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MATHS TUTORIAL 2

✓ Q1 $L^{-1}\left\{\frac{s^2}{(s^2-a^2)^2}\right\}$

✓ Q2 $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

✓ Q3 $L^{-1}\left\{\frac{s}{(s+2)^2(s+1)}\right\}$

Q4 $L^{-1}\left\{\frac{1}{s^3+a^3}\right\}$

✓ Q5 $L^{-1}\left\{\frac{1}{(s-1)^4(s+3)}\right\}$

✓ Q6 Evaluate $\int_0^\infty e^{-t}\left[\int_0^t \frac{\sin u}{u}\,du\right]dt$

✓ Q7 If $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$

then S.T. $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

Q8 Find $L^{-1}\left\{\frac{3s+2}{5s^2+4s+7}\right\}$

ANS 1. $L^{-1}\left\{\frac{s^2}{(s^2-a^2)^2}\right\}$

$= L^{-1}\left\{\frac{s}{s^2-a^2} \cdot \frac{s}{s^2-a^2}\right\}$

$= L^{-1}\left\{\bar{f}(s)\ \bar{g}(s)\right\}$ [let]

$\therefore f(t) = L^{-1}(\bar{f}(s)) = \cosh at = g(t)$

By convolution theorem

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = \int_0^t f(u).g(t-u)\,du$$

$$= \int_0^t \cosh(au).\cosh(at-au)\,du$$

$$= \int_0^t \frac{e^{au}+e^{-au}}{2} \times \frac{e^{at-au}+e^{au-at}}{2}\,du$$

$$= \frac{1}{4}\int_0^t e^{at} + e^{2au-at} + e^{at-2au} + e^{-at}\,du$$

$$= \frac{1}{4}\left[u(e^{at}+e^{-at}) + \frac{e^{2au-at}}{2a} - \frac{e^{at-2au}}{2a}\right]_0^t$$

$$= \frac{1}{4}\left[t(e^{at}+e^{-at}) + \frac{e^{at}}{2a} - \frac{e^{-at}}{2a} - \frac{e^{-at}}{2a} + \frac{e^{at}}{2a}\right]$$

$$= \frac{1}{4}\left[t(e^{at}+e^{-at}) + \frac{1}{a}(e^{at}-e^{-at})\right]$$

$$= \frac{1}{2a}\left[at.\cosh(at) + \sinh(at)\right] //$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2-a^2)^2}\right\} = \frac{1}{2a}\left[at.\cosh(at) + \sinh(at)\right]$$

ANS 2. To find : $L^{-1}\left\{\dfrac{s}{s^4+4a^4}\right\}$

$$= L^{-1}\left\{\frac{s}{s^4+4s^2a^2+4a^4-4s^2a^2}\right\}$$

$$= L^{-1}\left\{\frac{s}{(s^2+2a^2)^2-(2as)^2}\right\}$$

let $I = L^{-1}\left\{\dfrac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right\}$

$$\frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)} = \frac{A}{s^2+2a^2-2as} + \frac{B}{s^2+2a^2+2as}$$

putting $s=0$: $0 = A(2a^2) + B(2a^2)$

$\therefore A = -B \longrightarrow (i)$

putting $s=1$: $1 = A(\cancel{1}+2a+\cancel{2a^2}) - A(\cancel{1}+\cancel{2a^2}-2a)$

$1 = 4a\,A$

$\therefore A = -B = \dfrac{1}{4a}$

$$\therefore I = L^{-1}\left\{\frac{1}{4a[(s-a)^2+a^2]} + \left(\frac{-1}{4a}\right)\cdot\frac{1}{[(s+a)^2+a^2]}\right\}$$

$$= \frac{1}{4a}L^{-1}\left\{\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - e^{-at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right] \quad [\text{First Shift th}^{m}]$$

$$= \frac{1}{4a}\left[\frac{e^{at}\sin at}{a} - \frac{e^{-at}\sin at}{a}\right]$$

$$= \frac{1}{4a^2}\sin at\,(e^{at}-e^{-at})$$

Q3 To find $L^{-1}\left\{\frac{s}{(s+2)^2(s+1)}\right\}$

$$\bar{f}(s) = \frac{s}{(s+2)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$= \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+2)^2(s+1)}$$

put $s = -1$: $-1 = A$

put $s = 0$: $0 = 4A + 2B + C$

put $s = 1$: $1 = 9A + 6B + 2C$

$\therefore B = 1, \; C = 2$

$$\therefore \bar{f}(s) = \frac{-1}{s+1} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

$$L^{-1}(\bar{f}(s)) = -e^{-t} L^{-1}\left(\frac{1}{s}\right) + e^{-2t} L^{-1}\left(\frac{1}{s}\right) + e^{-2t} L^{-1}\left(\frac{2}{s^2}\right)$$

