Animal 15 (2021) 100074

Contents lists available at ScienceDirect

Animal
The international journal of animal biosciences

Lactation curve model with explicit representation of perturbations as a phenotyping tool for dairy livestock precision farming

A. Ben Abdelkrim [a,b,*], L. Puillet [a], P. Gomes [a,c], O. Martin [a]

[a] Université Paris-Saclay, INRAE, AgroParisTech, UMR Modélisation Systémique Appliquée aux Ruminants, 75005 Paris, France
[b] Université Paris-Saclay, INRAE, AgroParisTech, UMRGABI, 78350 Jouy-en-Josas, France
[c] NEOVIA, 56250 Saint-Nolff, France

ARTICLE INFO

Article history:
Received 24 January 2020
Received in revised form 1 September 2020
Accepted 8 September 2020
Available online 10 December 2020

Keywords:
Disturbances
Individual variability
Milk yield
Precision phenotyping
Resilience

ABSTRACT

In the context of dairy farming, ruminant females often face challenges inducing perturbations that affect their performance and welfare. A key issue is how to assess the effect of perturbations and provide metrics to quantify how animals cope with their environment. Milk production dynamics are good candidates to address this issue: i) they are easily accessible, ii) overall dynamics throughout lactation process are well described and iii) perturbations are visible through milk losses. In this study, a perturbed lactation model (**PLM**) with explicit representation of perturbations was developed. The model combines two components: i) the unperturbed lactation model that describes a theoretical lactation curve, assumed to reflect female production potential and ii) the perturbation model that describes all the deviations from the unperturbed lactation model with four parameters: starting date, intensity and shape (collapse and recovery). To illustrate the use of the PLM as a phenotyping tool, it was fitted on a data set of 319 complete lactations from 181 individual dairy goats. A total of 2 354 perturbations were detected, with an average of 7.40 perturbations per lactation. Loss of milk production for the whole lactation due to perturbations varied between 2 and 19% of the milk production predicted by the unperturbed lactation model. The number of perturbations was not the major factor explaining the loss of milk production, suggesting that there are different types of animal response to challenges. By incorporating explicit representation of perturbations in a lactation model, it was possible to determine for each female the potential milk production, characteristics of each perturbation and milk losses induced by perturbations. Further, it was possible to compare animals and analyze individual variability. The indicators produced by the PLM are likely to be useful to move from raw data to decision support tools in dairy production.

Implication

In the context of precision livestock farming, automatic data collections at commercial farms are now more and more available. In dairy systems, milk production measurements open interesting opportunities to estimate how females cope with their local environment and breeding purposes. Such estimates can be useful for on-farm management. To move from raw time series data sets to useful information, simple interpretive tools are required. Based on milk measurements, the present model allows to estimate the individual milk production potential (as if there was no perturbation), the characteristics of each perturbation occurring during lactation and the overall consequence of perturbations on milk losses.

Introduction

In dairy systems, it is well known that milk yield can be affected by events such as udder health problems (Rajala-Schultz et al., 1999), lameness (Huxley, 2013), meteorological changes (West, 2003) or feed quality (Friggens et al., 2016). Such problems induce perturbations in the course of the lactation process and result in a serrated shape pattern of the lactation curve. These perturbations can be seen as deviations of the lactation curve from its typical profile. Modelling the lactation curve is a long standing issue (Delage et al., 1953), and numerous authors have proposed mathematical models allowing the characterization of milk yield dynamics. The overall objective of lactation models is to reduce the variability in data by creating a profile, thereby being able to characterize an average animal milk production, or to compare the production of different animals. An important limitation of these modelling approaches is that short-term perturbations are ignored during the fitting procedure in order to extract an unperturbed phenotype, corresponding to a typical lactation curve (Adriaens et al., 2018). However, characterizing perturbations can be highly relevant

* Corresponding author.
E-mail address: ahmed.ben-abdelkrim@inrae.fr (A. Ben Abdelkrim).

https://doi.org/10.1016/j.animal.2020.100074

for better understanding the response to challenges of dairy females regarding their milk production and therefore for making management decisions (Windig et al., 2005; Elgersma et al., 2018). Furthermore, evaluating the effect of perturbations on animal performance could provide metrics to quantify how animals cope with their environment and develop management strategies to find a good balance between animal welfare and performance (Berghof et al., 2019; Adriaens et al., 2020; Poppe et al., 2020).

The need for incorporating perturbations into lactation curve models is also driven by the development of precision livestock farming. High throughput data have led to the development and use of statistical methods, such as smoothing methods, to capture and understand perturbations. Codrea et al. (2011) studied the effect of nutritional challenges on the lactation curve in dairy cows using differential smoothing procedures for quantifying biological perturbations in an animal performance. Results of this study highlighted the decline in milk yield during the challenge period for each cow and showed the presence of other deviations with unknown causes or unrelated to the feed restriction during the experiment. There are few other approaches to describe the shape of the lactation curves from animals faced with health problems. Lescourret and Coulon (1994) have shown the huge variability of milk production in response to mastitis in both the shape of the lactation curve and intensity of milk production. Adriaens et al. (2018) developed a novel methodology to predict quarter milk yield during clinical mastitis. The primary objective of the approaches cited above is not an explicit representation of perturbations. However, explicitly modelling perturbations could allow us to characterize them and use this information for phenotyping and benchmarking.

In this study, we developed a perturbed lactation model (**PLM**) that incorporates an explicit representation of perturbations and that converts individual raw time series data into biological meaningful parameters. The fitting procedure of PLM allows the detection and the characterization of perturbations in milk time series. The objective of the present paper is (1) to introduce the PLM model and the explicit representation of perturbations, (2) to describe the use of PLM to detect and characterize perturbations in milk yield time series with an example in dairy goats and (3) to illustrate the role of PLM as a phenotyping tool by analyzing the variability in perturbed lactation curves on the basis of the fitting results obtained on the dairy goat dataset.

Material and methods

The PLM is composed of a lactation model, denoted Y^*, describing the theoretical unperturbed dynamics of milk yield along the lactation, and a perturbation model, denoted π, describing deviations from the lactation model. The list of model parameters is provided in Table S1.

The dynamics of daily milk yield ($Y(t)$, in kg) during the lactation is thus given by:

$$Y(t) = Y^*(t) \cdot \pi(t)$$

where t is the time after parturition in days.

Unperturbed lactation model

Among the numerous mathematical models developed to study lactation curves, the incomplete Gamma function proposed by Wood (1967) has been widely used in different mammals (e.g., rabbit (Casado et al., 2006), sheep (Ruiz et al., 2000) and cattle (Beever et al., 1991)). This model gives a general expression for the dynamics of milk yield along the lactation. In this article, we have selected this model as an example to define the unperturbed lactation curve. Because the structure of PLM is generic, any other lactation model can be used (e.g., Cobby and Le Du (1978), Dhanoa (1981) or Wilmink (1987) models, see Fig. S1).

The Wood model is given by:

$$Y^*(t) = a \cdot t^b \cdot e^{-c \cdot t}$$

where $Y^*(t)$ is the unperturbed daily milk yield in kg, t is the time in days after parturition and a, b, c are positive parameters that determine the shape of the lactation curve. Values of these parameters can be used to calculate some essential features of the lactation curve such as the time of peak yield (b/c, in days), the lactation persistency, i.e., the extent to which peak yield is maintained ($-(b+1) \cdot ln(c)$ in kg/d), or the peak yield ($a \cdot (b/c)^b \cdot e^{-b}$ in kg) (France and Thornley, 1984).

Perturbation model

The perturbation model is based on the idea that each single perturbation i affecting lactation dynamics can be described as a transient proportional decrease in milk yield, through a sequence of collapse and recovery. Each perturbation can thus be modelled by way of a 3-compartment model (Fig. 1, panel a) representing the dynamics of the proportion of milk withdrawn from the theoretical, unperturbed yield.

