

MATH2001, Assignment 1, Summer Semester, 2024-2025

Due at 2:00pm 19 December. Each question marked out of 10 then homogeneously rescaled up to a total marks of 13. **Total marks:** $\frac{13}{60}(Q1 + Q2 + Q3 + Q4 + Q5 + Q6)$. Submit your assignment online via the Assignment 1 submission link in Blackboard.

Question 1: Exact IVP after Integrating Factors - [10 marks]

Consider the equation

$$\frac{\sin y}{y} - 2e^{-x} \sin x + \left(\frac{\cos y + 2e^{-x} \cos x}{y} \right) y' = 0.$$

- (a) Show that it is not exact, but it becomes exact when multiplying by the integrating factor $\mu = ye^x$.
- (b) Solve the exact equation subject to the initial condition $y(\pi) = \frac{\pi}{2}$. Present your solution as a relation defining y implicitly as a function of x .

Solution: (a)

$$\text{Let } P(x, y) = \frac{\sin y}{y} - 2e^{-x} \sin x$$

$$\text{Let } Q(x, y) = \frac{\cos y + 2e^{-x} \cos x}{y}$$

$$\text{Then } P(x, y) + Q(x, y)y' = 0 \text{ is exact } \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{\sin y}{y} - 2e^{-x} \sin x \right) \\ &= \frac{y \cos y - \sin y}{y^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{\cos y + 2e^{-x} \cos x}{y} \right) \\ &= \frac{-2e^{-x} \cos x - 2e^{-x} \sin x}{y} \\ &= \frac{-2e^{-x}(\cos x + \sin x)}{y} \neq \frac{\partial P}{\partial y} \end{aligned}$$

Let's now define new functions which are scaled by integrating factor $\mu = ye^x$.

$$\text{Let } \hat{P}(x, y) = \mu P(x, y) = e^x \sin y - 2y \sin x$$

$$\text{Let } \hat{Q}(x, y) = \mu Q(x, y) = e^x \cos y + 2 \cos x$$

$$\begin{aligned}
\frac{\partial \widehat{P}}{\partial y} &= \frac{\partial}{\partial y} (e^x \sin y - 2y \sin x) \\
&= e^x \cos y - 2 \sin x \\
\frac{\partial \widehat{Q}}{\partial x} &= \frac{\partial}{\partial x} (e^x \cos y + 2 \cos x) \\
&= e^x \cos y - 2 \sin x = \frac{\partial \widehat{P}}{\partial y}
\end{aligned}$$

Therefore, while the ODE

$$P(x, y) + Q(x, y)y' = 0$$

is not exact, multiplying the equation by an integrating factor, μ gives the ODE

$$\widehat{P}(x, y) + \widehat{Q}(x, y)y' = e^x \sin y - 2y \sin x + (e^x \cos y + 2 \cos x)y' = 0$$

which is exact.

Solution: (b)

Since the ODE $\widehat{P}(x, y) + \widehat{Q}(x, y)y' = 0$ is exact, $\exists f$ such that $f_x = \widehat{P}(x, y)$, $f_y = \widehat{Q}(x, y)$, and $f(x, y) = K$ implicitly defines y as a function of x .

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \widehat{P}(x, y) = e^x \sin y - 2y \sin x \\
\implies f(x, y) &= \int \widehat{P}(x, y) dx = \int e^x \sin y - 2y \sin x \, dx \\
&= e^x \sin y + 2y \cos x + g(y) \\
\frac{\partial f}{\partial y} &= \widehat{Q}(x, y) = \frac{\partial}{\partial y} (e^x \sin y + 2y \cos x + g(y)) \\
e^x \cos y + 2 \cos x &= e^x \cos y + 2 \cos x + g'(y) \\
\implies g'(y) &= 0 \Rightarrow g(y) = C \\
\therefore f(x, y) &= K = e^x \sin y + 2y \cos x + C
\end{aligned}$$

Since, K and C are arbitrary constants, we will incorporate them into one arbitrary constant, K . Now, we'll apply the initial condition, $y(\pi) = \pi/2$.

$$\begin{aligned}
f\left(\pi, \frac{\pi}{2}\right) &= K = e^\pi \sin \frac{\pi}{2} + 2\frac{\pi}{2} \cos \pi \\
&= e^\pi - \pi \\
\therefore f(x, y) &= e^\pi - \pi = e^x \sin y + 2y \cos x
\end{aligned}$$

implicitly defines y as a function of x .

