

Fractal in Action

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1 Class 10

2 Class 12

3 NCERT

Question

The centre of a circle is at $(2,0)$. If one end of a diameter is at $(6,0)$, then the other end is at :

- ① $(0,0)$
- ② $(4,0)$
- ③ $(-2,0)$
- ④ $(-6,0)$

Solution

Let

$$\mathbf{O} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1)$$

If \mathbf{O} divides AB in the ratio $k : 1$,

$$\mathbf{O} = \frac{(\mathbf{A} + k\mathbf{B})}{1 + k} \quad (2)$$

In this case, $\therefore k = 1$,

$$\mathbf{O} = \frac{(\mathbf{A} + \mathbf{B})}{2} \quad (3)$$

$$\implies \mathbf{B} = 2\mathbf{O} - \mathbf{A} \quad (4)$$

$$= 2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (5)$$

Question

$ABCD$ is a rectangle formed by the points $A(-1, -1)$, $B(-1, 6)$, $C(3, 6)$ and $D(3, -1)$. P, Q, R and S are mid-points of sides AB, BC, CD and DA respectively. Show that the diagonals of the quadrilateral $PQRS$ bisect each other.

Solution

From (2),

$$\mathbf{P} = \frac{\mathbf{A} + \mathbf{B}}{2}, \quad \mathbf{Q} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (6)$$

$$\mathbf{R} = \frac{\mathbf{C} + \mathbf{D}}{2}, \quad \mathbf{S} = \frac{\mathbf{D} + \mathbf{A}}{2} \quad (7)$$

Let \mathbf{O}_1 and \mathbf{O}_2 be the midpoints of PR and QS respectively

$$\mathbf{O}_1 = \frac{\mathbf{P} + \mathbf{R}}{2} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}}{4} \quad (8)$$

$$\mathbf{O}_2 = \frac{\mathbf{Q} + \mathbf{S}}{2} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}}{4} \quad (9)$$

Since

$$\mathbf{O}_1 = \mathbf{O}_2, \quad (10)$$

the diagonals bisect each other.

Question

AD is a median of $\triangle ABC$ with vertices $A(5, -6)$, $B(6, 4)$ and $C(0, 0)$. Length AD is equal to:

- ① $\sqrt{68}$
- ② $2\sqrt{15}$
- ③ $\sqrt{101}$
- ④ 10

Solution

The midpoint of **BC** is

$$\mathbf{D} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (11)$$

$$= \frac{1}{2} \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad (12)$$

Since

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 5 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix} \quad (13)$$

$$\Rightarrow \|\mathbf{A} - \mathbf{D}\| \triangleq \sqrt{(\mathbf{A} - \mathbf{D})^\top (\mathbf{A} - \mathbf{D})} \quad (14)$$

$$= \sqrt{\begin{pmatrix} 2 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ -8 \end{pmatrix}} = \sqrt{2^2 + 8^2} = \sqrt{68} \quad (15)$$

Question

If the distance between the points $(3, -5)$ and $(x, -5)$ is 15 units, then the values of x are

- ① $12, -18$
- ② $-12, 18$
- ③ $18, 5$
- ④ $-9, -12$

Solution

$$\mathbf{A} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} x \\ -5 \end{pmatrix} \quad (16)$$

$$\Rightarrow \mathbf{A} - \mathbf{B} = \begin{pmatrix} 3 - x \\ -5 - (-5) \end{pmatrix} = \begin{pmatrix} 3 - x \\ 0 \end{pmatrix} \quad (17)$$

$$\Rightarrow \|\mathbf{A} - \mathbf{B}\| = \sqrt{(3 - x \quad 0) \begin{pmatrix} 3 - x \\ 0 \end{pmatrix}} = \sqrt{(3 - x)^2} \quad (18)$$

$$\Rightarrow 15 = \pm(3 - x) \quad (19)$$

$$\Rightarrow x = -12, 18 \quad (20)$$

Question

Solve the following system of linear equations algebraically

$$\begin{aligned}2x + 5y &= -4 \\4x - 3y &= 5\end{aligned}\tag{21}$$

Solution

(21) can be expressed as

$$\begin{aligned}\mathbf{n}_1^\top \mathbf{x} &= c_1 \\ \mathbf{n}_2^\top \mathbf{x} &= c_2\end{aligned}\tag{22}$$

where

$$\mathbf{n}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \quad c_1 = 5\tag{23}$$

$$\mathbf{n}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad c_2 = -4\tag{24}$$

(22) gives the normal forms of the equations given in (21) where

$$\mathbf{n}_1, \mathbf{n}_2\tag{25}$$

are defined to be the normal vectors of the respective lines.

