CMSC657 FINAL PROJECT REPORT

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1. ABSTRACT

Magic State Distillation is a key technique in fault-tolerant quantum computing that allows the implementation of non-Clifford gates – namely the T gate – using only operations that are, themselves, fault-tolerant under stabilizer codes. Stabilizer codes can perform Clifford gates: HADAMARD, CNOT, and P, reliably and efficiently. However, Clifford gates alone are not universal for quantum computation. A non-Clifford gate is required to achieve full computational power and finding stabilizer codes that implement such gates fault tolerantly are rare. Moreover, both codes must have good distance (number of detectable errors) for the resources/qubits used.

The **Easten-Knill Theorem** states that no singular quantum error correcting code can implement both the Clifford group of gates and a Non-Clifford gate transversally (i.e fault tolerantly) simultaneously. Magic State Distillation solves this problem by preparing many noisy copies of a special quantum state called a ***magic state***. This allows one to use Clifford operations and measurements to purify these states, detecting and discarding those affected by noise.

Through iterative rounds of this purification, a small number of high-fidelity magic states are obtained, which can then be used in the computation to simulate fault-tolerant T gates. Although resource-intensive - requiring large overhead in qubits and operations - Magic State Distillation is currently the most practical method to achieve universal, error-corrected quantum computation. However, not all codes have equal yields for creating these magic states, and the utility of many qubit and prime qudit codes is derived from the fidelity and efficacy with which they can be used to distill magic states.

In our project, we explored Magic State Distillation yields for families of doubled quadratic residue (QR) based CSS codes as constructed by Jain and Albert [2]. There are three overlapping families described: doubly even QR CSS codes, weak triply even codes obtained via doubling QR codes and, triorthogonal codes obtained by doubling self dual codes. The latter two codes are promising candidates for a Bravyi-Haah style triorthogonal block distillation protocol, as the weak triply even codes requires no clifford corrections and both provide high order error suppression at high distances. For multiple qubit diagonal circuits (many CCZ gates, adders, etc.) the same codes can also be used in Campbell-Howard synthillation.

2. APPROACH

The following algorithm describes our coded Bravyi-Haah Magic State Distillation Algorithm used to analyze the CSS codes described:

Algorithm 1: Distillation$(H_X, z_{\log}, w_{\max})$

Input: H_X X-stabilizer matrix, $z_{\log}$ logical Z vector, maximum weight $w_{\max}$
Output: $w_{\min}$ (harmful undetected weight), $A_{w_{\min}}$ (multiplicity)

$w_{\min} \leftarrow$ None $A_{w_{\min}} \leftarrow 0$
for $w \in \{1, \ldots, w_{\max}\}$ **do**
  FOUND $\leftarrow$ False
  for each subset $S \subseteq \{1, \ldots, n\}$ with $|S| = w$ **do**
    Construct $e \in \{0,1\}^n$ by setting $e_i = 1$ iff $i \in S$ **if** $H_X e^T = 0$ **then**
      if $z_{\log} \cdot e = 1$ **then**
        FOUND $\leftarrow$ True
        if $w_{\min} =$ None **then**
          $w_{\min} \leftarrow w$
          $A_{w_{\min}} \leftarrow 1$
        else if $w = w_{\min}$ **then**
          $A_{w_{\min}} \leftarrow A_{w_{\min}} + 1$
  if FOUND **then**
    break

3. Hypothesis

Claim 3.1. ***Weakly triply-even*** *(*TE**) codes derived from quadratic-residue constructions will achieve strictly better* finite-size overhead *than both generic* ***doubled self-dual*** *constructions and the standard* ***Bravyi–Haah triorthogonal*** *distillation codes due to their high distances in n-qubit codes for small n, structure inherited from QR code weight distributions, and exceptionally low-weight X-stabilizers while still satisfying necessary triorthogonality conditions.*

4. Results and What Went Well

Our results demonstrate that QR-based weakly triply-even (TE*) codes exhibit strong logical error suppression at small physical error rates, consistent with our original hypothesis. In particular, preliminary simulations indicate that TE* constructions achieve high effective distance at relatively small block lengths, leading to a strong error-suppression exponent of the form $p_{\text{out}}(p) = O(p^d)$. This behavior supports the claim that QR-based TE* codes offer a practical advantage in the finite-size regime, rather than only asymptotic improvements.

The hypothesis that QR-based TE* codes outperform standard Bravyi-Haah constructions for realistic block sizes was supported by our initial simulation results. The strengths observed were the combination of high distance for small n, strong logical error mitigation, and the presence of transversal diagonal gates arising from the underlying divisibility conditions.

Ultimately, while our results demonstrate strong finite-size performance for QR-based TE* codes, substantial opportunities remain to extend both the analytical and numerical scope of the project toward more realistic and scalable fault-tolerant quantum computing architectures.

5. Potential Improvements

In this project, we focused exclusively on analyzing magic state distillation using the standard Bravyi–Haah protocol applied to weakly triply-even and related CSS code constructions. While this framework is well-established and provides a concrete mathematical baseline for evaluating distillation performance, it does not reflect the most resource-efficient protocols currently available.

