MATH2100 Assignment 1

Michael Kasumagic, 44302669
Professor Ivana Carrizo Molina

Due: 13:00, 18 August 2025

Problem 1

(a) **(3 marks)** Create the following functions:

$$f(x) = x - 3 + e^{2x} \qquad h(x) = x^4 + 2x^3 + x + 1$$

then compute and simplify $h(2) - f(4)$ to 6 s.d., and $h(f(x)) - f(h(x))$.

(b) **(2 marks)** Define a function that represents the distance from (x, y) to $(3, 4)$ and compute the value of the function at $(3, 5)$.

(c) **(3 marks)** Use `Solve` to find solutions for x, y, z in the system

$$\begin{cases} w + x + y + z = 4 \\ w + 2x + 3y + 4z = 5 \\ 2w + x - y - 2z = 0 \end{cases}$$

then determine solutions when $w = 0, 1$, and -1.

(d) **(11 marks)** Let $p(x) = -x^4 + 2x^3 - 5x^2 + 4x + 1$

- (i) Create $p(x)$ as a function.
- (ii) Using `Solve` and `NSolve` solve $p(x) = 0$.
- (iii) Plot $p(x)$ for $x \in [2, -2]$, using `PlotRange`$\rightarrow \{-50, 20\}$
- (iv) Find $p'(x)$ and identify the critical points
- (v) Find a numerical approximation of the integral of $p(x)$ from -1 to 1, to 3 s.f..
- (vi) Using different colours, plot $p'(x)$ and $p''(x)$ for $x \in [-2, 2]$. Then combine the plots with the one from (v) with `Show`.

Solution (a):

```
In[1] := f[x_] := x - 3 + Exp[2 x]
In[2] := h[x_] := x^4 + 2 x^3 + x + 1
In[3] := N[h[2] - f[4], 6]
Out[3] = -2946.96
In[4] := h[f[x]] - f[h[x]]
Out[4] = Exp[2 x] - Exp[2 (1 + x + 2 x^3 + x^4)] - 2 x^3 - x^4 +
          2 (-3 + Exp[2 x] + x)^3 + (-3 + Exp[2 x] + x)^4
```

Solution (b):

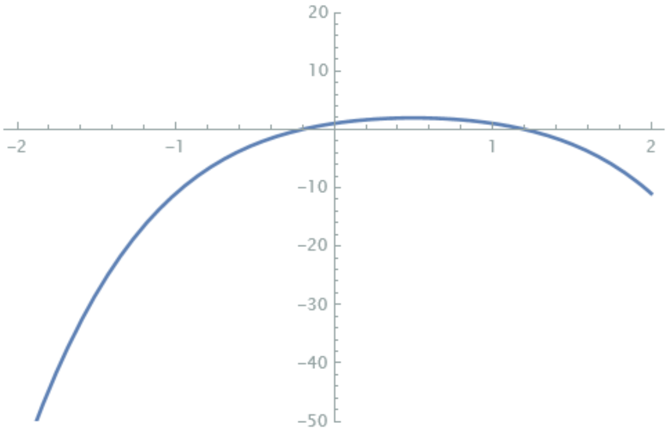
```
In[1] := dist[x_, y_] := Sqrt[(3 - x)^2 + (4 - y)^2]
In[2] := dist[-3, 5]
Out[2] = Sqrt[37]
```

Solution (c):

```
In[1] := sol1c =
          Solve[{w + x + y + z == 4, w + 2 x + 3 y + 4 z == 5,
            2 w + x - y - 2 z == 0}, {x, y, z}]
Out[1] = {{x -> -3 - w, y -> 17 - w, z -> -10 + w}}
In[2] := sol1c /. w -> 0
In[3] := sol1c /. w -> 1
In[4] := sol1c /. w -> -1
Out[2] = {{x -> -3, y -> 17, z -> -10}}
Out[3] = {{x -> -4, y -> 16, z -> -9}}
Out[4] = {{x -> -2, y -> 18, z -> -11}}
```

