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SE COMPS A-3

MATHS
TUTORIAL - 6

Q1 Find $Z\{K^2\}_{K\geq 0}$ hence find $Z\{K^2 e^{-aK}\}_{K\geq 0}$

Q2 Find $Z\left\{\left(\frac{1}{3}\right)^{|K|}\right\}$

Q3 Find $Z\{f(n) * g(K)\}$ $f(K) = 1/3^K$; $g(K) = 1/7^K$; $K \geq 0$

Q4 Find inverse Z transform of $F(z) = \dfrac{1}{(z-3)(z-2)}$ if R.O.C is

(i) $|z| < 2$

(ii) $2 < |z| < 3$

(iii) $|z| > 3$

Q5 Find $Z\{f(K) * g(K)\}_{K\geq 0}$; $f(K) = 4^K u(K)$; $g(K) = 5^K u(K)$.

ANS 1

Z - Transform is given by :-

$$Z\{f(K)\} = \sum_{K=0}^{\infty} f(K) \cdot z^{-K} \qquad (\because K \geq 0)$$

$$Z\{1\} = \sum_{K=0}^{\infty} (1)\, z^{-K} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

$$= \frac{1}{1 - 1/z} \qquad \left|\frac{1}{z}\right| < 1$$

$$\therefore F(z) = \frac{z}{z-1}$$

Now, $Z\{K^2 f(K)\} = \left(-z\dfrac{d}{dz}\right)^2 F(z)$.

$$\therefore\ Z\{K^2 \cdot 1\} = \left(-z\frac{d}{dz}\right)^2 F(z)$$

$$= \left(-z\frac{d}{dz}\right)^2\left(\frac{z}{z-1}\right)$$

$$= \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]$$

$$= \left(-z\frac{d}{dz}\right)\left[-z \times \frac{(z-1) - z(1)}{(z-1)^2}\right]$$

$$= \left(-z\frac{d}{dz}\right)\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z \times 2 \times (z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{z^2 - 2z + 1 - 2z^2 + 2z}{(z-1)^4}\right]$$

$$= -z\,\frac{(1-z^2)}{(z-1)^4}$$

$$= \frac{2\,(z+1)(z-1)}{(z-1)^4}$$

$$\therefore\ Z\{K^2\} = \frac{z(z+1)}{(z-1)^3} = E(z)$$

Now $Z\{f(K)\,e^{-aK}\} = E(e^a z)$

$$\therefore\ Z\{K^2 e^{-aK}\} = E(e^a z)$$

$$= \frac{e^a z\,(e^a z + 1)}{(e^a z - 1)^3}$$

$\therefore$ Z-transform of $K^2 e^{-aK}$ is $\dfrac{e^a z\,(e^a z + 1)}{(e^a z - 1)^3}$.

ANS 2

$$f(K) = \begin{cases} (1/3)^K & K > 0 \\ (1/3)^{-K} & K < 0 \end{cases}$$

Z-transform is given by :

$$Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K) \cdot z^{-K}$$

$$= \sum_{K=-\infty}^{-1} \left(\frac{1}{3}\right)^{-K} z^{-K} + \sum_{K=0}^{\infty} \left(\frac{1}{3}\right)^{K} z^{-K}$$

$$= \left[\cdots + \left(\frac{1}{3}\right)^3 z^3 + \left(\frac{1}{3}\right)^2 z^2 + \left(\frac{1}{3}\right) z\right]$$

$$+ \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \cdots\right]$$

$$= \frac{z}{3}\left[\frac{1}{3} + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right] + \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \cdots\right]$$

$\because$ $$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$\therefore$ $$Z\{f(K)\} = \frac{z}{3} \frac{1}{1 - z/3} + \frac{1}{1 - 1/3z} \qquad \left|\frac{z}{3}\right| < 1, \left|\frac{1}{3z}\right| < 1$$

