

Q1 Find Laplace transform of $f(t) = \sin^6 t$

ANS $\sin t = \frac{e^{it} - e^{-it}}{2i}$

$$\sin^6 t = \left(\frac{e^{it} - e^{-it}}{2i} \right)^6$$

$$= \frac{1}{64i^6} \left[e^{6it} - 6e^{4it} + 15e^{2it} - 20 + 15e^{-2it} - 6e^{-4it} + e^{-6it} \right]$$

$$= \frac{-1}{64} \left[(e^{6it} + e^{-6it}) - 6(e^{4it} + e^{-4it}) + 15(e^{2it} + e^{-2it}) - 20 \right]$$

$$= \frac{-1}{64} \left[2\cos 6t - 6\cos 4t + 30\cos 2t - 20 \right]$$

$$= \frac{-1}{32} \left[\cos 6t - 6\cos 4t + 15\cos 2t - 10 \right]$$

$$L\{\sin^6 t\} = \frac{-1}{32} L\left\{ \cos 6t - 6\cos 4t + 15\cos 2t - 10 \right\}$$

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$\therefore L\{\sin^6 t\} = \frac{-1}{32} \left[L\{\cos 6t\} - 6L\{\cos 4t\} + 15L\{\cos 2t\} - 10L\{1\} \right]$$

$$= \frac{-1}{32} \left[\frac{s}{s^2 + 36} - \frac{6s}{s^2 + 16} + \frac{15s}{s^2 + 4} - \frac{10}{s} \right]$$

$$= \frac{1}{32} \left[\frac{10}{s} + \frac{6s}{s^2 + 16} - \frac{15s}{s^2 + 4} - \frac{s}{s^2 + 36} \right] //$$

Q2 To find Laplace transform of $f(t) = t \sin 2t \cdot \cosh t$

let $f(t) = \sin 2t$

$$L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2 + 4} \rightarrow \textcircled{1}$$

$$L\{tf(t)\} = (-1)' \frac{d'}{ds} \left(\frac{2}{s^2 + 4} \right) = \frac{(-2)(-2s)}{(s^2 + 4)^2} = \frac{4s}{(s^2 + 4)^2} \rightarrow \textcircled{2}$$

$$L\{g(t)\} = \frac{4s}{(s^2 + 4)^2} = \bar{g}(s)$$

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

$$\therefore L\{g(t) \cosh t\} = \frac{1}{2} L\{g(t)e^t + g(t)e^{-t}\}$$

$$= \frac{1}{2} [\bar{g}(s-1) + \bar{g}(s+1)]$$

$$= \frac{1}{2} \left[\frac{4(s-1)}{((s-1)^2 + 4)^2} + \frac{4(s+1)}{((s+1)^2 + 4)^2} \right]$$

$$= \frac{2(s-1)}{((s-1)^2 + 4)^2} + \frac{2(s+1)}{((s+1)^2 + 4)^2}$$

$$\therefore L\{t \sin 2t \cdot \cosh t\} = \frac{2(s-1)}{((s-1)^2 + 4)^2} + \frac{2(s+1)}{((s+1)^2 + 4)^2}$$

$$= 2 \left[\frac{s-1}{((s-1)^2 + 4)^2} + \frac{s+1}{((s+1)^2 + 4)^2} \right] //$$

Q3 Find $L \left[\cos ht \cdot \int_0^t e^t \cos ht \, dt \right]$

let $f(t) = \cos ht$

$\therefore L[f(t)] = \frac{s}{s^2 - a^2} = \bar{f}(s) \quad [a = 1]$

$L\{e^t f(t)\} = \bar{f}(s-1) = \frac{s-1}{(s-1)^2 - a^2} = \frac{s-1}{s^2 - 2s + 1 - 1^2} = \frac{s-1}{s^2 - 2s} = \bar{g}(s)$

$L\{g(t)\} = \bar{g}(s)$

$\therefore L\left\{\int_0^t g(t) \, dt\right\} = \frac{1}{s} \bar{g}(s) = \frac{1}{s} \frac{(s-1)}{s(s-2)}$

