Experiment 5: Harmonic Oscillator
Part I. Spring Oscillator

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Investigating Damped and Undamped Harmonic Oscillation

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In mechanics, a mass suspended from a spring undergoes a harmonic motion. The objective of the current experiment was to apply the theoretical physics that was studied by observing and investigating the damped and undamped harmonic motion. A system of hanging mass and spring was used to measure the resonant frequency of the two harmonic motions. For analyzing the undamped harmonic oscillation, a mass was freely suspended from a spring attached to a force sensor and an initial force was applied to initiate the oscillation about an equilibrium position. For analyzing the damped harmonic oscillation the mass was suspended in an aluminum tube for providing a damping force due to Eddy currents produced. A graph was plotted between force sensor voltage and time for both cases. It was observed that the amplitude remained almost constant for undamped oscillation while it decreased by an almost constant factor for consecutive amplitudes in case of damped oscillation. The resonant frequency was found to be $.67 \pm .01 Hz$ for undamped oscillation and $.66 \pm .01 Hz$ for damped oscillation from the graphs. These values matched exactly with the calculated values of frequency from the formulas.

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Introduction

In mechanics, harmonic oscillation is the motion of a body about an equilibrium position due to its inertia and a restoring force acting on it. Harmonic oscillations are found in nature extensively and were studied by several scientists including Robert Hooke and Jean Fourier. If the only force acting on a body in harmonic oscillation is proportional to the displacement, then such a motion is called Simple Harmonic Motion and the oscillations are undamped. Such a motion has constant amplitude and frequency. If there is another force acting on the body that is proportional to the velocity of the body, the resulting oscillation is a damped oscillation.

The objective of the current experiment was to learn how to investigate damped and undamped harmonic oscillators and to find the resonant frequency. The first step of the experiment was finding the spring constant by plotting a graph of weight versus expansion of spring. For studying the undamped oscillation, a mass was freely suspended from the spring and an oscillation was initiated and the force sensor voltage readings were recorded. For creating a damped oscillation, the mass, which had magnets embedded, was suspended in an aluminum tube. The oscillation of the magnets in a non-magnetic material create eddy currents due to aluminum's inductance which provide the necessary damping force.

Methods

The first step of the experiment was to calculate the spring constant *k* of the spring being used in the experiment. A series of masses were weighed and then freely suspended from the spring and the distance of spring from ground was recorded in an excel sheet for each of the five masses. The masses were converted to force and a graph was plotted (Figure 5.1) for force versus distance and the equation of the best fit line was obtained. The slope of the line gives the spring constant *k.*

The force sensor was connected to PASCO and DAQ was setup. Since the experiment does not require the value of the force, the display was set to voltage. The sample rate was set to 40Hz. The force sensor was attached to a rigid stand. Then the spring was attached to the force sensor using a string. The mass which has embedded magnets was attached to the spring using another string as shown in Figure 5.1. The strings were used to avoid systematic uncertainty that may arise due to rotation of spring as it stretches. The strings decouple the rotation. Then a small amplitude was provided to the mass such that the coils of the spring did not touch on stretching. The recording was started on Capstone and readings were recorded for 20 seconds and then transferred to an Excel sheet. For the damped oscillation, the mass was suspended in an aluminum tube. The mass was positioned such that the oscillation would happen completely inside the tube. A small amplitude was provided and the readings were recorded for 20 seconds and transferred to another Excel sheet.

