MATH1061
Discrete Mathematics I

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Contents

Chapter 1

Week 1

1.1 Lecture 1

This course will run a little differently. Prior to every lecture, we must work through a set of pre-lecture problems. The goal of timetabled lectures is to discuss and learn from solving problems.

1.1.1 What is in this course?

Logic and set theory, methods of proof

Modern mathematics uses the language of set theory and the notation of logic.

$$((P \land \neg Q) \lor (P \land Q)) \land Q \equiv P \land Q$$

We will learn to read and analyse this. Historical, there was a big shift in recent history, there was a big effort to define and axiom-itise everything, such that math itself is defined rigorously. Symbolic logic is the basis for many areas of computer science. It helps us formulate mathematical ideas and proofs effectively and correctly!

Definition 1.1.1: Gödel's Incompleteness Theorem (1931)

There exists true statements which we can not prove!

Number Theory

Example 1.1.1 $(1 + \cdots + 100)$

A young Gauss had to add up all the numbers from 1 to 100 in primary school. What did he do?

$$\begin{array}{ccccccccccccc} & & 1 & + & 2 & + & \cdots & + & 98 & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 & & \\ \hline 100 & + & 100 & + & 100 & + & \cdots & + & 100 & + & 100 & + & 100 \end{array}$$

So...

$$1 + \cdots + 100 = \frac{101 \cdot 100}{2} = 5050$$

This generalises to $\forall n \in \mathbb{N}$. Two leaps of faith are needed though!

- The dots: We introduce the notation to deal with them.
- The equality of two equations invoving dots. We will use induction to deal with this!

Graph Theory

Example 1.1.2 (The Königsberg Bridge Problem)
Find a route through the city which crosses each of seven bridges exactly once, and returns you to your start location.

This is provably impossible! But how can we rigorously prove this? Euler solve this problem in 1935 and in doing so invented graph theory. We'll learn how eventually... :p

Counting and Probability

Both fundamental and beautifully applicable. We introduce the pigeonhole principle as a introduction to "counting."

Example 1.1.3 (The pigeonhole principle)
If you have n pigeons sitting in k pigeonholes, if $n > k$, then at least of the pigeonholes contains at least 2 pigeons.

Question 1

If you have socks of three different colours in your drawer, what is the minimum number of socks you need to pull out to guarantee a matching pair?

Solution: #socks $\equiv$ #pigeons and #colours $\equiv$ #holes. If #socks > #colors, a double must occur. Therefore, we need a minimum of 4 socks to guarantee a match.

Question 2: True or False?

In every group of five people, there are two people who have the same number of friends within the group.

Solution: True! #people $\equiv$ #pigeons and #friends $\equiv$ #holes. There are 5 possible values for the amount of friends one could have, $\{0, 1, 2, 3, 4\}$, but you can never have an individual with 0 friends, and 4 friends in the same group. So there are 5 people, and 4 possible #friend values (think "holes.") Therefore, by pigeonhole principle, the statement is true!

Question 3: True or False?

A plane is coloured blue and red. Is it possible to find exactly two points the same colour exactly one unit apart?

Note:-
We will answer this on Wednesday!

Recursion

Example 1.1.4 (The Tower of Hanoi)
Given: a tower of 8 discs in decreasing size on one of three pegs.
Problem: transfer the entire towert to one of the other pegs.
Rule 1: Move only one disc at a time.
Rule 2: Never move a larger disc onto a smaller disk.

1. Is there a solution?

2. What's the minimal number of moves necessary and sufficient for the task?

A key idea is to generalise! What if there are n discs? Let T_n be the minimal number of moves, then trivially $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, so what is $T_3 =?$. Is there a pattern? The winning strategy is

1. Move the $n-1$ smallest discs from peg A to B.
2. Move the big disc from A to C.
3. Move $n-!$ smallest discs from B to C

By induction we show that

$$T_n = 2T_{n-1} + 1.$$

So $T_3 = 7$, $T_4 = 15$, $T_5 = 31$, $T_6 = 63$. Remarkably, this is one less then the sqaure numbers! We will prove this fact by induction later in the course.

