MATH2001
Calculus & Linear Algebra II

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Contents

Chapter 1

Week 1

1.1 Lecture 1

In this course we will cover four major topics:

- Ordinary Differential Equations
- Linear Algebra
- Vector Calculus
- Integral Calculus

1.1.1 Solutions to First Order ODEs

We are comfortable solving three types of first order ODEs by now:

- Directly integrable: $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$

$$y(x) = \int f(x)\mathrm{d}x = F(x) + C$$

- Seperable: $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$

$$\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \iff \int \frac{1}{g(y)}\frac{\mathrm{d}y}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int f(x)\mathrm{d}x \iff G(y(x)) = F(x) + C$$

If G is invertible, then $y(x) = G^{-1}(F(x) + C)$

- Linear: $\frac{\mathrm{d}y}{\mathrm{d}x} = q(x) - p(x)y$

$$\text{Let } \mu = \exp\left(\int p(x)\mathrm{d}x\right) \implies \mu\frac{\mathrm{d}y}{\mathrm{d}x} + \mu p(x)y = \mu q(x) \iff \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu q(x) \iff y(x) = \frac{1}{\mu(x)}\int \mu q(x)\mathrm{d}x$$

In many applications, we need to solve an IVP. In general this is an equation of form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \quad y(x_0) = y_0.$$

In other words, we seek to find solutions to the ODE which pass through the point (x_0, y_0) in the x−y plane.

Example 1.1.1

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x, \quad y(0) = 1 \text{ has a unique solution:}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x \\
y(x) &= \frac{1}{2}x^2 + C \\
\text{Impose } y(0) &= 1 \\
\therefore 1 &= \frac{1}{2}(0)^2 + C \\
\therefore C &= 1 \\
\therefore y(x) &= \frac{1}{2}x^2 + 1
\end{aligned}$$

Example 1.1.2

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3xy^{1/3}, \quad y(0) = 0 \text{ has more than one solution:}$$

$$\begin{aligned}
y^{-1/3}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x \\
\int y^{-1/3}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= \int 3x\mathrm{d}x \\
\int y^{-1/3}\mathrm{d}y &= \int 3x\mathrm{d}x \\
\frac{3}{2}y^{2/3} + C_1 &= \frac{3}{2}x^2 + C_2 \\
y^{2/3} &= x^2 + C \\
\text{Impose } y(0) &= 0 \\
0^{2/3} &= 0^2 + C \\
\implies C &= 0 \\
\therefore y^{2/3} &= x^2 \\
\therefore y &= \pm x^3
\end{aligned}$$

This is problematic. Our inital value constraint hasn't allowed us to pick one particular solution.

Note:-

The previous IVP has multiple solutions because $f(x, y) = 3xy^{1/3}$ is not differentiable at $y = 0$.

Example 1.1.3

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x}, \quad y(0) = 1 \text{ has no solutions:}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{x} - \frac{1}{x}y \\
&= q(x) - p(x)y \\
\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 1
\end{aligned}$$

$$\begin{aligned}
\mu &= \exp\left(\int p(x)\mathrm{d}x\right) \\
&= \exp\left(\int \frac{1}{x}\mathrm{d}x\right) \\
&= \exp(\ln(x)) \\
&= x
\end{aligned}$$

$$\begin{aligned}
\mu\frac{\mathrm{d}y}{\mathrm{d}x} + \mu p(x)y &= \mu \\
x\frac{\mathrm{d}y}{\mathrm{d}x} + y &= x \\
\frac{\mathrm{d}}{\mathrm{d}x}(xy) &= x \\
xy &= \int x\mathrm{d}x \\
&= \frac{1}{2}x^2 + C
\end{aligned}$$

Impose $y(0) = 1$

$$\begin{aligned}
\therefore 0 \cdot 1 &= \frac{1}{2}(0)^2 + C \\
C &= 0 \\
\therefore y(x) &= \frac{1}{2}x
\end{aligned}$$

However, our general solution **does not** satisfy our inital value constraint, $y(0) = \frac{1}{2}(0) = 0 \neq 1$.

Note:-

Our IVP doesn't have a solution because $f(x,y) = \frac{x-y}{x}$ is not differentiable or continuous around $x = 0$.

We're kind of loosley referring to "existence and uniqueness" theorems, or Picard-Lindelöf Theorem, which generally states:

$$\text{The IVP } \frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) \quad y(x_0) = y_0$$

has a unique solution around x_0 if:

1. $f(x,y)$ is continuous around (x_0, y_0)
2. $f(x,y)$ is differentiable with respect to y around (x_0, y_0), ie $\frac{\partial f}{\partial y}$ is continuous aorund (x_0, y_0).

1.1.2 Existence and Uniqueness

Theorem 1.1.1

Consider the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \quad y(x_0) = y_0$$

We are concerned with the conditions under which a solution exists and is unique.

