$

1.4.2. Find the intersection $\mathbf{O}$ of the perpendicular bisectors of AB and AC.
Solution:
The intersection of (1.4.1.10) and (1.4.1.11), can be obtained as

$$\begin{pmatrix}5 & -7 & -25\\1 & 1 & -4\end{pmatrix} \xleftrightarrow{R_2\leftarrow 5R_2-R_1} \begin{pmatrix}5 & -7 & -25\\0 & 12 & 5\end{pmatrix} \tag{1.4.2.1}$$

$$\xleftrightarrow{R_1\leftarrow \frac{12}{7}R_1+R_2} \begin{pmatrix}\frac{60}{7} & 0 & \frac{-265}{7}\\0 & 12 & 5\end{pmatrix} \xleftrightarrow[R_1\leftarrow \frac{7}{60}R_1]{R_2\leftarrow \frac{1}{12}R_2} \begin{pmatrix}1 & 0 & \frac{-53}{12}\\0 & 1 & \frac{5}{12}\end{pmatrix} \tag{1.4.2.2}$$

$$\implies \mathbf{O} = \begin{pmatrix}\frac{-53}{12}\\\frac{5}{12}\end{pmatrix} \tag{1.4.2.3}$$

1.4.3. Verify that $\mathbf{O}$ satisfies (1.4.1.1). $\mathbf{O}$ is known as the circumcentre.
Solution: Substituing from (1.4.2.3) in (1.4.1.1), when

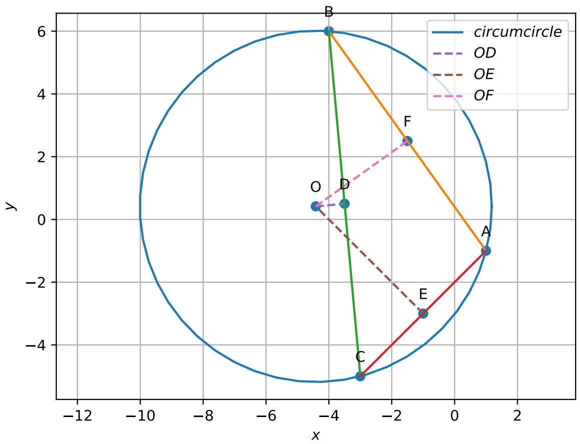

Fig. 1.4.5.1: Circumcircle of $\triangle ABC$ with centre $\mathbf{O}$.

substituted in the above equation,

$$\left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)^{\top}(\mathbf{B}-\mathbf{C})$$
$$= \left(\frac{1}{12}\begin{pmatrix}-53\\5\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix}\right)^{\top}\begin{pmatrix}-1\\11\end{pmatrix}$$
$$= \frac{1}{12}\begin{pmatrix}-11 & -1\end{pmatrix}\begin{pmatrix}-1\\11\end{pmatrix} = 0 \quad (1.4.3.1)$$

1.4.4. Verify that

$$OA = OB = OC \quad (1.4.4.1)$$

1.4.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$R = OA \quad (1.4.5.1)$$

This is known as the *circumradius*.
Solution: See Fig. 1.4.5.1.

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \quad (1.4.6.1)$$

Solution:

a) To find the value of $\angle BOC$:

$$\mathbf{B}-\mathbf{O} = \begin{pmatrix}\frac{5}{12}\\\frac{67}{12}\end{pmatrix}, \mathbf{C}-\mathbf{O} = \begin{pmatrix}\frac{17}{12}\\\frac{-65}{12}\end{pmatrix} \quad (1.4.6.2)$$

$$\implies (\mathbf{B}-\mathbf{O})^{\top}(\mathbf{C}-\mathbf{O}) = \frac{-4270}{144} \quad (1.4.6.3)$$

$$\implies \|\mathbf{B}-\mathbf{O}\| = \frac{\sqrt{4514}}{12}, \|\mathbf{C}-\mathbf{O}\| = \frac{\sqrt{4514}}{12} \quad (1.4.6.4)$$

Thus,

$$\cos BOC = \frac{(\mathbf{B}-\mathbf{O})^{\top}(\mathbf{C}-\mathbf{O})}{\|\mathbf{B}-\mathbf{O}\|\,\|\mathbf{C}-\mathbf{O}\|} = \frac{-4270}{4514} \quad (1.4.6.5)$$

$$\implies \angle BOC = \cos^{-1}\left(\frac{-4270}{4514}\right) \quad (1.4.6.6)$$

$$= 161.07536^\circ \text{ or } 198.92464^\circ \quad (1.4.6.7)$$

b) To find the value of $\angle BAC$:

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix}-5\\7\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix} \quad (1.4.6.8)$$

$$\implies (\mathbf{B}-\mathbf{A})^{\top}(\mathbf{C}-\mathbf{A}) = -8 \quad (1.4.6.9)$$

$$\|\mathbf{B}-\mathbf{A}\| = \sqrt{74}\,\|\mathbf{C}-\mathbf{A}\| = 4\sqrt{2} \quad (1.4.6.10)$$

Thus,

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^{\top}(\mathbf{C}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{A}\|} = \frac{-8}{4\sqrt{148}} \quad (1.4.6.11)$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{-8}{4\sqrt{148}}\right) \quad (1.4.6.12)$$

