Applications of Linear Algebra

SC 205, Discrete Mathematics

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2 | Abstract

Discrete Mathematics is a beautiful subject that consists of various fascinating domains which serve various purposes in real life.

Out of this vast subject we were fascinated by how concepts of Linear Algebra are beautifully used in various real life applications and how these concepts help to determine and solve this problems. We explored the fields of Markov Chain, Probability and Connectivity of Network and it's variety of applications.

3 | Stochastic Process

Random Variable

A random variable gives each result of an real experiment a real value.

Definition 3.1 (Random Variable). A random variable X induces a probability measure P_X on (<, B) using the function mapping. For any of the elementary events $(\infty, a), thisprobabilityisgivenl$

Take the experiment of throwing two unbiased coins. There are 4 possible possibilities.

$$HH, HT, TH, andTT$$

where H stands for a heads-up result and T for a tails-up result. A random variable 'Z' is assigned the value

$$Z(\phi) = \begin{cases} -1, & \phi! = HH, TT \\ 1, & otherwise \end{cases}$$

where ϕ is the space.

According to the induced probability P_X, $P_X(1)$ =$P_X(-1) = 0.5$.

3.1 | Stochastic Process

Through the stochastic process we give the result of a real experiment a time function.
A set of ordered random variables makes up a Stochastic Process.Every random variable is given a subscript to show the order.The ordering happens from observing the random variables over a period of time. X_t is an example of a random value. It is used to determine the outcome of the stochastic process.

Definition 3.2 (Stochastic Process). A Stochastic process is a collection of random variable $X_t; t\epsilon T$ defined on the same probability space.

Process takes on random values, X_{t1} =X_1,........., $X_{t+n} = X_n$ at times t_1,......., t_n.
The random variables x_1,......., x_n are specified by specifying their joint distribution
One can also choose the time points t_1,......., t_n. (where the process X_t is examined) in different ways.

1. The set T is called Index Set or Time Set or Time Domain
2. For each t ϵ T , X_t is a random variable and it denotes the state of the process art time t.

Example 3.1.

There are three brands of a phone L, M, N

- Given that a man previously purchased phone of brand L,
  - There is 60%chance that he would continue with brand L, 25% chance he will shift to brand M and 15% chance that he would shift to brand N.
- Given that a lady last purchased phone of brand M,
  - There is 40% chance that she would continue with brand M , 37% chance he would shift to L and 23% chance that he would shift to N.
- Given that a lady last purchased phone of brand N,
  - There is 18% chance that he would continue with brand N , 63% chance he will shift to brand L and 19% chances that he would shift to brand M , respectively.

Example 3.2.

A professor made an effort to arrive for class on time. If he is late one day,he is 93% sure to be on time the following day. If he is on time, then there is a 26%chance of him not being on time the following day.

The most crucial aspect in this example is how factors' or states' behaviours vary over time.

In Example 1: Behaviour of man to choose PHONE brand
In Example 2: Professor behaviour regarding ON TIME OR LATE in a class

Hence, we are interested in how a random variable changes over time.

Example 3.3.

No. of students enter in a class before time t.

$$X_0 \text{ be number of students entered in a class at t} = t_0$$
$$X_1 \text{ be number of students entered in a class at t} = t_0 + 1$$
$$\vdots$$
$$X_n \text{ be number of students entered in a class at t} = t_n$$

X_n ,n = 0,1,...,n is a stochastic process which has a countable number of potential values.

NOTATION:

$$X_n = \text{i}$$

At time n, the system will be in the i_{th} state.

3.2 | Stochastic Matrix

Definition 3.3 (Stochastic Matrix). Stochastic matrix is a square matrix whose elements are probabilities and whose columns add up to 1

Examples

Following matrices are Stochastic matrices:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix} \quad \begin{bmatrix} 0 & \frac{3}{4} \\ 1 & \frac{1}{4} \end{bmatrix}$$

Following matrices are not stochastic matrices :

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{2}{3} \end{bmatrix} \quad \begin{bmatrix} 0 & \frac{3}{4} \\ 2 & 1 \end{bmatrix}$$

A general 2x2 stochastic matrix can be written :

$$\begin{bmatrix} x & y \\ 1-x & 1-y \end{bmatrix}$$

where $0 \leq x \leq 1 and 0 \leq y \leq 1$.

4 | Markov Chain

4.1 | The Markov Property Or MemoryLess Property

Assume that such a system transitions from one state to another over time and that the system's status is checked on a regular basis. If only the process' current condition determines how it will develop in the future. When this is the case, a system has a Markovian property. According to [3] As a result, if the Stochastic Process holds a Markovian Property it is said to be a Markov process if the points in the time scale are represented by $t_0 < t_1 < \cdots < t_n$.

$$PX(t_n + 1) = x_n + 1|X(t_n) = x_n.......X(t1) = x1 = PX(t_n + 1) = x_n + 1|X(t_n) = x_n$$

4.2 | The Markov Chain / The Markov Model / The Markov Process

Those Stochastic process who satisfies markovian property are Markov Process.

Description: Assume that such a system transitions from a state to another over time and that the system's state is checked on a regular basis. Processes in which despite the exact state can't be anticipated with confidence, the probability can be calculated provided the previous state is known are called Markov Chains.
A set of random variable x_0,x_1,... x_n which follow the property described below is known as Markov Chain:

$$P[X(t_n) = j|X(t_n - 1) = i_n - 1, \ldots, X(t_1) = i_1, X(t_0) = i_0] = p(X(t_n) = j|X(t_n - 1) = i_n - 1)$$

For all integer times $n > m$ and states i$_0, i_1, .., i_{m-1}, i, j, i_n, s$

For Example

At time n, let X_n be the number of people having Ebola Virus
So at time n + 1, the number of people having those who have the virus at time is X_{n+1}.According to Markov Chain, the number of infected people at time n + 1 depends on the number of people infected at time n.
The Markov process **does not depend on** X$_n - 1; X_n - 2; \ldots; X_0$.

4.3 | Terminologies

[1]

4.3.1 | States

For a Markov Chain, the state space is denoted by 'S' where,

$$\text{S} = 1,2,3,\ldots,\text{n}$$

There are n possible states. At time 'n' the state is given by X_n.

4.3.2 | Trajectory

The series of states that the process has so far experienced is known as the trajectory of a Markov chain. The trajectory values are denoted as:

$$s_0; s_1; s_2; \ldots; s_n$$

We can say

$$x_0 = s_0; x_1 = s_1; x_2 = s_2; \ldots; x_3 = s_n$$

4.3.3 | Transition Probability

A Markov Chain can't exist in two states simultaneously. It can, however, switch between several states which is known as a transition from state sn to $sn + 1$.
Consider a Markov Chain with n possible states. The probability of the system to go to state 'i' from state 'j' is known as Transition Probability and is denoted by p_{ij}. It is defined as,

$$P_{ij} = PX_{n+1} = j|X_n = i$$

N-step probabilities : probabilities from state I to j after n-step time period, denoted by $p_{ij}($ n) or p_{ij}^n is defined as,

$$p_{ij}^n = pX_{n+1} = j|X_1 = i$$

4.3.4 | Transition Probability Matrix (TPM)

For a Markov Chain, the matrix P= $[p_{ij}]$ is called the transition matrix. 3mm
For Example
Consider a Markov chain model who has 3 states, The transition matrix has been form

$$\text{P} = \text{Preceding state}$$

$$\begin{bmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In this matrix, p_{32} is the probability that the system will perform a transition from state 2 to 3.
These are probabilities so they lie between '0' and '1'. For the transition from a give state 'j' we can say

$$p_{1j} + p_{2j} + p_{3j} + \ldots + p_{mj} = 1 \quad \text{for each j} = 1,2,\ldots,\text{m}$$

4.3.5 | Probability Vector

The components of a probability vector give a sum of '1' and are non negative in nature.
Following are the properties of a TPM

- It is square matrix
- Its column vectors are probability vectors
- $p_{ij} \geq 0$ for all i and j
- $\Sigma p_{ij} = 1$ (row wise)

4.3.6 | State Vector

A Markov chain can't reliably predict the system state at any given observation period. Typically, the best one can do is to provide probabilities for each of the potential states. In a Markov chain with three states, for instance, we could use a column vector 'X' to represent the system's potential state at a given observation time.

$$\text{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

in which x_1 is the probability that the system is in state 1, x_2 is the probability that the system is in state 2, x_3 is the probability that the system is in state 3.

