

# MATRICES

## In Geometry

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## 1 VECTORS

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \quad (1)$$

## 1.1 Sides

1.1.1. The direction vector of  $AB$  is defined as

$$\mathbf{B} - \mathbf{A} \quad (1.1.1.1)$$

Find the direction vectors of  $AB, BC$  and  $CA$ .

**Solution:**

a) The Direction vector of  $AB$  is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 - 1 \\ 6 - (-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \quad (1.1.1.2)$$

b) The Direction vector of  $BC$  is

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 - (-4) \\ -5 - 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \quad (1.1.1.3)$$

c) The Direction vector of  $CA$  is

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 - (-3) \\ -1 - (-5) \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.1.1.4)$$

1.1.2. The length of side  $BC$  is

$$c = \|\mathbf{B} - \mathbf{A}\| \triangleq \sqrt{(\mathbf{B} - \mathbf{A})^T (\mathbf{B} - \mathbf{A})} \quad (1.1.2.1)$$

where

$$\mathbf{A}^T \triangleq (1 \quad -1) \quad (1.1.2.2)$$

Similarly,

$$b = \|\mathbf{C} - \mathbf{B}\|, a = \|\mathbf{A} - \mathbf{C}\| \quad (1.1.2.3)$$

Find  $a, b, c$ .

a) From (1.1.1.2),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \quad (1.1.2.4)$$

$$\Rightarrow c = \|\mathbf{B} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \quad (1.1.2.5)$$

$$= \sqrt{\begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix}} = \sqrt{(5)^2 + (7)^2} \quad (1.1.2.6)$$

$$= \sqrt{74} \quad (1.1.2.7)$$

b) Similarly, from (1.1.1.3),

$$a = \|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix}} \quad (1.1.2.8)$$

$$= \sqrt{(1)^2 + (11)^2} = \sqrt{122} \quad (1.1.2.9)$$

and from (1.1.1.4),

c)

$$b = \|\mathbf{A} - \mathbf{C}\| = \sqrt{\begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}} \quad (1.1.2.10)$$

$$= \sqrt{(4)^2 + (4)^2} = \sqrt{32} \quad (1.1.2.11)$$

1.1.3. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \quad (1.1.3.1)$$

Are the given points in (1) collinear?

**Solution:** From (1),

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ -1 & 6 & -5 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -4 & -3 \\ 0 & 2 & -8 \end{pmatrix} \quad (1.1.3.2)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 2 & -8 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - \frac{2}{5}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & -\frac{48}{5} \end{pmatrix} \quad (1.1.3.3)$$

There are no zero rows. So,

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \quad (1.1.3.4)$$

Hence, the points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are not collinear. This is visible in Fig. 1.1.3.



Fig. 1.1.3:  $\triangle ABC$

1.1.4. The parametric form of the equation of  $AB$  is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \quad k \neq 0, \quad (1.1.4.1)$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \quad (1.1.4.2)$$

is the direction vector of  $AB$ . Find the parametric equations of  $AB, BC$  and  $CA$ .

**Solution:** From (1.1.4.1) and (1.1.1.2), the parametric

equation for  $AB$  is given by

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k \begin{pmatrix} -5 \\ 7 \end{pmatrix} \quad (1.1.4.3)$$

Similarly, from (1.1.1.3) and (1.1.1.4),

$$BC : \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + k \begin{pmatrix} 1 \\ -11 \end{pmatrix} \quad (1.1.4.4)$$

$$CA : \mathbf{x} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + k \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.1.4.5)$$

1.1.5. The normal form of the equation of  $AB$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (1.1.5.1)$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \quad (1.1.5.2)$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \quad (1.1.5.3)$$

Find the normal form of the equations of  $AB$ ,  $BC$  and  $CA$ .

**Solution:**

a) From (1.1.1.3), the direction vector of side  $BC$  is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \quad (1.1.5.4)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -11 \end{pmatrix} = \begin{pmatrix} -11 \\ -1 \end{pmatrix} \quad (1.1.5.5)$$

from (1.1.5.3). Hence, from (1.1.5.1), the normal equation of side  $BC$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \quad (1.1.5.6)$$

$$\Rightarrow \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} \quad (1.1.5.7)$$

$$\Rightarrow BC : \begin{pmatrix} 11 & 1 \end{pmatrix} \mathbf{x} = -38 \quad (1.1.5.8)$$

b) Similarly, for  $AB$ , from (1.1.1.2),

$$\mathbf{m} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \quad (1.1.5.9)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \quad (1.1.5.10)$$

and

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (1.1.5.11)$$

$$\Rightarrow AB : \mathbf{n}^\top \mathbf{x} = \begin{pmatrix} 7 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.1.5.12)$$

$$\Rightarrow \begin{pmatrix} 7 & -1 \end{pmatrix} \mathbf{x} = 2 \quad (1.1.5.13)$$

c) For  $CA$ , from (1.1.1.4),

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.1.5.14)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.1.5.15)$$

$$(1.1.5.16)$$

$$\Rightarrow \mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \quad (1.1.5.17)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = 2 \quad (1.1.5.18)$$

1.1.6. The area of  $\triangle ABC$  is defined as

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \quad (1.1.6.1)$$

where

$$\mathbf{A} \times \mathbf{B} \triangleq \begin{vmatrix} 1 & -4 \\ -1 & 6 \end{vmatrix} \quad (1.1.6.2)$$

Find the area of  $\triangle ABC$ .

