MATH2001
Calculus & Linear Algebra II

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Contents

Chapter 1

Week 1

1.1 Lecture 1

In this course we will cover four major topics:

- Ordinary Differential Equations
- Linear Algebra
- Vector Calculus
- Integral Calculus

1.1.1 Solutions to First Order ODEs

We are comfortable solving three types of first order ODEs by now:

- Directly integrable: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)$

$$y(x) = \int f(x)\mathrm{d}x = F(x) + C$$

- Seperable: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$

$$\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \iff \int \frac{1}{g(y)}\frac{\mathrm{d}y}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \int f(x)\mathrm{d}x \iff G(y(x)) = F(x) + C$$

$$\text{If } G \text{ is invertible, then } y(x) = G^{-1}(F(x) + C)$$

- Linear: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = q(x) - p(x)y$

$$\text{Let } \mu = \exp\left(\int p(x)\mathrm{d}x\right) \implies \mu\frac{\mathrm{d}y}{\mathrm{d}x} + \mu p(x)y = \mu q(x) \iff \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu q(x) \iff y(x) = \frac{1}{\mu(x)}\int \mu q(x)\mathrm{d}x$$

In many applications, we need to solve an IVP. In general this is an equation of form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \quad y(x_0) = y_0.$$

In other words, we seek to find solutions to the ODE which pass through the point (x_0, y_0) in the $x-y$ plane.

Example 1.1.1

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x, \quad y(0) = 1 \text{ has a unique solution:}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x \\
y(x) &= \frac{1}{2}x^2 + C \\
\text{Impose } y(0) &= 1 \\
\therefore 1 &= \frac{1}{2}(0)^2 + C \\
\therefore C &= 1 \\
\therefore y(x) &= \frac{1}{2}x^2 + 1
\end{aligned}$$

Example 1.1.2

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3xy^{1/3}, \quad y(0) = 0 \text{ has more than one solution:}$$

$$\begin{aligned}
y^{-1/3}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x \\
\int y^{-1/3}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= \int 3x\mathrm{d}x \\
\int y^{-1/3}\mathrm{d}y &= \int 3x\mathrm{d}x \\
\frac{3}{2}y^{2/3} + C_1 &= \frac{3}{2}x^2 + C_2 \\
y^{2/3} &= x^2 + C \\
\text{Impose } y(0) &= 0 \\
0^{2/3} &= 0^2 + C \\
\implies C &= 0 \\
\therefore y^{2/3} &= x^2 \\
\therefore y &= \pm x^3
\end{aligned}$$

This is problematic. Our inital value constraint hasn't allowed us to pick one particular solution.

Note:-

The previous IVP has multiple solutions because $f(x, y) = 3xy^{1/3}$ is not differentiable at $y = 0$.

Example 1.1.3

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x}, \quad y(0) = 1 \text{ has no solutions:}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{x} - \frac{1}{x}y \\
&= q(x) - p(x)y \\
\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y &= 1
\end{aligned}$$

$$
\begin{aligned}
\mu &= \exp\left(\int p(x)\mathrm{d}x\right) \\
&= \exp\left(\int \frac{1}{x}\mathrm{d}x\right) \\
&= \exp(\ln(x)) \\
&= x \\
\mu\frac{\mathrm{d}y}{\mathrm{d}x} + \mu p(x)y &= \mu \\
x\frac{\mathrm{d}y}{\mathrm{d}x} + y &= x \\
\frac{\mathrm{d}}{\mathrm{d}x}(xy) &= x \\
xy &= \int x\mathrm{d}x \\
&= \frac{1}{2}x^2 + C \\
\text{Impose } y(0) = 1 & \\
\therefore 0 \cdot 1 &= \frac{1}{2}(0)^2 + C \\
C &= 0 \\
\therefore y(x) &= \frac{1}{2}x
\end{aligned}
$$

However, our general solution **does not** satisfy our inital value constraint, $y(0) = \frac{1}{2}(0) = 0 \neq 1$.

Note:-

Our IVP doesn't have a solution because $f(x, y) = \frac{x-y}{x}$ is not differentiable or continuous around $x = 0$.