$$= 99.46232^\circ \quad (1.4.6.13)$$

From (1.4.6.13) and (1.4.6.7),

$$2 \times \angle BAC = \angle BOC \quad (1.4.6.14)$$

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \quad (1.4.7.1)$$

where

$$\theta = \angle BOC \quad (1.4.7.2)$$

Verify that

$$\mathbf{B}-\mathbf{O} = \mathbf{P}(\mathbf{C}-\mathbf{O}) \quad (1.4.7.3)$$

All codes for this section are available at

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codes/triangle/perp−bisect.py
```

1.5 Angle Bisector

1.5.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \tag{1.5.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Problem 1.1.2.
Solution: From the given information,

$$a = m + n, \tag{1.5.1.2}$$
$$b = n + p, \tag{1.5.1.3}$$
$$c = m + p \tag{1.5.1.4}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.5}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.6}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \tag{1.5.1.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.5.1.8}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.9}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.10}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.5.1.11}$$

Therefore,

$$\begin{aligned} p &= \frac{c + b - a}{2} = \frac{\sqrt{74} + \sqrt{32} - \sqrt{122}}{2} \\ m &= \frac{a + c - b}{2} = \frac{\sqrt{74} + \sqrt{122} - \sqrt{32}}{2} \\ n &= \frac{a + b - c}{2} = \frac{\sqrt{122} + \sqrt{32} - \sqrt{74}}{2} \end{aligned} \tag{1.5.1.12}$$

upon substituting from (1.1.2.7), (1.1.2.9) and (1.1.2.11).

1.5.2. Using section formula, find

$$\mathbf{D}_3 = \frac{m\mathbf{C} + n\mathbf{B}}{m + n}, \mathbf{E}_3 = \frac{n\mathbf{A} + p\mathbf{C}}{n + p}, \mathbf{F}_3 = \frac{p\mathbf{B} + m\mathbf{A}}{p + m} \tag{1.5.2.1}$$

1.5.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the *incentre* and *inradius* of $\triangle ABC$.

1.5.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the *incircle* of $\triangle ABC$.
Solution: See Fig. 1.5.4.1

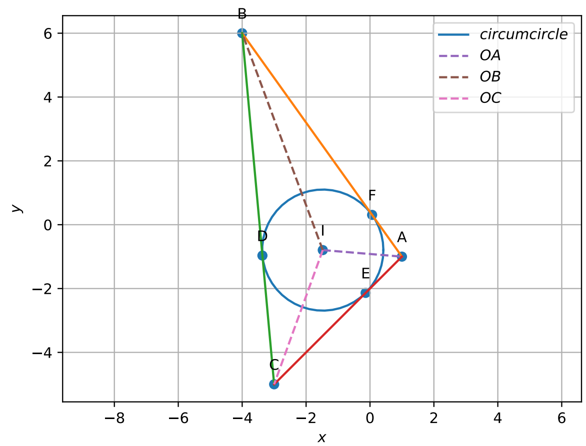

Fig. 1.5.4.1: Incircle of $\triangle ABC$

1.5.5. Using (1.1.7.1) verify that

$$\angle BAI = \angle CAI. \tag{1.5.5.1}$$

AI is the bisector of $\angle A$.

1.5.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.
All codes for this section are available at

codes/triangle/ang−bisect.py

1.6 Eigenvalues and Eigenvectors

The equation of the incircle is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1.6.1}$$

where

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\| - r^2, \tag{1.6.2}$$

$\mathbf{O}$ being the incentre and r the inradius. Here $\mathbf{I}$ is the identity matrix.

1.6.1. Compute

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\mathbf{V} \tag{1.6.1.1}$$

for $\mathbf{h} = \mathbf{A}$.

1.6.2. Find the roots of the equation

$$\left|\lambda\mathbf{I} - \mathbf{\Sigma}\right| = 0 \tag{1.6.2.1}$$

These are known as the eigenvalues of $\mathbf{\Sigma}$.

1.6.3. Find $\mathbf{p}$ such that

$$\mathbf{\Sigma}\mathbf{p} = \lambda\mathbf{p} \tag{1.6.3.1}$$

using row reduction. These are known as the eigenvectors of Σ.

1.6.4. Define

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{1.6.4.1}$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} \end{pmatrix} \tag{1.6.4.2}$$

1.6.5. Verify that

$$\mathbf{P}^\top = \mathbf{P}^{-1}. \tag{1.6.5.1}$$

$\mathbf{P}$ is defined to be an orthogonal matrix.

1.6.6. Verify that

$$\mathbf{P}^\top \mathbf{\Sigma}\mathbf{P} = \mathbf{D}, \tag{1.6.6.1}$$

This is known as the spectral (eigenvalue) decomposition of a symmetric matrix

1.6.7. The direction vectors of the tangents from a point $\mathbf{h}$ to the circle in (1.6.1) are given by

$$\mathbf{m} = \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{1.6.7.1}$$

1.6.8. The points of contact of the pair of tangents to the circle in (1.6.1) from a point $\mathbf{h}$ are given by

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{1.6.8.1}$$

where

$$\mu = -\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{1.6.8.2}$$

for $\mathbf{m}$ in (1.6.7.1). Compute the points of contact. You should get the same points that you obtained in the previous section.

All codes for this section are available at

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codes/triangle/tangpair.py
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2 Matrices

The matrix of the veritices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{2.1}$$

2.1 Vectors

2.1. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{2.1.1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{2.1.1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{2.1.1.3}$$

where the second matrix above is known as a *circulant* matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

2.2. Obtain the normal matrix of the sides of $\triangle ABC$

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.1.2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{2.1.2.2}$$

2.3. Obtain a, b, c.

Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \tag{2.1.3.1}$$

2.4. Obtain the constant terms in the equations of the sides of the triangle.

Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag}\left\{\left(\mathbf{N}^\top \mathbf{P}\right)\right\} \tag{2.1.4.1}$$

2.2 Median

2.2.1. Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.2.1.1}$$

2.2.2. Obtain the median direction matrix.

Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F} \end{pmatrix} \tag{2.2.2.1}$$

$$= \begin{pmatrix} \mathbf{A} - \frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B} - \frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C} - \frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \tag{2.2.2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{2.2.2.3}$$

2.2.3. Obtain the median normal matrix.

2.2.4. Obtian the median equation constants.

2.2.5. Obtain the centroid by finding the intersection of the medians.

2.3 Altitude

2.3.1. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix}\mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B}\end{pmatrix} \tag{2.3.1.1}$$

$$= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix} \tag{2.3.1.2}$$

2.3.2. Find the constants vector for the altitudes.

Solution: The desired vector is

$$\mathbf{c}_2 = \text{diag}\left\{\left(\mathbf{M}^\top\mathbf{P}\right)\right\} \tag{2.3.2.1}$$

