MATH1061
Discrete Mathematics I

Problem Set 1

Michael Kasumagic, sID#: 44302669

Due: 5pm, 16^{th} of Augsut, 2024

Question 1: *(10 marks)*

Use a truth table to determine whether the following statement is a contradiction, a tautology or neither. If it is a contradiction or a tautology, verify your answer using logical equivalences.

$$((p \leftrightarrow q) \wedge (p \oplus r)) \rightarrow (q \vee r)$$

Solution: Given the statement, we will construct a truth table.

p	q	r	$p \leftrightarrow q$	$p \oplus r$	$(p \leftrightarrow q) \wedge (p \oplus r)$	$q \vee r$	$((p \leftrightarrow q) \wedge (p \oplus r)) \rightarrow q \vee r$
T	T	T	T	F	F	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	T	T
F	T	F	F	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	F	F	F	T

Therefore, the given statement is a tautology.
Let's now verify this with a proof using logical equivalences. For the sake of brevity, we'll label the original statement A.

$$\text{A} \equiv ((p \leftrightarrow q) \wedge (p \oplus r)) \rightarrow (q \vee r)$$

First, we'll expand and simplify $p \leftrightarrow q$.

$$\begin{aligned}
p \leftrightarrow q &\equiv (p \rightarrow q) \wedge (q \rightarrow p) && \text{(Logical Equivalence of } \leftrightarrow) \\
&\equiv (\sim p \vee q) \wedge (\sim q \vee p) && \text{(Logical Equivalence of } \rightarrow) \\
&\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge \sim q) \vee (q \wedge p) && \text{(Distributivity)} \\
&\equiv (\sim p \wedge \sim q) \vee \perp \vee \perp \vee (q \wedge p) && \text{(Negation Law)} \\
&\equiv (\sim p \wedge \sim q) \vee (q \wedge p) && \text{(Identity Law)} \\
\therefore \text{A} &\equiv ((\sim p \wedge \sim q) \vee (q \wedge p) \wedge (p \oplus r)) \rightarrow (q \vee r) && \text{(Subsittute back into A)}
\end{aligned}$$

Next, we'll expand and simplify $p \oplus r$.

$$\begin{aligned}
p \oplus r &\equiv (p \vee r) \wedge \sim(p \wedge r) && \text{(Logical Equivalence of } \oplus) \\
&\equiv (p \vee r) \wedge (\sim p \vee \sim r) && \text{(Apply De Morgan's Law)} \\
&\equiv (p \wedge \sim p) \vee (p \wedge \sim r) \vee (r \wedge \sim p) \vee (r \wedge \sim r) && \text{(Distributivity)} \\
&\equiv \perp \vee (p \wedge \sim r) \vee (r \wedge \sim p) \vee \perp && \text{(Negation Law)} \\
&\equiv (p \wedge \sim r) \vee (r \wedge \sim p) && \text{(Identity Law)} \\
\therefore \text{A} &\equiv (((\sim p \wedge \sim q) \vee (q \wedge p)) \wedge ((p \wedge \sim r) \vee (r \wedge \sim p))) \rightarrow (q \vee r) && \text{(Subsittute back into A)}
\end{aligned}$$

Now, we'll attempt to distribute $p \leftrightarrow q$ over $p \oplus r$.

