60004190057
Junaid · Girkar

MATHS TUTORIAL - 3

1 P.T. $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-(1/4s)}$

2 P.T. $L\{ erf_c \sqrt{t} \} = \frac{1}{\sqrt{s+1}\,(1+\sqrt{1+s})}$

3. find $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

4. Find $L^{-1}\left[\frac{1}{s}\log\left(1-\frac{a^2}{s^2}\right)\right]$

5 Solve using L.T : $\frac{dx}{dt} + x = \sin\omega t$; $x(0) = 2$

6 Solve using L.T : $(D^2+4D+8)y = 1$ G.T: $y(0) = 0$; $y'(0) = 0$

7 Find L.T of f(t) where $f(t) = \begin{cases} t & 0 < t < \pi \\ \pi - t & \pi < t < 2\pi \end{cases}$

8 Find $L^{-1}\left[\frac{(1-\sqrt{s})^2 e^{-3s}}{s^4}\right]$

9 Find $L\{t^2 H(t-2) - \sinh t\, \delta(t-4)\}$

(ii) evaluate $\int_0^\infty t^2 e^{-t} \sin t\, \delta(t-2)\, dt$.

10 Solve by using L.T, $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = t$ with $y(0) = 1$; $y'(0) = 0$

ANS 1 To prove that : $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-(1/4s)}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

$$\therefore \cos\sqrt{t} = 1 - \frac{(\sqrt{t})^2}{2!} + \frac{(\sqrt{t})^4}{4!} - \frac{(\sqrt{t})^6}{6!} \cdots$$

$$\cos\sqrt{t} = 1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{6} \cdots$$

$$\therefore \frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} \cdots$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\Gamma\frac{1}{2}}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma\frac{3}{2}}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma\frac{5}{2}}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma\frac{7}{2}}{s^{7/2}} \cdots$$

$$= \frac{\Gamma\frac{1}{2}}{s^{1/2}} - \frac{1}{2!}\frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} + \frac{1}{4!}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} - \frac{1}{6!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{7/2}} \cdots$$

$$\therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!\,(4s)^2} - \frac{1}{3!\,(4s)^3} \cdots\right] \quad \left[\Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$$

$$\therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

FOR EDUCATIONAL USE

Sundaram

ANS 2 P.T $L\{erf_c \sqrt{t}\} = \dfrac{1}{\sqrt{s+1}\,(1+\sqrt{s+1})}$

we know that : $erf\, x + erf_c\, x = 1$

$\therefore erf_c\, x = 1 - erf\, x$

$\therefore erf_c \sqrt{t} = 1 - erf\sqrt{t}$

Taking laplace transform on both sides

$$L[erf_c \sqrt{t}] = L[1] - L[erf\sqrt{t}]$$

$$= \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$$

$$= \frac{\sqrt{s+1} - 1}{s\sqrt{s+1}}$$

$$= \frac{\sqrt{s+1} - 1}{s\sqrt{s+1}} \cdot \frac{\sqrt{s+1}+1}{\sqrt{s+1}+1}$$

$$= \frac{s+1-1}{s\sqrt{s+1}\,[1+\sqrt{s+1}]}$$

$$= \frac{1}{\sqrt{s+1}\,[1+\sqrt{s+1}]} \;//$$

Hence Proved.

Sundaram

FOR EDUCATIONAL USE

ANS 3 To find : $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

we know that if $L\{f(t)\} = \bar{f}(s)$

then $L\{t f(t)\} = -\frac{d}{ds}\bar{f}(s)$

$$L^{-1}\bar{f}(s) = \frac{-1}{t} L^{-1}\left(\frac{d}{ds}\bar{f}(s)\right)$$

$$\therefore L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+4/s^4} \quad \frac{-4}{s^3}\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{s^4+4}\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{4}{t}\cdot\frac{1}{4} L^{-1}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{t}\left[e^t L^{-1}\left[\frac{1}{s^2+1}\right] - e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right]\right]$$

$$= \frac{1}{t}\left[e^t \sin t - e^{-t}\sin t\right]$$

$$= \frac{2\sin t}{t}\left(\frac{e^t - e^{-t}}{2}\right), \quad = \frac{2\sin t \cdot \sinh t}{t}$$ //

