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MATHS – 3

Q1

a

ANS $L\{t^2(e^{3t} + \sinh 2t)\}$

Finding Laplace of $e^{3t} + \sinh 2t$

$$\therefore L\{e^{3t} + \sinh 2t\} = \frac{1}{s-3} + \frac{2}{s^2-4}$$

Now $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \phi(s)$

$$\therefore L[t^2(e^{3t} + \sinh 2t)] = \frac{d^2}{ds^2}\left[\frac{1}{s-3} + \frac{2}{s^2-4}\right]$$

$$= \frac{d}{ds}\left[\frac{-1}{(s-3)^2} + \frac{-2(2s)}{(s^2-4)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{(s-3)^2} + \frac{4s}{(s^2-4)^2}\right]$$

$$= -\left[\frac{-1(2)(s-3)}{(s-3)^4}\right] - 4\left[\frac{(s^2-4)^2 - s(2)(s^2-4)(2s)}{(s^2-4)^4}\right]$$

$$= \frac{2(s-3)}{(s-3)^4} + 4\left[\frac{4s^2(s^2-4) - (s^2-4)^2}{(s^2-4)^4}\right]$$

$$= \frac{2}{(s-3)^3} + \frac{16s^2}{(s^2-4)^3} - \frac{4}{(s^2-4)^2}$$

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b

ANS $L^{-1}\left(\frac{s}{s^4-1}\right) \Rightarrow \phi(s) = \frac{s}{s^4-1}$

$$\phi(s) = \frac{As+B}{s^2-1} + \frac{Cs+D}{s^2+1}$$

$$\therefore s = (As+B)(s^2+1) + (Cs+D)(s^2-1)$$

$$\therefore s = s^3(A+C) + s^2(B+D) + s(A-C) + B-D$$

$\therefore$ $A+C=0 \Rightarrow A=-C$

$B+D=0 \Rightarrow B=-D$

$A-C=1$

$B-D=0$

$\therefore$ $-2C=1$

$\therefore C = -1/2,\ A = 1/2,\ B=0,\ D=0$

$$\therefore \phi(s) = \frac{1}{2}\frac{s}{s^2-1} - \frac{1}{2}\frac{s}{s^2+1}$$

$$L^{-1}[\phi(s)] = \frac{1}{2}L^{-1}\left[\frac{s}{s^2-1}\right] - \frac{1}{2}L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= \frac{1}{2}\cosh t - \frac{1}{2}\cos t$$

$$\therefore L^{-1}\left(\frac{s}{s^4-1}\right) = \frac{1}{2}(\cosh t - \cos t)$$

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Q3

ANS $f(x) = \cos x$

Half range sine and cosine series of $\cos x$ is

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x . \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin nx \times \cos x \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin (n+1)x + \sin(n-1)x \, dx\right]$$

$$= \frac{-1}{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left\{\left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right] - \left[\frac{1}{n+1} + \frac{1}{n-1}\right]\right\}$$

$$= \frac{-1}{\pi}\left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{-1}{\pi}\left[\frac{(-1)^{n+1}-1}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right]$$

$$= \left[\frac{1-(-1)^{n+1}}{\pi}\right]\left[\frac{n-1+n+1}{n^2-1}\right]$$

$$= \frac{2\left(1-(-1)^{n+1}\right)}{\pi} \frac{n}{n^2-1}$$

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$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \cdot \sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[\frac{1-1}{2}\right] = 0$$

Half range sine series of $\cos x$ is given by

$$\cos x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=2}^{\infty} \frac{2\left[1-(-1)^{n+1}\right]}{\pi} \frac{n}{n^2-1} \sin nx$$

$$\therefore \cos x = \sum_{n=2}^{\infty} \frac{2\left[1-(-1)^{n+1}\right]}{\pi} \sin nx \cdot \frac{n}{n^2-1}$$

$$\cos x = \frac{2}{\pi}\left[\frac{\sin 2x \cdot 2}{2^2-1} + \frac{\sin 4x \cdot 4}{4^2-1} + \frac{\sin 6x \cdot 6}{6^2-1} + \cdots\right]$$

put $x = \frac{\pi}{4}$

$$\frac{1}{\sqrt{2}} = \frac{4}{\pi}\left[\frac{2}{2^2-1} - \frac{6}{6^2-1} + \frac{10}{5^2-1} + \cdots\right]$$

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$$\frac{1}{\sqrt{2}} = \frac{8}{\pi}\left[\frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} - \frac{7}{14^2-1} + \cdots\right]$$

$$\boxed{\frac{\pi}{8\sqrt{2}} = S}$$

$$\therefore \frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} + \cdots = \frac{\pi}{8\sqrt{2}} \quad //$$

Q4

a $f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

Fourier transform of $f(x)$ =

$$F(\alpha) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} (1-x^2)\, e^{i\alpha x} dx + \int_{-\infty}^{-2} 0 \cdot e^{i\alpha x} dx + \int_{2}^{\infty} 0 \cdot e^{i\alpha x} dx\right]$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} (1-x^2)\, e^{i\alpha x}\, dx\right]$$

