MATH1061
Advanced Multivariate Calculus & Ordinary Differential Equations

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Contents

Chapter 1

Week 1

1.1 Lecture 1

In this course, we will be looking at:

- functions of several variables and calculus
- vector calculus. Rates of change of vector valued functions and applications!
- differential equations
- MATLAB - Only 6 lab sessions. MATLAB will be incorprated into assignments.

An overview of the tools of applied mathematics

- Creating and studying models of phenomena in the world:
  - physics
  - chemistry
  - biology
  - ecology
  - economics
  - engineering.
- $\boxed{\text{natural world}} \xRightarrow{\text{simplification}} \boxed{\text{mathematical model}}$.
- $\boxed{\text{mathematical model}} \xRightarrow[\text{validation}]{\text{interpretation}} \boxed{\text{natural world}}$.
- Most importantly the $\boxed{\text{mathematical model}}$ offers predictive power.
- Modelling: identify key variables and processes.
- Formulation:
  - functions of several variables
  - ordinary differential equations (involving single variable rates of change)
  - WE WILL NOT TOUCH: partial differential equations (involving functions of several variables)
  - WE WILL NOT TOUCH: statistical models

Dimensional Analysis

Definition 1.1.1: Base Quantities

There exist base quantities (or dimensions) that provide units in terms of which the units of all other physical quantities can be expressed. Conventionally, these are: mass (M), length (L), time (T) (and temperature, electric current, amount of substance, luminous intensity).

Example 1.1.1 (A falling mass)
Suppose we conduct an experiment on the time, t, it takes an object of mass m, to fall a distance of x from rest in a vaccum (near the surface of the Earth).

In Australia we find that

$$x = 4.91t^2 \text{ (metres)},$$

Our friend in the USA finds that

$$x = 16.1t^2 \text{(feet)}.$$

It would be correct to write $x = ct^2$, where c is a physical quantity, depending on units, $c = \frac{1}{2}g$.

Some quantities have dimensions as a product $M^a L^b T^c$, where $a, b, c \in \mathbb{Z}$. Let $[y]$ denote the dimensions of y and $[x]$ the dimensions of x. Then $[x, y] = [x][y]$.

Example 1.1.2 (Finding dimensions of physical quantities)
Velocity $\left(\frac{dx}{dt}\right)$:

$$\left[\frac{dx}{dt}\right] = [x][t]^{-1} = LT^{-1}$$

Acceleration $\left(\frac{dx}{dt^2}\right)$:

$$\left(\frac{dx}{dt^2}\right) = [x][t]^{-2} = LT^{-2}$$

Force $m\left(\frac{d}{dt}\right)\left(\frac{dx}{dt}\right)$

$$[F] = [m][t]^{-1}[x][t]^{-1} = MLT^{-2}$$

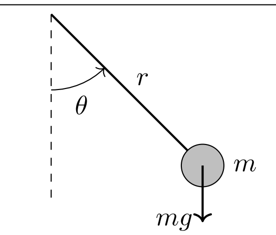
We call a quantity with dimensions $M^0L^0T^0$ **dimensionless**.
An equation that is true regardless of units is said to be **dimensionally homogeneous**. In such an equation, the dimensions of all terms must be the same.

Claim 1.1.1 Equations representing physical laws are dimensionally homogeneous.
To achieve this in our mathematical model we seek *all possible* dimensionless products among the variables. Such a collection is called **complete set**.

1.2 Lecture 2

A Simple Pendulum

Consider the simple pendulum, with mass m, length r released from angle of displacement θ, and acted upon by gravity g.



We want to find the relationship between the period, τ and θ, m, r and g. I.e. we want to find

$$\tau = f(r, m, \theta, g)$$

First, lets find the dimensions of the system.

$[\tau]$	$[r]$	$[m]$	$[\theta]$	$[g]$
T	L	M	1	LT^{-2}

Note:-

g is a dimensionless variable! In other words, its dimensions are $M^0L^0T^0$.