1. (existence, Peano's Theorem) If $f(x, y)$ is continuous in some rectangle

$$R = \{(x, y) \mid |x - x_0| < a,\ |y - y_0| < b\}$$

then the IVP has at least one solution.

2. (uniqueness, Picard's Theorem) If $f_y(x, y) := \frac{\partial f}{\partial y}$ is also continuous in R then there is some interval $|x - x_0| \leq h \leq a$ which contains at least one solution.

This result only tells us that a solution exists or is unique locally. Beyond R, we simply don't know.

Example 1.1.4

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x, \quad y(0) = 1$$

$$f(x, y) = x$$
$$f_y(x, y) = 0$$

These functions are both continuous over $\mathbb{R}^2$. Therefore there exists a unique solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3xy^{1/3}, \quad y(0) = 0 \text{ has more than one solution:}$$

$$f(x, y) = 3xy^{1/3}$$
$$f_y(x, y) = xy^{-2/3}$$

$f(x, y)$ is continuous over $\mathbb{R}^2$ so there exists at least one solution. However, f_y has a discontinuity at $y = 0$, so there may or may not be unique solutions (remember, its not an iff).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x}, \quad y(0) = 1 \text{ has no solutions:}$$

$$f(x, y) = \frac{x - y}{x}$$
$$f_y(x, y) = -\frac{1}{x}$$

$f(x, y)$ and f_y both have discontinuities when $x = 0$, so we don't know from this test if there are solutions, or if the solution is unique.

Note:-

These theorems are not if and only if's. They can fail. For example, take the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}x^{-2/3}, \quad y(0) = 1/$$

We see

$$f(x, y) = \frac{1}{3}x^{-2/3}, \qquad f_y(x, y) = 0$$

f has a discontinuity when $x = 0$, so the theorem's fail to identify if this IVP has solutions. However, this

IVP **does** have a unique solution,

$$y(x) = x^{1/3} + 1,$$

so we need to be careful we're using these theorems correctly. **If** f and f_y are continuous in some reigon **then** there exists a unique solution in that reigon.

Example 1.1.5

Solve these:

1. $y' = y^{2/3}, \quad y(0) = 1$

$$f(x, y) = y^{2/3}$$
$$f_y(x, y) = \frac{2}{3} y^{-1/3}$$

Therefore there exist at least one solution to the IVP.

$$\begin{aligned} y^{-2/3} y' &= 1 \\ \int y^{-2/3} \mathrm{d}y &= \int 1 \mathrm{d}x \\ 3y^{1/3} &= x + C \\ y^{1/3} &= \frac{1}{3}(x + C) \\ y &= \frac{1}{27}(x + C)^3 \\ \text{Impose } y(0) &= 1 \end{aligned}$$

Imposing the IVP and expanding the cubic expression, will reveal 3 values for C, the nicest of which is 3. The one which satisfies our IVP is

$$y(x) = \frac{1}{27}(x + 3)^3$$

Even though those other solutions exist, only one satisfies the IVP, hence this solution is unique.

2. $y' = (3x^2 + 4x + 2)/(2y - 2), \quad y(0) = 1$

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

Because of the discontinuity at $y = 1$, our exisitence theorem fails to identify if solutions exist.

$$\begin{aligned} y' &= \frac{3x^2 + 4x + 2}{2(y - 1)} \\ 2(y - 1)y' &= 3x^2 + 4x + 2 \\ 2\int y - 1 \mathrm{d}y &= \int 3x^2 + 4x + 2 \,\mathrm{d}x \\ y^2 - 2y &= x^3 + 2x^2 + 2x + C \\ y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + C + 1 \\ (y - 1)^2 &= x^3 + 2x^2 + 2x + C + 1 \\ \text{Impose } y(0) &= 1 \\ ((1) - 1)^2 &= (0)^3 + 2(0)^2 + 2(0) + C + 1 \\ \iff C &= -1 \\ \therefore (y - 1)^2 &= x^3 + 2x^2 + 2x \\ \therefore y(x) &= 1 \pm \sqrt{x^3 + 2x^2 + 2x} \end{aligned}$$

The IVP has two solutions.