The three compartments of the model are: A_i, the maximal proportion of milk potentially affected by the ith perturbation, P_i, the proportion of milk effectively affected by the ith perturbation and U_i, the proportion of milk unaffected by the ith perturbation. Given the structure of the compartmental model, forming a path from A_i to U_i through P_i, and given that the model is defined such that $A_i + P_i + U_i = 1$, the dynamics of P_i represents the proportional deviation in milk yield, i.e., the dynamics of P_i (see Fig. 1 panel b) describes the shape of an individual perturbation.

The perturbation model for a single perturbation i is defined by the following simple differential system:

$$\text{if } t_i \geq t_P : \begin{cases} \dfrac{dA_i}{dt} = -k_{1,i} \cdot A_i \\ \dfrac{dP_i}{dt} = +k_{1,i} \cdot A_i - k_{2,i} \cdot P_i \\ \dfrac{dU_i}{dt} = +k_{2,i} \cdot P_i \end{cases} \text{otherwise} : \begin{cases} \dfrac{dA_i}{dt} = 0 \\ \dfrac{dP_i}{dt} = 0 \\ \dfrac{dU_i}{dt} = 0 \end{cases}$$

with the following initial conditions at parturition time ($t = 0$):

$$\begin{cases} A_i(0) = k_{0,i} \\ P_i(0) = 0 \\ U_i(0) = 1 - k_{0,i} \end{cases}$$

and where t_{Pi} is the time of start of the ith perturbation, $k_{0,\,i}$ is the parameter of intensity of the ith perturbation ($k_{0,\,i} \in]0;\,1]$), $k_{1,\,i}$ is the parameter of collapse speed of the ith perturbation and $k_{2,\,i}$ is the parameter of recovery speed of the ith perturbation.

Assuming that $k_{1,\,i} \neq k_{2,\,i}$, the algebraic solution of this differential system is given by:

$$\begin{cases} A_i(t) = k_{0,i} \cdot e^{-k_{1,i} \cdot \Delta_i(t)} \\ P_i(t) = \dfrac{k_{0,i} \cdot k_{1,i}}{k_{1,i} - k_{2,i}} \cdot \left(e^{-k_{2,i} \cdot \Delta_i(t)} - e^{-k_{1,i} \cdot \Delta_i(t)}\right) \\ U_i(t) = 1 - \dfrac{k_{0,i}}{k_{1,i} - k_{2,i}} \cdot \left(k_{1,i} \cdot e^{-k_{2,i} \cdot \Delta_i(t)} - k_{2,i} \cdot e^{-k_{1,i} \cdot \Delta_i(t)}\right) \end{cases}$$

where $\Delta_i(t)$ is the elapsed time since the beginning of the ith perturbation and is given by:

$$\Delta_i(t) = \begin{cases} 0 & \text{if } t < t_{Pi} \\ t - t_{Pi} & \text{if } t \geq t_{Pi} \end{cases}$$

Finally, the perturbation model, including n individual perturbations affecting the lactation curve, is given by:

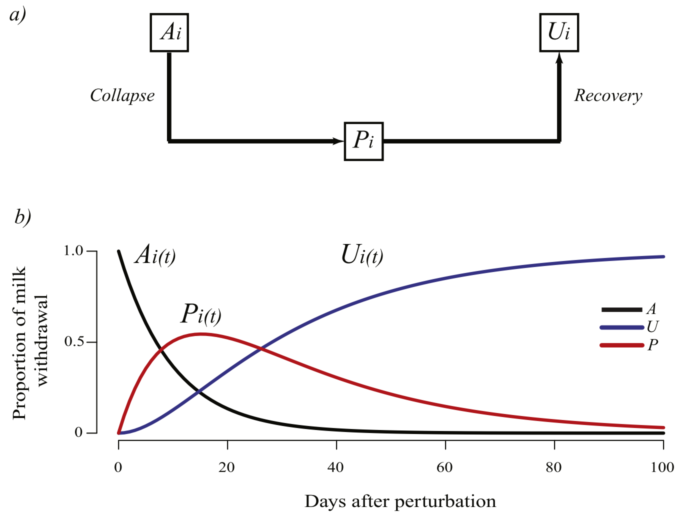

Fig. 1. Conceptual model of a single perturbation. *A*: proportion potentially affected by the perturbation, *P*: proportion effectively affected by the perturbation, *U*: proportion unaffected by the perturbation. (a) Model diagram and (b) solution dynamics.

$\pi(t) = \prod_{i=1}^{n}(1-P_i(t))$

Perturbed lactation model

The detailed algebraic formula of PLM with n individual perturbations is given by:

$$Y(t) = a\cdot t^b \cdot e^{-c\cdot t}\cdot\prod_{i=1}^{n}\left(1-\frac{k_{0,i}\cdot k_{1,i}}{k_{1,i}-k_{2,i}}\cdot\left(e^{-k_{2,i}\cdot\Delta_i(t)}-e^{-k_{1,i}\cdot\Delta_i(t)}\right)\right)$$

The model includes the three parameters of the Wood model (a, b, and c) to define the unperturbed lactation curve, one parameter to define the number of perturbations affecting the lactation curve (n) and four parameters per individual perturbation i (t_{Pi}, $k_{0,\,i}$, $k_{1,\,i}$ and $k_{2,\,i}$) so that the total number of parameters to define PLM is equal to $4+4\cdot n$. For a given perturbation, the parameter $k_{0,\,i}$ (perturbation intensity parameter) is the initial size of the compartment *A* and corresponds to the maximal proportion of milk loss at nadir of the perturbation. Parameters $k_{1,\,i}$ and $k_{2,\,i}$ (respectively, collapse and recovery speed parameters) are fractional rates of change of compartments *A*, *P* and *U* in the differential system. They correspond to daily changes of the proportion of milk affected by the perturbation.

A simulation of PLM with five perturbations over 300 days of lactation is shown in Fig. 2 as an illustration of the model behaviour.

Perturbations were considered individually so that a perturbation can occur within another one (for instance, P_3 in Fig. 2 at $t_{P3}=100$). Given that individual perturbations are proportional deviations multiplied between them, when a perturbation is added at a time point in the iteration, it affects the perturbed curve (i.e., unperturbed Wood and all previous perturbations at that time). Moreover, perturbations can be used to simulate the effect of pregnancy (see P_5 in Fig. 2 at $t_{P5}=225$) with the recovery parameter $k_{2,\,i}$ set to zero.

In practice, the number of perturbations and the model parameters being unknown, we adopted a fitting strategy in two steps: first, performing numerous repeated fittings to estimate the most frequent number of detected perturbations. Then, we fixed the number of perturbations to the value determined in the first step and performed the fitting procedure to estimate the remaining parameters of the model. Details of the fitting procedure are given in the Supplementary material and Fig. S2. The RMSE was calculated to indicate the goodness-of-fit of PLM_N (the perturbed curve with N perturbations). Additionally, the percentage of loss 'L' was calculated using the formula $L=1-S_N^*/S_N$ where S_N^* and S_N are, respectively, the total milk yield calculated with PLM_N^* (the unperturbed curve corrected that N perturbations) and PLM_N. To provide complementary information on lactation time series and refine the PLM outputs analysis, the model of Grossman et al. (1999) was also fit to the lactation data as described in Martin and Sauvant (2002). This fitting cuts the lactation period into three stages corresponding to early, middle and late stages (respectively, intervals $[t_0;t_1]$: increasing phase, $[t_1;t_2]$: plateau-like phase, and $[t_2;t_3]$: decreasing phase, where t_0 is the first day of lactation and t_3 the last day of lactation). This triphasic model, based on a smoothing logistic transition between intersecting straight lines, specifies the cut points of the three stages (instead of an *a priori* determined number of days in milk). These stages were used to classify detected perturbations along the lactation as either early-, middle- or late-stage perturbations, i.e., occurring, respectively, during the increasing, peak/plateau or decreasing phases of the lactation.

Dairy goat data set

In this study, we used data from 181 goats (94 Alpine and 87 Saanen) born between 2009 and 2017. Data concerned 319 lactations (126 primiparous and 193 multiparous; parity ranging from 1 to 7) including 80 773 milk records from the dairy goat herd of the INRAE-AgroParisTech Systemic Modelling Applied to Ruminants research unit (Paris, France) between 2015 and 2018. Records are shown in Fig. S3 by breed and parity. All lactations considered had at least one record in the first 5 days of lactation and a last record between 150 and 358 days of lactation (no extended lactation included).