Sundaram FOR EDUCATIONAL USE

NS 4 Find $L^{-1}\left[\frac{1}{s}\log\left(1-\frac{a^2}{s^2}\right)\right]$

Let $\bar{F}(s) = \log(1-a^2/s^2) = \log(s^2-a^2) - \log(s^2)$

$$\frac{d}{ds}\bar{F}(s) = \frac{d}{ds}\left[\log(s^2-a^2) - \log(s^2)\right] = \frac{2s}{s^2-a^2} - \frac{2s}{s^2}$$

$$= 2\left[\frac{s}{s^2-a^2} - \frac{1}{s}\right]$$

$$L^{-1}\left[\frac{d}{ds}\bar{F}(s)\right] = -t f(t)$$

$$\therefore L^{-1}\left[\frac{d}{ds}\bar{F}(s)\right] = 2\left[L^{-1}\frac{s}{s^2-a^2} - L^{-1}\frac{1}{s}\right]$$

$$-t \cdot f(t) = 2\left[\cosh(at) - 1\right]$$

$$\therefore f(t) = \frac{2}{t}\left[1-\cosh(at)\right]$$

$$L^{-1}\left[\frac{1}{s}\bar{F}(s)\right] = \int_0^t f(u)\,du$$

$$\therefore L^{-1}\left\{\frac{1}{s}\bar{F}(s)\right\} = \int_0^t \frac{2}{u}\left[1-\cosh(au)\right]du = \int_0^t \frac{2\,du}{u} - \frac{2\cosh(au)\,du}{u}$$

$$= \left[2\log u\right]_0^t - 2\left[\frac{\frac{u}{a}\sinh au - \cosh(au)}{u^2}\right]_0^t$$

$$= 2\log t - \frac{2t\sinh(at)}{a} - \cosh(at)$$

ANS 5 $\frac{dx}{dt} + x = \sin \omega t$, $x(0) = 2$

Let $L(x) = \bar{x}$

$\therefore L(x') + L(x) = L(\sin \omega t)$

$L(x') = s(\bar{x}) + x(0) = s\bar{x} - 2$

$\therefore s\bar{x} - 2 + \bar{x} = \frac{\omega}{s^2+\omega^2}$

$\therefore \bar{x}(s+1) = 2 + \frac{\omega}{s^2+\omega^2} = \frac{2s^2 + 2\omega^2 + \omega}{s^2+\omega^2}$

$\therefore \bar{x} = \frac{2s^2 + 2\omega^2 + \omega}{(s^2+\omega^2)(s+1)}$

$\frac{2s^2 + 2\omega^2 + \omega}{s^2+\omega^2(s+1)} = \frac{A}{(s^2+\omega^2)} + \frac{B}{(s+1)} = \frac{A(s+1) + B(s^2+\omega^2)}{(s^2+\omega^2)(s+1)}$

By Partial fractions:

$\bar{x} = \frac{2s^2 + 2\omega^2 + \omega}{(s^2+\omega^2)(s+1)} = \frac{2\omega^2+\omega+2}{1+\omega^2} \cdot \frac{1}{s+1} + \frac{\omega - \omega s}{(1+\omega^2)(s^2+\omega^2)}$

$= \frac{2\omega^2+\omega+2}{1+\omega^2} \cdot \frac{1}{s+1} - \frac{\omega}{1+\omega^2} \cdot \frac{s}{s^2+\omega^2} + \frac{\omega}{1+\omega^2} \cdot \frac{1}{s^2+\omega^2}$

Taking inverse laplace.

$x = \frac{2\omega^2+\omega+2}{1+\omega^2} L^{-1}\left(\frac{1}{s+1}\right) - \frac{\omega}{1+\omega^2} L^{-1}\left(\frac{s}{s^2+\omega^2}\right) + \frac{\omega}{1+\omega^2} L^{-1}\left(\frac{1}{s^2+\omega^2}\right)$

$= \frac{2\omega^2+\omega+2}{1+\omega^2} \cdot e^{-t} - \frac{\omega}{1+\omega^2} \cos \omega t + \frac{\omega}{1+\omega^2} \frac{1}{\omega} \sin \omega t.$

$= \frac{1}{1+\omega^2} \left[(2\omega^2+\omega+2) e^{-t} - \omega \cos \omega t + \sin \omega t \right]$

