MATRICES
In Geometry

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Contents

1 Vectors

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1}$$

1.1 Sides

1.1.1. The direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Find the direction vectors of AB, BC and CA.
Solution:

a) The Direction vector of AB is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4-1 \\ 6-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.1.2}$$

b) The Direction vector of BC is

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3-(-4) \\ -5-6 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.1.3}$$

c) The Direction vector of CA is

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1-(-3) \\ -1-(-5) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.1.4}$$

1.1.2. The length of side BC is

$$c = \|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{A})} \tag{1.1.2.1}$$

where

$$\mathbf{A}^\top \triangleq \begin{pmatrix} 1 & -1 \end{pmatrix} \tag{1.1.2.2}$$

Similarly,

$$b = \|\mathbf{C} - \mathbf{B}\|, \ a = \|\mathbf{A} - \mathbf{C}\| \tag{1.1.2.3}$$

Find a, b, c.

a) From (1.1.1.2),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \tag{1.1.2.4}$$

$$\implies c = \|\mathbf{B} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \tag{1.1.2.5}$$

$$= \sqrt{\begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix}} = \sqrt{(5)^2 + (7)^2} \tag{1.1.2.6}$$

$$= \sqrt{74} \tag{1.1.2.7}$$

b) Similarly, from (1.1.1.3),

$$a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix}} \tag{1.1.2.8}$$

$$= \sqrt{(1)^2 + (11)^2} = \sqrt{122} \tag{1.1.2.9}$$

and from (1.1.1.4),

c)

$$b = \|\mathbf{A} - \mathbf{C}\| = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}} \tag{1.1.2.10}$$

$$= \sqrt{(4)^2 + (4)^2} = \sqrt{32} \tag{1.1.2.11}$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{1.1.3.1}$$

Are the given points in (1) collinear?
Solution: From (1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \tag{1.1.3.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{5} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & \frac{-48}{5} \end{pmatrix} \tag{1.1.3.3}$$

There are no zero rows. So,

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{1.1.3.4}$$

Hence, the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear. This is visible in Fig. 1.

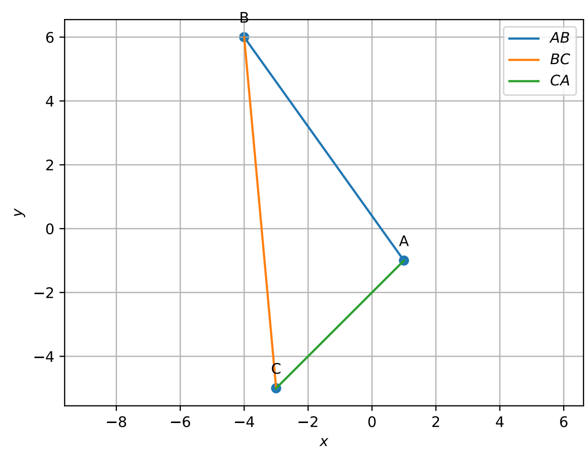

Fig. 1. $\triangle ABC$

1.1.4. The parametric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \quad k \neq 0, \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

is the direction vector of AB. Find the parametric equations of AB, BC and CA.
Solution: From (1.1.4.1) and (1.1.1.2), the parametric equation for AB is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.4.3}$$

Similarly, from (1.1.1.3) and (1.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.4.4}$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.4.5}$$

1.1.5. The normal form of the equation of AB is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.1}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \tag{1.1.5.2}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.3}$$

Find the normal form of the equations of AB, BC and CA.
Solution:

a) From (1.1.1.3), the direction vector of side $\mathbf{BC}$ is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.1.5.4}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} -11 \\ -1 \end{pmatrix} \tag{1.1.5.5}$$

from (1.1.5.3). Hence, from (1.1.5.1), the normal equation of side BC is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.1.5.6}$$

$$\implies \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \tag{1.1.5.7}$$

$$\implies BC: \quad \begin{pmatrix} 11 & 1 \end{pmatrix} \mathbf{x} = -38 \tag{1.1.5.8}$$

b) Similarly, for AB, from (1.1.1.2),

$$\mathbf{m} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.1.5.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \tag{1.1.5.10}$$

and

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.11}$$

$$\implies AB: \quad \mathbf{n}^\top \mathbf{x} = \begin{pmatrix} 7 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.12}$$

$$\implies \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} = 2 \tag{1.1.5.13}$$

c) For CA, from (1.1.1.4),

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.1.5.14}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.15}$$

$$\tag{1.1.5.16}$$

$$\implies \mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.1.5.17}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = 2 \tag{1.1.5.18}$$

1.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| \tag{1.1.6.1}$$

where

$$\mathbf{A} \times \mathbf{B} \triangleq \begin{vmatrix} 1 & -4 \\ -1 & 6 \end{vmatrix} \tag{1.1.6.2}$$

Find the area of $\triangle ABC$.
Solution: From (1.1.1.2) and (1.1.1.4),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{1.1.6.3}$$

$$\implies (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix} 5 & 4 \\ -7 & 4 \end{vmatrix} \tag{1.1.6.4}$$

$$= 5 \times 4 - 4 \times (-7) \tag{1.1.6.5}$$

$$= 48 \tag{1.1.6.6}$$

$$\implies \frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| = \frac{48}{2} = 24 \tag{1.1.6.7}$$

which is the desired area.

