School of Mathematics and Physics, UQ

MATH2001, Assignment 3, Summer Semester, 2024-2025

Due on 23 January at 14:00AEST. Each question is marked out of 10 then homogeneously rescaled up to a total marks of 13. **Total marks:** $\frac{13}{60}(Q1+Q2+Q3+Q4+Q5+Q6)$.
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Question 1: Polarising Integral

Evaluate the following integral by first converting the integral to polar coordinates

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 e^{x^2+y^2}\ \mathrm{d}y\mathrm{d}x$$

Solution:

From the limits of the integral, we can see that x ranges from 0 to 3, and y ranges from $-\sqrt{9-x^2}$, for some fixed x to 0. The y bound $-\sqrt{9-x^2}$ is particularly interesting, because it corresponds to a bottom semi-circle. Rearranging, we can find the equation of the circle,

$$y = -\sqrt{9-x^2}, \qquad y^2 = 9 - x^2, \qquad x^2+y^2 = 3^2 \implies r = 3$$

Now we can make the conversion,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The bounds of x correspond to the bounds of r. So, where x ranges from 0 to 3, r ranges from 0 to 3.
The bounds of y correspond with the bounds of θ. So, where y ranges from the bottom of the semi circle, with radius 3; so $\theta = -\pi/2$, to $y = 0$, which corresponds to $\theta = 0$.
The last thing we need to convert our integral, is to note that $e^{x^2+y^2}$ can be rewritten as e^{r^2}. Now, we can rewrite the integral.

$$I = \int_0^3 \int_{-\sqrt{9-x^2}}^0 e^{x^2+y^2}\ \mathrm{d}y\mathrm{d}x = \int_{-\pi/2}^0 \int_0^3 e^{r^2} r\ \mathrm{d}r\mathrm{d}\theta$$

$$\text{Let } u = r^2 \Rightarrow u' = 2r \iff \mathrm{d}u = 2r\mathrm{d}r \iff r\mathrm{d}r = \frac{1}{2}\mathrm{d}u$$

$$\begin{aligned}
\int_0^3 e^{r^2} r\ \mathrm{d}r &= \frac{1}{2}\int_{0^2=0}^{3^2=9} e^u\ \mathrm{d}u \\
&= \frac{1}{2}\Big[e^u\Big]_0^9 \\
&= \frac{1}{2}\left(e^9 - e^0\right) \\
&= \frac{1}{2}\left(e^9 - 1\right) \\
\therefore I &= \frac{1}{2}\left(e^9-1\right)\int_{-\pi/2}^0 \mathrm{d}\theta \\
&= \frac{1}{2}\left(e^9-1\right)\Big[\theta\Big]_{-\pi/2}^0 \\
\therefore I &= \frac{\pi}{4}\left(e^9-1\right) \approx 6363.3618
\end{aligned}$$

Question 2: Volume of a Bounded Region Within a Cylinder

Use a triple integral to determine the volume of a the region below $z = 6 - x$, above $z = -\sqrt{4x^2 + 4y^2}$, inside the cylinder $x^2 + y^2 = 3$ with $x \leq 0$.

Solution:

The upper bound of z is the plane $z = 6 - x$ and its lower bound is the upside-down cone, $z = -\sqrt{4x^2 + 4y^2} = -\sqrt{4\left(x^2 + y^2\right)}$.
The cylinder is described by the circle equation $x^2 + y^2 = r^2 = 3$, which implies that the radius of the cylinder is $\sqrt{3}$.
These facts make it really natural to express the integral using cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \mathrm{d}x\mathrm{d}y &= r\mathrm{d}r\mathrm{d}\theta \end{aligned}$$

r ranges from 0 to $\sqrt{3}$, the radius of the cylinder. θ ranges from $\pi/2$ to $3\pi/2$, since we're only dealing with the left side of the cylinder, $x \leq 0$. Finally, z will range from the bottom to the top of the region (the height, in a sense), namely from $-\sqrt{4r^2} = -2r$ to $6 - r\cos\theta$.
Therefore, the volume of the region of interest is given by the triple integral

$$V = \int_{\theta=\pi/2}^{3\pi/2} \int_{r=0}^{\sqrt{3}} \int_{z=-2r}^{6-r\cos\theta} r\ \mathrm{d}z\mathrm{d}r\mathrm{d}\theta.$$

Now, we'll evaluate the integral to find our volume.