Note:-

On Wednesday we start proberly.
Read:

Pages 23-36 (Epp, 4th) or
Pages 37-50 (Epp, 5th).

Watch the first video on UQ Extend, try the first quiz before Wednesday's lecture!

Chapter 2

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2.1 Random Examples

Definition 2.1.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say
$$\lim_{n\to\infty} s_n = s$$
where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$
$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 4

Is the set $x-$axis$\setminus\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Claim 2.1.1 Topology

Topology is cool

Example 2.1.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\bigcup\limits_{x\in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

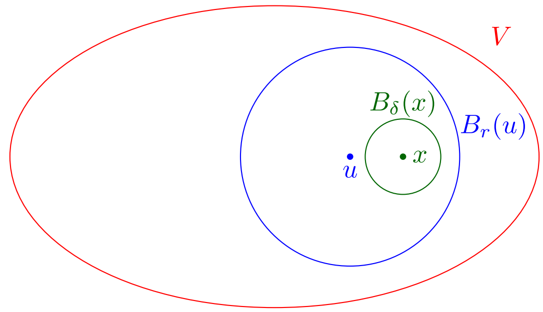
Closed Set:
- $X,\ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Theorem 2.1.1

If $x \in$ open set V then $\exists$ $\delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u,x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u,y) < r$ but

$$d(u,y) \le d(u,x) + d(x,y) < d + \delta < r$$

Corollary 2.1.1

By the result of the proof, we can then show...

Lenma 2.1.1

Suppose $\vec{v_1}, \ldots, \vec{v_n} \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 2.1.1

$1 + 1 = 2$.

2.2 Random

Definition 2.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|\ V \to \mathbb{R}_{\ge 0}$ satisfying

1. $\|x\| = 0 \iff x = 0\ \forall\ x \in V$
2. $\|\lambda x\| = |\lambda|\|x\|\ \forall\ \lambda \in \mathbb{R}(\text{or } \mathbb{C}),\ x \in V$
3. $\|x + y\| \le \|x\| + \|y\|\ \forall\ x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.

• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\ge 0}$) where 2 becomes $\|\lambda x\| = |\lambda|\|x\|$ $\forall\ \lambda \in \mathbb{C},\ x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

Example 2.2.1 (p−Norm)

$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\ge 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \Big(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\Big)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.

Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$

For $m = 1$ these p−norms are nothing but $|x|$. Now exercise

Question 5

Prove that triangle inequality is true if $p \geq 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property 3 for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\begin{aligned}
&\sum_i (x_i + y_i)^2 \leq \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2 \\
\implies &\sum_i (x_i^2 + 2x_iy_i + y_i^2) \leq \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2 \\
\implies &\left[\sum_i x_iy_i\right]^2 \leq \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]
\end{aligned}$$

So in other words prove $\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_iy_i$$

Note:-

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$
- $\langle \cdot, \cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y\rangle = r\langle x, y\rangle + \langle x', y\rangle \text{ and similarly for second slot}$$

Here in $\langle x, y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x - \lambda y, x - \lambda y\rangle &= \langle x, x - \lambda y\rangle - \lambda\langle y, x - \lambda y\rangle \\
&= \langle x, x\rangle - \lambda\langle x, y\rangle - \lambda\langle y, x\rangle + \lambda^2\langle y, y\rangle \\
&= \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle
\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y\rangle = \sum_i \overline{x_i}y_i$$

Then we still have $\langle x, x\rangle \geq 0$

2.3 Algorithms

Algorithm 1: what

```
Input: This is some input
Output: This is some output
/* This is a comment */
some code here;
x ← 0;
y ← 0;
if x > 5 then
    x is greater than 5 ;                    // This is also a comment
else
    x is less than or equal to 5;
end
foreach y in 0..5 do
    y ← y + 1;
end
for y in 0..5 do
    y ← y − 1;
end
while x > 5 do
    x ← x − 1;
end
return Return something here;
```