Solution

(22) can be expressed as

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix}^T \mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (26)$$

yielding the matrix equation

$$\begin{pmatrix} 2 & 5 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \quad (27)$$

Writing the augmented matrix for using Gauss elimination

$$\left(\begin{array}{cc|c} 2 & 5 & -4 \\ 4 & -3 & 5 \end{array} \right) \xleftarrow{R_2 \rightarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 2 & 5 & -4 \\ 0 & -13 & 13 \end{array} \right) \quad (28)$$

$$\left(\begin{array}{cc|c} 2 & 5 & -4 \\ 0 & -13 & 13 \end{array} \right) \xleftarrow{R_1 \rightarrow \frac{13}{5}R_1 + R_2} \left(\begin{array}{cc|c} \frac{26}{5} & 0 & \frac{13}{5} \\ 0 & -13 & 13 \end{array} \right) \quad (29)$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \quad (30)$$

Question

Find the ratio in which the point $C \left(\frac{8}{5}, y \right)$ divides the line segment joining the points $A(1, 2)$ and $B(2, 3)$. Also, find the value of y .

Solution

For collinearity,

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \quad (31)$$

Performing row reduction,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 8/5 \\ 2 & 3 & y \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & (2-1) & (\frac{8}{5}-1) \\ 0 & (3-2) & (y-3) \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 1 & y-3 \end{pmatrix} \quad (32)$$

$$\xleftrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & y - \frac{18}{5} \end{pmatrix} \Rightarrow y = \frac{18}{5} \quad (33)$$

in order to get a 0 row.

Question

The sum of the digits of a 2-digit number is 14. The number obtained by interchanging its digits exceeds the given number by 18. Find the number.

Solution

Let the digits of the number be x_1 (tens) and x_2 (units). Given

$$x_1 + x_2 = 14 \quad (34)$$

$$10x_2 + x_1 = 18 + 10x_1 + x_2 \quad (35)$$

$$\implies x_1 - x_2 = -2 \quad (36)$$

The above equations can be expressed in matrix form as

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \end{pmatrix} \quad (37)$$

$$(38)$$

Solution

If

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (39)$$

$$\mathbf{A}^T \mathbf{A} = \mathbf{I} \quad (40)$$

\mathbf{A} is then defined to be an orthogonal matrix.

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ -2 \end{pmatrix} \quad (41)$$

$$\implies 2\mathbf{I}\mathbf{x} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \quad (42)$$

$$\implies \mathbf{x} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \quad (43)$$

Topics covered so far

- ① Vectors
- ② Section Formula
- ③ Norm
- ④ Gauss Elimination
- ⑤ Rank
- ⑥ Orthogonal matrix

Question

If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$, then \vec{a} and \vec{b} are:

- ① Collinear vectors which are not parallel
- ② Parallel vectors
- ③ Perpendicular vectors
- ④ Unit vectors

Solution

Let

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad (44)$$

Applying concept of rank from (31)

$$\text{rank} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} = 2 \neq 1, \text{ Not parallel} \quad (45)$$

Applying condition for perpendicularity:

$$\mathbf{a}^\top \mathbf{b} = (2 \quad -1 \quad 1) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \implies \mathbf{a} \perp \mathbf{b} \quad (46)$$

Question

If α, β and γ are the angles which a line makes with positive directions of x, y and z axes respectively, then which of the following are not true?