$$\begin{aligned} V &= \int_{\theta=\pi/2}^{3\pi/2} \int_{r=0}^{\sqrt{3}} r \int_{z=-2r}^{6-r\cos\theta} 1\ \mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\ &= \int_{\theta=\pi/2}^{3\pi/2} \int_{r=0}^{\sqrt{3}} r \Big[z \Big]_{-2r}^{6-r\cos\theta} \mathrm{d}r\mathrm{d}\theta \\ &= \int_{\theta=\pi/2}^{3\pi/2} \int_{r=0}^{\sqrt{3}} r\left(6 - r\cos\theta + 2r\right)\ \mathrm{d}r\mathrm{d}\theta \\ &= \int_{\theta=\pi/2}^{3\pi/2} \int_{r=0}^{\sqrt{3}} 6r - r^2\cos\theta + 2r^2\ \mathrm{d}r\mathrm{d}\theta \\ &= \int_{\theta=\pi/2}^{3\pi/2} \left(\int_{r=0}^{\sqrt{3}} 6r\mathrm{d}r - \int_{r=0}^{\sqrt{3}} r^2\cos\theta\mathrm{d}r + \int_{r=0}^{\sqrt{3}} 2r^2\mathrm{d}r \right) \mathrm{d}\theta \\ &= \int_{\theta=\pi/2}^{3\pi/2} \left(6\int_{r=0}^{\sqrt{3}} r\mathrm{d}r - \cos\theta \int_{r=0}^{\sqrt{3}} r^2\mathrm{d}r + 2\int_{r=0}^{\sqrt{3}} r^2\mathrm{d}r \right) \mathrm{d}\theta \\ &= \int_{\theta=\pi/2}^{3\pi/2} \left(6\left[\frac{1}{2}r^2\right]_0^{\sqrt{3}} - \cos\theta\left[\frac{1}{3}r^3\right]_{r=0}^{\sqrt{3}} + 2\left[\frac{1}{3}r^3\right]_{r=0}^{\sqrt{3}} \right) \mathrm{d}\theta \\ &= \int_{\theta=\pi/2}^{3\pi/2} \left(6\left(\frac{3}{2} - \frac{0}{2}\right) - \cos\theta\left(\frac{\sqrt{27}}{3} - \frac{0}{3}\right) + 2\left(\frac{\sqrt{27}}{3} - \frac{0}{3}\right) \right) \mathrm{d}\theta \\ &= \int_{\theta=\pi/2}^{3\pi/2} \left(6\left(\frac{3}{2}\right) - \cos\theta\left(\frac{3\sqrt{3}}{3}\right) + 2\left(\frac{3\sqrt{3}}{3}\right) \right) \mathrm{d}\theta \end{aligned}$$

$$
\begin{aligned}
&= \int_{\theta=\pi/2}^{3\pi/2} 9 - \sqrt{3}\cos\theta + 2\sqrt{3}\ \mathrm{d}\theta \\
&= \int_{\theta=\pi/2}^{3\pi/2} 9\mathrm{d}\theta - \int_{\theta=\pi/2}^{3\pi/2} \sqrt{3}\cos\theta\mathrm{d}\theta + \int_{\theta=\pi/2}^{3\pi/2} 2\sqrt{3}\mathrm{d}\theta \\
&= 9\int_{\theta=\pi/2}^{3\pi/2} 1\mathrm{d}\theta - \sqrt{3}\int_{\theta=\pi/2}^{3\pi/2} \cos\theta\mathrm{d}\theta + 2\sqrt{3}\int_{\theta=\pi/2}^{3\pi/2} 1\mathrm{d}\theta \\
&= 9\Big[\theta\Big]_{\pi/2}^{3\pi/2} - \sqrt{3}\Big[\sin\theta\Big]_{\pi/2}^{3\pi/2} + 2\sqrt{3}\Big[\theta\Big]_{\pi/2}^{3\pi/2} \\
&= 9\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) - \sqrt{3}\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) + 2\sqrt{3}\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) \\
&= 9\pi - \sqrt{3}\,(-1-1) + 2\pi\sqrt{3} \\
&= 9\pi + 2\sqrt{3} + 2\pi\sqrt{3} \\
\therefore V &= 2\sqrt{3} + \left(9 + 2\sqrt{3}\right)\pi \approx 42.6212 \text{ units}^3
\end{aligned}
$$

Question 3: Line Integral Over Vector Field

Evaluate $\int_C \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r}$ where $\underset{\sim}{F} = (6x - 2y)\hat{\imath} + x^2\hat{\jmath}$ for each of the following curves.

