

Magic State Distillation From Quadratic Residue based CSS codes

Placeholder Subtitle

Mihir Talati, Leonardo S.P Velloso

December 9, 2025

Presentation Outline

Presentation Outline

- Background
 - Magic State Distillation Overview
 - Transversal Gates
 - Self-Dual Codes
 - Self-Dual Codes
 - Doubled QR Codes
 - Distillation Protocol
- Motivation and Approach
 - Methodology
 - Hypothesis
 - Yield
 - Scaling
- Simulation
 - Algorithm
 - Results
 - Analysis
- Conclusion
- Potential Future Work

Background

Magic State Distillation Overview

- Fault-Tolerant QC: In many stabilizer codes, Clifford gates are transversal while T is not
- Applying $T \rightarrow$ spreads error \rightarrow **NOT FAULT-TOLERANT**

Motivation

- Motivation:
 - i) Universal Gate Set → we want "net effect" of a T gate without actually using it
 - ii) MS Distillation → "refining" imperfect magic states
 - iii) Measurements + Clifford Operations → Better MS
 - iv) Higher fidelity

Transversal Gatesets Background

Transversal Gate

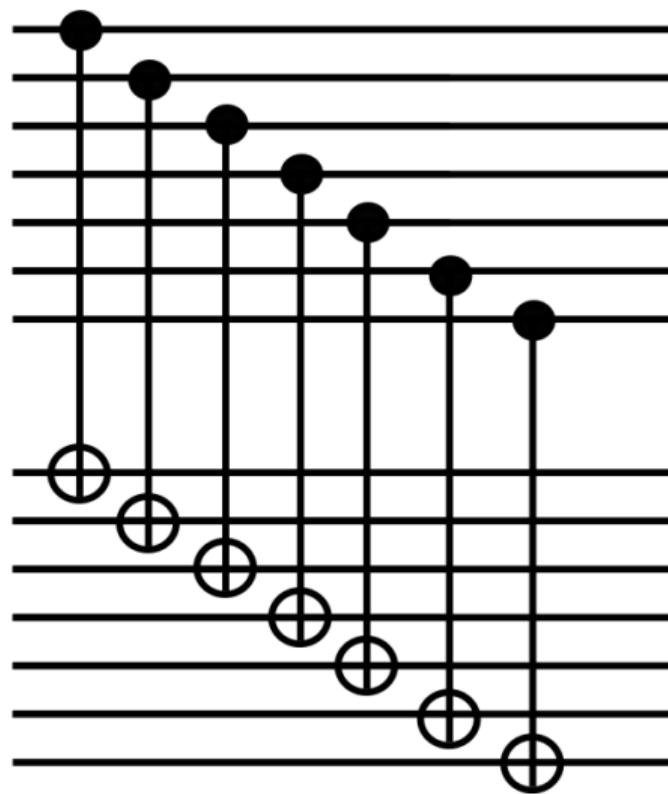
A quantum operation \mathcal{F} acting on m blocks (n qubits per block) is called transversal if it can be written as a tensor product of gates G_i , i.e

$$\mathcal{F} = \bigotimes_{i=1}^n G_i$$

where

- G_i acts on the i^{th} qubit of each block
- G_i acts on an m -qubit space (one qubit from each of the m blocks)
- No G_i ever touches more than one qubit in the same block

Example: CNOT Gate



Transversal Gatesets Background

- Properties for Fault-Tolerant Quantum Computing:
 - i) GPP – Applying gate avoids error blow-up inside the block
 - ii) GCP – makes the codes' correctable errors behave nicely through the gate

Constraints Imposed by Transversal Gates

- **Eastin–Knill Theorem:** No QECC can realize a universal gate set using only transversal gates.
- Transversal gates preserve the structure of stabilizer codes: they map Pauli errors to Pauli errors (GCP).

Constraints Continued

- Logical gates implementable transversally are restricted to a **finite subgroup** of the Clifford hierarchy.
- Non-Clifford logical gates (e.g. T) cannot be implemented transversally in standard codes \Rightarrow need magic states.