$$\begin{aligned}
& (p \leftrightarrow q) \wedge (p \oplus r) \\
\equiv\ & ((\sim p \wedge \sim q) \vee (q \wedge p)) \wedge ((p \wedge \sim r) \vee (r \wedge \sim p)) && \text{(Subsittute previous results)} \\
\equiv\ & ((\sim p \wedge \sim q) \wedge (p \wedge \sim r)) \vee ((\sim p \wedge \sim q) \wedge (r \wedge \sim p)) \\
& \vee ((q \wedge p) \wedge (p \wedge \sim r)) \vee ((q \wedge p) \wedge (r \wedge \sim p)) && \text{(Distributivity)} \\
\equiv\ & ((\sim p \wedge p) \wedge (\sim q \wedge \sim r)) \vee ((\sim p \wedge \sim p) \wedge (r \wedge \sim q)) \\
& \vee ((p \wedge p) \wedge (q \wedge \sim r)) \vee ((\sim p \wedge p) \wedge (r \wedge q)) && \text{(Associativity)} \\
\equiv\ & (\perp \wedge (\sim q \wedge \sim r)) \vee (\sim p \wedge (r \wedge \sim q)) \vee (p \wedge (q \wedge \sim r)) \vee (\perp \wedge (r \wedge q)) && \text{(Negation Law)} \\
\equiv\ & (\perp) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\perp) && \text{(Domination Law)} \\
\equiv\ & (\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) && \text{(Identity Law)} \\
\therefore \text{A} \equiv\ & ((\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r)) \rightarrow (q \vee r) && \text{(Subsittute back into A)}
\end{aligned}$$

Holy hell.
Next, we'll expand this resultant $\alpha \to \beta$ into $\sim\alpha \vee \beta$.

$$\begin{aligned}
\mathrm{A} &= ((\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r)) \to (q \vee r) && \text{(Previous Result)}\\
&= \sim((\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r)) \vee (q \vee r) && \text{(Logical Equivalence of } \to\text{)}\\
&= (\sim(\sim p \wedge \sim q \wedge r) \wedge \sim(p \wedge q \wedge \sim r)) \vee (q \vee r) && \text{(Apply De Morgan's Law)}\\
&= ((p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r)) \vee (q \vee r) && \text{(Apply De Morgan's Law)}\\
&= (q \vee r) \vee ((p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r)) && \text{(Commutativty)}
\end{aligned}$$

Finally, we'll distribute $(q \vee r)$ over the rest of the statement.

$$\begin{aligned}
\mathrm{A} &= (q \vee r) \vee ((p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r)) && \text{(Previous Result)}\\
&= ((q \vee r) \vee (p \vee q \vee \sim r)) \wedge ((q \vee r) \vee (\sim p \vee \sim q \vee r)) && \text{(Distributivity)}\\
&= (q \vee r \vee p \vee q \vee \sim r) \wedge (q \vee r \vee \sim p \vee \sim q \vee r) && \text{(Associativity)}
\end{aligned}$$

We're so close now!
To finish it off, we can use associativity to rearrange our statement, find the tautology, then expand it outward, until we show the whole statement evaluates to a tautology.

$$\begin{aligned}
\mathrm{A} &= (q \vee r \vee p \vee q \vee \sim r) \wedge (q \vee r \vee \sim p \vee \sim q \vee r)\\
&= (r \vee \sim r \vee p \vee q \vee q) \wedge (q \vee \sim q \vee p \vee r \vee r) && \text{(Associativity)}\\
&= (\top \vee p \vee q) \wedge (\top \vee p \vee r) && \text{(Nagation Law and Idempotent Law)}\\
&= (\top \vee q) \wedge (\top \vee r) && \text{(Domination Law)}\\
&= \top \wedge \top && \text{(Idempotent Law)}\\
\therefore \mathrm{A} &\equiv \top && \text{(Conclusion)}
\end{aligned}$$

And we're done! Therefore, we've proven, by logical equivalence, that the statement

$$((p \leftrightarrow q) \wedge (p \oplus r)) \to (q \vee r)$$

is logically equivalent to a tautology. $\square$

Question 2: *(10 marks)*

Using the laws of logical equivalence, show that the following statement is a tautology:

$$((a \to r) \land (b \to r) \land (\sim a \to b)) \to r$$

Solution: Again, we'll label the statement B for the sake of brevity.

$$\mathrm{B} \equiv ((a \to r) \land (b \to r) \land (\sim a \to b)) \to r$$

We'll start by expanding each $\to$ expression into its corresponding $\sim\lor$ expression.

