Questions 3.1–3.3 only use material from week 3.
Questions 3.4 onwards use material from the pre-class video for Monday 12 August (*Divisibility*).

3.1. For each of the following statements, either prove the statement or disprove it using a counterexample.

(a) The sum of any pair of even integers is even.
(b) There exist positive integers a, b and c such that $a^2 + b^2 = c^2$.
(c) There is an even integer n such that $5n - 4$ is prime.
(d) $\forall n \in \mathbb{Z}^+$, if $n \geq 4$ then $2n^2 - 5n + 2$ is composite.
(e) $\forall x, y \in \mathbb{R}$, if $y^2 > x^2$ then $y > x$.
(f) The difference of the squares of any two consecutive integers is odd.

3.2. Prove the following statement using a proof by contradiction. For real numbers m and n, if m is irrational and n is rational, then $m - n$ is irrational.

3.3. Prove the following statement, again by contradiction. For all $m \in \mathbb{N}$, if m, $m+2$ and $m+4$ are all prime, then $m = 3$.

You might remember this as a challenge problem from last week's lectures!

3.4. Find the unique factorisation (in standard form) of the integer 27 720.

3.5. For the following statement about the positive integers m, n and q, if the statement is true, prove it; if the statement is false, give a counterexample.

If $m \mid n$ and $n \mid q$ then $m^2 \mid nq$.

3.6. Prove the following statements.

(a) If k is any even integer and m is any odd integer, then $(k+2)^2 - (m-3)^2$ is divisible by 4.
(b) The difference between the cube of two consecutive integers leaves the remainder 1 when it is divided by 6.

3.7. (a) For all integers c, d and e, if $c \mid d$ and $c \nmid e$, then $c \nmid (d+e)$.
Use *proof by contradiction* to prove this statement.

(b) If y is *any* integer, prove that when y^2 is divided by 5, the remainder is always 0, 1 or 4, and never 2 or 3.
(Hint: Any integer can be written in the form $5m$, $5m+1$, $5m+2$, $5m+3$ *or* $5m+4$*, where* m *is also an integer.)*

3.8. For each of the following, determine whether the statement is true or false. If the statement is true, provide a proof; if the statement is false, provide a counterexample.

(a) For all integers a, b and c, if $a \mid b$ and $a \mid c$, then $a \mid (4b - 2c)$.
(b) For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$.
(c) For all integers a, b and c, if $a \mid bc$, then $a \mid b$ or $a \mid c$.

Here are two **puzzles** that you can think about during week 4. Feel free to ask your lecturer for more hints!

E. Scott and Ainsley are having trouble balancing their budgets, and so they decide to return to school. At the end of the school year, the teacher wants to test the prime division skills of the entire class of 25 students. They are all lined up in a queue, and Scott and Ainsley stand next to each other.

The teacher writes a natural number on the board. The first student in the queue says "That number is divisible by 1." Then the second student in the queue says "That number is divisible by 2," and so on, the kth student says "That number is divisible by k," until the final student in the queue says "That number is divisible by 25."

"Well done," the teacher exclaims, "except for Scott and Ainsley everyone gave a correct answer." Where did Scott and Ainsley stand in the queue?

F. Egyptian fractions: The ancient Egyptians had the habit of writing fractions in the form:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \dots,$$

where $a, b, c, \dots$ are whole numbers and *no two of them are equal to each other.* For instance,

$$\frac{3}{8} = \frac{1}{4} + \frac{1}{8} = \frac{1}{3} + \frac{1}{24} = \frac{1}{3} + \frac{1}{33} + \frac{1}{88}.$$

Write $\frac{9}{13}$ in Eqyptian form!

Extra practice questions from the textbook (Solutions at the back of the book.)

Epp 5th ed.:
Section 4.1, pp. 171–173: Questions 1, 3, 5, 10, 12, 14, 17, 21, 23, 25
Section 4.3, pp. 187–189: Questions 1, 3, 4, 6, 11, 12, 13, 15, 21, 24, 31, 35, 37.
Section 4.4, pp. 197–199: Questions 1, 2, 4, 6, 7, 8, 10, 12, 14, 15, 18, 19, 24, 25, 37a, 38a, 44.
Section 4.7, pp. 225–227: Questions 1, 3, 5, 8, 10, 12, 16, 17, 19, 21, 23, 26, 31a.
Section 4.8, pp. 233–235: Questions 1, 3, 6, 8, 10, 12, 16, 19, 22, 23, 24, 28, 29, 35, 37.

Epp 4th ed.:
Section 4.1, pp. 161–163: Questions 2, 4, 7, 9, 11, 14, 24, 25, 29, 31, 33, 35, 38, 39, 40, 43, 44, 45, 47, 54.
Section 4.2, pp. 168–170: Questions 1, 4, 6, 11, 12, 13, 15, 21, 24, 31, 35, 37.
Section 4.3, pp. 177–179: Questions 1, 2, 4, 6, 7, 8, 10, 12, 14, 15, 17, 18, 24, 25, 37a, 38ab, 44.
Section 4.6, pp. 205–207: Questions 1, 3, 5, 8, 10, 11, 13, 14, 15, 18, 19, 21, 23, 26, 27, 30, 31a.
Section 4.7, pp. 212–213: Questions 3, 5, 7, 9, 13, 16, 19, 20, 21, 25, 26, 32, 34.

Puzzle hints:

Stuck on the puzzles from week 3?

C. The pigeonhole principle (again!). Pick one person x. How many other people does x know? How many other people does x not know? What can you deduce from here?

D. Try replacing 2015 with smaller odd numbers (5, 7, 9, …). Look for patterns!