1.1.3 Method of Succesive Approximations

To start, we note that it is always possible to apply a variable shift and so that the IVP is expressed:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(0) = 0$$

Example 1.1.6

$$y' = 2(x-1)(y-1), y(1) = 2$$

$$\begin{aligned} \text{Let } \bar{x} &= x - 1 \\ \text{Let } \bar{y} &= y - 2 \\ \text{So } \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} \\ \implies \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} &= 2\bar{x}(\bar{y}+1),\ \bar{y}(0) = 0 \end{aligned}$$

Without loss of generality we will consider this problem where the inital point is at the origin. We can restate the previous theorem 1.1.1 as follows

Theorem 1.1.2

If f and f_y are continuous in some rectangle

$$R = \{(x, y) \mid |x| \leq a,\ |y| \leq b\},$$

then there is some interval $|x| \leq h \leq a$ which contains a uniuqe solution $y = \phi(x)$ of the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(0) = 0$$

Equivilance with integral equation

Let $y = \phi(x)$ be the solution to the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(0) = 0, \tag{1}$$

and note that the function $F(x) = f(x, \phi(x))$ is a continuous function of x only. We then have

$$\phi(x) = \int_0^x F(t)\mathrm{d}t = \int_0^x f(t, \phi(t))\mathrm{d}t. \tag{2}$$

Note that $\phi(0) = 0$. This is an example of an *integral equation*. Conversely, let $\phi(x)$ satisfy the integral equation (2). By the Fundamental Theorem of Integral Calculus, $\phi'(x) = f(x, \phi(x))$, which implies that $y = \phi(x)$ is a solution of the IVP (1). In other words, the IVP (1) and the integral equation (2) are equivalent, meaning that a solution of one is a solution of the other. Herein we work with (2).

Method of succesive approximations

The goal of the approach is to generate a sequence of functions $\phi_0, \phi_1, \ldots, \phi_n, \ldots$. Starting with the initial function $\phi_0(x) = 0$ (satisfying the initial condition of (1)), the sequence is generated iteratively by

$$\phi_{n+1}(x) = \int_0^x f(t, \phi_n(t))\mathrm{d}t. \tag{3}$$

Note that each ϕ_n satisfies $\phi_n(0) = 0$, but generally not the integral equation (2) itself. However, if there is a k, such that $\phi_{k+1}(x) = \phi_k(x)$, then $\phi_k(x)$ is a solution of the integral equation (2) and hence the IVP (1). Generally this does not occur, but we may instead consider *limit functions*.

There are 4 key points to consider:

1. Do all members of the sequence exist?
2. Does the sequence converge to a limit function ϕ?
3. What are the properties of ϕ?
4. If ϕ satisfies the IVP (1), are there other solutions?

Example 1.1.7

$$y' = 2x(y+1), \qquad y(0) = 0$$

$$\phi_0(x) = 0, \qquad f(x,y) = 2x(y+1)$$

$$\phi_1(x) = \int_0^x f(t,\phi_0(t))\mathrm{d}t = \int_0^x f(t,0)\mathrm{d}t = \int_0^x 2t(0+1)\mathrm{d}t = t^2\Big|_0^x = x^2$$

$$\phi_2(x) = \int_0^x f(t,\phi_1(t))\mathrm{d}t = \int_0^x f(t,t^2)\mathrm{d}t = \int_0^x 2t(t^2+1)\mathrm{d}t = \int_0^x 2t^3+2t\ \mathrm{d}t = \frac{1}{2}t^4+t^2\Big|_0^x = \frac{1}{2}x^4+x^2$$

Similarly,

$$\phi_3(x) = x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6$$

$$\phi_4(x) = x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8$$

Proposition.

$$\phi_n(x) = \sum_{i=1}^{n} \frac{1}{i!}x^{2i}$$

proof. By induction: True for $n=1$. Suppose True for $n=k$.

Then $\phi_{k+1}(x) = \int_0^x f(t,\phi_k(t))\mathrm{d}t = \int_0^x 2t\left(1+\sum_{i=1}^{k}\frac{1}{i!}t^{2i}\right)\mathrm{d}t = \int_0^x \left(2t+\sum_{i=1}^{k}\frac{2}{i!}t^{2i+1}\right)\mathrm{d}t = t^2+\sum_{i=2}^{k+1}\frac{1}{i!}t^{2i}\Big|_0^x$

$$\therefore \phi_{k+1} = x^2 + \sum_{i=2}^{k+1}\frac{1}{i!}x^{2i} = \sum_{i=1}^{k+1}\frac{1}{i!}x^{2i}$$

So the proposition is true $\forall n \in \mathbb{N}$. □

$$\lim_{n\to\infty}\phi_n(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i!}x^{2i} \text{ exists} \iff \text{the series converges}$$

Applying the ratio test between two successive terms, j and $j+1$, as j goes to infinty,

$$\lim_{j\to\infty}\left|\frac{\frac{x^{2j+2}}{(j+1)!}}{\frac{x^{2j}}{j!}}\right| = \lim_{j\to\infty}\left|\frac{x^{2j+2}}{(j+1)!}\cdot\frac{j!}{x^{2j}}\right| = \lim_{j\to\infty}\left|\frac{x^2}{j+1}\right| = 0$$

Therefore, the series converges!
Therefore, the limit, as $n\to\infty$ of ϕ_n exists.

