

STAT2003 Mathematical Probability
Semester 1 2025
Problem Set 1

Michael Kasumagic, 44302669
Tutorial Group #10
Due 1pm Friday 28 March 2025

Question 1

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{a, b, c\}$ and \mathcal{F} , the set of events, is the set of all possible subsets of Ω .

- (a) Enumerate all the events in \mathcal{F} .
- (b) Assume that for all $A \in \mathcal{F}$,

$$\mathbb{P}(A) = \frac{|A \setminus \{c\}|}{|\Omega \setminus \{c\}|}.$$

List out the values of $\mathbb{P}(A)$ for all events $A \in \mathcal{F}$.

- (c) How would you describe this probability space in words?
- (d) Prove that $\mathbb{P}(\cdot)$, as given, satisfies the 3 probability axioms.

Solution: (a)

$$\mathcal{F} = \left\{ \begin{array}{l} \emptyset, \{a\}, \{a, b\}, \{a, b, c\} \\ \{b\}, \{a, c\}, \\ \{c\}, \{b, c\}, \end{array} \right\}$$

Solution: (b)

Firstly, we note that

$$\Omega \setminus \{c\} = \{a, b, c\} \setminus \{c\} = \{a, b\}, \implies |\Omega \setminus \{c\}| = 2$$

With that, we can go ahead and define the probability function, $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$,

$$\mathbb{P} = \left\{ \begin{array}{l} \left(\emptyset, \frac{0}{2} = 0 \right), \quad \left(\{a\}, \frac{1}{2} \right), \quad \left(\{a, b\}, \frac{2}{2} = 1 \right), \quad \left(\{a, b, c\}, \frac{2}{2} = 1 \right) \\ \left(\{b\}, \frac{1}{2} \right), \quad \left(\{a, c\}, \frac{1}{2} \right), \\ \left(\{c\}, \frac{0}{2} = 0 \right), \quad \left(\{b, c\}, \frac{1}{2} \right), \end{array} \right\}$$

Solution: (c)

In words, I describe this probability function counts if a and b are present in the event, while c contributes nothing to the probability of an event occurring.

Solution: (d)

Axiom 1: $\mathbb{P}(A) \geq 0$

Proof. By exhaustion.

We defined $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ by listing all mappings, $A \mapsto p$.

From that list, we see that $\nexists A \in \mathcal{F} : \mathbb{P}(A) < 0$.

Therefore, $\forall A \in \mathcal{F}, \mathbb{P}(A) \geq 0$. □

Axiom 2: $\mathbb{P}(\Omega) = 1$

Proof. Directly.

$$\mathbb{P}(\Omega) = \mathbb{P}(\{a, b, c\}) = \frac{|\{a, b, c\} \setminus \{c\}|}{|\{a, b, c\} \setminus \{c\}|} = \frac{|\{a, b\}|}{|\{a, b\}|} = \frac{2}{2} = 1.$$

Therefore, $\mathbb{P}(\Omega) = 1$. □

Axiom 3: For any sequence of disjoint events, A_1, A_2, \dots we have $\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)$

Proof. Directly.

Suppose $\{A_i\}_{i=1}^n$ is a finite sequence, of length n , of disjoint sets in \mathcal{F} .

Then, because the sets are disjoint, the cardinality of the union, is simply the sum of the cardinality of the individual sets.

$$\begin{aligned} \mathbb{P}\left(\bigcup_{i=1}^n A_i\right) &= \frac{|\bigcup_{i=1}^n A_i \setminus \{c\}|}{|\Omega \setminus \{c\}|} \\ &= \frac{\sum_{i=1}^n |A_i \setminus \{c\}|}{|\Omega \setminus \{c\}|}. \end{aligned}$$

We can move the constant division into the summation,

$$= \sum_{i=1}^n \frac{|A_i \setminus \{c\}|}{|\Omega \setminus \{c\}|}.$$

And we note that this is the explicit definition of $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ we are given.

$$= \sum_{i=1}^n \mathbb{P}(A_i).$$

Therefore, for a sequence of disjoint events, $\{A_i\}_{i=1}^n$, $\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i)$. □

Question 2

Let the events B_i signify a cyclone of category i for $i = 1, \dots, 5$, with $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^5 B_i = \Omega$. Let the event A be the event of total property damage exceeding \$10 billion. Assume $\mathbb{P}(B_i) = (5 - i + 1)/15$, and assume $\mathbb{P}(A | B_i) = i/5$.

- (a) Calculate the probability of total property damage exceeding \$10 billion, $\mathbb{P}(A)$.
- (b) Are the events B_2 and B_3 independent? Prove your answer.
- (c) Are the events A and B_4 independent? Prove your answer.
- (d) Assume that A happened (the total property damage exceeded \$10 billion). What are the probabilities $\mathbb{P}(B_i | A)$ for $i = 1, \dots, 5$? Which category of cyclone is most likely (given property damage exceeding \$10 billion)?

