

Problem zero : Random Walk on the Integer Line and Computing Probability Distribution and RMS Displacement

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1 Introduction

Consider a one-dimensional symmetric random walk. At each time step, Bob flips a fair coin:

$$\text{Heads} \rightarrow +1, \quad \text{Tails} \rightarrow -1.$$

After N steps, Bob's position is denoted by p . The objective of this report is to compute the probability distribution of p and derive the RMS (root-mean-square) displacement after N steps.

2 Probability Distribution of the Position

After N steps, Bob can be at any point

$$p \in \{-N, -N+1, \dots, N-1, N\}, \quad p \in \mathbb{Z}.$$

The probability distribution satisfies the normalization condition:

$$\sum_{i=-N}^N P(p=i) = 1.$$

2.1 Number of Total Trajectories

Each step has 2 possibilities. Thus the total number of possible trajectories is:

$$T(N) = 2^N.$$

2.2 Trajectories Leading to Position $p = i$

Let N_+ be the number of $+1$ steps and N_- the number of -1 steps. Then

$$N = N_+ + N_-,$$

$$i = p = N_+ - N_-.$$

Solving these equations:

$$N_+ = \frac{N+i}{2}, \quad N_- = \frac{N-i}{2}.$$

Both must be integers, hence i must have the same parity as N .

The number of trajectories leading to position i is the number of ways to choose which of the N steps are the N_+ positive ones:

$$W(N, i) = \binom{N}{N_+} = \frac{N!}{\left(\frac{N+i}{2}\right)! \left(\frac{N-i}{2}\right)!}.$$

2.3 Probability Distribution

Thus the probability that Bob is at position $p = i$ after N steps is

$$P(p = i) = \frac{W(N, i)}{T(N)} = \frac{N!}{\left(\frac{N+i}{2}\right)! \left(\frac{N-i}{2}\right)!} \cdot \frac{1}{2^N}.$$

3 RMS Displacement

The RMS displacement is defined as

$$\text{RMS} = \sqrt{\mathbb{E}[p^2]},$$

where $\mathbb{E}[p^2]$ is the expected value of p^2 .

3.1 Expectation of p^2

Using the probability distribution:

$$\mathbb{E}[p^2] = \sum_{i=-N}^N i^2 P(p = i).$$

3.2 Final Expression for RMS Displacement

$$\text{RMS displacement after } N \text{ steps} = \sqrt{\sum_{i=-N}^N i^2 P(p = i)}.$$

This expression is evaluated in the Jupyter file "problem_zero.ibynb". It is observed to grow approximately as \sqrt{N}

4 Conclusion

Starting from first principles, we derived the full probability distribution of the position p after N steps of a symmetric random walk. Using this distribution, we computed the RMS displacement, which grows as \sqrt{N} , a hallmark of diffusive processes.