2.4 Perpendicular Bisector

2.4.1. Find the normal matrix for the perpendicular bisectors

Solution: The normal matrix is $\mathbf{M}_2$

2.4.2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix}\right\} \tag{2.4.2.1}$$

2.5 Angle Bisector

2.5.1. Find the points of contact.

Solution: The points of contact are given by

$$\begin{pmatrix}\frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & \frac{n}{b} & \frac{m}{c}\\ \frac{n}{a} & 0 & \frac{p}{c}\\ \frac{m}{a} & \frac{p}{b} & 0\end{pmatrix} \tag{2.5.1.1}$$

All codes for this section are available at

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codes/triangle/mat−alg.py
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3 Length

3.1 Compute the magnitude of the following vectors:

$$\mathbf{a} = \hat{i} + \hat{j} + \hat{k} \tag{3.1.1}$$

$$\mathbf{b} = 2\hat{i} - 7\hat{j} - 3\hat{k} \tag{3.1.2}$$

$$\mathbf{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{3}\hat{k} \tag{3.1.3}$$

Solution: Let

$$\mathbf{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2\\-7\\3\end{pmatrix}, \mathbf{c} = \begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\end{pmatrix} \tag{3.1.4}$$

Then

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^\top\mathbf{a}} = \sqrt{3}, \tag{3.1.5}$$

$$\|\mathbf{b}\| = \sqrt{\mathbf{b}^\top\mathbf{b}} = \sqrt{62}, \tag{3.1.6}$$

$$\|\mathbf{c}\| = \sqrt{\mathbf{c}^\top\mathbf{c}} = 1 \tag{3.1.7}$$

3.2 Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Solution:

$$\because \mathbf{x} = x\begin{pmatrix}1\\1\\1\end{pmatrix}, \|\mathbf{x}\| = 1 \implies x\sqrt{3} = 1 \tag{3.2.1}$$

$$\text{or, } x = \frac{1}{\sqrt{3}} \tag{3.2.2}$$

3.3 If $\mathbf{a} = \mathbf{b} + \mathbf{c}$, then is it true that $|\mathbf{a}| = |\mathbf{b}| + |\mathbf{c}|$? Justify your answer.

Solution: Let

$$\mathbf{b} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \mathbf{c} = \begin{pmatrix}2\\-1\\-2\end{pmatrix} \tag{3.3.1}$$

Then

$$\mathbf{a} = \mathbf{b} + \mathbf{c} = \begin{pmatrix}3\\1\\1\end{pmatrix} \tag{3.3.2}$$

$$\implies \|\mathbf{a}\| = \sqrt{11}, \|\mathbf{b}\| = \sqrt{14}, \|\mathbf{c}\| = 3. \tag{3.3.3}$$

Thus

$$\|\mathbf{a}\| \neq \|\mathbf{b}\| + \|\mathbf{c}\| \tag{3.3.4}$$

3.4 If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then $\lambda\vec{a}$ is a unit vector if

a) $\lambda = 1$
b) $\lambda = -1$
c) $a = |\lambda|$
d) $a = 1/|\lambda|$

4 Direction

4.1 For given vectors, $\mathbf{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\mathbf{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\mathbf{a} + \mathbf{b}$.

Solution:

$$\because \mathbf{a}+\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \tag{4.1.1}$$

$$\|\mathbf{a}+\mathbf{b}\| = \sqrt{2} \tag{4.1.2}$$

$$\implies \frac{\mathbf{a}+\mathbf{b}}{\|\mathbf{a}+\mathbf{b}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{4.1.3}$$

which is the desired the unit vector.

4.2 Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.
Solution: Let the required vector be

$$c\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}. \tag{4.2.1}$$

From the given information,

$$\left\| c\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right\| = 8 \tag{4.2.2}$$

$$\implies |c| = \frac{4\sqrt{30}}{15} \tag{4.2.3}$$

4.3 Find the unit vector in the direction of the vector $\mathbf{a} = \hat{i} + \hat{j} + 2\hat{k}$.

4.4 Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where **P** and **Q** are the points (1, 2, 3) and (4, 5, 6), respectively.

4.5 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + \hat{k}$.
Solution:

$$\because \mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{4.5.1}$$

$$\mathbf{a}+\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \implies \|\mathbf{a}+\mathbf{b}\| = \sqrt{10} \tag{4.5.2}$$

From problem 4.1, the unit vector in the direction of $\mathbf{a}+\mathbf{b}$ is

$$\frac{\mathbf{a}+\mathbf{b}}{\|\mathbf{a}+\mathbf{b}\|} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{4.5.3}$$

The desired vector can then be expressed as

$$\pm\frac{5}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \tag{4.5.4}$$

4.6 Find the direction cosines of the vector joining the points **A** (1, 2, –3) and **B**(–1, –2, 1), directed from **A** to **B**.
Solution: The unit vector in the direction of AB is

$$\frac{\mathbf{B}-\mathbf{A}}{\|\mathbf{B}-\mathbf{A}\|} = \frac{1}{3}\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} \tag{4.6.1}$$

and the direction cosines are the elements of the above vector.

4.7 Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.
Solution: Since all entries of the given vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.7.1}$$

are equal, it is equally inclined to the axes.

4.8 If a line has the direction ratios –18, 12, –4, then what are its direction cosines?
Solution: Let

$$\mathbf{A} = \begin{pmatrix} -18 \\ 12 \\ -4 \end{pmatrix} \tag{4.8.1}$$

Then the unit direction vector of the line is

$$\frac{\mathbf{A}}{\|\mathbf{A}\|} = \begin{pmatrix} \frac{-9}{11} \\ \frac{6}{11} \\ \frac{-2}{11} \end{pmatrix} \tag{4.8.2}$$

4.9 Find the direction cosines of the sides of a triangle whose vertices are $\begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix}$.
Solution: Let the vertices be

$$\mathbf{A} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -5 \\ -2 \end{pmatrix} \tag{4.9.1}$$

The direction vectors of the sides are,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix} = \mathbf{m_1}, \mathbf{B}-\mathbf{C} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix} = \mathbf{m_2}, \tag{4.9.2}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix} -8 \\ -10 \\ 2 \end{pmatrix} = \mathbf{m_3}, \tag{4.9.3}$$

The corresponding unit vectors are then obtained as

$$\begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \\ \frac{-3}{\sqrt{17}} \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{3}{\sqrt{17}} \\ \frac{2}{\sqrt{17}} \end{pmatrix}, \begin{pmatrix} \frac{-4}{\sqrt{42}} \\ \frac{-5}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{pmatrix} \tag{4.9.4}$$

4.10 Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.
Solution: The unit vector in the direction of the given vector is

$$\mathbf{A} = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{4.10.1}$$