**Solution:** From (1.1.1.2) and (1.1.1.4),

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}, \mathbf{A} - \mathbf{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (1.1.6.3)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix} 5 & 4 \\ -7 & 4 \end{vmatrix} \quad (1.1.6.4)$$

$$= 5 \times 4 - 4 \times (-7) \quad (1.1.6.5)$$

$$= 48 \quad (1.1.6.6)$$

$$\Rightarrow \frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| = \frac{48}{2} = 24 \quad (1.1.6.7)$$

which is the desired area.

1.1.7. Find the angles  $A, B, C$  if

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} \quad (1.1.7.1)$$

**Solution:**

a) From (1.1.1.2), (1.1.1.4), (1.1.2.7) and (1.1.2.11)

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -5 & 7 \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} \quad (1.1.7.2)$$

$$= -8 \quad (1.1.7.3)$$

$$\Rightarrow \cos A = \frac{-8}{\sqrt{74} \sqrt{32}} = \frac{-1}{\sqrt{37}} \quad (1.1.7.4)$$

$$\Rightarrow A = \cos^{-1} \frac{-1}{\sqrt{37}} \quad (1.1.7.5)$$

b) From (1.1.1.2), (1.1.1.3), (1.1.2.7) and (1.1.2.9)

$$(\mathbf{C} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 & -11 \end{pmatrix} \begin{pmatrix} 5 \\ -7 \end{pmatrix} \quad (1.1.7.6)$$

$$= 82 \quad (1.1.7.7)$$

$$\Rightarrow \cos B = \frac{82}{\sqrt{74} \sqrt{122}} = \frac{41}{\sqrt{2257}} \quad (1.1.7.8)$$

$$\Rightarrow B = \cos^{-1} \frac{41}{\sqrt{2257}} \quad (1.1.7.9)$$

c) From (1.1.1.3), (1.1.1.4), (1.1.2.9) and (1.1.2.11)

$$(\mathbf{A} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 4 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix} \quad (1.1.7.10)$$

$$= 40 \quad (1.1.7.11)$$

$$\Rightarrow \cos C = \frac{40}{\sqrt{32} \sqrt{122}} = \frac{5}{\sqrt{61}} \quad (1.1.7.12)$$

$$\Rightarrow C = \cos^{-1} \frac{5}{\sqrt{61}} \quad (1.1.7.13)$$

All codes for this section are available at

codes/triangle/sides.py

## 1.2 Median

1.2.1. If  $\mathbf{D}$  divides  $BC$  in the ratio  $k : 1$ ,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k + 1} \quad (1.2.1.1)$$

Find the mid points  $\mathbf{D}, \mathbf{E}, \mathbf{F}$  of the sides  $BC, CA$  and  $AB$  respectively.

**Solution:** Since  $\mathbf{D}$  is the midpoint of  $BC$ ,

$$k = 1, \quad (1.2.1.2)$$

$$\Rightarrow \mathbf{D} = \frac{\mathbf{C} + \mathbf{B}}{2} = \frac{1}{2} \begin{pmatrix} -7 \\ 1 \end{pmatrix} \quad (1.2.1.3)$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \quad (1.2.1.4)$$

$$\mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} = \frac{1}{2} \begin{pmatrix} -3 \\ 5 \end{pmatrix} \quad (1.2.1.5)$$

1.2.2. Find the equations of  $AD, BE$  and  $CF$ .

**Solution:**

a) The direction vector of  $AD$  is

$$\mathbf{m} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -9 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.2.1)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (1.2.2.2)$$

Hence the normal equation of median  $AD$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (1.2.2.3)$$

$$\Rightarrow \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \quad (1.2.2.4)$$

b) For  $BE$ ,

$$\mathbf{m} = \mathbf{E} - \mathbf{B} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (1.2.2.5)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (1.2.2.6)$$

Hence the normal equation of median  $BE$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \quad (1.2.2.7)$$

$$\Rightarrow \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 6 \end{pmatrix} = -6 \quad (1.2.2.8)$$

c) For median  $CF$ ,

$$\mathbf{m} = \mathbf{F} - \mathbf{C} = \begin{pmatrix} -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{15}{2} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.2.2.9)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad (1.2.2.10)$$

Hence the normal equation of median  $CF$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \quad (1.2.2.11)$$

$$\Rightarrow \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = -10 \quad (1.2.2.12)$$

1.2.3. Find the intersection  $\mathbf{G}$  of  $BE$  and  $CF$ .

**Solution:** From (1.2.2.8) and (1.2.2.12), the equations of

$BE$  and  $CF$  are, respectively,

$$\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 \end{pmatrix} \quad (1.2.3.1)$$

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -10 \end{pmatrix} \quad (1.2.3.2)$$