Solution: (a)

We'll use the law of total probability.

$$\begin{aligned}\mathbb{P}(A) &= \sum_{i=1}^5 \mathbb{P}(A | B_i) \mathbb{P}(B_i) \\ &= \mathbb{P}(A | B_1) \mathbb{P}(B_1) + \mathbb{P}(A | B_2) \mathbb{P}(B_2) + \mathbb{P}(A | B_3) \mathbb{P}(B_3) \\ &\quad + \mathbb{P}(A | B_4) \mathbb{P}(B_4) + \mathbb{P}(A | B_5) \mathbb{P}(B_5) \\ &= \left(\frac{1}{5}\right) \left(\frac{5}{15}\right) + \left(\frac{2}{5}\right) \left(\frac{4}{15}\right) + \left(\frac{3}{5}\right) \left(\frac{3}{15}\right) + \left(\frac{4}{5}\right) \left(\frac{2}{15}\right) + \left(\frac{5}{5}\right) \left(\frac{1}{15}\right) \\ &= \frac{5 + 8 + 9 + 8 + 5}{75} \\ &= \frac{35}{75} \\ &= \frac{7}{15} \\ \therefore \mathbb{P}(A) &= \frac{7}{15}\end{aligned}$$

Solution: (b)

B_2 and B_3 are not independent.

Proof. Directly.

We are given $B_i \cap B_j = \emptyset, \forall i, j \in \{1, 2, 3, 4, 5\}, i \neq j$.

Applying this, $\mathbb{P}(B_2 \cap B_3) = \mathbb{P}(\emptyset) = 0$.

Next, we'll compute $\mathbb{P}(B_2)\mathbb{P}(B_3) = \frac{4}{15} \cdot \frac{3}{15} = \frac{12}{15}$.

We note that $\mathbb{P}(B_2 \cap B_3) = 0 \neq \frac{12}{15} = \mathbb{P}(B_2)\mathbb{P}(B_3)$.

By definition, B_2, B_3 are independent $\iff \mathbb{P}(B_2 \cap B_3) = \mathbb{P}(B_2)\mathbb{P}(B_3)$

Therefore, B_2 and B_3 are not independent. □

Solution: (c)

A and B_4 are not independent.

Proof. Directly.

We'll compute $\mathbb{P}(A \cap B_4)$ using the conditional probability formula.

$$\mathbb{P}(A \cap B_4) = \mathbb{P}(A | B_4) \mathbb{P}(B_4) = \frac{4}{5} \cdot \frac{2}{15} = \frac{8}{75}.$$

$$\text{Next, we'll compute } \mathbb{P}(A)\mathbb{P}(B_4) = \frac{7}{15} \cdot \frac{2}{15} = \frac{14}{225}.$$

$$\text{We note that } \mathbb{P}(A \cap B_4) = \frac{8}{75} \neq \frac{14}{225} = \mathbb{P}(A)\mathbb{P}(B_4).$$

$$\text{By definition, } A, B_4 \text{ are independent} \iff \mathbb{P}(A \cap B_4) = \mathbb{P}(A)\mathbb{P}(B_4)$$

Therefore, A and B_4 are not independent. □

Solution: (d)

We'll start with a lot of repeated computation using the conditional probability formula.

$$\begin{aligned}\mathbb{P}(B_1 | A) &= \frac{\mathbb{P}(B_1 \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A | B_1) \mathbb{P}(B_1)}{\mathbb{P}(A)} = \frac{5/75}{7/15} = \frac{5}{35} \\ \mathbb{P}(B_2 | A) &= \frac{\mathbb{P}(B_2 \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A | B_2) \mathbb{P}(B_2)}{\mathbb{P}(A)} = \frac{8/75}{7/15} = \frac{8}{35} \\ \mathbb{P}(B_3 | A) &= \frac{\mathbb{P}(B_3 \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A | B_3) \mathbb{P}(B_3)}{\mathbb{P}(A)} = \frac{9/75}{7/15} = \frac{9}{35} \\ \mathbb{P}(B_4 | A) &= \frac{\mathbb{P}(B_4 \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A | B_4) \mathbb{P}(B_4)}{\mathbb{P}(A)} = \frac{8/75}{7/15} = \frac{8}{35} \\ \mathbb{P}(B_5 | A) &= \frac{\mathbb{P}(B_5 \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A | B_5) \mathbb{P}(B_5)}{\mathbb{P}(A)} = \frac{5/75}{7/15} = \frac{5}{35}\end{aligned}$$

Therefore, given that damage exceeded \$10 billion, a category 3 cyclone is the most likely to have occurred.

Question 3

Consider the equation,

$$x_1 + x_2 + x_3 = 6,$$

where we restrict solutions to be tuples of integers (x_1, x_2, x_3) with $x_i \geq 0$.

