TUTORIAL - 8

Q1 Find Fourier transform of $g(x) = e^{-x^2/2} \cdot \sin \alpha x$

Q2 Find Fourier transform of $f(x) = x\, e^{-9x^2}$

Q3 $h(x) = \begin{cases} 2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$g(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

Find fourier transform of $h * g$

ANS 1 Let $f(x) = e^{-x^2/2}$

Now, $F[f(x)]$ is given by :

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} - i\alpha x\right)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}} - \frac{i\alpha}{\sqrt{2}}\right)^2 - \left(\frac{i\alpha}{\sqrt{2}}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x - i\alpha}{\sqrt{2}}\right)^2}\, e^{-\alpha^2/2}\, dx$$

Let $x - i\alpha = t$ $\qquad x \to \infty \quad t \to \infty$

Differentiating w.r.t x $\qquad x \to -\infty \quad t \to -\infty$

$\therefore\ dx = dt$

$$\therefore\ F[f(x)] = \frac{1}{\sqrt{2\pi}} e^{-\alpha^2/2} \int_{-\infty}^{\infty} e^{-t^2/2}\, dt$$

$\because\ e^{-t^2/2}$ is an even function.

$$\therefore\ F[f(x)] = \frac{1}{\sqrt{2\pi}} \times 2 \times e^{-\alpha^2/2} \int_0^{\infty} e^{-t^2/2}\, dt.$$

$$= \sqrt{\frac{2}{\pi}}\, e^{-\alpha^2/2} \int_0^{\infty} e^{-t^2/2}\, dt$$

Let $\frac{t^2}{2} = m$ $\quad t \to 0,\ m \to 0$

$\quad t \to \infty,\ m \to \infty$

Differentiate w.r.t t

$$t\,dt = dm$$

$$\therefore\ dt = \frac{dm}{\sqrt{2m}}.$$

$$\therefore\ F[f(x)] = \sqrt{\frac{2}{\pi}} \times \frac{1}{\sqrt{2}} \times e^{-\alpha^2/2} \int_0^{\infty} e^{-m}\, m^{-1/2}\, dm$$

$$\because\ \int_0^{\infty} e^{-m} \cdot m^{n-1}\, dm = \Gamma n$$

$$\therefore\ F[f(x)] = \frac{1}{\sqrt{\pi}} e^{-\alpha^2/2}\, \Gamma \tfrac{1}{2}$$

$$= \frac{1}{\sqrt{\pi}} e^{-\alpha^2/2} \times \sqrt{\pi}$$

$$\therefore\ F[f(x)] = e^{-\alpha^2/2}$$

$$\therefore\ F[f(x).\sin ax] = \frac{1}{2i}\left[F(\alpha+a) - F(\alpha-a)\right]$$

$$= \frac{1}{2i}\left[e^{-\frac{(\alpha+a)^2}{2}} - e^{-\frac{(\alpha-a)^2}{2}}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{e^{\frac{(\alpha+a)^2}{2}}} - \frac{1}{e^{\frac{(\alpha-a)^2}{2}}}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{\alpha^2-2\alpha a+a^2/2} - e^{\alpha^2+2a\alpha+a^2/2}}{e^{\alpha^2-2a\alpha+a^2+\alpha^2+2a\alpha+a^2/2}}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{\left(\frac{\alpha^2+a^2}{2}-a\alpha\right)} - e^{\left(\frac{\alpha^2+a^2}{2}\right)+a\alpha}}{e^{\alpha^2+a^2}}\right]$$

$$= \frac{1}{2i}\left[\frac{e^{\frac{(a^2+\alpha^2)}{2}}}{e^{(\alpha^2+a^2)}}\left\{e^{-a\alpha} - e^{a\alpha}\right\}\right]$$

$$= \frac{1}{2i}\, e^{-(\alpha^2+a^2)/2}\left[e^{-a\alpha} - e^{a\alpha}\right]$$

$$= -e^{-(a^2+\alpha^2)/2}\,\frac{\left[e^{a\alpha} - e^{-a\alpha}\right]}{2i}$$

$$= \frac{-e^{-(\alpha^2+a^2)/2}\cdot \sinh a\alpha}{i}$$

$$= i\, e^{-(\alpha^2+a^2)/2}\cdot \sinh a\alpha$$

ANS 2

$$f(x) = x\, e^{-9x^2}$$

$$g(x) = e^{-9x^2}$$

$$F[g(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x).\, e^{i\alpha x}.\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-9x^2}\cdot e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-9\left(x^2-\frac{i\alpha x}{9}\right)}\, dx.$$