(i) C is the line segment from $(6, -3)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(6, 3)$.

(ii) C is the line segment from $(6, -3)$ to $(6, 3)$.

Solution: (i)
We'll break up the line integral into two for each line segment

$$\int_C \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r} = \int_{C_1} \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r} + \int_{C_2} \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r},$$

where C_1 is the segment from $(6, -3)$ to $(0, 0)$, and C_2 is the segment from $(0, 0)$ to $(6, 3)$. We can parameterise these line segments,
C_1:

$$x = 6 - 6t \qquad y = -3 + 3t, \ t \in [0, 1]$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -6 \Rightarrow \mathrm{d}x = -6\mathrm{d}t \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3 \Rightarrow \mathrm{d}y = 3\mathrm{d}t$$

C_2:

$$x = 6t \qquad y = 3t, \ t \in [0, 1]$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6 \Rightarrow \mathrm{d}x = 6\mathrm{d}t \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3 \Rightarrow \mathrm{d}y = 3\mathrm{d}t$$

Finally, utilising the fact that $\underset{\sim}{F} = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$, and $\int_C \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r} = \int_C P(x, y)\mathrm{d}x + \int_C Q(x, y)\mathrm{d}y$, we can break up the line integral into 4, single variable integrals.

$$\begin{aligned}
I = \int_C \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r} &= \int_{C_1} \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r} + \int_{C_2} \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r} \\
&= \int_{C_1} P(x, y)\mathrm{d}x + \int_{C_1} Q(x, y)\mathrm{d}y + \int_{C_2} P(x, y)\mathrm{d}x + \int_{C_2} Q(x, y)\mathrm{d}y \\
&= \int_{C_1} (6x - 2y)\mathrm{d}x + \int_{C_1} \left(x^2\right)\mathrm{d}y + \int_{C_2} (6x - 2y)\mathrm{d}x + \int_{C_2} \left(x^2\right)\mathrm{d}y
\end{aligned}$$

Now, we'll substitute the approriate x and y and the approriate bounds for C_1 and C_2, respectively.

$$\begin{aligned}
I &= \int_0^1 -6(6(6 - 6t) - 2(-3 + 3t))\mathrm{d}t + \int_0^1 3\left((6 - 6t)^2\right)\mathrm{d}t \\
&\quad + \int_0^1 6(6(6t) - 2(3t))\mathrm{d}t + \int_0^1 3\left((6t)^2\right)\mathrm{d}t \\
&= \int_0^1 (-252 + 252t)\mathrm{d}t + \int_0^1 \left(108 - 216t + 108t^2\right)\mathrm{d}t \\
&\quad + \int_0^1 180t\mathrm{d}t + \int_0^1 108t^2\mathrm{d}t \\
&= \int_0^1 (-252 + 252t) + \left(108 - 216t + 108t^2\right) + 180t + 108t^2 \ \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 108t^2 + 108t^2 + 252t - 216t + 180t - 252 + 108 \ \mathrm{d}t \\
&= \int_0^1 216t^2 + 216t - 144 \ \mathrm{d}t \\
&= \left[72t^3 + 108t^2 - 144t\right]_0^1 \\
&= 72(1)^3 + 108(1)^2 - 144(1) - 0 \\
&= 72 + 108 - 144 - 0 \\
\therefore I &= 36
\end{aligned}
$$

Solution: (ii)
We're going to follow a similar process as we did in (i). Except, this time, we only need to parameterise the one line segment, from (6,-3) to (6,3).

$$
\begin{aligned}
x &= 6 & y &= -3 + 6t, \ t \in [0,1] \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \Rightarrow \mathrm{d}x = 0\mathrm{d}t & \frac{\mathrm{d}y}{\mathrm{d}t} &= 6 \Rightarrow \mathrm{d}y = 6\mathrm{d}t
\end{aligned}
$$

Now, we'll toss these functions into the integral, and evaluate.

$$
\begin{aligned}
I = \int_C \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{r} &= \int_C (6x - 2y)\mathrm{d}x + \int_C x^2 \mathrm{d}y \\
&= \int_0^1 0(6(6) - 2(-3 + 6t))\mathrm{d}t + \int_0^1 6(6)^2 \mathrm{d}t \\
&= 0 + \int_0^1 216\mathrm{d}t \\
&= \int_0^1 216\mathrm{d}t \\
&= \Big[216t\Big]_0^1 \\
&= 216(1) - 216(0) \\
\therefore I &= 216
\end{aligned}
$$

Question 4: Finding Potential Function

Find the potential function $f(x, y)$ for the vector field

$$\underset{\sim}{F} = y^2(1 + \cos(x + y))\hat{\imath} + (2xy - 2y + y^2\cos(x + y) + 2y\sin(x + y))\hat{\jmath}$$

that satisfies $\nabla f = \underset{\sim}{F}$.

