SCHOOL OF MATHEMATICS AND PHYSICS

MATH1072
Assignment 3
Semester Two 2024

Submit your answers - along with this sheet - by 1pm on the 30th of September, using the blackboard assignment submission system. Assignments must consist of a single PDF.

You may find some of these problems challenging. Attendance at weekly tutorials is assumed.

Family name:

Given names:

Student number:

Marker's use only

Each question marked out of 3.

- Mark of 0: You have not submitted a relevant answer, or you have no strategy present in your submission.
- Mark of 1: Your submission has some relevance, but does not demonstrate deep understanding or sound mathematical technique.
- Mark of 2: You have the right approach, but need to fine-tune some aspects of your calculations.
- Mark of 3: You have demonstrated a good understanding of the topic and techniques involved, with well-executed calculations.

Q1a	Q1b:	Q1c:	Q1d:
Q2a:	Q2b:	Q2c:	Q3a:
Q3b:	Q3c:	Q3d:	Q4

Total (out of 33):

1. A population of Christmas beetles in Brisbane grows at a rate proportional to their current population. In the absence of external factors, the population will double in one week's time. On any given day, there is a net migration into the area of 10 beetles, 11 beetles are eaten by a local Magpie population, and 2 die of natural causes.

a) Write a initial value problem to describe the change in population at time t, given that the initial population of Christmas beetles is $P(0) = 100$. You **must** write all the parameters in your model **explicitly**.

b) Solve the initial value problem from part a) to find the population $P(t)$, at any time t.

c) Use MATLAB to plot your solution from part b) from time $t = 0$ to $t = 100$. Will the beetle population survive?

d) Use MATLAB to plot the solution to part a) from time $t = 0$ to $t = 100$ for 30 different initial populations of Christmas beetles from $P(0) = 20$ to $P(0) = 50$. Recall that initial population sizes must be integer valued. What can you say about the stability of the population of Christmas beetles from this plot?

2. Fluid flow

(In the following question, we write the cartesian variables as $x \equiv x_1$, $y \equiv x_2$, and $z \equiv x_3$. We also write the basis vectors as $\mathbf{i} \equiv \mathbf{e}_1$, $\mathbf{j} \equiv \mathbf{e}_2$, and $\mathbf{k} \equiv \mathbf{e}_3$.)

In fluid mechanics, fluids which are **incompressible** and **inviscid** are referred to as **ideal**.

Let $\mathbf{u} = \mathbf{u}(x, y, z, t) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$ be the velocity of an ideal fluid at an arbitrary point in space and time. Its motion is governed by the Euler equation,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F} - \frac{\nabla P}{\rho}. \tag{1}$$

Here P is the fluid's pressure, ρ its constant density, $\mathbf{F}$ is the external force on the fluid at any given point, and

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \equiv \sum_{i=1}^{3} u_i \frac{\partial \mathbf{u}}{\partial x_i}.$$

Suppose the system is simplified in three ways:

- The flow is **steady** ($\mathbf{u} = \mathbf{u}(x, y, z)$ is not changing with time)
- $\mathbf{u}$ is a conservative vector field (this occurs when the fluid is **irrotational**, though we won't elaborate on that here.)
- The external force is also conservative, with $\mathbf{F} = -\nabla V$, for some scalar potential $V = V(x_1, x_2, x_3)$.

a) Show that in this case, $(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla||\mathbf{u}||^2$.

(Hint, you can assume that Clairaut's theorem applies)

b) Hence show that the Euler equation simplifies to

$$\frac{1}{2}||\mathbf{u}||^2 + V + \frac{P}{\rho} = \text{constant}$$

c) If V is a constant, what can you say about the relationship between a fluid's speed and its pressure?

3. Consider Newton's second law of motion which states that,

$$F = ma \tag{2}$$

or, net force, F, is equal to mass m, times acceleration a.

a) Use equation (2) to write out the equation of motion of a particle of mass m, subject to a frictional force proportional to the square of the velocity $v(t)$, completely in terms of the particle's velocity $v(t)$.

b) Solve the first-order differential equation from part a) to find the particle's velocity $v(t)$ at time t, with initial velocity v_0.

c) Use your solution from part b) to solve for the position of the particle $x(t)$ at time t, with initial position x_0.

d) Use Euler's method with step size 0.1 to estimate the particle's position $x(0.5)$ at time $t = 0.5$ of your solution in part b). Take $m = 1$, $\beta = 2$, $x_0 = 0$, and $v_0 = 1$. Calculate the error in using Euler's method (round to the fourth decimal).

4. Evaluate the line integral

$$\int_C xe^y \, ds,$$

where C is the line segment from $(2, 0)$ to $(5, 4)$.