4a

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \left[(1-x^2)\frac{e^{i\alpha x}}{i\alpha} - (-2x)\frac{e^{i\alpha x}}{(i\alpha)^2} + (-2)\frac{e^{i\alpha x}}{(i\alpha)^3} \right]_{-1}^{1}$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \left\{ \left[0 + \frac{2e^{i\alpha}}{(i\alpha)^2} - \frac{2e^{i\alpha}}{(i\alpha)^3} \right] - \left[0 + \frac{(-2)e^{-i\alpha}}{(i\alpha)^2} - \frac{2e^{-i\alpha}}{(i\alpha)^3} \right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{(i\alpha)^2}(e^{i\alpha} + e^{-i\alpha}) - \frac{2}{(i\alpha)^3}(e^{i\alpha} - e^{-i\alpha}) \right]$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \left[\frac{2}{(i\alpha)^2}(2\cos\alpha) - \frac{2}{(i\alpha)^3}[2i\sin\alpha) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[4\left(\frac{\sin\alpha}{\alpha^3} - \frac{\cos\alpha}{\alpha^2} \right) \right]$$

$$= \frac{2\sqrt{2}}{\alpha^3\sqrt{\pi}}(\sin\alpha - \alpha\cos\alpha)$$

//

4b

ANS $f(x) = xe^{-ax}$

$$F_c(e^{-ax}) = \int_0^\infty e^{-ax}\cos sx\, dx = \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + \sin sx) \right]_0^\infty$$

$$= 0 - \frac{1}{a^2+s^2}(-a\cos 0 + 0) = \frac{a}{a^2+s^2}$$

Now $F_s(x e^{-ax}) = -\frac{d}{ds}\left[F_c(e^{-ax})\right] = -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right)$

$$= \frac{-(-1)\,2as}{(s^2+a^2)^2}$$

$$\boxed{= \frac{2as}{s^2+a^2}}$$

$$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \sin sx \, dx = \left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$= 0 - \frac{-s\cos 0}{a^2+s^2}$$

$$= \frac{s}{a^2+s^2}$$

$$F_c(x e^{-ax}) = \frac{d}{ds}\left\{F_s(e^{-ax})\right\} = \frac{d}{ds}\left(\frac{s}{a^2+s^2}\right)$$

$$= \frac{(a^2+s^2)(1) - s(2s)}{(a^2+s^2)^2}$$

$$\boxed{= \frac{a^2-s^2}{(a^2+s^2)^2}}$$

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Q6

Q $f(k) = k^2$

ANS

Z transform is given by $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) . z^{-k}$

$\therefore \; z\{1\} = \sum_{k=0}^{\infty} (1)\, z^{-k}$

$= \frac{1}{z^0} + \frac{1}{z^1} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$

$= \frac{1}{1 - 1/z} \quad \dots \left[\text{since } \left|\frac{1}{z}\right| < 1, \text{ its finite}\right]$

$F(z) = \frac{z}{z-1} \longrightarrow (i)$

Now, $Z\{k^2 f(k)\} = \left(-z\frac{d}{dz}\right)^2 F(z)$

$\therefore \; Z\{k^2 . 1\} = \left(-z\frac{d}{dz}\right)^2 F(z)$

$= \left(-z\frac{d}{dz}\right)^2 \left(\frac{z}{z-1}\right)$

$= \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]$

$= \left(-z\frac{d}{dz}\right)\left[-z \times \frac{(z-1)(1) - z(1)}{(z-1)^2}\right]$

$= \left(-z\frac{d}{dz}\right)\left[-z \times \frac{-1}{(z-1)^2}\right]$

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$$\therefore Z\{K^2 \cdot 1\} = \left(-z\frac{d}{dz}\right)\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\frac{d}{dz}\left[\frac{(z-1)^2(1) - 2(2)(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{z^2-2z+1-2z^2+2z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2+1}{(z-1)^4}\right]$$

$$= \frac{z(z+1)(z-1)}{(z-1)^4}$$

$$= \frac{z(z+1)}{(z-1)^3}$$

$$\therefore Z\{k^2\} = \frac{z(z+1)}{(z-1)^3} = \epsilon(z)$$

b

ANS To find : $Z\{f(K) = g(K)\}$

Given : $f(K) = \frac{1}{3^K}$ $\quad g(K) = \frac{1}{5^K}$

By convolution theorem : $Z\{f(K) * g(K)\} = Z\{f(K)\} * Z\{g(K)\}$

$$Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K) \cdot z^{-K} \qquad [\text{since } K \geq 0]$$

$$= \sum_{K=0}^{\infty} \frac{1}{3^K z^K}$$

$$= \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \cdots\right]$$

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6b

$\because \quad 1 + x + x^2 + x^3 + \dots = \dfrac{1}{1-x}$

$\therefore \quad Z\{f(K)\} = \dfrac{1}{1 - 1/3z} \qquad \left[\text{where } \left|\dfrac{1}{3z}\right| < 1\right]$

$= \dfrac{3z}{3z-1} \qquad \left[\dfrac{1}{3} < |z|\right]$

$\therefore \quad F(z) = Z\{f(K)\} = \dfrac{3z}{3z-1} \qquad \left[\dfrac{1}{3} < |z|\right]$

NOW, $\quad Z\{g(K)\} = \displaystyle\sum_{K=0}^{\infty} g(K)\, z^{-K}$

$= \displaystyle\sum_{K=0}^{\infty} \dfrac{1}{5^K z^K}$

$= \left[1 + \dfrac{1}{5z} + \dfrac{1}{(5z)^2} + \dfrac{1}{(5z)^3} + \dots\right]$

$\because \quad 1 + x + x^2 + x^3 + \dots = \dfrac{1}{1-x}$

$\therefore \quad Z\{g(K)\} = \dfrac{1}{1 - 1/5z} = \dfrac{5z}{5z-1} \qquad \left[\dfrac{1}{5} < |z|\right]$

$\therefore \quad Z\{f(K) * g(K)\} = \dfrac{3z}{3z-1} \times \dfrac{5z}{5z-1} = \dfrac{15z^2}{(3z-1)(5z-1)}$