Statistical analysis

Breed and parity are two well-known factors affecting lactation curve in dairy goats (Gipson and Grossman, 1990; Arnal et al., 2018). In order to evaluate PLM ability to characterize lactation time series, statistical analysis was performed on fitting results.

Fixed effects of breed (Saanen vs Alpine) and parity (1 vs 2 and more) were tested on parameters of Wood, with and without the changes made from PLM model. It was also tested on estimated peak milk yield, peak time, total milk yield over $[t_0;t_3]$, the number of perturbation and the rate milk loss using a mixed ANOVA model with goat as a random factor. Fixed effect of lactation stage (early vs middle vs late)

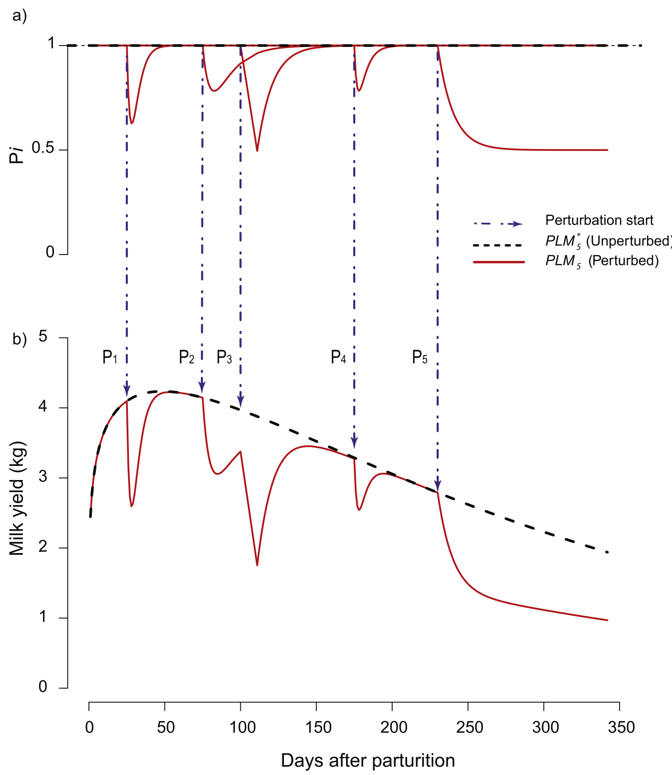

Fig. 2. Example of a simulation of the perturbed lactation model (PLM) including five perturbations. (a) Individual perturbations dynamics expressed as the proportion of unperturbed lactation curve (*Pi*) and (b) unperturbed and perturbed milk yield dynamics.

was tested on RMSE and on PLM parameters t_P, k_0, k_1, $k_2$with a mixed ANOVA model with parity as a fixed factor. Pearson linear correlations were calculated for PLM parameters: intra-class of breed and parity for a, b, c, N and L and intra-class of stage of lactation for t_P, k_0, k_1 and k_2. All statistical analyses were performed using R (R Development Core Team, 2018).

Results

Lactation duration ranged from $t_0 = 1.2 \pm 0.6$ to $t_3 = 270.3 \pm 40.8$ days in milk. Early, middle and late lactation stages determined with Grossman's model were 1.2 to 34.4, 34.4 to 171.0 and 171.0 to 270.3 days, respectively.

Fitting procedure

The fitting procedure converged for the 319 lactations and detected a total of 2 354 perturbations with an average of 7.4 perturbations per animal per lactation. Fig. 3 shows the fitting of PLM on data for one single lactation: panel (a) distribution of starting time of perturbations within 10 days classes; panel (b) shows the unperturbed and perturbed fitted lactation models plotted against data. The fitting results on individual lactations corresponding to the minimum and maximum values for the RMSE (0.1 and 0.4 kg, respectively) are provided in Fig. S4. The number of perturbations varied between 4 and 11, the percentage of milk loss between 2 and 19%, the total unperturbed milk yield was between 393 and 1 557 kg. During the first fitting steps, the Wood's parameters were stabilized on average after the detection of the first four perturbations (Fig. S5). This indicates the robustness of the unperturbed curve.

Table 1 compares the results of the Wood parameters estimation without considering perturbations (i.e., fitting procedure with PLM_0) with the results of the Wood parameters estimation with perturbations (i.e., fitting procedure with PLM_N). Regarding the goodness of fit, PLM_N had lower RMSE (0.2 ± 0.1 kg) than the PLM_0 (0.4 ± 0.1 kg), showing (0.2 ± 0.1 kg) an improvement of fitting quality.

Unperturbed lactation curve

Descriptive statistics of the parameters a, b and c for the unperturbed lactation curves (for both models: PLM_N^* and Wood model) are presented in Table 2 for the overall data set, breed and parity. The parameter a, which drives the general scaling of the curve, was not significantly different for the two breeds (Alpine: 2.49 ± 0.71; Saanen: 2.58 ± 0.73). Consequently, no significant breed effect was found for the peak milk or for the total unperturbed milk production. The same statistical effects were found with the Wood adjustment without perturbation. The parameter a was significantly affected by the parity, with first lactations having a lower value for parameter a than the two and more parities (Table 2). Consequently, there was a significant parity effect on the peak milk yield and on the total milk production. The parameter b, which drives the curvature of the lactation curve, was significantly affected by breed. Alpine goats exhibited higher values of b compared to Saanen goats (Alpine: 0.19 ± 0.08; Saanen: 0.16 ± 0.07). Parity also had a significant effect on the parameter b, with first lactations having a lower value for parameter b than lactations from animals with parity two and higher. Regarding the parameter c, which drives the rate of decrease of milk production after the peak, both parity and breed effects were highly significant. Alpine goats exhibited the same value for the parameter c as the Saanen goats (Alpine: 0.003 ± 0.001; Saanen: 0.003 ± 0.001). For this parameter, first lactations had a lower value than two and more lactations (Primiparous: 0.002 ± 0.001; Multiparous: 0.003 ± 0.001). The peak time of the unperturbed curve, resulting from both b and c parameters, was significantly affected by breed, with Saanen goats exhibiting a peak 14 days later in lactation than the Alpine goats.

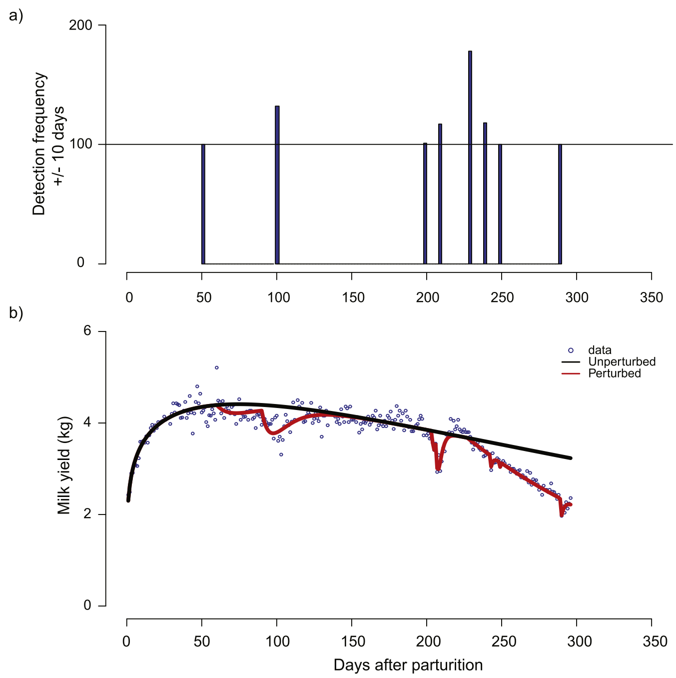

Fig. 3. Example of fitting result of the perturbed lactation on one goat lactation data set. (a) Frequency of detection of a single perturbation within ±10 days among 100 repeated fits (i.e., distribution of estimated perturbation starting times); (b) final unperturbed and perturbed lactation models plotted against data.