$h(t) = \frac{s-1}{s^3 - 2s^2} = \bar{h}(s)$

$\therefore L\{\cos ht \cdot h(t)\} = L\left\{h(t) \left[\frac{e^t + \bar{e}^t}{2} \right]\right\}$

$= \frac{1}{2} \left\{ L\{h(t)e^t\} + L\{h(t)\bar{e}^t\} \right\}$

$= \frac{1}{2} \left[\bar{h}(s-1) + \bar{h}(s+1) \right]$

$= \frac{1}{2} \left[\frac{(s-1)-1}{(s-1)^3 - 2(s-1)^2} + \frac{(s+1)-1}{(s+1)^3 - 2(s+1)^2} \right]$

$= \frac{1}{2} \left[\frac{s-2}{(s-1)^3 - 2(s-1)^2} + \frac{s}{(s+1)^3 - 2(s+1)^2} \right] //$

Q 4a

Find inverse Laplace transform using convolution theorem

$$\phi(s) = \frac{(s+2)^2}{(s^2+4s+8)^2}$$

$$L^{-1}\{\phi(s)\} = L^{-1}\left\{\frac{(s+2)^2}{((s+2)^2+4)^2}\right\}$$

$$= e^{-2t} L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$$

$$= e^{-2t} L^{-1}\left[\frac{s}{s^2+4} \times \frac{s}{s^2+4}\right]$$

$$= e^{-2t} L^{-1}[F(s) \cdot \bar{g}(s)]$$

By convolution theorem : $L^{-1}\{F(s)\bar{g}(s)\} = \int_0^t f(u) \cdot g(t-u) du$

$$= e^{-2t} \int_0^t f(u) \cdot g(t-u) du$$

$$f(u) = L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2u = g(u)$$

$$\therefore L^{-1}\{\phi(s)\} = e^{-2t} \int_0^t \cos 2u \cdot \cos(2t-2u) du$$

$$= \frac{e^{-2t}}{2} \int_0^t \cos(2u+2t-2u) + \cos(2u-2t+2u) du$$

$$= \frac{e^{-2t}}{2} \int_0^t \cos 2t + \cos(4u-2t) du$$

$$= \frac{e^{-2t}}{2} \left[u \cos 2t + \frac{\sin(4u-2t)}{4} \right]_0^t$$

$$= \frac{e^{-2t}}{2} \left[t \cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4} \right]$$

$$= \frac{e^{-2t}}{4} [2t \cos 2t + \sin 2t]$$

//

Q5b

Find $L^{-1} \left[\frac{s}{s^4 + s^2 + 1} \right]$

Let $\bar{f}(s) = \frac{s}{(s^2+1)^2 - s^2} = \frac{s}{(s^2+1+s)(s^2+1-s)}$

Since the difference is $2s$ in denominator

$$\therefore \bar{f}(s) = \frac{1}{2} \left[\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s} \right]$$

$$= \frac{1}{2} \left[\frac{1}{(s-1/2)^2 + 3/4} - \frac{1}{(s+1/2)^2 + 3/4} \right]$$

$$L^{-1} \left[\frac{1}{s^2 + a^2} \right] = \frac{\sin at}{a}; \quad L^{-1} \{ \bar{f}(s-a) \} = e^{at} L^{-1} \{ \bar{f}(s) \}$$

$$\therefore L^{-1} \{ \bar{f}(s) \} = \frac{1}{2} \left[\frac{e^{t/2} \sin \frac{\sqrt{3}}{2}t}{\sqrt{3}/2} - \frac{e^{-t/2} \sin \frac{\sqrt{3}}{2}t}{\sqrt{3}/2} \right]$$

$$= \frac{2}{\sqrt{3}} \times \frac{1}{2} \times \sin \left(\frac{\sqrt{3}}{2}t \right) (e^{t/2} - e^{-t/2})$$

$$= \frac{2}{\sqrt{3}} \sinh \left(\frac{t}{2} \right) \cdot \sin \left(\frac{\sqrt{3}t}{2} \right)$$

$$\therefore L^{-1} \left[\frac{s}{s^4 + s^2 + 1} \right] = \frac{2}{\sqrt{3}} \sinh \left(\frac{t}{2} \right) \cdot \sin \left(\frac{\sqrt{3}t}{2} \right) //$$