Question 2: Variation of Parameters - [10 marks]

Consider the non-homogeneous ODE

$$x^2 y'' - 3xy' + 4y = \ln x, \quad x > 0.$$

- (a) Show that $y_1 = x^2$ and $y_2 = x^2 \ln x$ are solutions to the corresponding homogeneous ODE $x^2 y'' - 3xy' + 4y = 0$.
- (b) Find the general solution of the non-homogeneous ODE.

Solution: (a)

Given the homogeneous ODE

$$x^2 y'' - 3xy' + 4y = 0,$$

consider the potential solution $y_1 = x^2$.

$$y_1 = x^2 \implies y_1' = 2x \implies y_1'' = 2.$$

Substitute these back into the ODE,

$$x^2 y'' - 3xy' + 4y = x^2(2) - 3x(2x) + 4(x^2) = 2x^2 - 6x^2 + 4x^2 = 0.$$

Therefore, $y_1 = x^2$ is a solution to the homogeneous ODE.

Now consider the potential solution $y_2 = x^2 \ln x$.

$$\begin{aligned} y_2 = x^2 \ln x &\implies y_2' = 2x \ln x + x^2(1/x) = 2x \ln x + x \\ &\implies y_2'' = 2 \ln x + 2x(1/x) + 1 = 2 \ln x + 3. \end{aligned}$$

Substitute these back into the ODE,

$$\begin{aligned} x^2 y'' - 3xy' + 4y &= x^2(2 \ln x + 3) - 3x(2x \ln x + x) + 4(x^2 \ln x) \\ &= 2x^2 \ln x + 3x^2 - 6x^2 \ln x - 3x^2 + 4x^2 \ln x = 2x^2 \ln x + 4x^2 \ln x - 6x^2 \ln x + 3x^2 - 3x^2 = 0. \end{aligned}$$

Therefore, $y_2 = x^2 \ln x$ is also a solution to the homogeneous ODE.

Solution: (b)

Given the second order, non-homogeneous ODE

$$x^2 y'' - 3xy' + 4y = \ln x, \quad x > 0,$$

and the solutions to the homogeneous case found in the prior question, we will solve this ODE using variation of parameters. To apply variation of parameters though, first we need the y'' term without a multiplying factor, so we'll divide the ODE through by x^2 ,

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \frac{\ln x}{x^2}$$

Since we determined the fundamental set of solutions y_1, y_2 in the previous question, let's go ahead and calculate the Wronskian.

$$\mathcal{W}(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = 2x^3 \ln x + x^3 - 2x^3 \ln x = x^3$$

After variation of parameters, our general solution will look like,

$$y = y_H + y_p = \alpha y_1(x) + \beta y_2(x) + u(x)y_1(x) + v(x)y_2(x)$$

We know (and have previously derived) the equations

$$u = - \int \frac{y_2 r}{W} dx \quad v = \int \frac{y_1 r}{W}$$

where $W = \mathcal{W}(y_1, y_2)(x)$, which we just calculated, and $r = \ln x/x^2$, the RHS of the ODE. We'll make the substitutions and solve for u and v .