- ① $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
- ② $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$
- ③ $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$
- ④ $\cos \alpha + \cos \beta + \cos \gamma = 1$

Solution

Let \mathbf{m} represent the unit direction vector of the line. Then,

$$\mathbf{m} = \begin{pmatrix} \cos \alpha \\ \cos \beta \\ \cos \gamma \end{pmatrix} \quad (47)$$

with

$$\|\mathbf{m}\| = 1 \quad (48)$$

Parametric Form

Also,

$$2x + 5y = -4 \quad (49)$$

$$\implies 2x = -4 - 5y \quad (50)$$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + y \begin{pmatrix} -\frac{5}{2} \\ 1 \end{pmatrix} \quad (51)$$

$$\mathbf{x} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} - \frac{5y}{2} \begin{pmatrix} 1 \\ -\frac{2}{5} \end{pmatrix} \quad (52)$$

$$= \mathbf{A} + k\mathbf{m} \quad (53)$$

\mathbf{m} is defined to be the direction vector of the line.

Question

\vec{a}, \vec{b} and \vec{c} are three mutually perpendicular unit vectors. If θ is the angle between \vec{a} and $(2\vec{a} + 3\vec{b} + 6\vec{c})$, find the value of $\cos \theta$.

Solution

Given:

$$\mathbf{a}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{c} = \mathbf{c}^\top \mathbf{a} = 0 \quad (54)$$

$$\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{c}\| = 1 \quad (55)$$

$$\cos \theta = \frac{\mathbf{a}^\top (2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c})}{\|\mathbf{a}\| \|2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}\|} \quad (56)$$

Now,

$$\mathbf{a}^\top (2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}) = 2\mathbf{a}^\top \mathbf{a} + 3\mathbf{a}^\top \mathbf{b} + 6\mathbf{a}^\top \mathbf{c} = 2 + 0 + 0 = 2 \quad (57)$$

$$\|\mathbf{a}\| \|2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}\| = \|2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}\| \quad (58)$$

Solution

From (14) norm definition:

$$(\|2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}\|)^2 = \|4\mathbf{a}^2\| + \|9\mathbf{b}^2\| + \|36\mathbf{c}^2\| = 49 \quad (59)$$

$$\implies \|2\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}\| = +7 \quad (60)$$

$$\implies \cos \theta = \frac{2}{7} \quad (61)$$

Question

Find the position vector of point **C** which divides the line segment joining points **A** and **B** having position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio 4 : 1 externally. Further, find $|\overrightarrow{AB}| : |\overrightarrow{BC}|$.

Solution

We know that

$$\mathbf{C} = \frac{4\mathbf{B} - \mathbf{A}}{4 - 1} \quad (62)$$

Simplify the above for \mathbf{C} .

Question

Two vertices of the parallelogram **ABCD** are given as **A**(−1, 2, 1) and **B**(1, −2, 5). If the equation of the line passing through **C** and **D** is $\frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$, then find the distance between the sides AB and CD . Hence, find the area of parallelogram $ABCD$.

Solution

Let the two parallel lines be

$$\mathbf{x} = \mathbf{A} + k_1 \mathbf{m} \quad (63)$$

$$\mathbf{x} = \mathbf{C} + k_2 \mathbf{m} \quad (64)$$

If \mathbf{P} be a point on the second line,

$$\mathbf{P} = \mathbf{C} + k_2 \mathbf{m} \quad (65)$$

$$(\mathbf{A} - \mathbf{P})^\top \mathbf{m} = 0 \quad (66)$$

From the above,

$$(\mathbf{A} - \mathbf{C})^\top \mathbf{m} - k_2 \|\mathbf{m}\|^2 = 0 \quad (67)$$

$$\implies k_2 = \frac{(\mathbf{A} - \mathbf{C})^\top \mathbf{m}}{\|\mathbf{m}\|^2} \quad (68)$$

Question

Find the equation of the line passing through the point of intersection of the lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x-1}{0} = \frac{y}{-3} = \frac{z-7}{2}$ and perpendicular to these given lines.