1.1.4 Exact First Order ODEs

Definition 1.1.1: Exact First Order ODE

Recall that if $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t, then z is a differentiable function of t, whose derivative is given by the chain rule:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Now suppose the equation

$$f(x, y) = C$$

defines y implicitly as a function of x. Then $y = y(x)$ can be show to satisfy a first order ODE obtained by using the chain rule above. In this case, $z = f(x, y(x)) = C$, so,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}C = 0 = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\implies f_x + f_y y' = 0$$

A first order ODE of form

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is called exact if there is a function $f(x, y)$ such that

$$f_x(x, y) = P(x, y) \quad \text{and} \quad f_y(x, y) = Q(x, y).$$

The solution is then given implicitly by the equation

$$f(x, y) = C,$$

where C can usually be determined by some intial condition.

Theorem 1.1.3 Test for Exactness

Let $P, Q, \dfrac{\partial P}{\partial y}, \dfrac{\partial Q}{\partial x}$ be continuous over some reigon of interest. Then

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is an exact ODE if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

everywhere in the reigon

Proof. 1. Prove: ODE is exact $\implies \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
Recall Clairout's Theorem,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ if both } f_{xy} \text{ and } f_{yx} \text{ are continuous in the reigon.}$$

$$\text{Suppose ODE is exact} \implies \exists f(x, y) : \frac{\partial f}{\partial x} = P(x, y), \frac{\partial f}{\partial y} = Q(x, y)$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial x}, \text{ by Clairout's Theorem.}$$

2. Prove: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \implies$ ODE is exact.

Suppose $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. We seek a function f such that $f_x = P, f_y = Q$.

$$\text{Take } f(x,y) = \int_{x_0}^{x} P(x',y)\mathrm{d}x' + \int_{y_0}^{y} Q(x_0,y') + C$$
$$f_x(x,y) = \frac{\partial}{\partial x}\left(\int_{x_0}^{x} P(x',y)\mathrm{d}x' + \cancel{\int_{y_0}^{y} Q(x_0,y')\mathrm{d}y'}\right) = P(x,y)$$
$$f_y(x,y) = \frac{\partial}{\partial y}\left(\cancel{\int_{x_0}^{x} P(x',y)\mathrm{d}x'} + \int_{y_0}^{y} Q(x_0,y')\mathrm{d}y'\right) = Q(x,y)$$

Therefore $P(x,y) + Q(x,y)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ is an exact ODE $\iff \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ everywhere in the reigon. □

Example 1.1.8

$$\text{Solve the ODE } 2x + e^y + xe^y y' = 0$$

$$\begin{aligned}
P(x,y) &= 2x + e^y \\
\frac{\partial P}{\partial y} &= e^y \\
Q(x,y) &= xe^y \\
\frac{\partial Q}{\partial x} &= e^y \\
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} &\Rightarrow \text{ODE is exact} \\
\therefore \exists f(x,y) : f_x(x,y) &= P = 2x + e^y \\
\text{and } f_y(x,y) &= Q = xe^y \\
\implies f = \int P\mathrm{d}x &= \int 2x + e^y \mathrm{d}x \\
&= x^2 + xe^y + g(y) \\
\implies f_y(x,y) = xe^y &= \frac{\partial}{\partial y}\left(x^2 + xe^y + g(y)\right) \\
xe^y &= xe^y + \frac{\mathrm{d}g}{\mathrm{d}y} \\
\implies \frac{\mathrm{d}g}{\mathrm{d}y} &= 0 \\
\therefore f(x,y) &= x^2 + xe^y + C
\end{aligned}$$

All solutions to ODE: $f(x,y) = k$.

$$\begin{aligned}
\iff x^2 + xe^y &= k' && (k' = k - C) \\
\iff y &= \ln\left(\frac{k' - x^2}{x}\right)
\end{aligned}$$

1.2 Lecture 2

1.2.1 Almost Exact ODEs and Integrating Factors

Suppose we have

$$P(x,y) + Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}.$$

This is not an exact ODE, but can we do anything with it anyway? Let's consider an "integrating factor" (not to be confused with integrating factors used when solving linear ODEs).

The general idea though, is to multiple the ODE by some function, $h(x, y)$ such that the resulting ODE

$$h(x,y)P(x,y) + h(x,y)Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact. We know that this new ODE is exact if and only if

$$\frac{\partial}{\partial y}(hP) = \frac{\partial}{\partial x}(hQ)$$

Let's find a h which accomplishes this:

$$\frac{\partial}{\partial y}(hP) = \frac{\partial}{\partial x}(hQ)$$
$$h_y P + hP_y = h_x Q + hQ_x \iff h_y P - h_x Q + h(P_y - Q_x) = 0$$

Solving for h in general requires us to solve this first order partial differential equation, which is very nasty and also outside the scope of this course.