- (a) How many solutions are there? Explain your answer.
- (b) Assume that we now choose a random tuple (x_1, x_2, x_3) such that each $x_i \in \{0, 1, 2, 3, 4, 5, 6\}$ is possible with equal probability and the outcome for x_i is independent of the outcome for x_j when $i \neq j$. What is the probability that this tuple is a solution to the equation?
- (c) The following Python code attempts to estimate your answer from (b).

```
import random
def trial():
    x1 = random.randint(0, 2)
    x2 = random.randint(0, 2)
    x3 = random.randint(0, 2)
    if x1 + x2 + x3 == 6:
        return 1
    else:
        return 0

N = 10**6
sum([trial() for _ in range(N)]) / N
```

The output of this code is about 0.037 (with actual numbers varying depending on the random numbers generated). This should not agree with your answer for (b). Correct the bug(s) in the code so that you get a result that approximately agrees with the correct answer for (b). Run your corrected code and show the output. Also briefly explain what happens in the corrected code.

Solution: (a)

There are 28 solutions.

```
>>> len([(a,b,c) for a in range(0,7) for b in range(0,7) \
... for c in range(0,7) if a + b + c == 6])
28
```

The first x has 7 choices (0,1,2,3,4,5,6).

If the first x chose 0, then the second x has 7 choices (0,1,2,3,4,5,6).

If the first x chose 1, then the second x has 6 choices (0,1,2,3,4,5).

If the first x chose 2, then the second x has 5 choices (0,1,2,3,4).

If the first x chose 3, then the second x has 4 choices (0,1,2,3).

If the first x chose 4, then the second x has 3 choices (0,1,2).

If the first x chose 5, then the second x has 2 choices (0,1).

If the first x chose 6, then the second x has 1 choice (0).

The third x always has one choice: 6 minus the sum of the first two x 's.

Solution: (b)

Let's find the cardinality of the sample space. Each of the three x 's has 7 options to choose from. Hence,

$$|\Omega| = 7^3 = 343.$$

Let S be the event that a tuple is a solution to the equation. In the previous question, we calculated that $|S| = 28$.

$$\therefore \mathbb{P}(S) = \frac{|S|}{|\Omega|} = \frac{28}{343} = \frac{4}{49} \approx 0.081633$$

Solution: (c)

The fixed code:

```
1 import random
2 def trial():
3     x1 = random.randint(0, 6)
4     x2 = random.randint(0, 6)
5     x3 = random.randint(0, 6)
6     if x1 + x2 + x3 == 6:
7         return 1
8     else:
9         return 0
10
11 N = 10**6
12 sum([trial() for _ in range(N)]) / N
```

Output:

0.081638

The faulty code was drawing x_i from the set $\{0, 1, 2\}$, not $\{0, 1, 2, 3, 4, 5, 6\}$. By adjusting the upper bound of `random.randint()` on lines 3, 4, and 5, we correctly draw x_i from the correct set, $\forall i \in \{1, 2, 3\}$.

Question 4

A standard deck of playing cards consists of 52 cards, divided into four suits: hearts ♡, diamonds ◇, clubs ♣, and spades ♠. Each suit contains 13 ranks: ace, numbers 2 through 10, jack, queen, and king.

You are dealt a hand of 6 cards at random from a full deck of 52 cards.

- (a) What is the probability that all six cards belong to the same suit?
- (b) What is the probability that the hand contains exactly one pair (two cards of the same rank, with the other four cards all having different ranks)?
- (c) What is the probability that the hand contains three cards of one rank and three cards of another rank (i.e., two trips)?

Solution: (a)

There are $\binom{52}{6} = \frac{52!}{6!(52-6)!} = 20,358,520$ total 6 card hands.

To get dealt a 6 card hand, with matching suits, there are:

- 4 choices for the suit.
- 13 choose 6 ways to select cards from that suit.

So there are $4 \cdot \binom{13}{6} = 6864$ 6 card hands, with all matching suits.

$$\therefore \mathbb{P}(6 \text{ cards, same suit}) = \frac{|\{6 \text{ cards, matching suit}\}|}{|\{6 \text{ card hands}\}|} = \frac{6864}{20,358,520} \approx 0.000337 = 0.0337\%$$

Solution: (b)

To get dealt a 6 card hand containing exactly one pair, we must:

- Choose 1 rank from 13.
- Choose 2 suits for the pair from 4.
- Choose 4 ranks from the remaining 12.
- Choose any suit for the remaining 4 cards.

So there are $\binom{13}{1} \binom{4}{2} \binom{12}{4} 4^4 = 13 \cdot 6 \cdot 495 \cdot 256 = 9,884,160$ hands with exactly 1 pair.

$$\therefore \mathbb{P}(6 \text{ cards, one pair}) = \frac{9,884,160}{20,358,520} \approx 0.485504 = 48.5504\%$$

Solution: (c)

To get dealt a 6 card hand with two trips, we must:

- Choose 2 rank from 13.
- Choose 3 suits for the first set of trips, from 4.
- Choose 3 suits for the second set of trips, from 4.

