Due **Friday 6 September at 5pm** on blackboard.

Marks will be deducted for sloppy working. Clearly state your assumptions and conclusions, and justify all steps in your work.

Q1 Prove the following statements:
(a) The sum of every five consecutive integers is always divisible by 5.
(a) Suppose n is an odd integer. The sum of every n consecutive integers is always divisible by n.

(10 marks)

Q2 (a) Compute the following quantities

$$\lfloor 3.6 \rfloor, \qquad \lceil \pi \rceil, \qquad \lceil e \rceil, \qquad \lceil e + 0.5 \rceil$$

(b) Prove or disprove the following statement: for all real numbers x,

$$\lceil x + 0.5 \rceil = \lceil x \rceil + 1$$

(5 marks)

Q3 (a) Use the definition, prove or disprove

$$3 \equiv -4 \bmod 7$$

(b) Use the definition, prove or disprove: for all integers x,

$$2x \equiv -14x \bmod 8$$

(c) Prove or disprove the following statement.
Suppose a, b, c, d are positive integers, $ac \equiv bc \bmod d$, then

$$a \equiv b \bmod d$$

(d) Prove or disprove the following statement.
Suppose a, b, c, d are positive integers, $ac \equiv bc \bmod d$ and $\gcd(c, d) = 1$, then

$$a \equiv b \bmod d$$

(Hint: you may use a fact we mentioned in Lecture 12.)

(10 marks)

Q4 Use the Euclidean algorithm, compute

$$\gcd(101, 24)$$

(5 marks)

(Continued on the following page...)

Q5 The *least common multiple* of the integers a, b, denoted as $\operatorname{lcm}(a, b)$, is defined as the smallest positive integer which is *divisible* by both a and b.
Let $a = 2^7 \cdot 3^2 \cdot 5$ and $b = 2^3 \cdot 3^3 \cdot 7$.
(a) Compute $\gcd(a, b)$.
(b) Compute $\operatorname{lcm}(a, b)$.
(c) Verify that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.
(d) Can you prove the statement $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$ for *arbitrary* positive integers a and b? (Hint: use the prime factorisation.)

(10 marks)

Q6 **MATH1061 only:** A sequence is defined recursively as

$$\begin{aligned} a_0 &= 1, \qquad a_1 = 2, \\ a_n &= 4a_{n-1} - 3a_{n-2}, \qquad n \geq 2 \end{aligned}$$

Prove the formula

$$a_n = \frac{3^n + 1}{2}$$

holds for all $n \geq 0$. *(10 marks)*

Q6 **MATH7861 only:**
The Fibonacci sequence is defined recursively as

$$\begin{aligned} a_0 &= 0, \qquad a_1 = 1, \\ a_n &= a_{n-1} + a_{n-2}, \qquad n \geq 2 \end{aligned}$$

Use strong mathematical induction, prove

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

Hint: you may use the fact that $\gamma = -\frac{1-\sqrt{5}}{2} = \frac{2}{1+\sqrt{5}}$ is related to the golden ratio and is a root of the equation $\gamma^2 + \gamma - 1 = 0$, but it's also possible to go without this fact.

(10 marks)