Individual unperturbed lactation curves obtained with PLM_N^* for increasing parities are shown in Fig. 4. Some of these individual adjusted curves were considered as atypical, in the sense they departed from the general shape of the Wood model (18 out of the 319 analyzed curves). Generally, these atypical curves come from the same goat in different parities or from primiparous that did not start the second lactation. Peaks of milk of the unperturbed lactation curve were on average increased by 27.47% between the first parity and the second parity, by 9.46% between the second parity and the third parity and decreased by 0.29% between the third parity and the fourth parity (Fig. 4). The total milk production for the unperturbed curve was increased by 32.55% between the first parity and the second parity, 5.20% between the second parity and the third parity and by 1.01% between the third parity and the fourth parity.

The Pearson linear correlation matrix by breed and parity between parameters of PLM_N^* is shown in Fig. 5 (panels a and b). A strong negative correlation was found between a and b (−0.65), indicating that high values of a (scaling of the lactation curve) were associated with low values of b (shaping the curve). A positive correlation was found between the parameters c and b (0.64) indicating a positive association between the shape of the curve and the rate of decrease of lactation, which is a well-known feature of Wood's model. Finally, a low negative correlation between c and a (−0.11) was found. These results are consistent with the well-known features of lactation curves: higher milk at peak yield being associated with higher speed of decline after peak. Several factors (e.g., breed, parity, seasonality and season of kidding) can affect characteristics of the lactation curve.

Table 1
Results of the fitting procedure applied on lactation curves of dairy goats.

Wood model[1]	1 (126[8])	2 + (193[8])	Total (319[8])
	Mean (SD)	Mean (SD)	Mean (SD)
a	1.88 (0.63)	2.39 (0.79)	2.14 (0.71)
b	0.22 (0.11)	0.24 (0.11)	0.23 (0.11)
c	0.004 (0.002)	0.004 (0.002)	0.004 (0.002)
RMSE[3] (kg/d)	0.31 (0.08)	0.44 (0.14)	0.38 (0.11)
Peak milk[4] (kg)	3.54 (0.55)	4.72 (0.72)	4.13 (0.64)
Peak time[5] (d)	63.85 (32.18)	56.81 (22.01)	60.33 (27.10)
Total milk (kg)	719.60 (149.14)	972.84 (204.34)	846.24 (176.69)
Perturbed lactation model (PLM)[2]			
a	2.16 (0.60)	2.77 (0.69)	2.53 (0.72)
b	0.17 (0.08)	0.19 (0.08)	0.18 (0.08)
c	0.003 (0.001)	0.003 (0.002)	0.003 (0.001)
RMSE[3] (kg/d)	0.18 (0.04)	0.25 (0.05)	0.22 (0.05)
Peak milk[4] (kg)	3.57 (0.47)	4.81 (0.71)	4.19 (0.59)
Peak time[5] (d)	63.51 (25.65)	69.46 (37.33)	66.49 (31.49)
S_N[6] (kg)	712.25 (147.60)	962.42 (201.67)	837.29 (174.58)
S_N^*[7] (kg)	766.28 (164.17)	1053.91 (232.29)	910.06 (198.19)
N	7.59 (1.30)	7.38 (1.47)	7.49 (1.39)
L (%)	6.02 (2.38)	7.43 (3.50)	6.73 (2.94)

N: mean of total number of perturbations, L: milk yield loss.
[1] Wood (1967): a, b and c: estimated Wood parameters.
[2] PLM based on Wood.
[3] RMSE of model fit.
[4] peak milk $= a\cdot\left(\frac{b}{c}\right)^{b}\cdot e^{-b}$.
[5] peak time $= \frac{a}{b}$.
[6] Total milk based on the PLM perturbed lactation curve: $S_N = \sum_{t_0}^{t_1} y(t)$.
[7] Total milk based on the PLM unperturbed lactation curve: $S_N^* = \sum_{t_0}^{t_1} y^*_{(t)}$.
[8] Number of lactation curves.

Number of perturbations and milk loss

The effects of parity and breed on the total number of perturbations were not significant. Total number of perturbations was 7.59 (SD = 1.30) for the primiparous, 7.38 (SD = 1.47) for the multiparous, 7.45 (SD = 1.41) for the Alpine and 7.47 (SD = 1.41) for the Saanen. By contrast, the rate of milk yield loss after perturbation was significantly affected by the parity. A Pearson linear correlation matrix by breed and parity between PLM_N^* estimates for the number of perturbations (N), percentage loss of milk yield (L) and goodness of fit (RMSE) was also carried out (Fig. 5, panels c and d). A positive correlation was found between RMSE and milk loss (0.38) suggesting that in highly perturbed curves (i.e., strongly deviating from the classical form), the concept of viewing a lactation curve as an unperturbed curve heckled by individual perturbations reaches its limit. However, weak negative correlations between the number of detected perturbations and RMSE (−0.16), and

Table 2
Comparison between breeds (SAA: Saanen and ALP: Alpine) and parity numbers of dairy goats across the models and variables.

Wood model[1]	SAA (143)			ALP (176)			P-value	
	1 (59[8])	2 + (84[8])	total (143[8])	1 (67[8])	2 + (109[8])	total (176[8])		
	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	Mean (SD)	Mean(SD)	Breed	Parity
a	1.84 (0.55)	2.44 (0.84)	2.20 (0.83)	1.92 (0.69)	2.34 (0.76)	2.17 (0.72)	NS	***
b	0.22 (0.11)	0.23 (0.11)	0.23 (0.11)	0.22 (0.11)	0.25 (0.12)	0.24 (0.12)	NS	NS
c	0.003 (0.002)	0.004 (0.002)	0.004 (0.002)	0.003 (0.001)	0.005 (0.002)	0.004 (0.02)	***	***
RMSE[3] (kg/d)	0.32 (0.87)	0.46 (0.15)	0.40 (0.15)	0.30 (0.08)	0.43 (0.12)	0.38 (0.13)	a,b	***
Peak milk[4] (kg)	3.56 (0.59)	4.69 (0.70)	4.25 (0.88)	3.53 (0.52)	4.75 (0.73)	4.26 (0.87)	NS	***
Peak time[5] (d)	74.28 (39.88)	60.23 (24.76)	67.26 (32.32)	54.66 (19.50)	54.17 (19.33)	54.42 (19.42)	a,b	a,b
Total milk (kg)	731.91 (150.04)	986.85 (223.17)	859.28 (186.63)	708.76 (148.61)	962.03 (188.91)	865.54 (168.78)	NS	***
Perturbed Lactation Model (PLM)[2]								
a	2.14 (0.49)	2.89 (0.71)	2.58 (0.73)	2.18 (0.68)	2.68 (0.66)	2.49 (0.71)	NS	***
b	0.16 (0.07)	0.16 (0.07)	0.16 (0.07)	0.17 (0.09)	0.20 (0.08)	0.19 (0.08)	***	NS
c	0.002 (0.001)	0.003 (0.001)	0.003 (0.001)	0.003 (0.001)	0.004 (0.001)	0.003 (0.001)	***	***
RMSE[3] (kg/d)	0.19 (0.05)	0.25 (0.04)	0.22 (0.05)	0.18 (0.03)	0.24 (0.06)	0.21 (0.06)	NS	***
Peak milk[4] (kg)	3.56 (0.44)	4.75 (0.68)	4.28 (0.83)	3.59 (0.50)	4.86 (0.73)	4.37 (0.90)	a,b	***
Peak time[5] (d)	77.73 (45.07)	67.81 (32.26)	68.89 (29.72)	57.80 (24.11)	60.56 (33.26)	59.50 (30.04)	***	NS
S_N[6] (kg)	723.99 (148.53)	976.64 (220.74)	850.34 (184.59)	701.91 (147.12)	951.36 (185.47)	826.62 (166.29)	NS	***
S_N^*[7] (kg)	780.74 (165.60)	1069.68 (255.55)	925.20 (210.56)	753.54 (163.07)	1041.65 (212.87)	897.59 (188.04)	NS	***
N	7.53 (1.28)	7.44 (1.51)	7.48 (1.41)	7.64 (1.33)	7.33 (1.45)	7.45 (1.41)	NS	NS
L (%)	6.19 (2.75)	7.51 (3.66)	6.97 (3.37)	5.87 (2.01)	7.36 (3.39)	6.79 (3.02)	NS	***