$$\begin{aligned} u(x) &= - \int \frac{y_2 r}{W} dx \\ &= - \int \frac{x^2 \ln x \frac{\ln x}{x^2}}{x^3} dx \\ &= - \int \frac{\ln^2 x}{x^3} dx \\ &= \frac{\ln^2 x}{2x^2} - \int \frac{\ln x}{x^3} dx && \text{(Integration by Parts)} \\ &= \frac{\ln^2 x}{2x^2} + \frac{\ln x}{2x^2} - \int \frac{1}{2x^3} dx && \text{(Integration by Parts)} \\ \therefore u(x) &= \frac{\ln^2 x}{2x^2} + \frac{\ln x}{2x^2} + \frac{1}{4x^2} \\ v(x) &= \int \frac{y_1 r}{W} dx \\ &= \int \frac{x^2 \frac{\ln x}{x^2}}{x^3} dx \\ &= \int \frac{\ln x}{x^3} dx \\ &= \frac{-\ln x}{2x^2} - \int \frac{-1}{2x^3} dx && \text{(Integration by Parts)} \\ &= \frac{-\ln x}{2x^2} + \frac{1}{2} \int \frac{1}{x^3} dx \\ \therefore v(x) &= \frac{-\ln x}{2x^2} - \frac{1}{4x^2} \end{aligned}$$

Bringing it all together, we find the general solution

$$y = \alpha x^2 + \beta x^2 \ln x + \left(\frac{\ln^2 x}{2x^2} + \frac{\ln x}{2x^2} + \frac{1}{4x^2} \right) x^2 + \left(\frac{-\ln x}{2x^2} - \frac{1}{4x^2} \right) x^2 \ln x,$$

which we can simplify to

$$y = \alpha x^2 + \beta x^2 \ln x + \frac{1 + \ln x}{4}.$$

Question 3: Orthogonal Basis and Projections - [10 marks]

Consider the inner product space $P_2(\mathbb{R})$ with inner product

$$\langle p_0 + p_1x + p_2x^2, q_0 + q_1x + q_2x^2 \rangle = p_0q_0 + p_1q_1 + p_2q_2, \quad p_0, p_1, p_2, q_0, q_1, q_2 \in \mathbb{R}.$$

Let $U = \{1 + 2x + x^2\}$.

- (a) Find U^\perp .
- (b) Determine an orthogonal basis for U^\perp .

Solution: (a)

$$\begin{aligned} \langle p_0 + p_1x + p_2x^2, q_0 + q_1x + q_2x^2 \rangle &= p_0q_0 + p_1q_1 + p_2q_2 \\ U &= \{1 + 2x + x^2\} \\ U^\perp &= \{p \in P_2(\mathbb{R}) \mid \langle p, u \rangle = 0, \forall u \in U\} \end{aligned}$$

U has one element, so we'll take it for u , and we'll take a general polynomial for p .

$$\begin{aligned} \langle p_0 + p_1x + p_2x^2, 1 + 2x + x^2 \rangle &= p_0 + 2p_1 + p_2 = 0 \\ \implies p_0 &= -2p_1 - p_2 \end{aligned}$$

So, a polynomial which belongs in U^\perp is constrained by this condition.

$$\implies p = (-2p_1 - p_2, p_1, p_2) \in P_2(\mathbb{R})$$

Finally, we'll utilise some vector addition, and scalar multiplication

$$\therefore p = (-2p_1, p_1, 0) + (-p_2, 0, p_2) = p_1(-2, 1, 0) + p_2(-1, 0, 1)$$

All polynomials which belong in U^\perp must be some linear combination of these vectors

$$\therefore U^\perp = \{\alpha(-2, 1, 0) + \beta(-1, 0, 1) \mid \alpha, \beta \in \mathbb{R}\}.$$

Solution: (b)

We've found a basis for U^\perp , namely

$$\{(-2, 1, 0), (-1, 0, 1)\}.$$

Since these vectors are linearly independent ($\nexists \alpha \in \mathbb{R} : (-2, 1, 0) = \alpha(-1, 0, 1)$), we can find an orthogonal basis for U^\perp by taking the cross product to find an orthogonal vector.

$$(-2, 1, 0) \times (-1, 0, 1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1, 2, 1)$$

So, we've actually determined two orthogonal bases for U^\perp ,

$$\{(-2, 1, 0), (1, 2, 1)\} \quad \text{and} \quad \{(-1, 0, 1), (1, 2, 1)\}.$$

Question 4: Orthogonality and Projections - [10 marks]

Consider the vector space \mathbb{R}^4 endowed with the inner product

$$\langle (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \rangle = u_1v_1 + 3u_2v_2 + u_3v_3 + 2u_4v_4.$$

Let $U = \text{span} \{ \underline{u}_1 = (-2, 1, 0, 1), \underline{u}_2 = (0, 1, 2, 3) \}$ be a subspace of \mathbb{R}^4 .