Solution: Given that $\nabla f = \underset{\sim}{F}$, we can deduce that

$$\frac{\partial f}{\partial x} = y^2(1 + \cos(x + y)) \qquad \frac{\partial f}{\partial y} = 2xy - 2y + y^2\cos(x + y) + 2y\sin(x + y)$$

To find our potential function $f(x, y)$ then, we'll start by integrating its partial derivative with respect to x, with respect to x.

$$\begin{aligned} f(x, y) &= \int \frac{\partial f}{\partial x}\mathrm{d}x = \int y^2(1 + \cos(x + y))\mathrm{d}x \\ &= y^2\int 1\mathrm{d}x + y^2\int \cos(x + y)\mathrm{d}x \\ \therefore f(x, y) &= xy^2 + y^2\sin(x + y) + g(y) \end{aligned}$$

Next, we'll differentiate this expression with respect to y, and compare it to the other expression of $\dfrac{\partial f}{\partial y}$, which is an equivalent.

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(xy^2 + y^2\sin(x + y) + g(y)\right) \\ &= 2xy + 2y\sin(x + y) + y^2\cos(x + y) + g'(y) \\ &= 2xy + 2y\sin(x + y) + y^2\cos(x + y) - 2y \\ \implies g'(y) &= -2y \\ \iff g(y) &= -2\int y\mathrm{d}y \\ &= -y^2 + A_i, \end{aligned}$$

where A_i is an arbitrary constant. Therefore,

$$f(x, y) = 2xy - y^2 + 2y\sin(x + y) + y^2\cos(x + y) + A_i.$$

Question 5: Flux Over Surface of Bounded Solid

Evaluate $\iint_S \underset{\sim}{F} \cdot \mathrm{d}S$ where $\underset{\sim}{F} = y\hat{\imath} + 2x\hat{\jmath} + (z-8)\hat{k}$ and S is the surface of the solid bounded by $4x + 2y + z = 8,\ z = 0,\ y = 0$ and $x = 0$ with the positive orientation. Note that all four surfaces of the solid are included in S.

Solution:
Since these bounds bound a solid region, we can apply the divergence theorem to find the flux over the surface of solid.

$$\iint_S \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{S} = \iiint_V (\nabla \cdot \underset{\sim}{F}) \mathrm{d}V$$

First, we'll calculate div $\underset{\sim}{F}$

$$\begin{aligned} \nabla \cdot \underset{\sim}{F} &= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial z}(z-8) \\ &= 0 + 0 + 1 \\ &= 1 \end{aligned}$$

Next, we're going to assess the vertices which bound the solid. First, we note that the three planes $x = 0,\ y = 0,\ z = 0$ all intersect at the vertex $(0, 0, 0)$. If we hold x and y at 0, z will reach its maximum of 8, so $(0, 0, 8)$ is a vertex. If x and z are held at 0, y reaches its maximum at 4, so $(0, 4, 0)$ is a vertex. Finally, if y and z are held at 0, x reaches its maximum at 2, so $(2, 0, 0)$ is the last vertex.
Thus, the solid, V, is bound by the verticies

$$(0, 0, 0), \quad (2, 0, 0), \quad (0, 4, 0), \quad (0, 0, 8)$$

The volume of the solid is given by halving the scalar triple product,

$$\frac{1}{2} \cdot \frac{1}{3} (\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c} = \frac{1}{6} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{vmatrix} = \frac{1}{6}(2 \cdot 4 \cdot 8) = \frac{64}{6} = \frac{32}{3}$$

By the divergence theorem then,

$$I = \iint_S \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{S} = \iiint_V (\nabla \cdot \underset{\sim}{F}) \mathrm{d}V = \iiint_V 1 \mathrm{d}V = \frac{32}{3} \approx 10.6667$$

Question 6: Divergence Theorem

Use the Divergence Theorem to evaluate $\iint_S \underset{\sim}{F} \cdot \mathrm{d}S$ where $\underset{\sim}{F} = 2xz\hat{\imath} + (1-4xy^2)\hat{\jmath} + (2z-z^2)\hat{k}$ and S is the surface of the solid bounded by $z = 6-2x^2-2y^2$ and the plane $z=0$. Note that both of the surfaces of this solid are included in S.

