SCHOOL OF MATHEMATICS AND PHYSICS

MATH1072
Assignment 3
Semester Two 2024

Submit your answers - along with this sheet - by 1pm on the 30th of September, using the blackboard assignment submission system. Assignments must consist of a single PDF.
You may find some of these problems challenging. Attendance at weekly tutorials is assumed.

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Marker's use only

Each question marked out of 3.

- Mark of 0: You have not submitted a relevant answer, or you have no strategy present in your submission.
- Mark of 1: Your submission has some relevance, but does not demonstrate deep understanding or sound mathematical technique.
- Mark of 2: You have the right approach, but need to fine-tune some aspects of your calculations.
- Mark of 3: You have demonstrated a good understanding of the topic and techniques involved, with well-executed calculations.

Q1a	Q1b:	Q1c:	Q1d:
Q2a:	Q2b:	Q2c:	Q3a:
Q3b:	Q3c:	Q3d:	Q4:

Total (out of 36):

Question 1: Inital Value Problem

A population of Christmas beetles in Brisbane grows at a rate proportional to their current population. In the absence of external factors, the population will double in one week's time. On any given day, there is a net migration into the area of 10 beetles, 11 beetles are eaten by a local Magpie population, and 2 die of natural causes.

(a) Write a initial value problem to describe the change in population at time t, given that the initial population of Christmas beetles is $P(0) = 100$. You **must** write all the parameters in your model **explicitly**.

(b) Solve the initial value problem from part (a) to find the population $P(t)$, at any time t.

(c) Use MATLAB to plot your solution from part (b) from time $t = 0$ to $t = 100$. Will the beetle population survive?

(d) Use MATLAB to plot the solution to part (a) from time $t = 0$ to $t = 100$ for 30 different initial populations of Christmas beetles from $P(0) = 20$ to $P(0) = 50$. Recall that initial population sizes must be integer valued. What can you say about the stability of the population of Christmas beetles from this plot?

Solution: (a) We'll answer this by breaking down the given description, and making additions to the model explicitly on each step.

"A population of Christmas beetles in Brisbane grows
at a rate proportaional to their current population."

So, the rate of change with resepct to time, $\frac{\mathrm{d}}{\mathrm{d}t}$, of the population of Christmas beetles in Brisbane, P, is proportaional to itself. Let the constant of proportionality be $k \in \mathbb{R}$. We can then write

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

"In the absense of external factors,"

This doesn't add anything to the model, but it shows that the Malthusian model we've recorded so far will be sufficent for our purposes.

"the populaiton will double in one week's time"

This helps us establish a frame of reference for t. Let t be the number of weeks that have passed since the start of recording. It also allows us to determine the value of the growth constant, k, since

$$2P(0) = P(1),\ P(1) = P(0)e^{k\cdot 1} \iff 2P(0) = P(0)e^{k} \iff 2 = e^{k} \iff k = \ln 2$$

So our model is now

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \ln 2 \cdot P$$

"On any given day, there is a net migration in the area of 10 beetles."

This means that every week 70 beetles migrate into the area. Regardless of $\ln 2 \cdot P$, 70 beetles are added.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \ln 2 \cdot P + 70$$

"On any given day, . . . 11 beetles are eaten by a local Magpie population,"

Similarly, this means that each week, 77 beetles will be lost from the population, which again affects the RHS of our model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \ln 2 \cdot P + 70 - 77$$

"On any given day, ... 2 die of natural causes."

Finally, 14 beetles will be removed from the population each week,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \ln 2 \cdot P + 70 - 77 - 14$$

Finally, this leaves us with a full expression of the IVP:

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = \ln 2 \cdot P - 21 \\ \\ P(0) = 100 \end{cases}$$

Solution: (b) This is a linear first-order ODE, so we can rewrite it in the form $P' + a(t)P = b(t)$.

$$\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} - \ln 2 \cdot P = -21 \tag{1}$$

where, $a(t) = -\ln 2$ and $b(t) = -21$. We seek the intergrating factor, $I(t)$, and note that

$$I(t) = \exp\left(\int a(t)\mathrm{d}t\right) = \exp\left(-\int \ln 2 \;\mathrm{d}t\right) = \exp\left(-\ln 2 \cdot t\right)$$