Signification codes: NS: not significant, $^{a,b}P < 0.05$, $^{***}P < 0.001$.
N: mean of total number of perturbations, *L*: milk yield loss.
[1] Wood (1967): *a*, *b* and *c*: estimated Wood parameters.
[2] Perturbed Lactation Model (PLM) based on Wood.
[3] RMSE of model fit.
[4] peak milk $= a \cdot \left(\frac{b}{c}\right)^b \cdot e^{-b}$.
[5] peak time $= \frac{a}{b}$.
[6] Total milk based on the PLM perturbed lactation curve: $S_N = \sum_{t_0}^{t_1} y(t)$.
[7] Total milk based on the PLM unperturbed lactation curve: $S_N^* = \sum_{t_0}^{t_1} y^*_{(t)}$.
[8] Number of lactation curves.

the number of perturbations and the milk loss (−0.20) were also found. Distributions of *N*, *L* and RMSE showed an even larger difference according to the parity than to the breeds. Rather than the total number of perturbations, these results show that it is the intensity of perturbations that contribute the most to the loss in milk yield over the lactation.

Perturbation timing and shape

Table 3 gives descriptive statistics on the parameters of PLM characterizing the 2 354 perturbations detected during the fitting procedure: time t_P, intensity k_0, collapse speed $k_1$and recovery speed k_2 according to the lactation stage determined with Grossman's model. Most of the perturbations were detected during the late and middle stages of lactation (respectively, $n = 1\,063$ and $n = 1\,054$) compared to those detected in the early stage ($n = 237$). The parameter $k_0$increased from early, middle to late lactation stage (Table 3). These results suggest that throughout the lactation process, perturbations become more intense. The parameter k_1 decreased from early to late stages of lactation. This suggests that perturbations tended to be sharper at the beginning of lactation, with a high speed of collapse and recovery, while they tended to be smoother when the lactation progressed.

The PLM parameter k_0, which drives the intensity of the perturbation, varied considerably between 0.001 and 1 (set as a boundary). The parameter k_1(which drives the collapse speed of the perturbation), and the parameter k_2(which drives the speed of recovery) varied between 0 and 10 (set as a boundary). These parameters tend to vary across lactation stages. A gradual increase in k_0 and a gradual decrease in k_1 and k_2 according to early, middle and late lactation stages were found (Table 3). In the late stage, 30.20% of the perturbations were detected with a parameter k_2 equal to 0, which implied a perturbation

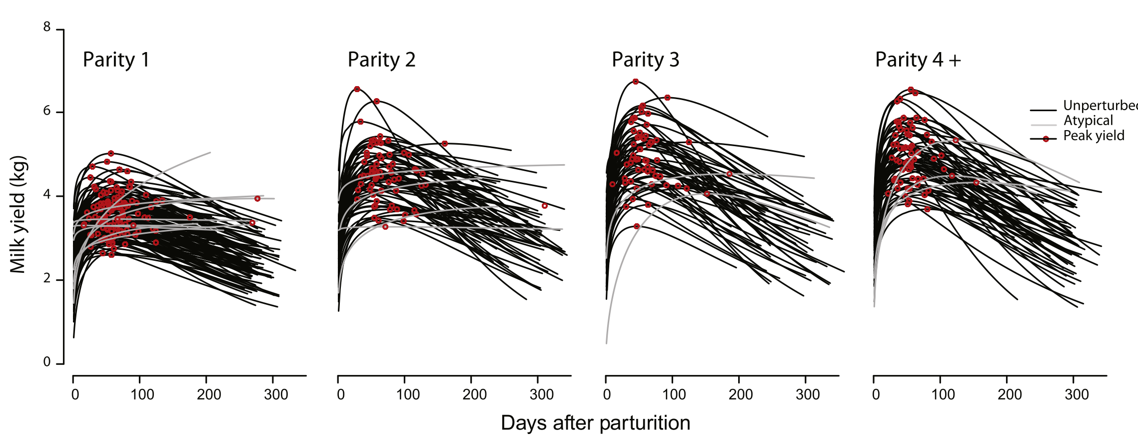

Fig. 4. Individual unperturbed curves extracted from data after removal of the estimated perturbations using perturbed lactation model (PLM) for increasing parity number (fit on 319 goat lactation data; atypical curves correspond to outlying estimates of the parameter *c* governing milk persistency).

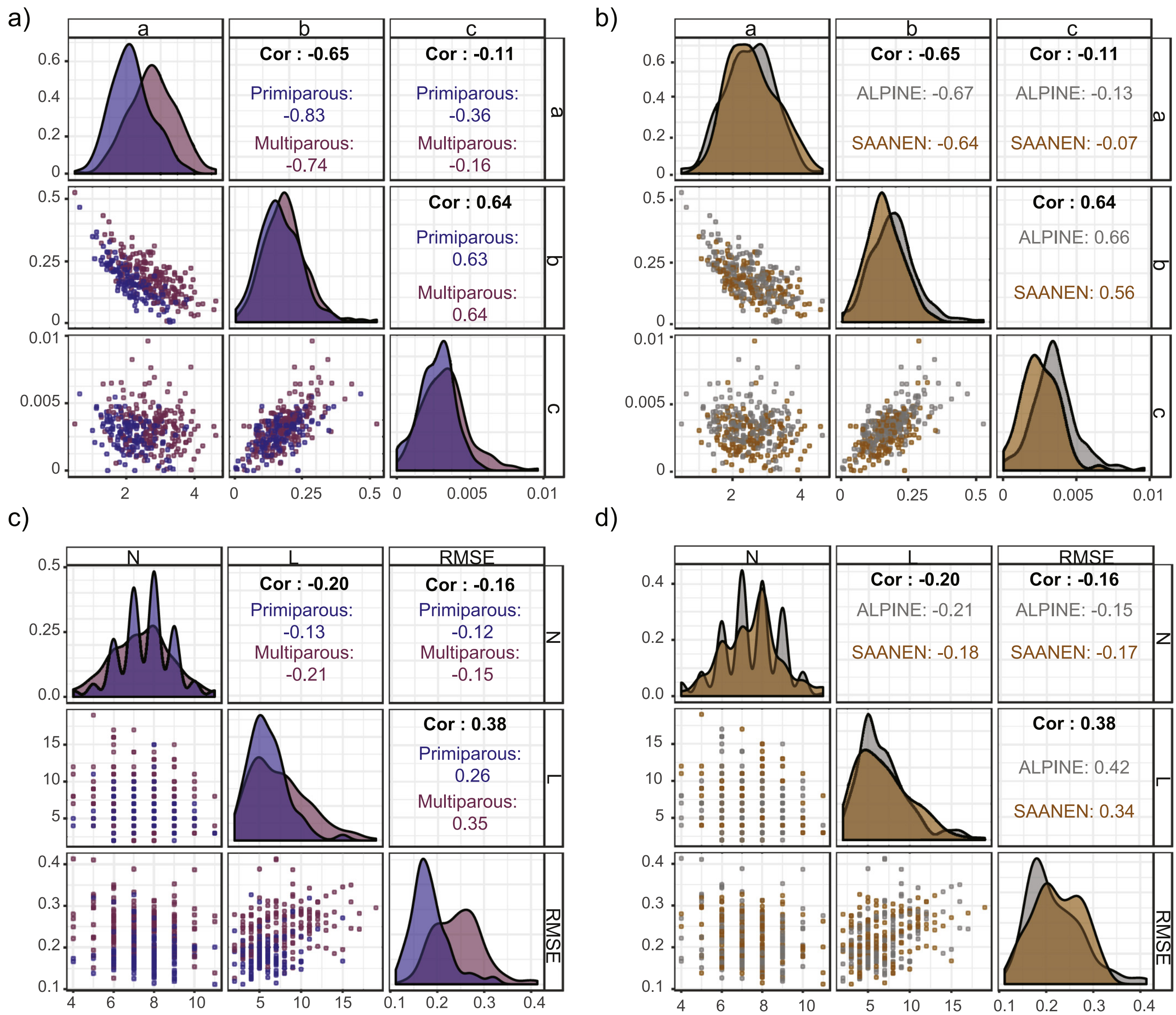

Fig. 5. Pearson linear correlation matrix of perturbed lactation model (PLM) parameters estimates. Panels (a) and (b): the *a*, *b* and *c* parameters defining the unperturbed curve (a: by parity and b: by breed of goat). Panels (c) and (d): the number of perturbations N, milk loss and RMSE (c: by parity and d: by breed of goat).