$$\begin{aligned}
(a \to r) &\equiv \sim a \lor r && \text{(Logical Equivalence of } \to\text{)}\\
(b \to r) &\equiv \sim b \lor r && \text{(Logical Equivalence of } \to\text{)}\\
(\sim a \to b) &\equiv a \lor b && \text{(Logical Equivalence of } \to\text{)}\\
\therefore \mathrm{B} &\equiv ((\sim a \lor r) \land (\sim b \lor r) \land (a \lor b)) \to r && \text{(Subsittute back into B)}
\end{aligned}$$

We will now apply the distributivity law to undistribute r from $(\sim a \lor r) \land (\sim b \lor r)$.

$$\begin{aligned}
(\sim a \lor r) \land (\sim b \lor r) &\equiv r \lor (\sim a \land \sim b) && \text{(Distributivity)}\\
\therefore \mathrm{B} &\equiv ((r \lor (\sim a \land \sim b)) \land (a \lor b)) \to r && \text{(Subsittute back into B)}\\
&\equiv ((a \lor b) \land (r \lor (\sim a \land \sim b))) \to r && \text{(Associativity)}
\end{aligned}$$

Next, we'll now distribute $(a \lor b)$ over $r \lor (\sim a \land \sim b)$.

$$\begin{aligned}
&(a \lor b) \land (r \lor (\sim a \land \sim b))\\
&\equiv ((a \lor b) \land r) \lor ((a \lor b) \land (\sim a \land \sim b)) && \text{(Distributivity)}\\
&\equiv ((a \land r) \lor (b \land r)) \lor (((a \lor b) \land \sim a) \land ((a \lor b) \land \sim b)) && \text{(Distributivity)}\\
&\equiv ((a \land r) \lor (b \land r)) \lor (((a \land \sim a) \lor (b \land \sim a)) \land ((a \land \sim b) \lor (b \land \sim b))) && \text{(Distributivity)}\\
&\equiv ((a \land r) \lor (b \land r)) \lor ((\bot \lor (b \land \sim a)) \land ((a \land \sim b) \lor \bot)) && \text{(Negation Law)}\\
&\equiv ((a \land r) \lor (b \land r)) \lor ((b \land \sim a) \land (a \land \sim b)) && \text{(Identity)}\\
&\equiv ((a \land r) \lor (b \land r)) \lor ((b \land \sim b) \land (a \land \sim a)) && \text{(Associativity)}\\
&\equiv ((a \land r) \lor (b \land r)) \lor (\bot \land \bot) && \text{(Negation Law)}\\
&\equiv ((a \land r) \lor (b \land r)) \lor \bot && \text{(Identity Law)}\\
&\equiv ((a \land r) \lor (b \land r)) && \text{(Identity Law)}\\
&\equiv r \land (a \lor b) && \text{(Distributivity)}\\
\therefore \mathrm{B} &\equiv (r \land (a \lor b)) \to r && \text{(Subsittute back into B)}
\end{aligned}$$

Finally, let's expand the $\to$ into its logically equivalent $\sim\lor$ form.

$$\begin{aligned}
\mathrm{B} &\equiv (r \land (a \lor b)) \to r\\
&\equiv \sim(r \land (a \lor b)) \lor r && \text{(Logical Equivalence of } \to\text{)}\\
&\equiv (\sim r \lor \sim(a \lor b)) \lor r && \text{(Applying De Morgan's Law)}\\
&\equiv (\sim r \lor r) \lor \sim(a \lor b) && \text{(Associativity)}\\
&\equiv \top \lor \sim(a \lor b)\\
\therefore \mathrm{B} &\equiv \top
\end{aligned}$$

Therefore, we've shown, by use of laws of logical equivalence, that the statement

$$((a \to r) \land (b \to r) \land (\sim a \to b)) \to r$$

is a tautology. $\square$

Question 3: (*5 marks*)

Show that the following argument is valid, by adding steps using the rules of inference and/or logical equivalences. Clearly label which rule you used in each step.

