4.1. (a) Evaluate 58 div 11 and 58 mod 11.

(b) Find integers q and r, with $0 \leqslant r < d$ and $n = dq + r$, in the case that $n = -95$ and $d = 11$.

4.2. (a) An integer n, when divided by 7, leaves remainder 4. What is the remainder (between 0 and 6 inclusive) when $3n$ is divided by 7?

(b) When the integer m is divided by 11, the remainder 9 is left. What remainder is left when $4m$ is divided by 22?

4.3. (a) Assume that $a, b, c, d \in \mathbb{Z}$. Find a condition on these variables that ensures x is a real number, where

$$\frac{ax+b}{cx+d} = 2.$$

Then prove that provided this condition holds, x is a rational number.

(b) Prove, or else disprove with a counter-example, each of the following statements:

(i) For all real numbers x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.

(ii) For all odd integers n, $\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)$.

4.4. (a) Prove that there exists a *unique* prime number of the form $n^2 + 2n - 3$, where $n \in \mathbb{Z}^+$.

Hint: Find some value of n for which the given quadratic is prime. Then you must show uniqueness. (Can you factorise the quadratic? What do you know about factors of a prime number?) To show uniqueness, you can use a contradiction argument: suppose that there are two different values of n yielding different primes p and q.

(b) Suppose instead that the quadratic in part (a) was $n^2 + 2n + 3$, with $n \in \mathbb{Z}^+$. Does there exist a unique prime number of this form? Explain clearly.

4.5. Use the Euclidean Algorithm to determine each of the following:

(a) $\gcd(14, 3003)$

(b) $\gcd(18, 1028)$

(c) $\gcd(221, 255)$

4.6. (a) Write $(x^2 - 1) + (2x^3 - 1) + (3x^4 - 1) + (4x^5 - 1) + (5x^6 - 1)$ in summation notation.

(b) Evaluate $\prod_{i=1}^{5} (-1)^i \cdot i$.

(c) Write $(y-1)(y^2-2)(y^3-3)(y^4-4)$ in product notation.

(d) Evaluate $\sum_{j=1}^{n} (-1)^j \frac{1}{2^j}$ when $n = 5$.

4.7 Use mathematical induction to prove the following statements:

(a) For every positive integer n, $n(n+1)$ is even.

(b) For every integer $n \geq 2$, $n^3 - n$ is a multiple of 6.

(c) For all integers $n \geq 2$, $\sqrt{n} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$.

(d) For all integers $n \geq 1$, $\sum_{i=1}^{n} (3i-1)^2 = \frac{1}{2} n(6n^2 + 3n - 1)$.

Here are two **puzzles** that you can think about during week 5. Feel free to ask your tutors or lecturer for more hints!

G. A hat contains 42 slips of paper, numbered $1, 2, \ldots, 42$ respectively. Two slips are drawn at random from the hat, and the difference of their numbers is written on a new slip which is put in the hat while the two old slips are destroyed. This procedure is repeated until the hat contains only one slip. Show that this last slip bears an odd number.

H. Suppose you have a pizza and a knife. Into how many pieces can you cut the pizza with n straight cuts? If you cut the pizza in the usual way, using diameters, you get $2n$ pieces. If all of your cuts are parallel, you get $n+1$ pieces.

What is the maximal number of pieces into which you can cut your pizza using n cuts?

Hint: This problem is similar to the Tower of Hanoi problem: you can first find a recursion for the maximal number, and then a closed form.

Extra practice questions from the textbook (Solutions at the back of the book.)

Epp 5th ed.:
Section 4.5, pp. 209–211: Questions 1, 3, 5, 7, 13, 16, 20, 23, 32, 37, 39.
Section 4.6, pp. 217–218: Questions 1, 3, 8, 13, 14, 15, 23, 26.
Section 4.10, pp. 255–257: Questions 9, 10, 13, 14, 22.
Section 5.1, pp. 273–275: Questions 1, 3, 5, 10, 11, 12, 14, 18abcd, 19, 20, 23, 27, 29, 31, 37, 40, 43, 46, 47, 49, 51, 53, 55, 56, 59, 62, 65, 66, 68, 69.
Section 5.2, pp. 286–289: Questions 2, 6, 8, 10, 13, 15, 20, 22, 24, 25, 28.
Section 5.3, pp. 297–300: Questions 8, 11, 13, 14, 16, 19.

Epp 4th ed.:
Section 4.4, pp. 189–190: Questions 1, 3, 5, 7, 13, 16, 20, 23, 28, 32, 38, 39, 42.
Section 4.5, p. 197: Questions 1, 3, 8, 10, 12, 14, 15, 23, 26, 30.
Section 4.8, p. 225: Questions 9, 10, 13, 14, 19.
Section 5.1, pp. 242–244: Questions 1, 3, 5, 10, 11, 12, 14, 18abcd, 19, 20, 23, 27, 29, 31, 37, 40, 43, 46, 47, 49, 51, 53, 55, 56, 59, 62, 65, 66, 68, 69.
Section 5.2, pp. 256–258: Questions 3, 6, 8, 10, 13, 15, 20, 22, 24, 25, 28.
Section 5.3, pp. 266–268: Questions 8, 11, 13, 14, 16, 19.

Puzzle hints:

Stuck on the puzzles from week 4?

E. If it were true that $12 \mid n$, then what other integers d in the range $1, \ldots, 25$ would also have to satisfy $d \mid n$? If it were true that $4 \mid n$ *and* $5 \mid n$, then what other integers d in the range $1, \ldots, 25$ would also have to satisfy $d \mid n$? Look for more relationships of this type.

F. Look at the example $\frac{3}{8} = \frac{1}{3} + \frac{1}{24}$. You can rewrite this as $\frac{3}{8} = \frac{8}{24} + \frac{1}{24}$; notice that 8 and 1 are both distinct common divisors of the common denominator 24.