100 Derivatives

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(Q1.) $\dfrac{d}{dx}\left(ax^2+bx+c\right)$

$$\Rightarrow \frac{d}{dx}ax^2+\frac{d}{dx}bx+\frac{d}{dx}c=2ax+b$$

(Q2.) $\dfrac{d}{dx}\left(\dfrac{\sin x}{1+\cos x}\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{u'v-uv'}{v^2}\\
u=\sin x &\Rightarrow u'=\cos x\\
v=1+\cos x &\Rightarrow v'=-\sin x\\
\Rightarrow \frac{df}{dx} &= \frac{\cos x(1+\cos x)+\sin^2 x}{(1+\cos x)^2}\\
&= \frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}\\
&= \frac{\cos x+1}{(1+\cos x)^2}\\
\therefore \frac{df}{dx} &= \frac{1}{(1+\cos x)}
\end{aligned}$$

(Q3.) $\dfrac{d}{dx}\left(\dfrac{1+\cos x}{\sin x}\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{u'v-uv'}{v^2}\\
u=1+\cos x &=\Rightarrow u'=-\sin x\\
v=\sin x &\Rightarrow v'=\cos x\\
\Rightarrow \frac{df}{dx} &= \frac{-\sin^2 x-\cos x-\cos^2 x}{\sin^2 x}\\
&= \frac{-\cos x-1(\sin^2 x+\cos^2 x)}{\sin^2 x}\\
&= \frac{-\cos x-1}{\sin^2 x}\\
&= \frac{-1}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\\
\therefore \frac{df}{dx} &= -\cot x\csc x-\csc^2 x
\end{aligned}$$

(Q4.) $\dfrac{d}{dx}\left(\sqrt{3x+1}\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}(3x+1)^{1/2} &= \frac{1}{2}(3x+1)^{-1/2}\frac{d}{dx}(3x+1)\\ &= \frac{1}{2\sqrt{3x+1}}(3)\\ \therefore \frac{df}{dx} &= \frac{3}{2\sqrt{3x+1}}\end{aligned}$$

(Q5.) $\dfrac{d}{dx}\left(\sin^3 x + \sin x^3\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}\sin^3 x + \frac{d}{dx}\sin x^3 &= 3\sin^2 x\frac{d}{dx}(\sin x) + \cos x^3\frac{d}{dx}(x^3)\\ \therefore \frac{df}{dx} &= 3\sin^2 x\cos x + 3x^2\cos x^3\end{aligned}$$

(Q6.) $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}x^{-4} &= -4x^{-5}\\ \therefore \frac{df}{dx} &= \frac{-4}{x^5}\end{aligned}$$

(Q7.) $\dfrac{d}{dx}\left((1+\cot x)^3\right)$

$$\Rightarrow 3(1+\cot x)^2\frac{d}{dx}(1+\cot x) = -3\csc^2 x(1+\cot x)^2$$

(Q8.) $\dfrac{d}{dx}\left(x^2(2x^3+1)^{10}\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}u\cdot v &= u'v+uv'\\ u = x^2 &\Rightarrow u' = 2x\\ v = (2x^3+1)^{10} &\Rightarrow v' = 10(2x^3+1)^9\frac{d}{dx}(2x^3+1)\\ \therefore \frac{dv}{dx} &= 60x^2(2x^3+1)^9\\ \therefore \frac{df}{dx} &= (2x)(2x^3+1)^{10} + 60x^4(2x^3+1)^9\\ &= 2x(2x^3+1)^9(2x^3+1+30x^3)\\ \therefore \frac{df}{dx} &= 2x(2x^3+1)^9(32x^3+1)\end{aligned}$$

(Q9.) $\dfrac{d}{dx}\left(\dfrac{x}{(x^2+1)^2}\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}\frac{u}{v} &= \frac{u'v=uv'}{v^2}\\ u = x &\Rightarrow u' = 1\\ v = (x^2+1)^2 &\Rightarrow v' = (4x^3+4x)\\ \therefore \frac{df}{dx} &= \frac{(x^2+1)^2 - x(4x^3+4x)}{(x^2+1)^4}\\ &= \frac{(x^2+1)^2 - 4x^2(x^2+1)}{(x^2+1)^4}\end{aligned}$$

$$= \frac{(x^2+1) - 4x^2}{(x^2+1)^3}$$
$$\therefore \frac{df}{dx} = \frac{1-3x^2}{(x^2+1)^3}$$

(Q10.) $\frac{d}{dx}\left(\frac{20}{1+5\exp(-2x)}\right)$

$$\Rightarrow \frac{d}{dx}\frac{u}{v} = \frac{u'v - uv'}{v^2}$$
$$u = 20 \Rightarrow u' = 0$$
$$v = 1 + 5\exp(-2x) \Rightarrow v' = -10\exp(-2x)$$
$$\therefore \frac{df}{dx} = \frac{200\exp(-2x)}{(1+5\exp(-2x))^2}$$

(Q11.) $\frac{d}{dx}\left(\sqrt{\exp(x)} + \exp(\sqrt{x})\right)$

$$\Rightarrow \frac{d}{dx}\sqrt{\exp(x)} + \frac{d}{dx}(\exp(\sqrt{x})) = \frac{1}{2}(\exp(x))^{-\frac{1}{2}}\frac{d}{dx}(\exp(x)) + \exp(\sqrt{x})\frac{d}{dx}\sqrt{x}$$
$$= \frac{\exp(x)}{2\sqrt{\exp(x)}} + \frac{\exp(\sqrt{x})}{2\sqrt{x}}$$
$$\therefore \frac{df}{dx} = \frac{\sqrt{\exp(x)}}{2} + \frac{\exp(\sqrt{x})}{2\sqrt{x}}$$

(Q12.) $\frac{d}{dx}\left(\sec^3(2x)\right)$

$$\Rightarrow 3\sec^2(2x)\frac{d}{dx}(\sec(2x)) = 3\sec^2(2x)\sec(2x)\tan(2x)\frac{d}{dx}(2x)$$
$$\therefore \frac{df}{dx} = 6\sec^3(2x)\tan(2x)$$

(Q13.) $\frac{d}{dx}\left(\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln(\sec x + \tan x)\right)$

$$\Rightarrow \frac{d}{dx}\left(\frac{\tan x}{2\cos x}\right) + \frac{d}{dx}\left(\frac{1}{2}\ln(\sec x + \tan x)\right) = \frac{d}{dx}\left(\frac{\sin x}{2\cos^2 x}\right) + \frac{d}{dx}\left(\frac{1}{2}\ln(\sec x + \tan x)\right)$$
$$\frac{d}{dx}\left(\frac{\sin x}{2\cos^2 x}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$
$$u = \sin x \Rightarrow u' = \cos x$$
$$v = 2\cos^2 x \Rightarrow v' = -4\cos x\sin x$$
$$\therefore \frac{d}{dx}\left(\frac{\sin x}{2\cos^2 x}\right) = \frac{2\cos^3 x + 4\cos x\sin^2 x}{4\cos^4 x}$$
$$= \frac{\cos^2 x + 2\sin^2 x}{2\cos^3 x}$$
$$= \frac{1}{2\cos x} + \frac{\sin^2 x}{\cos^3 x}$$
$$= \frac{1}{2}\sec x + \sec x\tan^2 x$$
$$\frac{d}{dx}\left(\frac{1}{2}\ln\left(\sec x + \tan x\right)\right) = \frac{1}{2\sec x + 2\tan x}\frac{d}{dx}\left(\sec x + \tan x\right)$$
$$= \frac{\sec x\tan x + \sec^2 x}{2\sec x + 2\tan x}$$

$$
\begin{aligned}
&= \frac{\sec x(\tan x + \sec x)}{2(\sec x + \tan x)} \\
&= \frac{1}{2}\sec x \\
\therefore \frac{df}{dx} &= \frac{1}{2}\sec x + \sec x \tan^2 x + \frac{1}{2}\sec x \\
&= \sec x + \sec x \tan^2 x \\
&= \sec x(1 + \tan^2 x) \\
&= \sec x(\sec^2 x) \\
\therefore \frac{df}{dx} &= \sec^3 x
\end{aligned}
$$

