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MATHS TUTORIAL 5

1. Find H.R.C.S for $f(x) = \begin{cases} x & 0 < x \le \pi/2 \\ \pi - x & \pi/2 \le x < \pi \end{cases}$

Hence find $\sum_{n=1}^{\infty} 1/n^4$

ANS We know that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n . \cos nx$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$$

$$a_0 = \frac{2}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right]$$

$$a_0 = \frac{2}{\pi}\left\{\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right\}$$

$$a_0 = \frac{2}{\pi} \times \frac{\pi^2}{4} = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx = \frac{2}{\pi}\left\{\int_0^{\pi/2} x.\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right\}$$

$$a_n = \frac{2}{\pi}\left\{\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi/2} + \left[\frac{(\pi - x)\sin nx}{n} - \frac{(-1)(-\cos nx)}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$a_n = \frac{2}{\pi}\left\{\frac{\pi/2 \sin n\pi/2}{n} + \frac{1}{n^2}\cos n\pi/2 - \frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \frac{\pi}{2}\frac{\sin n\pi/2}{n} + \frac{1}{n^2}\cos n\pi/2\right\}$$

$$a_n = \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \left(\frac{1 + (-1)^n}{n^2}\right)\right]$$

$$f(x) = \frac{\pi}{2} \times \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{4}{\pi n^2} \cos \frac{n\pi}{2} - \frac{2}{\pi}\left(\frac{1+(-1)^n}{n^2}\right)\right] \cos nx$$

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi n^2} \sum_{n=1}^{\infty} \left[2\cos \frac{n\pi}{2} - (1+(-1)^n)\right] \cos nx$$

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi} \left\{ \frac{-4\cos 2x}{2^2} - \frac{4\cos 6x}{6^2} - \frac{4\cos 10x}{10^2} \cdots \right\}$$

$$f(x) = \frac{\pi}{4} - \frac{8}{2^2\pi} \left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots \right]$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left\{ \frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots \right\}$$

$$\frac{2}{\pi} \int_0^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

By Parseuuls identity

$$\frac{1}{\pi} \left[\int_0^{\pi/2} x^2 dx + \int_{\pi/2}^{\pi} (\pi - x)^2 dx \right] = \frac{\pi^2}{16} + \frac{1}{2} \times \frac{4}{\pi^2} \left[\frac{16}{2^4} + \frac{16}{6^4} + \frac{16}{10^4} + \cdots \right]$$

$$\Rightarrow \frac{1}{\pi} \left[\left[\frac{x^3}{3}\right]_0^{\pi/2} + \left[\frac{(\pi - x)^3}{3}\right]_{\pi/2}^{\pi} \right] = \frac{\pi^2}{16} + \frac{2}{\pi^2} \times \frac{16}{2^4} \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \right]$$

$$\Rightarrow \frac{1}{\pi} \left[\frac{\pi^3}{24} + \frac{\pi^3}{24} \right] = \frac{\pi^2}{16} + \frac{2}{\pi^2} \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \right]$$

$$\frac{\pi^2}{12} - \frac{\pi^2}{16} = \frac{2}{\pi^2} \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \right]$$

$$\frac{\pi^2}{48} \times \frac{\pi^2}{2} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

let $S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

$$S = \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right] + \left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\right]$$

$$S = \frac{\pi^4}{96} + \frac{1}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots\right]$$

$$S = \frac{\pi^4}{96} + \frac{S}{16}$$

$$\therefore \frac{15S}{16} = \frac{\pi^4}{96}$$

$$\boxed{S = \frac{\pi^4}{90}} \quad \text{where } S = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Q2 Find complex form of F.S, for $f(x) = \sinh x$ in $(-l, l)$

ANS Complex form

$$f(x) = \sum_{-\infty}^{\infty} c_n \cdot e^{in\pi x/l} \qquad c_n = \frac{1}{2l}\int_{-l}^{l} f(x) \cdot e^{\frac{-in\pi x}{l}} \cdot dx$$

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$c_n = \frac{1}{2l}\left[\int_{-l}^{l} \frac{e^{x(1 - in\pi/l)} - e^{-x(1+in\pi/l)}}{2}\, dx\right]$$

$$c_n = \frac{1}{2l}\left[\int_{-l}^{l} \frac{e^{x\left(\frac{l-in\pi}{l}\right)} - e^{-x\left(\frac{l+in\pi}{l}\right)}}{2}\, dx\right]$$

$$c_n = \frac{1}{4l}\left[\frac{l e^{x(l-in\pi/l)} - e^{-(l-in\pi)}}{l - in\pi} - \frac{e^{-(l+in\pi)} - e^{(l+in\pi)}}{l + in\pi}\right]$$