- (a) Use Gram-Schmidt procedure to construct an orthonormal basis for U .
- (b) Find the orthogonal projection of $\underline{v} = (-1, 2, 5, 1)$ in U and U^\perp .

Solution: (a)

Let v_1, v_2 be the basis vectors of the orthogonal basis. Set v_1 to u_1 .

$$v_1 = u_1 = (-2, 1, 0, 1).$$

Constructing v_2 requires us to subtract the projection of u_2 onto v_1 , from u_2 .

$$v_2 = u_2 - \text{proj}_{v_1}(u_2) = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

Let's now compute these inner products, according to the definition given in the question,

$$\begin{aligned} \langle u_2, v_1 \rangle &= u_{2x}v_{1x} + 3u_{2y}v_{1y} + u_{2z}v_{1z} + 2u_{2w}v_{1w} \\ &= 0 \cdot -2 + 3 \cdot 1 \cdot 1 + 2 \cdot 0 + 2 \cdot 3 \cdot 1 \\ &= 0 + 3 + 0 + 6 \\ &= 9 \end{aligned}$$

$$\begin{aligned} \langle v_1, v_1 \rangle &= v_{1x}v_{1x} + 3v_{1y}v_{1y} + v_{1z}v_{1z} + 2v_{1w}v_{1w} \\ &= -2 \cdot -2 + 3 \cdot 1 \cdot 1 + 0 \cdot 0 + 2 \cdot 1 \cdot 1 \\ &= 4 + 3 + 0 + 2 \\ &= 9 \end{aligned}$$

$$\therefore v_1 = u_2 - \frac{9}{9}v_1 = (0, 1, 2, 3) - (-2, 1, 0, 1) = (2, 0, 2, 2)$$

Now, to make this into an orthonormal basis, we need to normalise the vectors. These normalised vectors, we'll label e_1 and e_2 .

$$\begin{aligned} e_1 &= \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{\langle v_1, v_1 \rangle}} = \frac{v_1}{\sqrt{v_{1x}v_{1x} + 3v_{1y}v_{1y} + v_{1z}v_{1z} + 2v_{1w}v_{1w}}} = \frac{v_1}{\sqrt{9}} = \frac{v_1}{3} = \left(-\frac{2}{3}, \frac{1}{3}, 0, \frac{1}{3} \right) \\ e_2 &= \frac{v_2}{\|v_2\|} = \frac{v_2}{\sqrt{\langle v_2, v_2 \rangle}} = \frac{v_2}{\sqrt{v_{2x}v_{2x} + 3v_{2y}v_{2y} + v_{2z}v_{2z} + 2v_{2w}v_{2w}}} = \frac{v_2}{\sqrt{16}} = \frac{v_2}{4} = \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \right) \end{aligned}$$

Therefore, we've constructed an orthonormal basis for U ,

$$\left\{ \left(-\frac{2}{3}, \frac{1}{3}, 0, \frac{1}{3} \right), \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \right) \right\}.$$

Solution: (b)