From (1.2.3.1) and (1.2.3.2) the augmented matrix is

$$\begin{pmatrix} 3 & 1 & -6 \\ 5 & -1 & -10 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 8 & 0 & -16 \\ 5 & -1 & -10 \end{pmatrix} \quad (1.2.3.3)$$

$$\xrightarrow{R_1 \leftarrow R_1/8} \begin{pmatrix} 1 & 0 & -2 \\ 5 & -1 & -10 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \quad (1.2.3.4)$$

$$\xrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \quad (1.2.3.5)$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (1.2.3.6)$$

1.2.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \quad (1.2.4.1)$$

**Solution:**

a) From (1.2.1.4) and (1.2.3.6),

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \mathbf{E} - \mathbf{G} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (1.2.4.2)$$

$$\Rightarrow \mathbf{G} - \mathbf{B} = 2(\mathbf{E} - \mathbf{G}) \quad (1.2.4.3)$$

$$\Rightarrow \|\mathbf{G} - \mathbf{B}\| = 2\|\mathbf{E} - \mathbf{G}\| \quad (1.2.4.4)$$

$$\text{or, } \frac{BG}{GE} = 2 \quad (1.2.4.5)$$

b) From (1.2.1.5) and (1.2.3.6),

$$\mathbf{F} - \mathbf{G} = \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \mathbf{G} - \mathbf{C} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad (1.2.4.6)$$

$$\Rightarrow \mathbf{G} - \mathbf{C} = 2(\mathbf{F} - \mathbf{G}) \quad (1.2.4.7)$$

$$\Rightarrow \|\mathbf{G} - \mathbf{C}\| = 2\|\mathbf{F} - \mathbf{G}\| \quad (1.2.4.8)$$

$$\text{or, } \frac{CG}{GF} = 2 \quad (1.2.4.9)$$

c) From (1.2.1.3) and (1.2.3.6),

$$\mathbf{G} - \mathbf{A} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \mathbf{D} - \mathbf{G} = \frac{1}{2} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (1.2.4.10)$$

$$\mathbf{G} - \mathbf{A} = 2(\mathbf{D} - \mathbf{G}) \quad (1.2.4.11)$$

$$\Rightarrow \|\mathbf{G} - \mathbf{A}\| = 2\|\mathbf{D} - \mathbf{G}\| \quad (1.2.4.12)$$

$$\text{or, } \frac{AG}{GD} = 2 \quad (1.2.4.13)$$

From (1.2.4.5), (1.2.4.9), (1.2.4.13)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \quad (1.2.4.14)$$

1.2.5. Show that  $\mathbf{A}, \mathbf{G}$  and  $\mathbf{D}$  are collinear.

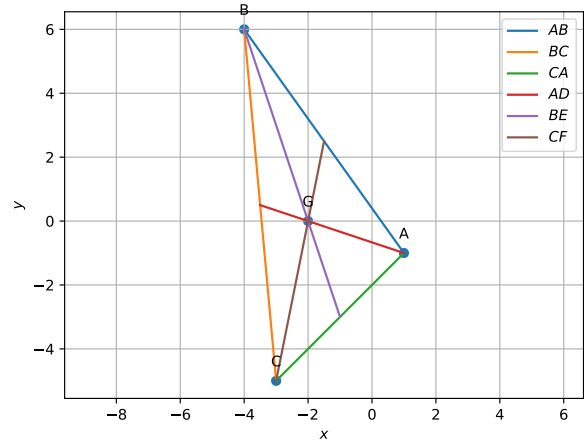


Fig. 1.2.5: Medians of  $\triangle ABC$  meet at  $\mathbf{G}$ .

**Solution:** Points  $\mathbf{A}, \mathbf{D}, \mathbf{G}$  are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \quad (1.2.5.1)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{7}{2} & -2 \\ 0 & -3 & -2 \end{pmatrix} \quad (1.2.5.2)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & -3 & -2 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - \frac{2}{3}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9}{2} & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad (1.2.5.3)$$

Thus, the matrix (1.2.5.1) has rank 2 and the points are collinear. Thus, the medians of a triangle meet at the point  $\mathbf{G}$ . See Fig. 1.2.5.

1.2.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \quad (1.2.6.1)$$

$\mathbf{G}$  is known as the *centroid* of  $\triangle ABC$ .

**Solution:**

$$\mathbf{G} = \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \end{pmatrix}}{3} \quad (1.2.6.2)$$

$$= \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

1.2.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \quad (1.2.7.1)$$

The quadrilateral  $AFDE$  is defined to be a parallelogram.