We're kind of loosley referring to "existence and uniqueness" theorems, or Picard-Lindelöf Theorem, which generally states:

$$\text{The IVP } \frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \quad y(x_0) = y_0$$

has a unique solution around x_0 if:

1. $f(x, y)$ is continuous around (x_0, y_0)
2. $f(x, y)$ is differentiable with respect to y around (x_0, y_0), ie $\frac{\partial f}{\partial y}$ is continuous aorund (x_0, y_0).

1.1.2 Existence and Uniqueness

Consider the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \quad y(x_0) = y_0$$

We are concerned with the conditions under which a solution exists and is unique.

1. (existence, Peano's Theorem) If $f(x, y)$ is continuous in some rectangle

$$R = \{(x, y) \mid |x - x_0| < a,\ |y - y_0| < b\}$$

then the IVP has at least one solution.

2. (uniqueness, Picard's Theorem) If $f_y(x, y) := \frac{\partial f}{\partial y}$ is also continuous in R then there is some interval $|x - x_0| \leq h \leq a$ which contains at least one solution.

This result only tells us that a solution exists or is unique locally. Beyond R, we simply don't know.

Example 1.1.4

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x, \quad y(0) = 1$$

$$f(x, y) = x$$
$$f_y(x, y) = 0$$

These functions are both continuous over $\mathbb{R}^2$. Therefore there exists a unique solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3xy^{1/3}, \quad y(0) = 0 \text{ has more than one solution:}$$

$$f(x, y) = 3xy^{1/3}$$
$$f_y(x, y) = xy^{-2/3}$$

$f(x, y)$ is continuous over $\mathbb{R}^2$ so there exists at least one solution. However, f_y has a discontinuity at $y = 0$, so there may or may not be unique solutions (remember, its not an iff).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x}, \quad y(0) = 1 \text{ has no solutions:}$$

$$f(x, y) = \frac{x - y}{x}$$
$$f_y(x, y) = -\frac{1}{x}$$

$f(x, y)$ and f_y both have discontinuities when $x = 0$, so we don't know from this test if there are solutions, or if the solution is unique.

Note:-

These theorems are not if and only if's. They can fail. For example, take the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}x^{-2/3}, \quad y(0) = 1/$$

We see

$$f(x, y) = \frac{1}{3}x^{-2/3}, \qquad f_y(x, y) = 0$$

f has a discontinuity when $x = 0$, so the theorem's fail to identify if this IVP has solutions. However, this IVP **does** have a unique solution,

$$y(x) = x^{1/3} + 1,$$

so we need to be careful we're using these theorems correctly. **If** f and f_y are continuous in some reigon **then** there exists a unique solution in that reigon.

Example 1.1.5

Solve these:

1. $y' = y^{2/3}, \quad y(0) = 1$

$$f(x, y) = y^{2/3}$$
$$f_y(x, y) = \frac{2}{3}y^{-1/3}$$

Therefore there exist at least one solution to the IVP.

$$\begin{aligned}
y^{-2/3}y' &= 1 \\
\int y^{-2/3}\mathrm{d}y &= \int 1\mathrm{d}x \\
3y^{1/3} &= x + C \\
y^{1/3} &= \frac{1}{3}(x+C) \\
y &= \frac{1}{27}(x+C)^3 \\
\text{Impose } y(0) &= 1
\end{aligned}$$

Imposing the IVP and expanding the cubic expression, will reveal 3 values for C, the nicest of which is 3. The one which satisfies our IVP is

$$y(x) = \frac{1}{27}(x+3)^3$$

Even though those other solutions exist, only one satisfies the IVP, hence this solution is unique.

2. $y' = (3x^2 + 4x + 2)/(2y - 2), \quad y(0) = 1$

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y-1)}$$

Because of the discontinuity at $y = 1$, our exisitence theorem fails to identify if solutions exist.

$$\begin{aligned}
y' &= \frac{3x^2 + 4x + 2}{2(y-1)} \\
2(y-1)y' &= 3x^2 + 4x + 2 \\
2\int y - 1\mathrm{d}y &= \int 3x^2 + 4x + 2 \ \mathrm{d}x \\
y^2 - 2y &= x^3 + 2x^2 + 2x + C \\
y^2 - 2y + 1 &= x^3 + 2x^2 + 2x + C + 1 \\
(y-1)^2 &= x^3 + 2x^2 + 2x + C + 1 \\
\text{Impose } y(0) &= 1 \\
((1) - 1)^2 &= (0)^3 + 2(0)^2 + 2(0) + C + 1 \\
\iff C &= -1 \\
\therefore (y-1)^2 &= x^3 + 2x^2 + 2x \\
\therefore y(x) &= 1 \pm \sqrt{x^3 + 2x^2 + 2x}
\end{aligned}$$

The IVP has two solutions.