The product of these varaibles takes the form:

$$\tau^a r^b m^c \theta^d g^e$$

and its dimensions are

$$[\tau^a r^b m^c \theta^d g^e] = [\tau^a][r^b][m^c][\theta^d][g^e] = T^a \cdot L^b \cdot M^c \cdot 1^d \cdot L^e T^{-2e} = M^c L^{b+e} T^{a-2e}.$$

Since we desire to solve represent a physical law, by claim 1.1.1, we know we're looking for a dimensionally homogeneous system.

$$\text{Let } M^c L^{b+e} T^{a-2e} = M^0 L^0 T^0$$

$$\text{Then} \quad \left.\begin{aligned} a - 2e &= 0 \\ b + e &= 0 \\ c &= 0 \end{aligned}\right\} e = \frac{a}{2}, b = -\frac{a}{2}, d \text{ is free.}$$

In other words, solve the linear system $A\underset{\sim}{x} = \underset{\sim}{0}$ (In other OTHER words, solve for the nullspace of $\mathcal{N}(A)$.)

$$\begin{aligned} \tau^a r^{-\frac{a}{2}} m^0 \theta^d g^{\frac{a}{2}} &= \left(\tau r^{-\frac{1}{2}} g^{\frac{1}{2}}\right)^a \theta^d \\ &= \left(\tau \sqrt{\frac{g}{r}}\right)^a \theta^d \end{aligned}$$

By setting (a, d) to $(1, 0)$, and $(0, 1)$, we obtain independent dimensionless products, Π_1 and Π_2.

$$\Pi_1 = \tau\sqrt{\frac{g}{r}}$$

$$\Pi_2 = \theta$$

The Buckingham Π-Theorem

Theorem 1.2.1 Buckingham Π-Theorem

An equation is dimensionally homogeneous if and only if it can be written in the form

$$f(\Pi_1, \Pi_2, \ldots, \Pi_n) = 0,$$

where f is some function satisfying certain conditions (outside out scope) and $\{\Pi_1, \Pi_2, \ldots, \Pi_n\}$ is a complete set of dimensionless products corresponding to some mathematical model.

Note:-

Π_k is dimensionless, i.e. $\forall \Pi_k, [\Pi_k] = 1$

The set (Π_k) can be obtained by giving all solutions to a linear system of exponents for the model. We've found the complete set of dimensionless products for the pendulum problem, $\left\{\tau\sqrt{g/r}, \theta\right\}$. Applying the Buckingham Π-theorem now, our mathematical model is of the form:

$$f(\Pi_1, \Pi_2) = 0 \implies f(\tau\sqrt{g/r}, \theta) = 0$$

We further assume that from f we can deduce

$$\tau\sqrt{\frac{g}{r}} = h(\theta) \qquad \text{by Implict Function Theorem.}$$

We'll describe implicit function theorem in detail later.

Note:-

If $\Pi = \{\Pi_1, \Pi_2, \Pi_3\}$, then $\Pi_1 = h(\Pi_1, \Pi_2)$. More generally, if $\Pi = \{\Pi_k \mid k \in \mathbb{N}, k \leq n\}$, then $\Pi_1 = h\left(\Pi_2, \Pi_3, \ldots, \Pi_n\right)$.

Archimedes' Law

The famous "Eureka!" leaping from the bathtub one ;P

Archimedes' law applies to bodies immersed in a fluid. Consider a box of mass m, which displaces V fluid, with constant density ρ. Suppose your class mate claims that this phenomenon is described by the equation

$$m\frac{d^2x}{dt^2} = mg - mVg$$

Let's verify this. . .

$$\begin{aligned}
[mVg] &= [m][V][g] \\
&= M \cdot L^3 \cdot LT^{-2} \\
&= ML^4T^{-2} \\
\left[m\frac{d^2x}{dt^2}\right] &= [m]\left[\frac{d^2x}{dt^2}\right] \\
&= MLT^{-2} \\
&\neq [mVg]
\end{aligned}$$

So we can conclude that this model is not dimensionally consistent.
Another classmate suggests the equation

$$m\frac{d^2x}{dt^2} = mb - \rho Vg.$$

We'll similarlly analyse this like

$$\begin{aligned}
[\rho Vg] &= [\rho][V][g] \\
&= ML^{-3} \cdot L^3 \cdot LT^{-2}
\end{aligned}$$

$$
\begin{aligned}
&= ML^1T^{-2} \\
&= MLT^{-2} \\
\left[m\frac{d^2x}{dt^2}\right] &= [m]\left[\frac{d^2x}{dt^2}\right] \\
&= MLT^{-2} \\
&= [\rho V g]
\end{aligned}
$$

Which is dimensionally consistent!