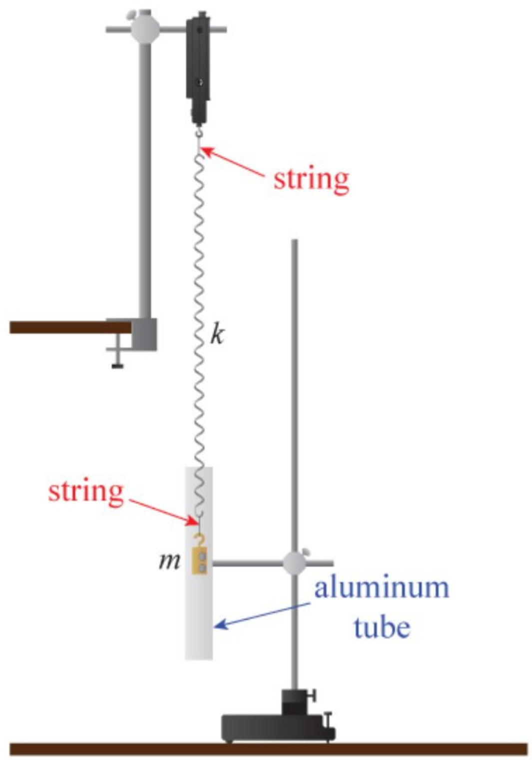

Figure 5.1 Mass with embedded magnets suspended from a spring attached to a force sensor for observing damped and undamped harmonic oscillation. The aluminum tube is used in the damped oscillation case because it provides a damping force proportional to velocity of *m* due to Eddy currents[1].

Analysis

First, the spring constant *k* of the spring being used in the experiment was determined by using Hooke's Law. Five masses were suspended from the spring and the values of the distance from floor to the end of spring were recorded. Figure 5.1 shows the graph obtained and the best fit line.

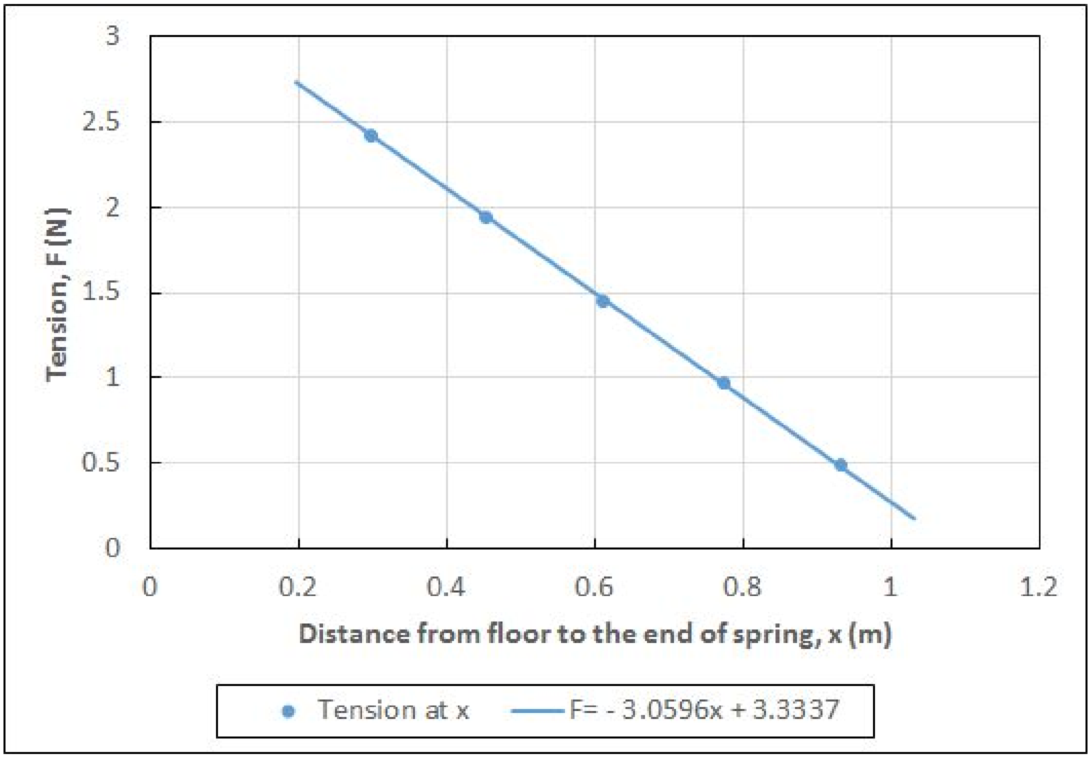

Figure 5.2 Determination of spring constant by suspending five masses (49.4g, 99.1g, 148.5g, 198.3g, 247.7g) freely form a spring. The best fit line has equation $F = -3.0596x + 3.3337$. The spring constant was found to be $k = 3.06 \pm .01\ N/m$

The resonant frequency f_0 for an undamped harmonic motion is given by Eqn. 5.1:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

where k is the spring constant = $3.06 \pm .01\ N/m$ and
m is the hanging mass = 172.5g $\pm .05g$

Substituting these values in Eqn. 5.1 and using the uncertainty propagation (using Eqn. ii.23 and ii.24):

$$\frac{\delta f_0}{f_0} = \sqrt{(\frac{\delta k}{2k})^2 + (\frac{\delta m}{2m})^2}$$

=> $\delta f_{0,\, predicted} = .670 \pm .001 Hz$

For measuring frequency of free oscillation the recorded data was plotted (Figure 5.3) and then positions of extrema were used . This was done by zooming at maximas and noting the time at which maxima occured. This was done for two maximas and the time difference was divided by the difference of the maximas.

