School of Mathematics and Physics, UQ

# MATH1071 Advanced Calculus & Linear Algebra I
# Semester 1 2025
# Problem Set 2


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Tutorial Group #8

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**Question 1: 5 marks**

Use the definition of limits, show that

$$\lim_{n\to\infty} \frac{1}{n^3} = 0$$

***Solution:***

**Definition** 1.1 (Limit of a Sequence). Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. The limit of $(a_n)_{n=1}^{\infty}$ equals $a \in \mathbb{R}$, written $\lim_{n\to\infty} a_n = a$, if $\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \implies |a_n - a| < \varepsilon$.

**Lemma.** $\lim\limits_{n\to\infty} \dfrac{1}{n^3} = 0$

*Proof.* Suppose $(a_n)_{n=1}^{\infty} := \left(\frac{1}{n^3}\right)_{n=1}^{\infty}$ is a sequence of real numbers, with a limit $a := 0 \in \mathbb{R}$. Suppose $\varepsilon > 0$. Let's consider

$$|a_n - a| = \left|\frac{1}{n^3} - 0\right| = \left|\frac{1}{n^3}\right| = \frac{1}{n^3} < \varepsilon \qquad (n \in \mathbb{N})$$

When $n \geq N$. Solving for $N$ now,

$$n^3 > \frac{1}{\varepsilon} \iff n > \sqrt[3]{\frac{1}{\varepsilon}}$$

$$\text{Choose } N = \left\lceil \sqrt[3]{\frac{1}{\varepsilon}} \right\rceil$$

Therefore, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, N = \text{ceil}\left(\sqrt[3]{1/\varepsilon}\right)$ we have

$$n \geq N = \left\lceil \sqrt[3]{\frac{1}{\varepsilon}} \right\rceil$$

$$n \geq N \geq \sqrt[3]{\frac{1}{\varepsilon}}$$

$$n^3 \geq N^3 \geq \frac{1}{\varepsilon}$$

$$\frac{1}{n^3} \leq \frac{1}{N^3} \leq \varepsilon$$

Verifying our choice of $N$, and completing the proof.

Therefore, by the $\varepsilon$−$N$ definition of the limit, $\lim\limits_{n\to\infty} \dfrac{1}{n^3} = 0$. $\square$

**Question 2: 10 marks**

Use suitable limit laws, find the limits for the following sequences. Please cite which laws you've used.

(a) $\displaystyle\lim_{n\to\infty} \frac{2n^3 + 4n}{7n^4 + 5n^2 - 1}$

(b) $\displaystyle\lim_{n\to\infty} \frac{\cos n + \sin n}{n}$

**Theorem** 2.1. (Sequence Limit Properties) Suppose $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, and $\lambda \in \mathbb{R}$ is fixed, then

(a) $\displaystyle\lim_{n\to\infty} (a_n + b_n) = a + b$

(b) $\displaystyle\lim_{n\to\infty} \lambda a_n = \lambda a$

(c) $\displaystyle\lim_{n\to\infty} a_n b_n = ab$

(d) $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a_n}{b_n}$, given $b \neq 0,\ b_n \neq 0, \forall n$

**Theorem** 2.2. (Squeeze Theorem) Suppose we have three sequences $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}$ such that $a_n \leq b_n \leq c_n$, and $a_n = c_n = L$. Then $b_n = L$.

***Solution:*** (a)
Just for fun, and for no particular reason, we'll divide through every term by $n^4$.

$$\text{Let } L := \lim_{n\to\infty} \frac{2n^3 + 4n}{7n^4 + 5n^2 - 1} = \lim_{n\to\infty} \frac{\frac{2n^3}{n^4} + \frac{4n}{n^4}}{\frac{7n^4}{n^4} + \frac{5n^2}{n^4} - \frac{1}{n^4}} = \lim_{n\to\infty} \frac{\frac{2}{n} + \frac{4}{n^3}}{7 + \frac{5}{n^2} - \frac{1}{n^4}}$$

