MATH1061
Discrete Mathematics I

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Contents

Chapter 1

Week 1

1.1 Lecture 1

This course will run a little differently. Prior to every lecture, we must work through a set of pre-lecture problems. The goal of timetabled lectures is to discuss and learn from solving problems.

What is in this course?

Logic and set theory, methods of proof

Modern mathematics uses the language of set theory and the notation of logic.

$$((P \wedge \sim Q) \vee (P \wedge Q)) \wedge Q \equiv P \wedge Q$$

We will learn to read and analyse this. Historical, there was a big shift in recent history, there was a big effort to define and axiom-itise everything, such that math itself is defined rigorously. Symbolic logic is the basis for many areas of computer science. It helps us formulate mathematical ideas and proofs effectively and cobbrectly!

Definition 1.1.1: Gödel's Incompleteness Theorem (1931)

There exists true statements which we can not prove!

Number Theory

Example 1.1.1 $(1 + \cdots + 100)$

A young Gauss had to add up all the numbers from 1 to 100 in primary school. What did he do?

$$\begin{array}{ccccccccccccc} & & 1 & + & 2 & + & \cdots & + & 98 & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 & & \\ \hline 100 & + & 100 & + & 100 & + & \cdots & + & 100 & + & 100 & + & 100 \end{array}$$

So...

$$1 + \cdots + 100 = \frac{101 \cdot 100}{2} = 5050$$

This generalises to $\forall n \in \mathbb{N}$. Two leaps of faith are needed though!

- The dots: We introduce the notation to deal with them.
- The equality of two equations invoving dots. We will use induction to deal with this!

Graph Theory

Example 1.1.2 (The Königsberg Bridge Problem)
Find a route through the city which crosses each of seven bridges exactly once, and returns you to your start location.

This is provably impossible! But how can we rigorously prove this? Euler solve this problem in 1935 and in doing so invented graph theory. We'll learn how eventually... :p

Counting and Probability

Both fundamental and beautifully applicable. We introduce the pigeonhole principle as a introduction to "counting."

Example 1.1.3 (The pigeonhole principle)
If you have n pigeons sitting in k pigeonholes, if $n > k$, then at least of the pigeonholes contains at least 2 pigeons.

Question 1

If you have socks of three different colours in your drawer, what is the minimum number of socks you need to pull out to guarantee a matching pair?

Solution: #socks $\equiv$ #pigeons and #colours $\equiv$ #holes. If #socks $>$ #colors, a double must occur. Therefore, we need a minimum of 4 socks to guarantee a match.

Question 2: True or False?

In every group of five people, there are two people who have the same number of friends within the group.

Solution: True! #people $\equiv$ #pigeons and #friends $\equiv$ #holes. There are 5 possible values for the amount of friends one could have, $\{0, 1, 2, 3, 4\}$, but you can never have an individual with 0 friends, and 4 friends in the same group. So there are 5 people, and 4 possible #friend values (think "holes.") Therefore, by pigeonhole principle, the statement is true!

Question 3: True or False?

A plane is coloured blue and red. Is it possible to find exactly two points the same colour exactly one unit apart?

Note:-
We will answer this on Wednesday!

Recursion

Example 1.1.4 (The Tower of Hanoi)
Given: a tower of 8 discs in decreasing size on one of three pegs.
Problem: transfer the entire towert to one of the other pegs.
Rule 1: Move only one disc at a time.
Rule 2: Never move a larger disc onto a smaller disk.

1. Is there a solution?

2. What's the minimal number of moves necessary and sufficient for the task?

A key idea is to generalise! What if there are n discs? Let T_n be the minimal number of moves, then trivially $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, so what is $T_3 =?$. Is there a pattern? The winning strategy is

1. Move the $n-1$ smallest discs from peg A to B.
2. Move the big disc from A to C.
3. Move $n-!$ smallest discs from B to C

By induction we show that

$$T_n = 2T_{n-1} + 1.$$

So $T_3 = 7$, $T_4 = 15$, $T_5 = 31$, $T_6 = 63$. Remarkably, this is one less then the sqaure numbers! We will prove this fact by induction later in the course.

Note:-

On Wednesday we start proberly.
Read:

Pages 23-36 (Epp, 4th) or
Pages 37-50 (Epp, 5th).

Watch the first video on UQ Extend, try the first quiz before Wednesday's lecture!

1.2 Lecture 2

Definition 1.2.1: Statement or Proposition

A sentence that is either true or false but not both.