$$= -e^{-t} + e^{-2t} + 2e^{-2t}\frac{t^1}{\lceil 2} \qquad \left[L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\lceil n}\right]$$

$$L^{-1}\{\bar{f}(s)\} = 2e^{-2t}\cdot t + e^{-2t} - e^{-t} \;//$$

Q4 To find : $L^{-1}\left\{\frac{1}{s^3+a^3}\right\}$

$$s^3 + a^3 = (s+a)(s^2 - as + a^2)$$

$$\bar{f}(s) = \frac{1}{(s+a)(s^2-as+a^2)} = \frac{A}{s+a} + \frac{Bs+C}{s^2-as+a^2}$$

$$= \frac{A(s^2-as+a^2) + (Bs+C)(s+a)}{s^2 - as + a^2}$$

$$\therefore \bar{f}(s) = \frac{(A+B)s^2 + (B-A)as + Aa^2 + Ca}{(s+a)(s^2-as+a^2)} = \frac{1}{(s+a)(s^2-as+a^2)}$$

$\therefore A+B=0 \Rightarrow A=-B$

$\therefore Aa^2+Ca=1 \Rightarrow \cancel{C=-Aa} = Ba$

$\therefore C+(B-A)a=0 \Rightarrow C+2aB=0 \Rightarrow C=-2aB=2Aa$

$\therefore Aa^2+a(2Aa)=1$

$3a^2A=1 \Rightarrow A=\frac{1}{3a^2}$; $B=\frac{-1}{3a^2}$; $C=\frac{2}{3a}$

$$\bar{f}(s) = \frac{1}{3a^2}\left[\frac{1}{s+a}\right] - \frac{1}{3a^2}\left[\frac{s-2a}{s^2-as+a^2}\right]$$

$$\bar{f}(s) = \frac{1}{3a^2}\left[\frac{1}{s+a} - \frac{(s-2a)}{s^2-as+a^2}\right]$$

$$L^{-1}(\bar{f}(s)) = \frac{1}{3a^2}\left[L^{-1}\left\{\frac{1}{s+a}\right\} - L^{-1}\left\{\frac{s-2a/_4 \mp 3a/2}{s^2-as+a^2/4+3a^2/4}\right\}\right]$$

$$L^{-1}(\bar{f}(s)) = \frac{1}{3a^2}\left[e^{-at}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left\{\frac{s-a/2}{(s-\frac{a}{2})^2+3a^2/4} - \frac{3a/2}{(s-\frac{a}{2})^2+3a^2/4}\right\}\right]$$

$$L^{-1}\{\bar{f}(s)\} = \frac{1}{3a^2}\left[e^{-at} - e^{at/2}L^{-1}\left\{\frac{s}{s^2+3a^2/4}\right\} + \sqrt{3}e^{at/2}\left\{L^{-1}\frac{\frac{\sqrt{3}a}{2}}{s^2+3a^2/4}\right\}\right]$$

$$L^{-1}\{\bar{f}(s)\} = \frac{1}{3a^2}\left[e^{-at} - e^{at/2}\left[\cos\frac{\sqrt{3}}{2}at - \sqrt{3}\sin\frac{\sqrt{3}}{2}at\right]\right]$$

$$= \frac{e^{-at}}{3a^2} - \frac{e^{at/2}}{3a^2}\cos\left(\frac{\sqrt{3}at}{2}\right) + \frac{e^{at/2}}{\sqrt{3}a^2}\sin\left(\frac{\sqrt{3}at}{2}\right) \quad //$$

Q5 To find : $L^{-1}\left\{\frac{1}{(s-1)^4(s+3)}\right\}$

$$L^{-1}\left\{\frac{1}{(s-1)^4(s+3)}\right\} = L^{-1}\left\{\frac{1}{(s-1)^4} \times \frac{1}{s+3}\right\} = L^{-1}\{\bar{f}(s).\bar{g}(s)\}$$

$$f(t) = L^{-1}\left\{\frac{1}{(s-1)^4}\right\} = e^t L^{-1}\left(\frac{1}{s^4}\right) = e^t . \frac{t^3}{6}$$

$$g(t) = e^{-3t} L^{-1}\left(\frac{1}{s}\right) = e^{-3t}$$

By convolution theorem

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = \int_0^t f(u).g(t-u).du = \int_0^t e^u \frac{u^3}{6} e^{-3(t-u)} du$$

$$= \frac{e^{-3t}}{6} \int_0^t e^{4u} . u^3 du$$

$$= \frac{e^{-3t}}{6}\left[\frac{u^3e^{4u}}{4} - \frac{3u^2e^{4u}}{16} + \frac{6ue^{4u}}{64} - \frac{6e^{4u}}{256}\right]_0^t$$

$$= \frac{e^{-3t}}{6}\left[\frac{t^3e^{4t}}{4} - \frac{3t^2e^{4t}}{16} + \frac{6te^{4t}}{64} - \frac{6e^{4t}}{256} + \frac{6}{256}\right]$$