[**State Vector**] For a given Markov Chain with m possible states, the column vector 'X' whose $i_t h$ component x_i represents the likelihood of the system to be in the i_{th} state is known as the state vector.
We will understand the above terminologies with the help of a few examples.

Example 4.1.

In the local marketplaces for its computers, Company L has a 48 % market share, while the other two businesses, M and N, each have an equal percentage. The following information, broken down by year, was revealed by a study conducted by a market research firm.

- Company L retains 60% of its customers and gain 7% from company M and 12% from company N
- Company M retains 85% of its customers and gain 18% from company L and 8% from company N
- Company N retains 80% of its customers and gain 22% from company L and 8% from company M.

Now, let's make TPM
No. of state in this system : 3 (Company A, Company B, Company c)
So, order of the TPM is 3 x 3

$$\begin{array}{cc} from->to & \begin{matrix} L & \quad M & \quad P \end{matrix} \\ \begin{matrix} CompanyL \\ CompanyM \\ CompanyP \end{matrix} & \begin{bmatrix} 0.60 & 0.18 & 0.22 \\ 0.07 & 0.85 & 0.08 \\ 0.12 & 0.08 & 0.80 \end{bmatrix} \end{array}$$

Example 4.2.

An new method of distributing homework based on probabilities is chosen by a statistics professor who doesn't wait to be predictable. He creates a graphic showing the transitions on the first day of the week. The diagram's nodes stand in for assignments that receive full credit (F), half credit (H), and no credit (N). The figure displays the day 1 transition probabilities.
Let's try to construct TPM No. of state in the system : 3 (F,H,N)

$$\begin{array}{cc} & \begin{matrix} F & \quad H & \quad N \end{matrix} \\ \begin{matrix} F \\ H \\ N \end{matrix} & \begin{bmatrix} 0.20 & 0.27 & 0.53 \\ 0.47 & 0.18 & 0.35 \\ 0.53 & 0.29 & 0.18 \end{bmatrix} \end{array}$$

4.4 | Prediction Using Markov Chain

But, first consider some notations:

Q^0 = initial probability vector of the states
Q^1 = probability vector of the states after 1 time period
.
.
.
Q^n =probability vector of the states after the n-time period
q_j^n = probability of the state j after the n-time period.

P = TPM after 1 time period.
P^2 = TPM after 2 time period.
.
.
.
P^n = TPM after n-time period. We, will learn Markov Chain theorem using example.

Example 4.3.

Now ,we take a research issue from [5]
Let there be a 3 chamber maze (numbered 1,2 and 3) where each chamber is painted in a different color.
A mouse is put inside one of these chambers who is checked on a regular interval.
As the mouse isn't under constant supervision its precise movements cannot be ascertained.
So we measure the transition in terms of probabilities.
The state where the Markov process begins is known as the initial state.
The beginning probability distribution is represented by the vector in the case when the starting state is determined by a random device that picks condition (chamber) j with a probability of $p_j^{(0)}$.

$$Q^{(0)} = \begin{bmatrix} q_1^{(0)} \\ q_2^{(0)} \\ . \\ . \\ q_n^{(0)} \end{bmatrix}$$

this is a probability vector.
If we consider the initial state to be chamber 1

$$Q^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The initial probability distribution would be as follows if the mouse had an equal chance of being placed in each of the three chambers:

$$Q^{(0)} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

If we consider the mouse to have preferences for a particular colour, lets say

$$p_{32} = \frac{1}{2}$$
$$p_{12} = \frac{1}{4}$$
$$p_{22} = \frac{1}{4}$$

Given below is the complete state transition diagram

$$\text{P} = \begin{bmatrix} & 1 & 2 & 3 \\ 1 & 0 & \frac{1}{4} & \frac{1}{6} \\ 2 & \frac{1}{3} & \frac{1}{4} & \frac{1}{2} \\ 3 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

After the first observation, the probability that the mouse will be in chamber j is p_j^1
Then the probability vector is

$$Q^{(1)} = \begin{bmatrix} \text{q}_1^1 \\ \text{q}_2^1 \\ . \\ . \\ \text{q}_n^1 \end{bmatrix}$$

represents the distribution for this situation.
In order to determine the chance that a mouse would be in chamber 3 and be q_3^1 after the 1^{st} observation, we now attempt to compute pX_3^1.
Using the knowledge of probabilities we can write an equation

$$q_1^{(1)} = q_1^{(0)} * p_{11} + q_2^{(0)} * p_{21} + q_3^{(0)} * p_{31}$$

So that , using matrix multiplication we can write

$$Q^{(1)} = Q^{(0)} * P$$

$$Q^{(1)} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \begin{bmatrix} & 1 & 2 & 3 \\ 1 & 0 & \frac{1}{4} & \frac{1}{6} \\ 2 & \frac{1}{3} & \frac{1}{4} & \frac{1}{2} \\ 3 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$
$$Q^{(1)} = \text{answer}$$

Now lets determine probability of mouse after 2^{nd} observation Similar to eq-1, we can write

$$q_1^{(2)} = q_1^{(1)} * p_{11} + q_2^{(1)} * p_{21} + q_3^{(1)} * p_{31}$$

So that , using matrix multiplication we can write

$$Q^{(2)} = Q^{(1)}P$$
$$Q^{(2)} = Q^{(0)}PP$$
$$Q^{(2)} = Q^{(0)}P^2$$

$$Q^{(2)} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \end{array}$$
$$Q^{(2)} = \text{answer}$$

represents the distribution for this situation. The probability j = 1, 2, and 3 fulfil the equation

$$P_j^{(n)} = p_{j1} * p_1^{(n-1)} + p_{j2} * p_2^{(n-1)} + p_{j3} * p_3^{(n-1)}$$

Given above is the probability of the mouse to be in chamber j after n^{th} observation.
It is the sum of three terms which individually constitute the multiplication of the probability to be in that chamber after $n - 1^{th}$ observation with the probability to arrive in chamber j from the previous chamber.
The above equations can be represented in the form of a matrix.

$$Q^{(n)} = PQ^{(n-1)}$$

where P is transition matrix of the chain.

$$\mathrm{Q}^{(1)} = Q^{(0)}P$$
$$\mathrm{Q}^{(2)} = Q^{(1)}P = Q^{(0)}P^2$$
$$\mathrm{Q}^{(3)} = Q^{(2)}P = Q^{(0)}P^3$$
$$.$$
$$.$$
$$\mathrm{Q}^{(n)} = Q^{(n-1)}P = Q^{(0)}P^n$$

