School of Mathematics and Physics, UQ

MATH2001, Assignment 1, Summer Semester, 2024-2025

Due at 2:00pm 19 December. Each question marked out of 10 then homogeneously rescaled up to a total marks of 13. **Total marks:** $\frac{13}{60}(Q1 + Q2 + Q3 + Q4 + Q5 + Q6)$.

Submit your assignment online via the Assignment 1 submission link in Blackboard.

(1) Consider the equation

$$\frac{\sin y}{y} - 2e^{-x}\sin x + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right) y' = 0.$$

(a) Show that it is not exact, but it becomes exact when multiplying by the integrating factor $\mu = ye^x$.

(b) Solve the exact equation subject to the initial condition $y(\pi) = \frac{\pi}{2}$. Present your solution as a relation defining y implicitly as a function of x.
(Do not solve the equation; only obtain the correct implicit equation that solves the given IVP).

Show all working.

(2) Consider the non-homogeneous ODE

$$x^2y'' - 3xy' + 4y = \ln x, \quad x > 0.$$

(a) Show that $y_1 = x^2$ and $y_2 = x^2 \ln x$ are solutions to the corresponding homogeneous ODE $x^2y'' - 3xy' + 4y = 0$.

(b) Find the general solution of the non-homogeneous ODE.

Show all working.

(3) Consider the inner product space $P_2(\mathbb{R})$ with inner product

$$\langle p_0 + p_1x + p_2x^2, q_0 + q_1x + q_2x^2 \rangle = p_0q_0 + p_1q_1 + p_2q_2, \quad p_0, p_1, p_2, q_0, q_1, q_2 \in \mathbb{R}.$$

Let $U = \{1 + 2x + x^2\}$.

(a) Find $U^\perp$.

(b) Determine an orthogonal basis for $U^\perp$.

Show all working.

(4) Consider the vector space $\mathbb{R}^4$ endowed with the inner product

$$\big\langle\, (u_1, u_2, u_3, u_4)\,, (v_1, v_2, v_3, v_4)\, \big\rangle = u_1 v_1 + 3u_2 v_2 + u_3 v_3 + 2u_4 v_4.$$

Let $U = \text{span}\{\mathbf{u}_1 = (-2, 1, 0, 1), \mathbf{u}_2 = (0, 1, 2, 3)\}$ be a subspace of $\mathbb{R}^4$.

(a) Use Gram-Schmidt procedure to construct an orthonormal basis for U.

(b) Find the orthogonal projection of $\mathbf{v} = (-1, 2, 5, 1)$ in U and $U^\perp$.

Show all working.

(5) Consider the vector space of 2×2 Matrices over the real numbers $M_{2,2}(\mathbb{R})$. The sets of matrices S and A such that $S^T = S$ and $A^T = -A$ define vector subspaces in $M_{2,2}(\mathbb{R})$. Call these subspaces respectively as $M^S_{2,2}(\mathbb{R}) = \{S \in M_{2,2}(\mathbb{R}) | S^T = S\}$ and $M^A_{2,2}(\mathbb{R}) = \{A \in M_{2,2}(\mathbb{R}) | A^T = -A\}$. Take the inner product over $M_{2,2}(\mathbb{R})$ defined by:

$$\langle \mathbf{v}, \mathbf{u} \rangle = \text{Tr}(\mathbf{v}^{\text{T}} \mathbf{u}) \ ,$$

$\forall\, \mathbf{v}, \mathbf{u} \in M_{2,2}(\mathbb{R})$. Are the subspaces $M^S_{2,2}(\mathbb{R})$ and $M^A_{2,2}(\mathbb{R})$ orthogonal according to the inner product defined above? Explain your answer.

(6) Let $P_2(\mathbb{R})$ have the inner product,

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x) q(x)\, dx, \quad \forall \mathbf{p}, \mathbf{q} \in P_2(\mathbb{R}).$$

Find the best approximation of $f(x) = x^2 + x^3$ by polynomials in $P_2(\mathbb{R})$. Show all working.

Hint: to simplify calculations, use the method of Solution 2 of section 12.1 of the workbook.