$\therefore$ $$Z\{f(K)\} = \frac{z}{3} \times \frac{3}{3-z} + \frac{3z}{3z-1} \qquad |z| < 3, \frac{1}{3} < |z|$$

$$= \frac{z}{3-z} + \frac{3z}{3z-1}$$

$$= \frac{3z^2 - z + 9z - 3z^2}{(3-z)(3z-1)} \qquad \frac{1}{3} < |z| < 3$$

$\therefore$ $$Z\{f(K)\} = \frac{8z}{(3-z)(3z-1)}$$ //

ANS 3 $\quad f(K) = \dfrac{1}{3^K} \quad ; \quad g(K) = \dfrac{1}{7^K} \qquad K \geq 0$

By convolution theorem,

$$Z\{f(K) * g(K)\} = Z\{f(K)\} * Z\{g(K)\} \qquad \text{——— (1)}$$

Now, $\quad Z\{f(K)\} = \sum_{K=0}^{\infty} f(K)\, z^{-K} \qquad \because K \geq 0$

$$= \sum_{K=0}^{\infty} \frac{1}{3^K} z^{-K}$$

$$\therefore \; Z\{f(K)\} = \sum_{K=0}^{\infty} \left(\frac{1}{3z}\right)^K = \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \cdots\right]$$

$$\because \; 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

$$\therefore \; Z\{f(K)\} = \frac{1}{1 - 1/3z} \qquad \left|\frac{1}{3z}\right| < 1$$

$$Z\{f(K)\} = \frac{3z}{3z-1} \qquad \frac{1}{3} < |z|$$

Let $\quad F(z) = \dfrac{3z}{3z-1} \qquad \dfrac{1}{3} < |z| \qquad \text{——— (2)}$

Now,

$$Z\{g(K)\} = \sum_{K=0}^{\infty} g(K)\, z^{-K} \qquad \because K \geq 0$$

$$= \sum_{K=0}^{\infty} \frac{1}{7^K} z^{-K}$$

$$= \sum_{K=0}^{\infty} \frac{1}{(7z)^K}$$

$$= \left[1 + \frac{1}{7z} + \frac{1}{(7z)^2} + \frac{1}{(7z)^3} + \dots \right]$$

$\because \quad 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$

$\therefore \quad Z\{g(k)\} = \frac{1}{1 - 1/7z} \qquad \left|\frac{1}{7z}\right| < 1$

Let $G(z) = \frac{7z}{7z-1} \qquad \frac{1}{7} < |z|$ —— ③

From ①

$$Z\{f(k)\} * Z\{g(k)\} = F(z) * G(z)$$

$$= \frac{3z}{3z-1} \times \frac{7z}{7z-1}$$

$\therefore \quad Z\{f(k) * g(k)\} = \frac{21z^2}{(3z-1)(7z-1)}$

ANS 4 $\quad F(z) = \frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$$= \frac{1}{z-3} - \frac{1}{z-2}$$

(i) $|z| < 2$

$\therefore \quad |z| < 2 < 3 \qquad \therefore \quad |z| < 3$

$\therefore \quad \left|\frac{z}{2}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$

$\therefore \quad F(z) = \frac{-1}{3} \frac{1}{1 - z/3} + \frac{1}{2} \frac{1}{1 - z/2}$

$\because \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$$\therefore F(z) = \frac{-1}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right]$$
$$+ \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$
$$= -\left[\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \cdots\right] + \left[\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \frac{z^3}{2^4} + \cdots\right]$$

$\therefore$ Coefficient of z^k is $(-3^{-k-1} + 2^{-k-1})$ $\quad k \geq 0$

$\therefore$ coefficient of z^{-k} is $(-3^{k-1} + 2^{k-1})$ $\quad k \leq 0$

$\therefore \quad z^{-1}\{F(z)\} = 2^{k-1} - 3^{k-1}$ where $k \leq 0$

(ii) $2 < |z| < 3$

$$\left|\frac{2}{z}\right| < 1 \qquad \left|\frac{z}{3}\right| < 1$$

$$\therefore F(z) = \frac{-1}{3}\frac{1}{(1 - z/3)} - \frac{1}{2}\frac{1}{(1 - z/2)}$$