(Q14.) $\dfrac{d}{dx}\left(\dfrac{x\exp(x)}{1+\exp(x)}\right)$

$$
\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{u'v - uv'}{v^2} \\
u = x\exp(x) \Rightarrow u' &= \alpha'\beta + \alpha\beta' \\
\alpha = x \Rightarrow \alpha' &= 1 \\
\beta = \exp(x) \Rightarrow \beta' &= \exp(x) \\
\therefore u' &= \exp(x) + x\exp(x) \\
v = 1 + \exp(x) \Rightarrow v' &= \exp(x) \\
\therefore \frac{df}{dx} &= \frac{(\exp(x) + x\exp(x))(1+\exp(x)) - (x\exp(x))(\exp(x))}{(1+\exp(x))^2} \\
&= \frac{\exp(x) + x\exp(x) + \exp(2x) + x\exp(2x) - x\exp(2x)}{(1+\exp(x))^2} \\
\therefore \frac{df}{dx} &= \frac{\exp(x) + x\exp(x) + \exp(2x)}{(1+\exp(x))^2}
\end{aligned}
$$

(Q15.) $\dfrac{d}{dx}\left(\exp(4x)\cos\left(\dfrac{x}{2}\right)\right)$

$$
\begin{aligned}
\Rightarrow \frac{d}{dx}(u \cdot v) &= u'v + uv' \\
u = \exp(4x) \Rightarrow u' &= 4\exp(4x) \\
v = \cos\left(\frac{x}{2}\right) \Rightarrow v' &= -\frac{1}{2}\sin\left(\frac{x}{2}\right) \\
\therefore \frac{df}{dx} &= 4\exp(4x)\cos\left(\frac{x}{2}\right) - \frac{1}{2}\exp(4x)\sin\left(\frac{x}{2}\right)
\end{aligned}
$$

(Q16.) $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt[4]{x^3-2}}\right)$

$$
\begin{aligned}
\Rightarrow (x^3-2)^{-1/4} &= -\frac{1}{4}(x^3-2)^{-5/4}\frac{d}{dx}(x^3-2) \\
&= \frac{-3x^2}{4\sqrt[4]{(x^3-2)^5}}
\end{aligned}
$$

(Q17.) $\dfrac{d}{dx}\left(\arctan\left(\sqrt{x^2-1}\right)\right)$

$$
\begin{aligned}
\Rightarrow \frac{1}{x^2}\frac{d}{dx}\left(\sqrt{x^2-1}\right) &= \frac{1}{x^2}\frac{d}{dx}\left(x^2-1\right)^{1/2} \\
&= \frac{1}{x^2}\cdot\frac{1}{2}(x^2-1)^{-1/2}\frac{d}{dx}(x^2-1)
\end{aligned}
$$

$$= \frac{1}{x^2} \cdot \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x$$

$$\therefore \frac{df}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

(Q18.) $\frac{d}{dx}\left(\frac{\ln x}{x^3}\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\ln x \cdot x^{-3}\right) &= \frac{d}{dx} u \cdot v \\
&= u'v + uv' \\
u = \ln x \Rightarrow u' &= \frac{1}{x} \\
v = x^{-3} = v' &= -3x^{-4} \\
\therefore \frac{df}{dx} &= \frac{1}{x^4} - \frac{3\ln x}{x^4} \\
&= \frac{1 - 3\ln x}{x^4}
\end{aligned}$$

(Q19.) $\frac{d}{dx}\left(x^x\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\exp(\ln x^x) = \exp(x \ln x)& \\
&= \exp(x\ln x)\frac{d}{dx}(x \ln x) \\
&= u'v + uv' \\
\frac{d}{dx}(x\ln x) &= \frac{d}{dx}(u \cdot v) \\
u = x \Rightarrow u' &= 1 \\
v = \ln x \Rightarrow v' &= \frac{1}{x} \\
\therefore \frac{d}{dx}(x\ln x) &= \ln x + 1 \\
\therefore \frac{df}{dx} &= \exp(x\ln x)(\ln x + 1) \\
&= x^x(\ln x + 1)
\end{aligned}$$

(Q20.) Find $\frac{dy}{dx}$ for $x^3 + y^3 = 6xy$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}x^3 + \frac{d}{dx}y^3 &= \frac{d}{dx}6xy \\
3x^2 + 3y^2\frac{dy}{dx} &= y\frac{d}{dx}(6x) + 6x\frac{dy}{dx} \\
3x^2 + 3y^2\frac{dy}{dx} &= 6y + 6x\frac{dy}{dx} \\
3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 \\
\frac{dy}{dx}(3y^2 - 6x) &= 6y - 3x^2 \\
\frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \\
\therefore \frac{dy}{dx} &= \frac{2y - x^2}{y^2 - 2x}
\end{aligned}$$

(Q21.) Find $\dfrac{dy}{dx}$ for $y\sin y = x\sin x$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(y\sin y) &= \frac{d}{dx}(x\sin x)\\
\sin y\frac{dy}{dx} + y\cos y\frac{dy}{dx} &= \sin x + x\cos x\\
\frac{dy}{dx}(\sin y + y\cos y) &= \sin x + x\cos x\\
\frac{dy}{dx} &= \frac{\sin x + x\cos x}{\sin y + y\cos y}
\end{aligned}$$

(Q22.) Find $\dfrac{dy}{dx}$ for $\ln\left(\dfrac{x}{y}\right) = \exp(xy^3)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\ln\left(\frac{x}{y}\right)\right) &= \frac{d}{dx}\left(\exp(xy^3)\right)\\
\frac{d}{dx}(\ln x - \ln y) &= \exp(xy^3)\frac{d}{dx}(xy^3)\\
\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} &= \exp(xy^3)(y^3 + 3xy^2\frac{dy}{dx})\\
\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} &= y^3\exp(xy^3) + 3xy^2\exp(xy^3)\frac{dy}{dx}\\
\frac{1}{x} - y^3\exp(xy^3) &= \frac{1}{y}\frac{dy}{dx} + 3xy^2\exp(xy^3)\frac{dy}{dx}\\
\therefore \frac{dy}{dx} &= \frac{\frac{1}{x} - y^3\exp(xy^3)}{\frac{1}{y} + 3xy^2\exp(xy^3)}\\
&= \frac{\frac{1}{x} - y^3\exp(xy^3)}{\frac{1}{y} + 3xy^2\exp(xy^3)}\cdot\frac{xy}{xy}\\
\therefore \frac{dy}{dx} &= \frac{y - xy^4\exp(xy^3)}{x + 3x^2y^3\exp(xy^3)}
\end{aligned}$$

(Q23.) Find $\dfrac{dy}{dx}$ for $x = \sec y$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(x) &= \frac{d}{dx}(\sec y)\\
1 &= \sec y\tan y\frac{dy}{dx}\\
\therefore \frac{dy}{dx} &= \frac{1}{\sec y\tan y}\\
&= \frac{\cos y}{\tan y}
\end{aligned}$$

$$\begin{aligned}
x &= \sec y\\
\Rightarrow \operatorname{arcsec} x &= \operatorname{arcsec}\sec y\\
\operatorname{arcsec} x &= y\\
\frac{d}{dx}\operatorname{arcsec} x &= \frac{d}{dx}y\\
\frac{1}{x\sqrt{x^2-1}} &= \frac{dy}{dx}
\end{aligned}$$

(Q24.) Find $\dfrac{dy}{dx}$ for $(x-y)^2 = \sin x + \sin y$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(x-y)^2 &= \frac{d}{dx}\sin x + \frac{d}{dx}\sin y \\
2(x-y)\frac{d}{dx}(x-y) &= \cos x + \cos y \frac{d}{dx} y \\
(2x-2y)\left(1-\frac{dy}{dx}\right) &= \cos x + \cos y \frac{dy}{dx} \\
2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} &= \cos x + \cos y \frac{dy}{dx} \\
2x - 2y - \cos x &= \cos y \frac{dy}{dx} + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} \\
\therefore \frac{dy}{dx} &= \frac{2x-2y-\cos x}{2x-2y+\cos y}
\end{aligned}$$

(Q25.) Find $\dfrac{dy}{dx}$ for $x^y = y^x$

$$\begin{aligned}
\Rightarrow \ln(x^y) &= \ln(y^x) \\
y\ln(x) &= x\ln(y) \\
\frac{\ln(x)}{x} &= \frac{\ln(y)}{y} \\
\frac{d}{dx}x^{-1}\ln(x) &= \frac{d}{dx}y^{-1}\ln(y) \\
\frac{-\ln x}{x^2} + \frac{1}{x^2} &= \frac{-\ln y}{y^2}\frac{dy}{dx} + \frac{1}{y^2}\frac{dy}{dx} \\
\frac{1-\ln x}{x^2} &= \left(\frac{1-\ln y}{y^2}\right)\frac{dy}{dx} \\
\frac{1-\ln x}{x^2} \cdot \frac{y^2}{1-\ln y} &= \frac{dy}{dx} \\
\therefore \frac{dy}{dx} &= \frac{y^2 - y^2\ln x}{x^2 - x^2\ln y}
\end{aligned}$$

