MATH1061
Discrete Mathematics I

Problem Set 3

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Question 1: *(10 marks)*

Prove the following set identitiesa

(1) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(2) $(A' \cap B)' \cap D = (D \setminus A') \cup (D \setminus B)$

Solution: (a)

$$\begin{aligned}
(A \cup B) \times C &= \{(x, c) \mid x \in A \cup B,\ c \in C\} && \text{(Definition of Cartesian Product)}\\
&= \{(x, c) \mid (x \in A \vee x \in B),\ c \in C\} && \text{(Definition of Union)}\\
&= \{(x, c) \mid (x \in A,\ c \in C) \vee (x \in B,\ c \in C)\} && \text{(Distributivity of } \wedge \text{ over } \vee\text{)}\\
&= \{(a, c) \mid a \in A,\ c \in C\} \cup \{(b, c) \mid b \in B,\ c \in C\} && \text{(Definition of Union)}\\
&= (A \times C) \cup (B \times C) && \text{(Definition of Cartesian Product)}
\end{aligned}$$

Which is what we wanted to show.

Solution: (b)

$$\begin{aligned}
(A' \cap B)' \cap D &= (A \cup B') \cap D && \text{(De Morgan's)}\\
&= D \cap (A \cup B') && \text{(Commutativity)}\\
&= (D \cap A) \cup (D \cap B') && \text{(Distributivity)}\\
&= (D \setminus A') \cup (D \setminus B) && \text{(Set Difference Law)}
\end{aligned}$$

Which is what we wanted to show.

Question 2: *(15 marks)*

Suppose $f : A \to B$ and $g : B \to A$ are functions, and ι_A is the identity function on A, ι_B is the identity function on B. In particular, $\iota_A(x) = x$, $\forall x \in A$ and similarly $\iota_B(x) = x$, $\forall x \in B$.

(1) Suppose $f \circ g = \iota_B$ and $g \circ f = \iota_A$. Prove that f and g are bijections.

(2) Suppose g is surjective and $f \circ g = \iota_B$. Prove that $g \circ f = \iota_A$.

Solution: (a)

Proposition. f is a bijection

Proof. Show that f is injective.

Suppose $x, y \in A$, $f(x) = f(y)$.
We can apply g to both sides, $g(f(x)) = g(f(y))$.
Which is the same as writing $(g \circ f)(x) = (g \circ f)(y)$.
Given that $g \circ f = \iota_A$, the previous expression can be rewritten $\iota_A(x) = \iota_A(y)$.
Evaluating the identity function, $x = y$.

Therefore, f is injective.

Show that f is surjective

Suppose $b \in B$. Take $a = g(b) \in A$, since $g : B \to A$.
Hence, $f(a) = f(g(b)) = (f \circ g)(b) = \iota_B(b) = b$.
Thus, $\forall b \in B, \exists a \in A : b = f(a)$.

Therefore, f is surjective.

f is surjective and injective.
Therefore, f is bijective. □

Proposition. g is a bijection

Proof. Show that g is injective.

Suppose $x, y \in B$, $g(x) = g(y)$.
We can apply f to both sides, $f(g(x)) = f(g(y))$.
Which is the same as writing $(f \circ g)(x) = (f \circ g)(y)$.
Given that $f \circ g = \iota_B$, the previous expression can be rewritten $\iota_B(x) = \iota_B(y)$.
Evaluating the identity function, $x = y$.

Therefore, g is injective.

Show that g is surjective

Suppose $a \in A$. Take $b = f(a) \in B$, since $f : A \to B$.
Hence, $g(b) = g(f(a)) = (g \circ f)(a) = \iota_A(a) = a$.
Thus, $\forall a \in A, \exists b \in B : a = g(b)$.

Therefore, f is surjective.

g is surjective and injective.
Therefore, g is bijective. □

Corollary. f is a bijection and g is a bijection.

Solution: (b)
Proposition. $g \circ f = \iota_A$

Proof. Suppose g is surrjective and $f \circ g = \iota_B$.

g is surrjective, which means $\forall a \in A, \exists b \in B : a = g(b)$.
Take any $\alpha \in A$. There is a $\beta \in B$ such that $\alpha = g(\beta)$.
Apply f to both sides, $f(\alpha) = f(g(\beta))$.
Apply g to both sides, $g(f(\alpha)) = g(f(g(\beta)))$.
Simplifying, $(g \circ f)(\alpha) = g((f \circ g)(\beta))$.
Simplify further, $(g \circ f)(\alpha) = g(\iota_B(\beta))$.
We see, $(g \circ f)(\alpha) = g(\beta)$.
Hence, $(g \circ f)(\alpha) = \alpha$, which is the definition of ι_A.