Now, we can multiply through (1) by $I(x)$

$$\exp\left(-\ln 2 \cdot t\right)\frac{\mathrm{d}P}{\mathrm{d}t} - \ln 2 \cdot P\exp\left(-\ln 2 \cdot t\right) = -21\exp\left(-\ln 2 \cdot t\right)$$

We can apply the product rule, $fg' + f'g = (fg)'$, to the LHS, where $f = \exp\left(-\ln 2 \cdot t\right)$, and $g = P$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(P\exp\left(-\ln 2 \cdot t\right)\right) = -21\exp\left(-\ln 2 \cdot t\right)$$

Finally, we can integrate both sides with respect to t.

$$\int \frac{\mathrm{d}}{\mathrm{d}t}\left(P\exp\left(-\ln 2 \cdot t\right)\right)\mathrm{d}t = \int -21\exp\left(-\ln 2 \cdot t\right)\mathrm{d}t \iff P\exp\left(-\ln 2 \cdot t\right) + c_1 = \frac{21}{\ln 2}\exp(-\ln 2 \cdot t) + c_2$$

Now we rearange for P

$$P = \frac{\frac{21}{\ln 2}\exp(-\ln 2 \cdot t) + \hat{c}}{\exp\left(-\ln 2 \cdot t\right)},$$

where $(\hat{c} = c_2 - c_1) \in \mathbb{R}$, and simplify, finding our general solution,

$$P = \frac{21}{\ln 2} + \hat{c}\exp\left(\ln 2 \cdot t\right) = \frac{21}{\ln 2} + \hat{c}2^t.$$

Now, we'll include the inital conditions in the model, to finish this off.

$$P(0) = 100 = \frac{21}{\ln 2} + \hat{c}2^0 = \frac{21}{\ln 2} + \hat{c} \iff \hat{c} = 100 - \frac{21}{\ln 2}$$

Therefore, the solution to this IVP is

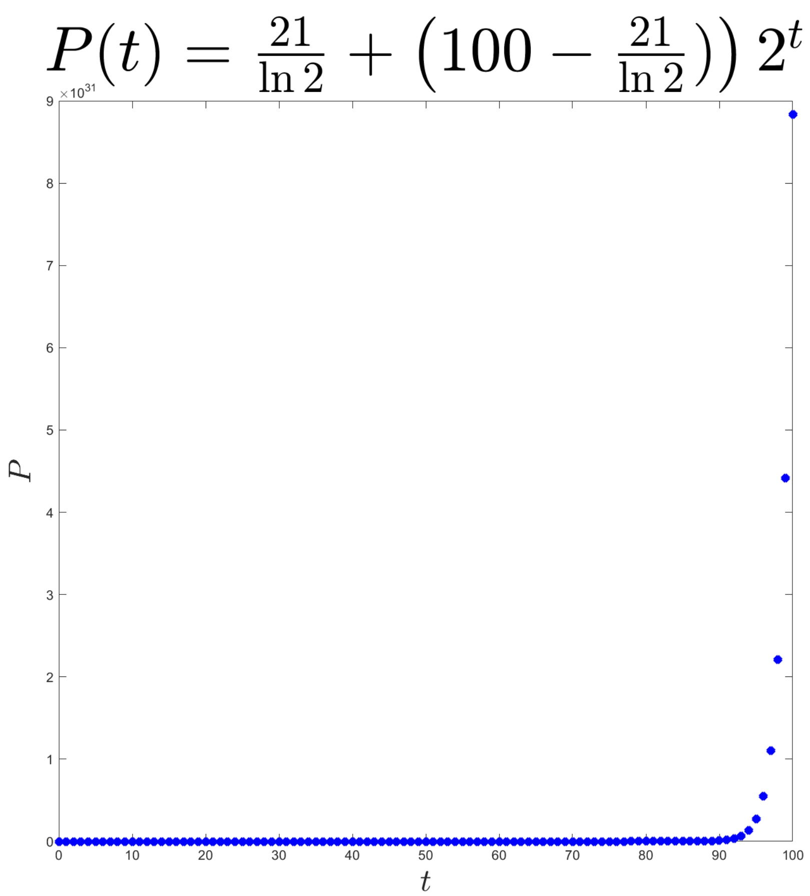
$$P(t) = \frac{21}{\ln 2} + \left(100 - \frac{21}{\ln 2}\right)2^t.$$

Solution: (c) We will plot our solution from part (b) from $t = 0$ to $t = 100$.

