SCHOOL OF MATHEMATICS AND PHYSICS, UQ

MATH1072
Assignment 1
Semester Two 2024

Submit your answers by 1pm on Monday, 12th August, using the Blackboard assignment submission system. Assignments must consist of a single PDF.
You may find some of these problems challenging. Attendance at weekly tutorials is assumed.

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Marker's use only

Each question marked out of 3.

- Mark of 0: You have not submitted a relevant answer, or you have no strategy present in your submission.
- Mark of 1: Your submission has some relevance, but does not demonstrate deep understanding or sound mathematical technique.
- Mark of 2: You have the right approach, but need to fine-tune some aspects of your calculations.
- Mark of 3: You have demonstrated a good understanding of the topic and techniques involved, with well-executed calculations.

Q1: Q2: Q3a:

Q3b:

Q3c:

Total (out of 15):

Question 1

Consider a sphere with radius r moving at speed v through a fluid of density ρ and viscosity μ. Use dimensional analysis to find a relationship for the drag force F, as a function of these other variables. i.e. determine a relationship of the form

$$F = f(r, v, \rho, \mu).$$

Solution: First, we'll note the relevant units and their dimensions.

$[F]$	$[r]$	$[v]$	$[\rho]$	$[\mu]$
$M^1L^1T^{-2}$	L^1	L^1T^{-1}	M^1L^{-3}	$M^1L^{-1}T^{-1}$

Now, we'll calculate the dimensional products,

$$\begin{aligned}
[F]^a\,[r]^b\,[v]^c\,[\rho]^d\,[\mu]^e &= \left(MLT^{-2}\right)^a (L)^b \left(LT^{-1}\right)^c \left(ML^{-3}\right)^d \left(ML^{-1}T^{-1}\right)^e \\
&= \left(M^aL^aT^{-2a}\right)\left(L^b\right)\left(L^cT^{-c}\right)\left(M^dL^{-3d}\right)\left(M^eL^{-e}T^{-e}\right) \\
&= M^{a+d+e}L^{a+b+c-3d-e}T^{-2a-c-e}
\end{aligned}$$

$$\text{Let this} = M^0L^0T^0 \text{ because we're assuming dimensional homogeneity.}$$

$$\Rightarrow \left.\begin{aligned} a+d+e &= 0 \\ a+b+c-3d-e &= 0 \\ -2a-c-e &= 0 \end{aligned}\right\} \Rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -3 & -1 & 0 \\ -2 & 0 & -1 & 0 & -1 & 0 \end{array}\right)$$

$$\underset{R_3\to R_3+2R_1}{\overset{R_2\to R_2-R_1}{\rightsquigarrow}} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & -2 & 0 \\ 0 & 0 & -1 & 2 & 1 & 0 \end{array}\right) \underset{R_2\to R_2+R_3}{\rightsquigarrow} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 & 0 \end{array}\right)$$

$$\underset{R_2\to R_2+R_3}{\rightsquigarrow} \left.\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 0 \end{array}\right)\right\} \Rightarrow \begin{aligned} e & & &= -a-d \\ c &= 2d+e &&= d-a \\ b &= c &&= d-a \end{aligned}$$

$$\begin{aligned}
\therefore [F]^a\,[r]^b\,[v]^c\,[\rho]^d\,[\mu]^e &= [F]^a\,[r]^{d-a}\,[v]^{d-a}\,[\rho]^d\,[\mu]^{-a-d} \\
&= [F]^a\,[r]^{-a}\,[r]^d\,[v]^{-a}\,[v]^d\,[\rho]^d\,[\mu]^{-a}\,[\mu]^{-d} \\
&= [F]^a\,[r]^{-a}\,[v]^{-a}\,[\mu]^{-a}\,[r]^d\,[v]^d\,[\rho]^d\,[\mu]^{-d} \\
&= \left([F]\,[r]^{-1}\,[v]^{-1}\,[\mu]^{-1}\right)^a \left([r]\,[v]\,[\rho]\,[\mu]^{-1}\right)^d \\
&= \left[\frac{F}{rv\mu}\right]^a \left[\frac{rv\rho}{\mu}\right]^d
\end{aligned}$$