Solution

Let the given lines be denoted by \mathbf{x}_1 and \mathbf{x}_2 respectively. From (53):

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \mathbf{A} + k_1 \mathbf{m}_1 \quad (69)$$

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = \mathbf{B} + k_2 \mathbf{m}_2 \quad (70)$$

Solution

Let the unknown line in its parametric form be denoted as follows from (53).

$$\mathbf{x}_3 = \mathbf{C} + k_3 \mathbf{m} \quad (71)$$

The two equations required to solve for the direction of line are

$$\mathbf{m}^\top \mathbf{m}_1 = 0 \quad (72)$$

$$\mathbf{m}^\top \mathbf{m}_2 = 0 \quad (73)$$

$$\implies (\mathbf{m}_1 \quad \mathbf{m}_2)^\top \mathbf{m} = 0 \quad (74)$$

yielding

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & 2 \end{pmatrix} \xleftarrow{R_1 \rightarrow 2R_2 + 3R_1} \begin{pmatrix} 3 & 0 & 13 \\ 0 & -3 & 2 \end{pmatrix} = 0 \quad (75)$$

$$\implies \begin{pmatrix} 3 & 0 & 13 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = 0 \implies \mathbf{m} = \begin{pmatrix} -\frac{13}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} \quad (76)$$

Solution

Equating (69) and (70),

$$\mathbf{A} + k_1 \mathbf{m}_1 = \mathbf{B} + k_2 \mathbf{m}_2 \quad (77)$$

$$(\mathbf{m}_1 \quad \mathbf{m}_2) \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} = \mathbf{B} - \mathbf{A} \quad (78)$$

From the above, k_1 and k_2 can be found by gauss elimination given in (29) and thus \mathbf{C} .

Question

Find the shortest distance between the lines whose vector equations are

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \kappa_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \kappa_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\end{aligned}\tag{79}$$

Solution

From (78) the lines will intersect if

$$\text{rank}(\mathbf{M} \quad \mathbf{B} - \mathbf{A}) = 2 \quad (80)$$

where

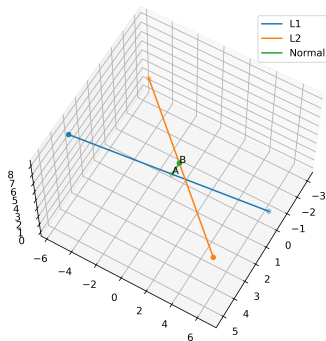
$$\mathbf{M} = (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (81)$$

Solution

If L_1, L_2 , do not intersect, let

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{A} + \kappa_1 \mathbf{m}_1 \\ \mathbf{x}_2 &= \mathbf{B} + \kappa_2 \mathbf{m}_2\end{aligned}\tag{82}$$

be points on L_1, L_2 respectively, that are closest to each other.



Figure

Solution

Then, from (82)

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} - \mathbf{B} + (\mathbf{m}_1 \quad \mathbf{m}_2) \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} \quad (83)$$

Also,

$$(\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_1 = (\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{m}_2 = 0 \quad (84)$$

$$\implies (\mathbf{x}_1 - \mathbf{x}_2)^\top (\mathbf{m}_1 \quad \mathbf{m}_2) = \mathbf{0} \quad (85)$$

$$\text{or, } \mathbf{M}^\top (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0} \quad (86)$$

$$\implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) + \mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{0} \quad (87)$$

from (83), yielding

$$\mathbf{M}^\top \mathbf{M} \begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \mathbf{M}^\top (\mathbf{B} - \mathbf{A}) \quad (88)$$

This is known as the *least squares solution*.