PS3_script1.m

```
P = @(t) (21/log(2)) + (100 - (21/log(2)))*2.^t;
t = 0:1:100;
y = P(t);
plot(t,y, 'bx',  "LineWidth", 4);
title("$P(t) = \frac{21}{\ln2} + \left( 100 - \frac{21}{\ln2}) \
   right)2^t$"...
        , "Interpreter", "latex", "FontSize", 48);
xlabel("$t$", "FontSize", 24, "Interpreter", "latex");
ylabel("$P$", "FontSize", 24, "Interpreter", "latex");
```

Output:



It seems that the beatle population will survive. According to our model, $P(100) \approx 8.8 \times 10^{31}$. Which means, after 100 weeks, a little under 2 years, the Christmas beetle population in Brisbane is predicted to weigh 8.8×10^{27}kg, about 4.5 times heavier then the planet Jupiter.

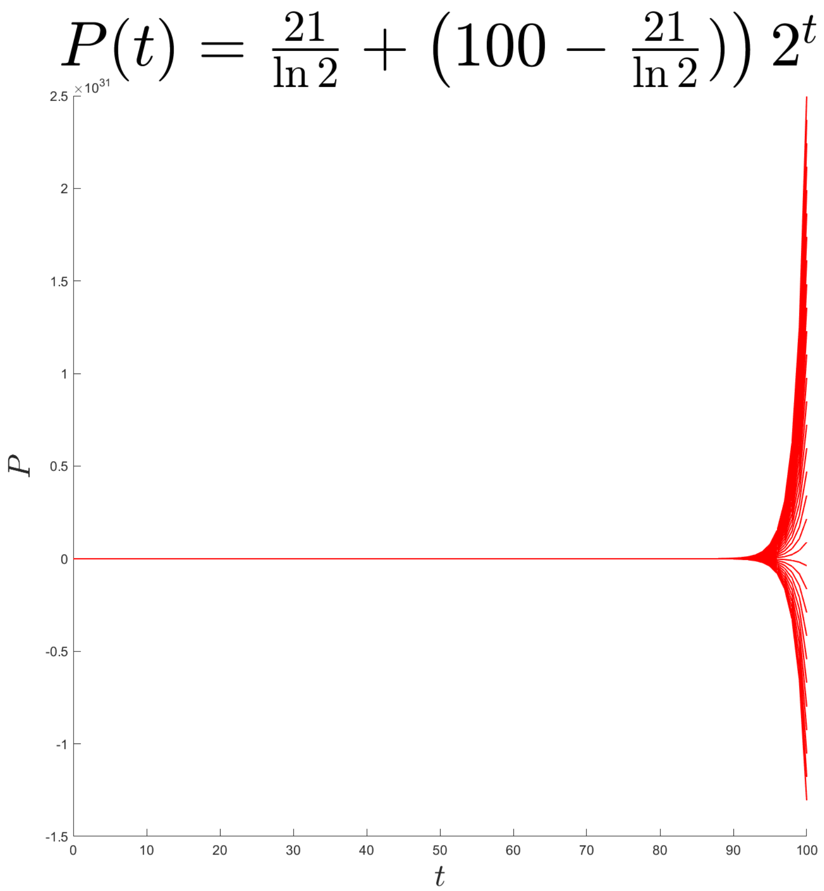
Solution: (d) We will plot our solution, but with inital population ranging between from $P_0 = 20$ to $P_0 = 50$.

PS3_script2.m

```
hold on;
for inital_pop = 20:1:50
    P = @(t) (21/log(2)) + (inital_pop - (21/log(2)))*2.^t;
    t = 0:1:100;
    y = P(t);
    plot(t,y, 'r',  "LineWidth", 1);
end

title("$P(t) = \frac{21}{\ln2} + \left( P_0 - \frac{21}{\ln2}) \right)2^t$"...
    , "Interpreter", "latex", "FontSize", 48);
xlabel("$t$", "FontSize", 24, "Interpreter", "latex");
ylabel("$P$", "FontSize", 24, "Interpreter", "latex");
```

Output:



We can clearly see that there is an unstable equilibrium! In fact, upon analysis of the ODE, we can conclude that there is an equilibrium when $p_0 = 21/\ln 2$, and confirm this:

$$P(t) = \frac{21}{\ln 2} + \left(\frac{21}{\ln 2} - \frac{21}{\ln 2}\right) 2^t = \frac{21}{\ln 2} \implies \frac{\mathrm{d}P}{\mathrm{d}t} = 0, \ \forall t$$

Of course, the inital population must be an integer, and so could never truly equal $21/\ln 2$ initally, but the equilibrium is still there, theoretically.