So, there are $\binom{13}{2} \binom{4}{3} \binom{4}{3} = 78 \cdot 4 \cdot 4 = 1248$

$$\therefore \mathbb{P}(6 \text{ cards, two trips}) = \frac{1248}{20,358,520} \approx 0.000061 = 0.0061\%$$

Question 5

Assume that you are putting r identical business cards randomly into n envelopes with $r \geq n$. As you put the business cards in the envelopes you don't notice if the envelope already has cards in it or not. Hence some envelopes can get multiple cards and other envelopes may end up without any cards at all. You are interested in the event that all envelopes are non empty. More specifically, denote the event that the i th envelope is empty as A_i . With this notation, you are interested in the event $B = \cap_{i=1}^n A_i^c$. The various items of this problem help you determine an expression for this probability.

(a) Determine a different real life situation (not business cards in envelopes) where knowing this probability is of interest.

(b) Denote,

$$p_k = \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}),$$

for any distinct collection of indices i_1, \dots, i_k with $k \leq n$. What is an expression for p_k ? Use counting arguments to arrive at this expression and explain your answer.

(c) Represent $\mathbb{P}(B)$ using one of De Morgan's laws and the inclusion-exclusion formula for n events. Do this using the probabilities p_k you computed above (for $k = 1, \dots, n$). Find the neatest expression possible.

(d) Assume we have $n = 20$ envelopes and $r = 60$ cards. Evaluate $\mathbb{P}(B)$. Use Python for a script of evaluating these probabilities. Present your code.

(e) Staying with $n = 20$, what is the minimal r needed to ensure that $\mathbb{P}(B) > 0.9$. Again use a Python script for this computation and present your code.

Solution: (a)

Suppose my server has n worker nodes, and is receiving r tasks (GET requests, database queries, etc.). Some worker nodes may receive multiple tasks, others may receive none. I am interested in the probability that every node is receiving at least one task, minimising my server's idle time.

Solution: (b)

This notation is describing the probability that k envelopes are empty. Given that I have n envelopes, and $r \geq n$ business cards, $k < n$, and the r business cards have filled the remaining $n - k$ envelopes.

p_k is a probability; so I need to count the number of ways we can fill n envelopes with r business cards (Ω), and I need to count the number of ways we can fill $n - k$ with r business cards.

r business cards are independently and uniform-randomly placed in n envelopes. Therefore $|\Omega| = n^r$.

r business cards are independent and uniform-randomly placed in $n - k$ envelopes, the k empty ones are ignored. $|P_k| = (n - k)^r$.

Therefore, $p_k = \frac{|P_k|}{|\Omega|} = \frac{(n - k)^r}{n^r} = \left(\frac{n - k}{n}\right)^r$

Solution: (c)

We'll start by applying De Morgan's Law to the given expression of $\mathbb{P}(B)$.

$$\mathbb{P}(B) = \mathbb{P}\left(\bigcap_{i=1}^n \bar{A}_i\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^n A_i\right)$$

The generalised inclusion-exclusion principle can allow us work out the probability of the union of events.

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k})$$

The summation counts through all the ways we can sequence empty envelopes, $\binom{n}{k}$, and $\mathbb{P}(A_{i_1} \cap \dots)$ is obviously our p_k ! We can make these substitutions.

$$\therefore \mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} p_k = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \left(\frac{n-k}{n}\right)^r$$

Finally, we can represent the probability that every envelope receives at least one business card,

$$\mathbb{P}(B) = 1 - \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \left(\frac{n-k}{n}\right)^r$$

Solution: (d)

My code:

```
1 from math import comb
2
3 def pr_allfilled(n, r):
4     return 1 - sum([
5         (-1) ** (k + 1) * comb(n, k) * ((n - k) / n) ** r \
6         for k in range(1, n+1)
7     ])
8
9 n = 20
10 r = 60
11
12 pb = pr_allfilled(n, r)
13 print(f"P(B) = {pb:.6f}")
```

Output:

P(B) = 0.360605

Solution: (e)

My code:

```
1 from math import comb
2
3 def pr_allfilled(n, r):
4     return 1 - sum([
5         (-1) ** (k + 1) * comb(n, k) * ((n - k) / n) ** r \
6         for k in range(1, n+1)
7     ])
8
9 n = 20
10 r = 60
11 while pr_allfilled(n, r) < 0.9:
12     r += 1
13
14 print(f"P(B) = {pr_allfilled(n,r):.6f}\nr      = {r}")
```

Output:

P(B) = 0.897151

r = 103

Question 6

Suppose a potential probability density function of a random variable X is given by

$$f(x) = \begin{cases} K(4x - x^2), & 0 < x < 3, \\ 0, & \text{otherwise,} \end{cases}$$

for some $k > 0$.

- (a) Find a value of K that makes $f(x)$ a density function or if there is not such value, argue why.
- (b) Determine the cumulative distribution function $F(x)$. Specify it for all $x \in \mathbb{R}$.
- (c) Find the mean of X , but try to reason if it is < 1.5 or > 1.5 , before computing it.
- (d) Find the variance and standard deviation of X .
- (e) Plot the density of X together with a density of a normal distribution with the same mean and variance.