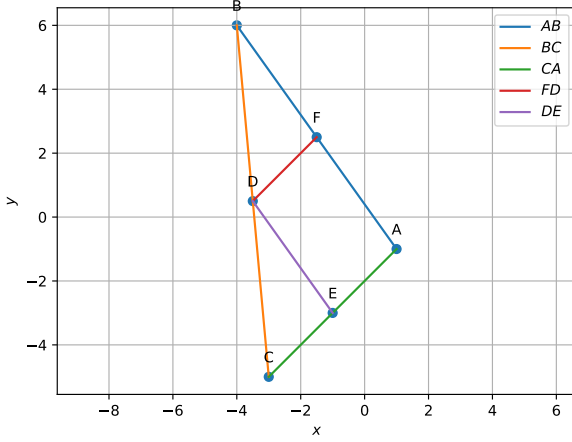


Fig. 1.2.7:  $AFDE$  forms a parallelogram in triangle  $ABC$

**Solution:**

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \end{pmatrix} \quad (1.2.7.2)$$

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} -7 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \end{pmatrix} \quad (1.2.7.3)$$

$$\Rightarrow \mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \quad (1.2.7.4)$$

See Fig. 1.2.7,

All codes for this section are available in

codes/triangle/medians.py  
codes/triangle/pgm.py

### 1.3 Altitude

1.3.1.  $\mathbf{D}_1$  is a point on  $BC$  such that

$$AD_1 \perp BC \quad (1.3.1.1)$$

and  $AD_1$  is defined to be the altitude. Find the normal vector of  $AD_1$ .

**Solution:** The normal vector of  $AD_1$  is the direction vector  $BC$  and is obtained from (1.1.1.3) as

$$\mathbf{n} = \begin{pmatrix} 1 \\ -11 \end{pmatrix} \quad (1.3.1.2)$$

1.3.2. Find the equation of  $AD_1$ .

**Solution:** The equation of  $AD_1$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (1.3.2.1)$$

$$\Rightarrow \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -12 \quad (1.3.2.2)$$

1.3.3. Find the equations of the altitudes  $BE_1$  and  $CF_1$  to the sides  $AC$  and  $AB$  respectively.

**Solution:**

a) From (1.1.1.4), the normal vector of  $CF_1$  is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \quad (1.3.3.1)$$

and the equation of  $CF_1$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \quad (1.3.3.2)$$

$$\Rightarrow \begin{pmatrix} -5 & 7 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} \right) = 0 \quad (1.3.3.3)$$

$$\Rightarrow \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = 20, \quad (1.3.3.4)$$

b) Similarly, from (1.1.1.2), the normal vector of  $BE_1$  is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1.3.3.5)$$

and the equation of  $BE_1$  is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \quad (1.3.3.6)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} -4 \\ 6 \end{pmatrix} \right) = 0 \quad (1.3.3.7)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2, \quad (1.3.3.8)$$

1.3.4. Find the intersection  $\mathbf{H}$  of  $BE_1$  and  $CF_1$ .

**Solution:** The intersection of (1.3.3.8) and (1.3.3.4), is obtained from the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 5 & -7 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} \quad (1.3.4.1)$$

which can be solved as

$$\begin{pmatrix} 1 & 1 & 2 \\ 5 & -7 & 20 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -12 & 10 \end{pmatrix} \quad (1.3.4.2)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{-12}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & \frac{17}{6} \\ 0 & 1 & \frac{-5}{6} \end{pmatrix} \quad (1.3.4.3)$$

yielding

$$\mathbf{H} = \frac{1}{6} \begin{pmatrix} 17 \\ -5 \end{pmatrix}, \quad (1.3.4.4)$$

See Fig. 1.3.4.1

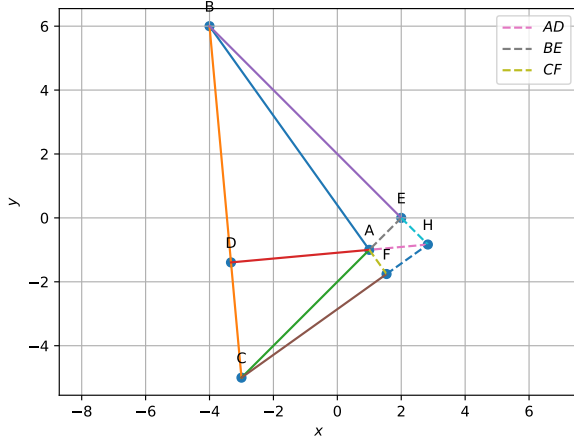


Fig. 1.3.4.1: Altitudes  $BE_1$  and  $CF_1$  intersect at  $\mathbf{H}$

1.3.5. Verify that

$$(\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = 0 \quad (1.3.5.1)$$

**Solution:** From (1.3.4.4),

$$\mathbf{A} - \mathbf{H} = -\frac{1}{6} \begin{pmatrix} 11 \\ 1 \end{pmatrix}, \quad \mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \quad (1.3.5.2)$$

$$\Rightarrow (\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = \frac{1}{6} \begin{pmatrix} 11 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix} = 0 \quad (1.3.5.3)$$

All codes for this section are available at

codes/triangle/altitude.py

## 1.4 Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of  $BC$  is

$$\left( \mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2} \right) (\mathbf{B} - \mathbf{C}) = 0 \quad (1.4.1.1)$$

Substitute numerical values and find the equations of the perpendicular bisectors of  $AB$ ,  $BC$  and  $CA$ .