Our results likely underestimate the achievable performance of the code families we studied given that we only performed the Bravyi-Haah Protocol. More recent distillation approaches incorporate additional circuit optimizations, adaptive measurement strategies, and correlated error handling that can lead to substantially improved overhead and performance. A major direction for improvement would be to implement a Monte Carlo based distillation simulation framework, rather than relying purely on Bravyi-Haah-style assumptions.

Such a simulation framework would make it possible to directly estimate acceptance probabilities, conditional logical error rates, and overall yield under realistic stochastic noise models. This would enable direct benchmarking of our doubled QR and weakly triply-even constructions against modern state-of-the-art distillation schemes under practical operating conditions.

Finally, our analysis assumed idealized conditions, including perfect Clifford operations and noiseless measurements. A natural extension would be to incorporate circuit-level noise models and study how error correlations propagate across multiple rounds of distillation. This would significantly strengthen the practical relevance of our results for near-term fault-tolerant quantum computing.

Ultimately, the most important direction for future improvement is the implementation of more efficient and realistic distillation protocols – particularly Monte Carlo – based and multi-round simulation strategies – to obtain tighter, experimentally relevant performance estimates.

6. CONTRIBUTIONS

6.1. **Leo.** Primarily responsible for writing the project reports, conducting background research, and collecting and organizing relevant references and technical background material.

6.2. **Mihir.** Primarily responsible for simulation code and running the experiment, utilizing prior familiarity with magic state distillation and quantum error-correcting codes.

6.3. **Overall.** Both partners jointly contributed to the preparation of the presentation, the interpretation of results, and overall project direction, and worked collaboratively throughout the entire project process.

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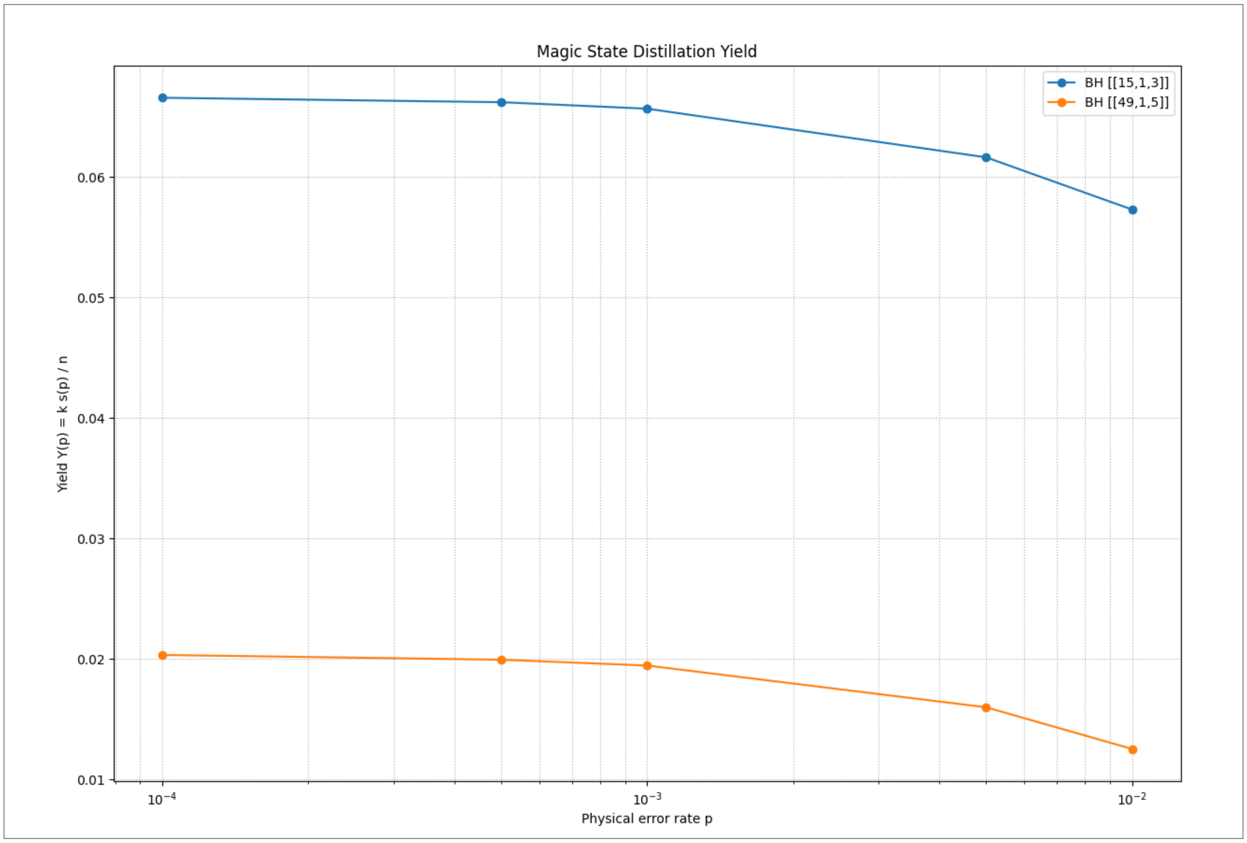
7. Appendix



Simulation Results for Magic State Distillation