1.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top \mathbf{C} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} \tag{1.1.7.1}$$

a) From (1.1.1.2), (1.1.1.4), (1.1.2.7) and (1.1.2.11)

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & 7 \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{1.1.7.2}$$

$$= -8 \tag{1.1.7.3}$$

$$\implies \cos A = \frac{-8}{\sqrt{74}\sqrt{32}} = \frac{-1}{\sqrt{37}} \tag{1.1.7.4}$$

$$\implies A = \cos^{-1} \frac{-1}{\sqrt{37}} \tag{1.1.7.5}$$

b) From (1.1.1.2), (1.1.1.3), (1.1.2.7) and (1.1.2.9)

$$(\mathbf{C} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 & -11 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix} \tag{1.1.7.6}$$

$$= 82 \tag{1.1.7.7}$$

$$\implies \cos B = \frac{82}{\sqrt{74}\sqrt{122}} = \frac{41}{\sqrt{2257}} \tag{1.1.7.8}$$

$$\implies B = \cos^{-1} \frac{41}{\sqrt{2257}} \tag{1.1.7.9}$$

c) From (1.1.1.3), (1.1.1.4), (1.1.2.9) and (1.1.2.11)

$$(\mathbf{A} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix} \tag{1.1.7.10}$$

$$= 40 \tag{1.1.7.11}$$

$$\implies \cos C = \frac{40}{\sqrt{32}\sqrt{122}} = \frac{5}{\sqrt{61}} \tag{1.1.7.12}$$

$$\implies C = \cos^{-1} \frac{5}{\sqrt{61}} \tag{1.1.7.13}$$

All codes for this section are available at

codes/triangle/sides.py

1.2 Median

1.2.1. If $\mathbf{D}$ divides BC in the ratio $k : 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k + 1} \tag{1.2.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.
Solution: Since $\mathbf{D}$ is the midpoint of BC,

$$k = 1, \tag{1.2.1.2}$$

$$\implies \mathbf{D} = \frac{\mathbf{C}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix}-7\\1\end{pmatrix} \tag{1.2.1.3}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} = \begin{pmatrix}-1\\-3\end{pmatrix} \tag{1.2.1.4}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix}-3\\5\end{pmatrix} \tag{1.2.1.5}$$

1.2.2. Find the equations of AD, BE and CF.

Solution: :

a) The direction vector of AD is

$$\mathbf{m} = \mathbf{D} - \mathbf{A} = \begin{pmatrix}\frac{-7}{2}\\\frac{1}{2}\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-9\\3\end{pmatrix} \equiv \begin{pmatrix}-3\\1\end{pmatrix} \tag{1.2.2.1}$$

$$\implies \mathbf{n} = \begin{pmatrix}1\\3\end{pmatrix} \tag{1.2.2.2}$$

Hence the normal equation of median AD is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{A}\right) = 0 \tag{1.2.2.3}$$

$$\implies \begin{pmatrix}1 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = -2 \tag{1.2.2.4}$$

b) For BE,

$$\mathbf{m} = \mathbf{E} - \mathbf{B} = \begin{pmatrix}-1\\-3\end{pmatrix} - \begin{pmatrix}-4\\6\end{pmatrix} = \begin{pmatrix}3\\-9\end{pmatrix} \equiv \begin{pmatrix}1\\-3\end{pmatrix} \tag{1.2.2.5}$$

$$\implies \mathbf{n} = \begin{pmatrix}3\\1\end{pmatrix} \tag{1.2.2.6}$$

Hence the normal equation of median BE is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{B}\right) = 0 \tag{1.2.2.7}$$

$$\implies \begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}3 & 1\end{pmatrix}\begin{pmatrix}-4\\6\end{pmatrix} = -6 \tag{1.2.2.8}$$

c) For median CF,

$$\mathbf{m} = \mathbf{F} - \mathbf{C} = \begin{pmatrix}\frac{-3}{2}\\\frac{5}{2}\end{pmatrix} - \begin{pmatrix}-3\\-5\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\\frac{15}{2}\end{pmatrix} \equiv \begin{pmatrix}1\\5\end{pmatrix} \tag{1.2.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix}5\\-1\end{pmatrix} \tag{1.2.2.10}$$

Hence the normal equation of median CF is

$$\mathbf{n}^\top\left(\mathbf{x}-\mathbf{C}\right) = 0 \tag{1.2.2.11}$$

$$\implies \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}-3\\-5\end{pmatrix} = -10 \tag{1.2.2.12}$$

1.2.3. Find the intersection $\mathbf{G}$ of BE and CF.
Solution: From (1.2.2.8) and (1.2.2.12), the equations of BE and CF are, respectively,

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}-6\end{pmatrix} \tag{1.2.3.1}$$

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x} = \begin{pmatrix}-10\end{pmatrix} \tag{1.2.3.2}$$

From (1.2.3.1) and (1.2.3.2) the augmented matrix is

$$\begin{pmatrix}3 & 1 & -6\\5 & -1 & -10\end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1+R_2} \begin{pmatrix}8 & 0 & -16\\5 & -1 & -10\end{pmatrix} \tag{1.2.3.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix}1 & 0 & -2\\5 & -1 & -10\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix}1 & 0 & -2\\0 & -1 & 0\end{pmatrix} \tag{1.2.3.4}$$

$$\xleftrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix}1 & 0 & -2\\0 & 1 & 0\end{pmatrix} \tag{1.2.3.5}$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix}-2\\0\end{pmatrix} \tag{1.2.3.6}$$