Question 2: Fluid Flow

In this question, we take the cartesian variables as $x \equiv x_1$, $y \equiv x_2$, and $z \equiv x_3$. We also take the basis vectors as $\hat{\imath} \equiv \underset{\sim}{e}_1$, $\hat{\jmath} \equiv \underset{\sim}{e}_2$, and $\hat{k} \equiv \underset{\sim}{e}_3$.

In fluid mechanics, fluids which are **incompressible** and **inviscid** are referred to as **ideal**. Let $\underset{\sim}{u} = \underset{\sim}{u}(x, y, z, t) = u_1\underset{\sim}{e}_1 + u_2\underset{\sim}{e}_2 + u_3\underset{\sim}{e}_3$ be the velocity of an ideal fluid at an arbitrary point in space and time. Its motion is governed by the Euler equation,

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} = \underset{\sim}{F} - \frac{\nabla P}{\rho}.$$

Here P is the fluid's pressure, ρ its constant density, $\underset{\sim}{F}$ is the external force on the fluid at any given point, and

$$(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} \equiv \sum_{i=1}^{3} u_i \frac{\partial \underset{\sim}{u}}{\partial x_i}.$$

Suppose the system is simplified in three ways:

- The flow is **steady**. $\underset{\sim}{u} = \underset{\sim}{u}(x, y, z)$ is not changing with time.
- $\underset{\sim}{u}$ is a conservative vector field. This occurs when the fluid is **irrotational**, though we won't elaborate on that here.
- The external force is also conservative, with $\underset{\sim}{F} = -\nabla V$, for some scalar potential $V = V(x_1, x_2, x_3)$.

(a) Show that in this case,

$$(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} = \frac{1}{2}\nabla||\underset{\sim}{u}||^2.$$

(Hint, you can assume that Clairaut's theorem applies)

(b) Hence show that the Euler equation simplifies to

$$\frac{1}{2}||\underset{\sim}{u}||^2 + V + \frac{P}{\rho} = \text{constant}$$

(c) If V is a constant, what can you say about the relationship between a fluid's speed and its pressure?

Solution: (a) We seek to show that $(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} = \frac{1}{2}\nabla\left|\left|\underset{\sim}{u}\right|\right|^2$.

Proof.

$$\begin{aligned}
\underset{\sim}{u} &= u_1\underset{\sim}{e_1} + u_2\underset{\sim}{e_2} + u_3\underset{\sim}{e_3} \\
\iff \left|\left|\underset{\sim}{u}\right|\right| &= \sqrt{u_1^2 + u_2^2 + u_3^2} \\
\iff \left|\left|\underset{\sim}{u}\right|\right|^2 &= u_1^2 + u_2^2 + u_3^2 \\
\iff \nabla\left|\left|\underset{\sim}{u}\right|\right|^2 &= \nabla\left(u_1^2 + u_2^2 + u_3^2\right) \\
&= \left(\frac{\partial}{\partial x_1}\underset{\sim}{e_1} + \frac{\partial}{\partial x_2}\underset{\sim}{e_2} + \frac{\partial}{\partial x_3}\underset{\sim}{e_3}\right)\left(u_1^2 + u_2^2 + u_3^2\right) \\
&= \frac{\partial\left(u_1^2 + u_2^2 + u_3^2\right)}{\partial x_1}\underset{\sim}{e_1} + \frac{\partial\left(u_1^2 + u_2^2 + u_3^2\right)}{\partial x_2}\underset{\sim}{e_2} + \frac{\partial\left(u_1^2 + u_2^2 + u_3^2\right)}{\partial x_3}\underset{\sim}{e_3}
\end{aligned}$$