Solution: (a)

We note that $\int_{-\infty}^0 0 dx = \int_3^{\infty} 0 dx = 0 \geq 0$

So, we are interested in finding $K \in \mathbb{R} : \int_0^3 K(4x - x^2) dx = 1, K(4x - x^2) \geq 0, \forall x \in (0, 3)$.

$$\begin{aligned} \int_0^3 K(4x - x^2) dx &= K \int_0^3 (4x - x^2) dx \\ &= K \left[2x^2 - \frac{1}{3}x^3 \right]_0^3 \\ &= K \left(2(3)^2 - \frac{1}{3}(3)^3 \right) \\ &= 9K = 1 \\ &\iff K = \frac{1}{9} \end{aligned}$$

Let's find the x -intercepts (we can ignore K when finding the intercepts here!)

$$-x^2 + 4x = (x)(4 - x) \iff \text{intercepts at } (0, 0) \text{ and } (4, 0).$$

Since the coefficient for x^2 is -1 , and the function has two real roots, we know that $\forall x \in [0, 4], (4x - x^2) \geq 0$. And since $(0, 3) \subset [0, 4]$, we know that $\forall x \in \mathbb{R}, f(x) = \frac{1}{9}(4x - x^2) \geq 0$. Therefore, we've found a value for K , namely $\frac{1}{9}$, that makes $f(x)$ a valid probability density function.

Solution: (b)

For values of $x \leq 0$, $F(x) = 0$.

For values of $x \geq 3$, $F(x) = 1$.

For values of $x \in (0, 3)$, $F(x) = \int_0^x f(t)dt$.

We can expand this out, and try and find an integral-less expression.

$$\begin{aligned}\int_0^x f(t)dt &= \int_0^x \frac{1}{9}(4t - t^2)dt \\ &= \frac{1}{9} \int_0^x (4t - t^2)dt \\ &= \frac{1}{9} \left[2t^2 - \frac{1}{3}t^3 \right]_0^x \\ &= \frac{1}{9} \left(2x^2 - \frac{1}{3}x^3 \right) \\ &= \frac{2}{9}x^2 - \frac{1}{27}x^3\end{aligned}$$

Therefore, our cumulative distribution function is

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{2}{9}x^2 - \frac{1}{27}x^3, & 0 < x < 3 \\ 1, & x \geq 3 \end{cases}$$

Solution: (c)

I have a feeling that the mean is greater than 1.5. Because when we figured out the x -intercepts, I noticed that the lower values between 3 and 4 were removed, probably skewing the expected value upward.

Let's compute $\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx$.

$$\begin{aligned}\int_{-\infty}^{\infty} xf(x)dx &= \frac{1}{9} \left(\int_{-\infty}^0 0dx + \int_0^3 (4x^2 - x^3)dx + \int_3^{\infty} 0dx \right) \\ &= \frac{1}{9} \left(0 + \left[\frac{4}{3}x^3 - \frac{1}{4}x^4 \right]_0^3 + 0 \right) \\ &= \frac{1}{9} \left(\frac{4}{3}(27) - \frac{1}{4}(81) \right) \\ &= \frac{1}{9} \left(36 - \frac{81}{4} \right) \\ &= \frac{36}{9} - \frac{9}{4} \\ &= \frac{63}{36} \\ \therefore \mu &= 1.75\end{aligned}$$

Solution: (d)

Variance computing time B)

We know that $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. We've already calculated the mean, so let's compute the second moment.

$$\mathbb{E}[X^2] = \frac{1}{9} \left(\int_{-\infty}^0 0dx + \int_0^3 (4x^3 - x^4)dx + \int_3^{\infty} 0dx \right)$$