Theorem 4.1. *Let P the transition matrix for a Markov Chain. The probability distribution (state vector) $Q^{(k)}$ after k steps is given by*

$$Q^{(k)} = Q^{(k-1)}P = Q^{(0)}P^k$$

where $Q^{(0)}$ is the initial probability distribution (state vector). P^k is also called Transition Probability Matrix after k^{th} time period

Q) In this illustration, we create a model of population migration between US cities and their surrounding suburbs. According to estimates, there were 58 million people living in cities while 142 million in the suburbs in the year 2000. Representing the above data using a matrix.

$$X_0 = \begin{bmatrix} 58 \\ 142 \end{bmatrix}$$

Sol) Let us take a look at the migration rate of the population from the cities to the suburbs and vice versa. There was a 0.96 percent chance that someone would remain in the city. Therefore, 0.04 was the likelihood of relocating to the suburbs. Considering the the flow from the suburb to the city The likelihood of someone going to the city was 0.01; the likelihood of them staying in the suburbs was 0.99. We will represent the above data using a matrix Q. Q is a stochastic matrix.

$$Q = \begin{bmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{bmatrix}$$

The element in column A and row B indicates the likelihood of relocating from site A to location B. The stochastic matrix is referred to as a matrix of transition probabilities in this context.
Consider the population distribution in 2001
City population = people who remained from 2000 + people who moved in from the suburbs
= (0.96 X 58) + (0.01 X 142)
= 57.1
City population = people who remained from 2000 + people who moved in from the suburbs = (0.96 X 58) + (0.99 X 142)
= 142.9 million

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Keeping 2000 as the initial state, let X_1 be the population in 2001, one year later. We can write

$$X_1 = PX_0$$

Assume that the matrix P, which represents the population flow, has remained constant across time. The population distribution X_2 after 2 years is given by

$$X_2 = PX_1$$

After 3 years the population distribution is given by

$$X_3 = PX_2$$

After n years

$$X_n = PX_{n-1}$$

The predictions of this model (to four decimal places) are

$$X_0 = \begin{bmatrix} 58 \\ 142 \end{bmatrix} X_1 = \begin{bmatrix} 57.1 \\ 142.9 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 56.245 \\ 143.755 \end{bmatrix} X_3 = \begin{bmatrix} 55.4325 \\ 144.5673 \end{bmatrix}$$

$$X_5 = \begin{bmatrix} 54.6611 \\ 145.3389 \end{bmatrix}$$

5 | Long Run Prediction

We now go over how to determine the long-term behaviour of specific Markov chains using eigenvalues and eigenvectors. Applications for forecasting the weather and demographics are provided.

5.1 | Regular Markov chain :

Take into account the transition matrix R and some of its succeeding powers as follows:

$$\text{R} = \begin{bmatrix} 0.5 & 0.7 \\ 0.5 & 0.3 \end{bmatrix} \quad R^2 = \begin{bmatrix} 0.5 & 0.7 \\ 0.5 & 0.3 \end{bmatrix} \quad R^3 = \begin{bmatrix} 0.58 & 0.588 \\ 0.42 & 0.412 \end{bmatrix} \quad R^3 = \begin{bmatrix} 0.58 & 0.588 \\ 0.42 & 0.412 \end{bmatrix}$$

If we were to compute power of R,we should see that the entries in R^n get closer and closer to the matrix R' $= \begin{bmatrix} 0.5867 & 0.5867 \\ 0.4167 & 0.4167 \end{bmatrix} = \begin{bmatrix} \frac{7}{12} & \frac{7}{12} \\ \frac{5}{12} & \frac{5}{12} \end{bmatrix}$

The vector $\mathbf{u} = \begin{bmatrix} \frac{7}{12} \\ \frac{5}{12} \end{bmatrix}$ is an eigenvector of matrix R when we consider the eigenvalue to be 1.

Vectors which satisfy the equation R$\mathbf{u} = \mathbf{u}$ are known as fixed point of the matrix R. Now we introduce some phenomenon [2]

Definition 5.1 (Fixed Point). A probability vector $\mathbf{u}$ is a fixed point of a given transition matrix A if, and only if, R$\mathbf{u} = \mathbf{u}$".

As n grew, we saw that the power Rn of the matrix R in the example above resembled the matrix R' more and more, whose column vectors were all equal to the fixed-point vector for R. Matrices which follow the above property are called *regular transition matrices*

Definition 5.2 (Regular). A probability vector $\mathbf{u}$ is a fixed point of a given transition matrix A if, and only if, R$\mathbf{u} = \mathbf{u}$".

Come back with me to the population movement theory. There, we discovered that a series of vectors, such as $x_0, x_1(= PX_0), x_2(= Px_1), x_3(= Px_2), \ldots P$, which express the annual population distribution. 'P' is a transition probability matrix.
If we consider, Regular Markov Chains, the series $X_0, x_1, x_2, \ldots$converges to some fixed vector x, where P$\mathbf{x} = \mathbf{x}$.The overall city population and total suburban population would then remain constant since the population movement would be in a **steady-state**. We then write

$$Q^{(0)}, Q^{(1)}, Q^{(2)}, \cdots \to Q'$$

Since such a vector $\mathbf{u}$ satisfies P$\mathbf{u} = \mathbf{u}$, it would be an eigenvector of P corresponding to eigenvalue 1. Knowledge of the existence and value of such a vector would give us information about the long-term behavior of the population distribution.

Theorem 5.1. *Consider a regular Markov chain having initial vector $Q^{(0)}$ and transition matrix P. Then*

1. *P has an unique fixed-point probability vector u whose components are positive.So that,* $Q^{(0)}, Q^{(1)}, Q^{(2)}, \cdots \to Q'$ *,where Q'=* $\boldsymbol{u}$ *satisfies* $P\boldsymbol{u} = \boldsymbol{u}$ *.Thus Q' is an eigenvector of P corresponding to* $\lambda = 1$*.*

2. $P, P^2, P^3, \cdots \to P'$ *,where P is a stochastic matrix.whose column vectors are equal to* $\boldsymbol{u}$*.*

The processes are expected to settle down to some predictable, stable behaviour regardless of how it starts. Although this isn't always the case, the theorem stated above shows that a steady and a long-range behaviour is predictable and possible when we use a Markov chain.
One such scenario is the mouse-maze experiment from example 1. The transition probabilities will stabilize at a value regardless of the initial chamber in which the mouse is placed, despite the fact that it seems the mouse will have certain preferences initially. The reason of the stabilization is that P is a regular matrix.
Let's consider some of important applications along with examples that describe the long-term prediction.

5.2 | Example Based On Long-Run Prediction :

5.2.1 | Weather Prediction : *Weather in tel-aviv*

[4]
K. R. Gabriel and J. Neumann have developed "A Markov Chain Model for Daily Rainfall Occurrence at Tel Aviv," Quart J. R. Met. Soc., 88(1962), 90–95. The probabilities used were based on data of daily rainfall in Tel-Aviv (Nahami Street) for 27 years from 1923 to 1950.
If there had been at least 0.1mm of rain on the given day, it would be classified as wet otherwise dry. For each month from November through April, which makes up the rainy season, a Markov chain was built. We talk about the chain created for November. The model makes the assumption that the likelihood of rain on any given day is solely dependent on whether the day before was wet or dry.
.