We'll take the partial derivatives. u_j is zeroed by x_i if $i \neq j$. And since u_i are functions of x_i, they are chain ruled

$$\begin{aligned}
\nabla \left|\left|\underset{\sim}{u}\right|\right|^2 &= 2u_1 \frac{\partial u_1}{\partial x_1} \underset{\sim}{e_1} + 2u_2 \frac{\partial u_2}{\partial x_2} \underset{\sim}{e_2} + 2u_3 \frac{\partial u_3}{\partial x_3} \underset{\sim}{e_3} \\
&= 2 \left(u_1 \frac{\partial u_1}{\partial x_1} \underset{\sim}{e_1} + u_2 \frac{\partial u_2}{\partial x_2} \underset{\sim}{e_2} + u_3 \frac{\partial u_3}{\partial x_3} \underset{\sim}{e_3} \right) \\
\therefore LHS = \frac{1}{2} \nabla \left|\left|\underset{\sim}{u}\right|\right|^2 &= u_1 \frac{\partial u_1}{\partial x_1} \underset{\sim}{e_1} + u_2 \frac{\partial u_2}{\partial x_2} \underset{\sim}{e_2} + u_3 \frac{\partial u_3}{\partial x_3} \underset{\sim}{e_3} \\
&= \sum_{i=1}^{3} \left(u_i \frac{\partial u_i}{\partial x_i} \underset{\sim}{e_i} \right) \\
&= \sum_{i=1}^{3} \left(u_i \frac{\partial \underset{\sim}{u}}{\partial x_i} \right) \\
&= (\underset{\sim}{u} \cdot \nabla) \underset{\sim}{u} = RHS
\end{aligned}$$

□

Solution: (b) We will start by taking the Euler equation, as in the question box above, and substituting the result we proved in (a)

$$\begin{aligned}
\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \nabla) \underset{\sim}{u} &= \underset{\sim}{F} - \frac{\nabla P}{\rho} \\
\iff \frac{\partial \underset{\sim}{u}}{\partial t} + \frac{1}{2} \nabla \left|\left|\underset{\sim}{u}\right|\right|^2 &= \underset{\sim}{F} - \frac{\nabla P}{\rho}
\end{aligned}$$

Next, since the external force is conservative, we'll substitute $\underset{\sim}{F} = -\nabla V$, where V is some scalar potential.

$$\begin{aligned}
\iff \frac{\partial \underset{\sim}{u}}{\partial t} + \frac{1}{2} \nabla \left|\left|\underset{\sim}{u}\right|\right|^2 &= -\nabla V - \frac{\nabla P}{\rho} \\
\iff \frac{1}{2} \nabla \left|\left|\underset{\sim}{u}\right|\right|^2 + \nabla V + \frac{\nabla P}{\rho} &= -\frac{\partial \underset{\sim}{u}}{\partial t}
\end{aligned}$$

Now, since the flow is steady, $\frac{\mathrm{d}\underset{\sim}{u}}{\mathrm{d}t} = 0$

$$\iff \frac{1}{2} \nabla \left|\left|\underset{\sim}{u}\right|\right|^2 + \nabla V + \frac{\nabla P}{\rho} = 0$$

Finally, we'll take out the ∇

$$\iff \nabla \left(\frac{1}{2} \left|\left|\underset{\sim}{u}\right|\right|^2 + V + \frac{P}{\rho} \right) = 0$$

The gradient of a function can only be 0 if the function itself is equal to some constant. Thus we've shown that, under these conditions, the Euler equation simplies to

$$\frac{1}{2} \left|\left|\underset{\sim}{u}\right|\right|^2 + V + \frac{P}{\rho} = \text{constant}$$

Solution: (c) If V is constant, we can simplify the model.

$$\frac{1}{2} \left|\left|\underset{\sim}{u}\right|\right|^2 + \frac{P}{\rho} = C$$

where C is some constant.

The two remaining terms are related in a such a way that, if fluid speed, $\left|\left|\underset{\sim}{u}\right|\right|$, were to increase, then the pressure, P, must decrease to compenstate, and keep the sum equal to the constant C; and vice versa. Thus, if V is constant, there is an inverse relationship between the speed of an ideal fluid and its pressure.

Question 3: First Order Differential Equation

Consider Newton's second law of motion which states that,

$$F = ma \tag{1}$$

or: net force, F, is equal to mass m, times acceleration a.