(Q26.) Find $\dfrac{dy}{dx}$ for $\arctan(x^2y) = x + y^3$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\arctan(x^2y) &= \frac{d}{dx}x + \frac{d}{dx}y^3 \\
\frac{1}{1+(x^2y)^2}\cdot\frac{d}{dx}(x^2y) &= 1 + 3y^2\frac{dy}{dx} \\
\frac{1}{1+(x^2y)^2}(2xy + x^2\frac{dy}{dx}) &= 1 + 3y^2\frac{dy}{dx} \\
\frac{2xy}{1+(x^2y)^2} + \frac{x^2}{1+(x^2y)^2}\frac{dy}{dx} &= 1 + 3y^2\frac{dy}{dx} \\
\frac{2xy}{1+(x^2y)^2} - 1 &= 3y^2\frac{dy}{dx} - \frac{x^2}{1+(x^2y)^2}\frac{dy}{dx} \\
2xy - 1 - (x^2y)^2 &= \left(3y^2 + 3y^2(x^2y)^2 - x^2\right)\frac{dy}{dx} \\
2xy - x^4y^2 - 1 &= \left(3y^2 + 3x^4y^4 - x^2\right)\frac{dy}{dx} \\
\therefore \frac{dy}{dx} &= \frac{2xy - x^4y^2 - 1}{3y^2 + 3x^4y^4 - x^2}
\end{aligned}$$

(Q27.) Find $\dfrac{dy}{dx}$ for $\dfrac{x^2}{x^2-y^2} = 3y$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\frac{x^2}{x^2-y^2} &= \frac{d}{dx}3y \\
\frac{(2x)(x^2-y^2)-(x^2)(2x-2y\frac{dy}{dx})}{(x^2-y^2)^2} &= 3\frac{dy}{dx} \\
-2xy^2+2x^2y\frac{dy}{dx} &= (x^2-y^2)^2\frac{dy}{dx} \\
-2xy^2+2x^2y\frac{dy}{dx} &= (3x^4-6x^2y^2+3y^4)\frac{dy}{dx} \\
-2xy^2 &= (3x^4-6x^2y^2+3y^4)\frac{dy}{dx} - 2x^2y\frac{dy}{dx} \\
-2xy^2 &= (3x^4-6x^2y^2+3y^4-2x^2y)\frac{dy}{dx} \\
\therefore \frac{dy}{dx} &= \frac{2xy^2}{6x^2y^2+2x^2y-3x^4-3y^4}
\end{aligned}$$

(Q28.) Find $\dfrac{dy}{dx}$ for $\exp\left(\dfrac{x}{y}\right) = x + y^2$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\exp\left(\frac{x}{y}\right) &= \frac{d}{dx}x + \frac{d}{dx}y^2 \\
\exp\left(\frac{x}{y}\right)\cdot\frac{d}{dx}xy^{-1} &= 1 + 2y\frac{dy}{dx} \\
\exp\left(\frac{x}{y}\right)\cdot\left(1y^{-1} - 1xy^{-2}\frac{dy}{dx}\right) &= 1 + 2y\frac{dy}{dx} \\
\frac{\exp(x/y)}{y} - \frac{x\exp(x/y)}{y^2}\frac{dy}{dx} &= 1 + 2y\frac{dy}{dx} \\
\frac{\exp(x/y)}{y} - 1 &= 2y\frac{dy}{dx} + \frac{x\exp(x/y)}{y^2}\frac{dy}{dx} \\
y\exp(x/y) - y^2 &= 2y^3\frac{dy}{dx} + x\exp(x/y)\frac{dy}{dx} \\
\therefore \frac{dy}{dx} &= \frac{y\exp(x/y)-y^2}{2y^3+x\exp(x/y)}
\end{aligned}$$

(Q29.) Find $\dfrac{dy}{dx}$ for $(x^2+y^2-1)^3 = y$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(x^2+y^2-1)^3 &= \frac{d}{dx}y \\
3(x^2+y^2-1)^2\cdot\frac{d}{dx}(x^2+y^2-1) &= \frac{dy}{dx} \\
6x(x^2+y^2-1)^2 + 6y(x^2+y^2-1)^2\frac{dy}{dx} &= \frac{dy}{dx} \\
6x(x^2+y^2-1)^2 &= \frac{dy}{dx} - 6y(x^2+y^2-1)^2\frac{dy}{dx} \\
\therefore \frac{dy}{dx} &= \frac{6x(x^2+y^2-1)^2}{1-6y(x^2+y^2-1)^2}
\end{aligned}$$

(Q30.) Find $\dfrac{d^2y}{dx^2}$ for $9x^2+y^2 = 9$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}9x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}9 \\
18x + 2y\frac{dy}{dx} &= 0
\end{aligned}$$

$$
\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{-9x}{y} \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{-9x}{y} \\
&= \frac{u'v - uv'}{v^2} \\
u = -9x \Rightarrow u' &= -9 \\
v = y \Rightarrow v' &= \frac{dy}{dx} = \frac{-9x}{y} \\
\therefore \frac{d^2y}{dx^2} &= \frac{(-9)(y) - (-9x)\left(\frac{-9x}{y}\right)}{y^2} \\
&= \frac{-9y - \frac{81x^2}{y}}{y^2} \\
&= \frac{-9y^2 - 81x^2}{y^3} \\
&= \frac{-9(y^2 + 9x^2)}{y^3} \\
&= \frac{-9(9)}{y^3} \\
\therefore \frac{d^2y}{dx^2} &= \frac{-81}{y^3}
\end{aligned}
$$

(Q31.) $\frac{d^2}{dx^2}\left(\frac{1}{9}\sec(3x)\right)$

$$
\begin{aligned}
\Rightarrow \frac{1}{9}\sec(3x)\tan(3x)\frac{d}{dx}(3x) &= \frac{1}{3}\sec(3x)\tan(3x) \\
&= u'v + uv' \\
u = \frac{1}{3}\sec(3x) \Rightarrow u' &= \sec(3x)\tan(3x) \\
v = \tan(3x) \Rightarrow v' &= 3\sec^2(3x) \\
\therefore \frac{d^2y}{dx^2} &= \sec(3x)\tan^2(3x) + \sec^3(3x)
\end{aligned}
$$

(Q32.) $\frac{d^2}{dx^2}\left(\frac{x+1}{\sqrt{x}}\right)$

$$
\begin{aligned}
\Rightarrow \frac{d^2}{dx^2}\left(\frac{x+1}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}\right) &= \frac{d^2}{dx^2}\left(\frac{x^{3/2} + x^{1/2}}{x}\right) \\
&= \frac{d^2}{dx^2}\left(x^{1/2} + x^{-1/2}\right) \\
&= \frac{d^2}{dx^2}(x^{1/2}) + \frac{d^2}{dx^2}(x^{-1/2}) \\
&= \frac{d}{dx}\left(\frac{1}{2}x^{-1/2}\right) + \frac{d}{dx}\left(\frac{-1}{2}x^{-3/2}\right) \\
&= \frac{-1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2} \\
&= \frac{-1}{4}x^{-3/2}x^{-2/2}x^{2/2} + \frac{3}{4}x^{-5/2} \\
&= \frac{-1}{4}x^{-5/2}x^{2/2} + \frac{3}{4}x^{-5/2} \\
&= \frac{-x^{2/2}}{4x^{-5/2}} + \frac{3}{4x^{-5/2}}
\end{aligned}
$$

$$= \frac{3 - x^{2/2}}{4x^{-5/2}}$$
$$= \frac{3 - x^{2/2}}{4x^{-5/2}}$$
$$= \frac{3 - x}{4\sqrt{x^5}}$$

(Q33.) $\frac{d^2}{dx^2}\left(\arcsin(x^2)\right)$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{1-x^4}}\frac{d}{dx}\left(x^2\right)\right) = \frac{d}{dx}\left(\frac{2x}{\sqrt{1-x^4}}\right)$$
$$= \frac{d}{dx}\left(\frac{2x}{\sqrt{1-x^4}}\right)$$
$$= \frac{u'v - uv'}{v^2}$$
$$u = 2x \Rightarrow u' = 2$$
$$v = \sqrt{1-x^4} \Rightarrow v' = \frac{-4x^3}{2\sqrt{1-x^4}}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{(2)(\sqrt{1-x^4}) - (2x)(\frac{-4x^3}{2\sqrt{1-x^4}})}{(\sqrt{1-x^4})^2}$$
$$= \frac{2\sqrt{1-x^4} + \frac{8x^4}{2\sqrt{1-x^4}}}{1-x^4}$$
$$= \frac{2\sqrt{1-x^4} + \frac{8x^4}{2\sqrt{1-x^4}}}{1-x^4} \cdot \frac{2\sqrt{1-x^4}}{2\sqrt{1-x^4}}$$
$$= \frac{4(1-x^4) + 8x^4}{2(1-x^4)(1-x^4)^{1/2}}$$
$$= \frac{2(1-x^4) + 4x^4}{(1-x^4)^{3/2}}$$
$$= \frac{2 + 2x^4}{(1-x^4)^{3/2}}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{2 + 2x^4}{\sqrt{(1-x^4)^3}}$$

