School of Mathematics and Physics, UQ

MATH1071 Advanced Calculus & Linear Algebra I
Semester 1 2025
Problem Set 1

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Tutorial Group #8

Due 3pm Friday 14 March 2025

Question 1: 10 marks

Given the following sets

$$A = \{2, 7, 17, 37\}, \qquad B = \{3, 37, 59\},$$
$$C = \{2, 3, 5, 17, 37, 59\}, \qquad D = \{37, 102\}$$

Determine if $B \subseteq C$ (1 mark), and write down the list of elements for each of the following sets (4 mark)

$$A \cup B, \qquad D \setminus A, \qquad B \cap C, \qquad B \times D.$$

Lastly, verify the following statement by listing the elements for both sides of the equality: (5 marks)

$$(A \cap B) \times D = (A \times D) \cap (B \times D).$$

Solution:
We note that, of the elements in B, 3 is an element of C, 37 is an element of C, and 59 is an element of C. Since all the elements of B are accounted for in C, we conclude that $B \subseteq C$.

$$\begin{aligned} A \cup B &= \{2, 7, 17, 37\} \cup \{3, 37, 59\} = \{2, 3, 7, 17, 37, 59\} \\ D \setminus A &= \{37, 102\} \setminus \{2, 7, 17, 37\} = \{102\} \\ B \cap C &= \{3, 37, 59\} \cap \{2, 3, 5, 17, 37, 59\} = \{3, 37, 59\} \\ B \times D &= \{3, 37, 59\} \times \{37, 102\} = \{(3, 37), (3, 102), (37, 37), (37, 102), (59, 37), (59, 102)\} \end{aligned}$$

$$\begin{aligned} (A \cap B) \times D &= (\{2, 7, 17, 37\} \cap \{3, 37, 59\}) \times \{37, 102\} \\ &= \{37\} \times \{37, 102\} \\ &= \{(37, 37), (37, 102)\} \\ (A \times D) \cap (B \times D) &= (\{2, 7, 17, 37\} \times \{37, 102\}) \cap (\{3, 37, 59\} \times \{37, 102\}) \\ &= \{(2, 37), (2, 102), (7, 37), (7, 102), (17, 37), (17, 102), (37, 37), (37, 102)\} \\ &\quad \cap \{(3, 37), (3, 102), (37, 37), (37, 102), (59, 37), (59, 102)\} \\ &= \{(37, 37), (37, 102)\} \\ &= (A \cap B) \times D \end{aligned}$$

Question 2: 5 marks

(a) (2 marks) Determine if f constitutes a binary operation on the set A. Please state your reasoning.

$$A = \{\text{QLD}, \text{VIC}, \text{NSW}\}, \qquad f : A \times A \to \mathbb{N},$$

where for each $a, b \in A$, we define $f(a, b) =$ the total number of driver licences issued in 2024 by states a and b combined.

(b) (3 marks) Determine if the following binary operation is associative. Please state your reasoning.

$$A = \{\text{States and territories in Australia}\}, \qquad f : A \times A \to A,$$

where for each $a, b \in A$, we define $f(a, b) =$ the state or territory with the larger population at the end of the year 2024.

We may assume that no states or territories have the same population at the end of the year 2024.

Solution: (a)
A binary operation is a function mapping the Cartesian product of a set with itself, to itself. Since the given function, f, is mapping between $A \times A$ to $\mathbb{N} \neq A$, we know that f does not constitute a binary operation on A.

Solution: (b)
A binary operation, $* : S \times S \to S$, is associative if it holds to:

$$(a * b) * c = a * (b * c), \quad \forall a, b, c \in S$$

We'll define a "magnitude" operator, $|\cdot| : S \to \mathbb{N}$, where $|s| \mapsto$ the population of s at the end of 2024, $\forall s \in S$.
Then, we can reformulate f using only mathematical notation. Let $f : A \times A \to A$ be a binary operatin on A, where $\forall a, b \in A$,

$$f(a, b) = \begin{cases} a, & |a| > |b| \\ b, & |b| > |a| \end{cases} .$$

Proof. We'll prove that the binary operation f on A is associative.

Suppose $a, b, c \in S$.
Without loss of generality, assume $|a| < |b| < |c|$.
We'll first consider, $f(f(a, b), c) = f(b, c)$, because $|b| > |a|$.
Then, $f(b, c) = c$, because $|c| > |b|$.
Now we'll consider $f(a, f(b, c)) = f(a, c)$, because $|c| > |b|$.
Then, $f(a, c) = c$, because $|c| > |a|$.
Hence, $f(f(a, b), c) = c = f(a, f(b, c))$.

Therefore, f on A is associative. □

Question 3: 10 marks

Let $\mathbb{Z}_{11} = \{0,1,2,3,4,5,6,7,8,9,10\}$, with modular arithmetic as the two binary operations. Find the additive and multiplicative inverse for each element in the set.

Solution:
Before we can find the additive and multiplicative inverses, we need to identify the additive and multiplicative identities.