(a) Use equation (1) to write out the equation of motion of a particle of mass m, subject to a frictional force proportional to the square of the velocity $v(t)$, completely in terms of the particle's velocity $v(t)$.

(b) Solve the first-order differential equation from part (a) to find the particle's velocity $v(t)$ at time t, with initial velocity v_0.

(c) Use your solution from part (b) to solve for the position of the particle $x(t)$ at time t, with initial position x_0.

(d) Use Euler's method with step size 0.1 to estimate the particle's position $x(0.5)$ at time $t = 0.5$ of your solution in part (b). Take $m = 1$, $k = 2$, $x_0 = 0$, and $v_0 = 1$. Calculate the error in using Euler's method, rounded to the fourth decimal.

Solution: (a) Suppose a particle has mass m and velocity v, which is itself a funciton of time t. Acceleration is just the derivative of velocity with respect to time, thus, the net force acting on it, can be expressed

$$F = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

Next, we consider the frictional force, F_{f}, which is proportaional to the velocity squared. Let k be the constant of proportionality.

$$F_{\mathrm{f}} = -kv^2$$

The sign is negative because friction acts against the motion of the particle. The only force acting on the particle, which means, by Newton's Second Law, net forces cancel out.

$$F - F_{\mathrm{f}} = 0 \iff F = F_{\mathrm{f}}$$

Subsituting the F's in terms of v's,

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -kv^2$$

Rearranging,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{k}{m}v^2$$

Solution: (b) We can start by noting that the ODE has an equilibrium when

$$v = 0 \implies \frac{\mathrm{d}v}{\mathrm{d}t}, \ \forall t$$

$v = 0$ is a singular solution to the ODE.
We will now proceed with the restriction $v \neq 0$.
This ODE is seperable, that is, it can be expressed

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f(t)g(v)$$

In our case,

$$f(t) = -\frac{k}{m}, \qquad g(v) = v^2$$

From this position, we can start to find the general solution.

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= f(t)g(v) \\
\frac{1}{g}\frac{\mathrm{d}v}{\mathrm{d}t} &= f(t) \\
\int \frac{1}{g}\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t &= \int f(t)\mathrm{d}t \\
\int v^{-2}\mathrm{d}v &= \int -\frac{k}{m}\mathrm{d}t \\
\frac{-1}{v} + c_1 &= -\frac{k}{m}\int \mathrm{d}t \\
&= -\frac{k}{m}t + c_2 \\
\frac{1}{v} &= \frac{k}{m}t + \hat{c},\ (\hat{c} = c_1 - c_2) \in \mathbb{R} \\
\therefore v &= \frac{1}{\frac{k}{m}t + \hat{c}} \\
&= \frac{m}{kt + \hat{c}m}
\end{aligned}$$

Now, we'll solve for $\hat{c}$, given our inital condition, $v(0) = v_0$

$$\begin{aligned}
v(0) = v_0 &= \frac{m}{k(0) + \hat{c}m} \\
\hat{c}m &= \frac{m}{v_0} \\
\hat{c} &= \frac{1}{v_0}
\end{aligned}$$

Next, we'll subtitute this back into our general solution.

$$v(t) = \frac{m}{kt + \frac{1}{v_0}m}$$

Finally, we'll simplify and present the particular solution to the ODE

$$v(t) = \frac{mv_0}{kv_0t + m}$$

Solution: (c) Position, $x(t)$ can be found by integrating velocity with respect to t.

$$x(t) = \int v(t)\mathrm{d}x = \int \frac{mv_0}{kv_0t + m}\mathrm{d}t = \frac{mv_0}{kv_0}\ln|kv_0t + m| + \hat{c} = \frac{m}{k}\ln|kv_0t + m| + \hat{c}$$

We'll use the inital condition $x(0) = x_0$ to solve for c,

$$x(0) = x_0 = \frac{m}{k}\ln|kv_0(0) + m| + \hat{c} = \frac{m}{k}\ln|m| + \hat{c} \iff \hat{c} = x_0 - \frac{m}{k}\ln|m|$$

Substituting back in,

$$x(t) = \frac{m}{k}\ln|kv_0t + m| + x_0 - \frac{m}{k}\ln|m| = \frac{m}{k}\left(\ln|kv_0t + m| - \ln|m|\right) + x_0$$