1. $a \wedge w \to p$
2. $\sim a \to l$
3. $\sim w \to m$
4. $\sim p$
5. $e \to \sim(l \vee m)$

$\therefore$ $\sim e$

Solution:

1.	$a \wedge w \to p$	
2.	$\sim a \to l$	
3.	$\sim w \to m$	
4.	$\sim p$	
5.	$e \to \sim(l \vee m)$	
6.	$(l \vee m) \to \sim e$	(Contrapositive of 5.)
7.	$\sim(a \wedge w)$	(Modus Tollens of 1. given 4.)
8.	$\sim a \vee \sim w$	(De Morgan's Law on 7.)
9.	$l \vee m$	(Constructive Dilemma, of 2., 3., given 8.)
10.	$\sim e$	(Modus Ponens of 6. given 9.)
$\therefore$	$\sim e$	

This shows that the argument is valid. Applying rules of inference and logical equivalence to the same given premises, we've drawn the same conclusion.Therefore the argument is valid. □

Question 4: *(10 marks)*

Let $P(x), Q(x), R(x)$ and $S(x)$ denote the following predicates with domain $\mathbb{Z}$:

$$\begin{aligned} P(x): &\quad x^2 = 9, \\ Q(x): &\quad x^2 = 6, \\ R(x): &\quad x \geq 0, \\ S(x): &\quad x \text{ is odd.} \end{aligned}$$

Determine whether each of the following statements is true or false, and give brief reasons.

(a) $\forall x \in \mathbb{Z}, P(x) \to S(x)$

(b) $\forall x \in \mathbb{Z}, P(x) \to R(x)$

(c) $\exists x \in \mathbb{Z} : P(x) \wedge R(x)$

(d) $\forall x \in \mathbb{Z}, Q(x) \to R(x)$

(e) $\forall x \in \mathbb{Z}, S(x) \to {\sim}Q(x)$

Solution: (a)

$$\forall x \in \mathbb{Z}, P(x) \to S(x) \equiv \forall x \in \mathbb{Z}, x^2 = 9 \to x \text{ is odd}$$

is true, because if $x^2 = 9$ then x is either $-\sqrt{9} = -3$ or $\sqrt{9} = 3$, both of which are odd.
An alternative view: the statement is true because if $x^2 = 9$, we can conclude that $|x||x| = 9$. 9's prime factorsation is $3 \cdot 3$, therefore $|x| = 3$. If $|x| = 3$, there are two cases which satisfy this: $x = 3$ and $x = -3$, both of which are odd.

Solution: (b)

$$\forall x \in \mathbb{Z}, P(x) \to R(x) \equiv \forall x \in \mathbb{Z}, x^2 = 9 \to x \geq 0$$

is false, because I can provide a counter example, namely, $x = -3$. If $x = -3$, then $x^2 = 9$, but $x \ngeq 0$. In fact, $x = -3 < 0$.

Solution: (c)

$$\exists x \in \mathbb{Z} : P(x) \wedge R(x) \equiv \exists x \in \mathbb{Z} : x^2 = 9 \wedge x \geq 0$$

is true, because I can provide an example $x \in \mathbb{Z}$, namely, $x = 3$. If $x = 3$, then $x^2 = 3^2 = 9$, and $x = 3 \geq 0$.

Solution: (d)

$$\forall x \in \mathbb{Z}, Q(x) \to R(x) \equiv \forall x \in \mathbb{Z}, x^2 = 6 \to x \geq 0$$

is true. $Q(x)$ is false $\forall x \in \mathbb{Z}$, because $\nexists x \in \mathbb{Z} : x^2 = 6$. No integer can satisfy this equation. You'd need to expand the domain of x into the reals to solve this, namely $r = \sqrt{6} \in \mathbb{R}$. Since $Q(x)$ is false $\forall x \in \mathbb{Z}$, it doesn't matter if $R(x)$ is true or false, because the statement, $Q(x) \to R(x) \equiv \bot \to R(x) \equiv \top \vee R(x) \equiv \top$, will evaluate to true always. The given statement is an example of a vacuous truth.

