**SCHOOL OF MATHEMATICS AND PHYSICS**

# MATH1072
## Assignment 4
## Semester Two 2024

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*Submit your answers - along with this sheet - by 1pm on the 21st of October, using the blackboard assignment submission system. Assignments must consist of a single PDF.*
You may find some of these problems challenging. Attendance at weekly tutorials is assumed.

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Marker's use only

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Each question marked out of 3.

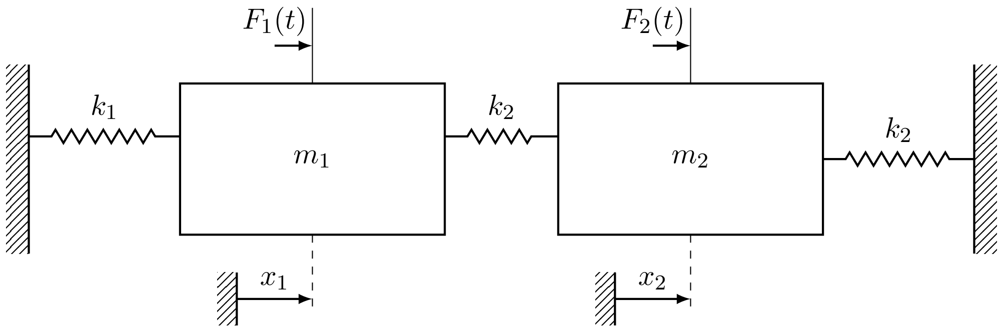
- Mark of 0: You have not submitted a relevant answer, or you have no strategy present in your submission.
- Mark of 1: Your submission has some relevance, but does not demonstrate deep understanding or sound mathematical technique.
- Mark of 2: You have the right approach, but need to fine-tune some aspects of your calculations.
- Mark of 3: You have demonstrated a good understanding of the topic and techniques involved, with well-executed calculations.

| | | | |
|---|---|---|---|
| Q1a | Q1b: | Q1c: | Q2a: |
| Q2b: | Q2c: | Q2d: | Q2e: |
| Q3: | | | |

Total (out of 27):

## Question 1: Spring-Mass System

Consider the spring-mass system described by the following image.




(a) Derive the system of second order ordinary differential equations that describes the spring-mass system.

(b) Write out the reduced system of ordinary differential equations in **vector form** that can be used to solve your system from part (a).

(c) Use the MATLAB function `ode45()` to solve your system from part (b) over time $[0, 200]$, with the following parameters:

$$F_1(t) = \sin(t),\ F_2(t) = e^{-t}$$
$$k_1 = 2,\ k_2 = 0.5$$
$$m_1 = 3,\ m_2 = 1$$

***Solution:*** (a)
We have here 2 masses, $m_1$ and $m_2$, which are joined by 3 springs . The spring connected a fixed support to $m_1$ has spring constant $k_1$. The spring connecting $m_1$ to $m_2$ and $m_2$ to a fixed support, has spring constant $k_2$. An arbitrary push force $F_1$, itself a function of time, is applied to $m_1$. Another push force, $F_2$, also a function of time, is applied to $m_2$. These forces will cause $m_1$ and $m_2$ to be displaced from their equilibria, and these displacements are given by $x_1(t)$ and $x_2(t)$, respectively.

Newton's second law of motion states that the sum of all forces applied to an object is equal to the object's acceleration times its mass,

$$F = ma.$$

Hooke's law states that the spring force is given by the spring's $k$-constant multipled by the distance from equilibrium,

$$F_s = kx$$

Each mass has 2 springs acting on it, so each mass's net force is comprised of 3 forces: 2 spring forces, and 1 arbitrary push force. So, we can write

$$m_1 a_1 = F_1 + F_{s1} + F_{s2}$$
$$m_2 a_2 = F_2 + F_{s3} + F_{s4}$$

The spring force between the fixed surface and $m_1$ is acting against the displacing force. The spring between $m_1$ and $m_2$, in contrast, is contributing to the displacing force, relative to the displacement between the two masses. Therefore,

$$
\begin{aligned}
F_{s1}(t) &= -k_1 x_1(t) \\
F_{s2}(t) &= k_2(x_2(t) - x_1(t)).
\end{aligned}
$$