Which means the final solution for position is

$$x(t) = x_0 + \frac{m}{k}\ln\left|\frac{kv_0t + m}{m}\right|$$

Solution: (d) Euler's method states (with notation adapted for this specific problem)

$$x_{n+1} = x_n + f(t_n, v_n)\Delta t$$

Given $\Delta t = 0.1$ and $x'(t) = f(t_n, v_n) = v(t)$, we can write

$$x_{n+1} = x_n + 0.1\frac{v_0 m}{m + kv_0 t_n}$$

Taking $m = 1$, $k = 2$, $x_0 = 0$ and $v_0 = 1$, we can write our solution, and Euler method as

$$v(t) = \frac{1}{2t+1}, \qquad x_{n+1} = x_n + \frac{1}{10 + 20t_n}$$

Now, given $\Delta t = 0.1$, the t values we're interested in are

$$t_0 = 0, \qquad t_1 = 0.1, \qquad t_2 = 0.2, \qquad t_3 = 0.3, \qquad t_4 = 0.4, \qquad t_5 = 0.5$$

We are no ready to make our Euler method itterations

$$\begin{aligned}
t_0 = 0.0 : x_0 &= 0 \qquad \text{(Given)}\\
t_1 = 0.1 : x_1 = x_0 + \frac{1}{10+20t_0} &= 0 + \frac{1}{10+20(0.0)} = \frac{1}{10} = 0.1\\
t_2 = 0.2 : x_2 = x_1 + \frac{1}{10+20t_1} &= \frac{1}{10} + \frac{1}{10+20(0.1)} = \frac{11}{60} \approx 0.1833\\
t_3 = 0.3 : x_3 = x_2 + \frac{1}{10+20t_2} &= \frac{11}{60} + \frac{1}{10+20(0.2)} = \frac{107}{420} \approx 0.2548\\
t_4 = 0.4 : x_4 = x_3 + \frac{1}{10+20t_3} &= \frac{107}{420} + \frac{1}{10+20(0.3)} = \frac{533}{1680} \approx 0.3173\\
t_5 = 0.5 : x_5 = x_4 + \frac{1}{10+20t_4} &= \frac{533}{1680} + \frac{1}{10+20(0.4)} = \frac{1879}{5040} \approx 0.3728
\end{aligned}$$

Therefore, using Euler's Method, we find that $x(0.5) \approx 0.3728$.
Let's now find the exact value of $x(0.5)$. We will use the $x(t)$ we found eariler, and substitute the fixed m, k, x_0, and v_0.

$$x(0.5) = \frac{1}{2}\ln|2(0.5)+1| = \frac{1}{2}\ln 2 \approx 0.3466$$

Finally, let's find the error. Let $\bar{x}$ be the value calcualted using Euler's method, and let x be the true position of the particle.

$$\varepsilon = |x(0.5) - \bar{x}(0.5)| = \left|\frac{1}{2}\ln 2 - \frac{1879}{5040}\right| \approx 0.0262$$

Question 4: Line Integral

Evaluate the line integral

$$\int_C xe^y \; \mathrm{d}s,$$

where C is the line segment from $(2, 0)$ to $(5, 4)$.

Solution:

Our first goal will be to find some $\underset{\sim}{r}(t)$ which correctly expresses the path C.
First, we note that when $t = 0$, $\underset{\sim}{r} = (2, 0)$.
Second, we note that at $t = 1$, $\underset{\sim}{r} = (5, 4)$.
Since, this is path is a straight line, we can use a linear interpolation,

$$\begin{aligned}
\underset{\sim}{r}(t) &= (x(t), y(t)) \\
&= (x_0 + t\Delta x, y_0 + t\Delta y) \\
&= (2 + t(5 - 2), 0 + t(4 - 0)) \\
&= (3t + 2, 4t)
\end{aligned}$$

Next, we should find the derivative of $\underset{\sim}{r}$ with respect for t.

$$\begin{aligned}
\frac{\mathrm{d}\underset{\sim}{r}}{\mathrm{d}t} &= \left(\frac{\mathrm{d}}{\mathrm{d}t} (3t + 2) \, , \frac{\mathrm{d}}{\mathrm{d}t} (4t) \right) \\
&= (3, 4)
\end{aligned}$$

Finally, we'll find the magnitude of the derivative.