(Q34.) $\frac{d^2}{dx^2}\left(\frac{1}{1+\cos x}\right)$

$$\Rightarrow \frac{d}{dx}\left(-1(1+\cos x)^{-2}\frac{d}{dx}(1+\cos x)\right) = \frac{d}{dx}\left(\sin x(1+\cos x)^{-2}\right)$$
$$= \frac{d}{dx}\left(\frac{\sin x}{(1+\cos x)^2}\right)$$
$$= \frac{u'v - uv'}{v^2}$$
$$u = \sin x \Rightarrow u' = \cos x$$
$$v = (1+\cos x)^2 \Rightarrow v' = -2\sin x(1+\cos x)$$
$$\therefore \frac{d^2y}{dx^2} = \frac{(\cos x)(1+\cos x)^2 + 2\sin^2 x(1+\cos x)}{(1+\cos x)^4}$$
$$= \frac{(\cos x)(1+\cos x)^2 + 2\sin^2 x(1+\cos x)}{(1+\cos x)^4}$$
$$= \frac{(\cos x)(1+\cos x) + 2\sin^2 x}{(1+\cos x)^3}$$

$$= \frac{\cos x + \cos^2 x + 2\sin^2 x}{(1+\cos x)^3}$$

(Q35.) $\dfrac{d^2}{dx^2}\left(x \arctan x\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\arctan x + \frac{x}{1+x^2}\right) &= \frac{d}{dx}(\arctan x) + \frac{d}{dx}\left(\frac{x}{1+x^2}\right) \\
&= \frac{1}{1+x^2} + \frac{u'v - uv'}{v^2} \\
u = x &\Rightarrow u' = 1 \\
v = 1+x^2 &\Rightarrow v' = 2x \\
\therefore \frac{d^2y}{dx^2} &= \frac{1}{1+x^2} + \frac{1+x^2-2x^2}{(1+x^2)^2} \\
&= \frac{1}{1+x^2}\cdot\frac{1+x^2}{1+x^2} + \frac{1-x^2}{(1+x^2)^2} \\
&= \frac{1+x^2}{(1+x^2)^2} + \frac{1-x^2}{(1+x^2)^2} \\
\therefore \frac{d^2y}{dx^2} &= \frac{2}{(1+x^2)^2}
\end{aligned}$$

(Q36.) $\dfrac{d^2}{dx^2}\left(x^4 \ln x\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(4x^3\ln x + x^3) &= \frac{d}{dx}(4x^3\ln x) + \frac{d}{dx}(x^3) \\
&= \frac{d}{dx}(4x^3\ln x) + \frac{d}{dx}(x^3) \\
&= 12x^2\ln x + 4x^2 + 3x^2 \\
&= 12x^2\ln x + 7x^2
\end{aligned}$$

(Q37.) $\dfrac{d^2}{dx^2}\left(\exp\left(-x^2\right)\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(-2x\exp(-x^2)\right) &= -2\exp(-x^2) + 4x^2\exp(-x^2) \\
&= 4x^2\exp(-x^2) - 2\exp(-x^2) \\
\therefore \frac{d^2y}{dx^2} &= 2\exp(-x^2)(2x^2-1)
\end{aligned}$$

(Q38.) $\dfrac{d^2}{dx^2}\left(\cos(\ln x)\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\frac{-\sin(\ln x)}{x}\right) &= \frac{u'v - uv'}{v^2} \\
u = -\sin(\ln x) &\Rightarrow u' = \frac{-\cos(\ln x)}{x} \\
v = x &= v' = 1 \\
\therefore \frac{d^2y}{dx^2} &= \frac{-\cos(\ln x) + \sin(\ln x)}{x^2} \\
&= \frac{\sin(\ln x) - \cos(\ln x)}{x^2}
\end{aligned}$$

(Q39.) $\dfrac{d^2}{dx^2}\left(\ln(\cos x)\right)$

$$\Rightarrow \frac{d}{dx}\left(\frac{-\sin x}{\cos x}\right) = \frac{d}{dx}(-\tan(x))$$
$$\therefore \frac{d^2y}{dx^2} = -\sec^2 x$$

(Q40.) $\dfrac{d}{dx}\left(\sqrt{1-x^2} + x\arcsin(x)\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}\left(\sqrt{1-x^2}\right) + \frac{d}{dx}\left(x\arcsin(x)\right) &= \frac{-2x}{2\sqrt{1-x^2}} + \arcsin x + \frac{x}{\sqrt{1-x^2}} \\ &= \frac{-2x}{2\sqrt{1-x^2}} + \arcsin x + \frac{2x}{2\sqrt{1-x^2}} \\ \therefore \frac{dy}{dx} &= \arcsin(x)\end{aligned}$$

(Q41.) $\dfrac{d}{dx}\left(x\sqrt{4-x^2}\right)$

$$\begin{aligned}&\Rightarrow u'v + uv' \\ &u = x \Rightarrow u' = 1 \\ &v = \sqrt{4-x^2} \Rightarrow v' = \frac{-x}{\sqrt{4-x^2}} \\ \therefore \frac{dy}{dx} &= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} \\ &= \frac{4-x^2}{\sqrt{4-x^2}} - \frac{x^2}{\sqrt{4-x^2}} \\ \therefore \frac{dy}{dx} &= \frac{4-2x^2}{\sqrt{4-x^2}}\end{aligned}$$

(Q42.) $\dfrac{d}{dx}\left(\dfrac{\sqrt{x^2-1}}{x}\right)$

$$\begin{aligned}&\Rightarrow \frac{u'v-uv'}{v^2} \\ &u = \sqrt{x^2-1} \Rightarrow u' = \frac{2x}{2\sqrt{x^2-1}} \\ &v = x \Rightarrow v' = 1 \\ \therefore \frac{dy}{dx} &= \frac{\frac{2x^2}{2\sqrt{x^2-1}} - \sqrt{x^2-1}}{x^2} \\ &= \frac{\frac{2x^2}{2\sqrt{x^2-1}} - \sqrt{x^2-1}}{x^2} \cdot \frac{2\sqrt{x^2-1}}{2\sqrt{x^2-1}} \\ &= \frac{-1}{x^2\sqrt{x^2-1}}\end{aligned}$$

(Q43.) $\dfrac{d}{dx}\left(\dfrac{x}{\sqrt{x^2-1}}\right)$

$$\begin{aligned}&\Rightarrow \frac{u'v-uv'}{v^2} \\ &u = x \Rightarrow u' = 1 \\ &v = \sqrt{x^2-1} \Rightarrow v' = \frac{2x}{2\sqrt{x^2-1}}\end{aligned}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{\sqrt{x^2-1}-\frac{2x^2}{2\sqrt{x^2-1}}}{x^2-1}\\ &= \frac{\sqrt{x^2-1}-\frac{2x^2}{2\sqrt{x^2-1}}}{x^2-1}\cdot\frac{2\sqrt{x^2-1}}{2\sqrt{x^2-1}}\\ &= \frac{-1}{\sqrt{(x^2-1)^3}}\end{aligned}$$

(Q44.) $\dfrac{d}{dx}\left(\cos(\arcsin x)\right)$

$$\begin{aligned}\Rightarrow -\sin(\arcsin x)\cdot\frac{d}{dx}(\arcsin x) &= -\sin(\arcsin x)\cdot\frac{1}{1-\sqrt{x^2}}\\ &= \frac{-x}{\sqrt{1-x^2}}\end{aligned}$$

(Q45.) $\dfrac{d}{dx}\left(\ln\left(x^2+3x+5\right)\right)$

$$\Rightarrow \frac{1}{x^2+3x+5}\cdot\frac{d}{dx}\left(x^2+3x+5\right)=\frac{2x+3}{x^2+3x+5}$$

(Q46.) $\dfrac{d}{dx}\left(\arctan^2 4x\right)$

$$\begin{aligned}\Rightarrow 2\arctan 4x\cdot\frac{d}{dx}\left(\arctan 4x\right) &= \frac{2\arctan 4x}{1+16x^2}\cdot\frac{d}{dx}4x\\ &= \frac{8\arctan 4x}{1+16x^2}\end{aligned}$$

