MATH1061
Advanced Multivariate Calculus & Ordinary Differential Equations

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Contents

Chapter 1

Week 1

1.1 Lecture 1

In this course, we will be looking at:

- functions of several variables and calculus
- vector calculus. Rates of change of vector valued functions and applications!
- differential equations
- MATLAB - Only 6 lab sessions. MATLAB will be incorprated into assignments.

An overview of the tools of applied mathematics

- Creating and studying models of phenomena in the world:
  - physics
  - chemistry
  - biology
  - ecology
  - economics
  - engineering.
- $\boxed{\text{natural world}} \xRightarrow{\text{simplification}} \boxed{\text{mathematical model}}$.
- $\boxed{\text{mathematical model}} \xRightarrow[\text{validation}]{\text{interpretation}} \boxed{\text{natural world}}$.
- Most importantly the $\boxed{\text{mathematical model}}$ offers predictive power.
- Modelling: identify key variables and processes.
- Formulation:
  - functions of several variables
  - ordinary differential equations (involving single variable rates of change)
  - WE WILL NOT TOUCH: partial differential equations (involving functions of several variables)
  - WE WILL NOT TOUCH: statistical models

Dimensional Analysis

Definition 1.1.1: Base Quantities

There exist base quantities (or dimensions) that provide units in terms of which the units of all other physical quantities can be expressed. Conventionally, these are: mass (M), length (L), time (T) (and temperature, electric current, amount of substance, luminous intensity).

Example 1.1.1 (A falling mass)

Suppose we conduct an experiment on the time, t, it takes an object of mass m, to fall a distance of x from rest in a vaccum (near the surface of the Earth).

In Australia we find that

$$x = 4.91t^2 \text{ (metres)},$$

Our friend in the USA finds that

$$x = 16.1t^2\text{(feet)}.$$

It would be correct to write $x = ct^2$, where c is a physical quantity, depending on units, $c = \frac{1}{2}g$.

Some quantities have dimensions as a product $M^aL^bT^c$, where $a, b, c \in \mathbb{Z}$. Let $[y]$ denote the dimensions of y and $[x]$ the dimensions of x. Then $[x, y] = [x][y]$.

Example 1.1.2 (Finding dimensions of physical quantities)

Velocity $\left(\frac{dx}{dt}\right)$:

$$\left[\frac{dx}{dt}\right] = [x][t]^{-1} = LT^{-1}$$

Acceleration $\left(\frac{dx}{dt^2}\right)$:

$$\left(\frac{dx}{dt^2}\right) = [x][t]^{-2} = LT^{-2}$$

Force $m\left(\frac{d}{dt}\right)\left(\frac{dx}{dt}\right)$

$$[F] = [m][t]^{-1}[x][t]^{-1} = MLT^{-2}$$

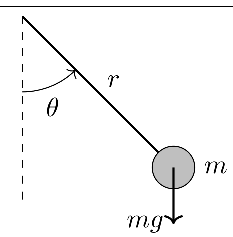
We call a quantity with dimensions $M^0L^0T^0$ **dimensionless**.
An equation that is true regardless of units is said to be **dimensionally homogeneous**. In such an equation, the dimensions of all terms must be the same.

Claim 1.1.1 Equations representing physical laws are dimensionally homogeneous.

To achieve this in our mathematical model we seek *all possible* dimensionless products among the variables. Such a collection is called **complete set**.

A Simple Pendulum

Consider the simple pendulum, with mass m, length r released from angle of displacement θ, and acted upon by gravity g.



Chapter 2

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2.1 Random Examples

Definition 2.1.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$$\lim_{n\to\infty} s_n = s$$

where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 1

Is the set $x-$axis$\setminus\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Claim 2.1.1 Topology

Topology is cool

Example 2.1.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\bigcup\limits_{x\in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

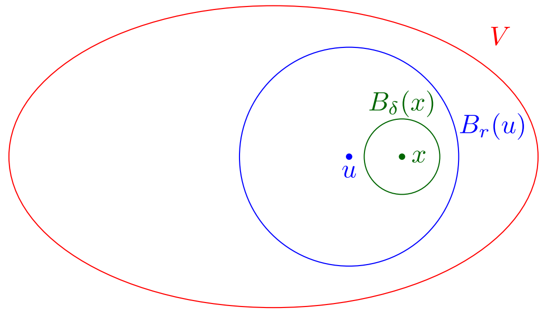
Closed Set:
- $X,\ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Theorem 2.1.1

If $x \in$ open set V then $\exists\ \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leq d(u, x) + d(x, y) < d + \delta < r$$

😄

Corollary 2.1.1

By the result of the proof, we can then show...

Lenma 2.1.1

Suppose $\vec{v_1}, \ldots, \vec{v_n} \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 2.1.1

$1 + 1 = 2$.

2.2 Random

Definition 2.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|\ V \to \mathbb{R}_{\geq 0}$ satisfying

1. $\|x\| = 0 \iff x = 0\ \forall\ x \in V$
2. $\|\lambda x\| = |\lambda|\|x\|\ \forall\ \lambda \in \mathbb{R}(\text{or } \mathbb{C}),\ x \in V$
3. $\|x + y\| \leq \|x\| + \|y\|\ \forall\ x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.
• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geq 0}$) where 2 becomes $\|\lambda x\| = |\lambda|\|x\|$ $\forall\ \lambda \in \mathbb{C},\ x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 2.2.1 ($p-$Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geq 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \Big(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\Big)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

Question 2

Prove that triangle inequality is true if $p \geq 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property 3 for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\sum_i (x_i + y_i)^2 \leq \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2$$
$$\implies \sum_i (x_i^2 + 2x_iy_i + y_i^2) \leq \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2$$
$$\implies \left[\sum_i x_iy_i\right]^2 \leq \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]$$

So in other words prove $\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_iy_i$$

Note:-

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle \cdot, \cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y\rangle = r\langle x, y\rangle + \langle x', y\rangle \text{ and similarly for second slot}$$

  Here in $\langle x, y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\ &= \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \\ &= \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y\rangle = \sum_i \overline{x_i}y_i$$

Then we still have $\langle x, x\rangle \geq 0$

2.3 Algorithms

Algorithm 1: what

```
Input: This is some input
Output: This is some output
/* This is a comment */

some code here;
x ← 0;
y ← 0;
if x > 5 then
    x is greater than 5 ;                         // This is also a comment
else
    x is less than or equal to 5;
end
foreach y in 0..5 do
    y ← y + 1;
end
for y in 0..5 do
    y ← y − 1;
end
while x > 5 do
    x ← x − 1;
end
return Return something here;
```