However, we can consider a simplier case, where h is a function of one of the varaibles x or y. Let's try $h = h(x)$:

$$\frac{\mathrm{d}h}{\mathrm{d}x} = h(x)\frac{P_y - Q_x}{Q} = h\hat{f}$$

Suppose $\hat{f}$ is a function of one variable, x. Then we are left with a first order seperable ODE which we can solve. Once we've solved for h, we can find $f(x, y)$ that solves the exact ODE we wanted to solve.

This is not a great technique, it often doesn't work and requires a lot of trial an error. For example, if $h = h(x)$ didn't yield an approriate $f(x, y)$, we could try $h = h(y)$ or $h = h(x) + h(y)$, which would also give us a seperable ODE to solve. If this technique does work, it hints to some underlying symmetry in the differential system we're solving.

Example 1.2.1

$$\text{Solve } (3xy + y^2) + (x^2 + xy)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$P(x,y) = 3xy + y^2 \qquad Q(x,y) = x^2 + xy$$
$$\frac{\partial P}{\partial y} = 3x + 2y \neq 2x + y = \frac{\partial Q}{\partial x}$$

So, this ODE is not exact. Can we multiply through by some integrating factor, h?

$$\text{Take } h = h(x) \neq 0$$
$$h(3xy + y^2) + h(x^2 + xy)\frac{\mathrm{d}y}{\mathrm{d}x} = \hat{p} + \hat{q}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact

$$\iff \frac{\partial \hat{p}}{\partial y} = \frac{\partial \hat{q}}{\partial x} \iff h(3x + 2y) = h_x(x^2 + xy) + h(2x + y) \iff h(x + y) = h_x x(x + y)$$

Supposing that $x+y \neq 0$, we can simplify and find

$$h = h'x$$

Supposing that $x \neq 0$, we can see

$$h' = \frac{1}{x}h$$

This is a seperable first order ODE we can simply solve,

$$\int \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\mathrm{d}x = \int \frac{1}{x}\mathrm{d}x \iff \int \frac{1}{h}\mathrm{d}h = \int \frac{1}{x}\mathrm{d}x \iff \ln|h| = \ln|x| + \hat{\alpha} \iff h(x) = \alpha x,\ \alpha = \exp(\hat{\alpha}).$$

We're free to choose $\alpha > 0$, so we'll take $\alpha = 1$ for simplicity, and then multiple our original ODE by our integrating factor $h = h(x) = x$. Check:

$$h(3xy+y^2) + h(x^2+xy)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \iff x(3xy+y^2) + x(x^2+xy)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\iff (3x^2y+xy^2) + (x^3+x^2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,\ P(x,y) = 3x^2y+xy^2,\ Q(x,y) = x^3+x^2y$$
$$\frac{\partial P}{\partial y} = 3x^2+2xy = 3x^2+2xy = \frac{\partial Q}{\partial x}$$

So this ODE is exact. Therefore, there exists some functoin, $f(x,y)$ such that $f_x = P$ and $f_y = Q$

$$\text{Take } f(x,y) = x^3y + \frac{1}{2}x^2y^2 \implies f_x = 3x^2y+xy^2 = P,\ f_y = x^3+x^2y$$

Therefore, the solution to our ODE is

$$f(x,y) = K \iff x^3y + \frac{1}{2}x^2y^2 = K \iff \frac{1}{2}x^2y^2 + x^3y - K = 0 \iff y = \frac{-x^2 \pm \sqrt{x^3+2K}}{x}$$

Purely for fun, we're going to apply an inital condition, $y(1) = 0$

$$\text{Then } 0 = \frac{-1 \pm \sqrt{1+2K}}{1} \iff 0 = K \text{ and we choose the positive branch}$$

So our final solution is

$$y(x) = \frac{x^2 + \sqrt{x^3}}{x} = \sqrt{x} - x$$

1.2.2 Hyperbolic Functions

Definition 1.2.1: Hyperbolic Functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2},$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2},$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{1 - e^{-2x}}{1 + e^{-2x}},$$

Corollary 1.2.1 Hyperbolic-Pythagorean Identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

This follows from direct calculation.

Note that the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ allows us to paramaterise the unit circle, namely by setting $x(t) = \cos t,\ y(t) = \sin t$, which gives us the equation of a unit circle, $\cos^2 t + \sin^2 t = x^2 + y^2 = 1$.