The statistics accumulated over the years for November were

A Given Day	Following Day Wet
Wet	117 out of 195
Dry	80 out of 615

Thus the probability of a wet day following a wet day is $\frac{117}{195} = 0.6$. The probability of a wet day following a dry day is $\frac{80}{615} = 0.13$. These probabilities lead to the following transition matrix for the weather pattern in November.

$$\text{P} = \begin{bmatrix} 0.6 & 0.13 \\ 0.4 & 0.87 \end{bmatrix}$$

P can be used to forecast the weather for any future day in November. For instance, if it's raining today, a Wednesday, let's calculate the likelihood that next Saturday won't be rainy. Saturday is three days from now. The elements of P^3, given below will provide the various probabilities for the weather on Saturday.

$$P^3 = \begin{bmatrix} 0.32 & 0.22 \\ 0.68 & 0.78 \end{bmatrix}$$

If it is a wet day, there is a 68% chance that Saturday will be dry. Considering the eigenvalue 1, the eigenvectors are non zero.

$$r \begin{bmatrix} 0.325 \\ 1 \end{bmatrix}$$

As Q is stochastic, the column vectors, which are eigenvectors of the long-term transition matrix Q give a sum of 1.
Therefore,$0.325r + r = 1$, giving r = 0.75 (to 2 decimal places). Thus

$$\text{P'} = \begin{bmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{bmatrix}$$

We can interpret this matrix as follows

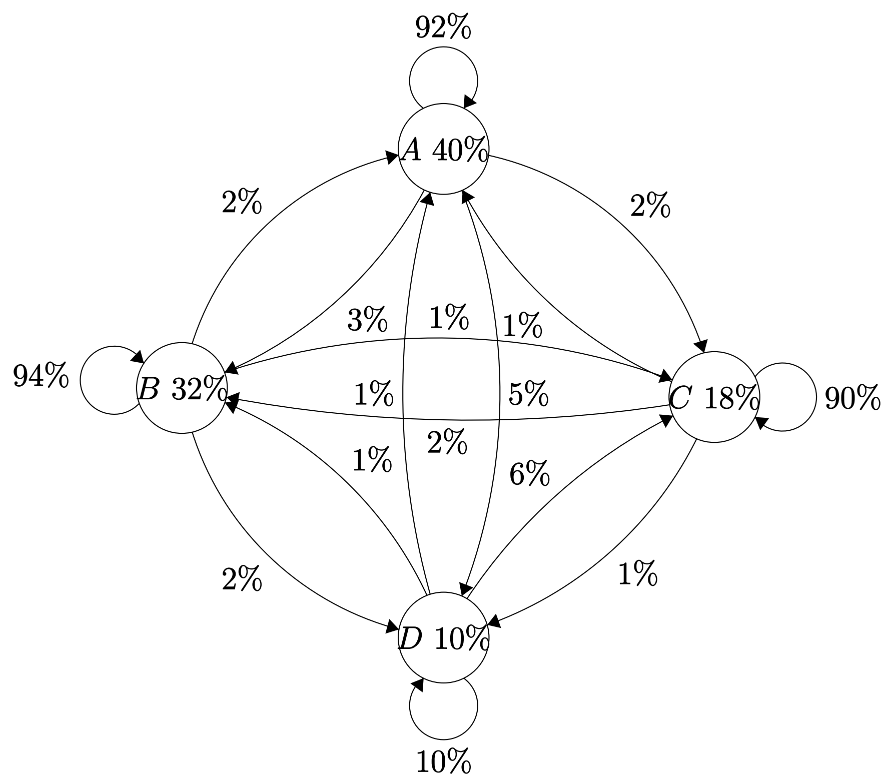
$$\begin{bmatrix} 0.25 & 0.25 \\ 0.75 & 0.75 \end{bmatrix}$$

This implies the following weather forecast for Tel Aviv in November.

Long – Range Forecast for Tel Aviv in November
0.25 probability wet
0.75 probability wet

5.2.2 | Brand Loyalty Or Market Penetration

In a certain market, there are four brands A,B,C,D .The Probability Transition Diagram is shown in figure 4. According to diagram , A man who last purchased product of Brand A,there is 92% chance that she would continue with brand A,2% ,2% and 1% chances that he would shift Brand B,Brand C and Brand D. and the present market share of the four brands A,B,C and D is 40%,32%,90% and 10%.
Based on the information supplied, we create a probability model that can aid in understanding the problem and determining the best strategy to address it.
Probability Model :



We will now generate the necessary The Transition Probability Matrix by closely studying this probability model.

$$\text{The Transition Probability Matrix (T.P.M) P} = \begin{bmatrix} 0.92 & 0.03 & 0.02 & 0.01 \\ 0.02 & 0.94 & 0.02 & 0.01 \\ 0.01 & 0.01 & 0.90 & 0.01 \\ 0.05 & 0.02 & 0.06 & 0.97 \end{bmatrix}$$

We also have a initial state matrix that may be stated in the following ways, in accordance with the problem description.

$$\text{The initial state vector is } Q^{(0)} = \begin{bmatrix} 0.40 \\ 0.32 \\ 0.18 \\ 0.10 \end{bmatrix}$$

Now that we have measures and metrics accessible for a certain market system, we can utilise them to forecast the future market system with the aid of the Markov chain and state vector concepts.

The MAtlab code for this is as follows **Matlab code :**

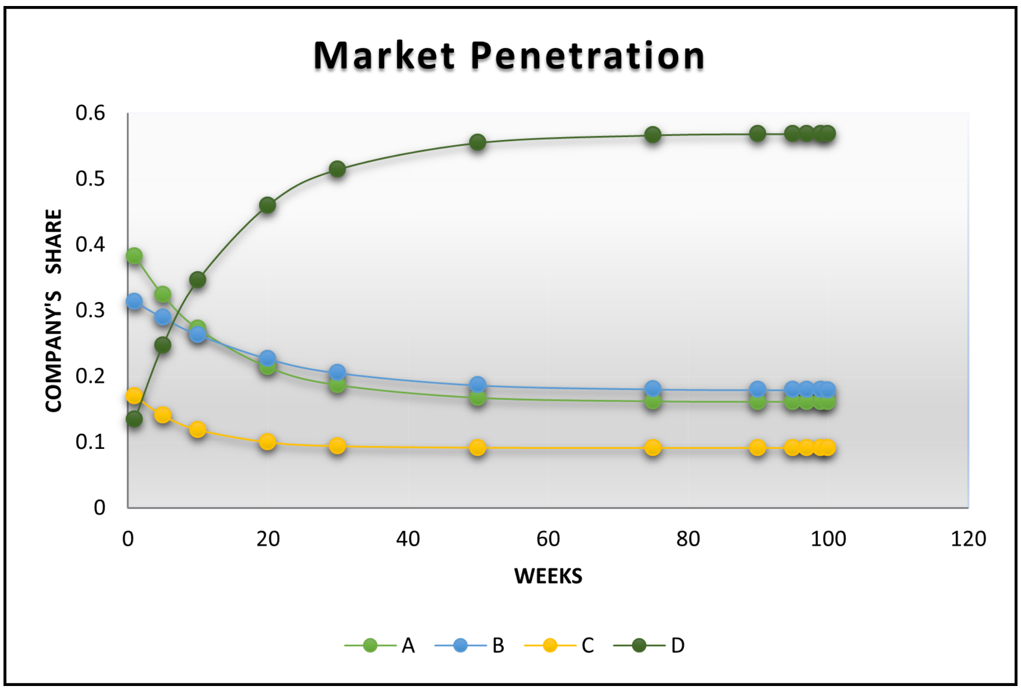
```

%Brand loyalty or Market penetration

%Transition Probability Matrix of given market

P = [...
    [0.92       0.03     0.02     0.01];...
    [0.02       0.94     0.02     0.01];...
    [0.01       0.01     0.90     0.01];...
    [0.05       0.02     0.06     0.97]];


% Intial Market share(Probability state vector) of four Brands

Q(0) = [0.40
    0.32
    0.18
    0.10];

%Market share after one week

Q(1) = P*Q(0);

%MArket share after two week

Q(2) = P*Q(1);

%MArket share after n week
prompt = "enter value of n";
n = input(prompt);
for i = 1:n
   Q(i) = P(i) * Q(0);
   disp(Q(i))
end
```