1.2.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.1}$$

Solution:

a) From (1.2.1.4) and (1.2.3.6),

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix}2\\-6\end{pmatrix}, \mathbf{E} - \mathbf{G} = \begin{pmatrix}1\\-3\end{pmatrix} \tag{1.2.4.2}$$

$$\implies \mathbf{G} - \mathbf{B} = 2\left(\mathbf{E} - \mathbf{G}\right) \tag{1.2.4.3}$$

$$\implies \|\mathbf{G} - \mathbf{B}\| = 2\,\|\mathbf{E} - \mathbf{G}\| \tag{1.2.4.4}$$

$$\text{or, } \frac{BG}{GE} = 2 \tag{1.2.4.5}$$

b) From (1.2.1.5) and (1.2.3.6),

$$\mathbf{F} - \mathbf{G} = \frac{1}{2}\begin{pmatrix}1\\5\end{pmatrix}, \mathbf{G} - \mathbf{C} \quad = \begin{pmatrix}1\\5\end{pmatrix} \tag{1.2.4.6}$$

$$\implies \mathbf{G} - \mathbf{C} = 2\left(\mathbf{F} - \mathbf{G}\right) \tag{1.2.4.7}$$

$$\implies \|\mathbf{G} - \mathbf{C}\| = 2\,\|\mathbf{F} - \mathbf{G}\| \tag{1.2.4.8}$$

$$\text{or, } \frac{CG}{GF} = 2 \tag{1.2.4.9}$$

c) From (1.2.1.3) and (1.2.3.6),

$$\mathbf{G} - \mathbf{A} = \begin{pmatrix}-3\\1\end{pmatrix}, \mathbf{D} - \mathbf{G} = \frac{1}{2}\begin{pmatrix}-3\\1\end{pmatrix} \tag{1.2.4.10}$$

$$\mathbf{G} - \mathbf{A} = 2\left(\mathbf{D} - \mathbf{G}\right) \tag{1.2.4.11}$$

$$\implies \|\mathbf{G} - \mathbf{A}\| = 2\,\|\mathbf{D} - \mathbf{G}\| \tag{1.2.4.12}$$

$$\text{or, } \frac{AG}{GD} = 2 \tag{1.2.4.13}$$

From (1.2.4.5), (1.2.4.9), (1.2.4.13)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.14}$$

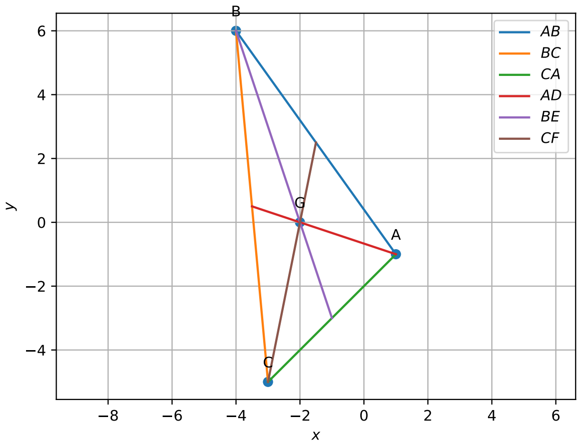

Fig. 2. Medians of $\triangle ABC$ meet at $\mathbf{G}$.

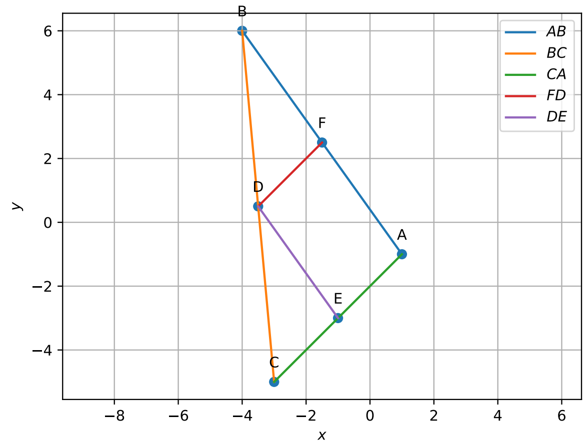

Fig. 3. $AFDE$ forms a parallelogram in triangle ABC

1.2.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear.
Solution: Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \tag{1.2.5.1}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ 0 & -3 & -2 \end{pmatrix} \tag{1.2.5.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & -3 & -2 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - \frac{2}{3}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.5.3}$$

Thus, the matrix (1.2.5.1) has rank 2 and the points are collinear. Thus, the medians of a triangle meet at the point $\mathbf{G}$. See Fig. 2.

1.2.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.2.6.1}$$

$\mathbf{G}$ is known as the centroid of $\triangle ABC$.
Solution:

$$\mathbf{G} = \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \end{pmatrix}}{3} \tag{1.2.6.2}$$

$$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

1.2.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

Solution:

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{-3}{2} \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.2.7.2}$$

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} \frac{-7}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{-7}{2} \end{pmatrix} \tag{1.2.7.3}$$

$$\implies \mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.4}$$

See Fig. 3,
All codes for this section are available in

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codes/triangle/medians.py
codes/triangle/pgm.py
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1.3 Altitude

1.3.1. $\mathbf{D}_1$ is a point on BC such that

$$AD_1 \perp BC \tag{1.3.1.1}$$

and AD_1 is defined to be the altitude. Find the normal vector of AD_1.
Solution: The normal vector of AD_1 is the direction vector BC and is obtained from (1.1.1.3) as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \tag{1.3.1.2}$$

1.3.2. Find the equation of AD_1.
Solution: The equation of AD_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{1.3.2.1}$$

$$\implies \begin{pmatrix} -1 & 11 \end{pmatrix}\mathbf{x} = \begin{pmatrix} -1 & 11 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -12 \tag{1.3.2.2}$$

