**SCHOOL OF MATHEMATICS AND PHYSICS**

# MATH1072
## Assignment 3
## Semester Two 2024

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*Submit your answers - along with this sheet - by 1pm on the 30th of September, using the blackboard assignment submission system. Assignments must consist of a single PDF.*
You may find some of these problems challenging. Attendance at weekly tutorials is assumed.

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Marker's use only

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Each question marked out of 3.

- Mark of 0: You have not submitted a relevant answer, or you have no strategy present in your submission.
- Mark of 1: Your submission has some relevance, but does not demonstrate deep understanding or sound mathematical technique.
- Mark of 2: You have the right approach, but need to fine-tune some aspects of your calculations.
- Mark of 3: You have demonstrated a good understanding of the topic and techniques involved, with well-executed calculations.

| | | | |
|---|---|---|---|
| Q1a | Q1b: | Q1c: | Q1d: |
| Q2a: | Q2b: | Q2c: | Q3a: |
| Q3b: | Q3c: | Q3d: | Q4: |

Total (out of 36):

## Question 1: Inital Value Problem

A population of Christmas beetles in Brisbane grows at a rate proportional to their current population. In the absence of external factors, the population will double in one week's time. On any given day, there is a net migration into the area of 10 beetles, 11 beetles are eaten by a local Magpie population, and 2 die of natural causes.

(a) Write a initial value problem to describe the change in population at time $t$, given that the initial population of Christmas beetles is $P(0) = 100$. You **must** write all the parameters in your model **explicitly**.

(b) Solve the initial value problem from part (a) to find the population $P(t)$, at any time $t$.

(c) Use MATLAB to plot your solution from part (b) from time $t = 0$ to $t = 100$. Will the beetle population survive?

(d) Use MATLAB to plot the solution to part (a) from time $t = 0$ to $t = 100$ for 30 different initial populations of Christmas beetles from $P(0) = 20$ to $P(0) = 50$. Recall that initial population sizes must be integer valued. What can you say about the stability of the population of Christmas beetles from this plot?

## Question 2: Fluid Flow

In this question, we take the cartesian variables as $x \equiv x_1$, $y \equiv x_2$, and $z \equiv x_3$. We also take the basis vectors as $\hat{\imath} \equiv \underset{\sim}{e}_1$, $\hat{\jmath} \equiv \underset{\sim}{e}_2$, and $\hat{k} \equiv \underset{\sim}{e}_3$.

In fluid mechanics, fluids which are **incompressible** and **inviscid** are referred to as **ideal**. Let $\underset{\sim}{u} = \underset{\sim}{u}(x, y, z, t) = u_1\underset{\sim}{e}_1 + u_2\underset{\sim}{e}_2 + u_3\underset{\sim}{e}_3$ be the velocity of an ideal fluid at an arbitrary point in space and time. Its motion is governed by the Euler equation,

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} = \underset{\sim}{F} - \frac{\nabla P}{\rho}.$$

Here $P$ is the fluid's pressure, $\rho$ its constant density, $\underset{\sim}{F}$ is the external force on the fluid at any given point, and

$$(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} \equiv \sum_{i=1}^{3} u_i \frac{\partial \underset{\sim}{u}}{\partial x_i}.$$

Suppose the system is simplified in three ways:

- The flow is **steady**. $\underset{\sim}{u} = \underset{\sim}{u}(x, y, z)$ is not changing with time.
- $\underset{\sim}{u}$ is a conservative vector field. This occurs when the fluid is **irrotational**, though we won't elaborate on that here.
- The external force is also conservative, with $\underset{\sim}{f} = -\nabla V$, for some scalar potential $V = V(x_1, x_2, x_3)$.

(a) Show that in this case,

$$(\underset{\sim}{u} \cdot \nabla)\underset{\sim}{u} = \frac{1}{2}\nabla||\underset{\sim}{u}||^2.$$

(Hint, you can assume that Clairaut's theorem applies)

(b) Hence show that the Euler equation simplifies to

$$\frac{1}{2}||\underset{\sim}{u}||^2 + V + \frac{P}{\rho} = \text{constant}$$

(c) If $V$ is a constant, what can you say about the relationship between a fluid's speed and its pressure?

## Question 3: First Order Differential Equation

Consider Newton's second law of motion which states that,

$$F = ma \tag{1}$$

or: net force, $F$, is equal to mass $m$, times acceleration $a$.

(a) Use equation (1) to write out the equation of motion of a particle of mass $m$, subject to a frictional force proportional to the square of the velocity $v(t)$, completely in terms of the particle's velocity $v(t)$.

(b) Solve the first-order differential equation from part (a) to find the particle's velocity $v(t)$ at time $t$, with initial velocity $v_0$.

(c) Use your solution from part (b) to solve for the position of the particle $x(t)$ at time $t$, with initial position $x_0$.

(d) Use Euler's method with step size 0.1 to estimate the particle's position $x(0.5)$ at time $t = 0.5$ of your solution in part (b). Take $m = 1$, $\beta = 2$, $x_0 = 0$, and $v_0 = 1$. Calculate the error in using Euler's method, rounded to the fourth decimal.

## Question 4: Line Integral

Evaluate the line integral

$$\int_C xe^y \, \mathrm{d}s,$$

where $C$ is the line segment from $(2, 0)$ to $(5, 4)$.