General Mathematics 2
Topic 3: Matrices

Michael Kasumagic

Term 3, 2025

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Chapter 1

Matrix Basics

1.1 Introduction

A matrix is an array or group of numbers. We display them in columns and rows, and surround them in square brackets. We typically denote them with capital letters, A, B, etc.

$$A = \begin{bmatrix} 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ 9 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 5 \\ 2 & 9 \\ 6 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 6 & 2 \\ 1 & 0 & 0 & 8 \\ 1 & 7 & 7 & 6 \end{bmatrix} \qquad E = \begin{bmatrix} 5 & 2 & 8 & 6 & 3 & 7 \\ 4 & 1 & 6 & 5 & 8 & 3 \\ 5 & 7 & 3 & 5 & 6 & 9 \end{bmatrix} \tag{1.1}$$

1.2 Identifying Matrix Sizes

The first property that we're going to understand is the order of a matrix, or its size. This will become increasingly important as we learn more about matrices.

Definition 1.2.1: Matrix Size/Order

We say an $m \times n$ matrix has m rows and n columns, or has order m by n.

Sadly, this is something we just need to memorise; there's no "reason" for this convetion, it's just an arbitary convention. Just memorise $m \times n$, rows by columns.

Example 1.2.1 (Order of a Matrix)

$$\begin{bmatrix} 7 & 8 & 9 \\ 5 & 6 & 7 \end{bmatrix}$$

I can count the individual numbers, and find that this matrix has 2 rows.
I can count the individual numbers, and find that this matrix has 3 columns.

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 7 & 8 & 9 \\ 5 & 6 & 7 \end{bmatrix} \end{array}$$

Therefore this matrix has order 2×3.

Question 1: Name those orders!

What are sizes of the matrices I've drawn in (1.1)?

Note:-

It's good to note here that an matrix can be arbitarily large. You could imagine a company might manipulate a matrix which contains some data for each of it's 10 million customers.

1.3 Identifying Matrix Elements

Next we learn how to denote and refer to specific elements from a given matrix.

Definition 1.3.1: Matrix Elements

An element of a matrix is a specific entry in a matrix.

Suppose you have a matrix A. We would denote the entry in row i, column j by a_{ij}.

Chapter 2

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Problem Solving and Modelling with Matrices

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Solutions

1.1.1(a) 1×1 (b) 2×2 (c) 3×2 (d) 4×4 (e) 3×6

Chapter 9

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9.1 Random Examples

Definition 9.1.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$$\lim_{n\to\infty} s_n = s$$

where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 2

Is the set $x-$axis$\setminus\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Claim 9.1.1 Topology

Topology is cool

Example 9.1.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\bigcup\limits_{x \in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

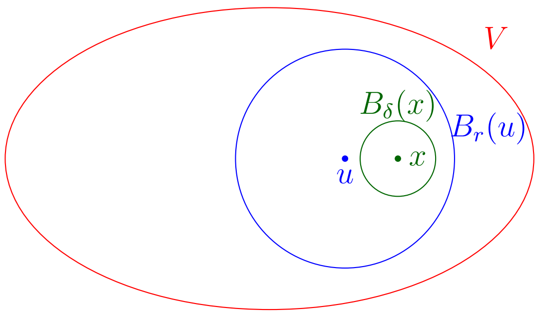
Closed Set:
- $X,\ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Theorem 9.1.1

If $x \in$ open set V then $\exists\ \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)
If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leq d(u, x) + d(x, y) < d + \delta < r$$

$\square$

Corollary 9.1.1

By the result of the proof, we can then show...

Lenma 9.1.1

Suppose $\vec{v_1}, \dots, \vec{v_n} \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 9.1.1

$1 + 1 = 2$.

9.2 Random

Definition 9.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|\ V \to \mathbb{R}_{\geq 0}$ satisfying

1. $\|x\| = 0 \iff x = 0\ \forall\ x \in V$
2. $\|\lambda x\| = |\lambda|\|x\|\ \forall\ \lambda \in \mathbb{R}(\text{or } \mathbb{C}),\ x \in V$
3. $\|x+y\| \leq \|x\| + \|y\|\ \forall\ x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.
• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geq 0}$) where 2 becomes $\|\lambda x\| = |\lambda|\|x\|\ \forall\ \lambda \in \mathbb{C},\ x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2+b^2}$

Example 9.2.1 ($p-$Norm)
$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geq 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \Big(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\Big)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

Question 3

Prove that triangle inequality is true if $p \geq 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property 3 for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\sum_i (x_i + y_i)^2 \leq \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2$$

$$\implies \sum_i (x_i^2 + 2x_iy_i + y_i^2) \leq \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2$$

$$\implies \left[\sum_i x_iy_i\right]^2 \leq \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]$$

So in other words prove $\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_i y_i$$

Note:-

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle \cdot, \cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y\rangle = r\langle x, y\rangle + \langle x', y\rangle \text{ and similarly for second slot}$$

Here in $\langle x, y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\ &= \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \\ &= \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y\rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x\rangle \geq 0$

9.3 Algorithms

Algorithm 1: what

Input: This is some input
Output: This is some output

```
/* This is a comment */
some code here;
x ← 0;
y ← 0;
if x > 5 then
    x is greater than 5 ;                    // This is also a comment
else
    x is less than or equal to 5;
end
foreach y in 0..5 do
    y ← y + 1;
end
for y in 0..5 do
    y ← y − 1;
end
while x > 5 do
    x ← x − 1;
end
return Return something here;
```