**Solution:** From (1.1.1.2), (1.1.1.3), (1.1.1.4), (1.2.1.3), (1.2.1.4) and (1.2.1.5),

$$\frac{\mathbf{B} + \mathbf{C}}{2} = \frac{1}{2} \begin{pmatrix} -7 \\ 1 \end{pmatrix}, \quad \mathbf{B} - \mathbf{C} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} \quad (1.4.1.2)$$

$$\frac{\mathbf{A} + \mathbf{B}}{2} = \frac{1}{2} \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \quad \mathbf{A} - \mathbf{B} = \begin{pmatrix} 5 \\ -7 \end{pmatrix} \quad (1.4.1.3)$$

$$\frac{\mathbf{C} + \mathbf{A}}{2} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \quad \mathbf{C} - \mathbf{A} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \quad (1.4.1.4)$$

$$(1.4.1.5)$$

yielding

$$(\mathbf{B} - \mathbf{C})^\top \left( \mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2} \right) = \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -1 & 11 \end{pmatrix} \begin{pmatrix} -7/2 \\ 1/2 \end{pmatrix} = 9 \quad (1.4.1.6)$$

$$(\mathbf{A} - \mathbf{B})^\top \left( \mathbf{x} - \frac{\mathbf{A} + \mathbf{B}}{2} \right) = \begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 5 & -7 \end{pmatrix} \begin{pmatrix} -3/2 \\ 5/2 \end{pmatrix} = -25 \quad (1.4.1.7)$$

$$(\mathbf{C} - \mathbf{A})^\top \left( \mathbf{x} - \frac{\mathbf{C} + \mathbf{A}}{2} \right) = \begin{pmatrix} -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -4 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 16 \quad (1.4.1.8)$$

Thus, the perpendicular bisectors are obtained from (1.4.1.1) as

$$BC: \begin{pmatrix} -1 & 11 \end{pmatrix} \mathbf{x} = 9 \quad (1.4.1.9)$$

$$CA: \begin{pmatrix} 5 & -7 \end{pmatrix} \mathbf{x} = -25 \quad (1.4.1.10)$$

$$AB: \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = -4 \quad (1.4.1.11)$$

1.4.2. Find the intersection  $\mathbf{O}$  of the perpendicular bisectors of  $AB$  and  $AC$ .

**Solution:**

The intersection of (1.4.1.10) and (1.4.1.11), can be obtained as

$$\begin{pmatrix} 5 & -7 & -25 \\ 1 & 1 & -4 \end{pmatrix} \xrightarrow{R_2 \leftarrow 5R_2 - R_1} \begin{pmatrix} 5 & -7 & -25 \\ 0 & 12 & 5 \end{pmatrix} \quad (1.4.2.1)$$

$$\xrightarrow{R_1 \leftarrow \frac{12}{7}R_1 + R_2} \begin{pmatrix} \frac{60}{7} & 0 & \frac{-265}{7} \\ 0 & 12 & 5 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{1}{12}R_2, R_1 \leftarrow \frac{7}{60}R_1} \begin{pmatrix} 1 & 0 & \frac{-53}{12} \\ 0 & 1 & \frac{5}{12} \end{pmatrix} \quad (1.4.2.2)$$

$$\Rightarrow \mathbf{O} = \begin{pmatrix} \frac{-53}{12} \\ \frac{5}{12} \end{pmatrix} \quad (1.4.2.3)$$

1.4.3. Verify that  $\mathbf{O}$  satisfies (1.4.1.1).  $\mathbf{O}$  is known as the circumcentre.

**Solution:** Substituting from (1.4.2.3) in (1.4.1.1), when





Fig. 1.4.5.1: Circumcircle of  $\triangle ABC$  with centre  $O$ .

substituted in the above equation,

$$\begin{aligned} \left( \mathbf{O} - \frac{\mathbf{B} + \mathbf{C}}{2} \right)^T (\mathbf{B} - \mathbf{C}) &= \left( \frac{1}{12} \begin{pmatrix} -53 \\ 5 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -7 \\ 1 \end{pmatrix} \right)^T \begin{pmatrix} -1 \\ 11 \end{pmatrix} \\ &= \frac{1}{12} \begin{pmatrix} -11 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 11 \end{pmatrix} = 0 \end{aligned} \quad (1.4.3.1)$$

1.4.4. Verify that

$$OA = OB = OC \quad (1.4.4.1)$$

1.4.5. Draw the circle with centre at  $O$  and radius

$$R = OA \quad (1.4.5.1)$$

This is known as the *circumradius*.

**Solution:** See Fig. 1.4.5.1.

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \quad (1.4.6.1)$$

**Solution:**

a) To find the value of  $\angle BOC$  :

$$\mathbf{B} - \mathbf{O} = \begin{pmatrix} \frac{5}{12} \\ \frac{17}{12} \end{pmatrix}, \mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{17}{12} \\ \frac{-65}{12} \end{pmatrix} \quad (1.4.6.2)$$

$$\Rightarrow (\mathbf{B} - \mathbf{O})^T (\mathbf{C} - \mathbf{O}) = \frac{-4270}{144} \quad (1.4.6.3)$$

$$\Rightarrow \|\mathbf{B} - \mathbf{O}\| = \frac{\sqrt{4514}}{12}, \|\mathbf{C} - \mathbf{O}\| = \frac{\sqrt{4514}}{12} \quad (1.4.6.4)$$