If instead, we set $x(t) = \cosh t,\ y(t) = \sinh t$, we can see

$$\cosh^2 t - \sinh^2 t = x^2 - y^2 = 1$$

which is the equation for a hyperbola.
Also following from direct calculation, similar to their trigonometric counterparts, the hyperbolic functions satisfy

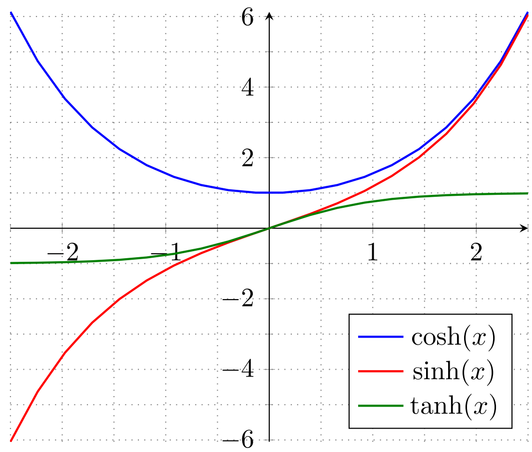
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\cosh x &= \frac{e^x - e^{-x}}{2} = \sinh x,\\ \frac{\mathrm{d}}{\mathrm{d}x}\sinh x &= \frac{e^x + e^{-x}}{2} = \cosh x\end{aligned}$$

Note that $\cosh(0) = 1,\ \cosh(x) \geq 1$ and $\cosh(x)$ is an even function ($\cosh(-x) = \cosh(x)$); $\sinh(0) = 0,\ \sinh(x)$ is an odd function $\sinh(-x) = -\sinh(x)$.

Example 1.2.2

Prove that:

1. $\cosh^2 x = \frac{1}{2}(\cosh(2x) + 1)$
2. $\sinh^2 x = \frac{1}{2}(\cosh(2x) - 1)$
3. $\sinh(2x) = 2\sinh x \cosh x$



Looking at the plots of the functions, we can deduce that

$$\begin{aligned}&\operatorname{dom}\cosh x = \mathbb{R} && \operatorname{dom}\sinh x = \mathbb{R} && \operatorname{dom}\tanh x = \mathbb{R}\\ &\operatorname{ran}\cosh x = [1, \infty) && \operatorname{ran}\sinh x = \mathbb{R} && \operatorname{ran}\tanh x = (-1, 1)\end{aligned}$$

Definition 1.2.2: Reciprocal Hyperbolic Functions

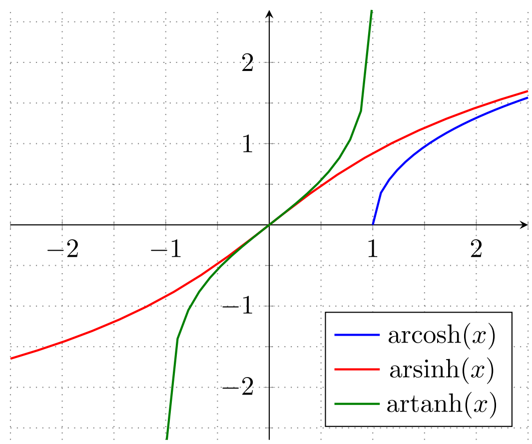
$$\coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)} = \frac{1+e^{-2x}}{1-e^{-2x}}$$
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x}-e^{-x}}$$
$$\operatorname{csch}(x) = \frac{1}{\operatorname{sech}(x)} = \frac{2}{e^{x}+e^{-x}}$$

1.2.3 Inverse Hyperbolic Functions

Definition 1.2.3: Inverse Hyperbolic Functions

If f is ... f^{-1} is denoted ...:

$\cosh(x)$	$\operatorname{arcosh}(x)$
$\sinh(x)$	$\operatorname{arsinh}(x)$
$\tanh(x)$	$\operatorname{artanh}(x)$



$$\operatorname{dom} \operatorname{arcosh} x = [1,\infty) \qquad \operatorname{dom} \operatorname{arsinh} x = \mathbb{R} \qquad \operatorname{dom} \operatorname{artanh} x = (-1,1)$$
$$\operatorname{ran} \operatorname{arcosh} x = [0,\infty) \qquad \operatorname{ran} \operatorname{arsinh} x = \mathbb{R} \qquad \operatorname{ran} \operatorname{artanh} x = \mathbb{R}$$

We have the following:

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \operatorname{arsinh} x + C$$
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \operatorname{arcosh} x + C,\ \ x > 1$$

Example 1.2.3

Show $\dfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arsinh} x) = \dfrac{1}{\sqrt{1+x^2}}$.

$$\operatorname{arsinh} x = y(x)$$
$$x = \sinh y$$

$$
\begin{aligned}
\iff \frac{\mathrm{d}}{\mathrm{d}x}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\sinh y) \\
\iff 1 &= \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \cosh y \\
\iff \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cosh y} \\
&= \frac{1}{\cosh(\operatorname{arsinh} x)} \\
&= \frac{1}{\sqrt{\cosh^2(\operatorname{arsinh} x)}} \\
&= \frac{1}{\sqrt{1+\sinh^2(\operatorname{arsinh} x)}} \\
&= \frac{1}{\sqrt{1+\sinh(\operatorname{arsinh} x)\sinh(\operatorname{arsinh} x)}} \\
&= \frac{1}{\sqrt{1+x\cdot x}} \\
&= \frac{1}{\sqrt{1+x^2}}
\end{aligned}
$$