Example 1.2.1

Statements:

- The number 6 is a number.
- $\pi > 3$
- Euler was born in 1707.

Not statements:

- How are you? (This is a question.)
- Stop! (This is a command.)
- She likes math. ("She" is not well defined.)
- $x^2 = 2x - 1$ (x is not well defined.)

Definition 1.2.2: Negation

Let p be a statement. The negation is of p is denoted $\sim p$ or $\neg p$ and is read "not p." It is defined as in the following truth table:

p	$\sim p$
T	F
F	T

Definition 1.2.3: Conjuction

Let p and q be statements. The conjuction of p and q is denoted $p \wedge q$ and is read "p and q." It is defined as in the following truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.2.4: Disjunction

Let p and q be statements. The disjunction of p and q is denoted $p \vee q$ and is read "p or q." It is defined as in the following truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 1.2.5: Logical Equivalence

Two statements, p and q are said to be logically equivalent if have identical truth values for every possible combination of truth values for their statement variables. This is denoted $p \equiv q$.

Example 1.2.2

$$\sim(\sim p) \equiv p.$$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Consider $P = \sim(p \wedge q)$, $Q = \sim p \wedge \sim q$ and $R = \sim p \vee \sim q$.

p	q	$\sim p$	$\sim q$	P	Q	R
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

$\therefore P \equiv R \not\equiv Q$.

Definition 1.2.6: Contradictions and Tautologies

A contradiction has truth values of false for every possible combination of its statement's truth values, and is denoted c or $\perp$. A tautology has truth values of true for every possible combination of its statement's truth values, and is denoted t or $\top$.

Example 1.2.3

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

$$\therefore p \wedge \sim p \equiv \top$$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

$$\therefore p \vee \sim p \equiv \bot$$

Important Laws of Logical Equivalence!

De Morgan's Law

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Commutativity

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$

Associativity

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$
$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

Distributivity

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Double Negative

$$\sim(\sim p) \equiv p$$

Idempotent

$$p \wedge p \equiv p$$
$$p \vee p \equiv p$$

Absorbtion

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

Identity Laws

$$p \wedge \top \equiv p$$
$$p \vee \bot \equiv p$$

Domination

$$p \vee \top \equiv \top$$
$$p \wedge \bot \equiv \bot$$

Negation Laws

$$p \vee \sim p \equiv \top$$
$$p \wedge \sim p \equiv \bot$$

Negations

$$\sim\top \equiv \bot$$
$$\sim\bot \equiv \top$$

Example 1.2.4

Prove that $((p \wedge \sim q) \vee (p \wedge q)) \wedge q \equiv p \wedge q$.

$$\begin{aligned} ((p \wedge \sim q) \vee (p \wedge q)) \wedge q &\equiv (p \wedge (\sim q \vee q)) \wedge q && \text{(Distributivity)} \\ &\equiv (p \wedge \top) \wedge q && \text{(Negation Law)} \\ &\equiv p \wedge q && \text{(Identity)} \end{aligned}$$

□

Questions

Question 4

Which of the following are statements?

(a) "Is it going to rain tomobbrow?"

(b) "She is happy."

(c) "23 July 2024 is a Tuesday"

(d) $x = 5y + 2$

(e) $65 < 2$

Solution: (a) No, a question. (b) No, "she" undefined. (c) Yes. (d) No, x, y undefined. (e) Yes.

Question 5

Let p, q, r be statements.

- $p =$ "it is cold."
- $q =$ "it is snowing."
- $r =$ "it is sunny."

Translate these to symbols:

(a) "It is not cold but it is snowing."

(b) "It is neither snowing nor cold, but it is sunny."

Translate these to English:

(c) $\sim p \wedge q$

(d) $(p \wedge q) \vee r$

Solution: (a) $\sim p \wedge q$ (b) $\sim p \wedge \sim q \wedge r$ (c) "It is not cold but it is snowing" (d) "It is either snowing and cold, or sunny, or it's both."

Question 6

Construct the truth table for $(p \wedge \sim q) \vee (q \wedge r)$

Solution:

p	q	r	$\sim q$	$p \wedge \sim q$	$q \wedge r$	$(p \wedge \sim q) \vee (q \wedge r)$
T	T	T	F	F	T	T
T	T	F	F	F	F	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	T	F	F	F	F	F
F	F	T	T	F	F	F
F	F	F	T	F	F	F

Question 7

Using De Morgan's Law, write down a statement which is logically equivalent to the negation of "5 is even and 6 is even."

