School of Mathematics and Physics, UQ

MATH2001/MATH7000 practice problems
Sheet 5

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(1) Let $C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Construct a matrix P such that $P^T C P = D$ where D is a diagonal matrix and find a general expression for the matrix C^n.

(2) Find an orthogonal matrix which diagonalizes the matrix $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(3) Determine whether the following matrix is orthogonal, and if so find its inverse.

$$A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

(4) Determine whether the following matrix is orthogonal, and if so find its inverse.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}.$$

(5) What conditions must a and b satisfy for the matrix;

$$C = \begin{pmatrix} a+b & b-a \\ a-b & b+a \end{pmatrix}$$

to be orthogonal?

(6) Write the following quadratic equations in the matrix form

$$\underset{\sim}{x}^T A \underset{\sim}{x} + K \underset{\sim}{x} + c = 0,$$

where A is a symmetric 2×2 matrix, $\underset{\sim}{x}$ is the coordinate vector $\begin{pmatrix} x \\ y \end{pmatrix}$, K is a 1×2 matrix, and c is a real number.

(a) $2x^2 - 3xy + 4y^2 - 7x + 2y + 7 = 0$,

(b) $x^2 - xy + 5x + 8y - 3 = 0$,

(c) $5xy = 8$,

(d) $4x^2 - 2y^2 = 7$,

(e) $y^2 + 7x - 8y - 5 = 0$.

(7) Suppose that $\mathbf{x}$ is a unit eigenvector of a matrix A corresponding to an eigenvalue 2. What is the value of $\mathbf{x}^t A\mathbf{x}$?

(8) Suppose that A is an $n \times n$ real symmetric matrix and

$$q(\mathbf{x}) = \mathbf{x}^t A\mathbf{x},$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that is expressed in column form. What can you say about the value of q if $\mathbf{x}$ is a unit eigenvector corresponding to an eigenvalue λ of A?

(9) Express the quadratic form $Q(x, y) = 5x^2 + 2y^2 + 4xy$ as a linear combination of two squares.

(10) A quadratic form $\underset{\sim}{x}^T A\underset{\sim}{x}$ is called positive definite if $\underset{\sim}{x}^T A\underset{\sim}{x} > 0$ for all $\underset{\sim}{x} \neq 0$, and a symmetric matrix A is called a positive definite matrix if the associated quadratic form is positive definite. Show that the quadratic form $f(x, y) = 5x^2 - 2xy + 5y^2$ is positive definite.

(11) Consider the conic section

$$2x^2 - 4xy - y^2 - 4x - 8y = -14.$$

Rotate and translate the coordinate axes to write it in standard form. Hence name the type of conic section.

(12) Consider the conic section

$$9x^2 - 4xy + 6y^2 - 10x - 20y = 5.$$

Rotate and translate the coordinate axes to write it in standard form. Hence name the type of conic section.

(13) Let A be a real $n \times n$ matrix. Show that if λ is a complex eigenvalue of A with eigenvector $\boldsymbol{v}$, then the complex conjugate $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{\boldsymbol{v}}$.

(14) Write down the conjugate transpose of the following matrices:
(a) $\begin{pmatrix} 2i & 1-i \\ 4 & 3+i \\ 5+i & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2i & 1-i & -1+i \\ 4 & 5-7i & -i \\ i & 3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 7i & 0 & -3i \end{pmatrix}$ (d) $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$.

(15) Find K, L, and M to make A a Hermitian matrix.

$$A = \begin{pmatrix} -1 & K & -i \\ 3-5i & 0 & M \\ L & 2+4i & 2 \end{pmatrix}$$

(16) Show that the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\theta} & e^{-i\theta} \\ ie^{i\theta} & -ie^{-i\theta} \end{pmatrix}$$

is unitary for every real value of θ.

(17) Let a and b be complex numbers.

(a) Show that all 2×2 matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are normal.

(b) Show that $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is unitarily diagonalisable by a matrix P with only *real* entries. Find P. This is an example of a complex matrix that can be *orthogonally* diagonalised.

(18) Find a unitary matrix P that diagonalises $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{pmatrix}$ and determine $P^{-1}AP$.