Therefore $g \circ f = \iota_A$. $\square$

Question 3: (*15 marks*)

(1) Show that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is in bijection with $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$. Deduce that $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

(2) Show that $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ is in bijection with $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$. Deduce that $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

(3) Is the set $\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$ (Cartesian product n times) countable? A yes or no would suffice.

Remark. In this this question, we'll use $\mathbb{Z}^+ =: \mathbb{N} = \{1, 2, 3, 4, \ldots\}$. We will denote the Cartesian product n times, $\mathbb{N} \times \cdots \times \mathbb{N}$, as $\mathbb{N}^n$. We'll also denote an injective function from set A to B, $f : A \rightarrowtail B$, and a bijective function $f : A \leftrightarrow B$.

Solution: (a)
Proposition. $\mathbb{N}^2$ is in bijection with $\mathbb{N}^3$.

Proof. We'll utilise the Schröder-Bernstein theorem, which states (with notation adapted for our specific problem)

$$f : \mathbb{N}^2 \rightarrowtail \mathbb{N}^3,\ g : \mathbb{N}^3 \rightarrowtail \mathbb{N}^2 \implies h : \mathbb{N}^2 \leftrightarrow \mathbb{N}^3$$

So, to prove that $\mathbb{N}^2$ is in bijection with $\mathbb{N}^3$, we'll find two functions which map from one set to the other, and show that those functions are injective.

Show that there exists an injective function $f : \mathbb{N}^2 \rightarrowtail \mathbb{N}^3$.

Let's propose the function $f : \mathbb{N}^2 \to \mathbb{N}^3$, defined by $f((a, b)) := (a, b, 0)$.
Suppose $(a_1, b_1), (a_2, b_2) \in \mathbb{N}^2$, and $f(a_1, b_1) = f(a_2, b_2)$.
Then $(a_1, b_1, 0) = (a_2, b_2, 0)$
Hence $a_1 = a_2,\ b_1 = b_2,\ 0 = 0$.

Therefore our proposed function $f : \mathbb{N}^2 \rightarrowtail \mathbb{N}^3$ is injective.

Show that there exists an injective function $g : \mathbb{N}^3 \rightarrowtail \mathbb{N}^2$.

Let's propose the function $g : \mathbb{N}^3 \to \mathbb{N}^2$, defined by $g((a, b, c)) := \left(2^a 3^b 5^c, 0\right)$.
Suppose $(a_1, b_1, c_1), (a_2, b_2, c_2) \in \mathbb{N}^3$ and $f((a_1, b_1, c_1)) = f((a_2, b_2, c_2))$.
Then $\left(2^{a_1} 3^{b_1} 5^{c_1}, 0\right) = \left(2^{a_2} 3^{b_2} 5^{c_2}, 0\right)$
Hence, $2^{a_1} 3^{b_1} 5^{c_1} = 2^{a_2} 3^{b_2} 5^{c_2}$.
Which implies that $2^{a_1} = 2^{a_2} \iff a_1 = a_2$,
$3^{b_1} = 3^{b_2} \iff b_1 = b_2$,
and $5^{c_1} = 5^{c_2} \iff c_1 = c_2$.

Therefore our proposed function $g : \mathbb{N}^3 \rightarrowtail \mathbb{N}^2$ is injective.

There exists an injection $f : \mathbb{N}^2 \rightarrowtail \mathbb{N}^3$, namely $f((a, b)) = (a, b, 0)$, and an injection $g : \mathbb{N}^3 \rightarrowtail \mathbb{N}^2$, namely $g((a, b, c)) = (2^a 3^b 5^c, 0)$.
Therefore, by Schröder-Bernstein theorem, there exists a bijection, $h : \mathbb{N}^2 \leftrightarrow \mathbb{N}^3$.
Therefore $\mathbb{N}^2$ is in bijection with $\mathbb{N}^3$. □

Corollary. $\mathbb{N}^3$ is countable.

Proof.

$\exists f : \mathbb{N}^2 \leftrightarrow \mathbb{N} \iff \left|\mathbb{N}^2\right| = |\mathbb{N}|$.
$\exists g : \mathbb{N}^2 \leftrightarrow \mathbb{N}^3 \iff \left|\mathbb{N}^2\right| = \left|\mathbb{N}^3\right|$.
$\therefore \left|\mathbb{N}^3\right| = |\mathbb{N}|$, by transitivity. This is the definition of countable.
Therefore $\mathbb{N}^3$ is countable.

