7.1. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 1$. Write down:

(a) the domain, codomain and range of f;
(b) the image of $\{x \in \mathbb{R} \mid -1 \le x \le 2\}$ under f;
(c) the preimage of $\{y \in \mathbb{R} \mid 0 \le y \le 3\}$ under f.

7.2. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = \left\lfloor \dfrac{1-6n}{3} \right\rfloor$.

(a) Is f one-to-one?
(b) Is f onto?
(c) Prove, or disprove with a counterexample, that for all $m, n \in \mathbb{Z}$, $f(mn) = f(m)f(n)$.

7.3. Let $A = \{1, 2\}$ and define $F\colon (A \times A) \to \mathbb{Z}$ by the rule $F((a, b)) = a(b + 1)$, for all $(a, b) \in A \times A$.

(a) Determine explicitly the values of $F((a, b))$ for every $(a, b) \in A \times A$.
(b) Is the function F one-to-one? Is it onto? (Explain your answers briefly.)
(c) What is the range of F? (Give your answer as a set.)

7.4. Determine whether the following functions are one-to-one, onto, bijections, or none of the above:

(a) $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = \lceil x \rceil$;
(b) $f\colon \mathbb{R} \to \mathbb{Z}$ where $f(x) = \lceil x \rceil$;
(c) $f\colon \mathbb{Z} \to \mathbb{R}$ where $f(x) = \lceil x \rceil$;
(d) $f\colon \mathbb{Z} \to \mathbb{Z}$ where $f(x) = \lceil x \rceil$.

7.5. Let $A = \{0, 1, 2, 3, 4\}$ and define functions $f\colon A \to A$ and $g\colon A \to A$ as

$$f(a) = (a+4)^2 \pmod 5 \quad \text{and} \quad g(a) = (a^2 + 3a + 1) \pmod 5.$$

Is $f = g$?

7.6. Define $f\colon \mathbb{Q} \to \mathbb{Z}$ by the rule $f(x) = 2\lfloor x + 1 \rfloor$, for all $x \in \mathbb{Q}$.

(i) Is the function f one-to-one? Prove this, or give a counter-example.
(ii) Is the function f onto? Prove this, or give a counter-example.
(iii) What is the range of f? (Give your answer as a set.)
(iv) The function $g : \mathbb{Z} \to \mathbb{Z}$ is given by the rule $g(x) = 2x - 1$, for all $x \in \mathbb{Z}$.
Determine the composition function $(g \circ f)(x)$, and calculate the value of $(g \circ f)(-\frac{3}{4})$.
(v) What is the range of $(g \circ f)$? (Give your answer as a set.)

7.7. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^3 - 1$, and $g\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = x + 5$.

(a) Write each of $f \circ g$ and $g \circ f$ as a polynomial.
(b) Determine which of the following functions is a bijection: f, g, $g \circ f$.
(c) For each bijection in (b), determine its inverse.
(d) Answer questions (b) and (c) again, but this time with f and g defined on the real numbers; that is, with $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$.

7.8. Let f and g be functions. Prove or disprove the following statements:

(a) If f and $f \circ g$ are one-one, then g is one-one.
(b) If f and $f \circ g$ are onto, then g is onto.

7.9. Find a one-to-one function $f\colon (\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+) \to \mathbb{Z}^+$.

7.10. Let $A = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$ and $B = \{x \in \mathbb{R} \mid 3 \leq x \leq 7\}$. Prove that $|A| = |B|$.

7.11. Let $\mathrm{Fun}(X, Y)$ denote the set of all functions from X to Y. Show that $|\mathrm{Fun}(X, \{0, 1\})| = |\mathcal{P}(X)|$ for all sets X.

7.12. Prove, disprove, or salvage if possible:

Let A and B be sets, with $|A| = |B|$, and let $f\colon A \to B$ be a function. Then f is injective if and only f is surjective.

Here are two **puzzles** that you can think about during week 8. Feel free to ask your tutors or lecturer for more hints!

M. In an intervarsity chess competition, each university enters a team of two players. Each participant plays all other players except their team-mate.

For each game, the winner receives one point and the loser receives nothing. If the game is drawn, however, then both players receive half a point each. At the end of the competition, UQ has a total winning score of 39 points, and all other universities have an equal number of points. How many universities participated in the competition?

N. The (finite) sequence $a_0, a_1, \ldots, a_{2024}$ has the following properties:

- $0 \leq a_n \leq 1$ for each $n \geq 0$;
- $a_n \geq (a_{n-1} + a_{n+1})/2$ for each $n \geq 1$.

Prove that $a_{2024} - a_{2023} \leq 1/2024$ for any such sequence. Can you an example of such a sequence with $a_{2024} - a_{2023} = 1/2024$?

Extra practice questions from the textbook (Solutions at the back of the book.)

Epp 5th ed.:
Section 7.4, pp. 484–486: Questions 2, 3, 8, 10, 26, 27.
Section 8.1, pp. 493–494: Questions 10, 13, 15, 16, 19.
Section 8.2, pp. 503–505: Questions 1, 3, 6, 9, 11, 12, 15, 18, 20, 23, 25, 27, 28, 31, 37, 38, 41.
Section 8.3, pp. 520–523: Questions 3, 5, 7, 8, 11, 25, 26, 28, 29, 36, 38, 40, 41, 44acdg.

Epp 4th ed.:
Section 7.4, pp. 439–441: Questions 2, 3, 8, 10, 26, 27.
Section 8.1, pp. 448–449: Questions 10, 13, 15, 16, 19.
Section 8.2, pp. 458–459: Questions 1, 3, 6, 9, 11, 12, 15, 18, 20, 23, 25, 27, 28, 31, 37, 38, 41.
Section 8.3, pp. 475–477: Questions 3, 5, 7, 8, 11, 20, 25, 26, 28, 29, 36, 38, 40, 41, 44acdg.

Puzzle hints:

Stuck on the puzzles from week 7?

K. 4096 – that's an unusual number. Have you seen it somewhere before?

L. Somebody must have answered "8", and somebody must have answered "0". What can you say about these two people?