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MATHS TERM TEST 2

Q1 Find Half Range Sine series for $f(x) = x(\pi - x)$ in $(0,\pi)$ and hence find the $\sum_{n=1}^{\infty} \frac{1}{n^6}$

ANS 1. $f(x) = x(\pi - x)$ in $[0,\pi]$

$[0,l] = [0,\pi]$

$\therefore l = \pi$

Half range sine series is given by $f(x) = \sum_{n=1}^{\infty} b_n . \sin nx$.

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x). \sin nx \, dx$

$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx \, dx$

$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$

$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$ $\to 0\ [\pi, 0]$

$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$

$= \frac{2}{\pi}\left[0 - \frac{2\cos n\pi}{n^3} - \left(0 - \frac{2}{n^3}\right)\right]$

$= \frac{2}{\pi}\left[\frac{2}{n^3} - \frac{2\cos n\pi}{n^3}\right]$

$= \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$ $[\because \cos n\pi = (-1)^n]$

$= \begin{cases} 0 & \text{for } n \text{ even} \\ 4/\pi n^3 & \text{for } n \text{ odd} \end{cases}$

$$\therefore f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[ \frac{1-(-1)^n}{n^3} \right] \sin nx$$

$$\therefore x(\pi - x) = \frac{8}{\pi} \left[ \frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} \ldots \right]$$

By Parseval's Identity,

$$\frac{1}{\pi} \int_0^{\pi} [f(x)]^2 dx = \frac{1}{2} \sum_{n=1}^{\infty} b_n^2 \quad \text{—— (1)}$$

Consider: $\frac{1}{\pi} \int_0^{\pi} [f(x)]^2 dx = \frac{1}{\pi} \int_0^{\pi} x^2 (\pi - x)^2 dx$

$$= \frac{1}{\pi} \int_0^{\pi} x^2 (\pi^2 - 2\pi x + x^2) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x^2\pi^2 - 2\pi x^3 + x^4 \, dx$$

$$= \frac{1}{\pi} \left[ \pi^2 \left[ \frac{x^3}{3} \right]_0^{\pi} - 2\pi \left[ \frac{x^4}{4} \right]_0^{\pi} + \left[ \frac{x^5}{5} \right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left\{ \frac{\pi^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5} \right]$$

$$= \pi^4 \left[ \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right]$$

$$= \frac{\pi^4}{30}$$

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Substitute in (1)

$$\therefore \frac{\pi^4}{30} = \frac{1}{2} \times \frac{1}{\pi} \sum_{n=1}^{\infty} \left[ \frac{1-(-1)^n}{n^3} \right]^2$$

$$\frac{\pi^4}{30} = \frac{8}{\pi^2} \left[ \frac{4}{1^6} + \frac{4}{3^6} + \frac{4}{5^6} \cdots \right]$$

$$\therefore \frac{\pi^6}{240 \times 4} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} \cdots$$

$$\text{let } S = \sum_{n=1}^{\infty} \frac{1}{n^6}$$

$$\therefore S = \left( \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} \cdots \right) + \left( \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \cdots \right)$$

$$S = \frac{\pi^6}{960} + \frac{1}{2^6} \left( \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots \right)$$

$$S = \frac{\pi^6}{960} + \frac{1}{64} S$$

$$\frac{63 S}{64} = \frac{\pi^6}{960}$$

$$\therefore S = \frac{\pi^6 \times 64}{960 \times 63} = \frac{\pi^6}{945}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

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Q2 Find the complex form of Fourier Series for $f(x) = e^{-x}$ in $(-2,2)$

ANS $f(x) = e^{-x}$ in $(-2, 2)$

Here $l = 2$

Complex form is given by :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi x}{l}}$$

$$= \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi x}{2}}$$

where $c_n = \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx$

$$= \frac{1}{4} \int_{-2}^{2} e^{-x} \cdot e^{-in\pi x/2}\, dx$$

$$= \frac{1}{4} \int_{-2}^{2} e^{-\left(\frac{2+in\pi}{2}\right)x}\, dx$$

$$= \frac{1}{4} \left[ \frac{e^{-\left(\frac{2+in\pi}{2}\right)x}}{-\left(\frac{2+in\pi}{2}\right)} \right]_{-2}^{2}$$

$$= \frac{1}{4} \left[ \frac{-2}{(2+in\pi)} e^{-(2+in\pi)} + \frac{2}{2+in\pi} e^{(2+in\pi)} \right]$$

$$= \frac{1}{2(2+in\pi)} \left[ e^{(2+in\pi)} - e^{-(2+in\pi)} \right]$$

$$= \frac{(2-in\pi)}{2(4+n^2\pi^2)} \left[ e^{2} \cdot e^{in\pi} - e^{-2} \cdot e^{-in\pi} \right]$$