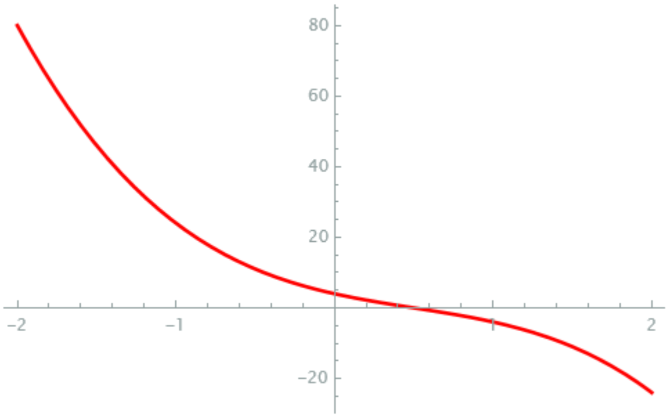
Solution (d):

```
(i)
In[1] := p[x_] := -x^4 + 2 x^3 - 5 x^2 + 4 x + 1

(ii)
In[2] := Solve[{p[x] == 0}, {x}]
Out[2] = {{x -> 1/2 (1 - Sqrt[-7 + 4 Sqrt[5]])},
          {x -> 1/2 (1 + Sqrt[-7 + 4 Sqrt[5]])},
          {x -> 1/2 (1 - i Sqrt[7 + 4 Sqrt[5]])},
          {x -> 1/2 (1 + i Sqrt[7 + 4 Sqrt[5]])}}
In[3] := NSolve[{p[x] == 0}, {x}]
Out[3] = {{x -> -0.197186}, {x -> 0.5 - 1.99651 I},
          {x -> 0.5 + 1.99651 I}, {x -> 1.19719}}

(iii)
In[4] := Plot[p[x], {x, -2, 2}, PlotRange -> {-50, 20}]
Out[4] =
```

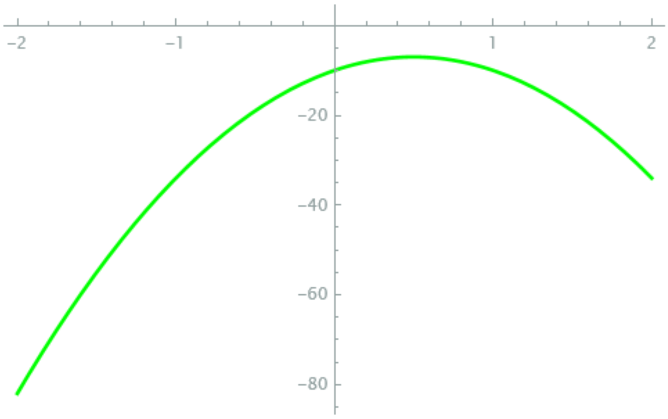



```
(iv)
In[5] := p'[x]
Out[5] = 4 - 10 x + 6 x^2 - 4 x^3
In[6] := NSolve[{p'[x] == 0}, {x}]
Out[6] = {{x -> 0.5}, {x -> 0.5 - 1.32288 i}, {x -> 0.5 + 1.32288 i}}

(v)
In[7] := NumberForm[NIntegrate[p[x], {x, -1, 1}], 3]
Out[7] = -1.73
```