Since the data in this problem is given for 1 week, once the Markovin theorem is applied on this problem, you can estimate the market system after 1 week.
But we have used this code and studied the market system for the next 100 weeks.

The output of the table is as follows.

Q^n	A	B	C	D
After 1 week	0.3822	0.3134	0.1702	0.1342
After 5 week	0.3237	0.2890	0.1407	0.2467
After 10 week	0.2723	0.2632	0.1187	0.3458
After 20 week	0.2140	0.2266	0.0996	0.4598
After 30 week	0.1868	0.2051	0.0936	0.5145
After 50 week	0.1674	0.1864	0.0912	0.5550
After 75 week	0.1621	0.1804	0.0909	0.5665
After 90 week	0.1614	0.1794	0.0909	0.5683
After 95 week	0.1613	0.1793	0.0909	0.5686
After 97 week	0.1612	0.1792	0.0909	0.5686
After 99 week	0.1612	0.1792	0.0909	0.5687
After 100 week	0.1612	0.1791	0.0909	0.5688

Using the data above as a starting point, the following graphic has been created, from which you can see how each company has grown.



5.2.3 | Charity Contribution

Let's assume ABC hospital operates on a charity basis. All expenses are paid by the government. Of late, the board of governors of the hospital has been complaining about the rise of the budget and instating that the hospital cut expenses.The major area of concern has been cost of keeping patients in the intensive care unit(ICU). The cost has averaged Rs.1000 per week per person compared to only Rs.500 per week per person for keeping patients in the wards.
History shows that of the patients in the ICU at the beginning of the week, 50% will remain there at the end of the week, and 50% will be moved to a WARD.of the patients in the WARDS at the beginning of the week,50% will remain there at the end of the week,10% will get worse and be transferred to the ICU, and 40% will become outpatients. Of the person who is outpatient at the beginning of the week,85% will remain outpatient at the end of the week, 10% will be admitted to WARDS, and 5% will be admitted to the ICU.
The board of governors(BOG) believes that the criteria for keeping patients in the ICU are too strict and has instructed the ICU staff to relax the criteria so that 40% of ICU patients will remain there at the end of the week, and 60% will be moved to a WARD. The staff insists that if this is done, 20% of the ward's patients will be going to ICU each week, and only 30% will be transferred to the Outpatients department. The percentage of the ward patient remaining will still be 50%. There will be no change in the outpatient's status.

Will the policy advocated by the board of governors save money?

Let the three-state ICU, wards, and outpatients are denoted by 1,2,3.
Cost of ICU = Rs.100 & Wards id Rs.500
Before Board Of Governors(BOG) policy:

$$\text{TPM is P} = \begin{bmatrix} 0.50 & 0.50 & 0 \\ 0.10 & 0.50 & 0.40 \\ 0.05 & 0.10 & 0.85 \end{bmatrix}$$

Assume that The steady-state probability distribution is Q' = $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$

Now,According to Regular Markov Chain Theorem;

$$PQ' = Q'$$

$$\begin{bmatrix} 0.50 & 0.50 & 0 \\ 0.10 & 0.50 & 0.40 \\ 0.05 & 0.10 & 0.85 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 0.1129 \\ 0.2419 \\ 0.6451 \end{bmatrix}$$

$$\text{Cost of the policy} = 0.1129 * 1000 + 0.2419 * 500$$
$$= 233.85$$

Based on Board Of Governors(BOG) policy :

$$\text{TPM is P} = \begin{bmatrix} 0.40 & 0.60 & 0 \\ 0.20 & 0.50 & 0.30 \\ 0.05 & 0.10 & 0.85 \end{bmatrix}$$

Assume that The steady-state probability distribution is Q' = $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$

Now,According to Regular Markov Chain Theorem;

$$PQ' = Q'$$

$$\begin{bmatrix} 0.40 & 0.60 & 0 \\ 0.20 & 0.50 & 0.30 \\ 0.05 & 0.10 & 0.85 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 0.1428 \\ 0.2857 \\ 0.5714 \end{bmatrix}$$

$$\begin{aligned} \text{Cost of the policy} &= 0.1428*1000 + 0.2857*500 \\ &= 285.65 \end{aligned}$$

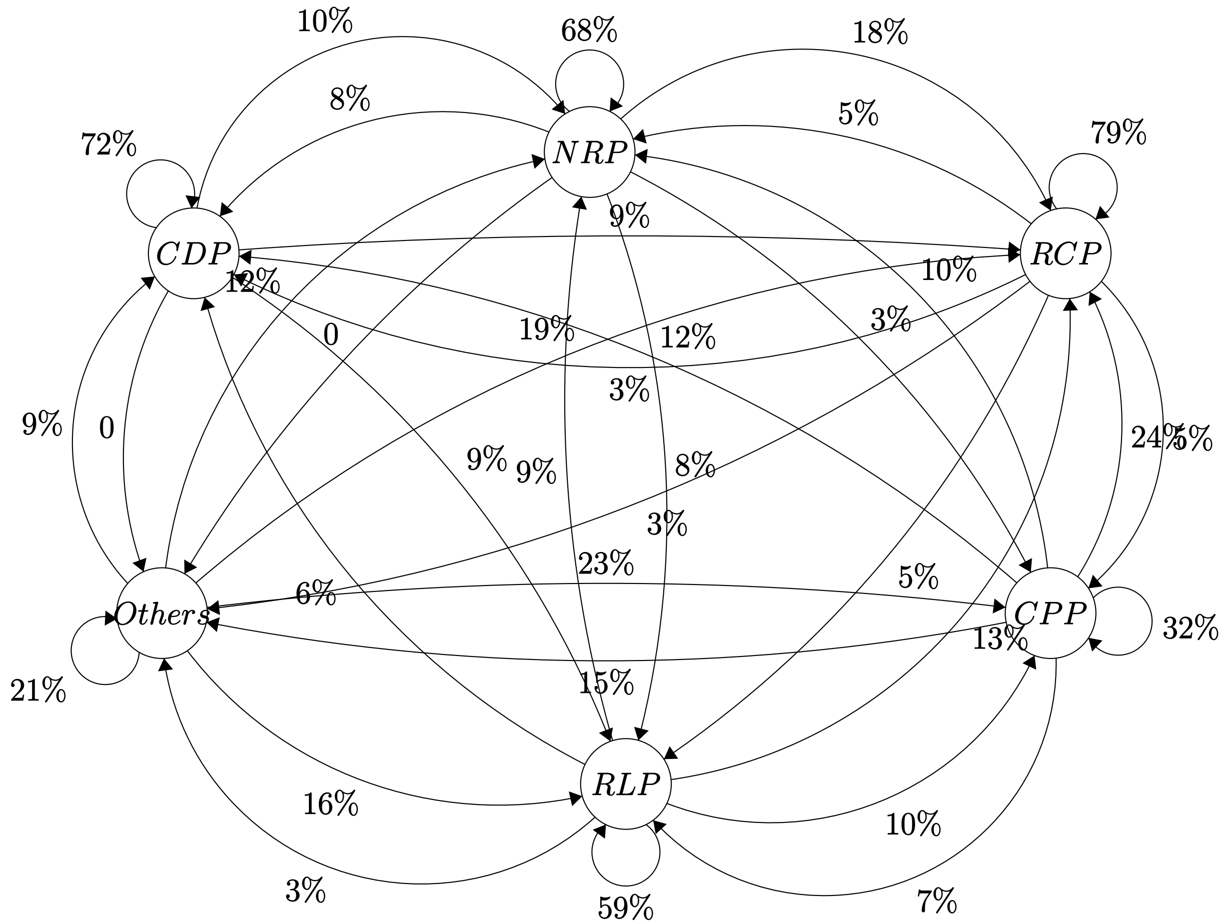
Since Expected cost of the BOG proposal is higher than the present condition.hence, proposal of the BOG will not save money.