Q2 Find complex form of F.S for $f(x) = \sinh x$ in $(-1, 1)$

ANS The complex form of F.S for $f(x)$ is given as

$$f(x) = \sum_{-\infty}^{\infty} c_n . e^{i\frac{n\pi x}{l}} \quad \text{where } c_n = \frac{1}{2l}\int_{c}^{c+2l} f(x) . e^{-i\frac{n\pi x}{l}} dx$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) . e^{-i(n\pi x/l)} dx$$

$$= \frac{1}{2l}\int_{-l}^{l} \frac{e^{x}-e^{-x}}{2} . e^{\frac{-in\pi x}{l}} dx$$

$$= \frac{1}{4l}\int_{-l}^{l} e^{x(1-\frac{in\pi}{l})} - e^{-x(1+\frac{in\pi}{l})} dx$$

$$= \frac{1}{4l}\left[\frac{e^{x(1-\frac{in\pi}{l})}}{1-in\pi/l} + \frac{e^{-x(1+\frac{in\pi}{l})}}{1+in\pi/l}\right]_{-l}^{l}$$

$$= \frac{1}{4}\left[\frac{e^{(l-in\pi)}}{l-in\pi} - \frac{e^{-(l-in\pi)}}{l-in\pi} - \frac{e^{l+in\pi}}{l+in\pi} + \frac{e^{-(l+in\pi)}}{l+in\pi}\right]$$

$$= \frac{1}{4}\left[\frac{e^{l}(-1)^n}{l-in\pi} - \frac{e^{-l}(-1)^n}{l-in\pi} - \frac{e^{l}(-1)^n}{l+in\pi} + \frac{e^{-l}(-1)^n}{l+in\pi}\right]$$

$$= \frac{(-1)^n}{4}\left[\frac{e^{l}-e^{-l}}{l-in\pi} - \frac{(e^{l}-e^{-l})}{l+in\pi}\right]$$

$$= \frac{(-1)^n}{2}\left[\frac{\sinh l}{l-in\pi} - \frac{\sinh l}{l+in\pi}\right]$$

$$= \frac{(-1)^n \sinh l}{2} \times \frac{2in\pi}{l^2+n^2\pi^2}$$

$$\boxed{\therefore c_n = \frac{(-1)^n \sinh l \times in\pi}{l^2+n^2\pi^2}}$$

$\therefore$ Complex form is given by $f(x) = i\pi \sinh l \sum_{-\infty}^{\infty} \frac{(-1)^n n}{l^2+n^2\pi^2} . e^{i(\frac{n\pi x}{l})}$ //

Q3 Show that $\left\{ \sin \frac{\pi x}{2l}, \sin \frac{3\pi x}{2l}, \sin \frac{5\pi x}{2l} \right\}$ is orthogonal over $(0,l)$. Hence find corresponding orthogonal set

ANS STEP 1 : Show that $\left\{ \sin \frac{\pi x}{2l}, \sin \frac{3\pi x}{2l}, \sin \frac{5\pi x}{2l} \right\}$ is orthogonal

CASE 1 : let $m \neq n$

$$\int_0^l f_m(x) \cdot f_n(x)\, dx$$

$$\therefore \int_0^l \sin\left(\frac{\pi x}{2l}\right) \cdot \sin\left(\frac{3\pi x}{2l}\right) = \frac{1}{2} \int_0^l \cos\left(\frac{\pi x}{l}\right) - \cos\left(\frac{2\pi x}{l}\right) dx$$

$$= \left[\frac{1}{2} \cdot \frac{\sin \pi x/l}{\pi/l} - \frac{1}{2} \frac{\sin 2\pi x/l}{2\pi/l} \right]_0^l$$

$$= \frac{1}{2} \cdot \frac{\sin \pi/l}{\pi/l} - \frac{1}{2} \frac{\sin 2\pi}{2\pi/l} - 0$$

$$\therefore \int_0^l \sin\left(\frac{\pi x}{2l}\right) \cdot \sin\left(\frac{3\pi x}{2l}\right) dx = 0$$

$$\therefore \int_0^l \sin \frac{3\pi x}{2l} \cdot \sin \frac{5\pi x}{2l} = \frac{1}{2} \int_0^l \cos \frac{\pi x}{l} - \cos \frac{4\pi x}{l}\, dx$$

$$= \frac{1}{2} \left[\frac{\sin \pi x/l}{\pi/l} - \frac{\sin 4\pi x/l}{4\pi/l} \right]_0^l$$

$$\therefore \int_0^l \sin \frac{3\pi x}{2l} \cdot \sin \frac{5\pi x}{2l}\, dx = 0$$

$$\therefore \int_0^l \sin \frac{5\pi x}{2l} \cdot \sin \frac{\pi x}{2l}\, dx = \frac{1}{2} \int_0^l \cos \frac{2\pi x}{l} - \cos \frac{3\pi x}{l}$$

$$= \frac{1}{2} \left[\frac{\sin 2\pi x/l}{2\pi/l} - \frac{\sin 3\pi x/l}{3\pi/l} \right]_0^l$$

$$= 0$$

Case 2 : $m=n$

$$\therefore \int_0^l \sin^2 \frac{\pi x}{2l}\, dx = \frac{1}{2}\int_0^l (1-\cos^{\pi x}/l)\, dx = \frac{1}{2}\left[x - \frac{\sin \pi x/l}{\pi/l}\right]_0^l$$

