MATH1072

Practical 6: Numerical Integration Methods for Solving Higher-Order and Coupled Systems of ODEs

Motivation: Congratulations! You have made it to your final practical for MATH1072. You saw in your last practical how to apply the MATLAB built-in ODE solvers to find numerical solutions to first-order linear ODEs. Often in real-world applications we have to solve more than one ODE at a time. For example, in our applications of population dynamics we derived *coupled systems of differential equations* to describe the complex dynamics of a population of one species dependent on the population of another. We will build on your knowledge of built-in ODE solvers by exploring how the same ODE solvers can solve systems of ODEs.

In fact, any higher-order explicit ODE can be written as a system of first-order ODEs! We will explore this as well.

1 Higher-order ODEs

Consider the initial value problem

$$\ddot{y} + \sin(y) = 0, \ y(0) = 1, \ \dot{y}(0) = 0. \tag{1}$$

We are able to use one of the MATLAB ODE solvers (e.g. `ode45` or `ode23` or `ode23t`) to solve this equation numerically. First, however, we need to convert the second order ODE into a system of two first order ODEs. We do this by defining a new variable (say v) by

$$\dot{y} = v \implies \ddot{y} = \dot{v}.$$

The second order ODE can then be written as,

$$\dot{v} = -\sin(y)$$

which leads to a *coupled system of first order ODEs* defined by,

$$\begin{aligned} \dot{v} &= -\sin(y), \\ \dot{y} &= v \\ v(0) = \dot{y}(0) &= 0,\ y(0) = 1. \end{aligned} \tag{2}$$

An ODE solver (let's use `ode45`) can then be used to solve this system in a way similar to that introduced in your last practical. The function representing the right hand side of the ODE that we give `ode45` must take variables t and (v, y). The usage of `ode45` requires the variables (v, y) be given as a single vector, which we shall call u. The function should output a column vector with two components, since the right hand side of the ODE has two parts. It may be helpful to see the ODE written as

$$\frac{d}{dt}\begin{pmatrix} v \\ y \end{pmatrix} = \begin{pmatrix} -\sin(y) \\ v \end{pmatrix}. \tag{3}$$

We've had some experience defining different types of implicit functions. Let's try to define the function in (3) as an implict function of the vector $u = (v, y)$ in MATLAB.

```
rhs = %define the implicit function representing the ODE
```

Try calling the value of your function at $t = 0$, $(v, y) = (2, \pi/4)$; $t = 1$, $(v, y) = (2, \pi/4)$; and $t = 1$, $(v, y) = (3, \pi/3)$.

```
rhs() %call the value of your function at the specified
   points
```

Note that for this example, the function `rhs` does not depend explicitly on the variable t. We still need to define the right hand side as a function involving t, because the ODE solver needs to be told which is the independent variable and which are the dependent variables.

Now, use `ode45` to generate data points for a numerical solution to the initial value problem (1) expressed as the system (2) for time interval $0 \leq t \leq 10$. Plot your solution y to the second-order ODE - make sure you add labels. Recall that, based on the setup we've performed, the first entry of your solution will be v and the second will be y. Convince yourself why this is true!

```
[T,Y] = ode45() %type in the correct arguments to run
   ode45 for the IVP
plot() %plot your solution
```

2 Systems of ODEs

Consider a satellite in a low-earth orbit where its projected 2D movement is governed by the system of two second-order ODEs,

$$\ddot{x} = -\frac{GM}{r^2} \cdot \frac{x}{r}$$
$$\ddot{y} = -\frac{GM}{r^2} \cdot \frac{y}{r}$$
$$x(0) = r,\ y(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = v$$

where $G = 4\pi^2$ and $M = 1.0$ and $r = \sqrt{x^2 + y^2}$.

a) Solve this system using `ode45` with initial conditions,

$$x(0) = 1,\ y(0) = 0$$
$$\dot{x}(0) = 0,\ \dot{y}(0) = \sqrt{GM}$$

over $0 \leq t \leq 10$. Plot the trajectory of x and y.

b) Solve the same system using `ode23` and `ode23t` (solvers meant for moderate to- stiff problems). Which one seems to be the most accurate based on your physical intuition?