□

Solution: (b)
Proposition. $\mathbb{N}^3$ is in bijection with $\mathbb{N}^4$.

Proof. We'll utilise the Schröder-Bernstein theorem, which states (with notation adapted for our specific problem)

$$f : \mathbb{N}^3 \rightarrowtail \mathbb{N}^4,\ g : \mathbb{N}^4 \rightarrowtail \mathbb{N}^3 \implies h : \mathbb{N}^3 \leftrightarrow \mathbb{N}^4$$

So, to prove that $\mathbb{N}^3$ is in bijection with $\mathbb{N}^4$, we'll find two functions which map from one set to the other, and show that those functions are injective.

Show that there exists an injective function $f : \mathbb{N}^3 \rightarrowtail \mathbb{N}^4$.

Let's propose the function $f : \mathbb{N}^3 \to \mathbb{N}^4$, defined by $f((a, b, c)) := (a, b, c, 0)$.
Suppose $(a_1, b_1, c_1), (a_2, b_2, c_1) \in \mathbb{N}^3$, and $f(a_1, b_1, c_1) = f(a_2, b_2, c_2)$.
Then $(a_1, b_1, c_1, 0) = (a_2, b_2, c_2, 0)$
Hence $a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2,\ 0 = 0$.

Therefore our proposed function $f : \mathbb{N}^3 \rightarrowtail \mathbb{N}^4$ is injective.

Show that there exists an injective function $g : \mathbb{N}^4 \rightarrowtail \mathbb{N}^3$.

Let's propose the function $g : \mathbb{N}^4 \to \mathbb{N}^3$, defined by $g((a, b, c, d)) := \left(2^a 3^b 5^c 7^d, 0, 0\right)$.
Suppose $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in \mathbb{N}^4$ and $f((a_1, b_1, c_1, d_1)) = f((a_2, b_2, c_2, d_2))$.
Then $\left(2^{a_1} 3^{b_1} 5^{c_1} 7^{d_1}, 0, 0\right) = \left(2^{a_2} 3^{b_2} 5^{c_2} 7^{d_2}, 0, 0\right)$
Hence, $2^{a_1} 3^{b_1} 5^{c_1} 7^{d_1} = 2^{a_2} 3^{b_2} 5^{c_2} 7^{d_2}$.
Which implies that $2^{a_1} = 2^{a_2} \iff a_1 = a_2$,
$3^{b_1} = 3^{b_2} \iff b_1 = b_2$,
$5^{c_1} = 5^{c_2} \iff c_1 = c_2$,
and $7^{d_1} = 7^{d_2} \iff d_1 = d_2$.

Therefore our proposed function $g : \mathbb{N}^4 \rightarrowtail \mathbb{N}^3$ is injective.

There exists an injection $f : \mathbb{N}^3 \rightarrowtail \mathbb{N}^4$, namely $f((a, b, c)) = (a, b, c, 0)$, and an injection $g : \mathbb{N}^4 \rightarrowtail \mathbb{N}^3$, namely $g((a, b, c, d)) = (2^a 3^b 5^c 7^d, 0, 0)$.
Therefore, by Schröder-Bernstein theorem, there exists a bijection, $h : \mathbb{N}^3 \leftrightarrow \mathbb{N}^4$.
Therefore $\mathbb{N}^3$ is in bijection with $\mathbb{N}^4$. □

Corollary. $\mathbb{N}^4$ is countable.

Proof.

$\exists f : \mathbb{N}^2 \leftrightarrow \mathbb{N} \iff \left|\mathbb{N}^2\right| = |\mathbb{N}|$.
$\exists g : \mathbb{N}^2 \leftrightarrow \mathbb{N}^3 \iff \left|\mathbb{N}^2\right| = \left|\mathbb{N}^3\right|$.
$\exists h : \mathbb{N}^3 \leftrightarrow \mathbb{N}^4 \iff \left|\mathbb{N}^3\right| = \left|\mathbb{N}^4\right|$.
$\therefore \left|\mathbb{N}^4\right| = |\mathbb{N}|$, by transitivity. This is the definition of countable.
Therefore $\mathbb{N}^4$ is countable.

□

Solution: (c) Yes.