$\exists i \in \mathbb{Z}_{11} : a + i = i + a = a,$ namely, $i = 0$. *Proof.* Suppose $a \in \mathbb{Z}_{11}$. $a + 0 = 0 + a = a$. □

$\exists e \in \mathbb{Z}_{11} a \cdot e = e \cdot a = a,$ namely, $e = 1$. *Proof.* Suppose $a \in \mathbb{Z}_{11}$. $a \cdot 1 = 1 \cdot a = a$. □

Now, for an element $x \in \mathbb{Z}_{11}$ to have an additive identity, $y \in \mathbb{Z}_{11}$, it must conform to the equation

$$x + y \equiv i \bmod 11$$

$\forall x \in \mathbb{Z}_{11}, \exists y \in \mathbb{Z}_{11} : x + y \equiv 0 \bmod 11,$ namely, $y \equiv -x \bmod 11$.

$x \in \mathbb{Z}_{11}$	0	1	2	3	4	5	6	7	8	9	10
y	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
$y \bmod 11 \in \mathbb{Z}_{11}$	0	10	9	8	7	6	5	4	3	2	1
$x + y$	0	0	0	0	0	0	0	0	0	0	0

For an element $x \in \mathbb{Z}_{11}$ to have a multiplicative identity, $y \in \mathbb{Z}_{11}$, it must conform to the equation

$$x \cdot y \equiv e \bmod 11$$

$\forall x \in \mathbb{Z}_{11}, \exists y \in \mathbb{Z}_{11} : x \cdot y \equiv 1 \bmod 11 \iff \exists k \in \mathbb{Z} : 1 \cdot xy = 11k,$ namely, $y = \frac{11k}{x}$.
From this result, we can immediately see that $x = 0 \in \mathbb{Z}_{11}$ does not have a multiplicative inverse in $\mathbb{Z}_{11}$, but every other element does.

$x \in \mathbb{Z}_{11}$	0	1	2	3	4	5	6	7	8	9	10
y		1	6	4	3	9	2	8	7	5	10
$x \cdot y$		1	1	1	1	1	1	1	1	1	1

Question 4: 15 marks

(a) (5 marks) Prove that the additive identity of a field is always unique. (Therefore we can drop the uniqueness condition from the field axioms.)

(b) (5 marks) Let $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. If we use modular arithmetic mod 4, are these still binary operations? Argue why there does not exist an additive identity in $\mathbb{Z}_8$.

(c) (5 marks) Let $\mathbb{R}$ be the set of real numbers, where multiplication is regarded as the (new) addition, and addition is regarded as the (new) multiplication. Is $\mathbb{R}$ still a field with the two operations swapped? Why?

Solution: (a)

Proof. Suppose a field, $\mathbb{F}$, has more than one unique additive identities, $i_1, i_2 \in \mathbb{F}$, an element $x \in \mathbb{F}$, and its inverse $y \in \mathbb{F}$.

Then $x + y = y + x = i_1$.
And $x + y = y + x = i_2$.
Hence $i_1 = i_2$. ※
Therefore, a field can't have more than one unique additive identities.

Therefore, a field has one unique additive identity. □

Solution: (b)

A binary operator on a set, S, is a function

$$f : S \times S \to S$$

which is closed, every pair of elements in S has a result in S.

If we endow $\mathbb{Z}_8$ with mod 4 arithmetic, the function may or may not be a binary operation. The question is a little vague; I'll explain. The key is considering what we mean by "modular arithmetic mod 4", and whether it meets the definition we laid out, or not.

If by "modular arithmetic mod 4" we mean

$$f : \mathbb{Z}_8 \times \mathbb{Z}_8 \to \mathbb{Z}_4,\ (x, y) \mapsto (x + y) \bmod 4, \forall x, y \in \mathbb{Z}_8$$

then this function is not a binary operation, since the codomain is not the same set as the domain (prior to taking the Cartesian product).

If by "modular arithmetic mod 4" we mean

$$f : \mathbb{Z}_8 \times \mathbb{Z}_8 \to \mathbb{Z}_8,\ (x, y) \mapsto (x + y) \bmod 4, \forall x, y \in \mathbb{Z}_8$$

then the function is a binary operation. Even though the entire codomain isn't being mapped to, every pair of elements has a result it can map to, which I show using this Cayley table,

f	0	1	2	3	4	5	6	7
0	0	1	2	3	0	1	2	3
1	1	2	3	0	1	2	3	0
2	2	3	0	1	2	3	0	1
3	3	0	1	2	3	0	1	2
4	0	1	2	3	0	1	2	3
5	1	2	3	0	1	2	3	0
6	2	3	0	1	2	3	0	1
7	3	0	1	2	3	0	1	2

If we accept the latter, even though arithmetic mod 4 on $\mathbb{Z}_8$ is a binary operator, there isn't an additive identity. This is because, again referring to the Cayley table above,

$$\nexists i \in \mathbb{Z}_8 : (k, i) \mapsto k, \forall k \in \{4, 5, 6, 7\} .$$

4, 5, 6 and 7 never map onto themselves; hence there is not an additive identity for $(\mathbb{Z}_8, +_{\text{mod}4})$.