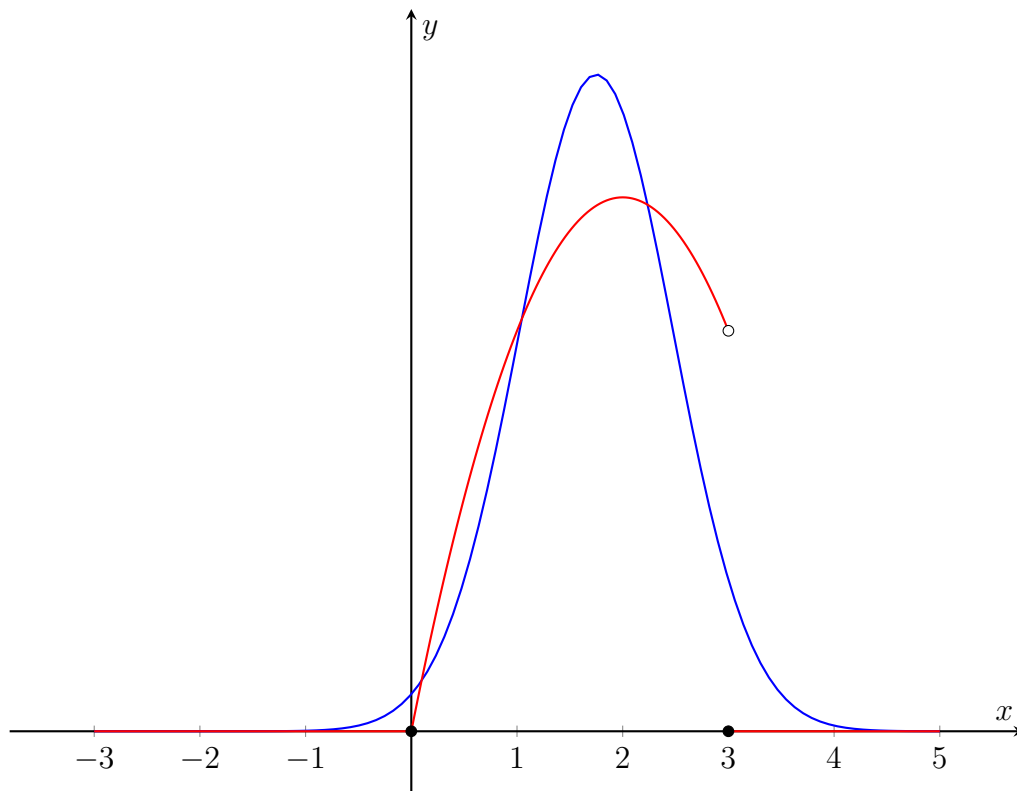
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$$\begin{aligned}
&= \frac{1}{9} \left(0 + \left[x^4 - \frac{1}{5}x^5 \right]_0^3 + 0 \right) \\
&= \frac{1}{9} \left((3)^4 - \frac{1}{5}(3)^5 \right) \\
&= \frac{81}{9} - \frac{243}{45} \\
&= \frac{162}{45} \\
\therefore \mathbb{E}[X^2] &= 3.6 \\
\therefore \text{Var}(X) &= 3.6 - (1.75)^2 \\
&= 3.6 - 3.0625 \\
&= 0.5375 \\
\sigma(X) &= \sqrt{\text{Var}(X)} \\
&= \sqrt{0.5375} \\
\therefore \sigma(X) &\approx 0.733144
\end{aligned}$$

Solution: (e)

My plot:

Normal Distribution, $\mu = 1.75$, $\sigma = 0.73$, and $f(x)$



Pretty good!

Question 7

Consider a random variable X , distributed with an exponential distribution with rate parameter λ (mean is $1/\lambda$). Assume that from X we obtain two discrete random variables, $Y_0 = \lfloor X \rfloor$ and $Y_1 = \lceil X \rceil$.

- (a) Prove that the n 'th moment of X is $n!/\lambda^n$, namely,

$$\mathbb{E}[X^n] = \frac{n!}{\lambda^n}, \quad \text{for } n = 0, 1, 2, \dots$$

Do so directly via (repeated) integration by parts.

- (b) Derive the moment generating function of X .
 (c) Use the moment generating function to verify the moment expressions from (a).
 (d) Recall the memoryless property of X , state it, and repeat its proof.
 (e) Determine the distribution of Y_0 and Y_1 , and present their probability mass functions. Hint:

$$\mathbb{P}(Y_0 = k) = \int_k^{k+1} \lambda e^{-\lambda x} dx.$$

- (f) Do the distributions of Y_0 and Y_1 fit within a paradigm of some known distribution family. If so, identify the distribution families and determine the parameters (in terms of λ).

Solution: (a)

$$\begin{aligned} \mathbb{E}[X^n] &= \int_0^\infty x^n \lambda e^{-\lambda x} dx \\ \text{Let } u &= x^n \Rightarrow du = nx^{n-1} \\ \text{Let } dv &= e^{-\lambda x} \Rightarrow v = \frac{-e^{-\lambda x}}{\lambda} \\ \therefore \int_0^\infty x^n \lambda e^{-\lambda x} dx &= \left. \frac{-x^n \lambda e^{-\lambda x}}{\lambda} \right|_0^\infty - \int_0^\infty \frac{-nx^{n-1} \lambda e^{-\lambda x}}{\lambda} dx \\ \lim_{x \rightarrow \infty} e^{-\lambda x} &= 0 \Rightarrow \left. \frac{-x^n e^{-\lambda x}}{\lambda} \right|_0^\infty = 0 \\ \therefore \int_0^\infty x^n \lambda e^{-\lambda x} dx &= \int_0^\infty \frac{nx^{n-1} \lambda e^{-\lambda x}}{\lambda} dx \\ &= \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x} dx \\ \text{If we let } I_n &= \int_0^\infty x^n e^{-\lambda x} dx \text{ such that } \mathbb{E}[X^n] = \lambda I_n \\ \text{We get } I_n &= \frac{n}{\lambda} I_{n-1} \\ I_0 &= \mathbb{E}[X] = \frac{1}{\lambda} \text{ by the exponential distribution} \\ \therefore I_1 &= \frac{1}{\lambda^2}, I_2 = \frac{2}{\lambda^3}, I_3 = \frac{2 \cdot 3}{\lambda^4}, \dots, I_n = \frac{n!}{\lambda^{n+1}} \end{aligned}$$

$$\therefore \mathbb{E}[X^n] = \lambda I_n = \frac{n!}{\lambda^n}$$

Solution: (b)

