Quantum Walk: Position–Qubit Dynamics Using Pauli-X

1 Introduction

We study a discrete-time quantum walk on the integer line, where Bob's position is represented by a basis state $|n\rangle$ and his internal coin (qubit) state by $|0\rangle$ or $|1\rangle$.

At each time step, Bob's position changes depending on the qubit state, and the qubit undergoes a Pauli-X operation. The goal is to compute the time evolution of the joint state over multiple steps.

2 System Setup

Bob begins at position $n = 0$, and the qubit starts in state $|0\rangle$:

$$|\psi(0)\rangle = |x = 0\rangle \otimes |0\rangle = |0\rangle|0\rangle.$$

The Pauli-X gate acts as:

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle.$$

Right movement corresponds to qubit $|1\rangle$ and left movement corresponds to $|0\rangle$.
Thus:

$$X(|0\rangle|0\rangle) = |1\rangle|1\rangle,$$
$$X(|1\rangle|1\rangle) = |0\rangle|0\rangle.$$

Bob oscillates between positions 0 and 1:

$$|n\rangle|q\rangle = \begin{cases} |0\rangle|0\rangle, & t \text{ even}, \\ |1\rangle|1\rangle, & t \text{ odd}. \end{cases}$$

3 General Coin State

Here, we use a Hadamard operation on the qubit instead of the Pauli-X operation. Let the qubit initially be in the arbitrary superposition:

$$|\chi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

The total state is

$$|\psi(0)\rangle = |n\rangle|\chi\rangle.$$

4 Step-by-Step Evolution

At $t = 1$

Movement according to coin value gives:

$$|\psi(1)\rangle = \alpha|n-1\rangle|0\rangle + \beta|n+1\rangle|1\rangle.$$

At $t = 2$

Applying H again:

$$|\psi(2)\rangle = \frac{\alpha}{\sqrt{2}}\left(|n-2\rangle|0\rangle + |n\rangle|1\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|n\rangle|0\rangle - |n+2\rangle|1\rangle\right).$$

Grouped by coin state:

$$|\psi(2)\rangle = \left(\frac{\beta}{\sqrt{2}}|n\rangle + \frac{\alpha}{\sqrt{2}}|n-2\rangle\right)|0\rangle + \left(\frac{\alpha}{\sqrt{2}}|n\rangle - \frac{\beta}{\sqrt{2}}|n+2\rangle\right)|1\rangle.$$

At $t = 3$

$$|\psi(3)\rangle = \frac{1}{\sqrt{8}}\Big[(\beta|n\rangle + \alpha|n-2\rangle)(|0\rangle + |1\rangle) + (\alpha|n\rangle - \beta|n+2\rangle)(|0\rangle - |1\rangle)\Big].$$

Expanding the expression:

$$\begin{aligned}|\psi(3)\rangle &= \frac{1}{2}\Big[(\beta|n-1\rangle + \alpha|n-3\rangle + \alpha|n-1\rangle - \beta|n+1\rangle)|0\rangle\Big] \\ &+ \frac{1}{2}\Big[(\beta|n+1\rangle + \alpha|n-1\rangle + \alpha|n+1\rangle + \beta|n+3\rangle)|1\rangle\Big].\end{aligned}$$

Final simplified form:

$$\begin{aligned}|\psi(3)\rangle &= \frac{1}{2}\left[(\beta+\alpha)|n-1\rangle + \alpha|n-3\rangle - \beta|n+1\rangle\right]|0\rangle \\ &+ \frac{1}{2}\left[(\beta-\alpha)|n+1\rangle + \alpha|n-1\rangle + \beta|n+3\rangle\right]|1\rangle.\end{aligned}$$

5 Conclusion

We derived the explicit time evolution of a quantum walk driven by the Pauli-X gate and the Hadamard gate. Starting from an arbitrary initial coin state, the position distribution spreads while the qubit state coherently mixes amplitudes. The computation outlines the mechanism by which interference patterns emerge in quantum walks. This walk is simulated in the file "problem_two.ipynb".