$$\begin{aligned}\text{proj}_U(v) &= \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 \\ \langle v, e_1 \rangle &= -\frac{2}{3}(-1) + 3\frac{1}{3}(2) + 0(5) + 2\frac{1}{3}(1) = \frac{2}{3} + 2 + 0 + \frac{2}{3} = \frac{10}{3} \\ \langle v, e_2 \rangle &= \frac{1}{2}(-1) + 3(0)(2) + \frac{1}{2}(5) + 2\frac{1}{2}(1) = \frac{-1}{2} + 0 + \frac{5}{2} + 1 = 3 \\ \therefore \text{proj}_U(v) &= \frac{10}{3} \left(-\frac{2}{3}, \frac{1}{3}, 0, \frac{1}{3} \right) + 3 \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \right) = \left(-\frac{20}{9}, \frac{10}{9}, 0, \frac{10}{9} \right) + \left(\frac{3}{2}, 0, \frac{3}{2}, \frac{3}{2} \right) \\ &= \left(\frac{-13}{18}, \frac{10}{9}, \frac{3}{2}, \frac{47}{18} \right) \\ \text{proj}_{U^\perp}(v) &= v - \text{proj}_U(v) = (-1, 2, 5, 1) - \left(\frac{-13}{18}, \frac{10}{9}, \frac{3}{2}, \frac{47}{18} \right) = \left(-\frac{31}{18}, \frac{8}{9}, \frac{7}{2}, -\frac{29}{18} \right)\end{aligned}$$

Question 5: Orthogonal Subspaces - [10 marks]

Consider the vector space of 2×2 Matrices over the real numbers $M_{2,2}(\mathbb{R})$. The sets of matrices S and A such that $S^T = S$ and $A^T = -A$ define vector subspaces in $M_{2,2}(\mathbb{R})$. Call these subspaces respectively $M_{2,2}^S(\mathbb{R}) = \{S \in M_{2,2}(\mathbb{R}) \mid S^T = S\}$ and $M_{2,2}^A(\mathbb{R}) = \{A \in M_{2,2}(\mathbb{R}) \mid A^T = -A\}$.

Take the inner product over $M_{2,2}(\mathbb{R})$ defined by:

$$\langle \underline{v}, \underline{u} \rangle = \text{Tr}(\underline{v}^T \underline{u}),$$

$\forall \underline{v}, \underline{u} \in M_{2,2}(\mathbb{R})$. Are the subspaces $M_{2,2}^S(\mathbb{R})$ and $M_{2,2}^A(\mathbb{R})$ orthogonal according to the inner product defined above? Explain your answer.

Solution:

Let's consider what elements of $M_{2,2}^A(\mathbb{R})$ and $M_{2,2}^S(\mathbb{R})$ look like, generally.

$$A = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \in M_{2,2}^A(\mathbb{R}).$$
$$S = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_{2,2}^S(\mathbb{R}).$$

$A^T = -A$ and $S^T = S$, as expected.

We'll now take the inner product of the two matrices, as defined above.

$$\left\langle \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right\rangle = \text{Tr} \left(\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) = \text{Tr} \left(\begin{pmatrix} -bx & -cx \\ ax & bx \end{pmatrix} \right) = -bx + bx = 0$$

For the sake of completeness, let's swap the arguments.

$$\left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \right\rangle = \text{Tr} \left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \right) = \text{Tr} \left(\begin{pmatrix} -bx & ax \\ -cx & bx \end{pmatrix} \right) = -bx + bx = 0$$

It's nice to know that the order of the arguments doesn't matter...

Since $\langle A, S \rangle = \langle S, A \rangle = 0$, $\forall A \in M_{2,2}^A(\mathbb{R})$, $S \in M_{2,2}^S(\mathbb{R})$, we can conclude that the two subspaces $M_{2,2}^A(\mathbb{R})$ and $M_{2,2}^S(\mathbb{R})$ are orthogonal under the inner product $\langle \underline{v}, \underline{u} \rangle = \text{Tr}(\underline{v}^T \underline{u})$.

Question 6: Least Squares Approximation - [10 marks]

Let $P_2(\mathbb{R})$ have the inner product,

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx, \quad \forall p, q \in P_2(\mathbb{R}).$$

Find the best approximation of $f(x) = x^2 + x^3$ by polynomials in $P_2(\mathbb{R})$.

Solution:

We seek to approximate $f(x) = x^2 + x^3$ with a polynomial $p(x) \in P_2(\mathbb{R})$ with a least squares approximation.

