MATH1072 - Dimensional analysis

1. The speed of sound in a gas depends on the pressure and the density. Note that in SI units, speed of sound has units $m \cdot s^{-1}$, pressure has units $kg \cdot m^{-1} \cdot s^{-2}$ and density has units $kg \cdot m^{-3}$. Use dimensional analysis to obtain an expression for the speed of sound in terms of pressure and density.
2. We want to know how the speed, v, of waves travelling along a string depends on the string mass m, length l, and tension Q. Hans claims that the relation is $v = kl\sqrt{\frac{Q}{m}}$, while Uschi claims that it is $v = k\sqrt{\frac{lQ}{m}}$, where k is a dimensionless constant. Whose claim is more credible?
3. Hans und Uschi Studios want to film a scene in which an earthquake strikes Brisbane and causes 1 William Street, standing at 260 metres tall, to collapse to the ground. In order to achieve this, the studio will construct a scale model of the building, one metre tall, and film the model collapsing. The resulting footage will look unrealistic, because the model will collapse too quickly. The studio can fix this by using slow-motion. Assume that the time to collapse only depends on the mass of the building, the height of the building, and the acceleration due to gravity. In order to make it look realistic, by what factor should the film be slowed down?
4. After a nuclear explosion, the radius (r) of the shock wave is assumed to be a function of the energy of the explosion (E, units of force times distance), the initial density of the atmosphere (ρ, units of mass per volume), the initial pressure of the atmosphere (P, units of force per area), and the time elapsed (t). Using the principles of dimensional analysis, determine the possible forms of $r = r(E, \rho, P, t)$.
5. **Challenge** - One approach to solving ODEs is through an appropriate change of variables. Dimensional analysis may provide a means to select a useful change of variables.
   Consider the equation
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2y} + \frac{y}{2x}. \tag{1}$$
   If we identify x and y as having different dimensions, say $[x] = X$, $[y] = Y$, then the equation is not dimensionally homogeneous. To enable dimensional homogeneity, consider instead
   $$\alpha\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\beta x}{2y} + \frac{\alpha y}{2x}.$$
   where $[\alpha] = X$, $[\beta] = Y^2X^{-1}$. Show that there exists a pair of dimensionless variables $t = t(x, y, \alpha, \beta)$ and $u = u(x, y, \alpha, \beta)$ such that (1) can be expressed as
   $$\frac{\mathrm{d}u}{\mathrm{d}t} = 1,$$
   and consequently determine the general solution of (1).