Let $a_n := \frac{2}{n} + \frac{4}{n^3}$ and $a := \lim_{n\to\infty} a_n$
Let $b_n := 7 + \frac{5}{n^2} - \frac{1}{n^4}$ and $b := \lim_{n\to\infty} b_n$.
We need to make sure that $b_n \neq 0$, $\forall n \in \mathbb{N}$

$$\begin{aligned}
n &> 0 \\
n^4 &> 0 \\
-n^4 &< 0 \\
\frac{n^2}{5} - n^4 &< 0 \\
\frac{5}{n^2} - \frac{1}{n^4} &> 0 \\
7 + \frac{5}{n^2} - \frac{1}{n^4} &> 0 \\
\therefore b_n &> 0,\ \forall n \in \mathbb{N}
\end{aligned}$$

Since, $b_n \neq 0,\ \forall n \in \mathbb{N},\ \lim_{n\to\infty} b_n \neq 0$. Therefore, we can apply Theorem 2.1(d)

$$L = \frac{a}{b}$$

Let's start by finding $a$

$$a = \lim_{n\to\infty} \frac{2}{n} + \frac{4}{n^3}$$

But this is just the sum of two other sequences!
Let $\alpha_n := \frac{2}{n}$ and $\alpha = \lim_{n\to\infty} \alpha_n$.
Let $\beta_n := \frac{4}{n^3}$ and $\beta := \lim_{n\to\infty} \beta_n$

$$\text{Then } a_n = \alpha_n + \beta_n$$

So we can apply Theorem 2.1(a)

$$a = \alpha + \beta = \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} \frac{4}{n^3}$$

So, let's find $\alpha$.

$$\alpha = \lim_{n\to\infty} \frac{2}{n}$$

2 is a fixed constant, so we can apply Theorem 2.1(b)

$$= 2 \lim_{n\to\infty} \frac{1}{n}$$

And $\lim_{n\to\infty} 1/n$ is trivially equal to 0. If I need further justification, I would direct you to Question 1, where the same argument holds, except choosing $N = \lceil 1/\varepsilon \rceil$.

$$\therefore \alpha = \lim_{n\to\infty} \frac{2}{n} = 2 \lim_{n\to\infty} \frac{1}{n} = 2 \cdot 0 = 0$$

Next, we'll find $\beta$.

$$\beta = \lim_{n\to\infty} \frac{4}{n^3}$$

4 is a fixed constant, so we can apply Theorem 2.1(b)

$$= 4 \lim_{n\to\infty} \frac{1}{n^3}$$

We proved in question 1 that $\lim_{n\to\infty} 1/n^3 = 0$, so

$$\therefore \beta = \lim_{n\to\infty} \frac{4}{n^3} = 4 \lim_{n\to\infty} \frac{1}{n^3} = 4 \cdot 0 = 0$$

Hence, we've found $a$,

$$a = \alpha + \beta = 0 + 0 = 0$$

Next we'll find $b$
$b$ is also the sum of three sequencess, so we can apply Theorem 2.1(a)
Let $\gamma_n := 7$ and $\gamma := \lim_{n\to\infty} \gamma_n$
Let $\delta_n := \frac{5}{n^2}$ and $\delta := \lim_{n\to\infty} \delta_n$
Let $\varepsilon_n := \frac{1}{n^4}$ and $\varepsilon := \lim_{n\to\infty} \varepsilon_n$

$$\text{Then } b_n = \gamma_n + \delta_n + \varepsilon_n$$

And we can apply Theorem 2.1(a)

$$b = \gamma + \delta + \varepsilon = \lim_{n\to\infty} 7 + \lim_{n\to\infty} \frac{5}{n^2} + \lim_{n\to\infty} \frac{1}{n^4}$$