without any recovery period. Among these perturbations, 85.39% had a k_0 value equal to 1, which implies a perturbation affecting 100% of the milk yield. These perturbations correspond to the drop in milk yield at the very end of lactation, before drying off. On the other hand, in the early and middle stages, the perturbations detected with a k_2 equal to 0 were 1.70 and 7.07%, respectively.

Table 3
Descriptive statistics of the disturbance parameters for the 2 354 disturbances detected by the disturbed lactation model at the early, middle and late of lactation (estimated by Grossman model) in dairy goats.

Perturbations	Stage of lactation (2 354)					
	Early (237)		Middle (1 054)		Late (1 063)	
	Mean	SD	Mean	SD	Mean	SD
tp: starting time	33.8	34.0	107	63.0	202	60.0
k_0: intensity	0.45	0.33	0.51	0.35	0.67	0.36
k_1: collapse speed	4.01	4.17	3.41	3.87	2.76	3.69
k_2: recovery speed	1.13	1.96	1.18	1.79	0.95	1.71

(1) k_0 corresponds to the maximal proportion of milk withdrawn at nadir of a perturbation, (2) k_1 and k_2 are fractional rates of the PLM differential system (unit is d-1) and correspond to the rates of change of the proportion of milk withdrawn during, respectively, the collapse and recovery phase of a perturbation.

Discussion

Combining an unperturbed curve model with models of individual perturbations

In this study, we described the PLM model proposed as a tool for extracting simultaneously perturbed and unperturbed lactation curves from daily milk time series. The key novelty feature of PLM is to combine an explicit representation of perturbations with a mathematical representation of the theoretical shape of the lactation curve.

Regarding the mathematical representation of the lactation curve, the structure of PLM is generic and any equation can be used to describe the general pattern of milk production throughout lactation (see Fig. S1 in Supplementary material showing illustration of results with other lactation models). The Wood model (Wood 1967) was chosen in this study as it is one of the most well-known and commonly used mathematical models of lactation curve. Behind the choice of considering a general pattern of lactation that is distorted by perturbations, the biological assumption is that the dairy female has a theoretical production potential (the unperturbed curve) corresponding to the expression of its genetics in a given environment. This genetic potential for milk production may not be fully expressed in the farm environment in part due to perturbations (the perturbed curve).

Regarding the representation of perturbations, we chose an explicit formalism with a compartmental structure for every single perturbation. With this conceptual choice, PLM overcomes limitations of recent models developed for capturing perturbations (Sadoul et al., 2015; Nguyen Ba et al., 2019; Revilla et al., 2019). It allows the capture of multiple perturbations with contrasted features: from a sharp and short drop (for instance due to a diarrhoea episode) to a long and slow decrease (for instance due to a subclinical infection). The PLM also allows to determine the time at which the perturbations occur during the lactation. This last point is of great interest to add value to on-farm data where challenges imposed to animals do not result from controlled trials and arise from the farm environment.

By combining a general model of the lactation curve with an explicit model of perturbations, PLM provides two key outputs: first, the unperturbed curve of the lactating female that reflects its production potential in a non-perturbed environment, and second the perturbed curve which reflects the production permitted by the farm environment. The PLM parameters ($k_{0,\ i}$, $k_{1,\ i}$ and $k_{2,\ i}$) provide the most useful information on the perturbed lactation curve including scale and shape for each perturbation. Indeed, by providing a perturbed curve, we give an estimate of the number of perturbations and for each perturbation an estimate of its time of start $t_{P,\ i}$, intensity $k_{0,\ i}$, collapse speed $k_{1,\ i}$ and recovery speed $k_{2,\ i}$. This not only allows PLM to be flexible in capturing different types of perturbations (e.g., gestation, drying off, disease), but also to produce metrics to compare the effect of these perturbations on milk yield.

Fitting perturbed lactation model to lactation data

Beyond the original concepts behind PLM, a key methodological issue was the development of the fitting algorithm. The number of parameters to be determined is important, including the Wood parameters of the unperturbed curve (3 parameters), and PLM parameters (4 parameters for each perturbation). To overcome the difficulty of estimating a high number of parameters, a 2-step algorithm was implemented. The first step of the procedure was to determine Wood parameters and the times when the perturbations start. The second step of the procedure was to determine PLM parameters. This 2-step algorithm was selected for three main reasons. The first one was related to the visual quality of the fitting results itself. Indeed, the obtained fitted curve is always very close to what would have been drawn after simply looking at the raw data and wondering what the lactation curve would be without perturbations. This proximity to what could have been inferred was considered intuitive, yet subjective. The second reason was related to the issue of finding the number of perturbations. The PLM procedure allows an automated determination of an optimal number of perturbations, without *a priori* estimates or use of an arbitrarily chosen stopping criterion. Preliminary results have shown that allowing a maximal number of 15 perturbations to be detected in the first step of the algorithm was enough for the considered data set. The third reason pertained to the model parameters identifiability issue (Muñoz-Tamayo et al., 2018). Since the fitting is based on a huge number of repeated fittings from which the systematically detected times of perturbations are retained, the 2-step fitting algorithm facilitates the practical identifiability of the model parameters. This was demonstrated by applying the overall fitting algorithm several times to the same data set. Given that obtained parameter estimates were the same between the different runs, not only it strengthens the convergence properties of the algorithm but also it guarantees model parameters identifiability.

Fitting results (see Fig. 6) have shown that, in some cases, parameter estimates characterizing an individual perturbation reached their initial upper boundaries (1 for parameter $k_{0,\ i}$ and 10 for parameters $k_{1,\ i}$ and $k_{2,\ i}$). This situation concerns perturbations with a narrow and deep peak shape. By construction, the value of the parameter $k_{0,\ i}$ (intensity of the perturbation) is a proportion and thus not supposed to exceed 1. For the parameters $k_{1,\ i}$ and $k_{2,\ i}$, a value of 10 already represents a very abrupt collapse or recovery, respectively. These results are therefore considered relevant. However, a next step may be to test the model on a larger data set to assess the need to broaden these boundaries.

Perspective of using perturbed lactation model as a phenotyping tool

The PLM was developed to improve the ability to phenotype animals by extracting biological meaningful information from raw data. The unperturbed curve fitted by PLM makes it possible to compare animals based on their potential of milk production. With this information, animals can be ranked based on the production level they would have achieved in a non-perturbed environment, instead of being ranked based on the measured production level assuming no perturbations were encountered. This ranking may be of interest for the farmer's breeding strategy, to identify animals that have both a high production potential and ability to cope with their environment or animals that are able to recover fast after a challenge.

The perturbed curve and the characteristics of each perturbation (time, intensity, collapse and recovery) open the perspective of working on perturbations as such and using this information for breeding and management. As a phenotyping tool, PLM can be useful for genetic selection. Studying characteristics of perturbations throughout many lactations of a large number of individuals and linking them to genetic or genomic information opens perspectives to evaluate their heritability and their potential genetic impact. PLM can also be a valuable tool for on-farm management. Linking perturbations with other information on the animals, such as lactation stage, parity, gestation stage, can help to detect sensitive periods where perturbations are more likely to occur. By cross-checking information on perturbations from all animals with information on the farm environment (for instance temperature, feed availability), it would be possible to detect synchronous occurrences of perturbations and link them to farm environmentor management practices during times of stress. With this better understanding of environmental effects on animal production, on-farm preventive measures could be made.