CSS Codes (Quick View)

- Two classical codes:

$$C_Z \subset C_X \subset \{0,1\}^n$$

- Encode:

$$k = \dim(C_X) - \dim(C_Z)$$

- Logical states:

$$|x + C_Z\rangle$$

- Error separation:

- $C_Z \rightarrow X$ (bit-flip) errors
- $C_X \rightarrow Z$ (phase) errors

- \Rightarrow Ideal for Pauli noise + distillation

Self-Dual Codes

- Self-dual condition:

$$C = C^\perp$$

- \Rightarrow Same code generates X and Z stabilizers
- \Rightarrow Perfect X/Z symmetry
- \Rightarrow Transversal Cliffords (e.g., H , sometimes CNOT)
- \Rightarrow Very useful for distillation

Doubled QR Codes: Construction

- Input ingredients:
 - Self-dual, doubly-even CSS code \Rightarrow X/Z symmetry, transversal Cliffords
 - QR-derived doubly-even CSS code \Rightarrow High distance
- Apply **doubling map**
- \Rightarrow Weakly triply-even, high-performance distillation codes

Doubled QR Codes: Construction Summary

- Resulting code \longrightarrow **weakly triply-even** (multiple of 8)
- Enables transversal T
- High Distance
- Low overhead

Codes Diagram

Table II
QUADRATIC-RESIDUE BASED WEAK TRIPLY EVEN CODES

extended QR	doubly even	triply even*
[8, 4, 4]	[[7, 1, 3]] [1]	[[15, 1, 3]] [2]
	[[17, 1, 5]] [3]	[[49, 1, 5]] [2]
[24, 12, 8]	[[23, 1, 7]] [4]	[[95, 1, 7]] [5]
[48, 24, 12]	[[47, 1, 11]] [7]	[[189, 1, 9]] [[283, 1, 11]]
[80, 40, 16]	[[79, 1, 15]]	[[441, 1, 13]] [[599, 1, 15]]
[104, 52, 20]	[[103, 1, 19]]	[[805, 1, 17]] [[1011, 1, 19]]
[168, 84, 24]	[[167, 1, 23]]	[[1345, 1, 21]] [[1679, 1, 23]]
[192, 96, 28]	[[191, 1, 27]]	[[2061, 1, 25]] [[2443, 1, 27]]
[200, 100, 32]	[[199, 1, 31]]	[[2841, 1, 29]] [[3239, 1, 31]]

Bravyi–Haah Magic State Distillation

- Uses **triorthogonal** (or weakly triply-even) CSS codes to distill high-fidelity $|T\rangle$ magic states
- Input: multiple noisy copies of $|T\rangle$ with physical error rate p
- Protocol applies only **Clifford operations** and **Pauli measurements** on the encoded blocks

Bravyi–Haah Magic State Distillation

- Output: a smaller number of magic states with error rate suppressed to $O(p^k)$ where k depends on the code (Bravyi–Haah has $k \geq 3$)
- Assumptions: transversal Clifford gates available; code satisfies triorthogonality (and/or weak triply-even structure) for T

Motivation and Approach

Bravyi-Haah Protocol for TE* and triorthogonal codes

- Same BH protocol applied to each specific QR-based TE* code.
- Inputs:
 - H_X matrix (rows = X stabilizers),
 - Logical-Z vector z_{log} ,
 - Physical error rate p on each T-state,
 - Noise model: i.i.d. Z noise for magic state injection.
- For each code:

$$s(p) = \Pr[H_X e^T = 0], \quad p_{\text{out}}(p) = \Pr[z_{\text{log}} \cdot e = 1 \mid \text{accepted}]$$

- We compute these numerically per-block.