(Q47.) $\dfrac{d}{dx}\left(\sqrt[3]{x^2}\right)$

$$\Rightarrow \frac{d}{dx}\left(x^{2/3}\right)=\frac{2}{3\sqrt[3]{x}}$$

(Q48.) $\dfrac{d}{dx}\left(\sin\left(\sqrt{x}\ln x\right)\right)$

$$\begin{aligned}\Rightarrow \cos\left(\sqrt{x}\ln x\right)\cdot\frac{d}{dx}\left(\sqrt{x}\ln x\right) &= \cos\left(\sqrt{x}\ln x\right)\left(\frac{\ln x}{2\sqrt{x}}+\frac{1}{\sqrt{x}}\right)\\ &= \frac{(2+\ln x)\cos\left(\sqrt{x}\ln x\right)}{2\sqrt{x}}\end{aligned}$$

(Q49.) $\dfrac{d}{dx}\left(\csc x^2\right)$

$$\Rightarrow -\csc x^2\cot x^2\cdot\frac{d}{dx}\left(x^2\right)=-2x\csc x^2\cot x^2$$

(Q50.) $\dfrac{d}{dx}\left(\dfrac{x^2-1}{\ln x}\right)$

$$\begin{aligned}&\Rightarrow \frac{u'v-uv'}{v^2}\\ &u=x^2-1\Rightarrow u'=2x\\ &v=\ln x\Rightarrow v'=\frac{1}{x}\\ \frac{dy}{dx} &= \frac{2x\ln x-\frac{x^2-1}{x}}{\ln^2 x}\\ &= \frac{2x^2\ln x-x^2+1}{x\ln^2 x}\end{aligned}$$

(Q51.) $\frac{d}{dx}\left(10^x\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}\left(\exp(x\ln 10)\right) &= \exp(x\ln 10)\frac{d}{dx}(x\ln 10)\\ &= \ln 10\cdot 10^x\end{aligned}$$

(Q52.) $\frac{d}{dx}\left(\sqrt[3]{x+\ln^2 x}\right)$

$$\begin{aligned}\Rightarrow \frac{1}{3\sqrt[3]{\left(x+\ln^2 x\right)^2}}\cdot\frac{d}{dx}\left(x+\ln^2 x\right) &= \frac{1}{3\sqrt[3]{\left(x+\ln^2 x\right)^2}}\cdot\left(1+\frac{2\ln x}{x}\right)\\ &= \frac{1+\frac{2\ln x}{x}}{3\sqrt[3]{\left(x+\ln^2 x\right)^2}}\\ &= \frac{x+2\ln x}{3x\sqrt[3]{\left(x+\ln^2 x\right)^2}}\end{aligned}$$

(Q53.) $\frac{d}{dx}\left(x^{3/4}-2x^{1/4}\right)$

$$\begin{aligned}\Rightarrow \frac{3}{4}x^{-1/4}-\frac{1}{2}x^{-3/4} &= \frac{3}{4}x^{-1/4}-\frac{1}{2}x^{-3/4}x^{2/4}x^{-2/4}\\ &= \frac{3}{4}x^{-1/4}-\frac{1}{2}x^{-1/4}x^{-2/4}\\ &= x^{-1/4}\left(\frac{3}{4}-\frac{2}{4}x^{-2/4}\right)\\ &= x^{-1/4}\left(\frac{3x^{1/2}}{4x^{1/2}}-\frac{2}{4x^{1/2}}\right)\\ &= x^{-1/4}\left(\frac{3x^{1/2}-2}{4x^{1/2}}\right)\\ &= \frac{3x^{1/2}-2}{4x^{1/2}x^{1/4}}\\ \therefore \frac{dy}{dx} &= \frac{3\sqrt{x}-2}{4\sqrt[4]{x^3}}\end{aligned}$$

(Q54.) $\frac{d}{dx}\left(\log_2\left(x\sqrt{1+x^2}\right)\right)$

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{\ln 2}\ln\left(x\sqrt{1+x^2}\right)\right) &= \frac{1}{\ln(2)x\sqrt{1+x^2}}\frac{d}{dx}\left(x\sqrt{1+x^2}\right)\\ &= \frac{1}{\ln(2)x\sqrt{1+x^2}}\left(\sqrt{1+x^2}+\frac{2x^2}{2\sqrt{1+x^2}}\right)\end{aligned}$$

(Q55.) $\frac{d}{dx}\left(\frac{x-1}{x^2-x+1}\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}\frac{u}{v} &= \frac{u'v-uv'}{v^2}\\ u=x-1 &\Rightarrow u'=1\\ v=x^2-x+1 &\Rightarrow 2x-1\\ \therefore \frac{dy}{dx} &= \frac{x^2-x+1-2x^2+2x+x-1}{(x^2-x+1)^2}\\ &= \frac{2x-x^2}{(x^2-x+1)^2}\end{aligned}$$

(Q56.) $\dfrac{d}{dx}\left(\dfrac{1}{3}\cos^3 x - \cos x\right)$

$$\begin{aligned}\Rightarrow \frac{d}{dx}\left(\frac{1}{3}\cos^3 x\right) - \frac{d}{dx}\left(\cos x\right) &= \sin x - \cos^2 x \sin x \\ &= \sin x\left(1-\cos^2 x\right) \\ &= \sin^3 x\end{aligned}$$

(Q57.) $\dfrac{d}{dx}\left(\exp(x\cos x)\right)$

$$\begin{gathered}\Rightarrow \exp(x\cos x)\cdot\frac{d}{dx}(x\cos x) = \exp(x\cos x)(u'v+uv') \\ u = x \Rightarrow u' = 1 \\ v = \cos x \Rightarrow v' = -\sin x \\ \therefore \frac{dy}{dx} = \exp(x\cos x)(\cos x - x\sin x)\end{gathered}$$

(Q58.) $\dfrac{d}{dx}\left(\left(x-\sqrt{x}\right)\left(x+\sqrt{x}\right)\right)$

$$\Rightarrow \frac{d}{dx}\left(x^2 - x\right) = 2x - 1$$

(Q59.) $\dfrac{d}{dx}\left(\operatorname{arccot}\left(\dfrac{1}{x}\right)\right)$

$$\begin{aligned}\Rightarrow \frac{-1}{1+\frac{1}{x^2}}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) &= \frac{-1}{1+\frac{1}{x^2}}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) \\ &= \frac{-1}{1+\frac{1}{x^2}}\cdot\frac{-1}{x^2} \\ &= \frac{1}{1+\frac{1}{x^2}}\cdot\frac{1}{x^2} \\ \therefore \frac{dy}{dx} &= \frac{1}{x^2+1}\end{aligned}$$

(Q60.) $\dfrac{d}{dx}\left(x\arctan x - \ln\sqrt{x^2+1}\right)$

$$\begin{gathered}\Rightarrow \frac{d}{dx}\left(x\arctan x\right) - \frac{d}{dx}\left(\ln\sqrt{x^2+1}\right) = (u'v+uv') - \left(\frac{1}{\alpha}\cdot\frac{d\alpha}{dx}\right) \\ u = x \Rightarrow u' = 1 \\ v = \arctan x \Rightarrow v' = \frac{1}{1+x^2} \\ \alpha = \sqrt{x^2+1} \Rightarrow \alpha' = \frac{2x}{2\sqrt{x^2+1}}\end{gathered}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \arctan x + \frac{x}{1+x^2} - \left(\frac{1}{\sqrt{x^2+1}}\cdot\frac{2x}{2\sqrt{x^2+1}}\right) \\ &= \arctan x + \frac{x}{1+x^2} - \frac{2x}{2x^2+2} \\ &= \arctan x\end{aligned}$$

(Q61.) $\dfrac{d}{dx}\left(\dfrac{x\sqrt{1-x^2}}{2}+\dfrac{\arcsin x}{2}\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\frac{x\sqrt{1-x^2}}{2}\right)+\frac{d}{dx}\left(\frac{\arcsin x}{2}\right) &= \frac{\sqrt{1-x^2}}{2}-\frac{x^2}{2\sqrt{1-x^2}}+\frac{1}{2\sqrt{1-x^2}}\\
&= \frac{\sqrt{1-x^2}}{2}+\frac{1-x^2}{2\sqrt{1-x^2}}\\
&= \frac{1-x^2}{\sqrt{1-x^2}}\\
\therefore \frac{dy}{dx} &= \sqrt{1-x^2}
\end{aligned}$$