ANS 6 $(D^2+4D+8)y = 1$; $y(0)=0$; $y'(0)=0$

$\therefore \; y''+4y'+8y = 1$

Let $L\{y(t)\} = \bar{y}(s)$

$L\{y''\} = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y}$

$L\{y'\} = s\bar{y} - y(0) = s\bar{y}$

Taking laplace on both sides

$L\{y''+4y'+8y\} = L\{1\}$

$\therefore s^2\bar{y} + 4s\bar{y} + 8\bar{y} = 1/s$

$\therefore \bar{y}(s^2+4s+8) = \frac{1}{s}$

$$\bar{y} = \frac{1}{s(s^2+4s+8)}$$

Taking laplace inverse

$$y = L^{-1}\left\{\frac{1}{s(s^2+4s+8)}\right\}$$

$$\frac{1}{s(s^2+4s+8)} = \frac{As+B}{s^2+4s+8} + \frac{C}{s} = \frac{(As+B)s + C(s^2+4s+8)}{s(s^2+4s+8)}$$

$1 = As^2 + Bs + Cs^2 + 4Cs + 8C$

$1 = (A+C)s^2 + (B+4C)s + 8C$

$\therefore C = 1/8$; $A = -1/8$; $B = -1/2$

$$\therefore \frac{1}{s(s^2+4s+8)} = \frac{-s/8 - 1/2}{s^2+4s+8} + \frac{1}{8s}$$

$$L^{-1}\left[\frac{-s/8 - 1/2}{s^2+4s+8} + \frac{1}{8s}\right] = \frac{1}{8}L^{-1}\left[\frac{1}{s} - \frac{(s+4)}{(s+2)^2+2^2}\right]$$

$$= \frac{1}{8} \; L^{-1}\left[\frac{1}{s}\right] - \left[L^{-1}\left\{\frac{s+2}{(s+2)^2+2^2}\right\} + L^{-1}\left\{\frac{2}{(s+2)^2+2^2}\right\}\right]$$

$$= \frac{1}{8}\left[1 - \left[e^{-2t}\cos 2t + e^{-2t}\sin 2t \right]\right]$$

ANS 7

$$f(t) = t \qquad 0 < t < \pi$$
$$f(t) = \pi - t \qquad \pi < t < 2\pi$$

Since $f(t)$ is a periodic function with period $a = 2\pi$, we have

$$L[f(t)] = \frac{1}{1-e^{-at}} \int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-at}} \int_0^{2\pi} e^{-st} \cdot f(t)\,dt$$

$$= \frac{1}{1-e^{-at}} \left[\int_0^{\pi} e^{-st} t\,dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t)\,dt \right]$$

$$= \frac{1}{1-e^{-at}} \left[\left[\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\pi} + \pi \left(\frac{e^{-st}}{-s} \right)_{\pi}^{2\pi} - \left[\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_{\pi}^{2\pi} \right]$$

$$= \frac{1}{1-e^{-at}} \left[\cancel{\frac{-\pi e^{-\pi s}}{s}} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \cancel{\frac{\pi e^{-\pi s}}{s}} - \frac{\pi e^{-2\pi s}}{s} + \frac{2\pi e^{-2\pi s}}{s} \right.$$
$$\left. + \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{1}{s^2} (1 - 2e^{-\pi s} + e^{-2\pi s}) - \frac{\pi e^{-\pi s} (1 - e^{-\pi s})}{s} \right]$$

$$= \frac{1}{s^2 (1-e^{-\pi s})(1+e^{-\pi s})} \left[(1-e^{-\pi s})^2 - \pi s e^{-\pi s} (1 - e^{-\pi s}) \right]$$

$$= \frac{1}{s^2 (1+e^{-\pi s})} \left[1 - e^{-\pi s} + \pi s e^{-\pi s} \right]$$

$$L\{f(t)\} = \frac{1 - (1+\pi s) e^{-\pi s}}{s^2 (1+e^{-\pi s})}$$

ANS Q8 Find $L^{-1}\left\{\frac{(1-\sqrt{s})^2 e^{-3s}}{s^4}\right\}$

let $\frac{(1-\sqrt{s})^2}{s^4} = \bar{f}(s)$

$L^{-1}\{\bar{f}(s)\} = f(t)$

$L^{-1}\left[\frac{(1-\sqrt{s})^2}{s^4}\right] = f(t)$

$f(t) = L^{-1}\left[\frac{1+s-2\sqrt{s}}{s^4}\right] = L^{-1}\left[\frac{1}{s^4} + \frac{1}{s^3} - \frac{2}{s^{7/2}}\right]$