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = \frac{e^t}{1536}\left[64t^3 - 48t^2 + 24t - 6 + 6e^{-4t}\right] \quad //$$

Q6 To evaluate : $\int_0^t e^{-t}\left[\int_0^t \frac{\sin u}{u}\, du\right] dt$

We know $\int_0^t e^{-st} f(t)\, dt = L(f(t)) = \bar{f}(s)$

$\therefore$ $s = 1$ and $f(t) = \int_0^t \frac{\sin u\, du}{u}$

Let $g(u) = \sin u$

$\therefore L(g(u)) = \frac{1}{s^2+1} \Rightarrow L\left\{\frac{g(u)}{u}\right\} = \int_s^\infty \bar{g}(s)\, ds$

$$L\left\{\frac{g(u)}{u}\right\} = \int_s^\infty \frac{1}{s^2+1}\, ds = \left[\tan^{-1}(s)\right]_s^\infty = \cot^{-1}(s)$$

$$L\left\{\frac{g(u)}{u}\right\} = \cot^{-1} s = \bar{h}(s) = L\{h(u)\}$$

$$L\left[\int_0^t h(u)\, du\right] = \frac{1}{s}\bar{h}(s) = \frac{1}{s}\cot^{-1} s$$

But $s = 1$

$$\therefore \int_0^\infty e^{-t}\left[\int_0^t \frac{\sin u\, du}{u}\right] dt = \frac{1}{1}\cot^{-1}(1) = \frac{\pi}{4}$$

$$\therefore \int_0^\infty e^{-t}\left[\int_0^t \frac{\sin u\, du}{u}\right] dt = \frac{\pi}{4}$$ //

Q7 Given : $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$

To Prove : $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

$$f(t) = \sqrt{\frac{4t}{\pi}} \Rightarrow \bar{f}(s) = \frac{1}{s^{3/2}}$$

$$L(f(t)) = \bar{f}(s)$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s)\,ds = \int_s^{\infty} s^{-3/2}\,ds$$

$$L\left\{\frac{2\sqrt{t/\pi}}{t}\right\} = \left[\frac{s^{-1/2}}{-1/2}\right]_s^{\infty} = -2\left[\frac{1}{\sqrt{s}}\right]_s^{\infty} = \frac{2}{\sqrt{s}} = 2L\left\{\frac{1}{\sqrt{\pi t}}\right\}$$

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}} \quad //$$

Q8 To find : $L^{-1}\left\{\frac{3s+2}{5s^2+4s+7}\right\}$

$$\bar{f}(s) = \frac{3s+2}{5s^2+4s+7} = \frac{3}{5}\left[\frac{s+2/3}{s^2+4s/5+7/5}\right]$$

$$= \frac{3}{5}\left[\frac{s+2/5+2/3-2/5}{s^2+4s/5+4/25+7/5-4/25}\right]$$

$$\bar{F}(s) = \frac{3}{5}\left[\frac{s+2/5}{(s+2/5)^2+31/25}\right] + \frac{4/5}{5\left[(s+2/5)^2+31/25\right]}$$

$$L^{-1}\bar{F}(s) = \frac{3}{5}L^{-1}\left[\frac{s+2/5}{(s+2/5)^2+31/25}\right] + \frac{4}{25}\left[\frac{1}{\left(s+\frac{2}{5}\right)^2+\frac{31}{25}}\right]$$

$$L^{-1}\bar{F}(s) = \frac{3}{5}e^{-2t/5}\left[L^{-1}\left(\frac{s}{s^2+31/25}\right)\right] + \frac{4}{25}\times\sqrt{\frac{25}{31}}\left[L^{-1}\left[\frac{\sqrt{31/25}}{s^2+31/25}\right]\right]e^{-2/5\,t}$$

$$L^{-1}\bar{F}(s) = \frac{3e^{-\frac{2t}{5}}}{5}\cos\left(\frac{\sqrt{31}\,t}{5}\right) + \frac{4}{5\sqrt{31}}e^{-\frac{2t}{5}}\sin\left(\frac{\sqrt{31}\,t}{5}\right)$$

$$L^{-1}\bar{F}(s) = \frac{e^{-\frac{2t}{5}}}{5}\left[3\cos\left(\frac{\sqrt{31}\,t}{5}\right) + \frac{4}{\sqrt{31}}\sin\left(\frac{\sqrt{31}\,t}{5}\right)\right]$$

$$L^{-1}\left\{\frac{3s+2}{5s^2+4s+7}\right\} = \frac{e^{-2t/5}}{5}\left[3\cos\left(\frac{\sqrt{31}\,t}{5}\right) + \frac{4}{\sqrt{31}}\sin\left(\frac{\sqrt{31}\,t}{5}\right)\right] //$$