We'll start by computing $\gamma$

$$\gamma = \lim_{n\to\infty} 7 = 7 \lim_{n\to\infty} 1 = 7 \cdot 1 = 7$$

In the first step, we applied Theorem 2.1(b), and in the second we note that $\lim_{n\to\infty} 1$ is trivially 1.
Next, we'll find $\delta$

$$\delta = \lim_{n\to\infty} \frac{5}{n^2} = 5 \lim_{n\to\infty} \frac{1}{n^2} = 5 \cdot 0 = 0$$

In the first step, we apply Theorem 2.1(b). In the second, we note that $\lim_{n\to\infty} 1/n^2$ is trivially equal to 0. If you're not convinced, apply Theorem 2.1(c) to $\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 0 \cdot 0 = 0$.
Finally, let's find $\varepsilon$

$$\varepsilon = \lim_{n\to\infty} \frac{1}{n^4} = 0$$

This is trivial again. You can either apply Theorem 2.1(c) twice, to find the limit is equal to $0 \cdot 0 \cdot 0$, or you can repeat my argument from question, but choosing $N = \left\lceil \sqrt[4]{1/n} \right\rceil$.
Thus, we can compute $b$ using Theorem 2.1(a)

$$b = \gamma + \delta + \varepsilon = 7 + 0 + 0 = 0$$

and we can proceed to find the limit we were looking for!

$$L = \frac{a}{b} = \frac{\alpha + \beta}{\gamma + \delta + \varepsilon} = \frac{0 + 0}{7 + 0 + 0} = \frac{0}{7} = 0$$

and conclude that

$$\lim_{n\to\infty} \frac{2n^3 + 4n}{7n^4 + 5n^2 - 1} = 0$$

***Solution:*** (b)
We'll start by applying Theorem 2.1(a) to break up the limit into two limits

$$\text{Let } L := \lim_{n\to\infty} \frac{\cos n + \sin n}{n} = \lim_{n\to\infty} \frac{\cos n}{n} + \lim_{n\to\infty} \frac{\sin n}{n}$$

Let $a_n := \frac{\cos n}{n}$, $a := \lim_{n\to\infty} a_n$.
Let $b_n := \frac{\sin n}{n}$, $b := \lim_{n\to\infty} b_n$.
$\therefore L = a + b$.
We'll work out $a$ and $b$ by using Theorem 2.2, and finding some sequences that may squeeze $a$ and $b$, respectively.

$$-1 \le \cos n \le 1 \qquad -1 \le \sin n \le 1$$

$$\frac{-1}{n} \le \frac{1}{\cos n} \le \frac{1}{n} \qquad \frac{-1}{n} \le \frac{1}{\sin n} \le \frac{1}{n}$$

$$\frac{-1}{n} \le a_n \le \frac{1}{n} \qquad \frac{-1}{n} \le b_n \le \frac{1}{n}$$

Note that, since $n \in \mathbb{N}$, $n$ is strictly positive, so we don't have to flip the equalities. Let's now find the limit of these sequences, and see if the sqeeuze $a_n$ and $b_n$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

That is a trivial limit we've already identified and worked with in previously.

$$\begin{aligned}\lim_{n\to\infty} \frac{-1}{n} &= -1 \lim_{n\to\infty} \frac{1}{n} \\ &= -1 \cdot 0 \\ &= 0\end{aligned}$$

We apply Theorem 2.1(b) to pull the constant fixed factor out, then compute the trivial limit again. As we can see, $\lim_{n\to\infty} 1/n = \lim_{n\to\infty} -1/n = 0$. Also, $-1/n \leq a_n \leq 1/n$ and $-1/n \leq b_n \leq 1/n$. Therefore, by Theorem 2.2, the sqeeze theorem,

$$a = 0, \qquad b = 0$$

Now, we can calculate the limit of interest,

$$\therefore L = a + b = 0 + 0 = 0$$

Therefore, we can conlude that the limit

$$\lim_{n\to\infty} \frac{\cos n + \sin n}{n} = 0$$

## Question 3: 10 marks

Suppose $(b_n)_{n=0}^{\infty}$ and $(c_n)_{n=0}^{\infty}$ are two convergent sequences with

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L.$$

Suppose there's another sequence $(a_n)_{n=0}^{\infty}$ where $b_n = a_{2n}$ and $c_n = a_{2n+1}$ for all $n \in \mathbb{Z}_{\geq 0}$. Use the definition of limits, show that $\lim_{n\to\infty} a_n = L$.