5.2.4 | Electoral Prediction

Assume, Presidential elections are taking place in a particular country. There are mainly five parties emerging in this election whose names are as follows:

1. Civic democratic Party (CDP)
2. National Republican Party (NRP)
3. Revolutionary Communist Party (RCP)
4. Conservative People's Party (CPP)
5. Radical Liberal Party (RLP)

An organization surveys its members to find out who they voted for in the most recent election and who they want to vote for in the following. By calculating the likelihood that voters from various parties may switch from one party to another, certain conclusions about potential future developments are obtained from this study
Whose probability model is as follows:



Now, We can create a TPM using this model

$$P = \begin{bmatrix} 0.72 & 0.06 & 0.03 & 0.12 & 0.06 & 0.09 \\ 0.10 & 0.68 & 0.05 & 0.10 & 0.09 & 0.12 \\ 0.09 & 0.18 & 0.79 & 0.24 & 0.13 & 0.19 \\ 0 & 0.03 & 0.05 & 0.32 & 0.10 & 0.23 \\ 0.09 & 0.05 & 0.05 & 0.07 & 0.59 & 0.16 \\ 0 & 0 & 0.03 & 0.15 & 0.03 & 0.21 \end{bmatrix}$$

TPM says that if someone votes for CDP in a particular election, then there is a 72% chance that he would vote for CDP in the next election, and similarly, there is a 10% chance that he would change his opinion and vote for NRP in the next election and 9%,0%,9% and 0% chance to vote for RCP, CPP, RLP, and Others respectively.
And Initial vote distribution is given :

$$Q^{(0)} = \begin{bmatrix} 0.50 \\ 0.22 \\ 0.05 \\ 0.11 \\ 0.09 \\ 0.03 \end{bmatrix}$$

$$\begin{aligned} Q^{(1)} &= P * Q^{(0)} \\ &= \begin{bmatrix} 0.3960 \\ 0.2248 \\ 0.1679 \\ 0.0602 \\ 0.1241 \\ 0.0270 \end{bmatrix} \end{aligned}$$

But we have used this code and predict the electoral and political situation for the next 25 election cycle.

Matlab code to solve this issue :

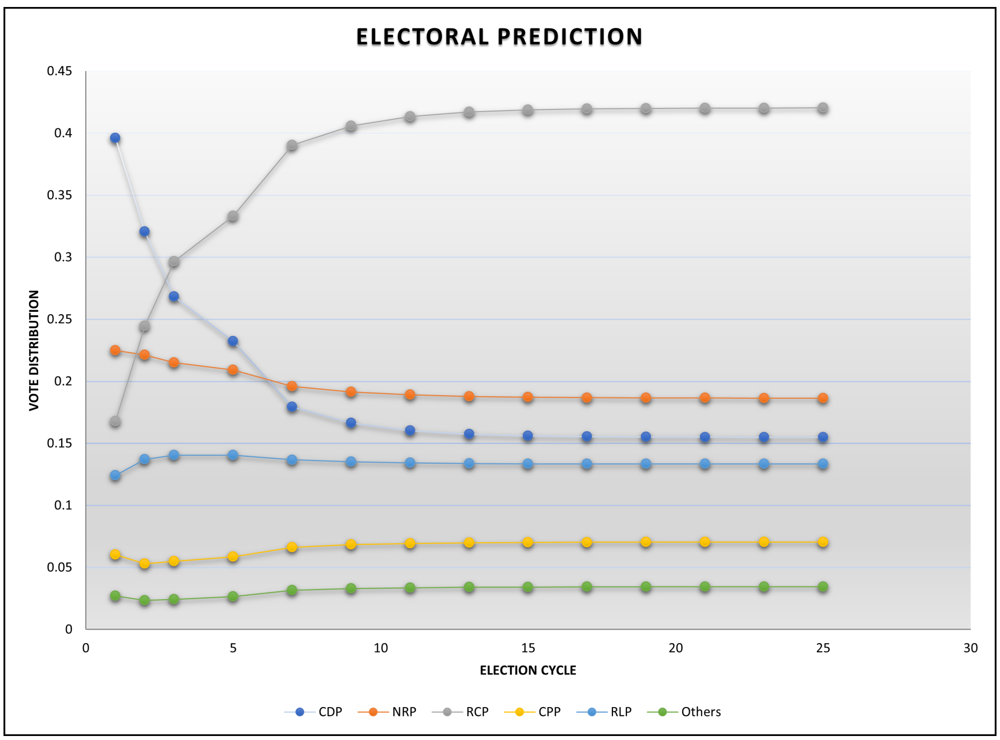
```
%Using the Markov chain to predict the outcome of an election in a
    given situation

%the transformation probabilit matrix is
P = [0.72       0.06    0.03    0.12    0.06    0.09;
0.10    0.68    0.05    0.10    0.09    0.12;
0.09    0.18    0.79    0.24    0.13    0.19;
0       0.03    0.05    0.32    0.10    0.23;
0.09    0.05    0.05    0.07    0.59    0.16;
0       0       0.03    0.15    0.03    0.21
];

%the initial vote distribution is
Q_0 = [0.50;
0.22;
0.05;
0.11;
0.09;
0.03];


%vote distribution after n election cycle
prompt = "enter value of n \n";
n = input(prompt);
for i = 1:n
    Q_i = P^i * Q_0;
    fprintf("vote distribution after %d election cycle\n",i);

    disp(Q_i);
end
```

The output of the table is as follows.