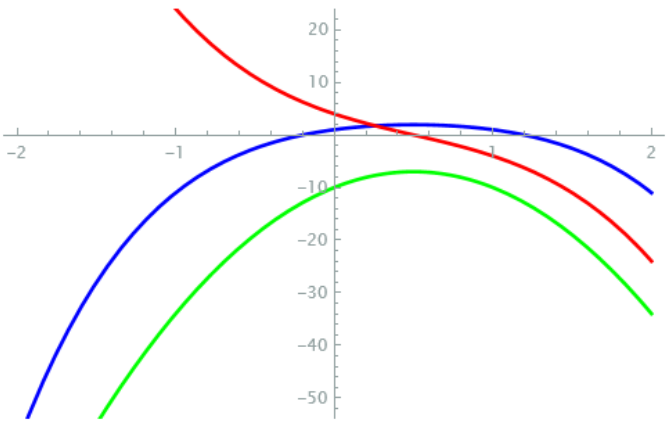
```
(vi)
In[8]  :=  Plot[p'[x], {x, -2, 2}, PlotStyle -> Red]
Out[8]  =
```




```
In[9]  :=  Plot[p''[x], {x, -2, 2}, PlotStyle -> Green]
Out[9]  =
```




```
In[10]  := Show[
            Plot[p[x], {x, -2, 2}, PlotStyle -> Blue],
            Plot[p'[x], {x, -2, 2}, PlotStyle -> Red],
            Plot[p''[x], {x, -2, 2}, PlotStyle -> Green],
            PlotRange -> {-50, 20}
           ]
 Out[10]=
```



Problem 2

(a) **(2 marks)** Let $f(y_1, y_2) = 2\sqrt{|y_1 + 2|}$. Determine whether $\frac{\partial f}{\partial y_1}$ exists.

(b) **(4 marks)** Examine the existence and uniqueness of solutions for the following IVP:

$$\begin{cases} \dot{y}_1 = \sqrt[4]{\left|\frac{(y_1+2)(y_2-1)}{2}\right|} \\ \dot{y}_2 = 2\sqrt{|y_1 + 2|} \end{cases} \qquad \begin{cases} y_1(0) = -2 \\ y_2(0) = 1 \end{cases}$$

(c) **(4 marks)** Consider the following vector-valued functions:

(i) Let

$$\mathbf{Y}_1(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

where

$$y_1(t) = \begin{cases} \frac{1}{4}t^2 - 2, & t \geq 0 \\ -\frac{1}{4}t^2 - 2, & t < 0 \end{cases} \quad \text{and} \quad y_2(t) = \begin{cases} \frac{1}{2}t^2 + 1, & t \geq 0 \\ -\frac{1}{2}t^2 + 1, & t < 0 \end{cases}.$$

Determine whether $\mathbf{Y}_1(t)$ satisfies the IVP from part (b) or whether it is only a solution to the system of ODEs.

(ii) $\forall t \in (-\infty, \infty)$ consider the constant vector function

$$\mathbf{Y}_2(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Does your analysis in part (b) support your conclusions?

Solution (a):

$$f(y_1, y_2) = \begin{cases} 2\sqrt{y_1 + 2} & , y_1 \geq -2 \\ 2\sqrt{-y_1 - 2} & , y_1 < -2 \end{cases}$$

$$\therefore \frac{\partial f}{\partial y_1} = \begin{cases} \dfrac{-1}{\sqrt{y_1 + 2}} & , y_1 \geq -2 \\ \dfrac{-1}{\sqrt{-y_1 - 2}} & , y_1 < -2 \end{cases}$$

$$\begin{aligned}
\lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h} &= \lim_{h \to 0^+} \frac{2\sqrt{-2+h+2} - 2\sqrt{-2+2}}{h} \\
&= \lim_{h \to 0^+} \frac{2\sqrt{h}}{h} = +\infty \\
\lim_{h \to 0^-} \frac{f(-2+h) - f(-2)}{h} &= \lim_{h \to 0^-} \frac{2\sqrt{-(-2+h)-2} - 2\sqrt{-(-2)-2}}{h} \\
&= \lim_{h \to 0^-} \frac{2\sqrt{-h}}{h} = -\infty \\
&\neq \lim_{h \to 0^+} \frac{f(-2+h) - f(-2)}{h}
\end{aligned}$$

Therefore, the $\frac{\partial f}{\partial y_1}$ does not exisit when $y_1 = -2$.
So, the partial derivative of f with respect to y_1 exists $\forall y_1, y_2 \in \mathbb{R}$, with $y_1 \neq -2$.