***Solution:***
Since $b_n$ and $c_n$ are convergent sequences,

$$\begin{gathered}\therefore \forall \varepsilon > 0, \exists N_1 \in \mathbb{Z}_{\geq 0} : n \geq N_1 \implies |b_n - L| < \varepsilon, \\ \forall \varepsilon > 0, \exists N_2 \in \mathbb{Z}_{\geq 0} : n \geq N_2 \implies |c_n - L| < \varepsilon.\end{gathered}$$

$a_n$ is made up of two subsequences. Even $n$'s take $b_n$'s value, while odd $n$'s take $c_n$'s value. We must show that

$$\forall \varepsilon > 0, \exists N \in \mathbb{Z}_{\geq 0} : n \geq N \implies |a_n - L| < \varepsilon$$

Therefore, choose $n = 2\max\{N_1, N_2\}$.
Case $n = 2k,\ k \in \mathbb{Z}$. $|a_{2N_1} - L| = |b_{N_1} - L| < \varepsilon$
Case $n = 2k+1,\ k \in \mathbb{Z}$. $|a_{2N_2} - L| = |c_{N_2} - L| < \varepsilon$

Therefore, with this choice of $N$, in either case, $a_n$ converges to $L$.

## Question 4: 15 marks

In class we studied the sequence $(a_n)_{n=0}^{\infty}$ where $a_0 = 1$ and $a_{n+1} = \frac{1}{a_n+1}$ for all $n$. We showed that the subsequence with even terms $(b_n)_{n=0}^{\infty}$ where $b_n = a_{2n}$ forms a bounded monotone decreasing sequence and concluded that it converges to the number $\phi = \frac{-1+\sqrt{5}}{2}$. The purpose of this exercise is to repeat this process for the subsequence with odd terms.

(a) Write out the first five terms of $(c_n)_{n=0}^{\infty}$ where $c_n = a_{2n+1}$.

(b) Find a recursion between the terms of $c_n$. (Hint: use the recursion for $a_n$ twice!)

(c) Show that $c_n \leq \phi$ for all $n$.

(d) Show that $c_n$ is monotone increasing.

(e) Find the limit of $\lim_{n\to\infty} c_n$.

***Solution:*** (a)
The first 5 terms of $c_n$:

$$\begin{aligned}
c_0 &= a_1 = \frac{1}{a_0+1} = \frac{1}{2} \approx 0.5 \\
c_1 &= a_3 = \frac{1}{a_2+1} = \frac{3}{5} \approx 0.6 \\
c_2 &= a_5 = \frac{1}{a_2+1} = \frac{8}{13} \approx 0.615385 \\
c_3 &= a_7 = \frac{1}{a_2+1} = \frac{21}{34} \approx 0.617647 \\
c_4 &= a_9 = \frac{1}{a_8+1} = \frac{55}{89} \approx 0.617978
\end{aligned}$$

***Solution:*** (b)
The easy part of developing the recursion relation is setting the starting point:

$$c_0 = 0.5$$

is clear from the term list above. Next, the relation itself,

$$c_n = a_{2n+1} = \frac{1}{a_{2n}+1} = \frac{1}{\frac{1}{a_{2n-1}+1}+1} = \frac{1}{\frac{1}{c_{n-1}+1}+1} = \frac{1}{\frac{1}{c_{n-1}+1}+\frac{c_{n-1}+1}{c_{n-1}+1}} = \frac{c_{n-1}+1}{c_{n-1}+2}$$