1.3.3. Find the equations of the altitudes BE_1 and CF_1 to the sides AC and AB respectively.

Solution:

a) From (1.1.1.4), the normal vector of CF_1 is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \tag{1.3.3.1}$$

and the equation of CF_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.3.3.2}$$

$$\implies \begin{pmatrix} -5 & 7 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} \right) = 0 \tag{1.3.3.3}$$

$$\implies \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = 20, \tag{1.3.3.4}$$

b) Similarly, from (1.1.1.2), the normal vector of BE_1 is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.3.3.5}$$

and the equation of BE_1 is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.3.3.6}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right) = 0 \tag{1.3.3.7}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \tag{1.3.3.8}$$

1.3.4. Find the intersection $\mathbf{H}$ of BE_1 and CF_1.

Solution: The intersection of (1.3.3.8) and (1.3.3.4), is obtained from the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 5 & -7 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} \tag{1.3.4.1}$$

which can be solved as

$$\begin{pmatrix} 1 & 1 & 2 \\ 5 & -7 & 20 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -12 & 10 \end{pmatrix} \tag{1.3.4.2}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{-12}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{17}{6} \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \tag{1.3.4.3}$$

yielding

$$\mathbf{H} = \frac{1}{6} \begin{pmatrix} 17 \\ -5 \end{pmatrix}, \tag{1.3.4.4}$$

See Fig. 4

1.3.5. Verify that

$$(\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = 0 \tag{1.3.5.1}$$

Solution: From (1.3.4.4),

$$\mathbf{A} - \mathbf{H} = -\frac{1}{6} \begin{pmatrix} 11 \\ 1 \end{pmatrix}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \tag{1.3.5.2}$$

$$\implies (\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = \frac{1}{6} \begin{pmatrix} 11 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix} = 0 \tag{1.3.5.3}$$