Solution (b):
Let $u := y_1 + 2$.
Let $v := y_2 - 1$.
Then we can rewrite the system like so:

$$\begin{cases} \dot{u} = \sqrt[4]{\left|\frac{uv}{2}\right|} = 2^{-1/4}(uv)^{1/4} \\ \dot{v} = 2\sqrt{|u|} = 2u^{1/2} \end{cases} \qquad \begin{cases} u(0) = 0 \\ v(0) = 0 \end{cases}$$

Note that we can drop the absolute here because for $t \geq 0$, u and v are strictly increasing. Next we'll consider the differential

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{u'}{v'} = \frac{2u^{1/2}}{2^{-1/4}(uv)^{1/4}} = 2^{5/4}u^{1/4}v^{-1/4}$$

We'll then separate and integrate,

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}u} &= 2^{5/4}u^{1/4}v^{-1/4} \\ v^{1/4}\frac{\mathrm{d}v}{\mathrm{d}u} &= 2^{5/4}u^{1/4} \\ \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{4}{5}v^{5/4}\right) &= 2^{5/4}u^{1/4} \\ \int \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{4}{5}v^{5/4}\right)\mathrm{d}u &= \int 2^{5/4}u^{1/4}\mathrm{d}u \\ \frac{4}{5}v^{5/4} &= \frac{4}{5}\cdot 2^{5/4}u^{5/4} + C \\ v^{5/4} &= 2^{5/4}u^{5/4} + C \\ v^{5/4} &= (2u)^{5/4} + C \\ v &= 2u + C \end{aligned}$$

Considering the initial condition, $u(0) = v(0) = 0$, we see that

$$0 = 2 \cdot 0 + C \iff C = 0$$

Let's now substitute $v = 2u$ back into u'.

$$u' = 2^{-1/4}(u \cdot 2u)^{1/4} = 2^{-1/4}2^{1/4}u^{1/2} = \sqrt{u},$$

Solving the IVP, $u' = \sqrt{u},\ u(0) = 0$ yields $u(t) = \frac{1}{4}t^2$, and we still have $v(t) = 2u(t)$.
We can finally substitute back into the original system,

$$y_1(t) = -2 + u(t), \qquad y_2(t) = 1 + 2u(t)$$

On Existence: $f(y_1, y_2)$ is continuous at its initial condition, (-2,1). Therefore, by Existence Theorem, at lease one solution exists.
On Uniqueness: For the solution to be unique, $\frac{\partial f}{\partial y_{1,2}}$ must be continuous, but as we established in 2(a), the partial derivative $\frac{\partial f}{\partial y_2}$ is discontinuous at $y_2 = -2$. Therefore, the Uniqueness Theorem fails and we can conlude that there are many non-unique solutions.

Problem 3

(a) **(2 marks)** Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2y(x) + 4\cos(2x)$ with IV $y(\pi/4) = 1$ by hand.

(b) **(2 marks)** Use `DSolve` to find the solution to this IVP. Compare the result with the solution you found by hand and plot the solution over the interval $[10, 10]$ and use the option `PlotRange->{12, 12}`.

(c) **(2 marks** Plot the vector field of the differential equation along with the particular solution corresponding to the initial condition $y(\pi/4) = 1$.

Solution (a):

$$\begin{aligned} y' - 2y &= 4\cos(2x) \\ y' + p(x)y &= q(x) \end{aligned}$$

Is a linear first order ODE, so let's find an integrating factor,

$$\begin{aligned} \mu(x) &= \exp\left(\int p(x)\mathrm{d}x\right) \\ &= \exp\left(\int -2\mathrm{d}x\right) \\ &= \exp(-2x) && \text{(Take +C=0)} \end{aligned}$$

