Magic State Distillation From Quadratic Residue based CSS codes

Placeholder Subtitle

Mihir Talati, Leonardo S.P Velloso

December 9, 2025

Presentation Outline

Presentation Outline

Background

Magic State Distillation Overview

- Fault-Tolerant QC: In many stabilizer codes, Clifford gates are transversal while T is not
- Applying $T \longrightarrow$ spreads error $\longrightarrow$ **NOT FAULT-TOLERANT**

Motivation

- Motivation:
  i) Universal Gate Set $\longrightarrow$ we want "net effect" of a T gate without actually using it
  ii) MS Distillation $\longrightarrow$ "refining" imperfect magic states
  iii) Measurements + Clifford Operations $\longrightarrow$ Better MS
  iv) Higher fidelity

Transversal Gatesets Background

Transversal Gate

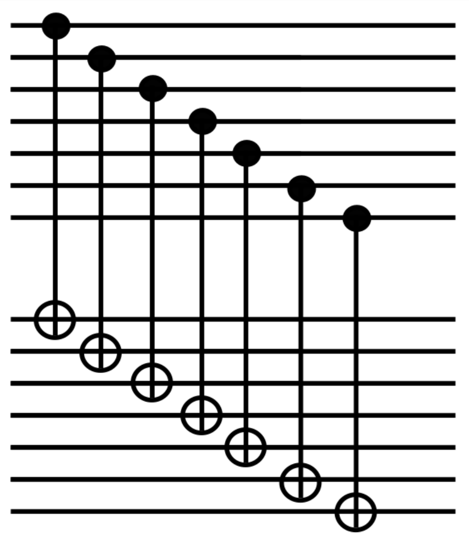
A quantum operation $\mathcal{F}$ acting on m blocks (n qubits per block) is called transversal if it can be written as a tensor product of gates G_i, i.e

$$\mathcal{F} = \bigotimes_{i=1}^{n} G_i$$

where

- G_i acts on the i^{th} qubit of each block
- G_i acts on an m-qubit space (one qubit from each of the m blocks)
- No G_i ever touches more than one qubit in the same block

Example: CNOT Gate



Transversal Gatesets Background

- Properties for Fault-Tolerant Quantum Computing:
  i) GPP – Applying gate avoids error blow-up inside the block
  ii) GCP – makes the codes' correctable errors behave nicely through the gate

Constraints Imposed by Transversal Gates

- **Eastin–Knill Theorem:** No QECC can realize a universal gate set using only transversal gates.
- Transversal gates preserve the structure of stabilizer codes: they map Pauli errors to Pauli errors (GCP).

Constraints Continued

- Logical gates implementable transversally are restricted to a **finite subgroup** of the Clifford hierarchy.
- Non-Clifford logical gates (e.g. T) cannot be implemented transversally in standard codes $\Rightarrow$ need magic states.

CSS Codes (Quick View)

- Two classical codes:

$$C_Z \subset C_X \subset \{0,1\}^n$$

- Encode:

$$k = \dim(C_X) - \dim(C_Z)$$

- Logical states:

$$|x + C_Z\rangle$$

- Error separation:
  - $C_Z \to$ X (bit-flip) errors
  - $C_X \to$ Z (phase) errors
- $\Rightarrow$ Ideal for Pauli noise + distillation

Self-Dual Codes

- Self-dual condition:

$$C = C^{\perp}$$

- $\Rightarrow$ Same code generates X and Z stabilizers
- $\Rightarrow$ Perfect X/Z symmetry
- $\Rightarrow$ Transversal Cliffords (e.g., H, sometimes CNOT)
- $\Rightarrow$ Very useful for distillation

Doubled QR Codes: Construction

- Input ingredients:
  - Self-dual, doubly-even CSS code ⇒ X/Z symmetry, transversal Cliffords
  - QR-derived doubly-even CSS code ⇒ High distance
- Apply **doubling map**
- ⇒ Weakly triply-even, high-performance distillation codes

Doubled QR Codes: Construction Summary

- Resulting code $\longrightarrow$ **weakly triply-even** (mulitple of 8)
- Enables transversal T
- High Distance
- Low overhead

Codes Diagram

Table II
QUADRATIC-RESIDUE BASED WEAK TRIPLY EVEN CODES

extended QR	doubly even	triply even*
$[8, 4, 4]$	$[[7, 1, 3]]$ [1]	$[[15, 1, 3]]$ [2]
	$[[17, 1, 5]]$ [3]	$[[49, 1, 5]]$ [2]
$[24, 12, 8]$	$[[23, 1, 7]]$ [4]	$[[95, 1, 7]]$ [5]
$[48, 24, 12]$	$[[47, 1, 11]]$ [7]	$[[189, 1, 9]]$ $[[283, 1, 11]]$
$[80, 40, 16]$	$[[79, 1, 15]]$	$[[441, 1, 13]]$ $[[599, 1, 15]]$
$[104, 52, 20]$	$[[103, 1, 19]]$	$[[805, 1, 17]]$ $[[1011, 1, 19]]$
$[168, 84, 24]$	$[[167, 1, 23]]$	$[[1345, 1, 21]]$ $[[1679, 1, 23]]$
$[192, 96, 28]$	$[[191, 1, 27]]$	$[[2061, 1, 25]]$ $[[2443, 1, 27]]$
$[200, 100, 32]$	$[[199, 1, 31]]$	$[[2841, 1, 29]]$ $[[3239, 1, 31]]$