4.11 Find the direction cosines of a line which makes equal angles with the coordinate axes.
Solution: Let α be the angle made by the line with the

axes. The unit direction vector can be expressed as

$$\mathbf{x} = \begin{pmatrix} \cos\alpha \\ \cos\alpha \\ \cos\alpha \end{pmatrix} \implies \|\mathbf{x}\| = 1 \tag{4.11.1}$$

$$\text{or, } \cos\alpha = \frac{1}{\sqrt{3}} \tag{4.11.2}$$

Thus the unit direction vector of the given line is

$$\mathbf{x} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.11.3}$$

4.12 Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

5 Scalar Product

5.1 Find the angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\overrightarrow{a} \cdot \overrightarrow{b} = \sqrt{6}$.
Solution: From the given information,

$$\|\mathbf{a}\| = \sqrt{3}, \|\mathbf{b}\| = 2, \mathbf{a}^\top \mathbf{b} = \sqrt{6} \tag{5.1.1}$$

$$\implies \cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{1}{\sqrt{2}} \tag{5.1.2}$$

$$\text{or, } \theta = 45^\circ \tag{5.1.3}$$

5.2 Find the angle between the the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.
Solution: Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \tag{5.2.1}$$

From problem 5.1,

$$\cos\theta = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{10}{\sqrt{14} \times \sqrt{14}} = \frac{5}{7} \tag{5.2.2}$$

5.3 Find $\left|\overrightarrow{a}\right|$ and $\left|\overrightarrow{b}\right|$, if $(\overrightarrow{a} + \overrightarrow{b}).(\overrightarrow{a} - \overrightarrow{b}) = 8$ and $\left|\overrightarrow{a}\right| = 8\left|\overrightarrow{b}\right|$.
Solution:

$$\because (\mathbf{a} + \mathbf{b})^\top (\mathbf{a} - \mathbf{b}) = 8, \|\mathbf{a}\| = 8 \|\mathbf{b}\|, \tag{5.3.1}$$

$$\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 = 8 \tag{5.3.2}$$

$$\implies \|8\mathbf{b}\|^2 - \|\mathbf{b}\|^2 = 8 \tag{5.3.3}$$

$$\implies \|\mathbf{b}\| = \frac{2\sqrt{2}}{3\sqrt{7}} \tag{5.3.4}$$

Thus,

$$\|\mathbf{a}\| = 8 \|\mathbf{b}\| = \frac{16\sqrt{2}}{3\sqrt{7}} \tag{5.3.5}$$

5.4 Evaluate the product $(3\overrightarrow{a} - 5\overrightarrow{b}) \cdot (2\overrightarrow{a} + 7\overrightarrow{b})$.
Solution:

$$\begin{aligned}(3\mathbf{a} - 5\mathbf{b})^\top (2\mathbf{a} + 7\mathbf{b}) &= 3\mathbf{a}^\top (2\mathbf{a} + 7\mathbf{b}) - 5\mathbf{b}^\top (2\mathbf{a} + 7\mathbf{b}) \\ &= 6\|\mathbf{a}\|^2 - 35\|\mathbf{b}\|^2 + 11\mathbf{a}^\top \mathbf{b}\end{aligned} \tag{5.4.1}$$

5.5 Find the magnitude of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.
Solution: Given

$$\|\mathbf{a}\| = \|\mathbf{b}\|, \cos\theta = \frac{1}{2}, \mathbf{a}^\top \mathbf{b} = \frac{1}{2}, \tag{5.5.1}$$

$$\implies \frac{1}{2} = \frac{\frac{1}{2}}{\|\mathbf{a}\|^2} \implies \|\mathbf{a}\| = \|\mathbf{b}\| = 1 \tag{5.5.2}$$

by using the definition of the scalar product.

5.6 Find $\left|\overrightarrow{x}\right|$, if for a unit vector $\overrightarrow{a}$, $(\overrightarrow{x} - \overrightarrow{a}) \cdot (\overrightarrow{x} + \overrightarrow{a})$=12.
Solution: From the given information,

$$(\mathbf{x} - \mathbf{a})^\top (\mathbf{x} + \mathbf{a}) = 12 \tag{5.6.1}$$

$$\implies \|\mathbf{x}\|^2 - \|\mathbf{a}\|^2 = 12 \tag{5.6.2}$$

$$\implies \|\mathbf{x}\| = \sqrt{13} \tag{5.6.3}$$

5.7 If the vertices A, B, C of a triangle ABC are (1,2,3), (-1,0,0), (0,1,2), respectively, then find $\angle ABC$.
Solution: From the given information,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \mathbf{C} - \mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{5.7.1}$$

$$\implies \angle ABC = \cos^{-1} \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{C} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{C} - \mathbf{B}\|} \tag{5.7.2}$$

$$= \cos^{-1} \frac{10}{\sqrt{102}} \tag{5.7.3}$$

$$\tag{5.7.4}$$

5.8 Find a unit vector perpendicular to each of the vector $\overrightarrow{a} + \overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$, where $\overrightarrow{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\overrightarrow{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.
Solution: Let the desired vector be $\mathbf{x}$. Then,

$$\begin{pmatrix} \mathbf{a} + \mathbf{b} & \mathbf{a} - \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \tag{5.8.1}$$

$$\tag{5.8.2}$$

$$\because \mathbf{a} + \mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.8.3}$$

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.8.4}$$

(5.8.2) can be expressed as

$$\left\{ \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\}^\top \mathbf{x} = 0 \tag{5.8.5}$$

$$\implies \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^\top \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \tag{5.8.6}$$

$$\implies \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^\top \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \tag{5.8.7}$$

$$\text{or, } \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^\top \mathbf{x} = 0 \tag{5.8.8}$$

which can be expressed as

$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & -2 \end{pmatrix} \xleftrightarrow[R_2 = \frac{R_2}{4}]{R_2 = 3R_2 - R_1} \begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & -2 \end{pmatrix} \tag{5.8.9}$$

$$\xleftrightarrow[R_1 = \frac{R_1}{3}]{R_1 = R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix} \tag{5.8.10}$$

yielding

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned} \implies \mathbf{x} = x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{5.8.11}$$

Thus, the desired unit vector is

$$\mathbf{x} = \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \tag{5.8.12}$$

5.9 If a unit vector $\overrightarrow{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\overrightarrow{a}$.
Solution: From the given information,

$$\mathbf{a} = \begin{pmatrix} \cos\frac{\pi}{3} \\ \cos\frac{\pi}{4} \\ \cos\theta \end{pmatrix} = \mathbf{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \cos\theta \end{pmatrix} \tag{5.9.1}$$

$$\because \|\mathbf{a}\| = 1, \tag{5.9.2}$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1 \tag{5.9.3}$$

$$\implies \cos\theta = \frac{1}{2} \tag{5.9.4}$$

$\because$ θ is an acute angle. Hence

$$\mathbf{a} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \tag{5.9.5}$$

5.10 If θ is the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}\cdot\mathbf{b} \geq 0$ only when
