School of Mathematics and Physics, UQ

MATH2001/MATH7000 practice problems
Sheet 3

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(1) Find a matrix in row echelon form equivalent to:

(a) $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{pmatrix}$

(2) Use the Gauss method to find A^{-1} if $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

(3) In this question, all matrices are square $n \times n$ ones.

(a) Show that the product of two upper triangular matrices (that is, two matrices with 0 everywhere below the main diagonal) is upper triangular, and show that the inverse of an upper triangular matrix is upper triangular.

(b) Repeat part **(a)**, but for lower triangular matrices. (So take transposes and use part **(a)**.)

(c) If L is lower triangular with 1 everywhere on the main diagonal, show that L^{-1} also has 1 everywhere on its main diagonal.

(4) Let $P_2(\mathbb{R})$ have inner product defined by

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx, \qquad \mathbf{p}, \mathbf{q} \in P_2(\mathbb{R}),$$

and let $\mathbf{u}, \mathbf{v} \in P_2(\mathbb{R})$ be given by

$$\mathbf{u} : x \mapsto 1, \qquad \mathbf{v} : x \mapsto x^2.$$

(a) Find $\langle \mathbf{u}, \mathbf{v} \rangle$.

(b) Find $d(\mathbf{u}, \mathbf{v})$.

(c) Find $||\mathbf{u}||$.

(d) Find $||\mathbf{v}||$.

(5) Show that

$$M_1 = \begin{pmatrix} 5 & -1 \\ 2 & -2 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}$$

are orthogonal with respect to the inner product $\langle A, B \rangle = \mathrm{Tr}(B^T A)$ on $M_{2,2}(\mathbb{R})$.

(6) Let $\mathbb{R}^4$ be endowed with the usual dot product, and let

$$S = \{(1,4,5,2), (2,1,3,0), (-1,3,2,2)\}.$$

Find a basis for $S^\perp$.

(7) Let $\mathbb{R}^4$ have inner product given by the dot product. Show that

$$\{(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}), (\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}), (-\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2})\}$$

is an orthonormal basis for $\mathbb{R}^4$.

(8) Let $\mathbb{R}^2$ have inner product given by the dot product, and let $\theta \in \mathbb{R}$. Find $(a, b) \in \mathbb{R}^2$ such that

$$\{(\cos\theta, \sin\theta), (a, b)\}$$

is an orthonormal basis for $\mathbb{R}^2$.

(9) Let $\mathbb{R}^4$ have inner product given by the dot product, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = (1,0,1,0), \qquad \mathbf{v}_2 = (1,1,1,1), \qquad \mathbf{v}_3 = (0,1,2,1).$$

(a) Show that S is linearly independent.

(b) Find an orthonormal basis for $\operatorname{span}(S)$.

(10) Let $P(\mathbb{R})$ have inner product given by

$$\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)dx.$$

Find the orthogonal projection of $r(x) = 4 + 3x - 2x^2$ onto $P_1(\mathbb{R})$.

(11) Let $\mathbb{R}^4$ be endowed with the usual dot product, and let

$$U = \operatorname{span}(\{(1,-1,5,2), (0,3,-2,-1), (2,2,-2,1), (1,0,3,5)\}).$$

Find the orthogonal projection of $(1,2,-2,-3)$ onto U.

(12) Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal basis for the real inner product space V. Show that

$$\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^{n} \langle \mathbf{u}, \mathbf{e}_i\rangle\langle \mathbf{e}_i, \mathbf{v}\rangle, \qquad \mathbf{u}, \mathbf{v} \in V.$$

(13) Let $P_1(\mathbb{R})$ have inner product given by

$$\langle p(x), q(x)\rangle = \int_{-1}^1 p(x)q(x)dx,$$

and let $S = \{1 - x, 1 + x\}$.

(a) Show that S is a basis for $P_1(\mathbb{R})$.

(b) Apply the Gram-Schmidt process to S to find an orthonormal basis β for $P_1(\mathbb{R})$.

(c) Find the coordinate vector of $r(x) = -2 - 3x$ relative to β.

(14) For $a > 0$, let $C[0, a]$ have inner product given by

$$\langle f(x), g(x)\rangle = \int_0^a f(x)g(x)dx.$$

(a) Find a such that

$$f(x) = \cos x + \sin x \qquad \text{and} \qquad g(x) = \cos x - \sin x$$

are orthogonal.

(b) For a given as in (a), find an orthonormal basis for $\text{span}(\{f(x), g(x)\})$.

(15) Let $P_2(\mathbb{R})$ be endowed with the standard inner product, and let $S = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$, where

$$f_1(x) = 3 + 4x + 5x^2, \qquad f_2(x) = 9 + 12x + 6x^2, \qquad f_3(x) = 1 - 7x + 25x^2.$$

(a) Show that S is a basis for $P_2(\mathbb{R})$.

(b) Apply the Gram-Schmidt process to S to find an orthonormal basis for $P_2(\mathbb{R})$.

(16) Let $\mathbb{R}^3$ have weighted dot product given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3)\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3.$$

Use the Gram-Schmidt process to transform $\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ into an orthonormal set.

(17) Find the least squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

(18) Use the least squares method to find the best straight line fit to the following data points:

$$(1, 0),\ (2, 0),\ (3, 1),\ (3, 2).$$

Graph the fitting line and plot the data points in the same coordinate system. Comment on whether the result for the line looks reasonable.