MATH1072 - Limits

1. Show that, for x and y not both equal to zero,
$$-1 \leq \frac{2xy}{x^2+y^2} \leq 1.$$
Then prove, using the Squeeze Theorem, that
$$\lim_{(x,y)\to(0,0)} \frac{xy\sin^2 y}{x^2+y^2} = 0.$$

2. Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist for the following choices of $f(x,y)$:
   (a) $f(x,y) = \dfrac{xy\sin(x+1)}{x^2+2y^2}$;
   (b) $f(x,y) = \dfrac{x^2-y^2}{x^2+xy}$;
   (c) $f(x,y) = \dfrac{xy^2}{x^2+y^4}$.

3. Evaluate
$$\lim_{(x,y)\to(0,0)} \frac{x^3+x^2+2y^2}{\sqrt{x^2+2y^2+1}-1}.$$

4. Consider $\lim_{(x,y)\to(0,0)} f(x,y) = \dfrac{x+y}{\sqrt{x^2+y^2}}$. Evaluate the limit if it exists, and confirm this with an $\epsilon-\delta$ proof, or show that the limit does not exist.

5. Consider $\lim_{(x,y)\to(0,0)} f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. Evaluate the limit if it exists, and confirm this with an $\epsilon-\delta$ proof, or show that the limit does not exist.

6. Develop an $\epsilon-\delta$ proof to establish that
$$\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+y^2+2} = 0.$$

7. Develop an $\epsilon-\delta$ proof to establish that
$$\lim_{(x,y)\to(0^+,0^+)} \frac{x^{3/2}y^{3/2}}{x^2+2y^2+x^2y^2} = 0.$$

8. a) Show that if $\lim_{\mathbf{h}\to\mathbf{0}} T(\mathbf{h}) = 0$, then $\lim_{t\to 0} T(t\mathbf{v}) = 0$ for all fixed $\mathbf{v} \neq \mathbf{0}$.
   b) Using the function
$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$
   show that the converse of the statement in part a) is false.

9. **Challenge** - Recall the definition for a limit: $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that, given $\epsilon > 0$ there exists $\delta > 0$ such that if $d((x,y),(a,b)) < \delta$ then $|f(x,y) - L| < \epsilon$, where

$$d((x,y),(a,b)) = \sqrt{(x-a)^2 + (y-b)^2}$$

is the *Euclidean* distance between (x,y) and (a,b) in $\mathbb{R}^2$.

There are many ways to define a distance. Here we consider the <u>*taxicab*</u> distance defined by

$$D((x,y),(a,b)) = |x-a| + |y-b|.$$

Prove that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ with respect to the taxicab distance in $\mathbb{R}^2$ if and only if $\lim_{(x,y)\to(a,b)} f(x,y) = L$ with respect to the Euclidean distance in $\mathbb{R}^2$.

10. **Challenge**

Consider a multivariable function $L : \mathbb{R}^n \to \mathbb{R}$ which is continuous at $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. Let $\eta_i : \mathbb{R} \to \mathbb{R}$ be a set of n functions such that

$$\lim_{t\to k} \eta_i(t) = a_i, \ \forall\ 1 \leq i \leq n. \tag{1}$$

Show that

$$\lim_{t\to k} L(\eta_1(t), \eta_2(t), \ldots, \eta_n(t)) = L(\mathbf{a}).$$