Show $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$.

$$
\begin{aligned}
\operatorname{arcosh} x &= y(x) \\
x &= \cosh y \\
\iff \frac{\mathrm{d}}{\mathrm{d}x}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\cosh y) \\
\iff 1 &= \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \sinh y \\
\iff \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sinh y} \\
&= \frac{1}{\sinh(\operatorname{arcosh} x)} \\
&= \frac{1}{\sqrt{\sinh^2(\operatorname{arcosh} x)}} \\
&= \frac{1}{\sqrt{\cosh^2(\operatorname{arcosh} x)-1}} \\
&= \frac{1}{\sqrt{\cosh(\operatorname{arcosh} x)\cosh(\operatorname{arcosh} x)-1}} \\
&= \frac{1}{\sqrt{x\cdot x-1}} \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}
$$

Example 1.2.4

Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+x^2}}$

$$
1+\sinh^2 t = \cosh^2 t
$$

$$
\begin{aligned}
&\text{Let } x = \sinh t \\
\implies &\frac{\mathrm{d}x}{\mathrm{d}t} = \cosh t \Rightarrow \mathrm{d}x = \cosh t \ \mathrm{d}t \\
\therefore \int \frac{\mathrm{d}x}{\sqrt{1+x^2}} &= \int \frac{\cosh t}{\sqrt{1+\sinh^2 t}} \mathrm{d}t \\
&= \int \frac{\cosh t}{\sqrt{\cosh^2 t}} \mathrm{d}t \\
&= \int \frac{\cosh t}{\cosh t} \mathrm{d}t \\
&= \int \mathrm{d}t \\
&= t + C \\
&= \operatorname{arsinh} x + C
\end{aligned}
$$

Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-1}}$

$$
\begin{aligned}
&\cosh^2 t - 1 = \sinh^2 t \\
&\text{Let } x = \cosh t \\
\implies &\frac{\mathrm{d}x}{\mathrm{d}t} = \sinh t \Rightarrow \mathrm{d}x = \sinh t \ \mathrm{d}t \\
\therefore \int \frac{\mathrm{d}x}{\sqrt{x^2-1}} &= \int \frac{\sinh t}{\sqrt{\cosh^2 t - 1}} \mathrm{d}t \\
&= \int \frac{\sinh t}{\sqrt{\sinh^2 t}} \mathrm{d}t \\
&= \int \frac{\sinh t}{\sinh t} \mathrm{d}t \\
&= \int \mathrm{d}t \\
&= t + C \\
&= \operatorname{arcosh} x + C, \ \ x \geq 1
\end{aligned}
$$

Example 1.2.5

Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{artanh} x) = \dfrac{1}{1-x^2}$

Using partial fractions, we also find that

$$\int \frac{\mathrm{d}x}{1-x^2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + C$$

In fact, we have the following identities

$$
\begin{aligned}
\operatorname{artanh} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\
\operatorname{arsinh} x &= \ln\left(x + \sqrt{x^2+1}\right) \\
\operatorname{arcosh} x &= \ln\left(x + \sqrt{x^2-1}\right)
\end{aligned}
$$

Example 1.2.6

Show that $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$

1.2.4 The Cateary Problem

1.2.5 Linear Second-Order Non-Homogenous ODEs and the Wronskian

1.3 Lecture 3

1.3.1 Variation of Parameters

We've seen that for a linear second-order, non-homogenous IVP,

$$y'' + p(x)y' + q(x)y = r(x), \quad y(x_0) = y_0$$

if p, q, r are continous on an open interval I, and the inital condition, $x_0 \in I$, then there exisits a solution to the IVP. The solution will be a linear combination of the solution in the homogenous case and the particular case, $y(x) = y_H(x) + y_P(x)$. Assuming the homogenous case is a linear combination of linearly independent ys, ie $W(y_1, y_2) \neq 0$, we can apply variation of parameters. The process is as follows:

1. Solve $y'' + p(x)y' + q(x)y = 0$, and obtain a fundamental set of solutions, y_1, y_2. Calculate the Wronskian, $W(y_1, y_2)(x) = W$.
2. Set $y_P = u(x)y_1(x) + v(x)y_2(x)$ and substitute into the ODE. We also impose the condition, $u'y_1 + v'y_2 = 0$. We can freely impose this condition because we have two functions, u, v, and only one equation they must satisfy, the ODE.
3. We obtain
$$u(x) = -\int \frac{y_2 r}{W} \mathrm{d}x, \qquad v(x) = \int \frac{y_1 r}{W} \mathrm{d}x.$$

This approach is a variation of the reduction of order, which prescribes taking some solution, y, of the associated ODE, and using it to find a particular solution.