I can always construct an injective function $f : \mathbb{N}^{n-1} \rightarrowtail \mathbb{N}^n,\ f((a_1, \ldots, a_{n-1})) = (a_1, \ldots, a_{n-1}, 0)$ and a second injective function $g : \mathbb{N}^n \rightarrowtail \mathbb{N}^{n-1},\ g(a_1, \ldots, a_n) = (\prod_{i=1}^{n} p_i^{a_i}, 0, \ldots, 0)$, which will inductively be in bijection with $\mathbb{N}$

Question 4: (*10 marks*)

Let A be the set of all logical statements. Define a relation on A: for $p, q \in A$, p is related to q if and only if $p \wedge q$ and $p \vee q$ have the same truth value.

Determine if the above relation is reflexive, symmetric, transitive. If your answer is yes for any of the three properties, please prove your answer; if your answer is no, please find a counterexample.

Solution:

Let $\sigma : A \to A$ be defined By

$$\begin{aligned}
\forall p, q \in A,\ p \,\sigma\, q &\iff p \wedge q \leftrightarrow p \vee q && \text{(Given)}\\
&\iff ((p \wedge q) \to (p \vee q)) \wedge ((p \vee q) \to (p \wedge q)) && \text{(Definition of } \leftrightarrow)\\
&\iff (\sim(p \wedge q) \vee (p \vee q)) \wedge (\sim(p \vee q) \vee (p \wedge q)) && \text{(Definition of } \to)\\
&\iff ((\sim p \vee \sim q) \vee (p \vee q)) \wedge ((\sim p \wedge \sim q) \vee (p \wedge q)) && \text{(De Morgan's)}\\
&\iff ((\sim p \vee p) \vee (\sim q \vee q)) \wedge ((\sim p \wedge \sim q) \vee (p \wedge q)) && \text{(Associativity)}\\
&\iff (\top \vee \top) \wedge ((\sim p \wedge \sim q) \vee (p \wedge q)) && \text{(Negation Law)}\\
&\iff \top \wedge ((\sim p \wedge \sim q) \vee (p \wedge q)) && \text{(Tautology)}\\
&\iff (\sim p \wedge \sim q) \vee (p \wedge q) && \text{(Identity)}\\
&\iff ((\sim p \vee p) \wedge (\sim q \vee p)) \wedge ((\sim p \vee q) \wedge (\sim q \vee q)) && \text{(Distributivity, twice)}\\
&\iff (\top \wedge (\sim q \vee p)) \wedge ((\sim p \vee q) \wedge \top) && \text{(Negation Laws)}\\
&\iff (\sim q \vee p) \wedge (\sim p \vee q) && \text{(Identity)}\\
&\iff (q \to p) \wedge (p \to q) && \text{(Definition of } \to)\\
&\iff p \leftrightarrow q && \text{(Definition of } \leftrightarrow)
\end{aligned}$$

Is σ reflexive?

Suppose $p \in A$.
Then $p \equiv$ True or $p \equiv$ False.
In either case, $p \leftrightarrow p$.
So $\forall p \in A,\ p \,\sigma\, p$

Therefore σ on A is reflexive.

Is σ symmetric?

Suppose $p, q \in A$.
Then $p \equiv$ True or $p \equiv$ False. And $q \equiv$ True or $q \equiv$ False.
Without any loss of generality, assume $p \equiv q$.
(if $p \not\equiv q$, then $p \not\leftrightarrow q$ hence $p \,\sigma\, q$ does not hold, and doesn't need to be considered.)
Suppose $p \,\sigma\, q$.
Then $p \leftrightarrow q$. Hence, $q \leftrightarrow p$, by commutativity of $\leftrightarrow$. Therefore, $q \,\sigma\, p$.

Therefore σ on A is symmetric.

Is σ transitive?

Suppose $p, q, r \in A$.
Then $p \equiv$ True or $p \equiv$ False. $q \equiv$ True or $q \equiv$ False. And $r \equiv$ True or $r \equiv$ False.
Without any loss of generality assume $p \equiv q$ and $q \equiv r$
(If $p \not\equiv q$ then $p \,\sigma\, q$ does not hold and need not be considered. If $q \not\equiv r$ then $q \,\sigma\, r$ does not hold and need not be considered.)
Suppose $p \,\sigma\, q$ and $q \,\sigma\, r$.
Then $p \leftrightarrow q$ and $q \leftrightarrow r$. Therefore $p \leftrightarrow r$, by transitivity of $\leftrightarrow$. Hence $p \,\sigma\, r$.

Therefore σ on A is transitive.

Therefore, σ is an equivilence relation on A.