Understanding the effects of the environment on-farm animals and understanding how they cope with challenges during crucial times could help to gain insights on resilience and robustness. These complex dynamic properties are highly desirable to face the changes occurring in the livestock sector (Dumont et al., 2014). While the conceptual framework to work on resilience and robustness is now well defined in animal sciences, we still need operational metrics (Friggens et al., 2017). Such metrics have been proposed for a single perturbation by Revilla et al. (2019) and Sadoul et al. (2015). The detection of perturbations in animal performance can provide a proxy to estimate the frequency and severity of disorders such as clinical mastitis (Erb et al., 1985). We found a low perturbation rate at the early lactation period. Normally, this period is known by the fragility of the animals facing problems such as metabolic diseases and mastitis. This can be explained by the definition of the duration of the early lactation period. Some authors, such as De Haas et al. (2008) or Urioste et al. (2012), arbitrarily fixed the duration of this period from 5 to 150 days. In this paper, the early lactation period was defined by Grossman's model and their duration varied from 1.2 to 34.4 days. Studying perturbations in lactation curves also makes it possible to compare animals facing the same stress and detect the ones with the greatest adaptive capacities.

To our knowledge, existing metrics for dropped milk yields per day in the lactation curve, as proposed by Elgersma et al. (2018) or Adriaens et al. (2020), are based on a variance approach applied to the whole curve. Fluctuations in milk yield are summarized with a single

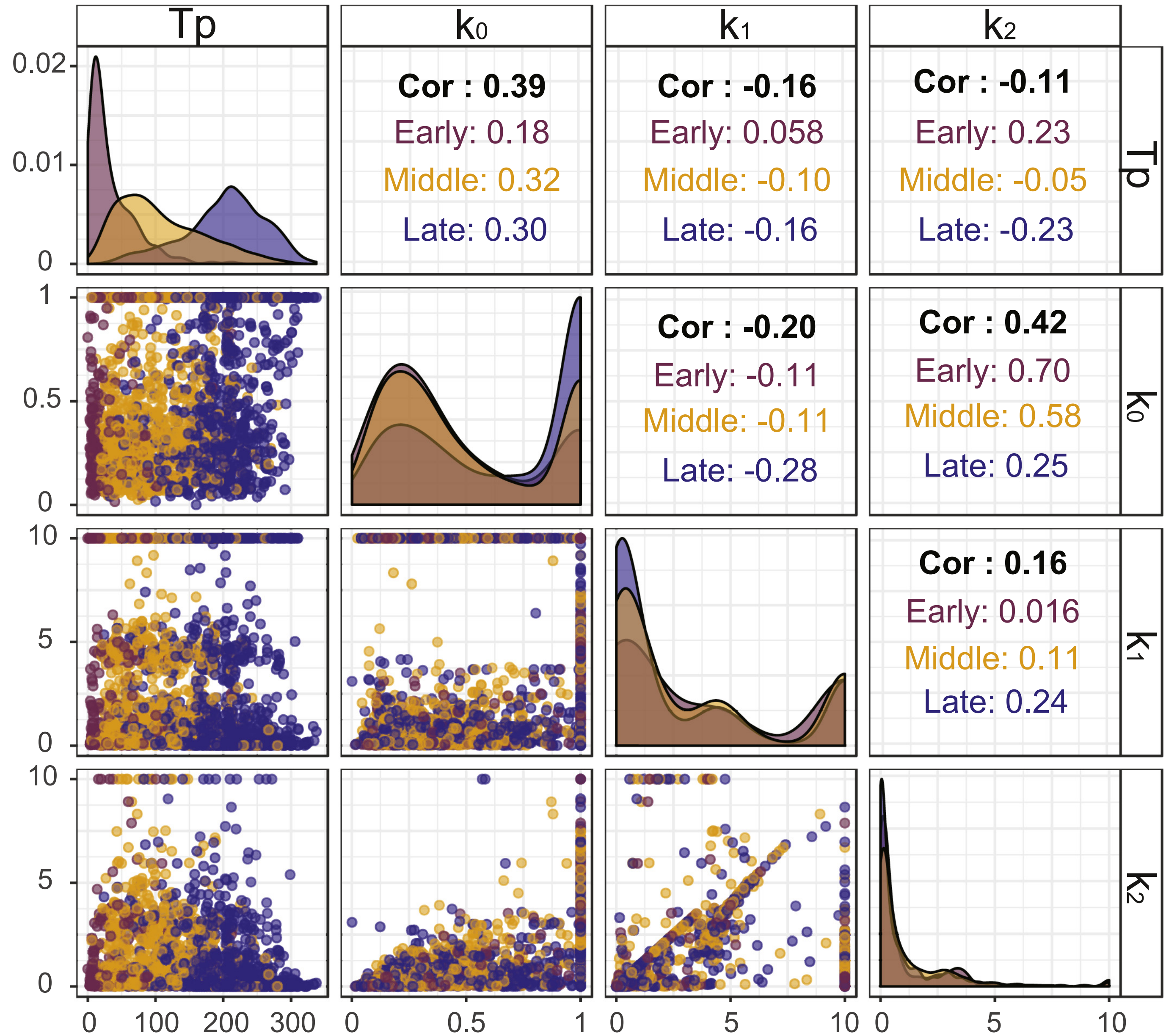

Fig. 6. Pearson linear correlation matrix on the perturbed lactation model (PLM) parameters by stage of lactation in dairy goat: tp: perturbations times detected; k_0: intenstity, k_1: collapse and k_2: recovery of perturbation.

statistical measure. Complementary to this type of approach, PLM can decompose the whole curve and characterize each perturbation, with metrics that are consistent with the concept of resilience of each and subsequent perturbation. The PLM model offers a way of quantifying the consequences of external factors and exploring hypotheses about the biological types of responses due to specific perturbations. With this respect, PLM is an interpretive tool providing information with a biological meaning. With the development of on-farm technology measurements, an interesting perspective for PLM is to be used to assess other biological time series data, such as BW changes, DM intake and hormones dynamics during lactation.

Perturbed lactation model limitations

A major limitation of PLM resides in its dependency on the quality of data. Indeed, if data are recorded with a low accuracy (due to technical problems of measurements), the outputs of PLM do not have consistency as detected perturbations have nothing to do with perturbations of the lactation curve, but are related to accuracy problem as intuitively identified when evaluating the curve shape. In addition, PLM has been developed with daily records. It will be necessary to evaluate if PLM can operate correctly with less frequent data.

Conclusion

By combining a general description of the lactation curve with an explicit representation of perturbations, the PLM model allows the characterization of two complementary aspects of milk production: the potential production in a non-limiting environment, reflecting genetic potential of a dairy female, and the deviations induced by the real farm conditions, reflecting the capacity of a dairy female to cope with the environment. Translating raw time series data into quantitative indicators makes it possible to compare the phenotypic responses of animals to challenges and therefore bring insights on their resilience to external factors. In that sense, PLM could be used as a valuable phenotyping tool and it contributes to provide decision solutions for dairy production that are grounded in a biologically meaningful framework.

Supplementary materials

Supplementary data to this article can be found online at https://doi.org/10.1016/j.animal.2020.100074.

Ethics approval

Not applicable.

Data and model availability statement

All relevant data are within the paper and its Supporting Information files. The data used in this paper and the R code for the Perturbed Lactation Model are public and accessible on the ZENODO data warehouse (https://zenodo.org/record/3241372#.XWOYxell9WY).

Author ORCIDs

Ben Abdelkrim Ahmed: 0000-0002-5721-0944 Puillet Laurence: 0000-0003-2358-7302 Gomes Pierre: 0000-0003-2641-0940 Martin Olivier: 0000-0001-7375-2850.

Author contributions

Ben Abdelkrim Ahmed: Conceptualization, methodology, visualization, validation, software, writing – original draft preparation; Puillet Laurence: Conceptualization, data curation, writing – review and editing; Gomes Pierre: Conceptualization, methodology, preliminary tests; Martin Olivier: Conceptualization, supervision, writing – review and editing.