We know that the MGF of a continuous random variable is given by

$$M(t) = \int_x e^{tx} f(x) dx.$$

So let's plug in $f(x)$ and derive this bad boy.

$$\begin{aligned} M(t) &= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{(t-\lambda)x} dx \\ &= \lambda \cdot \frac{1}{\lambda - t} \\ \therefore M(t) &= \frac{\lambda}{\lambda - t} \end{aligned}$$

Solution: (c)

Let's take some derivatives!

$$\begin{aligned} M(t) &= \frac{\lambda}{\lambda - t} \\ M'(t) &= \frac{\lambda}{(\lambda - t)^2} \\ M''(t) &= \frac{2\lambda}{(\lambda - t)^3} \\ M'''(t) &= \frac{2 \cdot 3 \cdot \lambda}{(\lambda - t)^4} \\ M''''(t) &= \frac{2 \cdot 3 \cdot 4 \cdot \lambda}{(\lambda - t)^5} \end{aligned}$$

Given this, we can generalise!

$$M^{(n)}(t) = \frac{n! \lambda}{(\lambda - t)^{n+1}}$$

Now, we'll evaluate this for $t = 0$,

$$\begin{aligned} M^{(n)}(0) &= \frac{n! \lambda}{(\lambda - 0)^{n+1}} \\ &= \frac{n! \lambda}{(\lambda)^{n+1}} \\ &= \frac{n! \lambda}{(\lambda)^{n+1}} \\ &= \frac{n!}{\lambda^n} \end{aligned}$$

Which is the same expression we got for $\mathbb{E}[X^n]$ in part (a)!

Solution: (d)

Theorem 2.2. (The Memoryless Property). Let $X \sim \text{Exp}(\lambda)$. Then

$$\mathbb{P}(X > x + y \mid X > x) = \mathbb{P}(X > y), \quad \forall x, y > 0.$$

Proof. Directly.

By the definition of conditional probability,

$$\mathbb{P}(X > x + y \mid X > x) = \frac{\mathbb{P}(X > x + y, X > x)}{\mathbb{P}(X > x)}.$$

$X > x + y$ is a continued event of $X > x$; If the former has happened, the latter necessarily has, because $X > x + y > x$. So the intersection of the sets, equals the bigger set, hence,

$$= \frac{\mathbb{P}(X > x + y)}{\mathbb{P}(X > x)}.$$

Next, we apply the exponential distribution's function,

$$= \frac{e^{\lambda(x+y)}}{e^{\lambda x}},$$

and simplify.

$$= e^{\lambda y}.$$

$$\therefore \mathbb{P}(X > x + y \mid X > x) = \mathbb{P}(X > y)$$

□

Solution: (e) We'll start with Y_0 .

$$\begin{aligned} Y_0 = k &\implies X \in [k, k + 1) \\ \therefore \mathbb{P}(Y_0 = k) &= \mathbb{P}(k \leq X < k + 1) \\ &= \int_k^{k+1} \lambda e^{-\lambda x} dx \\ &= -e^{-\lambda(k+1)} + e^{-\lambda k} \\ &= e^{-\lambda k}(1 - e^{-\lambda}), \quad k = 0, 1, 2, 3, \dots \end{aligned}$$

Now we follow a similar process for Y_1

$$\begin{aligned} Y_1 = k &\implies X \in (k - 1, k] \\ \therefore \mathbb{P}(Y_1 = k) &= \mathbb{P}(k - 1 < X \leq k) \\ &= \int_{k-1}^k \lambda e^{-\lambda x} dx \\ &= -e^{-k\lambda} + e^{-\lambda(k-1)} \\ &= (e^{\lambda} - 1)e^{-k\lambda} \\ &= (1 - e^{-\lambda})e^{-(k+1)\lambda}, \quad k = 0, 1, 2, 3, \dots \end{aligned}$$

Solution: (f)

Both Y_0 and Y_1 are similar to discrete geometric probability mass functions, with $p = e^{-\lambda}$. But, because e^{-1} is a factor, it's going to be the case that we're counting the number of successes until failure, as opposed to the number of failures until success, in the case of a "regular" geometric PMF.

$$Y_0 \sim \text{Geom}(e^{-\lambda}), (k \geq 0)$$

$$Y_1 \sim \text{Geom}(e^{-\lambda}), (k \geq 1)$$

Y_1 's k must be greater than or equal to 1, as a consequence of its domain.