The spring force between the $m_2$ and $m_1$ is acting against the displacing force, while the spring force between $m_2$ and the fixed surface acts along the displacement. Therefore,

$$
\begin{aligned}
F_{s3}(t) &= k_2(x_1(t) - x_2(t)) \\
F_{s4}(t) &= -k_2 x_2(t)
\end{aligned}
$$

Finally, we note that acceleration, $a$, is the second derivative of position, $x$, with respect to time, $\ddot{x}$. Bringing this all together, we can describe the spring-mass system witth the following system of second-order differential equations,

$$
\begin{aligned}
m_1\ddot{x}_1 &= F_1 - k_1 x_1 + k_2(x_2 - x_1) \\
m_2\ddot{x}_2 &= F_2 + k_2(x_1 - x_2) - k_2 x_2
\end{aligned}
$$

Which we can simplify to

$$
\begin{aligned}
m_1\ddot{x}_1 &= F_1 + (k_2 - k_1)x_1 + k_2 x_2 \\
m_2\ddot{x}_2 &= F_2 + k_2 x_1 - 2k_2 x_2
\end{aligned}
$$

***Solution:*** (b)
Let $v_1 = \dot{x}_1$ and $v_2 = \dot{x}_2$, and express the system we've derived in terms of these,

$$
\begin{aligned}
m_1\dot{v}_1 &= F_1 + (k_2 - k_1)x_1 + k_2 x_2 \\
m_2\dot{v}_2 &= F_2 + k_2 x_1 - 2k_2 x_2
\end{aligned}
$$

Let $\underset{\sim}{y}(t)$ be the state vector of the system,

$$
\underset{\sim}{y}(t) = \begin{pmatrix} x_1(t) \\ v_1(t) \\ x_2(t) \\ v_2(t) \end{pmatrix}.
$$

Taking the derivative,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\underset{\sim}{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1(t) \\ v_1(t) \\ x_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ \frac{1}{m_1}\left(F_1 + (k_2 - k_1)x_1 + k_2 x_2\right) \\ v_2(t) \\ \frac{1}{m_2}\left(F_2 + k_2 x_1 - 2k_2 x_2\right) \end{pmatrix},
$$

which is the reduced vector form of the system we sought.

***Solution:*** (c)
PS4_script1.m

```
% Defining our given parameters
k1 = 2;
k2 = 0.5;
m1 = 3;
m2 = 1;
t_range = [0 200];
```