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2, \quad \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$$

The best approximation of f is a projection onto $P_2(\mathbb{R})$. Therefore, the problem reduces to

$$\text{proj}_{P_2}(f) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 = \sum_{i=0}^2 \frac{\langle f, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

Where e_i are the basis vectors, $e_0 = 1$, $e_1 = x$, $e_2 = x^2$. The RHS here is just the standard projection formula. All that's left for us to do is to calculate all the inner products.

$$\begin{aligned} \langle f, e_0 \rangle &= \langle f, 1 \rangle = \int_0^1 (x^2 + x^3) \cdot 1 dx \\ &= \int_0^1 x^2 dx + \int_0^1 x^3 dx \\ &= \frac{1}{3} x^3 \Big|_0^1 + \frac{1}{4} x^4 \Big|_0^1 \\ \therefore \langle f, 1 \rangle &= \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \\ \langle e_0, e_0 \rangle &= \langle 1, 1 \rangle = \int_0^1 1 \cdot 1 dx \\ &= x \Big|_0^1 \\ \therefore \langle 1, 1 \rangle &= 1 \\ \langle f, e_1 \rangle &= \langle f, x \rangle = \int_0^1 (x^2 + x^3) x dx \\ &= \int_0^1 x^3 dx + \int_0^1 x^4 dx \\ &= \frac{1}{4} x^4 \Big|_0^1 + \frac{1}{5} x^5 \Big|_0^1 \\ \therefore \langle f, x \rangle &= \frac{1}{4} + \frac{1}{5} = \frac{9}{20} \\ \langle e_1, e_1 \rangle &= \langle x, x \rangle = \int_0^1 x^2 dx \\ &= \frac{1}{3} x^3 \Big|_0^1 \\ \therefore \langle x, x \rangle &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned}
\langle f, e_2 \rangle &= \langle f, x^2 \rangle = \int_0^1 (x^2 + x^3) x^2 \, dx \\
&= \int_0^1 x^4 \, dx + \int_0^1 x^5 \, dx \\
&= \frac{1}{5} x^5 \Big|_0^1 + \frac{1}{6} x^6 \Big|_0^1 \\
\therefore \langle f, x^2 \rangle &= \frac{1}{5} + \frac{1}{6} = \frac{11}{30} \\
\langle e_2, e_2 \rangle &= \langle x^2, x^2 \rangle = \int_0^1 x^4 \, dx \\
&= \frac{1}{5} x^5 \Big|_0^1 \\
\therefore \langle x^2, x^2 \rangle &= \frac{1}{5}
\end{aligned}$$

Now that we've calculated all these inner products, we can put it all together.

$$\begin{aligned}
\text{proj}_{P_2}(f) &= \sum_{i=0}^2 \frac{\langle f, \tilde{e}_i \rangle}{\langle \tilde{e}_i, \tilde{e}_i \rangle} \tilde{e}_i = \frac{\langle f, e_0 \rangle}{\langle e_0, e_0 \rangle} e_0 + \frac{\langle f, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle f, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 \\
&= \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle f, x \rangle}{\langle x, x \rangle} x + \frac{\langle f, x^2 \rangle}{\langle x^2, x^2 \rangle} x^2 \\
&= \frac{\frac{7}{12}}{1} + \frac{\frac{20}{1}}{\frac{1}{3}} x + \frac{\frac{11}{30}}{\frac{1}{5}} x^2 \\
&= \frac{7}{12} + \frac{27}{20} x + \frac{11}{6} x^2 \\
&= \alpha_0 + \alpha_1 x + \alpha_2 x^2 \\
\implies (\alpha_0, \alpha_1, \alpha_2) &= \left(\frac{7}{12}, \frac{27}{20}, \frac{11}{6} \right)
\end{aligned}$$

Therefore, the projection of f onto $P_2(\mathbb{R})$ under the inner product we've defined, and the best approximation of f by a polynomial in $P_2(\mathbb{R})$

$$\text{proj}_{P_2}(f) = p(x) = \frac{7}{12} + \frac{27}{20}x + \frac{11}{6}x^2$$

\therefore there is a being in the room with me rn.