By applying Buckingham-Π theorem, we can see that

$$\Pi = \left\{\frac{F}{rv\mu}, \frac{rv\rho}{\mu}\right\},$$

$$f\left(\Pi_1, \Pi_2\right) = 0,$$

$$f\left(\frac{F}{rv\mu}, \frac{rv\rho}{\mu}\right) = 0.$$

Finally, by implict function theorem, we can conclude that

$$F = f(r, v, \rho, \mu) = rv\mu\; h\left(\frac{rv\rho}{\mu}\right)$$

Question 2

When disturbed, a buoy floating in the ocean will oscillate up and down at a frequency f. Assume this frequency depends on the buoy's mass m, its diameter at the waterline d, and the specific weight γ (force exerted by gravity per unit volume) of the water. If d and γ are assumed constant and m is halved, use dimensional analysis to determine how f will change.

Solution: First, we note the relevant units and their dimensions.

$[f]$	$[m]$	$[d]$	$[\gamma]$
T^{-1}	M^1	L^1	$M^1L^{-2}T^{-2}$

Now, we will find the dimensional product,

$$\begin{aligned}[f]^\alpha[m]^\beta[d]^\delta[\gamma]^\varepsilon &= \left(T^{-1}\right)^\alpha\left(M^1\right)^\beta\left(L^1\right)^\delta\left(M^1L^{-2}T^{-2}\right)^\varepsilon\\ &= \left(T^{-\alpha}\right)\left(M^\beta\right)\left(L^\delta\right)\left(M^\varepsilon L^{-2\varepsilon}T^{-2\varepsilon}\right)\\ &= M^{\beta+\varepsilon}L^{\delta-2\varepsilon}T^{-\alpha-2\varepsilon}.\end{aligned}$$

Since, we assume, the system is dimensionally homogeneous, we can set this product to $M^0L^0T^0$, then find and solve the linear system,

$$\left.\begin{array}{rcl}\beta+\varepsilon &=& 0\\ \delta-2\varepsilon &=& 0\\ -\alpha-2\varepsilon &=& 0\end{array}\right\}\Rightarrow\begin{array}{rcl}\beta &=& -\varepsilon\\ \delta &=& 2\varepsilon\\ \varepsilon &=& -\frac{\alpha}{2}\end{array}\begin{array}{c}\Rightarrow\\ \Rightarrow\\ \end{array}\begin{array}{rcl}\beta &=& \frac{\alpha}{2}\\ \delta &=& -\alpha\\ &&\end{array}.$$

Let's now substitute these values, back into our dimensional product,

$$\begin{aligned}[f]^\alpha[m]^\beta[d]^\delta[\gamma]^\varepsilon &= [f]^\alpha[m]^{\frac{\alpha}{2}}[d]^{-\alpha}[\gamma]^{-\frac{\alpha}{2}}\\ &= \left[fm^{\frac{1}{2}}d^{-1}\gamma^{-\frac{1}{2}}\right]^\alpha\\ &= \left[\frac{f\sqrt{m}}{d\sqrt{\gamma}}\right]^\alpha.\end{aligned}$$

This shows that our system has one dimensionless product, namely

$$\Pi_1 = \frac{f\sqrt{m}}{d\sqrt{\gamma}}.$$

We can apply Buckingham-Π theorem here, which shows that

$$f_\Pi\left(\frac{f\sqrt{m}}{d\sqrt{\gamma}}\right) = 0.$$

By implict function theorem, we can show that

$$\frac{f\sqrt{m}}{d\sqrt{\gamma}} = k \Rightarrow f = k\frac{d\sqrt{\gamma}}{\sqrt{m}},$$

where k is some dimensionless constant, which ensures dimensional homogeneity.
Finally, we hold d and γ constant, but halve the mass,

$$f = k\frac{d\sqrt{\gamma}}{\sqrt{m/2}} = k\frac{d\sqrt{\gamma}}{\sqrt{1/2}\sqrt{m}} = \sqrt{2}k\frac{d\sqrt{\gamma}}{\sqrt{m}},$$

which allows us to conclude that halving m, but holding d and γ constant, will affect the frequency, by scaling it by a factor of $\sqrt{2}$.

Question 3

Consider the function

$$f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}.$$

The domain D of f is given by $\mathbb{R}^2 \setminus \{(0,0)\}$.