Solution: (e)

$$\forall x \in \mathbb{Z}, S(x) \to {\sim}Q(x) \equiv \forall x \in \mathbb{Z}, x \text{ is odd} \to x^2 \neq 6$$

is true, because if x is odd, or any element in the set of integers for that matter, then it will hold that $x^2 \neq 6$. In fact, no integer can satisfy the equation $x^2 = 6$, you'd need to expand the domain of x into the real numbers to find a solution, namely $r = \sqrt{6} \in \mathbb{R}$. So, Q(x) is false $\forall x \in \mathbb{Z}$, which means $\forall x \in \mathbb{Z}, {\sim}Q(x)$ is true. So the whole statement, $S(x) \to {\sim}Q(x) \equiv S(x) \to \top \equiv {\sim}S(x) \vee \top \equiv \top$ is always true. Another example of a vacuous truth.

Question 5: (*10 marks*)

Find values of a, b and r that show that the following statement is not a tautology, and show your working.

$$r \to ((a \to r) \wedge (b \to r) \wedge (\sim a \to b))$$

Solution: We can do this with a truth table.
For the sake of brevity, let C $= r \to ((a \to r) \wedge (b \to r) \wedge (\sim a \to b))$

a	b	r	$\sim a$	$a \to r$	$b \to r$	$\sim a \to b$	r	$(a \to r) \wedge (b \to r) \wedge (\sim a \to b)$	C
T	T	T	F	T	T	T	T	T	T
T	T	F	F	F	F	T	F	F	T
T	F	T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F	F	T
F	F	T	T	T	T	F	T	F	F
F	F	F	T	T	T	F	F	F	T

$\therefore$ $(a, b, r) = (\text{False}, \text{False}, \text{True})$ is an example configuration of variable truth values which causes the statement to evaluate to false. This, in turn, shows that the statement C is not a tautology.

I'm paranoid that proving this by truth table is insuffcent, so let's also work through this, just in case, and consider the statement

$$r \to ((a \to r) \wedge (b \to r) \wedge (\sim a \to b)) \equiv \sim r \vee ((a \to r) \wedge (b \to r) \wedge (\sim a \to b))$$

is true when r is false, or the ands are true.

1. $a \to r \equiv \sim a \vee r$: Is true if a is false or r is true.
2. $b \to r \equiv \sim b \vee r$: Is true if b is false or r is true.
3. $\sim a \to b \equiv a \vee b$: Is true if a is true or b is true.

By analysing this statement, we can see a weakness. 3. and (2. and 1.) will clash. They can't both be true at the same time, if we select the right configuration of truth values. We will set r to true, because this moves the "responsibility" of making the statement false, from r to the ands, where we've identified a weakness.

$$\text{Let } r \text{ be true.}$$
$$\text{Then } 1. \wedge 2. \wedge 3. \text{ must be true.}$$
$$\therefore 1. \equiv \text{True}, 2. \equiv \text{True}, 3. \equiv \text{True}$$
$$\text{True} \equiv a \to r \equiv a \to \text{True} \equiv \sim a \vee \text{True} \equiv \sim a$$
$$\text{So let } a \text{ be false.}$$
$$\text{True} \equiv b \to r \equiv b \to \text{True} \equiv \sim b \vee \text{True} \equiv \sim b$$
$$\text{So let } b \text{ be false.}$$
$$\text{True} \equiv \sim a \to b \equiv a \vee b \equiv \text{False} \vee \text{False} \equiv \text{False} \qquad ※$$

This is a contradiction! So our assumption that $1. \wedge 2. \wedge 3.$ is true must be wrong, and the statement "$r \to ((a \to r) \wedge (b \to r) \wedge (\sim a \to b))$ is a tautology" causes a contradiction. Therefore, we can conclude that $r \to ((a \to r) \wedge (b \to r) \wedge (\sim a \to b))$ is not a tautology, and can provide a counterexample, namely $(a, b, r) = (\text{False}, \text{False}, \text{True})$. $\square$