$\because (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$$\therefore F(z) = \frac{-1}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right] - \frac{1}{2}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right]$$
$$= -\left[\frac{1}{3}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right)\right] + \frac{1}{2}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right)\Bigg]$$

$$= \left\{\left[\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \cdots\right] + \left[\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \cdots\right]\right\}$$

$\therefore$ coefficient of z^k in first series is -3^{-k-1} $\quad k \geq 0$

$\therefore$ coefficient of z^{-k} in 2nd series is -2^{k-1} $\quad k \geq 1$

$\therefore$ coefficient of z^{-k} in first series is -3^{k-1} $\quad k \leq 0$

$$\therefore z^{-1}[f(z)] = \begin{cases} -3^{k-1} & k \leq 0 \\ -2^{k-1} & k \geq 1 \end{cases}$$

(iii) $|z| > 3$

$|z| > 3 > 2$ $\qquad$ $|z| > 2$

$\therefore \left|\frac{z}{3}\right| > 1$ $\qquad$ $\left|\frac{z}{2}\right| > 1$

$\therefore \left|\frac{3}{z}\right| < 1$ $\qquad$ $\left|\frac{2}{z}\right| < 1$

$$\therefore F(z) = \frac{1}{z}\left(\frac{1}{1-3/z}\right) - \frac{1}{z}\left(\frac{1}{1-2/z}\right)$$

$$= \frac{1}{z}(1-3/z)^{-1} - \frac{1}{z}(1-2/z)^{-1}$$

$$\because (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$\therefore F(z) = \frac{1}{z}\left[\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \dots\right) - \left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \dots\right)\right]$$

$$= \left[\left(\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \frac{3^3}{z^4} + \dots\right) - \left(\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \dots\right)\right]$$

Coefficient of z^{-k} in 1st series is 3^{k-1} $\quad k \geq 1$
Coefficient of z^{-k} in 2nd series is -2^{k-1} $\quad k \geq 1$

$$\therefore Z^{-1}\{F(z)\} = 3^{k-1} - 2^{k-1} \quad k \geq 1$$

ANS 5 $\quad f(n) = 4^k u(k)$ $\qquad g(k) = 5^k u(k)$ $\qquad k \geq 0$

$$u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Z - transform is given by $Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K).\, z^{-K}$

$$Z\{u(K)\} = \sum_{K=0}^{\infty} z^{-K}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$\because (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$\therefore Z\{u(K)\} = \frac{1}{(1 - 1/z)} \qquad \left|\frac{1}{z}\right| < 1$$

$$\therefore Z\{u(K)\} = \frac{z}{z-1} = F(z) \qquad |z| > 1$$

Now, $Z\{a^K f(K)\} = F\left(\frac{z}{a}\right)$

$$\therefore Z\{4^K f(n)\} = Z\{4^K u(K)\}$$

$$= F\left(\frac{z}{4}\right)$$

$$= \frac{z/4}{z/4 - 1} \qquad \left|\frac{z}{4}\right| > 1$$

$$F(z) = \frac{z}{z-4} \qquad |z| > 4 \quad \text{—— (1)}$$

$$\therefore\ Z\{g(K)\} = Z\{5^K u(K)\}$$

$$= F\left(\frac{z}{5}\right)$$

$$= \frac{z/5}{z/5 - 1} \qquad \left|\frac{z}{5}\right| > 1$$

$$\therefore\ G(z) = \frac{z}{z-5} \qquad |z| > 5 \quad \text{——} \ (2)$$

By convolution Theorem,

$$Z\{f(K) * g(K)\} = Z\{f(K)\} * Z\{g(K)\}$$

$$= F(z) * G(z)$$

From (1) and (2)

$$Z\{f(K) * g(K)\} = \frac{z}{z-4} \times \frac{z}{z-5}$$

$$\therefore\ Z\{f(K) * g(K)\} = \frac{z^2}{(z-4)(z-5)} \qquad |z| > 5$$

$$\therefore\ Z\{f(K) * g(K)\} \text{ is } \frac{z^2}{(z-4)(z-5)}$$