MATH2001 Tutorial 1

1. Find the general solution to the equation

$$y' + xy = xy^{-1}$$

using two approaches:

(a) The ODE is separable.

(b) Multiple both sides of the ODE by ye^{x^2} to make it exact.

Solution: (a)

$$\begin{aligned}
y' &= xy^{-1} - xy \\
&= x(y^{-1} - y) \\
\frac{1}{y^{-1} - y} y' &= x \\
\int \frac{1}{y^{-1} - y} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x &= \int x \mathrm{d}x \\
\int \frac{1}{y^{-1} - y} \mathrm{d}y &= \int x \mathrm{d}x
\end{aligned}$$

Lets focus on the LHS

$$\begin{aligned}
\int \frac{1}{y^{-1} - y} \mathrm{d}y &= \int \frac{1}{y^{-1} - y} \cdot \frac{y}{y} \, \mathrm{d}y \\
&= \int \frac{y}{1 - y^2} \mathrm{d}y \\
&= \int \frac{y}{(1-y)(1+y)} \mathrm{d}y \\
\frac{y}{(1-y)(1+y)} &= \frac{\alpha}{1-y} + \frac{\beta}{1+y} \\
y &= \alpha(1+y) + \beta(1-y) \\
&= \alpha + \alpha y + \beta - \beta y \\
&= (\alpha + \beta) + (\alpha - \beta) y \\
\therefore 0 = \alpha + \beta &\Leftrightarrow \alpha = -\beta \\
\therefore 1 = \alpha - \beta &\Leftrightarrow \alpha = 1 + \beta \\
\iff -\beta = 1 + \beta &\Leftrightarrow \beta = -\frac{1}{2} \\
&\Leftrightarrow \alpha = \frac{1}{2}
\end{aligned}$$

Bringing it together

$$\begin{aligned}
\int \frac{1}{y^{-1} - y} \mathrm{d}y &= \frac{1}{2} \int \frac{1}{1-y} \mathrm{d}y - \frac{1}{2} \int \frac{1}{1+y} \mathrm{d}y \\
\therefore \int \frac{1}{y^{-1} - y} \mathrm{d}y = -\frac{1}{2} \ln(1-y) - \frac{1}{2} \ln(1+y) &= \frac{1}{2} x^2 + C = \int x \mathrm{d}x \\
-\frac{1}{2} \ln\left(1 - y^2\right) &= \frac{1}{2} x^2 + C \\
\ln\left(1 - y^2\right) &= C - x^2 \\
1 - y^2 &= \exp\left(C - x^2\right) \\
y^2 &= 1 - \frac{K}{\exp\left(x^2\right)}, \quad K = \exp(C) > 0
\end{aligned}$$

Solution: (b)

$$
\begin{aligned}
\text{Let } h = h(x, y) &= y \exp\left(x^2\right) \\
hy' + hxy &= hxy^{-1} \\
\exp\left(x^2\right) yy' + \exp\left(x^2\right) xy^2 &= \exp\left(x^2\right) x
\end{aligned}
$$

2. Find the gernal solution to the ODE

$$(x^2 - 2x)y' = 2(x-1)y.$$

Then consider the IVP with $y(x_0) = y_0$ and deterine all inital conditions (x_0, y_0) such that the IVP has (a) no solutions, (b) more than one solution, (c) preciesly one solution.

Solution:

$$\begin{aligned}
y' &= \frac{2(x-1)}{x(x-2)}y \\
\frac{1}{y}y' &= \frac{2(x-1)}{x(x-2)} \\
\int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= \int \frac{2(x-1)}{x(x-2)}\mathrm{d}x \\
\int \frac{1}{y}\mathrm{d}y &= 2\int \frac{(x-1)}{x(x-2)}\mathrm{d}x \\
\frac{(x-1)}{x(x-2)} &= \frac{A}{x} + \frac{B}{x-2} \\
&= A(x-2) + Bx \\
&= (A+B)x - 2A \\
A &= \frac{1}{2} \\
B &= \frac{1}{2} \\
\int \frac{1}{y}\mathrm{d}y &= \int \frac{1}{x}\mathrm{d}x + \int \frac{1}{x-2}\mathrm{d}x\mathrm{d}x \\
\ln y &= \ln x + \ln(x-2) + C \\
\ln y &= \ln(x^2 - 2x) + C \\
y &= \exp(\ln(x^2 - 2x) + C) \\
y &= K(x^2 - 2x), \quad K = \exp(C) > 0
\end{aligned}$$

Consider the IVP $y(x_0) = y_0$

$$\begin{aligned}
y_0 &= K(x_0^2 - 2x_0) \\
\implies K &= \frac{y_0}{x_0^2 - 2x_0}, \quad x_0 \neq 0,\ x_0 \neq 2, y_0 > 0
\end{aligned}$$

So we have final solution

$$y = \frac{y_0(x^2 - 2x)}{x_0^2 - 2x_0}$$

Consider $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = \frac{2(x-1)}{x(x-2)}y$ with $\frac{\partial f}{\partial y} = \frac{2(x-1)}{x(x-2)}$

(a) IVP has no solutions if and only if $f(x,y)$ has discontinuities in a rectangle in some rectangle around the solution. $(x_0, y_0) = (0, y_0)$ and $(2, y_0)$ will make this happen.

(b) IVP has more than one solution if and only if $f(x,y)$ is continuous in a rectangle, but f_y is not.

(c) IVP has preciesly one solution if and only if $f(x,y)$ and its derivative with respect to y are continuous in a local rectangle. (x_0, y_0) where $x_0 \neq 0$, $x_0 \neq 2$ is required.