Question

Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

Solution

The equation of the circle is given by

$$\|\mathbf{x} - \mathbf{O}\|^2 = r^2 \quad (89)$$

which can be expressed as

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (90)$$

for

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = -\mathbf{O}, f = \|\mathbf{O}\|^2 - r^2, \quad (91)$$

Let (53) be the equation of the tangent from the point \mathbf{h} . Then, the intersection of (53) and (90) can be expressed as

$$(\mathbf{h} + \mu \mathbf{m})^\top \mathbf{V} (\mathbf{h} + \mu \mathbf{m}) + 2\mathbf{u}^\top (\mathbf{h} + \mu \mathbf{m}) + f = 0 \quad (92)$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V} \mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \quad (93)$$

For (93) to have exactly one root, the discriminant

Solution

$$\left\{ \mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \right\}^2 - g(\mathbf{h}) \mathbf{m}^\top \mathbf{V}\mathbf{m} = 0 \quad (94)$$

and

$$\mu = -\frac{\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \quad (95)$$

is obtained. (94) can be expressed as

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \mathbf{m} - g(\mathbf{h}) \mathbf{m}^\top \mathbf{V}\mathbf{m} = 0 \quad (96)$$

$$\implies \mathbf{m}^\top \boldsymbol{\Sigma} \mathbf{m} = 0 \quad (97)$$

where

$$\boldsymbol{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h}) \mathbf{V} \quad (98)$$

Using the eigenvalue decomposition

$$\mathbf{P}^\top \boldsymbol{\Sigma} \mathbf{P} = \mathbf{D}, \quad (99)$$

in (97),

Solution

$$\mathbf{m}^\top \mathbf{P} \mathbf{D} \mathbf{P}^\top \mathbf{m} = 0 \quad (100)$$

$$\implies \mathbf{v}^\top \mathbf{D} \mathbf{v} = 0 \quad (101)$$

where

$$\mathbf{v} = \mathbf{P}^\top \mathbf{m} \quad (102)$$

(101) can be expressed as

$$\lambda_1 v_1^2 - \lambda_2 v_2^2 = 0 \quad (103)$$

$$\implies \mathbf{v} = \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm \sqrt{|\lambda_1|} \end{pmatrix} \quad (104)$$

after some algebra. From (104) and (102) we obtain

$$\mathbf{m} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm \sqrt{|\lambda_1|} \end{pmatrix} \quad (105)$$

SVD

Perform the eigendecompositions

$$\mathbf{M}\mathbf{M}^\top = \mathbf{U}\mathbf{D}_1\mathbf{U}^\top \quad (106)$$

$$\mathbf{M}^\top\mathbf{M} = \mathbf{V}\mathbf{D}_2\mathbf{V}^\top \quad (107)$$

The following expression is known as *singular value decomposition*

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \quad (108)$$

where $\mathbf{\Sigma}$ is diagonal with

$$\mathbf{\Sigma} \triangleq \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} \quad (109)$$

Substituting in (88),

$$\mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top\boldsymbol{\kappa} = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top(\mathbf{B} - \mathbf{A}) \quad (110)$$

$$\implies \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^\top\boldsymbol{\kappa} = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top(\mathbf{B} - \mathbf{A}) \quad (111)$$

$$\implies \boldsymbol{\kappa} = \left(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^\top\right)^{-1}\mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top(\mathbf{B} - \mathbf{A}) \quad (112)$$

$$\implies \boldsymbol{\kappa} = \mathbf{V}\mathbf{\Sigma}^{-2}\mathbf{V}^\top\mathbf{V}\mathbf{\Sigma}\mathbf{U}^\top(\mathbf{B} - \mathbf{A}) \quad (113)$$

$$\implies \boldsymbol{\kappa} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^\top(\mathbf{B} - \mathbf{A}) \quad (114)$$

where $\mathbf{\Sigma}^{-1}$ is obtained by inverting the nonzero elements of $\mathbf{\Sigma}$. From (82),

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} + \kappa_1\mathbf{m}_1 - \mathbf{B} - \kappa_2\mathbf{m}_2 \quad (115)$$

$$= \mathbf{A} - \mathbf{B} + \mathbf{M}\boldsymbol{\kappa} \quad (116)$$

which, upon substitution from (108) yields

$$\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{A} - \mathbf{B} + \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^\top(\mathbf{B} - \mathbf{A}) \quad (117)$$

$$= (\mathbf{A} - \mathbf{B})\left(\mathbf{I} - \mathbf{U}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{U}^\top\right) \quad (118)$$

Conic Section

Let \mathbf{q} be a point such that the ratio of its distance from a fixed point \mathbf{F} and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \quad (119)$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \quad (120)$$

The locus of \mathbf{q} is known as a conic section. The line L is known as the directrix and the point \mathbf{F} is the focus. e is defined to be the eccentricity of the conic.