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$$= \frac{(2-in\pi)}{4^2+n^2\pi^2}(-1)^n\left[\frac{e^2-e^{-2}}{2}\right] \quad \left[\because e^{in\pi}=(-1)^n\right]$$

$$C_n = \frac{(-1)^n(2-in\pi)}{4^2+n^2\pi^2}\sinh(2)$$

$$\therefore f(x) = \sinh(2)\,.\sum_{n=-\infty}^{\infty}\frac{(-1)^n(2-in\pi)}{4+n^2\pi^2}\,e^{\frac{in\pi x}{2}}$$

$$\therefore e^{-x} = \sinh(2)\,.\sum_{n=-\infty}^{\infty}\frac{(-1)^n(2-in\pi)}{4+n^2\pi^2}\,e^{\frac{in\pi x}{2}}\,.$$

Q3 Show that the set of functions $\{\sin(2n+1) \quad n=0,1,2\ldots\}$ are orthogonal on $[0, \pi/2]$ and hence construct an orthonormal set from this on $[0, \pi/2]$

ANS Given : $f_n(x) = \sin(2n+1) \qquad n = 0, 1, 2\ldots$

$\therefore f_m(x) = \sin(2m+1) \qquad m = 0, 1, 2\ldots$

range = $[0, \pi/2]$

$\therefore$ For orthogonal : $\int_0^{\pi/2} f_m(x) . f_n(x) . dx = 0$

$$\therefore LHS = \int_0^{\pi/2} \sin(2n+1)x . \sin(2m+1)x \; dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos(2m-2n)x + \cos(2m+2n+2)x \; dx$$

$$= \frac{1}{2}\left[\frac{\sin(2m-2n)x}{2(m-n)} + \frac{\sin(2m+2n+2)}{2(m+n+1)}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\frac{\sin 2x(m-n)}{m-n} + \frac{\sin 2x(m+n+1)}{m+n+1}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\frac{\sin(m-n)\pi}{m-n} + \frac{\sin(m+n+1)\pi}{m+n+1} - \frac{\sin 0}{m-n} - \frac{\sin 0}{m+n+1}\right]$$

$\because \sin n\pi = 0$ and $\sin 0 = 0$

$\therefore LHS = \frac{1}{4}[0] = 0$

$= RHS$

Hence the function $\{\sin(2n+1) \quad n = 0, 1, 2\ldots$ is orthogonal on $[0, \pi/2]\}$

For orthonormal : $\int_0^{\pi/2} f_n(x) \cdot f_n(x)\, dx = 1$

$\therefore$ LHS $= \int_0^{\pi/2} \sin(2n+1)x \cdot \sin(2n+1)x\, dx$

$= \int_0^{\pi/2} \sin^2 (2n+1)x\, dx$

$= \frac{1}{2} \int_0^{\pi/2} 1 - \cos 2(2n+1)x\, dx$

$= \frac{1}{2} \left[ x - \frac{\sin 2(2n+1)x}{2(2n+1)} \right]_0^{\pi/2}$

$= \frac{1}{2} \left[ \frac{\pi}{2} - 0 - \frac{\sin(2n+1)\pi + \sin 0}{2(2n+1)} \right]$

But $\sin n\pi = \sin 0 = 0$

$\therefore$ LHS $= \frac{\pi}{4}$

$\therefore \int_0^{\pi/2} \sin(2n+1)x \cdot \sin(2n+1)x\, dx = \frac{\pi}{4}$

Dividing both sides by $\pi/4$

$\int_0^{\pi/2} \frac{\sin(2n+1)x}{\sqrt{\pi/4}} \cdot \frac{\sin(2n+1)x}{\sqrt{\pi/4}} = 1$

$\therefore$ For orthonormal set on $[0, \pi/2]$, $f_n(x) = \frac{\sin(2n+1)x}{\sqrt{\pi/4}}$

Q4 a. Find Fourier series for $f(x) = x + x^2$ in $(-\pi, \pi)$ hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

ANS $f(x) = x + x^2$ is the sum of odd function $f_1(x) = x$ and even function $f_2(x) = x^2$.
Hence, Fourier expansion of $f(x)$ is the sum of the Fourier expansions of $f_1(x)$ and $f_2(x)$.

Since $f_1(x) = x$ is an ODD function $[\because a_n = 0]$

Let $f_1(x) = x = \sum_{n=1}^{\infty} b_n \cdot \sin nx.$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x.\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x(-\cos nx)}{n} - \frac{1(-\sin nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi(-1)(-1)^n}{n} - 0\right\} - \{0\}\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{2(-1)^{n+1}}{n}$$

$$\therefore f_1(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.\sin nx.$$

FOR EDUCATIONAL USE

Now, since $f_2(x) = x^2$ is an even function $[\because b_n = 0]$

let $f_2(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos nx$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - (2x)\frac{(-\cos nx)}{n^2} + \frac{2(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 + \frac{2\pi \cos n\pi}{n^2} - 0\right\} - \{0\}\right]$$

$$= \frac{4\cdot(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$$\therefore f_2(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$\therefore x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

For deduction, we consider the series of $x^2$ only.

$$\therefore x^2 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cos nx \longrightarrow (1)$$

(i) putting $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{-1\cos\pi}{1^2} + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{—— (2)}$$

(ii) putting $x = 0$ in (1)

$$0 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad \text{—— (3)}$$

(iii) Adding (2) and (3)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right)$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$