Let's use Buckingham Π-theorem to establish the general form any correct model must take:

Consider the product $F^a\rho^b g^c V^d m^e$

$$
\begin{aligned}
\left[F^a\rho^b g^c V^d m^e\right] &= (MLT^{-2})^a(ML^{-3})^b(LT^{-2})^c(L^3)^d(M)^e \\
&= M^aL^aT^{-2a}\cdot M^bL^{-3b}\cdot L^cT^{-2c}\cdot L^{3d}\cdot M^e \\
&= M^{a+b+e}L^{a-3b+c+3d}T^{-2a-2c} \\
\text{Let } M^0L^0T^0 &= M^{a+b+e}L^{a-3b+c+3d}T^{-2a-2c} \\
\Rightarrow \quad \left.\begin{array}{rcl} a+b+e &=& 0 \\ a-3b+c+3d &=& 0 \\ -2a-2c &=& 0 \end{array}\right\} & \begin{array}{rcl} c &=& -a \\ b &=& -a-e \\ d &=& -a-e \end{array} \\
\text{So } F^a\rho^b g^c V^d m^e &= F^a\rho^{-a-e}g^{-a}V^{-a-e}m^e \\
&= F^a\rho^{-a}g^{-a}V^{-a}\rho^{-e}V^{-e}m^e \\
&= \left(F\rho^{-1}g^{-1}V^{-1}\right)^a\left(\rho^{-1}V^{-1}m\right)^e \\
&= \left(\frac{F}{\rho g V}\right)^a\left(\frac{m}{\rho V}\right)^e \\
\therefore \Pi &= \left\{\frac{F}{\rho g V}, \frac{m}{\rho V}\right\}.
\end{aligned}
$$

Now that we've deduced Π, we know that any valid physcial law must take the form:

$$
\begin{aligned}
\frac{F}{\rho g V} &= h\left(\frac{m}{\rho V}\right) \\
\implies F &= \rho g V h\left(\frac{m}{\rho V}\right)
\end{aligned}
$$

Note:-

Generally, when proceeding with the linear algebra portion of this procedure, keep in mind the power of your desired dependent variable, and try to express all other powers in terms of it. For example, in the pendulum example, and in the Archimedes' law exmaple too, we expressed the other variables in terms of a, because this was the power of the desired dependent variables τ and F, respectively.

1.3 Lecture 3

Drag Force

Consider a sphere of radius r, moving through a viscous fluidd. We wish to model the drag force, F, dependent on the relevant variables η, the viscosity, v, the velocity, and r the radius.

Consider the product $F^a\eta^b v^c r^d$

$$
\begin{aligned}
[F]^a[\eta]^b[v]^c[r]^d &= \left(MLT^{-2}\right)^a\left(ML^{-1}T^{-1}\right)^b\left(LT^{-1}\right)^c(L)^d \\
&= \left(M^aL^aT^{-2a}\right)\left(M^bL^{-b}T^{-b}\right)\left(L^cT^{-c}\right)\left(L^d\right) \\
&= M^{a+b}L^{a-b+c+d}T^{-2a-b-c} \\
\text{Let } M^0L^0T^0 &= M^{a+b}L^{a-b+c+d}T^{-2a-b-c}
\end{aligned}
$$

$$\Rightarrow \left.\begin{array}{rcl} a+b &=& 0 \\ a-b+c+d &=& 0 \\ -2a-b-c &=& 0 \end{array}\right\} \begin{array}{rcl} b &=& -a \\ c &=& -a \\ d &=& -a \end{array}$$

$$\begin{aligned} \text{So } F^a\eta^b v^c r^d &= F^a\eta^{-a}v^{-a}r^{-a} \\ &= \left(F\eta^{-1}v^{-1}r^{-1}\right)^a \\ &= \left(\frac{F}{\eta v r}\right)^a \\ \therefore \Pi &= \left\{\frac{F}{\eta v r}\right\} \end{aligned}$$

So from the Buckingham Π-theorem, we know that many valid physcial law must take the form:

$$f\left(\frac{F}{\eta v r}\right) = 0$$

$$\implies \frac{F}{\eta v r} = k$$

In other words, $F = k\eta v r$, where k is some dimensionless constant.