Example 1.3.1

Derivation of $u(x)$ and $v(x)$ of the variation of parameters.

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

Suppose we solved the homogenous case, $y'' + py' + qy = 0$.

$$\begin{aligned} \implies \exists y_1(x), y_2(x) : W(y_1, y_2)(x) \neq 0,\ y_H(x) = Ay_1(x) + By_2(x) \\ y_P(x) = u(x)y_1(x) + v(x)y_2(x) \\ \therefore y_P' = u'y_1 + uy_1' + v'y_2 + vy_2 \end{aligned} \tag{2}$$

Impose that $u'y_1 + v'y_2 = 0$, then

$$\begin{aligned} y_P' &= uy_1' + vy_2' \\ \therefore y_P'' &= u'y_1' + uy_1'' + v'y_2' + vy_2'' \end{aligned}$$

We'll now substitute (2)'s derivatives back into (1), and find

$$(u'y_1' + uy_1'' + v'y_2' + vy_2'') + p(uy_1' + vy_2') + q(uy_1 + vy_2) = r$$

Consider $uy_1'' + puy_1' + quy_1$ and $vy_2'' + pvy_2' + qvy_2$, and note that they are solutions to the homogenous case, and are therefore equal to 0. So we can simply cancel them out, and are left with:

$$u'y_1' + v'y_2' = r$$

In fact, the entire system has been reduced to the system of equations

$$\begin{cases} u'y_1 + v'y_2 = 0 \\ u'y_1' + v'y_2' = r \end{cases}$$

$$\iff \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ r(x) \end{pmatrix}$$

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = \hat{W},\ \det \hat{W} = \det W(y_1, y_2)(x) = W \neq 0 \implies \hat{W} \text{ is invertible.}$$

$$\hat{W}^{-1} = \frac{1}{\det \hat{W}} \begin{pmatrix} y_2'(x) & -y_2(x) \\ -y_1' & y_1(x) \end{pmatrix}$$

$$\therefore \begin{pmatrix} u' \\ v' \end{pmatrix} = \hat{W}^{-1} \begin{pmatrix} 0 \\ r(x) \end{pmatrix} = \frac{1}{\det \hat{W}} \begin{pmatrix} y_2'(x) & -y_2(x) \\ -y_1' & y_1(x) \end{pmatrix} \begin{pmatrix} 0 \\ r(x) \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} -y_2 r \\ y_1 r \end{pmatrix}$$

$$\iff \begin{cases} u' = \dfrac{-y_2 r}{W} \\ v' = \dfrac{y_1 r}{W} \end{cases} \iff \begin{cases} u = -\displaystyle\int \frac{y_2 r}{W} \mathrm{d}x \\ v = \displaystyle\int \frac{y_1 r}{W} \mathrm{d}x \end{cases}$$

Example 1.3.2

Solve

$$y'' - 4y' + 5y = \frac{2e^{2x}}{\sin x}$$

using variation of parameters.

$$y = y_H + y_P$$

Let's ansatz that $y_H = e^{\lambda x}$

$$\iff \lambda^2 - 4 + 5 = 0 \iff \lambda_{1,2} = 2 \pm i \iff y_H = Ae^{2x} \cos x + Be^{2x} \sin x$$

$$W = \det W(y_1, y_2)(x) = \det \begin{pmatrix} e^{2x} \cos x & e^{2x} \sin x \\ 2e^{2x} \cos x - e^{2x} \sin x & 2e^{2x} \sin x + e^{2x} \cos x \end{pmatrix} = e^{4x} \neq 0$$

Find $y_P = uy_1 + vy_2$

$$u(x) = -\int \frac{y_2 r}{W} \mathrm{d}x = -\int \frac{e^{2x} \sin x \frac{2e^{2x}}{\sin x}}{e^{4x}} \mathrm{d}x = -2 \int 1 \ \mathrm{d}x = -2x$$

$$v(x) = \int \frac{y_1 r}{W} \mathrm{d}x = \int \frac{e^{2x} \cos x \frac{2e^{2x}}{\sin x}}{e^{4x}} \mathrm{d}x = 2 \int \cot x \ \mathrm{d}x = 2 \ln |\sin x|$$

$$\implies y_P = 2 \ln |\sin x| \, e^{2x} \sin x - 2xe^{2x} \cos x$$

$$\implies y = Ae^{2x} \cos x + Be^{2x} \sin x + 2 \ln |\sin x| \, e^{2x} \sin x - 2xe^{2x} \cos x$$

Example 1.3.3

Solve

$$y'' - 4y' + 5y = \frac{2e^{2x}}{\sin x}$$

using reduction of order.

$$y = y_H + y_P$$