consider $x^2 - \frac{i\alpha x}{9}$

$$x^2 - \frac{i\alpha x}{9} + \left(\frac{\alpha}{18}\right)^2 - \left(\frac{i\alpha}{18}\right)^2$$

$$\left(x - \frac{i\alpha}{9}\right)^2 + \frac{\alpha^2}{324}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-9\left(x - \frac{i\alpha}{9}\right)^2 - \frac{9\alpha^2}{324}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-9\left(x - \frac{i\alpha}{9}\right)^2} \cdot e^{-9\alpha^2/324} \, dx$$

$$= e^{-\left(\frac{\alpha}{6}\right)^2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-9\left(x - \frac{i\alpha}{9}\right)^2} \, dx$$

Put $x - \frac{i\alpha}{9} = t$

$dx = dt$

$$= e^{-\left(\frac{\alpha}{6}\right)^2} \times \frac{1 \times 2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-9t^2} \cdot dt$$

Put $9t^2 = u$

$9 \times 2t \times dt = du$

$$dt = \frac{du}{18t} = \frac{du}{18\sqrt{\frac{u}{9}}} = \frac{du}{6\sqrt{u}}$$

$$= e^{-\left(\frac{\alpha}{6}\right)^2} \times \frac{1}{\sqrt{2\pi}} \times 2 \int_0^\infty e^{-u} \cdot \frac{1}{6\sqrt{u}}\, du$$

$$= e^{-(\alpha/6)^2} \times \sqrt{\frac{2}{\pi}} \times \frac{1}{6} \int_0^\infty e^{-u}\, u^{1/2-1}\, du$$

$$= e^{-\left(\frac{\alpha}{6}\right)^2} \times \sqrt{\frac{2}{\pi}} \times \frac{1}{6}\, \Gamma 1/2$$

$$= e^{-\left(\frac{\alpha}{6}\right)^2} \times \frac{\sqrt{2}}{\cancel{\sqrt{\pi}}} \times \frac{1}{6} \times \cancel{\sqrt{\pi}}$$

$$F[g(x)] = \frac{e^{-\left(\frac{\alpha}{6}\right)^2}}{3\sqrt{2}}$$

$$F[x\, g(x)] = (-i)^1 \frac{d}{d\alpha} \frac{e^{-\left(\frac{\alpha}{6}\right)^2}}{3\sqrt{2}} = \frac{-i}{3\sqrt{2}}\, e^{-\left(\frac{\alpha}{6}\right)^2} \times -\cancel{2}\frac{\alpha}{6} \times \frac{1}{\cancel{6}\,3}$$

$$F[x e^{-9x^2}] = \frac{i\alpha}{54\sqrt{2}}\, e^{-\left(\frac{\alpha}{6}\right)^2}$$

ANS 3

$$h(x) = \begin{cases} 2 , & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F[h * g] = F[h(x) * g(x)]$$

$$H(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{1} 2\, e^{i\alpha x}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x}}{i\alpha} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{i\alpha}}{i\alpha} - \frac{1}{i\alpha} \right]$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{(e^{i\alpha} - 1)}{i\alpha}$$

$$G(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x).\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} \cdot e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1-i\alpha)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1-i\alpha)x}}{-(1-i\alpha)} \right]_0^{\infty}$$

$$= \frac{-1}{\sqrt{2\pi}\ (1-i\alpha)} (0-1)$$

$$= \frac{1}{\sqrt{2\pi}\ (1-i\alpha)}$$

$F[h * g] = F[h(x) * g(x)] = H(\alpha) * G(\alpha)$ ... convolution theorem

$$= \sqrt{\frac{2}{\pi}}\, \frac{(e^{i\alpha} - 1)}{i\alpha} * \frac{1}{\sqrt{2\pi}\ (1-i\alpha)}$$

$$= \frac{1}{\pi} \frac{(e^{i\alpha}-1)}{i\alpha + \alpha^2}$$

$$F[h*g] = \frac{1}{\pi} \frac{e^{i\alpha}-1}{\alpha(\alpha+i)}$$

$$\boxed{F[h*g] = \frac{1}{\pi\alpha} \frac{(e^{i\alpha}-1)(\alpha-i)}{(\alpha^2+1)}}$$