Note:-

This result, that drag force is proportional to velocity is known as Stokes' Law, and in fact, the constant $k = 6\pi$.

A Mixing Model

Initially, a tank of water with v_0 litres has m_0 grams of salt dissolved in it. Brine with n grams of salt per litre runs into the tank at a rate of x litres per minute. The contents are constantly stirred (so you can assume that the concentration of salt is always uniform throughout the tank) and water runs out of the tank at the rate of y litres per minute. Let $s(t)$ denote the amount of salt in the tank at time t (measured in minutes). Determine the equation which governs the net rate of change of salt, checking that all terms are dimensionally consistent.

Let $c(t)$ denote the density of salt at time t.
Let $v(t)$ denote the volume of water in the tank at time t.

$$\begin{aligned} \text{We have } c(t) &= \frac{s(t)}{v(t)} \\ \text{And } v(t) &= v_0 + (x-y)t \\ \implies c(t) &= \frac{s(t)}{v_0 + (x-y)t}. \end{aligned}$$

Salt enters at a rate of nx grams/minute and leaves at a rate of $yc(t)$ per minute.

$$yc(t) = \frac{ys(t)}{v_0 + (x-y)t}.$$

Therefore, the net rate of change is

$$\frac{ds}{dt} = nx - \frac{ys}{v_0 + (x-y)t}, s(0) = m_0.$$

Now lets check the model for dimensional homogeneity!

$$
\begin{aligned}
\left[\frac{ds}{dt}\right] &= MT^{-1} \\
[nx] &= ML^{-3} \cdot L^3T^{-1} \\
&= MT^{-1} \\
[v_0] &= L^3 \\
[(x-y)t] &= L^3T^{-1}T \\
&= L^3 \\
\Rightarrow \left[\frac{ys}{v_0+(x-y)t}\right] &= \frac{ML^3T^{-1}}{L^3} \\
&= \frac{ML^3T^{-1}}{L^3} \\
&= MT^{-1} \\
\therefore \left[\frac{ds}{dt}\right] = [nx] &= \left[\frac{ys}{v_0+(x-y)t}\right]
\end{aligned}
$$

So, the system is dimensionally homogeneous.

Scaling

Question, can we scale experiments in a laboratory to ensure that the observed effects are consistent?
We can use dimensionless variables, and try to preserve their values. Some examples of dimensionless variables in fluid dynamics include

$$
\begin{aligned}
&\text{Reynold's Number,} \qquad Re = \frac{\rho l v}{\eta} \\
&\text{Froude's Number,} \qquad Fr = \frac{v}{\sqrt{gl}} \\
&\text{Mach Number,} \qquad M = \frac{v}{c}
\end{aligned}
$$

where, ρ is the fluid denisty, l is the length of an object, v is veloctiy, η is the viscosity of the fluid, g is gravitational acceleration, and c is the speed of sound.

A Ship Model

Suppose our model involves Fr, which we seek to keep fixed.
The true boat has hull length $l = 20\text{m}$, at speed $v = 10\text{ms}^{-1}$
We can model this with a boat of hull length $l^* = 0.2m$, i.e. $l^* = l/100$.
What is v^*?

$$
\begin{aligned}
Fr &= \frac{v}{\sqrt{gl}} = \frac{v^*}{\sqrt{gl^*}} \\
\therefore v^* &= v\sqrt{\frac{gl^*}{gl}} \\
&= v\sqrt{\frac{l^*}{l}} \\
&= 10\sqrt{\frac{0.2}{20}} \\
\therefore v^* &= 1\text{ms}^{-1}
\end{aligned}
$$

Chapter 2

Week 2

2.1 Lecture 4