All codes for this section are available at

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codes/triangle/altitude.py
```

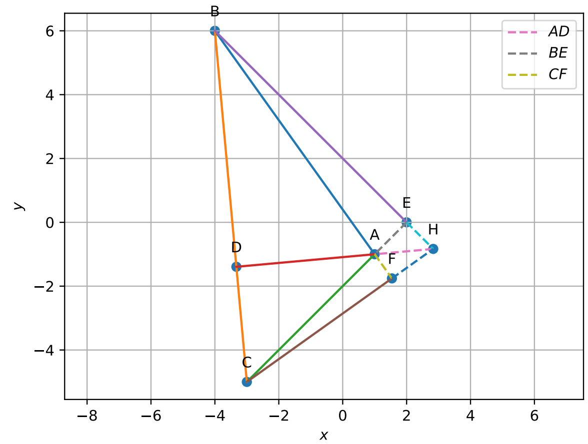

Fig. 4. Altitudes BE_1 and CF_1 intersect at $\mathbf{H}$

1.4 Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2} \right) (\mathbf{B} - \mathbf{C}) = 0 \tag{1.4.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.
Solution: From (1.1.1.2), (1.1.1.3), (1.1.1.4), (1.2.1.3), (1.2.1.4) and (1.2.1.5),

$$\frac{\mathbf{B} + \mathbf{C}}{\mathbf{2}} = \frac{1}{2} \begin{pmatrix} -7 \\ 1 \end{pmatrix}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \tag{1.4.1.2}$$

$$\frac{\mathbf{A} + \mathbf{B}}{\mathbf{2}} = \frac{1}{2} \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix} \tag{1.4.1.3}$$

$$\frac{\mathbf{C} + \mathbf{A}}{\mathbf{2}} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{1.4.1.4}$$

(1.4.1.5)

yielding

$$(\mathbf{B} - \mathbf{C})^\top \left(\frac{\mathbf{B} + \mathbf{C}}{2} \right) = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} = 9 \tag{1.4.1.6}$$

$$(\mathbf{A} - \mathbf{B})^\top \left(\frac{\mathbf{A} + \mathbf{B}}{2} \right) = \begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix} = -25 \tag{1.4.1.7}$$

$$(\mathbf{C} - \mathbf{A})^\top \left(\frac{\mathbf{C} + \mathbf{A}}{2} \right) = \begin{pmatrix} -4 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 16 \tag{1.4.1.8}$$

Thus, the perpendicular bisectors are obtained from (1.4.1.1) as

$$BC: \quad \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = 9 \tag{1.4.1.9}$$

$$CA: \quad \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = -25 \tag{1.4.1.10}$$

$$AB: \quad \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -4 \tag{1.4.1.11}$$

1.4.2. Find the intersection $\mathbf{O}$ of the perpendicular bisectors of AB and AC.
Solution:

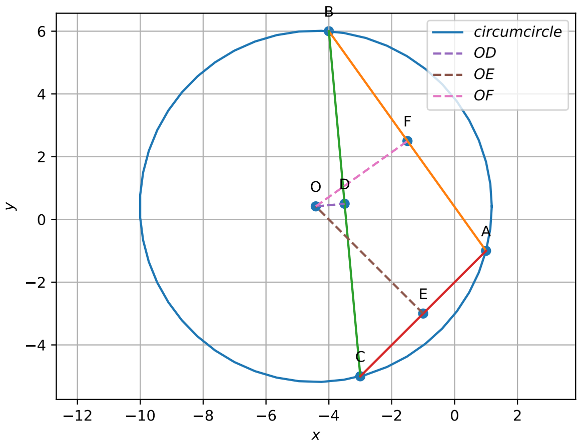

Fig. 5. Circumcircle of $\triangle ABC$ with centre $\mathbf{O}$.

The intersection of (1.4.1.10) and (1.4.1.11), can be obtained as

$$\begin{pmatrix} 5 & -7 & -25 \\ 1 & 1 & -4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} 5 & -7 & -25 \\ 0 & 12 & 5 \end{pmatrix} \tag{1.4.2.1}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{12}{7}R_1 + R_2} \begin{pmatrix} \frac{60}{7} & 0 & \frac{-265}{7} \\ 0 & 12 & 5 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{7}{60}R_1]{R_2 \leftarrow \frac{1}{12}R_2} \begin{pmatrix} 1 & 0 & \frac{-53}{12} \\ 0 & 1 & \frac{5}{12} \end{pmatrix} \tag{1.4.2.2}$$

$$\implies \mathbf{O} = \begin{pmatrix} \frac{-53}{12} \\ \frac{5}{12} \end{pmatrix} \tag{1.4.2.3}$$

1.4.3. Verify that $\mathbf{O}$ satisfies (1.4.1.1). $\mathbf{O}$ is known as the circumcentre.
Solution: Substituing from (1.4.2.3) in (1.4.1.1), when substituted in the above equation,

$$\left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)^\top (\mathbf{B} - \mathbf{C})$$
$$= \left(\frac{1}{12}\begin{pmatrix} -53 \\ 5 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -7 \\ 1 \end{pmatrix}\right)^\top \begin{pmatrix} -1 \\ 11 \end{pmatrix}$$
$$= \frac{1}{12}\begin{pmatrix} -11 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 11 \end{pmatrix} = 0 \tag{1.4.3.1}$$

1.4.4. Verify that

$$OA = OB = OC \tag{1.4.4.1}$$

1.4.5. Draw the circle with centre at $\mathbf{O}$ and radius

$$R = OA \tag{1.4.5.1}$$

This is known as the <u>circumradius</u>.
Solution: See Fig. 5.

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \tag{1.4.6.1}$$

Solution:

a) To find the value of $\angle BOC$:

$$\mathbf{B} - \mathbf{O} = \begin{pmatrix} \frac{5}{12} \\ \frac{67}{12} \end{pmatrix}, \mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{17}{12} \\ \frac{-65}{12} \end{pmatrix} \tag{1.4.6.2}$$

$$\implies (\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O}) = \frac{-4270}{144} \tag{1.4.6.3}$$

$$\implies \|\mathbf{B} - \mathbf{O}\| = \frac{\sqrt{4514}}{12}, \|\mathbf{C} - \mathbf{O}\| = \frac{\sqrt{4514}}{12} \tag{1.4.6.4}$$

Thus,

$$\cos BOC = \frac{(\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O})}{\|\mathbf{B} - \mathbf{O}\| \|\mathbf{C} - \mathbf{O}\|} = \frac{-4270}{4514} \tag{1.4.6.5}$$

$$\implies \angle BOC = \cos^{-1}\left(\frac{-4270}{4514}\right) \tag{1.4.6.6}$$

$$= 161.07536^\circ \text{ or } 198.92464^\circ \tag{1.4.6.7}$$

b) To find the value of $\angle BAC$:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \tag{1.4.6.8}$$