Thus,

$$\cos BOC = \frac{(\mathbf{B} - \mathbf{O})^T (\mathbf{C} - \mathbf{O})}{\|\mathbf{B} - \mathbf{O}\| \|\mathbf{C} - \mathbf{O}\|} = \frac{-4270}{4514} \quad (1.4.6.5)$$

$$\Rightarrow \angle BOC = \cos^{-1} \left( \frac{-4270}{4514} \right) \quad (1.4.6.6)$$

$$= 161.07536^\circ \text{ or } 198.92464^\circ \quad (1.4.6.7)$$

b) To find the value of  $\angle BAC$  :

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}, \mathbf{C} - \mathbf{A} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} \quad (1.4.6.8)$$

$$\Rightarrow (\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = -8 \quad (1.4.6.9)$$

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{74}, \|\mathbf{C} - \mathbf{A}\| = 4\sqrt{2} \quad (1.4.6.10)$$

Thus,

$$\cos BAC = \frac{(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} = \frac{-8}{4\sqrt{148}} \quad (1.4.6.11)$$

$$\Rightarrow \angle BAC = \cos^{-1} \left( \frac{-8}{4\sqrt{148}} \right) \quad (1.4.6.12)$$

$$= 99.46232^\circ \quad (1.4.6.13)$$

From (1.4.6.13) and (1.4.6.7),

$$2 \times \angle BAC = \angle BOC \quad (1.4.6.14)$$

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (1.4.7.1)$$

where

$$\theta = \angle BOC \quad (1.4.7.2)$$

Verify that

$$\mathbf{B} - \mathbf{O} = \mathbf{P}(\mathbf{C} - \mathbf{O}) \quad (1.4.7.3)$$

All codes for this section are available at

codes/triangle/perp-bisect.py

### 1.5 Angle Bisector

1.5.1. Let  $D_3, E_3, F_3$ , be points on  $AB, BC$  and  $CA$  respectively such that

$$BD_3 = BF_3 = m, CD_3 = CE_3 = n, AE_3 = AF_3 = p. \quad (1.5.1.1)$$

Obtain  $m, n, p$  in terms of  $a, b, c$  obtained in Problem 1.1.2.

**Solution:** From the given information,

$$a = m + n, \quad (1.5.1.2)$$

$$b = n + p, \quad (1.5.1.3)$$

$$c = m + p \quad (1.5.1.4)$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.5.1.5)$$

$$\Rightarrow \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (1.5.1.6)$$

Using row reduction,

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \quad (1.5.1.7)$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array} \right) \quad (1.5.1.8)$$

$$\xrightarrow{\begin{matrix} R_3 \leftarrow R_3 + R_2 \\ R_1 \leftarrow R_1 - R_2 \end{matrix}} \left( \begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array} \right) \quad (1.5.1.9)$$

$$\xrightarrow{\begin{matrix} R_2 \leftarrow 2R_2 - R_3 \\ R_1 \leftarrow 2R_1 + R_3 \end{matrix}} \left( \begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array} \right) \quad (1.5.1.10)$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \quad (1.5.1.11)$$

Therefore,

$$\begin{aligned} p &= \frac{c + b - a}{2} = \frac{\sqrt{74} + \sqrt{32} - \sqrt{122}}{2} \\ m &= \frac{a + c - b}{2} = \frac{\sqrt{74} + \sqrt{122} - \sqrt{32}}{2} \\ n &= \frac{a + b - c}{2} = \frac{\sqrt{122} + \sqrt{32} - \sqrt{74}}{2} \end{aligned} \quad (1.5.1.12)$$

upon substituting from (1.1.2.7), (1.1.2.9) and (1.1.2.11).

1.5.2. Using section formula, find

$$D_3 = \frac{mC + nB}{m + n}, E_3 = \frac{nA + pC}{n + p}, F_3 = \frac{pB + mA}{p + m} \quad (1.5.2.1)$$

1.5.3. Find the circumcentre and circumradius of  $\triangle D_3E_3F_3$ .

These are the *incentre* and *inradius* of  $\triangle ABC$ .

1.5.4. Draw the circumcircle of  $\triangle D_3E_3F_3$ . This is known as the *incircle* of  $\triangle ABC$ .

**Solution:** See Fig. 1.5.4.1

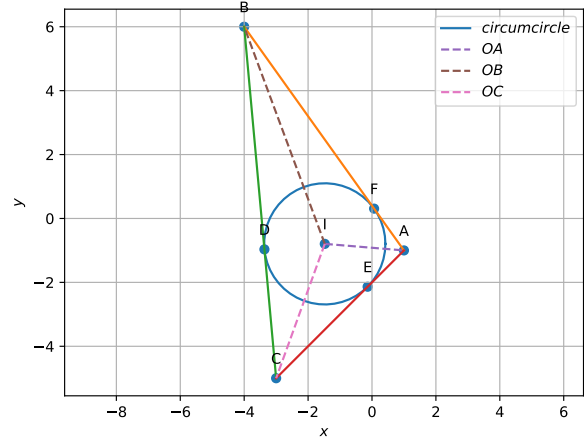


Fig. 1.5.4.1: Incircle of  $\triangle ABC$

1.5.5. Using (1.1.7.1) verify that

$$\angle BAI = \angle CAI. \quad (1.5.5.1)$$

$AI$  is the bisector of  $\angle A$ .