(a) Using MATLAB, plot the surface $z = f(x,y)$ around $(x,y) = (0,0)$ in D.

(b) Show that $|\cos^3\theta \sin\theta - \cos\theta \sin^3\theta| \leq \frac{1}{4}$

(c) Determine $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ if it exists and confirm this with an ε-δ proof, or show that the limit does not exist.

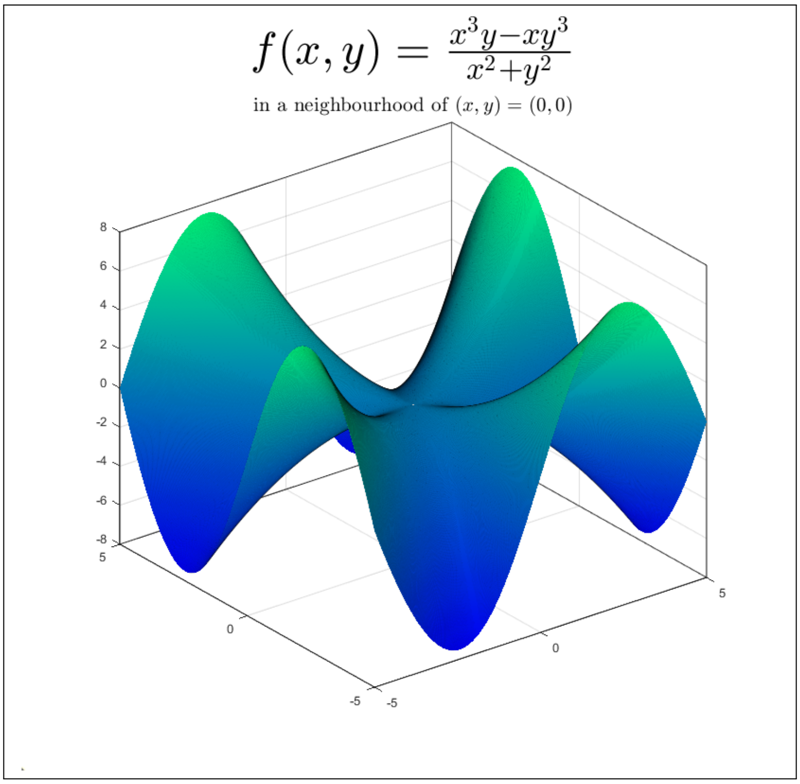
Solution: (a)

question3a.m

```matlab
f = @(x, y) (y*x^3 - x*y^3) / (x^2 + y^2);
[x, y] = meshgrid(-5:0.05:5, -5:0.05:5);
z = arrayfun(f, x, y);
surf(x, y, z);

% Let's make it look pretty! :)
title("\(f(x,y)=\frac{x^3y-xy^3}{x^2+y^2}\)", ...
    "FontSize", 36, "Interpreter", "latex")
subtitle("in a neighbourhood of \((x,y)=(0,0)\)", ...
    "FontSize", 16, "Interpreter", "latex")
posX=50; posY=50; width=800; height=800;
set(gcf, "Position", [posX, posY, width, height]);
colormap("winter");
box on;
```

Output:



Solution: (b)

$$
\begin{aligned}
\cos^3\theta\sin\theta - \cos\theta\sin^3\theta &= \sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) \\
&= \sin\theta\cos\theta\cos 2\theta \\
&= \left(\frac{\sin(\theta+\theta)+\sin(\theta-\theta)}{2}\right)\cos 2\theta \\
&= \left(\frac{\sin(2\theta)+0}{2}\right)\cos 2\theta \\
&= \frac{1}{2}\sin 2\theta\cos 2\theta \\
&= \frac{1}{2}\left(\frac{\sin(2\theta+2\theta)+\sin(2\theta-2\theta)}{2}\right) \\
&= \frac{1}{2}\left(\frac{\sin(4\theta)+\sin(0)}{2}\right) \\
&= \frac{1}{4}\sin 4\theta \\
|\sin\theta| &\le 1 \\
|\sin 4\theta| &\le 1 \\
\left|\frac{1}{4}\sin 4\theta\right| &\le \frac{1}{4} \\
\therefore \left|\cos^3\theta\sin\theta - \cos\theta\sin^3\theta\right| &\le \frac{1}{4}
\end{aligned}
$$

□

Solution: (c) First, we'll investigate two particular paths, namely $y = 0$ and $x = 0$.

