School of Mathematics and Physics, UQ

MATH2001/MATH7000 practice problems
Sheet 4

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(1) In $C[0, 2\pi]$, the set

$$\beta_n = \left\{\tfrac{1}{\sqrt{2\pi}}\right\} \cup \left\{\tfrac{1}{\sqrt{\pi}} \cos kx \mid k = 1, \ldots, n\right\} \cup \left\{\tfrac{1}{\sqrt{\pi}} \sin kx \mid k = 1, \ldots, n\right\},$$

where $n \in \mathbb{N}_0$, is orthonormal with respect to the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^{2\pi} f(x)g(x)dx.$$

Let $W_n = \operatorname{span}(\beta_n)$, and let $\mathbf{f}, \mathbf{g} \in C[0, 2\pi]$ such that $\mathbf{f} : x \mapsto x$ and $\mathbf{g} : x \mapsto x^2$.

(a) For arbitrary n, find the orthogonal projection of $\mathbf{f}$ onto W_n.

(b) Find the least squares approximation of $\mathbf{g}$ by an element of W_3.

(2) Find the Fourier series of $\sin^3(x)$.

(3) Find the least squares approximation of e^x over the interval $[0, 1]$ by a polynomial of degree at most 1.

(4) Describe the effect of the following linear transformations on the unit square.

(a) $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

(5) Let $\delta : M_{2,2}(\mathbb{R}) \longrightarrow \mathbb{R}$ be a function with the following three properties.

(i) δ is a linear function of each row of the matrix when the other row is held fixed.

(ii) If the two rows of $A \in M_{2,2}(\mathbb{R})$ are identical, then $\delta(A) = 0$.

(iii) If I is the 2×2 identity matrix, then $\delta(I) = 1$.

Prove that $\delta(A) = \det(A)$ for all $A \in M_{2,2}(\mathbb{R})$

(6) Find the eigenvalues and eigenvectors of the following matrices.

(a) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$

(7) Find matrices which diagonalize the following matrices:

(a) $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(8) Diagonalize the matrix $A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$.

(9) Which of the following matrices can be diagonalized?

(a) $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 & -1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **(c)** $\begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}$.

(10) Solve the recurrence relation $x_{n+1} = x_n + 2x_{n-1}$, given that $x_0 = 1$ and $x_1 = 3$.

(11) Let $A = \begin{pmatrix} 2 & 3 & 6 \\ 0 & 5 & 12 \\ 0 & 0 & -1 \end{pmatrix}$.

(a) Find a non-singular matrix P which diagonalizes A.

Background for (b): If an $n \times n$ matrix A has n linearly independent eigenvectors, we have seen that $A = PDP^{-1}$, where D is a diagonal matrix of eigenvalues. We can write

$$D = \lambda_1 E_{11} + \lambda_2 E_{22} + \ldots + \lambda_n E_{nn},$$

where, for example, in the 3×3 case

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then have $A = \lambda_1 S_1 + \lambda_2 S_2 + \ldots + \lambda_n S_n$, where $S_i = PE_{ii}P^{-1}$. This expansion is called the *spectral decomposition* of A.

(b) Use your answer in part (a) to calculate the matrices S_1, S_2 and S_3 for A, and hence write down the spectral decomposition of A.

Background for (c): Since there are three linearly independent eigenvectors for A, they form a basis of $\mathbb{R}^3$. The matrix S_i *projects* onto the subspace of $\mathbb{R}^3$ spanned by the eigenvector $\boldsymbol{v}_i$ corresponding to eigenvalue λ_i. This means that $S_i\boldsymbol{v}_i = \boldsymbol{v}_i$ and $S_i\boldsymbol{v}_j = \mathbf{0}$ (for $i \neq j$). In other words, if you take any vector

$$\boldsymbol{w} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3$$

in $\mathbb{R}^3$, multiplying by S_i extracts the $\boldsymbol{v}_i$ component. Using the above expansion of $\boldsymbol{w}$, we have $S_i\boldsymbol{w} = a_i\boldsymbol{v}_i$. We call S_i a *projection matrix*. It turns out that a matrix S is a projection if and only if $S^2 = S$, making them very easy to identify.

(c) Verify that the matrices S_1, S_2 and S_3 obtained in part (b) are indeed projection matrices, then use this fact to write the vector $\boldsymbol{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ as a linear combination of eigenvectors of A.