Therfore,

$$c_0 = \frac{1}{2}, \qquad c_{n+1} = \frac{c_n+1}{c_n+2}$$

***Solution:*** (c)
Let's consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{x+1}{x+2}$$

We're looking for a point which maps back to itself. I.e. if the reccurance relation $c_n$ ever reached this value, it would repeatedly map back onto itself. We'll call this point $\alpha$.

$$\alpha = f(\alpha) = \frac{\alpha+1}{\alpha+2} \iff \alpha(\alpha+2) = \alpha+1 \iff \alpha^2 + 2\alpha = \alpha + 1 \iff \alpha^2 + \alpha - 1 = 0$$

Applying the quadratic formula, to solve for $\alpha$,

$$\alpha = \frac{-(1) \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

and we'll take the positive square root, which is the larger number. Therefore, $\alpha = \frac{-1+\sqrt{5}}{2} = \phi$.

So, since $c_n$ starts below $\phi$, and $c_n$ is increasing, and if $c_n$ ever *reached* $\phi$, it would remain at $\phi$ forever, we can conclude

$$c_n \leq \phi$$

***Solution:*** (d)
For $c_n$ to be monotone increasing, we must show that $c_n \leq c_{n+1},\ \forall n \geq 0$. i.e.,

$$c_n \leq c_{n+1} = \frac{c_n+1}{c_n+2} \tag{1}$$

Let's work with the inequality, see if we can find find a fact which we certainly know is true

$$\begin{aligned} c_n &\leq \frac{c_n+1}{c_n+2} \\ c_n\left(c_n+2\right) &\leq c_n+1 \\ c_n^2 + 2c_n &\leq c_n + 1 \\ c_n^2 + c_n - 1 &\leq 0 \end{aligned}$$

We've already solved this quadratic!

$$\therefore c_n \leq \frac{-1+\sqrt{5}}{2} \tag{2}$$

And we know that this is true, we proved this fact in the previous part. In other words, (1) $\iff$ (2). (2). Therefore (1). Since the equality holds, we've proven that the sequence $c_n$ is monotone increasing.

***Solution:*** (e)
**Theorem** 4.1. (Monotone Convergence Theorem) A monotone sequence converges if and only if it is bounded.

In (d) we proved that $c_n$ is a monotone increasing sequence. In (c) we proved that $c_n$ is bounded, i.e. $c_n \leq \phi$. From these two facts, along with Theorem 4.1, it follows that

$$\lim_{n\to\infty} c_n = \phi = \frac{-1+\sqrt{5}}{2}$$

**Question 5: 10 marks**

Show that a convergent sequence is always bounded. In other words, given a sequence $(a_n)_{n=0}^{\infty}$ and assume that $\lim_{n\to\infty} a_n = L$. Show that there exists a number $M$ such that $|a_n| < M$ for all $n$.

***Solution:***

*Proof.* Directly, by construction.

Suppose $(a_n)_{n=0}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} a_n = L$.
We will construct a global bounding value, $A$.

Then, $\forall \varepsilon > 0,\ \exists N \in \mathbb{Z}_{\geq 0} : n \geq N \implies |a_n - L| < \varepsilon$.
Therefore, $\forall n \geq N$, $|L| - \varepsilon < |a_n| < |L| + \varepsilon$.
So, the "tail" of the sequence is bounded.

Cosnider the sequence $(a_0, a_1, \ldots, a_{N-1})$. In other words, the subsequence made up of $a_n$'s terms up-to, but not including $a_N$.
This subsequence is finite, since $N$ is an integer,
therefore, $A_0 = \max\{|a_0|, |a_1|, \ldots, |a_{N-1}|\}$ is well-defined.

Take $A = \max\{A_0,\ |L| + \varepsilon\}$.
Our construction guarantees that $a_n \leq A,\ \forall n \in \mathbb{Z}_{\geq 0}$.

Therefore, a convergent seuqnce, $a_n$, is bounded. □

Note that you can arbitrarily choose $\varepsilon > 0$, and the constrution holds.