Declaration of interest

None.

Acknowledgements

We gratefully acknowledge the team at the INRAE UMR 791 Modélisation Systémique Appliquée aux Ruminants (Paris, France) experimental installation for the care of the animals and their work to provide robust performance data. Special thanks to Dr. R. Muñoz-Tamayo for his conscientious reading and meticulous corrections of the manuscript. This work was carried out with the financial support of the ANR – Agence Nationale de la Recherche – The French National Research Agency under the "Deffilait project" (ANR; project: ANR-15-CE20-0014). This article was deposited as a pre-print has been reviewed and recommended by Peer Community In Animal Science (https://animsci.peercommunityin.org; https://doi.org/10.24072/pci.animsci.100001).

Financial support statement

This work was supported by ANR - Agence Nationale de la Recherche - The French National Agence de recherche; Projet Deffilait; ANR-15-CE20-0014, France and INRAE UMR 791 Modélisation Systémique Appliquée aux Ruminants.

References

Adriaens, I., Huybrechts, T., Aernouts, B., Geerinckx, K., Piepers, S., De Ketelaere, B., Saeys, W., 2018. Method for short-term prediction of milk yield at the quarter level to improve udder health monitoring. Journal of Dairy Science 101, 10327–10336.

Adriaens, I., Friggens, N.C., Ouweltjes, W., Scott, H., Aernouts, B., Statham, J., 2020. Productive life span and resilience rank can be predicted from on-farm first-parity sensor time series but not using a common equation across farms. Journal of Dairy Science 103, 7155–7171.

Arnal, M., Robert-Granie, C., Larroque, H., 2018. Diversity of dairy goat lactation curves in France. Journal of Dairy Science 101, 11040–11051.

Beever, D.E., Rook, A.J., France, J., Dhanoa, M.S., Gill, M., 1991. A review of empirical and mechanistic models of lactational performance by the dairy cow. Livestock Production Science 29, 115–130.

Berghof, T., Bovenhuis, H., Mulder, H., 2019. Body weight deviations as indicator for resilience in layer chickens. Frontiers in Genetics 10, 1216.

Casado, C., Piquer, O., Cervera, C., Pascual, J.J., 2006. Modelling the lactation curve of rabbit does: towards a model including fit suitability and biological interpretation. Livestock Science 99, 39–49.

Cobby, J.M., Le Du, Y.L.P., 1978. On fitting curves to lactation data. Animal Science 26 (suppl. 2), 127–133.

Codrea, M.C., Højsgaard, S., Friggens, N.C., 2011. Differential smoothing of time-series measurements to identify disturbances in performance and quantify animal response characteristics: an example using milk yield profiles in dairy cows. Journal of Animal Science 89, 3089–3098.

De Haas, Y., Ouweltjes, W., Ten Napel, J., Windig, J.J., De Jong, G., 2008. Alternative somatic cell count traits as mastitis indicators for genetic selection. Journal of Dairy Science 91, 2501–2511.

Delage, J., Leroy, A.M., Poly, J., 1953. Une étude sur les courbes de lactation. Annales de Zootechnie 2, 225–267.

Dhanoa, M.S., 1981. A note on an alternative form of the lactation model of wood. Animal Production 32, 349–351.

Dumont, B., González-García, E., Thomas, M., Fortun-Lamothe, L., Ducrot, C., Dourmad, J.Y., Tichit, M., 2014. Forty research issues for the redesign of animal production systems in the 21st century. Animal 8, 1382–1393.

Elgersma, G.G., de Jong, G., van der Linde, R., Mulder, H.A., 2018. Fluctuations in milk yield are heritable and can be used as a resilience indicator to breed healthy cows. Journal of Dairy Science 101, 1240–1250.

Erb, H.N., Smith, R.D., Oltenacu, P.A., Guard, C.L., Hillman, R.B., Powers, P.A., Smith, M.C., White, M.E., 1985. Path model of reproductive disorders and performance, milk fever, mastitis, milk yield, and culling in holstein cows. Journal of Dairy Science 68, 3337–3349.

France, J., Thornley, J.H., 1984. Mathematical models in agriculture. Butterworths, London, UK.

Friggens, N.C., Duvaux-Ponter, C., Etienne, M.P., Mary-Huard, T., Schmidely, P., 2016. Characterizing individual differences in animal responses to a nutritional challenge: toward improved robustness measures. Journal of Dairy Science 99, 2704–2718.

Friggens, N.C., Blanc, F., Berry, D.P., Puillet, L., 2017. Review: deciphering animal robustness. A synthesis to facilitate its use in livestock breeding and management. Animal 11, 2237–2251.

Gipson, T., Grossman, M., 1990. Lactation curves in dairy goats: a review. Small Ruminant Research 3, 383–396.

Grossman, M., Hartz, S.M., Koops, W.J., 1999. Persistency of lactation yield: a novel approach. Journal of Dairy Science 82, 2192–2197.

Huxley, J.N., 2013. Impact of lameness and claw lesions in cows on health and production. Livestock Science 156, 64–70.

Lescourret, F., Coulon, J.B., 1994. Modeling the impact of mastitis on milk production by dairy cows. Journal of Dairy Science 77, 2289–2301.

Martin, O., Sauvant, D., 2002. Meta analysis of input/output kinetics in lactating dairy cows. Journal of Dairy Science 85, 3363–3381.

Muñoz-Tamayo, R., Puillet, L., Daniel, J.B., Sauvant, D., Martin, O., Taghipoor, M., Blavy, P., 2018. Review: to be or not to be an identifiable model. Is this a relevant question in animal science modelling? Animal 12, 701–712.

Nguyen Ba, H., Van Milgen, J., Taghipoor, M., 2019. A procedure to quantify the feed intake response of growing pigs to perturbations. Animal 13, 1–8.

Poppe, M., Veerkamp, R.F., Van Pelt, M.L., Mulder, H.A., 2020. Exploration of variance, autocorrelation, and skewness of deviations from lactation curves as resilience indicators for breeding. Journal of Dairy Science 103, 1667–1684.

R Development Core Team, 2018. R: A language environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria.

Rajala-Schultz, P.J., Gröhn, Y.T., McCulloch, C.E., Guard, C.L., 1999. Effects of clinical mastitis on milk yield in dairy cows. Journal of Dairy Science 82, 1213–1220.

Revilla, M., Friggens, N.C., Broudiscou, L.P., Lemonnier, G., Blanc, F., Ravon, L., Mercat, M.J., Billon, Y., Rogel-Gaillard, C., Le Floch, N., Estellé, J., Muñoz-Tamayo, R., 2019. Towards the quantitative characterisation of piglets' robustness to weaning: a modelling approach. Animal 13, 1–11.

Ruiz, R., Oregui, L.M., Herrero, M., 2000. Comparison of models for describing the lactation curve of latxa sheep and an analysis of factors affecting milk yield. Journal of Dairy Science 83, 2709–2719.

Sadoul, B., Martin, O., Prunet, P., Friggens, N.C., 2015. On the use of a simple physical system analogy to study robustness features in animal sciences. PLoS ONE 10, e0137333.

Urioste, J.I., Franzén, J., Windig, J.J., Strandberg, E., 2012. Genetic relationships among mastitis and alternative somatic cell count traits in the first 3 lactations of Swedish Holsteins. Journal of Dairy Science 95, 3428–3434.

West, J.W., 2003. Effects of heat-stress on production in dairy cattle. Journal of Dairy Science 86, 2131–2144.

Wilmink, J.B.M., 1987. Adjustment of test-day milk, fat and protein yield for age, season and stage of lactation. Livestock Production Science 16, 335–348.

Windig, J.J., Calus, M.P.L., Veerkamp, R.F., 2005. Influence of herd environment on health and fertility and their relationship with milk production. Journal of Dairy Science 88, 335–347.

Wood, P., 1967. Algebraic model of the lactation curve in cattle. Nature 216, 164.