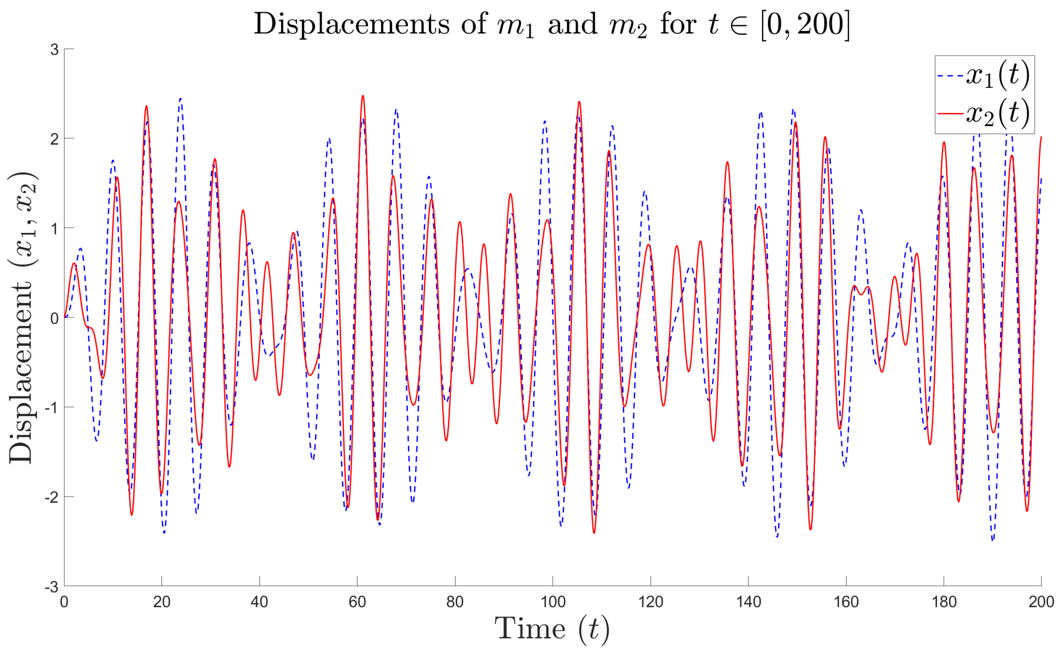
```
% The inital conditions
% Inital position and velocity are both 0
% i.e. The system starts at equilibrium
y0 = [0; 0; 0; 0];

% Defining the system, according to what we derived in the
% previous question. Note that y(1) = x_1(t), y(3) = x_2(t)
% and y(2) = v_1(t) = \dot{x}_1(t), y(4) = v_2(t) = \dot{x}_2(t).
system = @(t, y) [
  y(2);
  (sin(t) + k2 * (y(3) - y(1)) - k1 * y(1)) / m1;
  y(4);
  (exp(-t) + k2 * (y(1) - y(3)) - 2 * k2 * y(3)) / m2;
];

[t, y] = ode45(system, t_range, y0);

% Plot the results. The displacement results are stored in
% y(:, 1 and 3). Whereas the velocity results are stored
% in y(:, 2 and 4). We'll plot and present the displacements
% of m_1 and m_2 over time. The question was a little vauge
plot(t, y(:, 1), 'b--', 'LineWidth', 1.5); hold on;
plot(t, y(:, 3), 'r-', 'LineWidth', 1.5);
set(gca, 'FontSize', 14);
box off;
title('Displacements of $m_1$ and $m_2$ for $t\in[0,200]$', ...
  'FontSize', 36, 'Interpreter', 'latex');
xlabel('Time $(t)$', 'FontSize', 36, 'Interpreter', 'latex');
ylabel('Displacement $(x_1, x_2)$', 'FontSize', 36, ...
  'Interpreter', 'latex');
legend('$x_1(t)$', '$x_2(t)$', 'FontSize', 36, ...
  'Interpreter', 'latex');
```

Output:



## Question 2

We will explore what it means for the linear combination of solutions to a second-order differential equation, resulting from an application of the *superposition principle*, to be a general solution to the corresponding initial value problem. That is, we will explore what it means for a set of solutions to form a **fundamental set of solutions**.

(a) Suppose $y_1(t)$ and $y_2(t)$ are solutions to the second-order differential equation,

$$p(t)y'' + q(t)y' + r(t)y = 0.$$

Use the superposition principle to find a general solution in terms of constant coefficients $c_1$ and $c_2$.

(b) Consider the initial conditions for the second-order differential equation given by,

$$y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

Apply these initial conditions to your solution from (a) and solve for the constants $c_1$ and $c_2$ using Cramer's rule:
Given the system of linear equations, $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$, then

$$c_1 = \frac{\begin{vmatrix} y_0 & b_1 \\ y_0' & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} a_1 & y_0 \\ a_2 & y_0' \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

where $|\cdot|$ denotes the determinant of the matrix. The quantity in the denominator is called the **Wronskian**.

(c) Use the Wronskian for $c_1$ and $c_2$ to develop a condition required for this initial value problem to be solvable. The set of solutions that satisfy this condition are called a **fundamental set of solutions**.