$f(t) = L^{-1}\left[\frac{1}{s^4}\right] + L^{-1}\left[\frac{1}{s^3}\right] - 2\,L^{-1}\left[\frac{1}{s^{7/2}}\right]$

$= \frac{t^3}{3!} + \frac{t^2}{2!} - \frac{2 \times t^{5/2}}{\Gamma 7/2}$

Sundaram

FOR EDUCATIONAL USE

$$= \frac{t^3}{6} + \frac{t^2}{2} - \frac{2t^{5/2}}{5/2 \cdot 3/2 \cdot 1/2 \sqrt{\pi}}$$

$$f(t) = \left[\frac{t^3}{6} + \frac{t^2}{2} - t^{5/2}\left(\frac{16}{15\sqrt{\pi}}\right)\right]$$

$$L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\,H(t-a)$$

$a = 3$

$$L^{-1}\left[\frac{(1-\sqrt{s})^2 e^{-3s}}{s^4}\right] = f(t-3).\,H(t-3)$$

$$= \left[\frac{(t-3)^3}{6} + \frac{(t-3)^2}{2} - \left(\frac{(t-3)^{5/2} \times 16}{15\sqrt{\pi}}\right)\right]$$

ANS 9 i] $f(t) = t^2$ and $a = 2$

$\therefore f(t+2) = (t+2)^2 = t^2 + 4t + 4$

$$L[f(t+2)] = L[t^2 + 4t + 4] = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$$

$$L[t^2 . H(t-2)] = e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$$

let $g(t) = \cosh t$ and $a = 4$

$$\therefore L[g(t)\,\delta(t-a)] = e^{-as}\,g(a) = e^{-4s}\cosh 4$$

Sundaram®

FOR EDUCATIONAL USE

Hence we have to find $L[t^2(4-2) - \cosh t\ \delta(t-4)]$

$$= f(t) - g(t)$$

$$= e^{-2as}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s}\cosh 4$$

ii] We have to find $\int_0^\infty t^2\, e^{-t} \sin t .\ \delta(t-2)\, dt \longrightarrow (1)$

By theorem: $L[f(t).\ \delta(t-a)] = e^{-as} f(a)$

$$\therefore \int_0^\infty e^{-st}.\ f(t).\ \delta(t-a)\, dt = e^{-as} f(a)$$

comparing with (1), we get

$f(t) = t^2 \sin t$, $s = 1$, $a = 2$

$$\therefore \int_0^\infty e^{-st}.\ f(t).\ \delta(t-a)\, dt = e^{-2} .\ 2^2 \sin 2 = 4\, e^{-2} \sin 2$$

ANS 10 $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = t$, $y(0) = 1$, $y'(0) = 0$

$$\therefore y'' + y' - 2y = t \quad \text{---} (1)$$

Let $L(y) = \bar{y}$

$\therefore L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s$

$L[y'] = s\bar{y} - y(0) = s\bar{y} - 1$

Applying laplace transform to (1)

$\therefore L(y'') + L(y') - 2L(y) = L(t)$

$$\therefore s^2\bar{y} - s + s\bar{y} - 1 - 2\bar{y} = \frac{1}{s^2}$$

Sundaram
FOR EDUCATIONAL USE

$$\therefore (s^2+s-2)\bar{y} = \frac{1}{s^2} + (1+s)$$

$$\bar{y} = \frac{1+s^2(s+1)}{s^2(s^2+s-2)}$$

$$\bar{y}(s) = \frac{1+s^2+s^3}{s^2(s^2+s-2)}$$

$$\bar{y}(s) = \frac{1+s^2+s^3}{s^2(s+2)(s+1)}$$

Using Partial fraction

$$\bar{y}(s) = \left[\frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4(s+2)} + \frac{1}{s-1}\right]$$

Taking laplace inverse on both sides

$$y = \frac{-1}{2}t - \frac{1}{4} + \frac{e^{-2t}}{4} + e^t$$

$$\therefore y = e^t + \frac{e^{-2t}}{4} - \frac{t}{2} + \frac{e^t - 1}{4}$$

$$y = \frac{1}{4}(e^{-2t} + 4e^t - 2t - 1)$$ //

FOR EDUCATIONAL USE