$T = \frac{t_n - t_1}{n-1}$ where T is the time period and t_1 and t_n are the time for first and nth extrema.

$f_0 = 1/T$

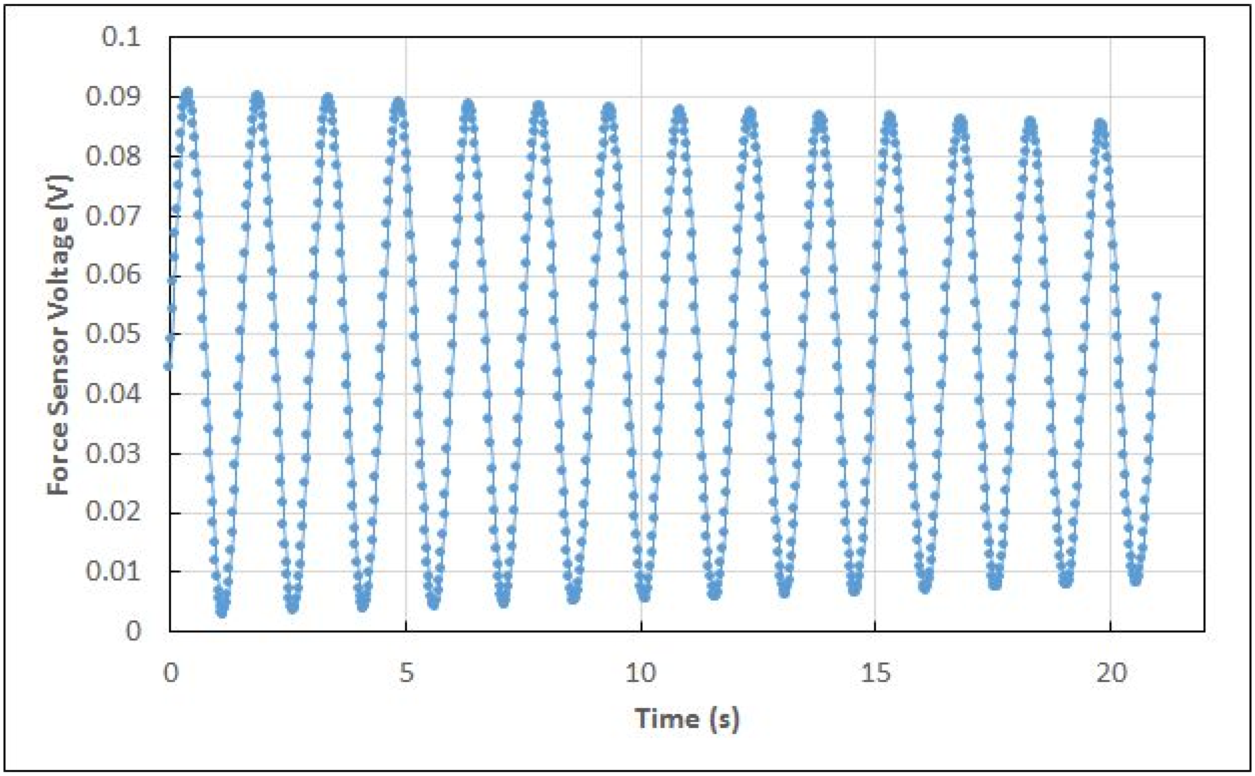

Figure 5.3 Simple harmonic oscillation of a mass suspended from a spring and given a small amplitude. The x-axis shows the time of reading and the y-axis is the Force Sensor Voltage in volts at that time.