(d) Consider the second-order differential equation given by,

$$2t^2y'' + ty' - 3y = 0, \ t > 0.$$

Given that $y_1(t) = t^{-1}$ is a solution to the second-order differential equation, find another solution $y_2(t)$ using the reduction of order method by assuming $y_2(t) = u(t)y_1(t)$. For ease of exposition, require that $u(1) = 0$ and $u'(1) = \frac{5}{2}$.

(e) Show that the solutions $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

***Solution:*** (a)
By the superposition principle, the general solution to a homogenous second order differential equation, such as $y'' + p(t)y' + q(t)y = 0$, $y(t)$, is the sum of two linearly independent solutions, $y_1(t)$ and $y_2(t)$. The general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t). \tag{1}$$

So, transform the equation by dividing across by $p(t)$.

$$p(t)y'' + q(t)y' + r(t)y = 0$$
$$y'' + \frac{q(t)}{p(t)}y' + \frac{r(t)}{p(t)}y = 0$$

Now, because these are arbitrary functions of $t$, let

$$P(t) = \frac{q(t)}{p(t)} \qquad\qquad Q(t) = \frac{r(t)}{p(t)}$$

$$\therefore y'' + P(t)y' + Q(t)y = 0$$

This is now in a form where we can apply the superposition principle, and therefore the solution is the same as (1).

***Solution:*** (b)
Let's substitute the inital condition $y(t_0) = y_0$ into (1).

$$y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = y_0.$$

Now, let's substitute the inital condition $y'(t_0) = y_0'$ into the derivative of (1).

$$y'(t_0) = c_1 y_1'(t) + c_2 y_2'(t) = y_0'.$$

$$\text{Let} \begin{cases} a_1 = y_1(t_0) \\ b_1 = y_2(t_0) \\ a_2 = y_1'(t_0) \\ b_2 = y_2'(t_0) \end{cases} \qquad \text{Then} \begin{cases} a_1 c_1 + b_1 c_2 = y_0 \\ a_2 c_1 + b_2 c_2 = y_0' \end{cases}$$

which can be expressed as the linear system

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

Applying Cramer's rule, and making back substitutions,

$$c_1 = \frac{\begin{vmatrix} y_0 & b_1 \\ y_0' & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{y_0 b_2 - y_0' b_1}{a_1 b_2 - a_2 b_1} = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

$$c_2 = \frac{\begin{vmatrix} a_1 & y_0 \\ a_2 & y_0' \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1 y_0' - a_2 y_0}{a_1 b_2 - a_2 b_1} = \frac{y_0' y_1(t_0) - y_0 y_1'(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

***Solution:*** (c)
The second-order homogenous system is solvable if and only if the Wronskian of this system,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0.$$

This condition is required, because if it isn't met, $c_1$ and $c_2$ are not defined, and hence there is no general solution to the homogenous second-order system of ODEs.

***Solution:*** (d)
Given that $y_1(t) = t^{-1}$ and $y_2(t) = u(t) y_1(t)$, we can see that

$$\begin{aligned} & y_2(t) = u(t) t^{-1} \\ \implies & y_2'(t) = u'(t) t^{-1} - u(t) t^{-2} \\ \implies & y_2''(t) = u''(t) t^{-1} - 2u'(t) t^{-2} + 2u(t) t^{-3} \end{aligned}$$

We can substitute this solution back into the ODE, $2t^2y'' + ty' - 3y = 0,\ t > 0.$

$$\begin{aligned}
2t^2y_2'' &= 2t^2\left(u''(t)t^{-1} - 2u'(t)t^{-2} + 2u(t)t^{-3}\right) = 2tu''(t) - 4u'(t) + 4u(t)t^{-1} \\
ty_2' &= t\left(u'(t)t^{-1} - u(t)t^{-2}\right) = u'(t) - u(t)t^{-1} \\
-3y_2 &= -3\left(u(t)t^{-1}\right) = -3u(t)t^{-1}
\end{aligned}$$

Now, we can express the system in terms of $u$ and its derivatives,

$$\begin{aligned}
0 &= 2tu''(t) - 4u'(t) + 4u(t)t^{-1} + u'(t) - u(t)t^{-1} - 3u(t)t^{-1} \\
&= 2tu''(t) - 3u'(t)
\end{aligned}$$