Bravyi–Haah Magic State Distillation

- Uses **triorthogonal** (or weakly triply-even) CSS codes to distill high-fidelity $|T\rangle$ magic states
- Input: multiple noisy copies of $|T\rangle$ with physical error rate p
- Protocol applies only **Clifford operations** and **Pauli measurements** on the encoded blocks

Bravyi–Haah Magic State Distillation

- Output: a smaller number of magic states with error rate suppressed to $O(p^k)$ where k depends on the code (Bravyi–Haah has $k \geq 3$)
- Assumptions: transversal Clifford gates available; code satisfies triorthogonality (and/or weak triply-even structure) for T

Motivation and Approach

Bravyi-Haah Protocol for TE* and triorthogonal codes

- Same BH protocol applied to each specific QR-based TE* code.
- Inputs:
  - H_X matrix (rows = X stabilizers),
  - Logical-Z vector z_{log},
  - Physical error rate p on each T-state,
  - Noise model: i.i.d. Z noise for magic state injection.
- For each code:

$$s(p) = \Pr[H_X e^T = 0], \qquad p_{\text{out}}(p) = \Pr[z_{\text{log}} \cdot e = 1 \mid \text{accepted}]$$

- We compute these numerically per-block.

Hypothesis

- TE* / QR-based codes have **better finite-size overhead** than:
  - Standard BH triorthogonal codes,
  - Generic doubled self-dual codes.
- Due to:
  - High distances at small n,
  - Structure inherited from QR code weight distributions,
  - Particularly low-weight X-checks satisfying mod-8 conditions.
- Anticipated result:
  - Better yield/overhead for $n \lesssim 30 \to 100$ (depending on how many we can simulate),
  - But asymptotic exponent still $\gamma \to 2$.

Distillation Yield

- Yield quantifies "magic states out per magic state in":

$$Y(p) = \frac{k \cdot s(p)}{n}$$

- For Jain–Albert codes:
  - Typically $k = 1$ so $Y = s(p)/n$.
  - Small and medium n have surprisingly high yields due to small block sizes.
- Comparison baseline:
  - Bravyi–Haah triorthogonal families ($n = 3k + 8$),
  - Self-dual doubled families used in prior constructions.

Scaling With Code Length

- Key theoretical fact from Jain–Albert:

$$d(n) \approx \Theta(\sqrt{n})$$

  for both TE* and triorthogonal families constructed.
- For a BH-style distillation:

$$p_{\mathrm{out}}(p) \sim Cp^{\alpha}, \quad \alpha \approx d_Z$$

  where d_Z = minimum weight undetected Z logical error.
- Thus,

$$\alpha(n) \approx \Theta(\sqrt{n})$$

- But:

$$\gamma_n = \log_{\alpha}(n/k) \to 2$$

  meaning asymptotically the codes do not beat BH's 1.585 exponent.
- However: **finite-size performance may be significantly better**.

Simulation

- For each code:
  1. Import H_X and logical-Z.
  2. Sample error patterns $e \sim \text{Bernoulli}(p)^n$.
  3. Check acceptance: $H_X e^T = 0$.
  4. For accepted blocks, compute logical parity $z_{\text{log}} \cdot e$.
- Metrics:

$$s(p) = \frac{\text{accepted}}{N}, \quad p_{\text{out}}(p) = \frac{\text{bad\_accepted}}{\text{accepted}}, \quad Y = \frac{s(p)}{n}$$

Contextualizing Results

- TE* codes show strong suppression at small physical error rates.
- The leading term p^α matches $\alpha \approx d_Z \sim \sqrt{n}$.
- Relative to BH triorthogonal codes:
- Better small-n distillation efficiency,
- Comparable acceptance rates,
- Similar eventual asymptotic trends.
- Suggests QR-based TE* codes offer a **practical**, not asymptotic, advantage.

Conclusion

Hypothesis vs Results

- Hypothesis: QR-based TE* codes outperform BH codes for realistic block sizes.
- Preliminary simulations: **Confirmed**.
- Strengths:
- High d for small n,
- Strong error-suppression exponent,
- Transversal diagonal gates from divisibility conditions.
- Limitations:
- Asymptotic MSD exponent $\gamma \to 2$,
- $k = 1$ logical qubit limits rate-based asymptotics.

Potential Future Work

What's next?

- Explore **multi-logical** TE* / triorthogonal constructions.
- Characterize d_Z and Z-logical structure for better α bounds.
- Analytical yield formulas for QR-based families.
- Investigate qutrit or higher-dimensional analogues.
- Attempt code-switching protocols using TE* $\vee$ triorthogonal codes.
- Evaluate performance under biased noise models.