Solution: (c)
Since this might get notationally confusing, let's clear that up right away. We'll define the "new" addition and multiplication like so,

$$\oplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ x \oplus y \mapsto x \cdot y,\ \forall x, y \in \mathbb{R}$$
$$\otimes : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ x \otimes y \mapsto x + y,\ \forall x, y \in \mathbb{R}$$

Now the question is, does the algebraic structure, $\mathbb{F} := (\mathbb{R}, \oplus, \otimes)$ form a field?

We'll start by determining whether or not $(\mathbb{R}, \oplus)$ forms an Abelian group. Since normal multiplication, $\cdot$, is closed, associative, and communitative on $\mathbb{R}$, so is $\oplus$. We need to consider the existence of an identity element.

$$\exists i \in \mathbb{R} : \forall x \in \mathbb{R}, x \oplus i = x, \text{ namely, } i = 1.$$

Proof. Suppose $x \in \mathbb{R}$. $x \oplus 1 = 1 \oplus x = 1 \cdot x = x \quad \square$

Next, we'll consider the existence of an inverse for each element.

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x \oplus y = 1$. This is false. We prove with a counterexample.

Proof. Suppose $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x \oplus y = 1$. Take $x = 0 \in \mathbb{R}$.

$$\text{Then } 0 \oplus y = 1 \iff 0 \cdot y = 1 \iff y = \frac{1}{0} ※$$
$$\therefore \forall x \in \mathbb{R},\ \nexists y \in \mathbb{R} : x \oplus y = 1 \quad \square$$

Therefore, the algebraic structure $(\mathbb{R}, \oplus)$ does not have an inverse for each element.
Therefore, the algebraic structure $\mathbb{F}$ is not a field.

The reason $(\mathbb{R}, \oplus, \otimes)$ does not form a field, but $(\mathbb{R}, +, \cdot)$ does is because the additive identity, 0, is critically removed from the set before we consider whether or not it forms an Abelian group. i.e. $(\mathbb{R}, \cdot)$ does not form an Abelian group, but $(\mathbb{R} \setminus \{0\}, \cdot)$ does.

Question 5: 10 marks

(a) (5 marks) Using the formal definition, prove that if $a_n \leq b_n$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} b_n = \infty$.

(b) (5 marks) Using the previous item, show that

$$\lim_{n\to\infty} (n - 7 + n^5 + 3^n) = \infty.$$

Solution: (a)
Definition 5.1.1 (Sequence Limits to Infinity). The sequence $\{a_n\}_{n=1}^{\infty}$ *goes to infinity*, denoted $\lim_{n\to\infty} a_n = \infty$, if $\forall M > 0,\ \exists N \in \mathbb{N} : a_N > M$.

Theorem 5.1.1. If $a_n \leq b_n,\ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \infty$ then $\lim_{n\to\infty} b_n = \infty$.

Proof. Suppose $\{a_n\}_{n=1}^{\infty},\ \{b_n\}_{n=1}^{\infty}$ are sequences with $\lim_{n\to\infty} a_n = \infty$ and $a_n \leq b_n,\ \forall n \in \mathbb{N}$.
Let M be an arbitrarily large real number.
Let N be the index such that the element $M < a_N$.
Since $\lim_{n\to\infty} a_n = \infty$, by definition 5.1.1, N is guaranteed to exist.
Since, $a_n \leq b_n,\ \forall n \in \mathbb{N}$, we know $a_N \leq b_N$.
Therefore, $M < a_N \leq b_N$.
Therefore, $\lim_{n\to\infty} b_n = \infty$ □

Solution: (b)
Corollary. $\lim_{n\to\infty}(n - 7 + n^5 + 3^n) = \infty$.

Proof. Let $\{a_n\}_{n=1}^{\infty} : a_n \mapsto n - 7 + n^5 + 3^n,\ \{b_n\}_{n=1}^{\infty} : b_n \mapsto 3^n,\ \forall n \in \mathbb{N}$.
We start by noting that $3^n \leq 3^n + n^5$.
Then, $3^n \leq 3^n + n^5 + n$.
Finally, $3^n \leq 3^n + n^5 + n - 7 \leq 3^n + n^5 + n$.
We see that $3^n \leq 3^n + n^5 + n - 7$,
thus, $b_n \leq a_n,\ \forall n \in \mathbb{N}$.
Let M be an arbitrarily large real number.
Choose $N = \lceil M \rceil$.
Then $b_N = 3^N = 3^{\lceil M \rceil} > M$.
Therefore by Definition 5.1.1, $\lim_{n\to\infty} b_n = \infty$.
Therefore by Theorem 5.1.1, $\lim_{n\to\infty} b_n = \infty,\ \because b_n \leq a_n$, and $\lim_{n\to\infty} b_n = \infty$.
Therefore, $\lim_{n\to\infty}(n - 7 + n^5 + 3^n) = \infty$ □