Multiply through,

$$\begin{aligned} \mu(x)y' - \mu(x)2y &= \mu(x)4\cos(2x) \\ e^{-2x}y' - 2e^{-2x}y &= 4e^{-2x}\cos(2x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-2x}y\right) &= 4e^{-2x}\cos(2x) \\ \int \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-2x}y\right)\mathrm{d}x &= \int 4e^{-2x}\cos(2x)\mathrm{d}x \\ e^{-2x}y &= 4\left(\frac{e^{-2x}}{(-2)^2 + 2^2}\left(-2\cos(2x) + 2\sin(2x)\right) + C\right) \\ &= \left(\frac{e^{-2x}}{2}\left(-2\cos(2x) + 2\sin(2x)\right) + C\right) \\ &= e^{-2x}\left(\sin(2x) - \cos(2x)\right) + C \\ \therefore y(x) &= Ce^{2x} + \sin(2x) - \cos(2x) \end{aligned}$$

Now, we'll consider the inital value

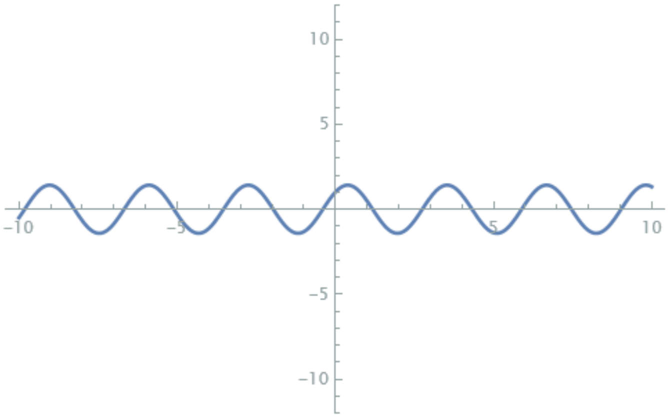
$$\begin{aligned} y\left(\frac{\pi}{4}\right) = 1 &= Ce^{2\left(\frac{\pi}{4}\right)} + \sin\left(2\left(\frac{\pi}{4}\right)\right) - \cos\left(2\left(\frac{\pi}{4}\right)\right) \\ &= Ce^{\frac{\pi}{2}} + \sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) \\ &= Ce^{\frac{\pi}{2}} + 1 - 0 \\ \therefore Ce^{\pi/2} &= 0 \iff C = 0 \end{aligned}$$

Hence, the final solution,

$$y(x) = \sin(2x) - \cos(2x)$$

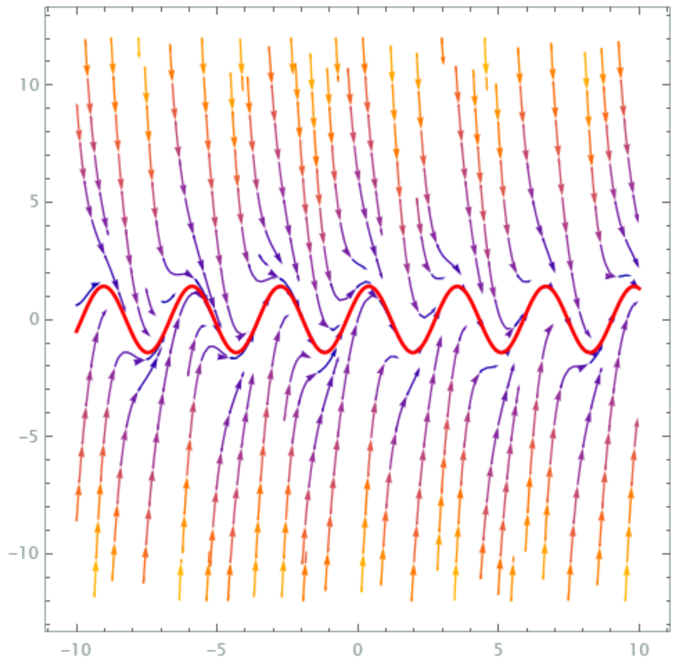
Solution (b):

```
In[1]  :=  sol3b =
            DSolve[{y'[x] == -2 y[x] + 4 Cos[2 x], y[Pi/4] == 1}, y[x], x]
Out[1]  =  {{y[x] -> Cos[2 x] + Sin[2 x]}}
In[2]  :=  Plot[y[x] /. sol3b, {x, -10, 10}, PlotRange -> {-12, 12}]
Out[2]  =
```



