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1. 10. 2020

MATHS TUTORIAL 4

FOURIER SERIES

Q1 Find fourier series expansion of

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0 & \pi \le x \le 2\pi \end{cases}$$

Hence deduce that $\frac{1}{2} = \frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \cdots$

Q2 Find fourier series expansion of $f(x) = |x|$ in $(-\pi, \pi)$

Q3 Obtain fourier series expansion of

$$f(x) = \begin{cases} \pi x & , 0 \le x \le 1 \\ \pi(2-x) & , 1 \le x \le 2 \end{cases}$$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

ANS 1

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\left[\int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} 0 dx\right]$$

$$= \frac{-1}{2\pi}\left[\cos x\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{1}{\pi}}$$

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The above method is not applicable for $n = 1$,

$\therefore$ from (2),

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x\cos x\, dx + \int_{\pi}^{2\pi} 0.\cos x\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x.\cos x.dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$\boxed{a_1 = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin x.\sin nx\, dx + \int_{\pi}^{2\pi} 0.\sin nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x.\sin nx\, dx + 0$$

$$= \frac{-1}{2\pi}\left[\int_0^{\pi} \cos(n+1)x - \cos(n-1)x\, dx\right.$$

$$= \frac{-1}{2\pi}\left[\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right]_0^{\pi}$$

$$b_n = \frac{-1}{2\pi}(0) = 0 \quad \text{when } n \neq 1$$

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For $n=1$, the previous method fails

$\therefore$ ~~Even~~

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx + 0$$

$$= \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$b_1 = \frac{1}{2}$$

Substituting values of a_0, a_n and b_n

$$I = \frac{1}{\pi} + \left[0 - \frac{2\cos 2x}{3\pi} + 0 - \frac{2}{15\pi}\cos 4x + 0 - \frac{2}{35\pi}\cos 6x + \cdots\right]$$

$$+ \frac{\sin x}{2} + 0$$

$$= \frac{1}{\pi}\left[1 - 2\sum \frac{1.\cos 2nx}{4n^2-1} + \frac{\pi}{2}\sin x\right]$$

let $n=0$, $\therefore I = 0$

$$\therefore 0 = \frac{1}{\pi} - \frac{2}{\pi}\sum \frac{1}{4n^2-1} + 0$$

$$\frac{1}{\pi} = \frac{2}{\pi}\left[\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right]$$

$$\therefore \frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$$

Hence Proved.

Q2

ANS

$f(x) = |x| \quad \text{in } (-\pi, \pi)$

$f(-x) = |-x| = |x|$

$\therefore \; f(x) = f(-x) \leftarrow$ Even function.

Fourier cosine series, $b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + 0$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} |x| \cos nx \, dx$$

since $|x| = x$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \frac{\pi^2 2}{2\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\frac{x \sin nx}{n} - \frac{1(-\cos nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\cos n\pi - 1}{n^2} \right]$$

$$= \frac{2}{n^2 \pi} (\cos n\pi - 1)$$

$$a_n = \frac{2}{n^2 \pi} \left[(-1)^n - 1 \right]$$

For even n: 0

For odd n: $\frac{-4}{n^2 \pi}$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi} \cos nx . dx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} dx$$

this is applicable ONLY when n is odd

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos (2n-1)x}{(2n-1)^2} dx$$

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ANS 3 Here $2l = 2$

$\therefore l = 1$

let $f(x) = a_0 + \sum a_n . \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$

$f(x) = a_0 + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x \longrightarrow (1)$

$$a_0 = \frac{1}{2l} \int_0^{2l} f(x) dx$$

$$= \frac{1}{2} \left[\int_0^1 \pi x dx + \int_1^2 \pi(2-x) dx \right]$$

$$= \frac{1}{2} \left[\pi \left[\frac{x^2}{2}\right]_0^1 + \pi \left[2x - \frac{x^2}{2} \right]_1^2 \right]$$

$$= \frac{\pi}{2} \left[\frac{1}{2} + \left(4 - 2 - 2 + \frac{1}{2}\right) \right]$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx$$

$$= \int_0^1 \pi . x . \cos n\pi x dx + \int_1^2 \pi(2-x) \cos n\pi x \, dx$$

$$= \pi \left[\left\{ x \left(\frac{\sin n\pi x}{n\pi} \right) - 1 \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) \right\}_0^1 + \left\{ (2-x) \left[\frac{\sin n\pi x}{n\pi} \right] - (-1) \left[-\frac{\cos n\pi x}{n^2\pi^2} \right]_1^2 \right\} \right]$$

$$= \pi \left[\left\{ x \left(\frac{\sin n\pi x}{n\pi} \right) - 1 \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) \right\} + \left\{ (2-x) \left(\frac{\sin n\pi x}{n\pi} \right) - (-1) \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) \right\}_1^2 \right]$$

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$$= \pi\left[\left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right\} + \left\{-\frac{1}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2}\right\}\right]$$

$$a_n = \frac{2\pi}{n^2\pi^2}\left[\cos n\pi - 1\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 \pi x \cdot \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[\left\{x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\left\{-\frac{\cos n\pi}{n\pi}\right\} + \left\{\frac{\cos n\pi}{n\pi}\right\}\right] = 0$$

Putting these values in (1)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$$