If we let $v(t) = u'(t)$, we can rewrite this

$$2tv'(t) = 3v(t) \implies \frac{v'(t)}{v(t)} = \frac{3}{2t}$$

Integrating both sides, we find that

$$\ln|v(t)| = \frac{3}{2}\ln|t| + C$$

Therefore

$$u'(t) = v(t) = C_1t^{3/2}$$

Integrating again, we can find $u(t)$.

$$u(t) = \int C_1t^{3/2}\mathrm{d}x = \frac{2C_1}{5}t^{5/2} + C_2$$

We can finally apply the given inital conditions and solve for the general solution of $u(t)$,

$$\begin{aligned}
u(1) = 0 &\implies 0 = \frac{2C_1}{5}(1)^{5/2} + C_2 = \frac{2C_1}{5} + C_2 \implies C_2 = -\frac{2}{5}C_1 \\
u'(1) = \frac{5}{2} &\implies \frac{5}{2} = C_1(1)^{3/2} = C_1 \implies C_1 = \frac{5}{2} \implies C_2 = -1 \\
u(t) &= \frac{2}{5}\cdot\frac{5}{2}t^{5/2} - 1 = t^{5/2} - 1
\end{aligned}$$

Now, we can substitute this $u(t)$ back into our solution for $y_2(t)$,

$$y_2(t) = u(t)t^{-1} = \left(t^{5/2} - 1\right)t^{-1} = t^{3/2} - t^{-1}.$$

***Solution:*** (e)
This $y_1$ and $y_2$ form a fundamental set of solutions if and only if they are linearly independent, if and only if their Wronskian is greater then 0 for all $t$.

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} t^{-1} & t^{3/2} - t^{-1} \\ -t^{-2} & \frac{3}{2}t^{1/2} + t^{-2} \end{vmatrix} = \frac{3}{2}t^{-1/2} + t^{-3} + t^{-1/2} - t^{-3} = \frac{5}{2}t^{-1/2} = \frac{5}{2\sqrt{t}}$$

For all $t > 0$, $\frac{5}{2}t^{-1/2} \neq 0$. We know this is true, because as $t$ approaches 0 from the right side (the only side from which it could approach, by definition) $\frac{5}{2}t^{-1/2}$ goes to infinity. As $t$ goes to infinity, $\frac{5}{2}t^{-1/2}$ asymptotically approaches 0, but never gets there. Therefore, this $y_1$ and $y_2$ form a fundamental set of solutions.

## Question 3: Second-Order Inhomogeneous DE

Solve the following initial value problem for the second-order inhomogeneous differential equation,

$$y'' - 4y' - 12y = 2e^{5t}, \; y(0) = \frac{8}{7}, \; y'(0) = -\frac{1}{7}.$$

***Solution:*** The general solution for this IVP will take the form

$$y(t) = y_h(t) + y_p(t),$$

where $y_h$ and $y_p$ are solutions to some homogenous and particular equation, respectively.

$$\begin{gathered} y_h(t) = e^{\lambda t}, \quad \text{for some } \lambda \in \mathbb{R}. \\ \text{Then } {y_h}'(t) = \lambda e^{\lambda t} \quad \text{and} \quad {y_h}''(t) = \lambda^2 e^{\lambda t}. \\ \text{So } \lambda^2 e^{\lambda t} - 4\lambda e^{\lambda t} - 12e^{\lambda t} = 0 \text{ is a homogenous equation we can solve.} \\ \left(\lambda^2 - 4\lambda - 12\right) e^{\lambda t} = 0. \end{gathered}$$

Since $\forall x \in \mathbb{R}, \; e^x > 0$, and $\lambda t \in \mathbb{R}$, we can safely through divide by $e^{\lambda t}$.