Solution: "5 is even and 6 is even." $\equiv p \wedge q$. The solution we want is the negation, $\sim(p \wedge q)$, which, by De Morgan's Law is the same as $\sim p \vee \sim q$ which in English is "5 is odd or 6 is odd."

Question 8

Show that

$$\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \equiv p$$

using a truth table, and by laws of logical equivalence.

p	q	$\sim p$	$\sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$	$(\sim p \wedge q) \vee (\sim p \wedge \sim q)$	$\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q))$
T	T	F	F	F	F	F	T
T	F	F	T	F	F	F	T
F	T	T	F	T	F	T	F
F	F	T	T	F	T	T	F

$\therefore \sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \equiv p$ by exhaustion.

$$\begin{aligned}
\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) &\equiv \sim(\sim p \wedge q) \wedge \sim(\sim p \wedge \sim q) && \text{(De Morgan's Law)}\\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{(De Morgan's Law)}\\
&\equiv p \vee (q \wedge \sim q) && \text{(Distributivity)}\\
&\equiv p \vee \top && \text{(Negation Law)}\\
&\equiv p && \text{(Identity)}
\end{aligned}$$

□

$\therefore \sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \equiv p$ by logical equivalence.

1.3 Lecture 3

Definition 1.3.1: Conditional Statement

Let p and q be statement variables. The conditional form p to q is denoted $p \to q$, and read as "if p, then q," or "p implies q." It is defined by the following truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p is called the hypothesis.
q is called the conclusion.

Example 1.3.1

Suppose I make you the following promise:

"If you do your homework then you get a chocolate."

(a) You do not do your homework and you get a chocolate.

(b) You do your homework and you get a chocolate.

(c) You do your homework and you do not get a chocolate.

(d) You do not do your homework and you do not get a chocolate.

I only lied in scenario (c), which corresponds with $(p, q) = (F, T)$.

Note:-

$$p \to q \equiv \sim p \lor q$$

p	q	$\sim p$	$\sim p \lor q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	F

Definition 1.3.2: Contrapositive

The contrapositive of $p \to q$ is $\sim q \to \sim p$.

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$$p \to q \equiv \sim q \to \sim p$$

Example 1.3.2

The contrapositive of

"If you do your homework then you get a chocolate."

Is the equivalent

"If you did not get a chocolate then you did not finish your homework."

Negation of the Conditional Statement

The negation of $p \to q$ is given by $p \land \sim q$ and can be proved logcically.

$$\begin{aligned} p \to p &\equiv \sim p \lor q \\ \sim(p \to p) &\equiv \sim(\sim p \lor q) \\ &\equiv \sim(\sim p) \land \sim q \\ &\equiv p \land \sim q \end{aligned}$$

Example 1.3.3

The negation of

"If today is Monday, then tomorrow is my birthday"

Is

"Today is Monday but tomorrow is not my birthday."

Definition 1.3.3: Biconditional Statement

Let p and q be statement variables. The biconditional statement of p and q, denoted $p \leftrightarrow q$, and read "p if and only if q" is defined by the following truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
T	T	T

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Questions

Question 9

Which of the following sentences have the same meaning as "If I am worried then I did not sleep"?

(a) if I am worried then I do not sleep.

(b) if I am not worried then I do sleep.

(c) If I do not sleep then I am worried.

(d) I am worried and I do sleep.

(e) If I do sleep then I am not worried.

(f) I am worried or I do not sleep

(g) I do not sleep or I am not worried.

Solution:

- Original: $p \to \sim q$
- (a): $p \to \sim q$, equivalent.
- (b): $\sim p \to q$, not equivalent.
- (c): $\sim q \to p$, not equivalent.
- (d): $p \wedge q$, not equivalent.
- (e): $q \to \sim p$, equivalent, the contrapositive.
- (f): $p \vee \sim q$, not equivalent.
- (g): $\sim q \vee \sim p$, equivalent, logically equivalent.

Question 10

Express the operations $\vee$, $\to$, and $\leftrightarrow$ using only $\sim$ and $\wedge$.

Solution:

$$\begin{aligned} p \vee q &\equiv \sim(\sim(p \vee q)) \\ &\equiv \sim(\sim p \wedge \sim q) \\ p \to q &\equiv \sim p \vee q \\ &\equiv \sim(\sim(\sim p \vee q)) \end{aligned}$$

$$
\begin{aligned}
&\equiv \sim(p \wedge \sim q) \\
p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) \\
&\equiv (\sim p \vee q) \wedge (\sim q \vee p) \\
&\equiv \sim(\sim(\sim p \vee q)) \wedge \sim(\sim(\sim q \vee p)) \\
&\equiv \sim(p \wedge \sim q) \wedge \sim(q \wedge \sim p)
\end{aligned}
$$

Question 11: Challenge

Consider the NAND operation $p \barwedge q \equiv \sim(p \wedge q)$ can you express $\wedge$, $\vee$, $\sim$, and $\to$ using only $\barwedge$ operations? Can you express using only $\sim$ and $\oplus$?