(Q62.) $\dfrac{d}{dx}\left(\dfrac{\sin x-\cos x}{\sin x+\cos x}\right)$

$$\Rightarrow \frac{u'v-uv'}{v^2}:$$

$$\begin{aligned}
u = \sin x-\cos x &\Rightarrow u' = \cos x+\sin x\\
v = \sin x+\cos x &\Rightarrow v' = \cos x-\sin x\\
&\qquad\;\; = -(\sin x-\cos x)
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{(\cos x+\sin x)^2+(\sin x-\cos x)^2}{(\sin x+\cos x)^2}\\
&= \frac{\cos^2 x+2\sin x\cos x+\sin^2 x+\sin^2 x-2\sin x\cos x+\cos^2 x}{(\sin x+\cos x)^2}\\
&= \frac{\cos^2 x+\sin^2 x+\sin^2 x+\cos^2 x}{(\sin x+\cos x)^2}\\
&= \frac{2}{(\sin x+\cos x)^2}
\end{aligned}$$

(Q63.) $\dfrac{d}{dx}\left(4x^2(2x^3-5x^2)\right)$

$$\Rightarrow \frac{d}{dx}\left(8x^5-20x^4\right) = 40x^4-80x^3$$

(Q64.) $\dfrac{d}{dx}\left(\sqrt{x}(4-x^2)\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(4\sqrt{x}-\sqrt{x^5}\right) &= \frac{2}{\sqrt{x}}-\frac{5\sqrt{x^3}}{2}\\
&= \frac{2}{\sqrt{x}}-\frac{5\sqrt{x^3}}{2}\\
&= \frac{4-5x^2}{2\sqrt{x}}
\end{aligned}$$

(Q65.) $\dfrac{d}{dx}\left(\sqrt{\dfrac{1+x}{1-x}}\right)$

$$\begin{aligned}
\Rightarrow \left(\frac{1+x}{1-x}\right)^{1/2} &= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2}\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)\\
&= \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\left(\frac{2}{(1-x)^2}\right)\\
&= \frac{1}{(1-x)^2}\sqrt{\frac{1-x}{1+x}}
\end{aligned}$$

$$= \frac{1}{\sqrt{(1-x)^3}\sqrt{1+x}}$$

(Q66.) $\dfrac{d}{dx}\left(\sin(\sin x)\right)$

$$\Rightarrow \cos(\sin x)\frac{d}{dx}\sin x = \cos x \cos(\sin x)$$

(Q67.) $\dfrac{d}{dx}\left(\dfrac{1+\exp(2x)}{1-\exp(2x)}\right)$

$$\begin{aligned}
&\Rightarrow \frac{u'v-uv'}{v^2}: \\
u = 1+\exp(2x) &\Rightarrow u' = 2\exp(2x) \\
v = 1-\exp(2x) &\Rightarrow v' = -2\exp(2x) \\
\frac{u'v-uv'}{v^2} &= \frac{2\exp(2x)(1-\exp(2x)) + 2\exp(2x)(1+\exp(2x))}{(1-\exp(2x))^2} \\
&= \frac{2\exp(2x)(1-\exp(2x)+1+\exp(2x))}{(1-\exp(2x))^2} \\
&= \frac{4\exp(2x)}{(1-\exp(2x))^2}
\end{aligned}$$

(Q68.) $\dfrac{d}{dx}\left(\dfrac{x}{1+\ln x}\right)$

$$\begin{aligned}
&\Rightarrow \frac{u'v-uv'}{v^2}: \\
u = x &\Rightarrow u' = 1 \\
v = 1+\ln x &\Rightarrow v' = \frac{1}{x} \\
\therefore \frac{dy}{dx} &= \frac{\ln x}{(1+\ln x)^2}
\end{aligned}$$

(Q69.) $\dfrac{d}{dx}\left(x^{\frac{x}{\ln x}}\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\exp\left(\frac{x}{\ln x}\ln x\right)\right) &= \frac{d}{dx}\left(\exp x\right) \\
&= \exp(x) \\
&= \exp\left(\frac{x}{\ln x}\ln x\right) \\
&= \exp\left(\ln x^{\frac{x}{\ln x}}\right) \\
&= x^{\frac{x}{\ln x}}
\end{aligned}$$

(Q70.) $\dfrac{d}{dx}\left(\ln\sqrt{\dfrac{x^2-1}{x^2+1}}\right)$

$$\begin{aligned}
\Rightarrow \sqrt{\frac{x^2+1}{x^2-1}}\cdot\frac{d}{dx}\left(\sqrt{\frac{x^2-1}{x^2+1}}\right) &= \sqrt{\frac{x^2+1}{x^2-1}}\cdot\frac{1}{2}\cdot\sqrt{\frac{x^2+1}{x^2-1}}\cdot\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) \\
&= \frac{1}{2}\cdot\frac{x^2+1}{x^2-1}\cdot\frac{u'v-uv'}{v^2} \\
u = x^2-1 &\Rightarrow u' = 2x \\
v = x^2+1 &\Rightarrow v' = 2x
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{1}{2} \cdot \frac{x^2+1}{x^2-1} \cdot \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} \\
&= \frac{1}{2} \cdot \frac{x^2+1}{x^2-1} \cdot \frac{2x(x^2+1-x^2+1)}{(x^2+1)^2} \\
&= \frac{1}{2} \cdot \frac{x^2+1}{x^2-1} \cdot \frac{4x}{(x^2+1)^2} \\
&= \frac{1}{2} \cdot \frac{1}{x^2-1} \cdot \frac{4x}{x^2+1} \\
&= \frac{2x}{x^4-1}
\end{aligned}$$

(Q71.) $\dfrac{d}{dx}\left(\arctan(2x+3)\right)$

$$\begin{aligned}
\Rightarrow \frac{1}{1+(2x+3)^2} \cdot \frac{d}{dx}(2x+3) &= \frac{2}{1+(2x+3)^2} \\
&= \frac{1}{2x^2+6x+5}
\end{aligned}$$

(Q72.) $\dfrac{d}{dx}\left(\cot^4 2x\right)$

$$\begin{aligned}
\Rightarrow 4\cot^3 2x \cdot \frac{d}{dx}(\cot 2x) &= 4\cot^3 2x(-\csc^2 2x)\frac{d}{dx}(2x) \\
&= -8\cot^3 2x \csc^2 2x
\end{aligned}$$

(Q73.) $\dfrac{d}{dx}\left(\dfrac{x^2}{1+\frac{1}{x}}\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\frac{x^3}{x+1}\right) &= \frac{u'v-uv'}{v^2} \\
u = x^3 &\Rightarrow u' = 3x^2 \\
v = x+1 \Rightarrow v' &= 1 \\
\therefore \frac{dy}{dx} &= \frac{3x^2(x+1) - x^3}{(x+1)^2} \\
&= \frac{2x^3+3x^2}{(x+1)^2}
\end{aligned}$$

(Q74.) $\dfrac{d}{dx}\left(\exp\left(\dfrac{x}{1+x^2}\right)\right)$

$$\begin{aligned}
\Rightarrow \exp\left(\frac{x}{1+x^2}\right) \cdot \frac{d}{dx}\left(\frac{x}{1+x^2}\right) &= \exp\left(\frac{x}{1+x^2}\right) \cdot \frac{u'v-uv'}{v^2} \\
u = x &\Rightarrow u' = 1 \\
v = 1+x^2 &\Rightarrow v' = 2x \\
\therefore \frac{dy}{dx} &= \exp\left(\frac{x}{1+x^2}\right) \cdot \frac{(1+x^2)-2x^2}{(1+x^2)^2} \\
&= \frac{1-x^2}{(1+x^2)^2}\exp\left(\frac{x}{1+x^2}\right)
\end{aligned}$$

(Q75.) $\dfrac{d}{dx}\left(\arcsin^3 x\right)$

$$\Rightarrow 3\arcsin^2 x \cdot \frac{d}{dx}(\arcsin x) = \frac{3\arcsin^2 x}{\sqrt{1-x^2}}$$

(Q76.) $\dfrac{d}{dx}\left(\dfrac{1}{2}\sec^2 x - \ln\sec x\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(\frac{1}{2}\sec^2 x\right) - \frac{d}{dx}(\ln\sec x) &= \sec x\frac{d}{dx}(\sec x) - \frac{1}{\sec x}\frac{d}{dx}(\sec x)\\
&= \sec x\sec x\tan x - \frac{1}{\sec x}\sec x\tan x\\
&= \sec^2 x\tan x - \tan x\\
&= \left(\frac{1}{\cos^2 x} - 1\right)\tan x\\
&= \left(\frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}\right)\tan x\\
&= \left(\frac{1-\cos^2 x}{\cos^2 x}\right)\tan x\\
&= \left(\frac{\sin^2 x}{\cos^2 x}\right)\tan x\\
&= \tan^3 x
\end{aligned}$$