$$\implies (\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = -8 \tag{1.4.6.9}$$

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{74} \, \|\mathbf{C} - \mathbf{A}\| = 4\sqrt{2} \tag{1.4.6.10}$$

Thus,

$$\cos BAC = \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} = \frac{-8}{4\sqrt{148}} \tag{1.4.6.11}$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{-8}{4\sqrt{148}}\right) \tag{1.4.6.12}$$

$$= 99.46232^\circ \tag{1.4.6.13}$$

From (1.4.6.13) and (1.4.6.7),

$$2 \times \angle BAC = \angle BOC \tag{1.4.6.14}$$

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.4.7.1}$$

where

$$\theta = \angle BOC \tag{1.4.7.2}$$

Verify that

$$\mathbf{B} - \mathbf{O} = \mathbf{P}(\mathbf{C} - \mathbf{O}) \tag{1.4.7.3}$$

All codes for this section are available at

codes/triangle/perp−bisect.py

1.5 Angle Bisector

1.5.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \tag{1.5.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Problem 1.1.2.

Solution: From the given information,

$$a = m + n, \tag{1.5.1.2}$$
$$b = n + p, \tag{1.5.1.3}$$
$$c = m + p \tag{1.5.1.4}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.5}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.6}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.5.1.7}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.8}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.9}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.5.1.10}$$

Therefore,

$$\begin{aligned} p &= \frac{c + b - a}{2} = \frac{\sqrt{74} + \sqrt{32} - \sqrt{122}}{2} \\ m &= \frac{a + c - b}{2} = \frac{\sqrt{74} + \sqrt{122} - \sqrt{32}}{2} \\ n &= \frac{a + b - c}{2} = \frac{\sqrt{122} + \sqrt{32} - \sqrt{74}}{2} \end{aligned} \tag{1.5.1.11}$$

upon substituting from (1.1.2.7), (1.1.2.9) and (1.1.2.11).

1.5.2. Using section formula, find

$$\mathbf{D}_3 = \frac{m\mathbf{C} + n\mathbf{B}}{m + n}, \mathbf{E}_3 = \frac{n\mathbf{A} + p\mathbf{C}}{n + p}, \mathbf{F}_3 = \frac{p\mathbf{B} + m\mathbf{A}}{p + m} \tag{1.5.2.1}$$

1.5.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the incentre and inradius of $\triangle ABC$.

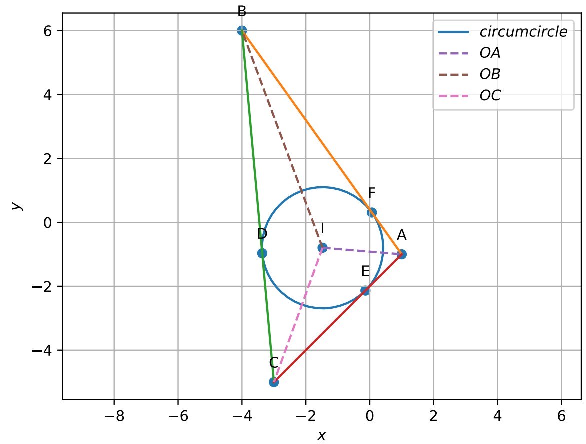

Fig. 6. Incircle of $\triangle ABC$

1.5.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the incircle of $\triangle ABC$.

Solution: See Fig. 6

1.5.5. Using (1.1.7.1) verify that

$$\angle BAI = \angle CAI. \tag{1.5.5.1}$$

AI is the bisector of $\angle A$.

1.5.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.

All codes for this section are available at

codes/triangle/ang−bisect.py

1.6 Eigenvalues and Eigenvectors

The equation of the incircle is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1.6.1}$$

where

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\| - r^2, \tag{1.6.2}$$

$\mathbf{O}$ being the incentre and r the inradius. Here $\mathbf{I}$ is the identity matrix.

1.6.1. Compute

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\mathbf{V} \tag{1.6.1.1}$$

for $\mathbf{h} = \mathbf{A}$.

1.6.2. Find the roots of the equation

$$\left|\lambda\mathbf{I} - \mathbf{\Sigma}\right| = 0 \tag{1.6.2.1}$$

These are known as the eigenvalues of $\mathbf{\Sigma}$.

1.6.3. Find $\mathbf{p}$ such that

$$\mathbf{\Sigma}\mathbf{p} = \lambda\mathbf{p} \tag{1.6.3.1}$$

using row reduction. These are known as the eigenvectors of Σ.

1.6.4. Define

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{1.6.4.1}$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} \end{pmatrix} \tag{1.6.4.2}$$

1.6.5. Verify that

$$\mathbf{P}^\top = \mathbf{P}^{-1}. \tag{1.6.5.1}$$

$\mathbf{P}$ is defined to be an orthogonal matrix.

1.6.6. Verify that

$$\mathbf{P}^\top \mathbf{\Sigma} \mathbf{P} = \mathbf{D}, \tag{1.6.6.1}$$

This is known as the spectral (eigenvalue) decomposition of a symmetric matrix

1.6.7. The direction vectors of the tangents from a point $\mathbf{h}$ to the circle in (1.6.1) are given by

$$\mathbf{m} = \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{1.6.7.1}$$

1.6.8. The points of contact of the pair of tangents to the circle in (1.6.1) from a point $\mathbf{h}$ are given by

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{1.6.8.1}$$

where

$$\mu = -\frac{\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top\mathbf{V}\mathbf{m}} \tag{1.6.8.2}$$

for $\mathbf{m}$ in (1.6.7.1). Compute the points of contact. You should get the same points that you obtained in the previous section.