The first and 14th extrema were used. $T = (19.8 - 0.4)/13 = 1.490 \pm .025s$

The calculated frequency was:
$f_{0,\ calculated} = \frac{1}{T} = .67 \pm .01 Hz$
$\delta f_0 = f_0 \frac{\delta T}{T}$
.025s is the time interval of taking voltage readings. This is the uncertainty in determining the exact time of the extremum. By using error propagation the uncertainty was $.01 Hz$ for $\delta f_{0,\ calculated}$.

Thus the value obtained of calculated frequency from the graph is same as the predicted frequency that was calculated using the formula.
Similarly, the voltage versus time data was plotted for damped oscillation (Figure 5.4). However, for this graph the oscillations were first centered around 0 by subtracting .04V from the voltage

readings.

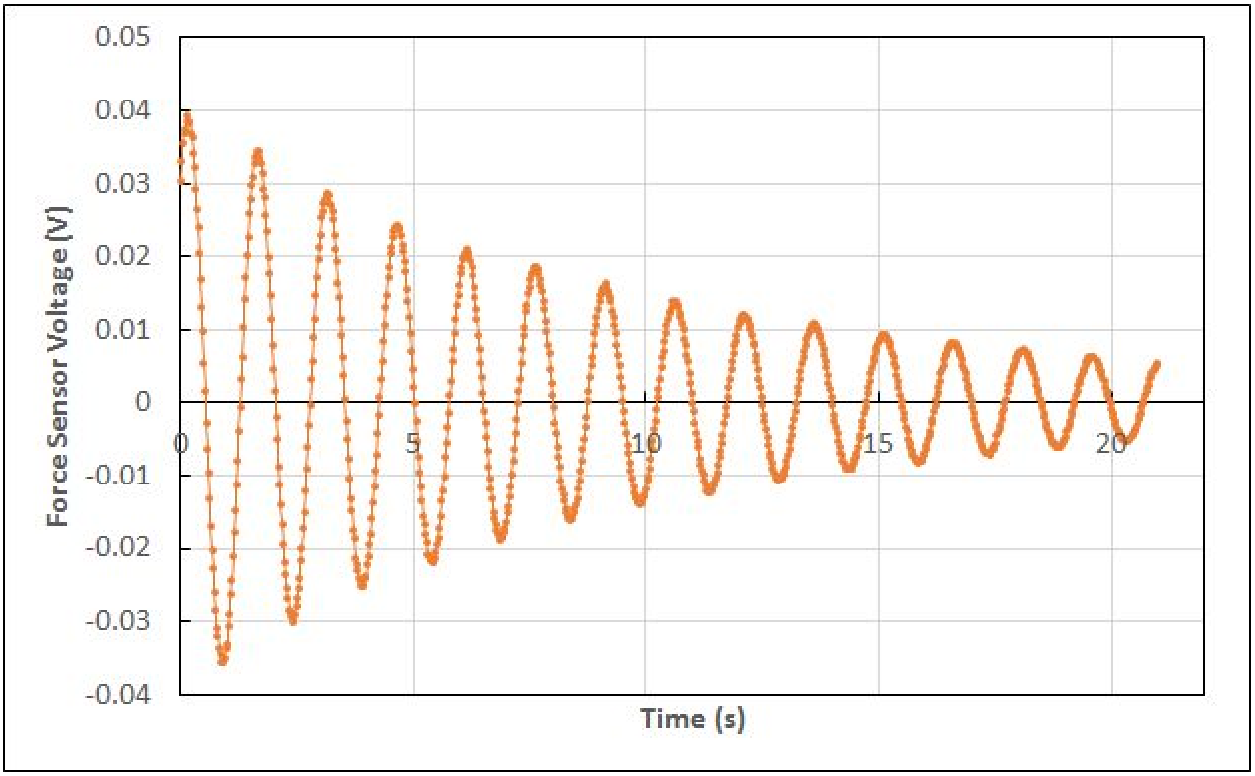

Figure 5.4 Damped harmonic oscillation of a mass suspended from a spring in an aluminum tube. The x-axis shows the time of reading and the y-axis is the Force Sensor Voltage in volts at that time. The offset (.04V) was subtracted from each voltage reading to center it around 0.

f_{damped} was calculated by zooming at maximas and noting the time at which maxima occured. This was done for two maximas and the time difference was divided by the difference of the maximas.