$$\left|\left| \frac{\mathrm{d}\underset{\sim}{r}}{\mathrm{d}t} \right|\right| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

We note that

$$\mathrm{d}s = \left|\left| \frac{\mathrm{d}\underset{\sim}{r}}{\mathrm{d}t} \right|\right| \mathrm{d}t = 5\mathrm{d}t$$

and proceed to make the subsitutions into the line integral we seek to evaluate

$$\begin{aligned}
\int_C x \exp(y) \; \mathrm{d}s &= \int_0^1 x(t) \exp(y(t)) 5 \; \mathrm{d}t \\
&= \int_0^1 5(3t + 2) \exp(4t) \; \mathrm{d}t \\
&= 5 \int_0^1 (3t + 2) \exp(4t) \; \mathrm{d}t \\
&= 5 \left(2 \int_0^1 \exp(4t) \, \mathrm{d}t + 3 \int_0^1 t \exp(4t) \, \mathrm{d}t \right) \\
&= 5 \, (2A + 3B)
\end{aligned}$$

Let's first evaluate the integral A, we'll use a $u-$substitution and the identity $f'(x) = \exp(x) = f(x)$.

$$\begin{aligned}
A &= \int_0^1 \exp(4t)\mathrm{d}t \\
\text{Let } u &= 4t \\
\text{Then } \frac{\mathrm{d}u}{\mathrm{d}t} &= 4 \implies \mathrm{d}t = \frac{1}{4}\mathrm{d}u \\
\therefore A &= \frac{1}{4}\int_{4\cdot 0}^{4\cdot 1} \exp(u)\mathrm{d}u \\
&= \frac{1}{4}\cdot \exp(u)\Big|_0^4 \\
&= \frac{1}{4}\left(\exp(4) - \exp(0)\right) \\
\therefore A = \int_0^1 \exp(4t)\mathrm{d}t &= \frac{1}{4}\left(\exp(4) - 1\right)
\end{aligned}$$

Next, let's evaluate the integral B, where we'll use integration by parts, and the previous result.

$$\begin{aligned}
B &= \int_0^1 t\exp(4t)\mathrm{d}t \\
&= \int_0^1 uv'\mathrm{d}t \\
&= uv|_0^1 - \int_0^1 u'v\mathrm{d}t \\
u = t &\Rightarrow u' = 1 \\
v' = \exp(4t) &\Rightarrow v = \frac{1}{4}\exp(4t) \\
uv|_0^1 - \int_0^1 u'v\mathrm{d}t &= t\frac{1}{4}\exp(4t)\Big|_0^1 - \int_0^1 1\cdot\frac{1}{4}\exp(4t)\mathrm{d}t \\
&= \frac{1}{4}\left(1\exp(4\cdot 1) - 0\exp(4\cdot 0)\right) - \frac{1}{4}\int_0^1 \exp(4t)\mathrm{d}t \\
&= \frac{1}{4}\exp(4) - \frac{1}{4}\left(\frac{1}{4}\left(\exp(4) - 1\right)\right) && \text{(Previous Result)} \\
&= \frac{4}{16}\exp(4) - \frac{1}{16}\exp(4) + \frac{1}{16} \\
\therefore B = \int_0^1 t\exp(4t)\mathrm{d}t &= \frac{3}{16}\exp(4) + \frac{1}{16}
\end{aligned}$$

Let's bring it all together now

$$\begin{aligned}
\int_C x\exp(y)\ \mathrm{d}s &= 5(2A + 3B) \\
&= 5\left(2\left(\frac{1}{4}\left(\exp(4) - 1\right)\right) + 3\left(\frac{3}{16}\exp(4) + \frac{1}{16}\right)\right) \\
&= 5\left(\left(\frac{1}{2}\left(\exp(4) - 1\right)\right) + \frac{9}{16}\exp(4) + \frac{3}{16}\right) \\
&= 5\left(\frac{8}{16}\exp(4) + \frac{9}{16}\exp(4) + \frac{3}{16} - \frac{8}{16}\right) \\
&= 5\left(\frac{17}{16}\exp(4) - \frac{5}{16}\right) \\
\therefore \int_C x\exp(y)\mathrm{d}s &= \frac{85}{16}\exp(4) - \frac{25}{16} \approx 288.4902
\end{aligned}$$