TUTORIAL - 7

Q1 Find the fourier transform of $f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| \ge 1 \end{cases}$

Q2 Find the fourier transform of $f(x) = x^2 e^{-x^2/2}$

Q3 The fourier cosine transform of $f(x) = \begin{cases} \sqrt{17/2} & 0 < t < 5 \\ 0 & t > 5 \end{cases}$

ANS 1 $f(x) = \begin{cases} 1 - x^2 & |x| \le 1 & \text{i.e. } -1 < x < 1 \\ 0 & |x| > 1 & x > 1 \ \& \ x < -1 \end{cases}$

Fourier transform of $f(x)$ is given by :

$$F(\alpha) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$\therefore F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-x^2)\, e^{i\alpha x} dx + \int_{-\infty}^{-2} 0 . e^{i\alpha x} dx + \int_{2}^{\infty} 0 . e^{i\alpha x} . dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{(1-x^2)\, e^{i\alpha x}}{i\alpha} - \frac{(-2x)\, e^{i\alpha x}}{(i\alpha)^2} + \frac{(-2)\, e^{i\alpha x}}{(i\alpha)^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \frac{2e^{i\alpha}}{(i\alpha)^2} - \frac{2e^{i\alpha}}{(i\alpha)^3} - 0 - \frac{(-2)e^{-i\alpha}}{(i\alpha)^2} + \frac{2e^{-i\alpha}}{(i\alpha)^3} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{(i\alpha)^2} (e^{i\alpha} + e^{-i\alpha}) - \frac{2}{(i\alpha)^3} (e^{i\alpha} - e^{-i\alpha}) \right]$$

$$\because \quad e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha$$
$$e^{i\alpha} - e^{-i\alpha} = 2i\sin\alpha$$

$$\therefore \; F(\alpha) = \frac{1}{\sqrt{2\pi}}\left[\frac{2}{(i\alpha)^2}(2\cos\alpha) - \frac{2}{(i\alpha)^3}(2i\sin\alpha)\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{\alpha^2}(2\cos\alpha) + \frac{2}{i\alpha^3}(2i\sin\alpha)\right]$$
$$= \frac{2\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sin\alpha}{\alpha^3} - \frac{\cos\alpha}{\alpha^2}\right]$$
$$= \sqrt{\frac{2}{\pi}} \times \frac{2}{\alpha^3}\left[\sin\alpha - \alpha\cos\alpha\right]$$

$\therefore$ Fourier transform of $f(x)$ is $\sqrt{\frac{2}{\pi}} \times \frac{2}{\alpha^3}\left[\sin\alpha - \alpha\cos\alpha\right]$

ANS 2 $\quad f(x) = x^2 e^{-x^2/2}$

Let $g(x) = e^{-x^2/2}$

Now, $F[g(x)]$ is given by :

$$F[g(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x) \cdot e^{i\alpha x}\, dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{i\alpha x} \cdot dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} - i\alpha x\right)} dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}} - \frac{i\alpha}{\sqrt{2}}\right)^2 - \left(\frac{i\alpha}{\sqrt{2}}\right)^2\right]} dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{x - i\alpha}{\sqrt{2}}\right)^2} \cdot e^{-\frac{\alpha^2}{2}}\, dx$$

let $x - i\alpha = t$ $\quad x \to -\infty \quad t \to -\infty$

Differentiating w.r.t x $\quad x \to \infty \quad t \to \infty$

$\therefore dx = dt$

$$\therefore F[g(x)] = \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2} \int_{-\infty}^{\infty} e^{-t^2/2} dt$$

$\because e^{-t^2/2}$ is an Even function

$$\therefore F[g(x)] = 2 \times \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2} \int_0^{\infty} e^{-t^2/2} dt$$

$$= \sqrt{\frac{2}{\pi}} e^{-\alpha^2/2} \int_0^{\infty} e^{-t^2/2} dt$$

let $\frac{t^2}{2} = m$ $\quad t \to 0 \quad m \to 0$

$\quad t \to \infty \quad m \to \infty$

Differentiating w.r.t t

$t\,dt = dm$

$\therefore dt = \frac{dm}{\sqrt{2m}}$

$$\therefore F[g(x)] = \sqrt{\frac{2}{\pi}} e^{-\alpha^2/2} \times \frac{1}{2} \int_0^{\infty} m^{-1/2} e^{-m} dm$$

$$\because \int_0^{\infty} e^{-m} m^{n-1} dm = \Gamma n$$

$$\therefore F[g(x)] = \frac{1}{\sqrt{\pi}} e^{-\alpha^2/2} \times \Gamma\frac{1}{2} = \frac{1}{\sqrt{\pi}} e^{-\alpha^2/2} \times \sqrt{\pi}$$

$$\therefore F[g(x)] = e^{-\alpha^2/2}$$

Now, $F[x^n g(x)] = (-1)^n \frac{d^n}{d\alpha^n} F(\alpha)$

$\therefore F[x^2 e^{-x^2/2}] = (-i)^2 \frac{d^2}{d\alpha^2} F[g(x)]$

$= (-1) \frac{d}{d\alpha}\left[\frac{d}{d\alpha} e^{-\alpha^2/2}\right]$

$= (-1) \frac{d}{d\alpha}\left[e^{-\alpha^2/2}\left(-\frac{2\alpha}{2}\right)\right]$

$= \frac{d}{d\alpha}\left[\alpha e^{-\alpha^2/2}\right]$

$= \alpha e^{-\alpha^2/2}\left(\frac{-2\alpha}{2}\right) + e^{-\alpha^2/2}(1)$

$\therefore F[x^2 e^{-x^2/2}] = e^{-\alpha^2/2}(1-\alpha^2)$

$\therefore$ Fourier transform of $f(x)$ is : $e^{-\alpha^2/2}(1-\alpha^2)$

ANS 3 Fourier cosine transform of $f(x)$ is given by :

$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) . \cos\alpha x . dx$

$= \sqrt{\frac{2}{\pi}}\left[\int_0^5 \sqrt{\frac{\pi}{2}} \cos\alpha x . dx + \int_5^\infty 0 . \cos\alpha x . dx\right]$

$= \sqrt{\frac{2}{\pi}} \times \sqrt{\frac{\pi}{2}} \times \int_0^5 \cos\alpha x . dx$

$= \left[\frac{\sin\alpha x}{\alpha}\right]_0^5$

$= \frac{\sin 5\alpha}{\alpha}$

$\therefore$ The Fourier Cosine transform of $f(x)$ is $\frac{\sin 5\alpha}{\alpha}$