The first and 14th extrema were used. $T = (19.6 - 0.15)/13 = 1.496 \pm .025s$

$f_{damped,\, calculated} = \frac{1}{T} = .66 \pm .01Hz$

$\delta f_{damped} = f_{damped}\frac{\delta T}{T}$

.025s is the time interval of taking voltage readings. This is the uncertainty in determining the exact time of the extremum. By using error propagation the uncertainty was $.01Hz$ for δf_{damped}.

In case of damped oscillation, the resonant frequency f_{damped} is also defined as:

$$f_{\text{damped}} \equiv f_o\sqrt{1 - \frac{1}{4Q^2}},$$

Where Q is the Quality Factor.

The damping time τ is defined as the time taken by the damped oscillator's amplitude to decrease by a factor of 1/e.

Deriving τ

The differential equation that represents the damped motion:

$$m\ddot{x} = -kx - b\dot{x}.$$

Trying $x(t) = Ae^{i\omega t}$

$$\begin{aligned} -\omega^2 m A e^{i\omega t} &= -kAe^{i\omega t} - i\omega b A e^{i\omega t} \\ m\omega^2 &= k + i\omega b. \end{aligned}$$

By solving this quadratic equation, we obtain

$$\omega = \frac{ib}{2m} \pm \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Substituting ω in the trial solution,

$$\begin{aligned} x(t) &= Ae^{i\left(\frac{ib}{2m} + \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\right)t} \\ &= Ae^{i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\,t} \times e^{-bt/2m}. \end{aligned}$$

By the aforementioned definition of τ

$$\tau = \frac{2m}{b}$$

Deriving relationship between Q, k, m and b

We know that,

$$f_{\text{damped}} \equiv f_o\sqrt{1 - \frac{1}{4Q^2}},$$

and,

$$f_{\text{damped}} = \frac{\omega_{\text{damped}}}{2\pi} \equiv \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = f_o\sqrt{1 - \frac{b^2}{4km}}$$

Equating these two equations,

$$\sqrt{1 - \frac{1}{4Q^2}} = \sqrt{1 - \frac{b^2}{4km}}$$

Squaring and simplifying,

$$\frac{1}{4Q^2} = \frac{b^2}{4km}$$

$$Q^2 = \frac{km}{b^2}$$

$$Q = \frac{\sqrt{km}}{b}$$

Uncertainty propagation (using Eqn. ii.23 and ii.24):

$$\frac{\delta Q}{Q} = \sqrt{\left(\frac{\delta k}{2k}\right)^2 + \left(\frac{\delta m}{2m}\right)^2 + \left(\frac{\delta b}{b}\right)^2}$$

Q can be found if we know τ by first finding *b* and then substituting the values in the above equation

The successive ratio of amplitudes for the damped oscillation were calculated in Excel and a graph of ratio versus extremum number were plotted (Figure 5.5).