a) $0 < \theta < \frac{\pi}{2}$
b) $0 \leq \theta \leq \frac{\pi}{2}$
c) $0 < \theta < \pi$
d) $0 \leq \theta \leq \pi$
Solution:

$$\because \mathbf{a}^\top\mathbf{b} = \cos\theta \|\mathbf{a}\| \|\mathbf{b}\|, \tag{5.10.1}$$

$$\mathbf{a}^\top\mathbf{b} \geq 0 \implies \cos\theta \geq 0 \tag{5.10.2}$$

$$\therefore 0 \leq \theta \leq \frac{\pi}{2}, \frac{3\pi}{2} \leq \theta \leq 2\pi. \tag{5.10.3}$$

5.11 Find the angle between x-axis and the line joining points (3,-1) and (4,-2).
Solution: The direction vector of the given line is

$$\mathbf{C} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{5.11.1}$$

Hence, the desired angle is given by

$$\cos\theta = \frac{\mathbf{C}^\top\mathbf{e}_1}{\|\mathbf{C}\| \|\mathbf{e}_1\|} = -\frac{1}{\sqrt{2}} \tag{5.11.2}$$

$$\implies \theta = 135° \tag{5.11.3}$$

5.12 The slope of a line is double of the slope of another line. If tangent of the angle between them is 1/3, find the slopes of the lines.
Solution: The direction vectors of the lines can be expressed as

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ m \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ 2m \end{pmatrix} \tag{5.12.1}$$

If the angle between the lines be θ,

$$\tan\theta = \frac{1}{3} \implies \cos\theta = \frac{3}{\sqrt{10}} \tag{5.12.2}$$

Thus,

$$\frac{3}{\sqrt{10}} = \frac{\mathbf{m}_1^\top\mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \tag{5.12.3}$$

$$= \frac{2m^2 + 1}{\sqrt{m^2+1}\sqrt{4m^2+1}} \tag{5.12.4}$$

$$\implies \frac{9}{10} = \frac{4m^4 + 4m^2 + 1}{4m^4 + 5m^2 + 1} \tag{5.12.5}$$

$$\text{or, } 4m^4 - 5m^2 + 1 = 0 \tag{5.12.6}$$

yielding

$$m = \pm\frac{1}{2}, \pm 1 \tag{5.12.7}$$

5.13 Find angle between the lines, $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$.
Solution: From the given equations, the normal vectors can be expressed as

$$\mathbf{n}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{5.13.1}$$

The angle between the lines can then be expressed as

$$\cos\theta = \frac{\mathbf{n}_1^T\mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{\sqrt{3}}{2} \tag{5.13.2}$$

$$\text{or, } \theta = 30° \tag{5.13.3}$$

5.14 The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

5.15 Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ is the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if
a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

5.16 If θ is the angle between any two vectors $\mathbf{a}$ and $\mathbf{b}$, then $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}\times\mathbf{b}|$ when θ is equal to
a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

5.17 A vector $\mathbf{r}$ has a magnitude 14 and direction ratios 2, 3, -6. Find the direction cosines and components of $\mathbf{r}$, given that $\mathbf{r}$ makes an acute angle with x-axis.

5.18 Find the angle between the vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$.

5.19 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the three vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$ and $|\mathbf{a}| = 2$, $|\mathbf{b}|=3$, $|\mathbf{c}|=5$, the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is
a) 0
b) 1

c) -19
d) 38

5.20 If **a**, **b**, **c** are unit vectors such that **a**+**b**+**c**=0, then the value of $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}$ is

a) 1
b) 3
c) $\frac{-3}{2}$
d) None of these

5.21 The angles between two vectors $\mathbf{a}, \mathbf{b}$ with magnitude $\sqrt{3}, 4$ respectively, and $\mathbf{a}\cdot\mathbf{b} = 2\sqrt{3}$ is

a) $\frac{\pi}{6}$
b) $\frac{\pi}{3}$
c) $\frac{\pi}{2}$
d) $\frac{5\pi}{2}$

5.22 The vector $\mathbf{a}+\mathbf{b}$ bisects the angle between the non-collinear vectors **a** and **b** if ______.

5.23 The vectors $\mathbf{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{k}$ are the adjancent sides of a parallelogram. The acute angle between its diagonals is ______.

5.24 If **a** is any non-zero vector, then $(\mathbf{a}\cdot\hat{i})\hat{i}+(\mathbf{a}\cdot\hat{j})\hat{j}+(\mathbf{a}\cdot\hat{k})\,\hat{k}$ equals ______.

5.25 If **a** and **b** are adjacent sides of a rhombus, then $\mathbf{a}\cdot\mathbf{b}$=0.

5.26 Find the angle between the lines

$$\overrightarrow{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and} \tag{5.26.1}$$
$$\overrightarrow{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k}) \tag{5.26.2}$$

5.27 Find the angle between the lines whose direction cosines are given by the equations $l+m+n = 0$, $l^2+m^2-n^2 = 0$.

5.28 If a variable line in two adjacent positions has directions cosines l, m, n and $l+\delta l, m+\delta m, n+\delta n$, show that the small angle $\delta\theta$ between the two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2 \tag{5.28.1}$$

5.29 The sine of the angle between the straight line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and the plane $2x - 2y + z = 5$ is

a) $\frac{10}{6\sqrt{5}}$
b) $\frac{4}{5\sqrt{2}}$
c) $\frac{2\sqrt{3}}{5}$
d) $\frac{\sqrt{2}}{10}$

5.30 The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

a) $\frac{\sqrt{3}}{2}$
b) $\frac{\sqrt{2}}{3}$
c) $\frac{2}{7}$
d) $\frac{3}{7}$

5.31 The angle between the line $\overrightarrow{r} = (5\hat{i}-\hat{j}-4\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ and the plane $\overrightarrow{r}\cdot(3\hat{i}-4\hat{j}-\hat{k})+5 = 0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

5.32 The angle between the planes $\overrightarrow{r}\cdot(2\hat{i}-3\hat{j}+\hat{k}) = 1$ and $\overrightarrow{r}\cdot(\hat{i}-\hat{j}) = 4$ is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

5.33 Let **a** and **b** be two unit vectors and θ is the angle between them. Then $\mathbf{a}+\mathbf{b}$ is a unit vector if

a) $\theta = \frac{\pi}{4}$
b) $\theta = \frac{\pi}{3}$
c) $\theta = \frac{\pi}{2}$
d) $\theta = \frac{2\pi}{3}$

5.34 The value of $\hat{i}\cdot(\hat{j}\times\hat{k}) + \hat{j}\cdot(\hat{i}\times\hat{k}) + \hat{k}\cdot(\hat{i}\times\hat{j})$ is

a) 0
b) -1
c) 1
d) 3

5.35 If θ is the angle between any two vectors **a** and **b**, then $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}\times\mathbf{b}|$ when θ is equal to

a) 0
b) $\frac{\pi}{4}$
c) $\frac{\pi}{2}$
d) π

6 Orthogonality