1.5.6. Verify that  $BI, CI$  are also the angle bisectors of  $\triangle ABC$ . All codes for this section are available at

codes/triangle/ang-bisect.py

## 1.6 Eigenvalues and Eigenvectors

The equation of the incircle is given by

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1.6.1)$$

where

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\|^2 - r^2, \quad (1.6.2)$$

$\mathbf{O}$  being the incentre and  $r$  the inradius. Here  $\mathbf{I}$  is the identity matrix.

1.6.1. Compute

$$\Sigma = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^T - g(\mathbf{h})\mathbf{V} \quad (1.6.1.1)$$

for  $\mathbf{h} = \mathbf{A}$ .

1.6.2. Find the roots of the equation

$$|\lambda \mathbf{I} - \Sigma| = 0 \quad (1.6.2.1)$$

These are known as the eigenvalues of  $\Sigma$ .

1.6.3. Find  $\mathbf{p}$  such that

$$\Sigma \mathbf{p} = \lambda \mathbf{p} \quad (1.6.3.1)$$

using row reduction. These are known as the eigenvectors of  $\Sigma$ .

1.6.4. Define

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (1.6.4.1)$$

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} \end{pmatrix} \quad (1.6.4.2)$$

1.6.5. Verify that

$$\mathbf{P}^T = \mathbf{P}^{-1}. \quad (1.6.5.1)$$

$\mathbf{P}$  is defined to be an orthogonal matrix.

1.6.6. Verify that

$$\mathbf{P}^T \Sigma \mathbf{P} = \mathbf{D}, \quad (1.6.6.1)$$

This is known as the spectral (eigenvalue) decomposition of a symmetric matrix

1.6.7. The direction vectors of the tangents from a point  $\mathbf{h}$  to the circle in (1.6.1) are given by

$$\mathbf{m} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm \sqrt{|\lambda_1|} \end{pmatrix} \quad (1.6.7.1)$$

1.6.8. The points of contact of the pair of tangents to the circle in (1.6.1) from a point  $\mathbf{h}$  are given by

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \quad (1.6.8.1)$$

where

$$\mu = -\frac{\mathbf{m}^T (\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \quad (1.6.8.2)$$

for  $\mathbf{m}$  in (1.6.7.1). Compute the points of contact. You should get the same points that you obtained in the previous section.

All codes for this section are available at

codes/triangle/tangpair.py

## 2 MATRICES

The matrix of the vertices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \quad (2.1)$$

### 2.1 Vectors

2.1. Obtain the direction matrix of the sides of  $\triangle ABC$  defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \quad (2.1.1.1)$$

**Solution:**

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \quad (2.1.1.2)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (2.1.1.3)$$

where the second matrix above is known as a *circulant* matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

2.2. Obtain the normal matrix of the sides of  $\triangle ABC$

**Solution:** Considering the rotation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (2.1.2.1)$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R} \mathbf{M} \quad (2.1.2.2)$$

2.3. Obtain  $a, b, c$ .

**Solution:** The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^T \mathbf{M})} \quad (2.1.3.1)$$

2.4. Obtain the constant terms in the equations of the sides of the triangle.

**Solution:** The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag}\{(\mathbf{N}^T \mathbf{P})\} \quad (2.1.4.1)$$

### 2.2 Median

2.2.1. Obtain the mid point matrix for the sides of the triangle

**Solution:**

$$(\mathbf{D} \quad \mathbf{E} \quad \mathbf{F}) = \frac{1}{2} \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad (2.2.1.1)$$

2.2.2. Obtain the median direction matrix.

**Solution:** The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A} - \mathbf{D} & \mathbf{B} - \mathbf{E} & \mathbf{C} - \mathbf{F} \end{pmatrix} \quad (2.2.2.1)$$

$$= \begin{pmatrix} \mathbf{A} - \frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B} - \frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C} - \frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \quad (2.2.2.2)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \quad (2.2.2.3)$$

2.2.3. Obtain the median normal matrix.

- 2.2.4. Obtain the median equation constants.  
 2.2.5. Obtain the centroid by finding the intersection of the medians.

### 2.3 Altitude

- 2.3.1. Find the normal matrix for the altitudes

**Solution:** The desired matrix is

$$\mathbf{M}_2 = (\mathbf{B} - \mathbf{C} \quad \mathbf{C} - \mathbf{A} \quad \mathbf{A} - \mathbf{B}) \quad (2.3.1.1)$$

$$= (\mathbf{A} \quad \mathbf{B} \quad \mathbf{C}) \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \quad (2.3.1.2)$$

- 2.3.2. Find the constants vector for the altitudes.

**Solution:** The desired vector is

$$\mathbf{c}_2 = \text{diag} \{(\mathbf{M}^\top \mathbf{P})\} \quad (2.3.2.1)$$

### 2.4 Perpendicular Bisector

- 2.4.1. Find the normal matrix for the perpendicular bisectors

**Solution:** The normal matrix is  $\mathbf{M}_2$

- 2.4.2. Find the constants vector for the perpendicular bisectors.