Question 8

Assume that a weight of a manufactured item is uniformly distributed between 19.25 and 20.75 Kg, with a mean of 20Kg and a standard deviation of σ . A manufacturing facility creates 20 items daily with the manufacturing of items assumed independent. An item is considered faulty if its weight is not within the range $[20 - \sigma, 20 + \sigma]$.

- (a) Determine the cumulative distribution function $F(x)$ of a manufactured item. Specify it for all $x \in \mathbb{R}$.
- (b) Determine σ .
- (c) Use $F(x)$ to determine the probability of an item being faulty.
- (d) Determine the distribution of the number of faulty items per day.
- (e) What is the mean and variance of the number of faulty items per day?
- (f) Numerically determine the probability of having 17 or less faulty items per day. Use a calculator only (no software).

Solution: (a)

My interpretation of the information given is that $\forall x \notin (19.25, 20.75), f(x) = 0$.

Therefore, $\forall x \in (-\infty, 19.25], F(x) = 0$.

Therefore, $\forall x \in [20.75, \infty), F(x) = 1$.

Because the weight is uniformly distributed for $x \in (19.25, 20.75)$, the CDF will be linear in this range.

$$\text{Ansatz } F(x) = mx + c$$

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{1 - 0}{20.75 - 19.25} \\ &= \frac{1}{1.5} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} F(19.25) = 0 &= \frac{2}{3}(19.25) + c \\ \iff c &= -\frac{77}{6} \end{aligned}$$

$$\therefore F(x) = \begin{cases} 0, & x \leq 19.25 \\ \frac{2}{3}x - \frac{77}{6}, & 19.25 < x < 20.75 \\ 1, & x \geq 20.75 \end{cases}$$

Solution: (b)

$$\sigma = \sqrt{\text{Var}(X)} = \sqrt{\frac{(19.25 - 20.75)^2}{12}} \approx 0.433013$$

Solution: (c)

Let's first define the boundary of a faulty item.

$$[20 - \sigma, 20 + \sigma] = [20 - 0.433, 20 + 0.433] = [19.567, 20.433] \subset (19.25, 20.75)$$

Because the faulty region is a subset of the interval, we can simply apply

$$\mathbb{P}(\text{not faulty}) = \frac{20.433 - 19.567}{20.75 - 19.25} = \frac{0.866}{1.5} = 0.577\dot{3}$$

Finally, we take the complement of this probability,

$$\mathbb{P}(\text{faulty}) = 1 - \mathbb{P}(\text{not faulty}) = 1 - 0.577\dot{3} = 0.422\dot{6} \approx 42.2667\%$$

Solution: (d)

Since we have a fixed number of trials per day, 20, and a probability of failure/success, (42.2667%, 57.7333%), a binomial distribution is a no-brainer. Therefore, if X is the number of faulty items per day,

$$X \sim \text{Binomial}(20, 0.4227)$$

Note we are using $p = 0.4227$, since we're counting the faulty items as "successes" here.

Solution: (e)

Binomial distributions have some very neat formulas.

$$\mathbb{E}[X] = np = 20 \cdot 0.4227 = 8.454$$

$$\text{Var}(X) = np(1 - p) = 20 \cdot 0.4227(0.5773) = 4.8805$$

Solution: (f)

We'll calculate $\mathbb{P}(X > 17) = \mathbb{P}(x=18) + \mathbb{P}(x=19) + \mathbb{P}(x=20)$, and then take the complement to calculate $\mathbb{P}(x \leq 17) = 1 - \mathbb{P}(X > 17)$.

$$\begin{aligned} \mathbb{P}(X > 17) &= \mathbb{P}(x=18) + \mathbb{P}(x=19) + \mathbb{P}(x=20) \\ &= \binom{20}{18} (0.4227)^{18} (0.5773)^2 + \binom{20}{19} (0.4227)^{19} (0.5773)^1 + \binom{20}{20} (0.4227)^{20} (0.5773)^0 \\ &= 190 \cdot 0.00018560 \cdot 0.33327529 + 20 \cdot 0.0000000785 \cdot 0.5773 + 1 \cdot 0.0000000332 \cdot 1 \\ &= 0.01175262 + 0.00000091 + 0.00000003 \\ &= 0.011754 \end{aligned}$$

$$\therefore \mathbb{P}(X \leq 17) = 1 - 0.011754 = 0.988245$$

I feel compelled to apologise. I don't think this was my best work. I misread my assignment planner/put the wrong date in following the cyclone situation and all that.

I submitted an assignment at midnight on Thursday night. Casually. For no reason in particular, after I submitted it, I thought "hey! while im on gradescope, let's check in on STAT2003. It's due on Monday, I can't wait to dedicate the next three days to working on it!"

My heart completely sank when I saw "due in 12 hours." I sighed deeply, grabbed a couple red bulls from my fridge and locked in.

If I get this in on time, it'll be a miracle.

Please accept my appologies! I'll try and stay on-track with my planner for the next one!