6.1 Name the type of quadrilateral formed, if any, by the following points,and give reasons for your answer

a) $A(-1,-2), B(1,0), (C-1,2), D(-3,0)$
b) $A(-3,5), B(-3,1), C(0,3), D(-1,-4)$
c) $A(4,5), B(7,6), C(4,3), D(1,2)$

Solution: See Table 6.1, Fig. 6.1.1, Fig. 6.1.2. and Fig. 6.1.3. In b), forming the collinearity matrix

$$\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{B}\end{pmatrix} = \begin{pmatrix}6 & -3\\ -4 & 2\end{pmatrix} \xleftrightarrow{R_2\to R_2+\frac{2}{3}R_1} \begin{pmatrix}6 & -3\\ 0 & 0\end{pmatrix} \tag{6.1.1}$$

which is a rank 1 matrix. Hence, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear.

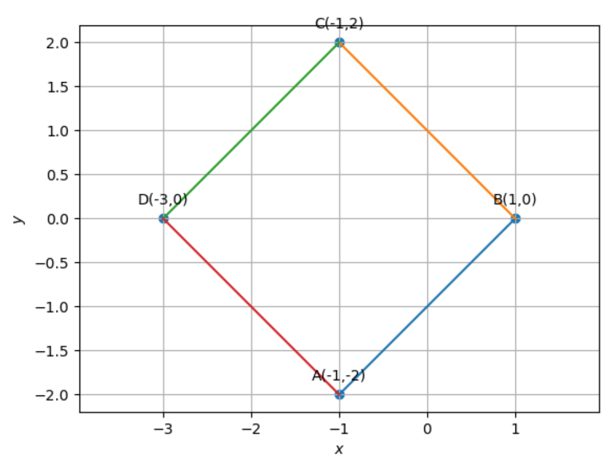

Fig. 6.1.1

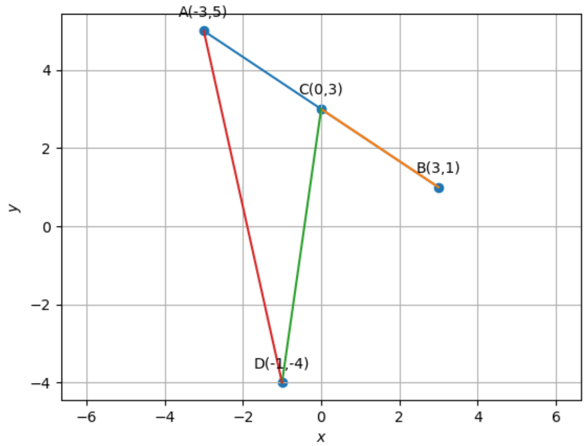

Fig. 6.1.2

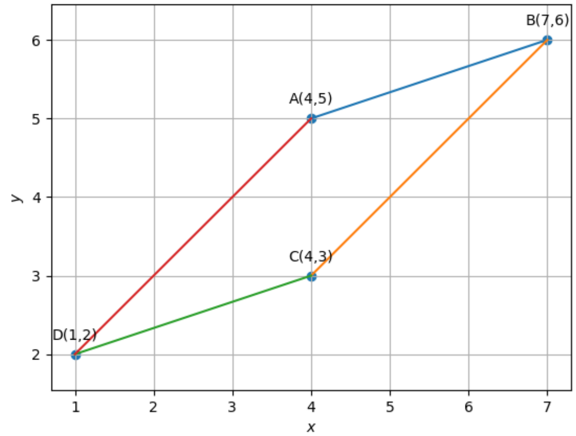

Fig. 6.1.3

	$\mathbf{B}-\mathbf{A} = \mathbf{C}-\mathbf{D}$?	$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{B}) = 0$?	$(\mathbf{C}-\mathbf{A})^\top(\mathbf{D}-\mathbf{B}) = 0$	**Geometry**
a)	Yes	Yes	Yes	Square
b)	No	-	-	Triangle
c)	Yes	No	No	Parallelogram

TABLE 6.1

6.2 Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.
Solution: Let

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{6.2.1}$$

The projection of $\mathbf{A}$ on $\mathbf{B}$ is defined as the foot of the perpendicular from $\mathbf{A}$ to $\mathbf{B}$ and obtained in (D.1.3). Substituting numerical values,

$$\mathbf{C} = \frac{10}{19}\begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix} \tag{6.2.2}$$

6.3 Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.
Solution: The given points are

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.3.1}$$

Since

$$\mathbf{A}^\top\mathbf{B} = 0, \tag{6.3.2}$$

from (D.1.3), the projection vector is the origin. See Fig. 6.3.1.

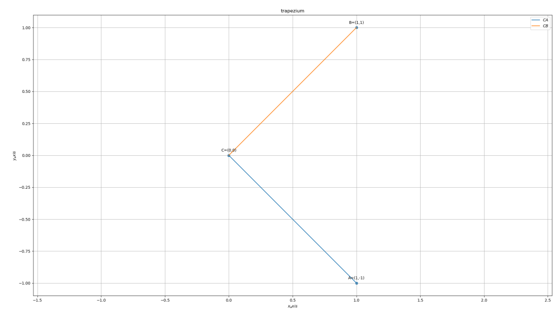

Fig. 6.3.1

6.4 Show that each of the given three vectors is a unit vector: $\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}), \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}), \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$. Also,show that they are mutually perpendicular to each other.
Solution:

$$\mathbf{A} = \begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \tag{6.4.1}$$

is an orthogonal matrix satisfying (D.5.1), which verifies the given conditions.

6.5 If $\overrightarrow{a} = 2\hat{i}+2\hat{j}3\hat{k}, \overrightarrow{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\overrightarrow{c} = 3\hat{i}+\hat{j}$ are such that $\overrightarrow{a}+\lambda\overrightarrow{b}$ is perpendicular to $\overrightarrow{c}$,then find the value of λ.
Solution:

$$\because (\mathbf{a}+\lambda\mathbf{b})^\top\mathbf{c} = 0, \tag{6.5.1}$$

$$\lambda = -\frac{\mathbf{a}^\top\mathbf{c}}{\mathbf{b}^\top\mathbf{c}} = 8, \tag{6.5.2}$$

upon substituting numerical values.

6.6 Show that $\left|\overrightarrow{a}\right|\overrightarrow{b}+\left|\overrightarrow{b}\right|\overrightarrow{a}$ is perpendicular to $\left|\overrightarrow{a}\right|\overrightarrow{b}-\left|\overrightarrow{b}\right|\overrightarrow{a}$, for any two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.
Solution:

$$\|\mathbf{a}\|\,\mathbf{b}+\|\mathbf{b}\|\,\mathbf{a} = \|\mathbf{a}\|\,\|\mathbf{b}\|\left(\frac{\mathbf{b}}{\|\mathbf{b}\|}+\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) \tag{6.6.1}$$

$$\|\mathbf{a}\|\,\mathbf{b}-\|\mathbf{b}\|\,\mathbf{a} = \|\mathbf{a}\|\,\|\mathbf{b}\|\left(\frac{\mathbf{b}}{\|\mathbf{b}\|}-\frac{\mathbf{a}}{\|\mathbf{a}\|}\right) \tag{6.6.2}$$

$$\implies (\|\mathbf{a}\|\,\mathbf{b}+\|\mathbf{b}\|\,\mathbf{a})^\top(\|\mathbf{a}\|\,\mathbf{b}-\|\mathbf{b}\|\,\mathbf{a}) = 0 \tag{6.6.3}$$

from (D.2.1).

6.7 If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are unit vectors such that $\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c} = \overrightarrow{0}$, find the value of $\overrightarrow{a}.\overrightarrow{b}+\overrightarrow{b}.\overrightarrow{c}+\overrightarrow{c}.\overrightarrow{a}$.

Solution:

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2 = 0$$
$$\implies \|\mathbf{a}\|^2+\|\mathbf{b}\|^2+\|\mathbf{c}\|^2+2(\mathbf{a}^\top\mathbf{b}+\mathbf{b}^\top\mathbf{c}+\mathbf{c}^\top\mathbf{a}) = 0$$
$$\implies 3+2(\mathbf{a}^\top\mathbf{b}+\mathbf{b}^\top\mathbf{c}+\mathbf{c}^\top\mathbf{a}) = 0$$
$$\implies \mathbf{a}^\top\mathbf{b}+\mathbf{b}^\top\mathbf{c}+\mathbf{c}^\top\mathbf{a} = -\frac{3}{2} \tag{6.7.1}$$

6.8 If either vector $\vec{a}=0$ or $\vec{b}=0$, then $\vec{a}.\vec{b}$=0. But the converse need not be true. Justify your answer with an example.
Solution:

$$\mathbf{a}=\begin{pmatrix}1\\1\end{pmatrix}, \mathbf{b}=\begin{pmatrix}1\\-1\end{pmatrix} \tag{6.8.1}$$
$$\implies \mathbf{a}^\top\mathbf{b}=0 \tag{6.8.2}$$

6.9 Show that the vectors $2\hat{i}-\hat{j}+\hat{k}, \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ from the vertices of a right angled triangle.
Solution:

$$\mathbf{A}=\begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{B}=\begin{pmatrix}1\\-3\\-5\end{pmatrix}, \mathbf{C}=\begin{pmatrix}3\\-4\\-4\end{pmatrix}, \tag{6.9.1}$$
$$\implies \mathbf{B}-\mathbf{C}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}, \mathbf{C}-\mathbf{A}=\begin{pmatrix}1\\-3\\-5\end{pmatrix}, \tag{6.9.2}$$
$$\text{or, } (\mathbf{B}-\mathbf{C})^\top(\mathbf{C}-\mathbf{A})=0 \tag{6.9.3}$$

6.10 Show that the points A, B and C with position vectors, $3\hat{i}-4\hat{j}-4\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}-3\hat{j}-5\hat{k}$, respectively, form the vertices of a right angled triangle.
Solution:

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}-1\\3\\5\end{pmatrix}, \mathbf{C}-\mathbf{B}=\begin{pmatrix}-1\\-2\\-6\end{pmatrix}, \mathbf{C}-\mathbf{A}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}, \tag{6.10.1}$$
$$\implies (\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})=0 \tag{6.10.2}$$

Hence, $\triangle ABC$ is right angled at **A**.

6.11 Let $\mathbf{a}=\hat{i}+4\hat{j}+2\hat{k}, \mathbf{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\mathbf{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector **d** which is perpendicular to both **a** and **b**, and $\mathbf{c}\cdot\mathbf{d}$=15.
Solution: From the given information,

$$\mathbf{a}^\top\mathbf{d}=0 \tag{6.11.1}$$
$$\mathbf{b}^\top\mathbf{d}=0 \tag{6.11.2}$$