$$\begin{gathered} \text{Hence } \lambda^2 - 4\lambda - 12 = 0, \\ \Rightarrow (\lambda - 6)(\lambda + 2) = 0. \\ \therefore \lambda \in \{-2, 6\}. \end{gathered}$$

Therefore the solution to the homogenous equation, by the superposition principle, is the sum of these two solutions,

$$\Rightarrow y_h(t) = \alpha e^{-2t} + \beta e^{6t}.$$

Since, $y'' - 4y' - 12y = 2e^{5t}$, we can safely assume a particular solution $y_p(t) = Ae^{5t}$.

$$\Rightarrow y_p'(t) = 5Ae^{5t} \quad \text{and} \quad y_p''(t) = 25Ae^{5t}.$$

Let's now substitute $y_p(t)$ and its derivatives back into the original inhomogeneous equation.

$$\begin{aligned} \left(25Ae^{5t}\right) - 4\left(5Ae^{5t}\right) - 12\left(Ae^{5t}\right) &= 2e^{5t} \\ 25Ae^{5t} - 20Ae^{5t} - 12Ae^{5t} &= 2e^{5t} \\ (25A - 20A - 12A)\, e^{5t} &= 2e^{5t} \\ -7Ae^{5t} &= 2e^{5t} \\ -7A &= 2 \\ A &= -\frac{2}{7} \\ \therefore y_p(t) &= -\frac{2}{7} e^{5t} \end{aligned}$$

Substituting the $y_p$ and $y_h$ we've found back into the general solution $y(t)$,

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= \alpha e^{-2t} + \beta e^{6t} - \frac{2}{7}e^{5t}. \end{aligned}$$

Into the general solution now, substitute the inital condition, $y(0) = \frac{8}{7}$.

$$\begin{aligned} \frac{8}{7} &= \alpha e^{-2\cdot 0} + \beta e^{6\cdot 0} - \frac{2}{7}e^{5\cdot 0} \\ &= \alpha e^{0} + \beta e^{0} - \frac{2}{7}e^{0} \\ &= \alpha + \beta - \frac{2}{7} \\ \therefore \alpha + \beta &= \frac{8}{7} + \frac{2}{7} = \frac{10}{7} \end{aligned} \tag{1}$$

Take the derivative of the general solution,

$$\begin{aligned} y'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha e^{-2t} + \beta e^{6t} - \frac{2}{7}e^{5t}\right) \\ &= -2\alpha e^{-2t} + 6\beta e^{6t} - \frac{10}{7}e^{5t}. \end{aligned}$$

Now substituting the inital condition, $y'(0) = -\dfrac{1}{7}$,

$$\begin{aligned} -\frac{1}{7} &= -2\alpha e^{-2\cdot 0} + 6\beta e^{6\cdot 0} - \frac{10}{7}e^{5\cdot 0} \\ &= -2\alpha e^{0} + 6\beta e^{0} - \frac{10}{7}e^{0} \\ &= -2\alpha + 6\beta - \frac{10}{7}. \\ \therefore -2\alpha + 6\beta &= -\frac{1}{7} + \frac{10}{7} = \frac{9}{7} \end{aligned} \tag{2}$$

We will now solve the linear system, $A\underset{\sim}{x} = \underset{\sim}{b}$, formed by (1) and (2),

$$A\underset{\sim}{x} = \underset{\sim}{b} := \begin{pmatrix} 1 & 1 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10/7 \\ 9/7 \end{pmatrix}.$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 10/7 \\ -2 & 6 & 9/7 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 10/7 \\ 1 & -3 & -9/14 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 10/7 \\ 0 & -4 & -29/14 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 10/7 \\ 0 & 1 & 29/56 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 51/56 \\ 0 & 1 & 29/56 \end{array}\right) \implies (\alpha, \beta) = \left(\frac{51}{56}, \frac{29}{56}\right).$$

Finally, let's substitute $\alpha$ and $\beta$ back into the gernal solution, and present our final solution.

$$y(t) = \frac{51}{56}e^{-2t} + \frac{29}{56}e^{6t} - \frac{2}{7}e^{5t}.$$