Solution: Divergence theorem states that

$$\iint_S \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{S} = \iiint_V (\nabla \cdot \underset{\sim}{F})\mathrm{d}V$$

We'll start by calculating div $\underset{\sim}{F}$,

$$\begin{aligned}\nabla \cdot \underset{\sim}{F} &= \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(1-4xy^2) + \frac{\partial}{\partial z}(2z-z^2)\\ &= 2z - 8xy + 2 - 2z\\ &= 2 - 8xy\end{aligned}$$

We'll use cylindrical coordinates,

$$x^2+y^2=r^2, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = h, \quad \mathrm{d}V = r\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}h$$

r will range between 0 and $0 = 6-2r^2 \iff r^2 = 3 \iff r = \sqrt{3}$.
θ will range between 0 and 2π, the full circular rotation.
h will range between 0 and the bounding surface $6-2x^2-2y^2 = 6-2(x^2+y^2) = 6-2r^2$.

$$\therefore \iiint_V (\nabla \cdot \underset{\sim}{F})\mathrm{d}V = \iiint_V (2-8xy)\mathrm{d}V = \iiint_V (2-8r^2\cos\theta\sin\theta)r\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}h$$

Let's go ahead and evaluate this integral

$$\begin{aligned}I = \iint_S \underset{\sim}{F} \cdot \mathrm{d}\underset{\sim}{S} &= \iiint_V (\nabla \cdot \underset{\sim}{F})\mathrm{d}V\\ &= \iiint_V (2-8r^2\cos\theta\sin\theta)r\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}h\\ &= \iiint_V 2r - 8r^3\cos\theta\sin\theta\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}h\\ &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} 2r - 8r^3\cos\theta\sin\theta\ \mathrm{d}h\mathrm{d}r\mathrm{d}\theta\\ &= \int_0^{2\pi}\int_0^{\sqrt{3}} \Big[2hr - 8hr^3\cos\theta\sin\theta\Big]_0^{6-2r^2}\ \mathrm{d}r\mathrm{d}\theta\\ &= \int_0^{2\pi}\int_0^{\sqrt{3}} \big(2(6-2r^2)r - 8(6-2r^2)r^3\cos\theta\sin\theta - \big(2(0)r - 8(0)r^3\cos\theta\sin\theta\big)\big)\,\mathrm{d}r\mathrm{d}\theta\\ &= \int_0^{2\pi}\int_0^{\sqrt{3}} \big(2(6-2r^2)r - 8(6-2r^2)r^3\cos\theta\sin\theta\big)\,\mathrm{d}r\mathrm{d}\theta\\ &= \int_0^{2\pi}\int_0^{\sqrt{3}} \big(12r - 4r^3 + 16r^5\cos\theta\sin\theta - 48r^3\cos\theta\sin\theta\big)\,\mathrm{d}r\mathrm{d}\theta\\ &= \int_0^{2\pi} \Big[6r^2 - r^4 + \frac{8}{3}r^6\cos\theta\sin\theta - 12r^4\cos\theta\sin\theta\Big]_0^{\sqrt{3}}\ \mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left(6(\sqrt{3})^2 - (\sqrt{3})^4 + \frac{8}{3}(\sqrt{3})^6 \cos\theta \sin\theta - 12(\sqrt{3})^4 \cos\theta \sin\theta \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} (18 - 9 + 72 \cos\theta \sin\theta - 108 \cos\theta \sin\theta) \, \mathrm{d}\theta \\
&= \int_0^{2\pi} 9 - 36 \cos\theta \sin\theta \ \mathrm{d}\theta \\
&= \int_0^{2\pi} 9 - 18 \cdot 2 \cos\theta \sin\theta \ \mathrm{d}\theta \\
&= \int_0^{2\pi} 9 - 18 \sin 2\theta \ \mathrm{d}\theta \\
&= \Big[9\theta + 9 \cos 2\theta \Big]_0^{2\pi} \\
&= 9(2\pi) + 9\cos(2(2\pi)) - 9(0) - 9\cos(2(0)) \\
&= 18\pi + 9 - 0 - 9 \\
\therefore I &= 18\pi \approx 56.5487
\end{aligned}
$$