All codes for this section are available at

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codes/triangle/tangpair.py
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2 Matrices

The matrix of the veritices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{8.1}$$

2.1 Vectors

2.1. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{2.1.1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{2.1.1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{2.1.1.3}$$

where the second matrix above is known as a circulant matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

2.2. Obtain the normal matrix of the sides of $\triangle ABC$

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.1.2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{2.1.2.2}$$

2.3. Obtain a, b, c.

Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\operatorname{diag}(\mathbf{M}^\top\mathbf{M})} \tag{2.1.3.1}$$

2.4. Obtain the constant terms in the equations of the sides of the triangle.

Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \operatorname{diag}\left\{\left(\mathbf{N}^\top\mathbf{P}\right)\right\} \tag{2.1.4.1}$$

2.2 Median

2.2.1. Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.2.1.1}$$

2.2.2. Obtain the median direction matrix.

Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F} \end{pmatrix} \tag{2.2.2.1}$$

$$= \begin{pmatrix} \mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \tag{2.2.2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{2.2.2.3}$$

2.2.3. Obtain the median normal matrix.

2.2.4. Obtian the median equation constants.

2.2.5. Obtain the centroid by finding the intersection of the medians.

2.3 Altitude

2.3.1. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B} \end{pmatrix} \tag{2.3.1.1}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{2.3.1.2}$$

2.3.2. Find the constants vector for the altitudes.

Solution: The desired vector is

$$\mathbf{c}_2 = \operatorname{diag}\left\{\left(\mathbf{M}^\top\mathbf{P}\right)\right\} \tag{2.3.2.1}$$

2.4 Perpendicular Bisector

2.4.1. Find the normal matrix for the perpendicular bisectors

Solution: The normal matrix is $\mathbf{M}_2$

2.4.2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \operatorname{diag}\left\{\mathbf{M}_2^\top\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right\} \tag{2.4.2.1}$$

2.5 Angle Bisector

2.5.1. Find the points of contact.
Solution: The points of contact are given by

$$\begin{pmatrix}\frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}0 & \frac{n}{b} & \frac{m}{c} \\ \frac{n}{a} & 0 & \frac{p}{c} \\ \frac{m}{a} & \frac{p}{b} & 0\end{pmatrix} \tag{2.5.1.1}$$

Appendix A
Points on a Line

A.1. The equation of a line is given by

$$y = mx + c \tag{A.1.1}$$

$$\implies \begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}x \\ mx + c\end{pmatrix} = \begin{pmatrix}0 \\ c\end{pmatrix} + x\begin{pmatrix}1 \\ m\end{pmatrix} \tag{A.1.2}$$

yielding (1.1.4.1).

A.2. (A.1.1) can also be expressed as

$$y - mx = c \tag{A.2.1}$$

$$\implies \begin{pmatrix}-m & 1\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = c \tag{A.2.2}$$

yielding (1.1.5.1).

A.3. From (1.1.4.1), if $\mathbf{A}, \mathbf{D}$ and $\mathbf{C}$ are on the same line,

$$\mathbf{D} = \mathbf{A} + q\mathbf{m} \tag{A.3.1}$$

$$\mathbf{C} = \mathbf{D} + p\mathbf{m} \tag{A.3.2}$$

$$\implies p(\mathbf{D} - \mathbf{A}) + q(\mathbf{D} - \mathbf{C}) = 0, \quad p, q \neq 0 \tag{A.3.3}$$

$$\implies \mathbf{D} = \frac{p\mathbf{A} + q\mathbf{C}}{p + q} \tag{A.3.4}$$

yielding (1.2.1.1) upon substituting

$$k = \frac{p}{q}. \tag{A.3.5}$$

$(\mathbf{D} - \mathbf{A}), (\mathbf{D} - \mathbf{C})$ are then said to be <u>linearly dependent</u>.

A.4. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear, from (1.1.5.1),

$$\mathbf{n}^\top \mathbf{A} = c \tag{A.4.1}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{A.4.2}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{A.4.3}$$

which can be expressed as

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top \mathbf{n} = c\begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix} \tag{A.4.4}$$

$$\implies \begin{pmatrix}1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top \begin{pmatrix}\mathbf{n} \\ -c\end{pmatrix} = \mathbf{0} \tag{A.4.5}$$

yielding (1.1.3.1). Rank is defined to be the number of linearly indpendent rows or columns of a matrix.