(Q77.) $\dfrac{d}{dx}(\ln\ln\ln x)$

$$\begin{aligned}
\Rightarrow \frac{1}{\ln\ln x}\cdot\frac{d}{dx}(\ln\ln x) &= \frac{1}{\ln\ln x}\cdot\frac{1}{\ln x}\cdot\frac{d}{dx}(\ln x)\\
&= \frac{1}{\ln\ln x}\cdot\frac{1}{\ln x}\cdot\frac{1}{x}\\
&= \frac{1}{x\ln(x)\ln(\ln x)}
\end{aligned}$$

(Q78.) $\dfrac{d}{dx}\left(\pi^3\right)$

$$\Rightarrow 0,\ \text{Ok}$$

(Q79.) $\dfrac{d}{dx}\left(\ln\left(x+\sqrt{1+x^2}\right)\right)$

$$\begin{aligned}
\Rightarrow \frac{1}{x+\sqrt{1+x^2}}\cdot\frac{d}{dx}(x+\sqrt{1+x^2}) &= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot\frac{d}{dx}\left(1+x^2\right)\right)\\
&= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)\\
&= \frac{1}{x+\sqrt{1+x^2}}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)\\
&= \frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

(Q80.) $\dfrac{d}{dx}(\operatorname{arsinh} x)$

$$\begin{aligned}
\Rightarrow \frac{1}{\sqrt{1+x^2}}&,\ \text{by identity.}\\
\Rightarrow y &= \operatorname{arsinh} x\\
\sinh y &= x\\
\frac{d}{dx}(\sinh y &= x)\\
\cosh y\frac{dy}{dx} &= 1
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\cosh y} \\
\cosh^2 t - \sinh^2 t &= 1 \\
\cosh^2 t &= 1 + \sinh^2 t \\
\cosh t &= \sqrt{1 + \sinh^2 t} \\
\therefore \frac{dy}{dx} &= \frac{1}{\sqrt{1 + \sinh^2 y}} \\
&= \frac{1}{\sqrt{1 + \sinh^2 \operatorname{arsinh} x}} \\
&= \frac{1}{\sqrt{1 + x^2}}
\end{aligned}$$

(Q81.) $\dfrac{d}{dx}\left(\exp x \sinh x\right)$

$$\begin{aligned}
&u'v + uv' : \\
&u = \exp x \Rightarrow u' = \exp x \\
&v = \sinh x \Rightarrow v' = \cosh x \\
\therefore \frac{dy}{dx} &= \exp x \sinh x + \exp x \cosh x \\
&= \exp x(\cosh x + \sinh x) \\
&= \exp(2x)
\end{aligned}$$

(Q82.) $\dfrac{d}{dx}\left(\operatorname{sech}\left(\dfrac{1}{x}\right)\right)$

$$\begin{aligned}
\Rightarrow -\operatorname{sech}\frac{1}{x}\tanh\frac{1}{x}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) &= -\operatorname{sech}\frac{1}{x}\tanh\frac{1}{x}\cdot\frac{d}{dx}\left(\frac{1}{x}\right) \\
&= \frac{1}{x^2}\operatorname{sech}\frac{1}{x}\tanh\frac{1}{x}
\end{aligned}$$

(Q83.) $\dfrac{d}{dx}\left(\cosh \ln x\right)$

$$\begin{aligned}
\Rightarrow \sinh \ln x \cdot \frac{d}{dx}(\ln x) &= \frac{1}{x}\sinh \ln x \\
&= \frac{1}{x}\cdot\frac{\exp(\ln x) - \exp(-\ln x)}{2} \\
&= \frac{x - \exp(\ln\left(\frac{1}{x}\right))}{2x} \\
&= \frac{x - \frac{1}{x}}{2x} \\
&= \frac{x^2 - 1}{2x^2}
\end{aligned}$$

(Q84.) $\dfrac{d}{dx}\left(\ln \cosh x\right)$

$$\begin{aligned}
\Rightarrow \frac{1}{\cosh x}\cdot\frac{d}{dx}(\cosh x) &= \frac{\sinh x}{\cosh x} \\
&= \tanh x
\end{aligned}$$

(Q85.) $\dfrac{d}{dx}\left(\dfrac{\sinh x}{1+\cosh x}\right)$

$$\begin{aligned}
&\Rightarrow \frac{u'v-uv'}{v^2}:\\
u &= \sinh x \Rightarrow u' = \cosh x\\
v &= 1+\cosh x \Rightarrow v' = \sinh x\\
\therefore \frac{dy}{dx} &= \frac{\cosh x + \cosh^2 x - \sinh^2 x}{(1+\cosh x)^2}\\
&= \frac{1+\cosh x}{(1+\cosh x)^2}\\
&= \frac{1}{1+\cosh x}
\end{aligned}$$

(Q86.) $\dfrac{d}{dx}\left(\operatorname{artanh}(\cos x)\right)$

$$\begin{aligned}
\Rightarrow \frac{1}{1-\cos^2 x}\cdot\frac{d}{dx}(\cos x) &= \frac{-\sin x}{\sin^2 x}\\
&= \frac{-1}{\sin x}\\
&= -\csc x
\end{aligned}$$

(Q87.) $\dfrac{d}{dx}\left(x\operatorname{artanh} x + \ln\sqrt{1-x^2}\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(x\operatorname{artanh} x) + \frac{d}{dx}\left(\frac{1}{2}\ln\left(1-x^2\right)\right) &= u'v+uv' + \frac{1}{2\left(1-x^2\right)}\cdot\frac{dy}{dx}\left(1-x^2\right)\\
u = x &\Rightarrow u' = 1\\
v = \operatorname{artanh} x &\Rightarrow v' = \frac{1}{1-x^2}\\
\therefore \frac{dy}{dx} &= \operatorname{artanh} x + \frac{x}{1-x^2} + \frac{-2x}{2\left(1-x^2\right)}\\
&= \operatorname{artanh} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\\
&= \operatorname{artanh} x
\end{aligned}$$

(Q88.) $\dfrac{d}{dx}\left(\operatorname{arsinh}(\tan x)\right)$

$$\begin{aligned}
\Rightarrow \frac{1}{\sqrt{1+\tan^2 x}}\cdot\frac{d}{dx}(\tan x) &= \frac{\sec^2 x}{\sqrt{\sec^2 x}}\\
&= \frac{\sec^2 x}{\sec x}\\
&= \sec x
\end{aligned}$$

(Q89.) $\dfrac{d}{dx}\left(\arcsin(\tanh x)\right)$

$$\begin{aligned}
\Rightarrow \frac{1}{\sqrt{1-\tanh^2 x}}\cdot\frac{d}{dx}(\tanh x) &= \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}}\\
&= \frac{\operatorname{sech}^2 x}{\operatorname{sech} x}\\
&= \operatorname{sech} x
\end{aligned}$$

(Q90.) $\dfrac{d}{dx}\left(\dfrac{\operatorname{artanh} x}{1-x^2}\right)$

$$\begin{aligned}
&\Rightarrow \frac{u'v-uv'}{v^2}: \\
u = \operatorname{artanh} x &\Rightarrow u' = \frac{1}{1-x^2} \\
v = 1-x^2 &\Rightarrow v' = -2x \\
\therefore \frac{dy}{dx} &= \frac{\frac{1-x^2}{1-x^2} + 2x \operatorname{artanh} x}{\left(1-x^2\right)^2} \\
&= \frac{1 + 2x \operatorname{artanh} x}{\left(1-x^2\right)^2}
\end{aligned}$$

(Q91.) $\dfrac{d}{dx}\left(x^3\right)$, using the definition of the derivative.

$$\begin{aligned}
&\Rightarrow \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\
&= 3x^2 + 3x0 + 0^2 \\
&= 3x^2
\end{aligned}$$

(Q92.) $\dfrac{d}{dx}\left(\sqrt{3x+1}\right)$, using the definition of the derivative.

$$\begin{aligned}
&\Rightarrow \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1} + \sqrt{3x+1}}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\
&= \lim_{h\to 0} \frac{3x + 3h + 1 - 3x - 1}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} \\
&= \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\
&= \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}} \\
&= \frac{3}{2\sqrt{3x+1}}
\end{aligned}$$

(Q93.) $\dfrac{d}{dx}\left(\dfrac{1}{2x+5}\right)$, using the definition of the derivative.