Solution (c):

```
In[1]  :=  Show[
            StreamPlot[{1, -2 y + 4 Cos[2 x]}, {x, -10, 10}, {y, -12, 12}],
            Plot[y[x] /. sol3b, {x, -10, 10}, PlotStyle -> {Thick, Red}]
           ]
Out[1]  =
```



Problem 4

(a) **(3 marks)** What are the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} a^2 & ab \\ a & b \end{pmatrix}$$

where a and b are real numbers?

(b) **(2 marks)** Find the general solution to

$$\dot{\mathbf{y}} = A\mathbf{y}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

with A as in part (a).

Solution (a):

$$\begin{aligned} 0 = \det(A - \lambda I) = \begin{vmatrix} a^2 - \lambda & ab \\ a & b - \lambda \end{vmatrix} &= (a^2 - \lambda)(b - \lambda) - (ab)(a) \\ &= a^2 b - a^2\lambda - b\lambda + \lambda^2 - a^2 b \\ &= \lambda^2 - a^2\lambda - b\lambda \\ &= \lambda(\lambda - a^2 - b) \\ &= \lambda(\lambda - a^2 - b) \\ &= 0 \\ \iff \text{eigenvals}\, A &= \left\{0, a^2 + b\right\} \end{aligned}$$

Take $\lambda_1 = 0$:

$$\begin{aligned} A\mathbf{x} = \lambda\mathbf{x} \iff \begin{pmatrix} a^2 & ab \\ a & b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} &= 0\begin{pmatrix} u \\ v \end{pmatrix} \\ \begin{pmatrix} a^2 u + abv \\ au + bv \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$\implies au = -bv \implies u = -\frac{b}{a}v,\ a \neq 0.$ Take $v = a$

$$\therefore \mathbf{v}_1 = \begin{pmatrix} -b \\ a \end{pmatrix}$$

Take $\lambda_2 = a^2 + b$

$$\begin{aligned} A\mathbf{x} = \lambda\mathbf{x} \iff \begin{pmatrix} a^2 & ab \\ a & b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} &= \left(a^2 + b\right)\begin{pmatrix} u \\ v \end{pmatrix} \\ \begin{pmatrix} a^2 u + abv \\ au + bv \end{pmatrix} &= \begin{pmatrix} a^2 u + bu \\ a^2 v + bv \end{pmatrix} \\ \begin{pmatrix} abv - bu \\ au - a^2 v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$\implies u = av,\ a \neq 0.$ Let $v = 1$

$$\therefore \mathbf{v}_2 = \begin{pmatrix} a \\ 1 \end{pmatrix}$$

$$\therefore \text{ eigenvals } A = \left\{0, a^2 + b\right\}$$
$$\therefore \text{ eigenvecs } A = \left\{\begin{pmatrix} -b \\ a \end{pmatrix}, \begin{pmatrix} a \\ 1 \end{pmatrix}\right\}$$

Solution (b):

$$\begin{aligned} \text{General solution: } y(t) &= \alpha \mathbf{v}_1 e^{\lambda_1 t} + \beta \mathbf{v}_2 e^{\lambda_2 t} \\ &= \alpha \begin{pmatrix} -b \\ a \end{pmatrix} e^{0t} + \beta \begin{pmatrix} a \\ 1 \end{pmatrix} e^{\left(a^2+b\right)t}, \ \ \alpha, \beta \in \mathbb{R} \\ &= \alpha \begin{pmatrix} -b \\ a \end{pmatrix} + \beta \begin{pmatrix} a \\ 1 \end{pmatrix} e^{\left(a^2+b\right)t} \end{aligned}$$