Hypothesis

- TE* / QR-based codes have **better finite-size overhead** than:
 - Standard BH triorthogonal codes,
 - Generic doubled self-dual codes.
- Due to:
 - High distances at small n ,
 - Structure inherited from QR code weight distributions,
 - Particularly low-weight X-checks satisfying mod-8 conditions.
- Anticipated result:
 - Better yield/overhead for $n \lesssim 30 \rightarrow 100$ (depending on how many we can simulate),
 - But asymptotic exponent still $\gamma \rightarrow 2$.

Distillation Yield

- Yield quantifies “magic states out per magic state in”:

$$Y(p) = \frac{k \cdot s(p)}{n}$$

- For Jain–Albert codes:
 - Typically $k = 1$ so $Y = s(p)/n$.
 - Small and medium n have surprisingly high yields due to small block sizes.
- Comparison baseline:
 - Bravyi–Haah triorthogonal families ($n = 3k + 8$),
 - Self-dual doubled families used in prior constructions.

Scaling With Code Length

- Key theoretical fact from Jain–Albert:

$$d(n) \approx \Theta(\sqrt{n})$$

for both TE* and triorthogonal families constructed.

- For a BH-style distillation:

$$p_{\text{out}}(p) \sim Cp^{\alpha}, \quad \alpha \approx d_Z$$

where d_Z = minimum weight undetected Z logical error.

- Thus,

$$\alpha(n) \approx \Theta(\sqrt{n})$$

- But:

$$\gamma_n = \log_{\alpha}(n/k) \rightarrow 2$$

meaning asymptotically the codes do not beat BH's 1.585 exponent.

- However: **finite-size performance may be significantly better.**

Simulation

Algorithm Overview [WIP]

- For each code:
 1. Import H_X and logical-Z.
 2. Sample error patterns $e \sim \text{Bernoulli}(p)^n$.
 3. Check acceptance: $H_X e^T = 0$.
 4. For accepted blocks, compute logical parity $z_{\log} \cdot e$.

- Metrics:

$$s(p) = \frac{\text{accepted}}{N}, \quad p_{\text{out}}(p) = \frac{\text{bad_accepted}}{\text{accepted}}, \quad Y = \frac{s(p)}{n}$$

- Complexity:
 - Monte Carlo: $O(Nrn)$ for r stabilizers.
 - Exhaustive enumeration for small n to compute α exactly.

Pretty Graphs [WIP]

- Intended figures:
 - p_{out} vs. p for each QR-based TE* code.
 - Yield $Y(p)$ vs. n .
 - Extracted error exponent α vs. block size.
 - Comparison to BH-[[$3k + 8, k$]] curves.
- Expected trends:
 - TE* codes outperform BH codes for $n < 200$.
 - Sharp drop in p_{out} due to large α .
 - Slow decrease in success rate due to dense X-checks.

Contextualizing Results

- TE* codes show strong suppression at small physical error rates.
- The leading term p^α matches $\alpha \approx d_Z \sim \sqrt{n}$.
- Relative to BH triorthogonal codes:
 - Better small- n distillation efficiency,
 - Comparable acceptance rates,
 - Similar eventual asymptotic trends.
- Suggests QR-based TE* codes offer a **practical**, not asymptotic, advantage.

Conclusion

Hypothesis vs Results

- Hypothesis: QR-based TE* codes outperform BH codes for realistic block sizes.
- Preliminary simulations: **Confirmed**.
- Strengths:
 - High d for small n ,
 - Strong error-suppression exponent,
 - Transversal diagonal gates from divisibility conditions.
- Limitations:
 - Asymptotic MSD exponent $\gamma \rightarrow 2$,
 - $k = 1$ logical qubit limits rate-based asymptotics.

Potential Future Work

What's next?

- Explore **multi-logical** TE^* / triorthogonal constructions.
- Characterize d_Z and Z -logical structure for better α bounds.
- Analytical yield formulas for QR-based families.
- Investigate qutrit or higher-dimensional analogues.
- Attempt code-switching protocols using $\text{TE}^* \vee$ triorthogonal codes.
- Evaluate performance under biased noise models.