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,0) &= \lim_{(x,y)\to(0,0)} \frac{0-0}{x^2+0} \\
&= \lim_{(x,y)\to(0,0)} \frac{0}{2x} \\
&= \lim_{(x,y)\to(0,0)} \frac{0}{2} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} f(0,x) &= \lim_{(x,y)\to(0,0)} \frac{0-0}{0+y^2} \\
&= \lim_{(x,y)\to(0,0)} \frac{0}{2y} \\
&= \lim_{(x,y)\to(0,0)} \frac{0}{2} \\
&= 0
\end{aligned}
$$

These appear to approach a definied limit, 0.
Next, we'll investigate all paths of form (r, θ),

$$
\begin{aligned}
\lim_{(r,\theta)\to(0,0)} f(r\cos\theta, r\sin\theta) &= \lim_{(r,\theta)\to(0,0)} \frac{r^4\cos^3\theta\sin\theta - r^4\cos\theta\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\
&= \lim_{(r,\theta)\to(0,0)} \frac{r^4\left(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta\right)}{r^2\left(\cos^2\theta+\sin^2\theta\right)} \\
&= \lim_{(r,\theta)\to(0,0)} \frac{r^2\left(\frac{1}{4}\sin 4\theta\right)}{1} && \text{(Result from question 3b.)} \\
&= \lim_{(r,\theta)\to(0,0)} \frac{r^2\sin 4\theta}{4} \\
&= 0\sin 0 \\
&= 0
\end{aligned}
$$

So, it seems we've determined the limit exists and is equal to 0.
Let's now prove the limit with an ε-δ proof.

$$
\lim_{(x,y)\to(a,b)} f(x,y) = L \iff \forall\varepsilon > 0, \exists\delta > 0 : 0 < \sqrt{(x-a)^2+(y-b)^2} < \delta \implies |f(x,y) - L| < \varepsilon.
$$

Namely,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \iff \forall\varepsilon > 0, \exists\delta > 0 : 0 < \sqrt{x^2+y^2} < \delta \implies |f(x,y)| < \varepsilon.$$

Given $\varepsilon > 0$, we choose $\delta = \sqrt{\varepsilon}/2$. Let's suppose $0 < \sqrt{x^2+y^2} < \delta$. We can see

$$\begin{aligned}
\left|\frac{x^3y - xy^3}{x^2+y^2}\right| &= \frac{\left|xy(x^2-y^2)\right|}{x^2+y^2} \\
&= \frac{|xy|\left|x^2-y^2\right|}{x^2+y^2}. \\
|xy| &= |x|\,|y|\,. \\
\left|x^2-y^2\right| &= |x+y|\,|x-y| \\
&\le (|x|+|y|)\,(|x|+|y|) \\
&= (|x|+|y|)^2\,. \\
x^2+y^2 &< \delta^2. \\
|x| \le \sqrt{x^2+y^2} &< \delta. \\
|y| \le \sqrt{x^2+y^2} &< \delta. \\
\frac{|xy|\left|x^2-y^2\right|}{x^2+y^2} &\le \frac{|x|\,|y|\,(|x|+|y|)^2}{x^2+y^2} \\
&< \frac{\delta\cdot\delta\,(\delta+\delta)^2}{\delta^2} \\
&= \frac{\delta^2\,(2\delta)^2}{\delta^2} \\
&= 4\delta^2 \\
&= 4\left(\frac{\sqrt{\varepsilon}}{2}\right)^2 \\
&= 4\left(\frac{\varepsilon}{4}\right) \\
&= \varepsilon.
\end{aligned}$$

Therefore, we've shown that for all $\varepsilon > 0$, we can choose $\delta = \sqrt{\varepsilon}/2$, and if given $0 < \sqrt{x^2+y^2} < \delta$, then $|f(x,y)| < \varepsilon$. So, we've proven that the limit, $\lim_{(x,y)\to(0,0)} f(x,y)$, exists and is equal to 0. □