$$\mathbf{c}^\top\mathbf{d}=15 \tag{6.11.3}$$

yielding

$$\begin{pmatrix}\mathbf{a}^\top\\\mathbf{b}^\top\\\mathbf{c}^\top\end{pmatrix}\mathbf{d}=\begin{pmatrix}0\\0\\15\end{pmatrix} \tag{6.11.4}$$
$$\implies \begin{pmatrix}1&4&2\\3&-2&7\\2&-1&4\end{pmatrix}\mathbf{d}=\begin{pmatrix}0\\0\\15\end{pmatrix} \tag{6.11.5}$$

Forming the augmented matrix,

$$\left(\begin{array}{ccc|c}1&4&2&0\\3&-2&7&0\\2&-1&4&15\end{array}\right)\xleftrightarrow[R_3\leftarrow R_3-2R_1]{R_2\leftarrow R_2-3R_1}\left(\begin{array}{ccc|c}1&4&2&0\\0&-14&1&0\\0&-9&0&15\end{array}\right)$$
$$\xleftrightarrow{R_3\leftarrow R_3-\frac{9}{14}R_2}\left(\begin{array}{ccc|c}1&4&2&0\\0&-14&1&0\\0&0&-\frac{9}{14}&15\end{array}\right) \tag{6.11.6}$$

yielding

$$\mathbf{d}=\begin{pmatrix}\frac{160}{3}\\-\frac{5}{3}\\-\frac{70}{3}\end{pmatrix} \tag{6.11.7}$$

upon back substitution.

6.12 Prove that $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=|\mathbf{a}|^2+|\mathbf{b}|^2$, if and only if $\mathbf{a},\mathbf{b}$ are perpendicular, given $\mathbf{a}\neq\mathbf{0},\mathbf{b}\neq\mathbf{0}$.
Solution:

$$\because (\mathbf{a}+\mathbf{b})^\top(\mathbf{a}+\mathbf{b})=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2, \tag{6.12.1}$$
$$\|\mathbf{a}\|^2+\|\mathbf{b}\|^2+2\mathbf{a}^\top\mathbf{b}=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2 \tag{6.12.2}$$
$$\implies \mathbf{a}^\top\mathbf{b}=0 \tag{6.12.3}$$

6.13 $ABCD$ is a rectangle formed by the points $\mathbf{A}(-1,-1), \mathbf{B}(-1,4), \mathbf{C}(5,4)$ and $\mathbf{D}(5,-1)$. $\mathbf{P},\mathbf{Q},\mathbf{R}$ and $\mathbf{S}$ are the mid-points of AB, BC, CD and DA respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer.
Solution: See Fig. 6.13.1. From (D.4.3), $PQRS$ is a parallelogram.

$$\mathbf{P}=\frac{3}{2}, \mathbf{Q}=\begin{pmatrix}2\\4\end{pmatrix}, \mathbf{R}=\begin{pmatrix}5\\\frac{3}{2}\end{pmatrix}, \mathbf{S}=\begin{pmatrix}2\\-1\end{pmatrix} \tag{6.13.1}$$
$$\implies (\mathbf{Q}-\mathbf{P})^\top(\mathbf{R}-\mathbf{Q})\neq 0 \tag{6.13.2}$$
$$(\mathbf{R}-\mathbf{P})^\top(\mathbf{S}-\mathbf{Q})=0 \tag{6.13.3}$$

Therefore $PQRS$ is a rhombus.

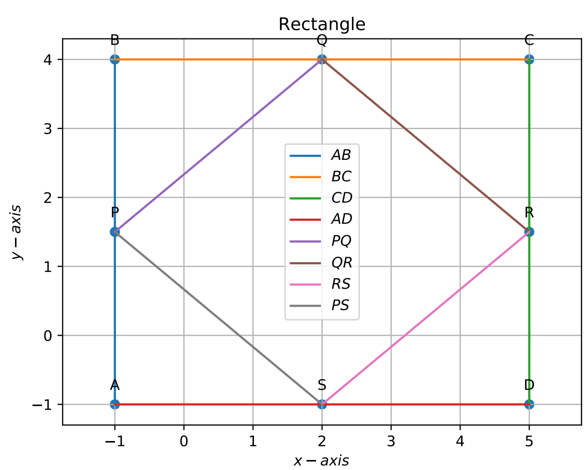

Fig. 6.13.1

6.14 Without using the Baudhayana theorem, show that the points $A(4,4), B(3,5)$ and $C(-1,-1)$ are the vertices of a right angled triangle. See Fig. 6.14.1.

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix}-5\\-5\end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix}1\\-1\end{pmatrix} \tag{6.14.1}$$

$$\implies (\mathbf{C}-\mathbf{A})^\top (\mathbf{A}-\mathbf{B}) = 0 \tag{6.14.2}$$

Thus, $AB \perp AC$.

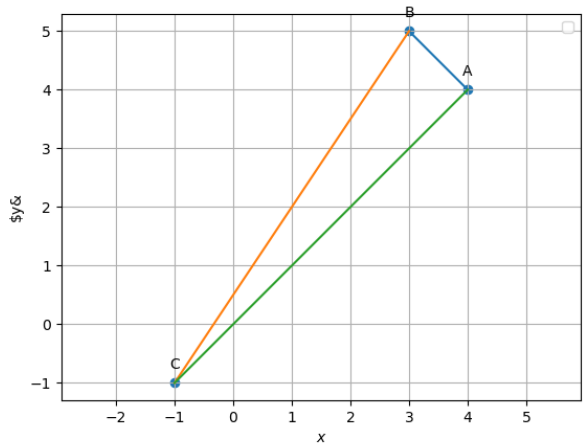

Fig. 6.14.1

6.15 The line through the points $(h,3)$ and $(4,1)$ intersects the line $7x-9y-19=0$ at a right angle. Find the value of h.
Solution: The direction vectors of the given lines are

$$\begin{pmatrix}4-h\\-2\end{pmatrix}, \begin{pmatrix}9\\7\end{pmatrix} \tag{6.15.1}$$

$$\implies \begin{pmatrix}9 & 7\end{pmatrix}\begin{pmatrix}4-h\\-2\end{pmatrix} = 0 \tag{6.15.2}$$

$$\implies h = \frac{22}{9} \tag{6.15.3}$$

See Fig. 6.15.1.

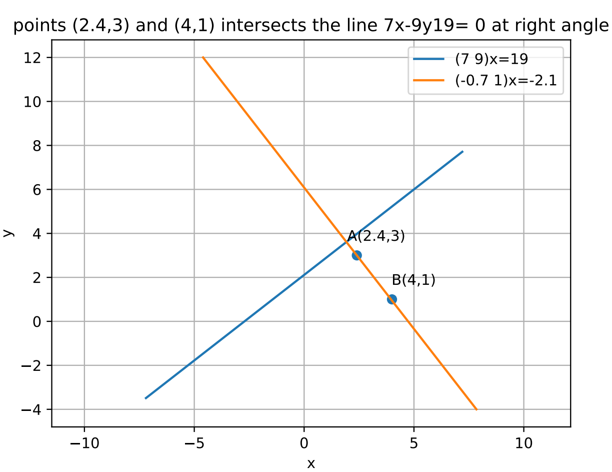

Fig. 6.15.1

6.16 In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

a) $7x+5y+6z+30=0$ and $3x–y–10z+4=0$
b) $2x+y+3z–2=0$ and $x–2y+5=0$
c) $2x–2y+4z+5=0$ and $3x–3y+6z–1=0$
d) $2x–y+3z–1=0$ and $2x–y+3z+3=0$
e) $4x+8y+z–8=0$ and $y+z–4=0$

Solution: See Table 6.16.

TABLE 6.16

$\mathbf{n}_1$	$\mathbf{n}_1$	$\mathbf{n}_1^\top\mathbf{n}_2$	$\|\mathbf{n}_1\|$	$\|\mathbf{n}_2\|$	Angle
$\begin{pmatrix}7\\5\\6\end{pmatrix}$	$\begin{pmatrix}3\\-1\\-10\end{pmatrix}$	-44	$\sqrt{110}$	$\sqrt{110}$	$\cos^{-1}-\frac{2}{5}$
$\begin{pmatrix}2\\1\\3\end{pmatrix}$	$\begin{pmatrix}1\\-2\\0\end{pmatrix}$	0			perpendicular
$\begin{pmatrix}2\\-2\\4\end{pmatrix}$	$\begin{pmatrix}3\\-3\\6\end{pmatrix}$	36	$\sqrt{24}$	$\sqrt{54}$	parallel
$\begin{pmatrix}2\\-1\\3\end{pmatrix}$	$\begin{pmatrix}2\\-1\\3\end{pmatrix}$	14	$\sqrt{14}$	$\sqrt{14}$	parallel
$\begin{pmatrix}4\\8\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	9	9	$\sqrt{2}$	45°

6.17 Show that the line joining the origin to the point $P(2,1,1)$ is perpendicular to the line determined by the points $A(3,5,–1), B(4,3,–1)$.
Solution:

$$(\mathbf{A}-\mathbf{B})^\top \mathbf{P} = \begin{pmatrix}-1 & 2 & 0\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = 0 \quad \square \tag{6.17.1}$$

6.18 If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both these are $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.
Solution:

$$\mathbf{P} = \begin{pmatrix} l_1 & l_2 & m_1n_2 - m_2n_1 \\ m_1 & m_2 & n_1l_2 - n_2l_1 \\ n_1 & n_2 & l_1m_2 - l_2m_1 \end{pmatrix} \tag{6.18.1}$$

satisfies (D.5.1). Hence, the three vectors are mutually perpendicular.

6.19 If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Solution: From the given information,

$$\mathbf{m}_1 = \begin{pmatrix} -3 \\ 2k \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} \tag{6.19.1}$$

$$\implies \begin{pmatrix} -3 & 2k & 2 \end{pmatrix}^{\top} \begin{pmatrix} 3k \\ 1 \\ -5 \end{pmatrix} = 0 \tag{6.19.2}$$

$$\implies k = -\frac{10}{7} \tag{6.19.3}$$

See Fig. 6.19.1

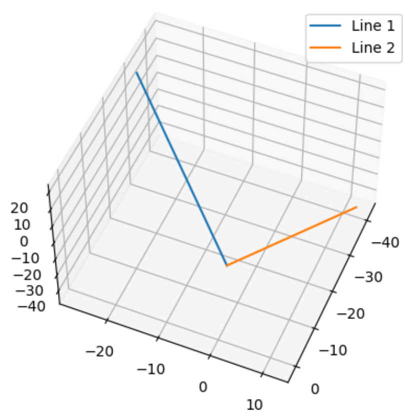

Fig. 6.19.1: lines represented for the given points and direction vector with k=$\frac{-10}{7}$

6.20 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\mathbf{c} \cdot \mathbf{d}$=15 is equally inclined to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

6.21 If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are mutually perpendicular vectors of equal magnitudes,show that the $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is equally inclined to $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

6.22 Check whether$(5, -2), (6, 4)$ and $(7, -2)$ are the vertices of an isosceles triangle.

7 Miscellaneous

8 Linear Forms

8.1 Equation of a Line

Find the equation of line

8.1

8.2 passing through the point (– 4, 3) with slope $\frac{1}{2}$.

8.3 passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with slope m.

Solution:

8.4 passing through $\mathbf{A} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ and inclined with the x-axis at an angle of 75°.

Solution:

8.5 intersecting the x-axis at a distance of 3 units to the left of origin with slope of -2.

