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MATHS

TUTORIAL - 2

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Q1 Using method of Lagrangian multipliers solve the following problem.

Optimize $z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

Subject to $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$

$x_1, x_2, x_3 \geq 0$

ANS We have $f(x_1, x_2, x_3) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 15$

$h_2(x_1, x_2, x_3) = 2x_1 - x_2 + 2x_3 - 20$

Now we construct Lagranges function.

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) - \lambda_1 h_1(x_1, x_2, x_3) - \lambda_2 h_2(x_1, x_2, x_3)$$

$$= 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

Calculating partial derivative w.r.t each variable,

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15)$$

$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20)$$

For obtaining stationary point,

$\frac{\partial L}{\partial x_1} = 0$; $\frac{\partial L}{\partial x_2} = 0$; $\frac{\partial L}{\partial x_3} = 0$; $\frac{\partial L}{\partial \lambda_1} = 0$; $\frac{\partial L}{\partial \lambda_2} = 0$

$\therefore \quad 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \quad \text{—(1)}$

$4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{—(2)}$

$2x_3 - \lambda_1 - 2\lambda_2 = 0 \quad \text{—(3)}$

$x_1 + x_2 + x_3 = 15 \quad \text{—(4)}$

$2x_1 - x_2 + 2x_3 = 20 \quad \text{—(5)}$

Equation (3) × 4 + Equation (1)

$8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 + 8x_3 - 4\lambda_1 - 8\lambda_2 = 0$

$4(2x_1 - x_2 + 2x_3) = 5\lambda_1 + 10\lambda_2$

Now, from Equation (5): $2x_1 - x_2 + 2x_3 = 20$

$\therefore 5\lambda_1 + 10\lambda_2 = 80 \quad \text{—(6)}$

Now, multiply (1) by 2, (2) by 3, (3) by 2 and add

$16x_1 - 8x_2 - 2\lambda_1 - 4\lambda_2 + 12x_2 - 12x_1 - 3\lambda_1 + 3\lambda_2 + 4x_3 - 2\lambda_1 - 4\lambda_2 = 0$

$\therefore \quad 4x_1 + 4x_2 + 4x_3 - 7\lambda_1 - 5\lambda_2 = 0$

$\therefore \quad 4(x_1 + x_2 + x_3) - 7\lambda_1 - 5\lambda_2 = 0$

$\therefore \quad 4(x_1 + x_2 + x_3) = 7\lambda_1 + 5\lambda_2$

Now, from Equation (4): $x_1 + x_2 + x_3 = 15$

$\therefore 60 = 7\lambda_1 + 5\lambda_2 \quad \text{—(7)}$

Solving Equation (6) and Equation (7)

$5\lambda_1 + 10\lambda_2 = 80$

$7\lambda_1 + 5\lambda_2 = 60 \quad (\times 2)$

$(-)$

$14\lambda_1 + 10\lambda_2 = 120$

$-9\lambda_1 = -40$

$$\therefore \lambda_1 = \frac{40}{9} \qquad \therefore \lambda_2 = \frac{80 - 5\left(\frac{40}{9}\right)}{10} = \frac{52}{9}$$

Now from Equation (3)

$$x_3 = \frac{\lambda_1 + 2\lambda_2}{2} = \frac{40/9 + 104/9}{2} = 8$$

Adding Equation (1) and Equation (2)

$$4x_1 - 2\lambda_1 - \lambda_2 = 0$$

$$\therefore x_1 = \frac{2\lambda_1 + \lambda_2}{4} = \frac{2(40/9) + 52/9}{4} = \frac{11}{3}$$

From Equation (4), $x_2 = 15 - x_1 - x_3 = 15 - \frac{11}{3} - 8 = \frac{10}{3}$

$$\boxed{\therefore x_1 = \frac{11}{3}, \quad x_2 = \frac{10}{3}, \quad x_3 = 8, \quad \lambda_1 = \frac{40}{9}, \quad \lambda_2 = \frac{52}{9}}$$

Now, $\frac{\partial h_1}{\partial x_1} = 1$, $\frac{\partial h_2}{\partial x_2} = 1$, $\frac{\partial h_3}{\partial x_3} = 1$

$\frac{\partial h_2}{\partial x_1} = 2$, $\frac{\partial h_2}{\partial x_2} = -1$, $\frac{\partial h_2}{\partial x_3} = 2$

$\frac{\partial^2 L}{\partial x_1^2} = 8$, $\frac{\partial^2 L}{\partial x_1 \partial x_2} = -4$, $\frac{\partial^2 L}{\partial x_1 \partial x_3} = 0$

$\frac{\partial^2 L}{\partial x_2 \partial x_1} = -4$, $\frac{\partial^2 L}{\partial x_2^2} = 4$, $\frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$

$\frac{\partial^2 L}{\partial x_3 \partial x_1} = 0$, $\frac{\partial^2 L}{\partial x_3 \partial x_2} = 0$, $\frac{\partial^2 L}{\partial x_3^2} = 2$

Now, $H^B = \begin{bmatrix} O & P \\ P' & Q \end{bmatrix}$

$O \rightarrow$ Null matrix of order $2\times 2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}_{2\times 3}$$

$$P' = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}_{3\times 2}$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H^B = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Now, $\because m = 2 \quad \therefore 2m+1 = 5$

$\therefore$ we have to check sign of principal minor of index 5.

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$$\Delta_5 = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & +2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix}$$

$R_1 \leftrightarrow R_3$, $R_4 \rightarrow R_4 - R_1$ $= - \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & -3 & -12 & 8 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix}$

$R_5 \rightarrow R_5 - R_1$ $= - \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & -3 & -12 & 8 & 0 \\ 0 & 0 & -8 & 4 & 2 \end{vmatrix}$

$R_4 \leftrightarrow R_2$, $R_4 \rightarrow R_4 - 2R_3$ $= \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & -3 & -12 & 8 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & -8 & 4 & 2 \end{vmatrix}$

$R_5 \rightarrow R_5 + 8R_3$ $= \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & -3 & -12 & 8 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 12 & 10 \end{vmatrix}$

$R_5 \rightarrow R_5 + 4R_4$

$$\therefore \Delta_5 = \begin{vmatrix} 1 & 2 & 8 & -4 & 0 \\ 0 & -3 & -12 & 8 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{vmatrix}$$

$\because$ It is an upper triangular matrix

$\therefore$ Determinant will be product of diagonal elements.

$$\therefore \Delta_5 = 1(-3) \times 1 \times (-3) \times 10$$
$$= 90$$

$$\therefore \Delta_5 > 0$$

$\because$ Sign of principal minors of order 5 is positive

i.e. $(-1)^2 [(-1)^m]$

$\therefore x_0 (11/3, 10/3, 8)$ is a point of minima.

$$\therefore Z_{min} = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1 x_2$$
$$= 4\left(\frac{11}{3}\right)^2 + 2\left(\frac{10}{3}\right)^2 + (8)^2 - 4\left(\frac{11}{3}\right)\left(\frac{10}{3}\right)$$
$$= \frac{4 \times 121}{9} + \frac{2 \times 100}{9} + 64 - \frac{440}{9}$$

$$\boxed{\therefore Z_{min} = \frac{820}{9}}$$