1.1.3 Method of Succesive Approximations

Note:-

WATCH THE BONUS CONTENT FOR THIS CHAPTER'S CONTENT

1.1.4 Exact First Order ODEs

Definition 1.1.1: Exact First Order ODE

Recall that if $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t, then z is a differentiable function of t, whose derivative is given by the chain rule:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

Now suppose the equation

$$f(x, y) = C$$

defines y implicitly as a function of x. Then $y = y(x)$ can be show to satisfy a first order ODE obtained by using the chain rule above. In this case, $z = f(x, y(x)) = C$, so,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}C = 0 = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\implies f_x + f_y y' = 0$$

A first order ODE of form

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is called exact if there is a function $f(x, y)$ such that

$$f_x(x, y) = P(x, y) \quad \text{and} \quad f_y(x, y) = Q(x, y).$$

The solution is then given implicitly by the equation

$$f(x, y) = C,$$

where C can usually be determined by some intial condition.

Theorem 1.1.1 Test for Exactness

Let $P, Q, \dfrac{\partial P}{\partial y}, \dfrac{\partial Q}{\partial x}$ be continuous over some reigon of interest. Then

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is an exact ODE if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

everywhere in the reigon

Proof. 1. Prove: ODE is exact $\implies \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
Recall Clairout's Theorem,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ if both } f_{xy} \text{ and } f_{yx} \text{ are continuous in the reigon.}$$

$$\text{Suppose ODE is exact} \implies \exists f(x, y) : \frac{\partial f}{\partial x} = P(x, y), \frac{\partial f}{\partial y} = Q(x, y)$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial x}, \text{ by Clairout's Theorem.}$$

2. Prove: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \implies$ ODE is exact.

Suppose $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. We seek a function f such that $f_x = P, f_y = Q$.

$$\begin{aligned}
\text{Take } f(x,y) &= \int_{x_0}^{x} P(x',y)\mathrm{d}x' + \int_{y_0}^{y} Q(x_0,y') + C \\
f_x(x,y) &= \frac{\partial}{\partial x}\left(\int_{x_0}^{x} P(x',y)\mathrm{d}x' + \cancel{\int_{y_0}^{y} Q(x_0,y')\mathrm{d}y'}\right) = P(x,y) \\
f_y(x,y) &= \frac{\partial}{\partial y}\left(\cancel{\int_{x_0}^{x} P(x',y)\mathrm{d}x'} + \int_{y_0}^{y} Q(x_0,y')\mathrm{d}y'\right) = Q(x,y)
\end{aligned}$$

Therefore $P(x,y) + Q(x,y)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ is an exact ODE $\iff \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ everywhere in the reigon. □

Example 1.1.6

Solve the ODE $2x + e^y + xe^y y' = 0$

$$\begin{aligned}
P(x,y) &= 2x + e^y \\
\frac{\partial P}{\partial y} &= e^y \\
Q(x,y) &= xe^y \\
\frac{\partial Q}{\partial x} &= e^y \\
\frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} \Rightarrow \text{ODE is exact} \\
\therefore \exists f(x,y) : f_x(x,y) &= P = 2x + e^y \\
\text{and } f_y(x,y) &= Q = xe^y \\
\implies f = \int P\mathrm{d}x &= \int 2x + e^y \mathrm{d}x \\
&= x^2 + xe^y + g(y) \\
\implies f_y(x,y) = xe^y &= \frac{\partial}{\partial y}\left(x^2 + xe^y + g(y)\right) \\
xe^y &= xe^y + \frac{\mathrm{d}g}{\mathrm{d}y} \\
\implies \frac{\mathrm{d}g}{\mathrm{d}y} &= 0 \\
\therefore f(x,y) &= x^2 + xe^y + C
\end{aligned}$$

All solutions to ODE: $f(x,y) = k$.

$$\begin{aligned}
\iff x^2 + xe^y &= k' && (k' = k - C) \\
\iff y &= \ln\left(\frac{k' - x^2}{x}\right)
\end{aligned}$$