$$= \frac{1}{2}(l)$$

$$\therefore \int_0^l \sin^2 \frac{\pi x}{2l}\, dx = \frac{l}{2} > 0$$

$$\therefore \int_0^l \sin^2 \frac{3\pi x}{2l}\, dx = \frac{1}{2}\int_0^l 1 - \cos\left(\frac{3\pi x}{l}\right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 3\pi x/l}{3\pi/l}\right]_0^l$$

$$= \frac{l}{2}$$

$$\therefore \int_0^l \sin^2\left(\frac{3\pi x}{2l}\right) dx = \frac{l}{2} > 0$$

$$\therefore \int_0^l \sin^2\left(\frac{5\pi x}{2l}\right) dx = \frac{1}{2}\int_0^l (1-\cos 5\pi x/l)\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 5\pi x/l}{5\pi/l}\right]_0^l$$

$$= \frac{l}{2}$$

$$\therefore \int_0^l \sin^2\left(\frac{5\pi x}{2l}\right) dx = \frac{l}{2} > 0$$

$\therefore$ From case 1 and case 2, $\left\{\sin \frac{\pi x}{2l}, \sin \frac{3\pi x}{2l}, \sin \frac{5\pi x}{2l}\right\}$

is orthogonal on $[0, l]$

To find orthogonal set of functions on $[0, \pi]$

From case 2 : when $m = n$

$$\Rightarrow \int_0^l \sin^2 \frac{\pi x}{2l} dx = \int_0^l \sin^2 \frac{3\pi x}{2l} dx = \int_0^l \sin^2 \frac{5\pi x}{2l} = \frac{l}{2}$$

Multiply both sides by $2/l$

$$\therefore \int_0^l \left(\sqrt{\frac{2}{l}} \sin \frac{\pi x}{2l}\right)^2 dx = \int_0^l \left(\sqrt{\frac{2}{l}} \sin \frac{3\pi x}{2l}\right)^2 = \int_0^l \left(\sqrt{\frac{2}{l}} \sin \frac{5\pi x}{2l}\right)^2 dx = 1$$

$$\therefore \left\{\sqrt{\frac{2}{l}} \sin \frac{\pi x}{2l}, \sqrt{\frac{2}{l}} \sin \frac{3\pi x}{2l}, \sqrt{\frac{2}{l}} \sin \frac{5\pi x}{2l}\right\}$$ is the orthonormal set

Q4 Find complex form of Fourier series for $f(x) = \cosh 3x + \sinh 3x$ in $(-\pi, \pi)$

ANS The complex form of fourier series for $\cosh 3x + \sinh 3x$ in $(-\pi, \pi)$ $f(x) = \sum_{-\infty}^{\infty} c_n \cdot e^{in\pi x/l}$

where $c_n = \frac{1}{2l} \int_c^{c+2l} f(x) \cdot e^{-i\left(\frac{n\pi x}{l}\right)}$

Here $c = -\pi$, $l = \pi$

$$\therefore c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cosh 3x + \sinh 3x) \cdot e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{3x} + e^{-3x} + e^{3x} - e^{-3x}}{2} \cdot e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{3x} \cdot e^{-inx} dx$$

$$\therefore C_{in} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x(3-in)} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{x(3-in)}}{3-in} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{3\pi} . e^{-in\pi} - e^{-3\pi} . e^{in\pi}}{3-in} \right]$$

$$= \frac{(-1)^n}{2\pi} \frac{e^{3\pi} - e^{-3\pi}}{3-in}$$

$$= \frac{(-1)^n \sinh(3\pi)}{(3-in)\pi}$$

$\therefore$ The complex form of Fourier series is

$$\therefore f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh(3\pi)(3+in)}{(9+n^2)\pi} . e^{-inx}$$

$$\therefore f(x) = \frac{\sinh(3\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n (3+in) e^{-inx}}{9+n^2}$$

Q5 Find H.R.C.S for $f(x) = (x-1)^2$ in $(0,1)$

$f(x) = (x-1)^2 = x^2 - 2x + 1$ in $(0,1)$ where $l = 1$

$\therefore$ H.R.C.S for $f(x)$ in $(0, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n . \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n . \cos(n\pi x) dx$$

$$\therefore a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{l} \int_0^l x^2 - 2x + 1 \, dx$$

$$= 2 \left[\frac{x^3}{3} - \frac{2x^2}{2} + x \right]_0^1$$

$$\boxed{\therefore a_0 = 2/3}$$

$$a_n = \frac{2}{l}\int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\therefore\ a_n = 2\int_0^1 (x^2-2x+1)\cos(n\pi x)\,dx$$

$$= 2\left[(x^2-2x+1)\frac{\sin(n\pi x)}{n\pi} + \frac{(2x-2).\cos(n\pi x)}{(n\pi)^2} + \frac{2\sin(n\pi x)}{(n\pi)^3}\right]_0^1$$

$$= 2\left[0+0+0-\frac{(-2)}{(n\pi)^2}\right]$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}}$$

Substitute values of a_0 and a_n in $f(x)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.\cos(n\pi x)$$

$$\therefore f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi x)$$

$$\boxed{f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi x)}{n^2}}$$