Vote distribution report up to 25 election cycle

$Q^{(n)}$	CDP	NRP	RCP	CPP	RLP	Others
After 1 E.C.	0.3960	0.2248	0.1679	0.0602	0.1241	0.0270
After 2 E.C.	0.3207	0.2213	0.2445	0.0530	0.1370	0.0235
After 3 E.C.	0.2682	0.2152	0.2968	0.0549	0.1405	0.0243
After 5 E.C.	0.2322	0.2091	0.3334	0.0585	0.1404	0.0265
After 7 E.C.	0.1793	0.1960	0.3904	0.0662	0.1367	0.0315
After 9 E.C.	0.1663	0.1915	0.4058	0.0684	0.1351	0.0330
After 11 E.C.	0.1602	0.1891	0.4134	0.0694	0.1342	0.0337
After 13 E.C.	0.1573	0.1878	0.4170	0.0699	0.1338	0.0341
After 15 E.C.	0.1560	0.1871	0.4188	0.0702	0.1336	0.0342
After 17 E.C.	0.1554	0.1868	0.4196	0.0703	0.1335	0.0343
After 19 E.C.	0.1551	0.1867	0.4200	0.0704	0.1335	0.0344
After 21 E.C.	0.1550	0.1866	0.4202	0.0704	0.1335	0.0344
After 23 E.C.	0.1549	0.1865	0.4203	0.0704	0.1335	0.0344
After 25 E.C.	0.1549	0.1865	0.4204	0.0704	0.1335	0.0344

Here is a chart that was created from the above table; by studying it, you can obtain a general understanding of how each political party fared in the vote over the period of 25 elections.



6 | Eigen Value

We define the eigenvector of matrix 'A' as a non-zero matrix that gives the product a scalar product of itself when multiplied with 'A.'
The scalar term which defines the scalar product is known as an eigenvalue.

Assume a n x n matrix 'A'. There exists a λ, for a matrix 'X' which satisfies the condition..

$$AX = \lambda X \quad ...(1)$$

where 'X' is an eigenvector and λ is an eigenvalue.
Further solving equation 1, we get

$$(A - \lambda)X = 0$$

As X is a non zero matrix, we get

$$det[A - \lambda I_n] = 0 \quad \ldots (2)$$

Equation 2 is known as characteristic equation.

Eigenvalues and eigenvectors are unique scalars and vectors associated with matrices and lead to unique coordinate systems. Eigenvalues and eigenvectors have a wide variety of applications ranging from uses in natural and social sciences to branches of engineering. An important property of eigenvalue is If λ is an eigenvalue for A, then

The concept still holds true if A has multiple eigenvalues $\lambda_1, \lambda_2, \lambda_3....\lambda_n$.

6.1 | Eigen Spaces

For a given lambda, the space that contains all the eigenvectors is known as eigenspaces.

Theorem 6.1. *Let A be an n x n matrix and λ an eigenvalue of A. The set of all eigenvectors corresponding to λ, together with the zero vector, is a subspace of R^n. This subspace is called the eigenspace of λ.*

Proof. Let A be an n x n matrix and λ an eigenvalue of A. The set of all eigenvectors corresponding to λ, together with the zero vector, is a subspace of R^n. This subspace is called the eigenspace of λ.
Let x_1 and x_2 be vectors in V and c be a scalar. Then,

$$Ax_1 = \lambda x_1$$
$$Ax_2 = \lambda x_2$$

Hence, solve a error here

Thus $x_1 + x_2$ is an eigenvector corresponding to A (or is the zero vector). V is closed under addition.
Further since, $Ax_1 = \lambda x_1$

$$cAx_1 = c\lambda x_1$$

$$\text{A(cx}_1) = lambda(cx_1)$$

Therefore, cx_1 is an eigenvector corresponding to λ (or is the zero vector). V is closed under scalar multiplication.
Thus V is a subspace of R^n. □

6.2 | Iteration

For various types of matrices, we can use a variety of techniques to calculate eigenvalues and eigenvectors
One well known method is power method which uses iteration. It uses the largest absolute value to calculate the eigenvalue and the eigenvector associated with it.

6.2.1 | Dominant Eigenvalues and Vectors

Will do it tommorow

6.2.2 | Rayleigh's Power Method

For a set of linear simultaeous equations, the largest eigenvalue and its corresponding eigenvector is found using Rayleigh's Power Method.

6.2.3 | Power Matrix

Select an arbitrary non zero column vector x_0 having n components.
Iteration 1

Compute Ax_0

Scale Ax_0 to get Ax_1

Compute $\frac{Ax_1x_1}{x_1x_1}$

Iteration 2

Compute Ax_1

Scale Ax_1 to get Ax_2

Compute $\frac{Ax_2x_2}{x_2x_2}$

Then,

$$x_0, x_1, x_2 \ldots \text{ converges to the dominant vector.}$$

and,

$$\frac{Ax_1x_1}{x_1x_1}, \frac{Ax_2x_2}{x_2x_2}, \ldots \text{ converges to the dominant eigenvector.}$$

if A has n linearly dependent eigenvectors and x_0 has a non zero component in the direction of a dominant eigenvector.
Here is matlab code on Power Methode

```
A = [0  1  0  0;
     1  0  1  1;
     0  1  0  1
     0  1  1  0];
x  =[1;0;0;1];
choice1 = menu('Pick the one','Largest','Smallest','Near to ...\n'
    );

if choice1 == 1
    B=A;
elseif choice1 == 2
        B=inv(A);

else
    lambda0 = input('Enter the value LAMBDA which is near to ....
        \n');
    d = A - lambda0.*eye(size(A));
    B = inv(d);
end
iter = 1;
maxerr = 1e-4;
err = 10000;
lambda1 = Inf;

fprintf('\t Iter  \t Eigenvalue \t Eigenvector   \n');
fprintf('\t ========================================== \n');
while all(err>maxerr)
    xo = x;
    Y = B*x;
    eigenvalue = max(abs(Y));
    eigenvector = Y./eigenvalue;
    x = eigenvector;
    err = abs(sum(xo-x));
    lambda1 = eigenvalue;
    disp([iter lambda1 x']);
```

```
    iter = iter + 1;
end
fprintf('Method converge in %d iteration \n',iter-1);
disp('===============')
if choice1 == 1
    fprintf('The Largest eigenvalue is %5.5f \n',lambda1);
elseif choice1 == 2
    fprintf('The Smallest eigenvalue is %5.5f \n',1/lambda1);
elseif choice1 == 3
    fprintf('The Nearest eigenvalue to %5.3f is %5.5f \n',lambda0,
       lambda0+1/lambda1);

end
```

6.2.4 | Perron–Frobenius theorem

Oskar Perron(1907) and Georg Frobenius(1912) gave the Perron-Fobenius Theorem. According to the theorem, there exists a unique eigenvalue which is real in nature and largest amongst the set of eigenvalues, for a real and posititve square matrix. The corresponding eigenvector only has positive entries.
This theorem also applies to certain classes of non-negative method.

Suppose A $\geq$ 0 is an irreducible square matrix. Then

1. λ (A) is an eigenvalue of A of (algebraic, hence also geometric) multiplicity one, and (a suitably scaled version v^* of) the corresponding eigenvector has strictly positive entries;

2. The only non-negative eigenvectors of A are multiples of v^* ;

3. If

$$|\lambda \epsilon C : \lambda is an eigenvalue of A such that |\lambda| = \lambda(A)| = k$$

   then the set of eigenvalues of A is invariant under a rotation about the origin by 2π ik

6.2.5 | Graph Theory

Graph theory represents the relationship between an object with another object through a graph.
It is represented as

$$\text{G} = (\text{V, E})$$

where V is the set of vertices and E is the set of edges. These vertices are connected with each other through edges.

6.2.6 | DiGraph

A digraph, or a directed graph is a graph oin which the edges have particular directions corresponding to their respective mappings.

