Magic State Distillation From Quadratic Residue based CSS codes

Placeholder Subtitle

Mihir Talati, Leonardo S.P Velloso

December 7, 2025

Presentation Outline

Presentation Outline

Background

Transversal Gatesets Background

Transversal Gate

A quantum operation $\mathcal{F}$ acting on m blocks (n qubits per block) is called transversal if it can be written as a tensor product of gates G_i, i.e

$$\mathcal{F} = \bigotimes_{i=1}^{n} G_i$$

where

- G_i acts on the ith qubit of each block
- G_i acts on an m-qubit space (one qubit from each of the m blocks)
- No G_i ever touches more than one qubit in the same block

Transversal Gatesets Background

- Properties for Fault-Tolerant Quantum Computing:
    i) GPP – Applying gate avoids error blow-up inside the block
    ii) GCP – makes the codes' correctable errors behave nicely through the gate

Constraints Imposed by Transversal Gates

- **Eastin–Knill Theorem:** No QECC can realize a universal gate set using only transversal gates.

- Transversal gates preserve the structure of stabilizer codes: they map Pauli errors to Pauli errors (GCP).

- Logical gates implementable transversally are restricted to a **finite subgroup** of the Clifford hierarchy.

- Non-Clifford logical gates (e.g. T) cannot be implemented transversally in standard codes $\Rightarrow$ need magic states.

CSS Code Formalization

- Built from two classical linear codes

$$C_Z \subset C_X \subset \{0,1\}^n.$$

- Encodes $k = \dim(C_X) - \dim(C_Z)$ logical qubits into n physical qubits.
- Logical basis states are uniform superpositions over cosets:

$$|x + C_Z\rangle.$$

- Error structure splits cleanly through stabilizers:
  - C_Z corrects bit-flip (X) errors
  - C_X corrects phase-flip (Z) errors
- Useful in distillation: Pauli-type noise handled independently, supports some Clifford Gates

Self-Dual Codes

- A CSS code is **self-dual** if its classical code satisfies

$$C = C^{\perp}$$

- Then the X and Z-stabilizers come from the same linear code
- This creates strong X/Z symmetry in the error structure
- Such symmetry allows more Clifford gates to be transversal (e.g., H, sometimes CNOT), which is useful for distillation

Doubled QR Codes: Construction Summary

- Begin with two ingredients:
  - a **self-dual, doubly-even** (Hamming weight is multiple of 4) CSS code (gives X/Z symmetry and transversal Cliffords)
  - a **QR-derived doubly-even** CSS code (provides high distance)
- Apply a **doubling map** to combine these two codes
- The resulting code is **weakly triply-even** (mulitple of 8), enabling a transversal logical T gate.
- Final product: a family of $[n, 1, d]$ codes with high distance and lower qubit overhead for transversal Clifford and T

Codes Diagram

Bravyi–Haah Magic State Distillation

- Uses **triorthogonal** (or weakly triply-even) CSS codes to distill high-fidelity $|T\rangle$ magic states
- Input: multiple noisy copies of $|T\rangle$ with physical error rate p
- Protocol applies only **Clifford operations** and **Pauli measurements** on the encoded blocks
- Output: a smaller number of magic states with error rate suppressed to $O(p^k)$ where k depends on the code (Bravyi–Haah has $k \geq 3$)
- Assumptions: transversal Clifford gates available; code satisfies triorthogonality (or weak triply-even structure) for logical T
- Target: preparation of high-fidelity logical $|T\rangle$ states for universal FTQC.

Motivation and Approach

Bravyi-Haah Protocol for TE* and triorthogonal codes

- Same BH protocol applied to each specific QR-based TE* code.
- Inputs:
    - H_X matrix (rows = X stabilizers),
    - Logical-Z vector $z_{\log}$,
    - Physical error rate p on each T-state,
    - Noise model: i.i.d. Z noise for magic state injection.
- For each code:

$$s(p) = \Pr[H_X e^T = 0], \qquad p_{\text{out}}(p) = \Pr[z_{\log} \cdot e = 1 \mid \text{accepted}]$$

- We compute these numerically per-block.

Hypothesis

- TE* / QR-based codes have **better finite-size overhead** than:
  - Standard BH triorthogonal codes,
  - Generic doubled self-dual codes.
- Due to:
  - High distances at small n,
  - Structure inherited from QR code weight distributions,
  - Particularly low-weight X-checks satisfying mod-8 conditions.
- Anticipated result:
  - Better yield/overhead for $n \lesssim 30 \rightarrow 100$ (depending on how many we can simulate),
  - But asymptotic exponent still $\gamma \rightarrow 2$.

Distillation Yield

- Yield quantifies "magic states out per magic state in":

$$Y(p) = \frac{k \cdot s(p)}{n}$$

- For Jain–Albert codes:
  - Typically $k = 1$ so $Y = s(p)/n$.
  - Small and medium n have surprisingly high yields due to small block sizes.
- Comparison baseline:
  - Bravyi–Haah triorthogonal families ($n = 3k + 8$),
  - Self-dual doubled families used in prior constructions.

Scaling With Code Length

- Key theoretical fact from Jain–Albert:

$$d(n) \approx \Theta(\sqrt{n})$$

for both TE* and triorthogonal families constructed.

- For a BH-style distillation:

$$p_{\text{out}}(p) \sim Cp^{\alpha}, \quad \alpha \approx d_Z$$

where d_Z = minimum weight undetected Z logical error.

- Thus,

$$\alpha(n) \approx \Theta(\sqrt{n})$$

- But:

$$\gamma_n = \log_{\alpha}(n/k) \to 2$$

meaning asymptotically the codes do not beat BH's 1.585 exponent.

- However: **finite-size performance may be significantly better**.

Simulation

Algorithm Overview [WIP]

- For each code:
    1. Import H_X and logical-Z.
    2. Sample error patterns $e \sim \text{Bernoulli}(p)^n$.
    3. Check acceptance: $H_X e^T = 0$.
    4. For accepted blocks, compute logical parity $z_{\log} \cdot e$.

- Metrics:

$$s(p) = \frac{\text{accepted}}{N}, \quad p_{\text{out}}(p) = \frac{\text{harmful and accepted}}{\text{accepted}}, \quad Y = \frac{s(p)}{n}$$

- Y is a proxy for approximate per code distillation yield

- Complexity:
    - Monte Carlo: $O(Nrn)$ for r stabilizers.
    - Exhaustive enumeration for small n to compute α exactly.

Pretty Graphs

- Placeholder description of the simulation setup and noise models.
- Mention computational tools or libraries anticipated for the study.

Contextualizing Results

- Placeholder interpretation of simulated performance metrics.
- Discussion points comparing outcomes to expectations or baselines.

Conclusion

HYpothesis vs Results

- Placeholder statement of the working hypothesis before analysis.
- Criteria used to judge success or failure.

Potential Future Work

What's next?

- Placeholder list of follow-up experiments and protocol refinements.
- Suggestions for code design or distillation strategy improvements.