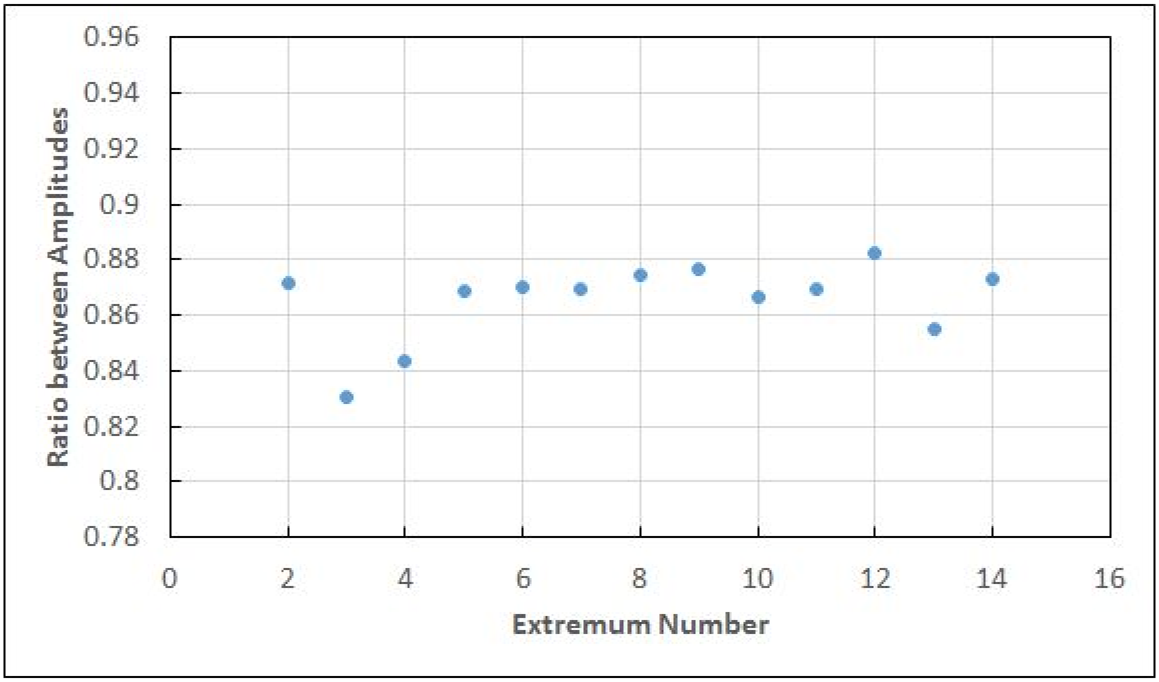

Figure 5.5 Analysis of amplitudes of the damped harmonic oscillation shown in figure 5.4 The ratio of successive amplitudes ranges from .83 to .88

The points plotted on Figure 5.5 show that the ratio of successive amplitudes ranges from .83 to .88 which is expected.

Each ratio measurement was converted to τ by using the formula:

$$\tau = -\frac{T}{ln[\frac{V(t+T)}{V(t)}]}$$

The mean damping time was: $\tau = 10.4 \pm .3s$

The statistical uncertainty in the mean value was found using the procedure in ii.1.6 of the lab manual by calculating the standard deviation and dividing by the square root of number of observations.

From the above derivation, $\tau = \frac{2m}{b}$

Thus $b = \frac{2m}{\tau} = .033 \pm .001 kg/s$

Since, $Q = \frac{\sqrt{km}}{b}$

Substituting the values of k, m and b

$Q = 21.9 \pm .9$

Using the value of Q and f_0,

$$f_{\text{damped}} \equiv f_o\sqrt{1 - \frac{1}{4Q^2}},$$

$$f_{damped} = .67\sqrt{1 - \frac{1}{4(21.9)^2}} = .66 \pm .01Hz$$

Uncertainty propagation:

$$\frac{\delta f_{damped}}{f_{damped}} = \sqrt{(\frac{\delta f_0}{f_0})^2 + (\frac{\delta Q}{Q})^2}$$

The value of measured f_{damped} and calculated f_{damped} using the formula is exactly the same and the calculated uncertainty is also the same in both cases.

Conclusion

This experiment was performed to study simple harmonic motion and damped harmonic motion. A spring-mass system was used to create an oscillator and an aluminum tube was used to provide the damping force. The free and damped oscillation frequencies were calculated using formulas and these values were compared to the experimentally calculated values. The values obtained were $f_{0,\ predicted} = .670 \pm .001Hz$ and $f_{0,\ calculated} = .67 \pm .01Hz$. It can be clearly seen from the values that the predicted and experimental values of frequency of free oscillation are equivalent till two decimal places. The frequency value calculated using the formula for damped oscillation was $f_{damped} = .66 \pm .01Hz$ which was exactly the same as the frequency calculated from Figure 5.4. Thus the relationship between frequency, spring constant and mass was verified.

The mean damping time for the damped oscillation was found to be $\tau = 10.4 \pm .3s$ and the Q-factor was calculated as $Q = 21.9 \pm .9$

One possible source of systematic error in this experiment is the spring. In our experiment the spring constant was small and thus the uncertainty in calculating it was high. Due to this systematic error, the calculation of $f_{0,\ predicted}$ may shift towards lower side. To avoid such a systematic error a spring with a higher spring constant can be used.

Bibliography

1. Campbell, W. C. *et al*. Physics 4AL: Mechanics Lab Manual (ver. May 15, 2016). (Univ. California Los Angeles, Los Angeles, California).