**Solution:** The desired vector is

$$\mathbf{c}_3 = \text{diag} \{ \mathbf{M}_2^\top (\mathbf{D} \quad \mathbf{E} \quad \mathbf{F}) \} \quad (2.4.2.1)$$

### 2.5 Angle Bisector

- 2.5.1. Find the points of contact.

**Solution:** The points of contact are given by

$$\begin{pmatrix} \frac{m\mathbf{C}+n\mathbf{B}}{m+n} & \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \end{pmatrix} = (\mathbf{A} \quad \mathbf{B} \quad \mathbf{C}) \begin{pmatrix} 0 & \frac{n}{b} & \frac{m}{c} \\ \frac{n}{a} & 0 & \frac{p}{c} \\ \frac{m}{a} & \frac{p}{b} & 0 \end{pmatrix} \quad (2.5.1.1)$$

All codes for this section are available at

codes/triangle/mat-alg.py

## APPENDIX A POINTS ON A LINE

- A.1. The equation of a line is given by

$$y = mx + c \quad (A.1.1)$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} + x \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (A.1.2)$$

yielding (1.1.4.1).

- A.2. (A.1.1) can also be expressed as

$$y - mx = c \quad (A.2.1)$$

$$\Rightarrow \begin{pmatrix} -m & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \quad (A.2.2)$$

yielding (1.1.5.1).

- A.3. From (1.1.4.1), if  $\mathbf{A}$ ,  $\mathbf{D}$  and  $\mathbf{C}$  are on the same line,

$$\mathbf{D} = \mathbf{A} + q\mathbf{m} \quad (A.3.1)$$

$$\mathbf{C} = \mathbf{D} + p\mathbf{m} \quad (A.3.2)$$

$$\Rightarrow p(\mathbf{D} - \mathbf{A}) + q(\mathbf{D} - \mathbf{C}) = 0, \quad p, q \neq 0 \quad (A.3.3)$$

$$\Rightarrow \mathbf{D} = \frac{p\mathbf{A} + q\mathbf{C}}{p + q} \quad (A.3.4)$$

yielding (1.2.1.1) upon substituting

$$k = \frac{p}{q}. \quad (A.3.5)$$

$(\mathbf{D} - \mathbf{A})$ ,  $(\mathbf{D} - \mathbf{C})$  are then said to be *linearly dependent*.

- A.4. If  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  are collinear, from (1.1.5.1),

$$\mathbf{n}^\top \mathbf{A} = c \quad (A.4.1)$$

$$\mathbf{n}^\top \mathbf{B} = c \quad (A.4.2)$$

$$\mathbf{n}^\top \mathbf{C} = c \quad (A.4.3)$$

which can be expressed as

$$(\mathbf{A} \quad \mathbf{B} \quad \mathbf{C})^\top \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (A.4.4)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \begin{pmatrix} \mathbf{n} \\ -c \end{pmatrix} = \mathbf{0} \quad (A.4.5)$$

yielding (1.1.3.1). Rank is defined to be the number of linearly independent rows or columns of a matrix.

APPENDIX B  
TANGENTS TO A CIRCLE

The equation of the *incircle* is given by

$$\|\mathbf{x} - \mathbf{O}\|^2 = r^2 \quad (\text{B.1})$$

which can be expressed as (1.6.1) using (1.6.2). In Fig. 1.5.4.1, Let (1.6.8.1) be the equation of  $AB$ . Then, the intersection of (1.6.8.1) and (1.6.1) can be expressed as

$$(\mathbf{h} + \mu\mathbf{m})^\top \mathbf{V}(\mathbf{h} + \mu\mathbf{m}) + 2\mathbf{u}^\top (\mathbf{h} + \mu\mathbf{m}) + f = 0 \quad (\text{B.2})$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V}\mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \quad (\text{B.3})$$

For (B.3) to have exactly one root, the discriminant

$$\left\{ \mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \right\}^2 - g(\mathbf{h}) \mathbf{m}^\top \mathbf{V}\mathbf{m} = 0 \quad (\text{B.4})$$

and (1.6.8.2) is obtained. (B.4) can be expressed as

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \mathbf{m} - g(\mathbf{h}) \mathbf{m}^\top \mathbf{V}\mathbf{m} = 0 \quad (\text{B.5})$$

$$\implies \mathbf{m}^\top \mathbf{\Sigma} \mathbf{m} = 0 \quad (\text{B.6})$$

for  $\mathbf{\Sigma}$  defined in (B.6). Substituting (1.6.6.1) in (B.6),

$$\mathbf{m}^\top \mathbf{P} \mathbf{D} \mathbf{P}^\top \mathbf{m} = 0 \quad (\text{B.7})$$

$$\implies \mathbf{v}^\top \mathbf{D} \mathbf{v} = 0 \quad (\text{B.8})$$

where

$$\mathbf{v} = \mathbf{P}^\top \mathbf{m} \quad (\text{B.9})$$

(B.8) can be expressed as

$$\lambda_1 v_1^2 - \lambda_2 v_2^2 = 0 \quad (\text{B.10})$$

$$\implies \mathbf{v} = \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm \sqrt{|\lambda_1|} \end{pmatrix} \quad (\text{B.11})$$

after some algebra. From (B.11) and (B.9) we obtain (1.6.7.1).