Complexity analysis of Randomized Quicksort

input array $A\{x_1, x_2, x_3, \dots x_n\}$

and X_{ij} be the indicator random variable indicating whether two elements x_i & x_j are compared or not.

The question is to determine the number of counts, a given x_i & x_j be compared.

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

Comparison of x_j & x_i would happen only under the following two conditions:

1. A sub-problem of quick sort contains x_i & x_j.
2. Either x_i or x_j is chosen as a pivot element.

$$\therefore \quad X_{ij} = \begin{cases} 1 & \text{comparison.} \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr(x_i \text{ is compared } x_j)$$

$\therefore$ $Pr[x_i \text{ is compared to } x_j]$

$$= Pr[x_i \text{ is chosen as a pivot}] + Pr[x_j \text{ is chosen as a pivot}]$$

$$= \frac{1}{j-i+1} + \frac{1}{j-i+1}$$

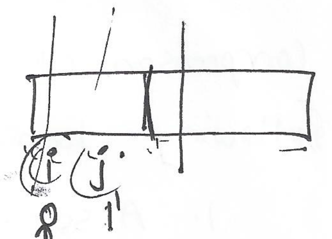
it can be observed that the pivot is chosen from a set which has $j-i+1$ elements. In addition both the events are equally likely. $\rightarrow$ $2-0+1$ $\boxed{3}$

This implies that

$$\Pr[x_i \text{ is compared to } x_j] = \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}$$

In other words, the probability $x_{ij} = 1$ is $\frac{2}{j-i+1}$

$$\therefore E(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[x_{ij}]$$



$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$\frac{1}{j-1+1} = \frac{1}{}$$

Substitution of $j-i=k$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k}$$

$$= 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{1}{k}$$

$$= \sum_{i=1}^{n-1} O(n \log n)$$

$$= O(n \log n)$$

Recall that $\frac{1}{k}$ is a harmonic series whose time complexity is $O(\log n)$. Therefore the expected runtime of randomized quick sort is $\underline{O(n \log n)}$.