6.2.7 | Gould Index

The Gould Index has been made by Edson Gould.
Edson Gould has developed the Gould index which is used as a measure of the accessibility of the corresponding eigenvalues. It describes how strongly the vertices are connected to other vertices.
.

7 | Connectivity of a Network

Eigenvalues integrated along with graph theory have beautiful applications in real life, one of them being analyzing the connectivity of a network.
These networks include airline routes, sea routes, roads, rivers and canals.
To predict the accessibility of such a network, geographers use the Gould Index.
The Gould index of the network described by a matrix 'A' is obtained from the corresponding eigenvector of the dominant eigenvalue of 'A'. Here 'A' is a n x n square matrix.
A network is a two way path. So, if there exists an edge from vertex 'u' to vertex 'v' there also exists an edge from 'v' to 'u'. This implies that adjacency matrix corresponding to that network will be symmetric.

Consider an augmented adjacency matrix 'Z'

$$Z = A + I_n$$

'Z' is also symmetric in nature and has n linearly independent eigenvectors.
From the Perrom- Frobenius theorem, Z has a dominant eigenvalue. The corresponding eigenspace is 1D as all the components of the basis vector are positive.
The accessibility of the cities, corresponding to the vertices is given by the components of the eigenvector. The eigenvalue gives a measure of the overall accessibility of the network.

7.1 | Connectivity of Transportation Network :

Let us consider the four places connected and their connecting graph is shown below, and their respective adjacency matrix and the another matrix required to calculate the dominant eigen value and their respective eigen vector . The required equation is $B = A + I_n$ where B is augmented adjacency matrix and A is adjacency matrix

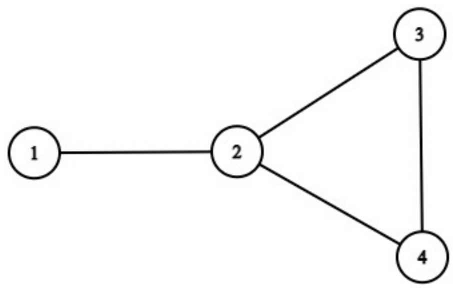
$$\text{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \text{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Finally, we get
$\lambda = 3.17$ and it's vector components x = (0.46,1,0.85,0.85)
From the graph we can conclude that vertex 2 has the highest connectivity and 1 has the lowest, while vertex 3 and 4 has the same connectivity. Now for normalizing the vector we divide this eigenvectors by the sum of the components. The component of the resulting eigen vectors are called the Gould Indices of Accessibility of the vertices. So, we have

$$(0.46,1,0.85,0.85)/0.46+1+0.85+0.85 = (0.15,0.32,0.27,0.27)$$

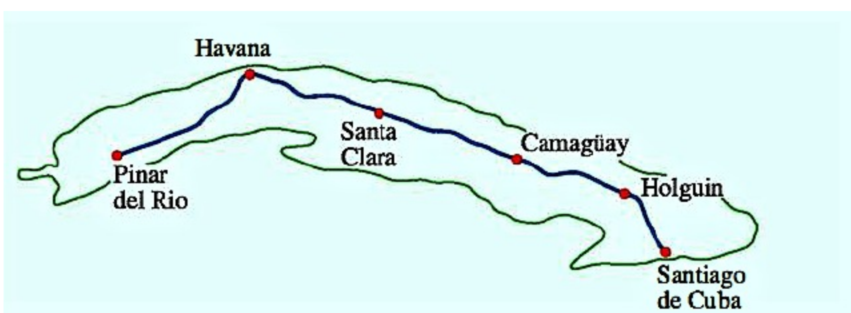
This vectors gives the indices of various vertex.

Vertex	Gould Index
1	0.15
2	0.32
3	0.27
4	0.27



7.2 | Connectivity of Cuba :-

Let us consider the map of Cuba with main cities and their major highways connecting them. The map is given in the figure. Then determine the connectivity of cities.
Solution :-
The adjacency matrix and the augmented are to be found and can be denoted as given below.



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Now with help of the power method we calculate the dominant eigenvalue and it's eigen vector. Finally, we get $\lambda = 2.8$ and it's vector components x = (0.4 5 ,0.8 0 , 1.00,1.00,

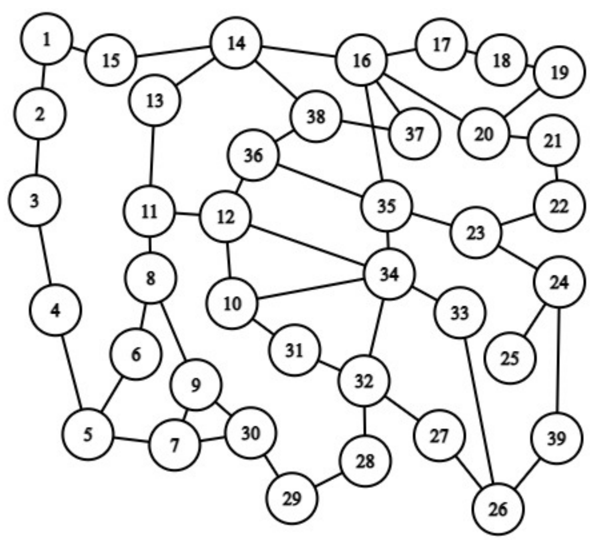
0.8 0,0.45) Now normalizing the vectors by dividing the eigenvectors by the sum of their components,
So , we get
(0.45,0.80,1.00,1.00,0.80,0.45)/0. 45+0.80+1.00+1.00+0.80+0.45 =(0.10,0.18,0.22,0.22,0.18,0.10)
This vectors gives the indices of various vertex.

City	Gould Index
Panar del Rio	0.10
Havana	0.18
Santa Clara	0.22
Camaguay	0.22
Holguin	0.18
Santiago De Cuba	0.10

The overall connectivity of this network is 2.80.
We can observe that Havana is not the most connected city on the island. We can observe the symmetry in the table that fit with the network of road.

7.3 | Moscow as a capital of Russia :-

Geographers had found that Moscow was assumed a dominant position in central Russia because of its strategic location on medieval trade routes.
The vertices of the graph as shown in the figure below shows the centres of population in Russia in the medieval age and the edge represents major trade routes. Some off the vertices are Moscow (35), Kiev(4), Novgorod(1). The accessibility index of the cities were computed. It will form 39 X 39 matrix , for finding the dominant eigen value, so , we found out using computer program.



City	Gould Index
Kozelsk(10)	0.0837
Kolomna(34)	0.0788
Vyazma(12)	0.0722
Bryansk(8)	0.0547
Mtsensk(30)	0.0493
Moscow (35)	0.0490
Dorogobusch(11)	0.0477

We have listed the highest indices in the table given above. Moscow ranks 6^{th} in the accessibility. We arrived at the conclusion that in the analysis it was found that sociological and political factors other than it's location on the trade routes must have been important in Moscow's rise.

7.4 | Comparison of Air Routes :-

The components of the dominant eigen vector of the modified adjacency matrix which gives internal information about comparison of the accessibility of it's vertices. The dominant eigen values of the adjacency matrix of digraph gives the information of the connectedness. The higher the dominant eigenvalue of the digraph the greater is it's connectedness. The below given table gives information of internal airline networks of eight countries based on their dominant eigen values.

City	Gould Index
USA	6
France	5.423
UK	4.610
India	4.590
Canada	4.511
Former USSR	3.855
Sweden	3.301
Türkiye (formerly Turkey)	2.903

,

8 | References

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