Solution:

8.6 Find the equation of the line which satisfy the given conditions: Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.

Solution:

8.7 Find the equation of line passing through the points $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

Solution:

8.8 Find the equation of line whose perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive x-axis is 30°.

Solution:

8.9

8.10

8.11

8.12

8.13

8.14

8.15

8.16 $P(a, b)$ is the mid-point of the line segment between axes. Show that the equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$

Solution:

8.17 Point $\mathbf{R}\,(h, k)$ divides a line segment between the axes in the ratio 1: 2. Find the equation of the line.

8.18

8.19 Find the equation of the line parallel to the line 3x-4y+2=0 and passing through the point (-2,3).

8.20 Find the equation of line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3

Solution:

8.21 Prove that the line through the point(x_1, y_1) and parallel to the line Ax+By+C=0 is A$(x - x_1)$+B$(y - y_1)$=0.

Solution:

8.22 Find the equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and} \tag{8.22.1}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \tag{8.22.2}$$

Solution:

8.23 Find the vector equation of the line passing through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and parallel to the planes $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}^{\top} \mathbf{x} = 5$ and $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}^{\top} \mathbf{x} = 6$.

Solution:

8.24

8.25

8.26

8.27 The perpendicular from the origin to the line $y = mx + c$ meets it at the point $(-1, 2)$. Find the values of m and c.

Solution:

8.28 Find the equation of the lines through the point (3, 2) which make an angle of 45°with the line x–2y = 3.

Solution:

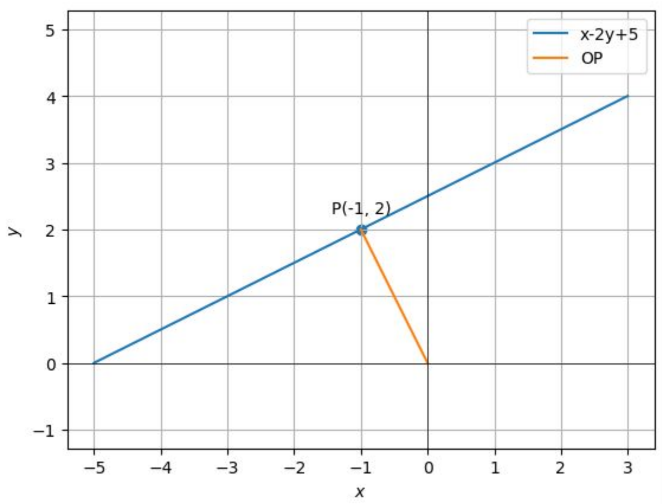

Fig. 8.27.1: Graph

Appendix A
Points on a Line

A.1. The equation of a line is given by

$$y = mx + c \tag{A.1.1}$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} + x \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{A.1.2}$$

yielding (1.1.4.1).

A.2. (A.1.1) can also be expressed as

$$y - mx = c \tag{A.2.1}$$

$$\implies \begin{pmatrix} -m & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \tag{A.2.2}$$

yielding (1.1.5.1).

A.3. From (1.1.4.1), if $\mathbf{A}, \mathbf{D}$ and $\mathbf{C}$ are on the same line,

$$\mathbf{D} = \mathbf{A} + q\mathbf{m} \tag{A.3.1}$$

$$\mathbf{C} = \mathbf{D} + p\mathbf{m} \tag{A.3.2}$$

$$\implies p(\mathbf{D} - \mathbf{A}) + q(\mathbf{D} - \mathbf{C}) = 0, \quad p, q \neq 0 \tag{A.3.3}$$

$$\implies \mathbf{D} = \frac{p\mathbf{A} + q\mathbf{C}}{p + q} \tag{A.3.4}$$

yielding (1.2.1.1) upon substituting

$$k = \frac{p}{q}. \tag{A.3.5}$$

$(\mathbf{D} - \mathbf{A}), (\mathbf{D} - \mathbf{C})$ are then said to be *linearly dependent*.

A.4. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear, from (1.1.5.1),

$$\mathbf{n}^\top \mathbf{A} = c \tag{A.4.1}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{A.4.2}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{A.4.3}$$

which can be expressed as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{A.4.4}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \begin{pmatrix} \mathbf{n} \\ -c \end{pmatrix} = \mathbf{0} \tag{A.4.5}$$

yielding (1.1.3.1). Rank is defined to be the number of linearly indpendent rows or columns of a matrix.

Appendix B
Tangents to a Circle

The equation of the *incircle* is given by

$$\|\mathbf{x}-\mathbf{O}\|^2 = r^2 \tag{B.1}$$

which can be expressed as (1.6.1) using (1.6.2). In Fig. 1.5.4.1, Let (1.6.8.1) be the equation of AB. Then, the intersection of (1.6.8.1) and (1.6.1) can be expressed as

$$(\mathbf{h}+\mu\mathbf{m})^\top \mathbf{V}(\mathbf{h}+\mu\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h}+\mu\mathbf{m}) + f = 0 \tag{B.2}$$

$$\implies \mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) + g(\mathbf{h}) = 0 \tag{B.3}$$

For (B.3) to have exactly one root, the discriminant

$$\left\{\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right\}^2 - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{B.4}$$

and (1.6.8.2) is obtained. (B.4) can be expressed as

$$\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\,\mathbf{m} - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{B.5}$$

$$\implies \mathbf{m}^\top\mathbf{\Sigma}\mathbf{m} = 0 \tag{B.6}$$

for $\mathbf{\Sigma}$ defined in (B.6). Substituting (1.6.6.1) in (B.6),

$$\mathbf{m}^\top\mathbf{P}\mathbf{D}\mathbf{P}^\top\mathbf{m} = 0 \tag{B.7}$$

$$\implies \mathbf{v}^\top\mathbf{D}\mathbf{v} = 0 \tag{B.8}$$

where

$$\mathbf{v} = \mathbf{P}^\top\mathbf{m} \tag{B.9}$$

(B.8) can be expressed as

$$\lambda_1 v_1^2 - \lambda_2 v_2^2 = 0 \tag{B.10}$$

$$\implies \mathbf{v} = \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{B.11}$$

after some algebra. From (B.11) and (B.9) we obtain (1.6.7.1).

Appendix C
Matrices

Appendix D
2 × 1 vectors

D.1. Mathematically, the projection of $\mathbf{A}$ on $\mathbf{B}$ is defined as

$$\mathbf{C} = k\mathbf{B}, \text{ such that } (\mathbf{A}-\mathbf{C})^\top\mathbf{C} = 0 \tag{D.1.1}$$

yielding

$$(\mathbf{A}-k\mathbf{B})^\top\mathbf{B} = 0 \tag{D.1.2}$$

$$\text{or, } k = \frac{\mathbf{A}^\top\mathbf{B}}{\|\mathbf{B}\|^2} \implies \mathbf{C} = \frac{\mathbf{A}^\top\mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} \tag{D.1.3}$$

D.2. If $\mathbf{A}, \mathbf{B}$ are unit vectors,

$$(\mathbf{A}-\mathbf{B})^\top(\mathbf{A}+\mathbf{B})$$

$$\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 = 0 \tag{D.2.1}$$

D.3. If $ABCD$ be a parallelogram,

$$\mathbf{B}-\mathbf{A} = \mathbf{C}-\mathbf{D} \tag{D.3.1}$$

D.4. If $PQRS$ is formed by joining the mid points of $ABCD$,

$$\mathbf{P} = \frac{1}{2}(\mathbf{A}+\mathbf{B}),\ \mathbf{Q} = \frac{1}{2}(\mathbf{B}+\mathbf{C}) \tag{D.4.1}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{C}+\mathbf{D}),\ \mathbf{S} = \frac{1}{2}(\mathbf{D}+\mathbf{A}) \tag{D.4.2}$$

$$\implies \mathbf{P}-\mathbf{Q} = \mathbf{S}-\mathbf{R}. \tag{D.4.3}$$

Hence, $PQRS$ is a parallelogram from (D.3.1).

D.5. If

$$\mathbf{A}^\top\mathbf{A} = \mathbf{I}, \tag{D.5.1}$$

then $\mathbf{A}$ is an *orthogonal* matrix.