A.5. Consequently, points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{C} - \mathbf{B}) \tag{A.5.1}$$

$$= (p + q)\mathbf{B} - p\mathbf{A} - q\mathbf{C} = 0 \tag{A.5.2}$$

$$\implies p = 0, q = 0 \tag{A.5.3}$$

A.6. In Fig. 7

$$AF = BF, AE = BE, \tag{A.6.1}$$

and the medians BE and CF meet at $\mathbf{G}$. Show that

$$\frac{GB}{GE} = \frac{GC}{GF} = 2 \tag{A.6.2}$$

Solution: From (1.2.1.1),

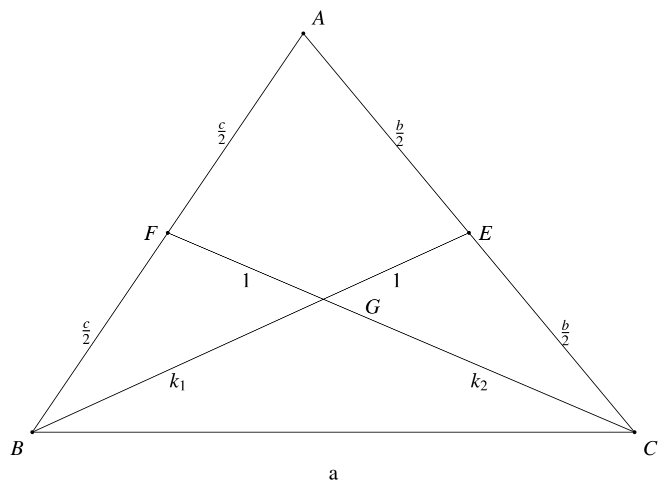

Fig. 7. $k_1 = k_2 = 2$.

$$\mathbf{G} = \frac{k_1\mathbf{E} + \mathbf{B}}{k_1 + 1} = \frac{k_2\mathbf{F} + \mathbf{C}}{k_2 + 1} \tag{A.6.3}$$

$$\implies \frac{k_1\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right) + \mathbf{B}}{k_1 + 1} = \frac{k_2\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) + \mathbf{C}}{k_2 + 1} \tag{A.6.4}$$

$$\implies (k_2 + 1)\{k_1(\mathbf{A} + \mathbf{C}) + 2\mathbf{B}\} = (k_1 + 1)\{k_2(\mathbf{A} + \mathbf{B}) + 2\mathbf{C}\} \tag{A.6.5}$$

which can be expressed as

$$\{2 + k_2 - k_1k_2\}\mathbf{B} - (k_2 - k_1)\mathbf{A} - \{k_1 + 2 - k_1k_2\}\mathbf{C} = 0 \tag{A.6.6}$$

and is of the form (A.5.3) with

$$p = k_2 - k_1, q = k_1 + 2 - k_1k_2. \tag{A.6.7}$$

Thus, from (A.5.3)

$$k_2 - k_1 = 0, \tag{A.6.8}$$

$$k_1 + 2 - k_1k_2 = 0 \tag{A.6.9}$$

Thus, from (A.6.9)

$$k_1 = k_2 \tag{A.6.10}$$

and substituting the above in (A.6.9) results in the quadratic

$$k_1^2 - k_1 - 2 = 0 \tag{A.6.11}$$

$$\implies (k_1 - 2)(k_1 + 1) = 0 \tag{A.6.12}$$

admitting $k_1 = k_2 = 2$ as the only possible solution.

A.7. Substituting $k_1 = 2$ in (A.6.3)

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{A.7.1}$$

A.8. In Fig. 8, AG is extended to join BC at $\mathbf{D}$. Show that AD is also a median.

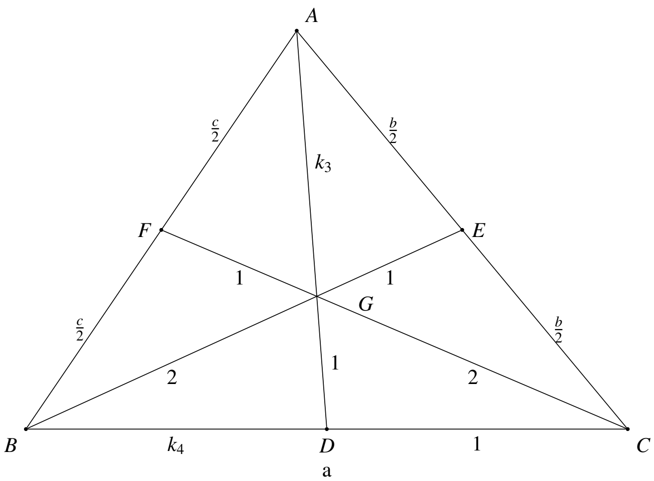

Fig. 8. $k_3 = 2, k_4 = 1$

Solution: Considering the ratios in Fig. 8,

$$\mathbf{G} = \frac{k_3\mathbf{D} + \mathbf{A}}{k_3 + 1} \tag{A.8.1}$$

$$\mathbf{D} = \frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1} \tag{A.8.2}$$

Substituting from (A.7.1) in the above,

$$(k_3 + 1)\left(\frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3}\right) = k_3\left(\frac{k_4\mathbf{C} + \mathbf{B}}{k_4 + 1}\right) + \mathbf{A} \tag{A.8.3}$$

$$\implies (k_3 + 1)(k_4 + 1)(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 3\{k_3(k_4\mathbf{C} + \mathbf{B}) + (k_4 + 1)\mathbf{A}\} \tag{A.8.4}$$

which can be expressed as

$$\begin{aligned}(k_3k_4 + k_3 - 2k_4 - 2)\,\mathbf{A}\\ - (-k_3k_4 - k_4 + 2k_3 - 1)\,\mathbf{B}\\ - (-k_3 - k_4 - 1 + 2k_3k_4)\,\mathbf{C} = \mathbf{0}\end{aligned} \tag{A.8.5}$$

Comparing the above with (A.5.3),

$$p = -k_3k_4 - k_4 + 2k_3 - 1, q = -k_3 - k_4 - 1 + 2k_3k_4 \tag{A.8.6}$$

yielding

$$-k_3k_4 - k_4 + 2k_3 - 1 = 0 \tag{A.8.7}$$

$$-k_3 - k_4 - 1 + 2k_3k_4 = 0 \tag{A.8.8}$$

Subtracting (A.8.7) from (A.8.8),

$$3k_3(k_4 - 1) = 0 \tag{A.8.9}$$

$$\implies k_4 = 1 \tag{A.8.10}$$

which upon substituting in (A.8.7) yields

$$k_3 = 2 \tag{A.8.11}$$

Appendix B
Tangents to a Circle

The equation of the incircle is given by

$$\|\mathbf{x} - \mathbf{O}\|^2 = r^2 \tag{B.1}$$

which can be expressed as (1.6.1) using (1.6.2). In Fig. 6, Let (1.6.8.1) be the equation of AB. Then, the intersection of (1.6.8.1) and (1.6.1) can be expressed as

$$(\mathbf{h} + \mu\mathbf{m})^\top \mathbf{V}(\mathbf{h} + \mu\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h} + \mu\mathbf{m}) + f = 0 \tag{B.2}$$

$$\implies \mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{B.3}$$

For (B.3) to have exactly one root, the discriminant

$$\left\{\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\right\}^2 - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{B.4}$$

and (1.6.8.2) is obtained. (B.4) can be expressed as

$$\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\,\mathbf{m} - g(\mathbf{h})\,\mathbf{m}^\top\mathbf{V}\mathbf{m} = 0 \tag{B.5}$$

$$\implies \mathbf{m}^\top\mathbf{\Sigma}\mathbf{m} = 0 \tag{B.6}$$

for $\mathbf{\Sigma}$ defined in (B.6). Substituting (1.6.6.1) in (B.6),

$$\mathbf{m}^\top\mathbf{P}\mathbf{D}\mathbf{P}^\top\mathbf{m} = 0 \tag{B.7}$$

$$\implies \mathbf{v}^\top\mathbf{D}\mathbf{v} = 0 \tag{B.8}$$

where

$$\mathbf{v} = \mathbf{P}^\top\mathbf{m} \tag{B.9}$$

(B.8) can be expressed as

$$\lambda_1 v_1^2 - \lambda_2 v_2^2 = 0 \tag{B.10}$$

$$\implies \mathbf{v} = \begin{pmatrix}\sqrt{|\lambda_2|}\\ \pm\sqrt{|\lambda_1|}\end{pmatrix} \tag{B.11}$$

after some algebra. From (B.11) and (B.9) we obtain (1.6.7.1).