- 1 For $e = 1$, the conic is a parabola
- 2 For $e < 1$, the conic is an ellipse
- 3 For $e > 1$, the conic is a hyperbola

Conic

The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus \mathbf{F} is given by

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (121)$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top, \quad (122)$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \quad (123)$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (124)$$

Conic

Using Definition 46, for any point \mathbf{x} on the conic,

$$\begin{aligned}\|\mathbf{x} - \mathbf{F}\|^2 &= e^2 \frac{(\mathbf{n}^\top \mathbf{x} - c)^2}{\|\mathbf{n}\|^2} \\ \implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) &= e^2 (\mathbf{n}^\top \mathbf{x} - c)^2 \\ \implies \|\mathbf{n}\|^2 (\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2) & \\ = e^2 \left(c^2 + (\mathbf{n}^\top \mathbf{x})^2 - 2c\mathbf{n}^\top \mathbf{x} \right) & \\ = e^2 \left(c^2 + (\mathbf{x}^\top \mathbf{n} \mathbf{n}^\top \mathbf{x}) - 2c\mathbf{n}^\top \mathbf{x} \right) & \quad (125)\end{aligned}$$

which can be expressed as (90) after simplification.

Conic

The eccentricity, directrices and foci of (90) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \quad (126)$$

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1,$$

$$c = \begin{cases} \frac{e \mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 (\mathbf{u}^\top \mathbf{n})^2 - \lambda_2 (e^2 - 1) (\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e (e^2 - 1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2 \mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases} \quad (127)$$

$$\mathbf{F} = \frac{c e^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \quad (128)$$

Conic

From (122), using the fact that \mathbf{V} is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top \mathbf{V} = \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \right)^\top \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \right) \quad (129)$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4 \mathbf{n} \mathbf{n}^\top \mathbf{n} \mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (130)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (131)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2 (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (132)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + (e^2 - 2) \|\mathbf{n}\|^2 (\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}) \quad (133)$$

which can be expressed as

$$\mathbf{V}^2 + (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{V} - (e^2 - 1) \|\mathbf{n}\|^4 \mathbf{I} = 0 \quad (134)$$

Using the Cayley-Hamilton theorem, (134) results in the characteristic equation,

$$\lambda^2 - (2 - e^2) \|\mathbf{n}\|^2 \lambda + (1 - e^2) \|\mathbf{n}\|^4 = 0 \quad (135)$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - (2 - e^2) \left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + (1 - e^2) = 0 \quad (136)$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \quad (137)$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = (1 - e^2) \lambda_2 \quad (138)$$

From (138), the eccentricity of (90) is given by (126). Multiplying both sides of (122) by \mathbf{n} ,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n} \mathbf{n}^\top \mathbf{n} \quad (139)$$

$$= \|\mathbf{n}\|^2 (1 - e^2) \mathbf{n} \quad (140)$$

$$= \lambda_1 \mathbf{n} \quad (141)$$

$$(142)$$

from (138). Thus, λ_1 is the corresponding eigenvalue for \mathbf{n} .

Conic

From (142), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \quad (143)$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\|\mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \quad (144)$$

from (138) . From (123) and (138),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \quad (145)$$

$$\Rightarrow \|\mathbf{F}\|^2 = \frac{(ce^2\mathbf{n} - \mathbf{u})^\top (ce^2\mathbf{n} - \mathbf{u})}{\lambda_2^2} \quad (146)$$

$$\Rightarrow \lambda_2^2\|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \quad (147)$$

Also, (124) can be expressed as

$$\lambda_2\|\mathbf{F}\|^2 = f + c^2e^2 \quad (148)$$

Conic

From (147) and (148),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 (f + c^2 e^2) \quad (149)$$

$$\implies \lambda_2 e^2 (e^2 - 1) c^2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \quad (150)$$

yielding (127)