5.1. A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1 = 2$ and $a_k = a_{k-1}/k$ for all integers $k \geqslant 2$.

Use mathematical induction to prove that the expression $a_n = 2/n!$, for all $n \in \mathbb{N}$, is a general form for the sequence.

5.2. The Lucas sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, $a_2 = 3$ and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geqslant 3$.

Use strong mathematical induction to prove that $a_n \leqslant \left(\frac{7}{4}\right)^n$, for all $n \in \mathbb{N}$.

5.3. Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2 \ldots$. Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$.

5.4. The Fibonacci sequence, $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is defined recursively as

$$\begin{aligned} &a_0 = 0 \\ &a_1 = 1 \\ &a_n = a_{n-2} + a_{n-1} \quad \text{for } n \geq 2 \end{aligned}$$

Prove by mathematical induction that for all integers $n \geq 0$,

$$a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5} \times 2^n}$$

5.5. Given a sequence of numbers defined by $T_1 = 1$ and $T_{n+1} = 1 + (T_1 \cdot T_2 \cdot T_3 \cdots T_n)$ for all $n \geq 1$, prove that $T_{n+1} = (T_n)^2 - T_n + 1$ for all $n > 1$.

Here are two **puzzles** that you can think about during week 6. Feel free to ask your tutors or lecturer for more hints!

I. The list

$$\begin{array}{c} 10234567 \\ 10234576 \\ 10234657 \\ \vdots \\ 76543210 \end{array}$$

contains all the 8-digit numbers which use every digit 0, 1, 2, 3, 4, 5, 6 and 7 exactly one each, listed in order from smallest to largest. What is the 20000th number on the list?

J. There are 12 balls, identical in appearance. One of the balls is either heavier or lighter than the others. If I give you a balance and allow you to make 3 weighings, how could you arrange to discover which ball is the odd one out, and whether it is heavier or lighter than the others? (A balance allows you to compare whether some collection of balls on the left has a smaller, equal or greater weight than some other collection of balls on the right.)

Extra practice questions from the textbook (Solutions at the back of the book.)

Epp 5th ed.:
Section 5.4, pp. 310–314: Questions 1, 4, 5, 11, 17, 30.
Section 5.6, pp. 337–340: Questions 1, 3, 5, 7, 9, 11, 13, 26, 27, 32, 34.
Section 5.7, pp. 350–352: Questions 1ab, 2ac, 3, 5, 10, 12, 18, 19, 24, 43, 45, 50.

Epp 4th ed.:
Section 5.4, pp. 277–279: Questions 1, 4, 5, 11, 17, 24.
Section 5.6, pp. 302–304: Questions 1, 3, 5, 7, 9, 11, 13, 26, 27, 32, 34.
Section 5.7, pp. 314–316: Questions 1ab, 2ac, 3, 5, 10, 12, 18, 19, 24, 43, 45, 50.

Puzzle hints:

Stuck on the puzzles from week 5?

G. What happens to the *sum* of the slips remaining in the hat after each step?

H. Consider the nth cut. What is the largest number of *other* cuts that it can intersect? What is the largest number of *regions* (pieces of pizza) that it can pass through?