Solution: Generating expressions using only NANDs:

$$
\begin{aligned}
\sim p &\equiv \sim(p \wedge p) \\
&\equiv p \barwedge p \\
p \wedge q &\equiv (p \wedge q) \wedge (p \wedge q) \\
&\equiv \sim(\sim(p \wedge q) \wedge \sim(p \wedge q)) \\
&\equiv \sim((p \barwedge q) \wedge (p \barwedge q)) \\
&\equiv (p \barwedge q) \barwedge (p \barwedge q) \\
p \vee q &\equiv \sim(\sim(p \vee q)) \\
&\equiv \sim(\sim p \wedge \sim q) \\
&\equiv \sim p \barwedge \sim q \\
&\equiv (p \barwedge p) \barwedge (q \barwedge q) \\
p \to q &\equiv \sim p \vee q \\
&\equiv \sim(\sim(\sim p \vee q)) \\
&\equiv \sim(p \wedge \sim q) \\
&\equiv p \barwedge \sim q \\
&\equiv p \barwedge (q \barwedge q)
\end{aligned}
$$

These sick fucks had me testing and observing truth tables for two hours. I was suspicious at times, but I assumed there must be a solution...
There is not. No matter how many XOR and NOT operations you apply, ultimately, you will always have 2 Falses and 2 Trues, or 4 Falses, or 4 Trues. **AHHHHHHHH**.

Question 12

Show that $\sim(p \to q) \not\equiv \sim p \to \sim q$.

Solution: By counterexample. Suppose $p = \text{True}$ & $q = \text{True}$

$$
\begin{aligned}
\sim(p \to q) &\equiv \sim(\text{True} \to \text{True}) \\
&\equiv \sim(\text{True}) \\
&\equiv \text{False} \\
\sim p \to \sim q &\equiv \sim\text{True} \to \sim\text{True} \\
&\equiv \text{False} \to \text{False} \\
&\equiv \text{True} \not\equiv \sim(p \to q)
\end{aligned}
$$
□

Question 13

Which of the following sentences have the opposite truth valuse as "If I am worried then I did not sleep"?

(a) if I am worried then I do not sleep.

(b) if I am not worried then I do sleep.

(c) If I do not sleep then I am worried.

(d) I am worried and I do sleep.

(e) If I do sleep then I am not worried.

(f) I am worried or I do not sleep

(g) I do not sleep or I am not worried.

Solution:

- Original: $p \to \sim q$
- (a): $p \to \sim q$, No, equivalent statement.
- (b): $\sim p \to q$, No, True 3/4 times, same as our statement. Not possible to be opposite.
- (c): $\sim q \to p$, No, True 3/4 times.
- (d): $p \wedge q$, Yes! $p \to \sim q \equiv \sim p \vee \sim q \equiv \sim(p \wedge q)$, exactly the opposite.
- (e): $q \to \sim p$, No, equivalent statement.
- (f): $p \vee \sim q$, No, True 3/4 times.
- (g): $\sim q \vee \sim p$, No, equivalent statement.

Question 14

Show that

$$p \to (q \vee r) \equiv (p \wedge \sim q) \to r.$$

Solution:

$$\begin{aligned} p \to (q \vee r) &\equiv \sim p \vee (q \vee r) \\ &\equiv (\sim p \vee q) \vee r \\ &\equiv \sim(\sim(\sim p \vee q)) \vee r \\ &\equiv \sim(p \wedge \sim q) \vee r \\ &\equiv (p \wedge q) \to r \end{aligned}$$

Question 15

Let n be a positive integer. Find conditions that are:

(a) necessary, but not sufficent for n to be a multiple of 10.

(b) sufficient but not necessary for n to be divisible by 10.

(c) necessary and sufficient for n to be divisible by 10.

Solution:

(a) n is a multiple of $10 \to n$ is necessarily even.

(b) n is a multiple of $50 \to$ sufficient to conclude that n is divisible by 10.

(c) n's last digit is a $0 \leftrightarrow n$ is divisible by 10.

Chapter 2

Week 2

2.1 Lecture 4