$$\begin{aligned}
&\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2(x+h)+5} - \frac{1}{2x+5}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2(x+h)+5} - \frac{1}{2x+5}}{h} \cdot \frac{2(x+h)+5}{2(x+h)+5}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{1 - \frac{2(x+h)+5}{2x+5}}{h\,(2(x+h)+5)} \cdot \frac{2x+5}{2x+5} \\
&= \lim_{h\to 0} \frac{2x+5-2x-2h-5}{h\,(2(x+h)+5)\,(2x+5)} \\
&= \lim_{h\to 0} \frac{-2h}{h\,(2(x+h)+5)\,(2x+5)} \\
&= \lim_{h\to 0} \frac{-2}{(2(x+h)+5)\,(2x+5)} \\
&= \frac{-2}{(2x+5)\,(2x+5)} \\
&= \frac{-2}{(2x+5)^2}
\end{aligned}
$$

(Q94.) $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$, using the definition of the derivative.

$$
\begin{aligned}
&\Rightarrow \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \cdot \frac{(x+h)^2}{(x+h)^2} \\
&= \lim_{h\to 0} \frac{1 - \frac{(x+h)^2}{x^2}}{h(x+h)^2} \cdot \frac{x^2}{x^2} \\
&= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{x^2 - x^2 - 2xh - h^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2xh - h^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} \\
&= \frac{-2x}{x^4} \\
&= \frac{-2}{x^3}
\end{aligned}
$$

(Q95.) $\dfrac{d}{dx}(\sin x)$, using the definition of the derivative.

$$
\begin{aligned}
&\Rightarrow \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\
&= \cos(x)
\end{aligned}
$$

Evaluating the limits $\lim_{h\to 0} \frac{\cos(h)-1}{h}$ and $\lim_{h\to 0} \frac{\sin(h)}{h}$ is a litte controversial here. Using L'Hopital's rule, would require us to differentiate $\sin(x)$, which is the question we're trying to answer. We have a couple ways to evaluate these limits, without ustilisting differentation. $\forall h \in \mathbb{R}, -h^2+1 \leq \frac{\sin(h)}{h} \leq h^2+1$, so by squeeze theorem, the limit of $\frac{\sin(h)}{h}$ as h goes to 0 is 1. $\frac{\cos(h)-1}{h}$ is a little tricker and utilises the Taylor series of $\cos(h)$, namely

$$\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} \cdots .$$

Minus 1 from both sides, and cancelling the denominator h with the polynomial terms, we're left with

$$\lim_{h\to 0} -\frac{h}{2!} + \frac{h^3}{4!} - \frac{h^6}{6!} \cdots ,$$

which trivially evaluates to 0. We could also evaluate these limits using our calculators, and see that

h	$\frac{\cos(h)-1}{h}$	h	$\frac{\sin(h)}{h}$
0.1	$-0.049958\ldots$	0.1	$0.998334\ldots$
0.01	$-0.004999\ldots$	0.01	$0.999983\ldots$
0.001	$-0.000499\ldots$	0.001	$0.999999\ldots$
0.0001	$-0.000049\ldots$	0.0001	$0.999999\ldots$
0.00001	$-0.000005\ldots$	0.00001	$0.999999\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	0	1

.

But, this method, while it works, just rubs me the wrong way.

(Q96.) $\frac{d}{dx}(\sec x)$, using the definition of the derivative.

$$\begin{aligned}
&\Rightarrow \lim_{h\to 0} \frac{\sec(x+h)-\sec(x)}{h} \\
&= \lim_{h\to 0} \frac{\sec(x+h)-\sec(x)}{h} \cdot \frac{\cos(x+h)\cos(x)}{\cos(x+h)\cos(x)} \\
&= \lim_{h\to 0} \frac{\cos(x)-\cos(x+h)}{h\cos(x+h)\cos(x)} \\
&= \lim_{h\to 0} \frac{\cos(x)-\cos(x)\cos(h)+\sin(x)\sin(h)}{h\cos(x+h)\cos(x)} \\
&= \lim_{h\to 0} \frac{\cos(x)(1-\cos(h))+\sin(x)\sin(h)}{h\cos(x+h)\cos(x)} \\
&= \lim_{h\to 0} \frac{1}{\cos(x+h)\cos(x)} \cdot \left(\lim_{h\to 0} \frac{\cos(x)(1-\cos(h))}{h} + \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h} \right) \\
&= \frac{1}{\cos(x)\cos(x)} \cdot \left(\cos(x) \lim_{h\to 0} \frac{1-\cos(h)}{h} + \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h} \right) \\
&= \frac{1}{\cos(x)\cos(x)} \cdot (\cos(x)\cdot 0 + \sin(x)\cdot 1) \\
&= \frac{\sin(x)}{\cos(x)\cos(x)} \\
&= \sec(x)\tan(x)
\end{aligned}$$

(Q97.) $\frac{d}{dx}(\arcsin x)$, using the definition of the derivative.

$$\begin{aligned}
&\Rightarrow \lim_{h\to 0} \frac{\arcsin(x+h)-\arcsin x}{h} \\
&= \lim_{h\to 0} \frac{\arcsin(x+h)-\arcsin x}{h}
\end{aligned}$$

(Q98.) $\dfrac{d}{dx}(\arctan x)$, using the definition of the derivative.

$$\Rightarrow$$

(Q99.) $\dfrac{d}{dx}(f(x)g(x))$, using the definition of the derivative.

$$\begin{aligned}
&\Rightarrow \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\
&=\lim_{h\to 0}\frac{g(x+h)(f(x+h)-f(x))+f(x)(g(x+h)-g(x))}{h}\\
&=\lim_{h\to 0}g(x+h)\cdot\lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}+f(x)\lim_{h\to 0}\frac{(g(x+h)-g(x))}{h}\\
&=g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&=g(x)f'(x)+f(x)g'(x)\\
&=f'(x)g(x)+f(x)g'(x)
\end{aligned}$$

(Q100.) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)$, using the definition of the derivative.

$$\begin{aligned}
&\Rightarrow \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\\
&=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\cdot\frac{g(x+h)g(x)}{g(x+h)g(x)}\\
&=\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x+h)g(x)}\\
&=\lim_{h\to 0}\frac{f(x+h)g(x)-g(x)f(x)+g(x)f(x)-f(x)g(x+h)}{h}\cdot\lim_{h\to 0}\frac{1}{g(x+h)g(x)}\\
&=\lim_{h\to 0}\frac{g(x)\left(f(x+h)-f(x)\right)+f(x)\left(g(x)-g(x+h)\right)}{h}\cdot\frac{1}{g(x)g(x)}\\
&=\lim_{h\to 0}\frac{g(x)\left(f(x+h)-f(x)\right)-f(x)\left(g(x+h)-g(x)\right)}{h}\cdot\frac{1}{g(x)^2}\\
&=\frac{1}{g(x)^2}\left(\lim_{h\to 0}\frac{g(x)\left(f(x+h)-f(x)\right)}{h}-\lim_{h\to 0}\frac{f(x)\left(g(x+h)-g(x)\right)}{h}\right)\\
&=\frac{1}{g(x)^2}\left(g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\\
&=\frac{1}{g(x)^2}\left(g(x)f'(x)-f(x)g'(x)\right)\\
&=\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}
\end{aligned}$$

(Q101.) $\dfrac{d}{dx}\left({}^3x\right)$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}\left(x^{x^x}\right)&=\frac{d}{dx}\left(\exp\left(\ln x^{x^x}\right)\right)\\
&=\frac{d}{dx}\left(\exp\left(x^x\ln x\right)\right)\\
&=\frac{d}{dx}\left(\exp\left(\exp\left(\ln x^x\right)\ln x\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dx}\left(\exp\left(\exp\left(x\ln x\right)\ln x\right)\right) \\
&= x^{x^x}\frac{d}{dx}\left(\exp\left(x\ln x\right)\ln x\right) \\
&= x^{x^x}(u'v + uv') \\
u &= \exp\left(x\ln x\right) \\
&= x^x \\
u' &= \exp\left(x\ln x\right)\frac{d}{dx}(x\ln x) \\
&= x^x(\alpha'\beta + \alpha\beta') \\
\alpha = x &\Rightarrow \alpha' = 1 \\
\beta = \ln x &\Rightarrow \beta' = \frac{1}{x} \\
\therefore u' &= \exp\left(x\ln x\right)\left(\ln x + 1\right) \\
&= x^x(\ln x + 1) \\
v = \ln x &\Rightarrow v' = \frac{1}{x} \\
\therefore \frac{dy}{dx} &= x^{x^x}\left(x^x(\ln x + 1)\ln x + x^x\frac{1}{x}\right) \\
&= x^{x^x}x^x\left((\ln x + 1)\ln x + \frac{1}{x}\right) \\
&= x^{x+x^x}\left(\ln^2 x + \ln x + \frac{1}{x}\right)
\end{aligned}
$$