Universtiy of Queensland
School of Mathematics and Physics

MATH2302 Discrete Mathematics II
Semester 2 2025
Problem Set 3

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Applied Class #1
Due 3pm Friday 17 October 2025

Question 1: 10 marks

For each of the following words: (i) draw the polygon represented by the word; (ii) determine whether the resulting surface is orientable or non-orientable; (iii) compute the Euler characteristic; (iv) use ii. and iii. to identify the surface in the form that it is given in the classification theorem.

(a) $a\,b\,c\,a^{-1}\,b\,c^{-1}$

(b) $a\,b\,c\,a^{-1}\,d\,b^{-1}\,c^{-1}\,d^{-1}$

Solution: (a)

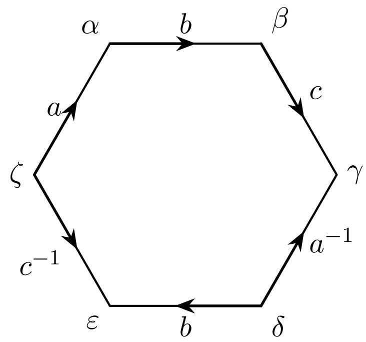

Figure 1: A drawing of the polygon represented by the word $abca^{-1}bc^{-1}$

A word describes an orientable surface if and only if every symbol appears with both signs.
a appears with both signs. c appears with both signs.
However, b only appears with one sign.
Therefore, this word describes a non-orientable surface.

$F = 1$ face, trivially.
The word has length $6 = 2 \cdot 3 \iff n = 3$.
$E = n = 3$ edges.
Calculating the vertices now,
Let's start with a and a^{-1}. We note that $\alpha \leftrightarrow \gamma$ and $\zeta \leftrightarrow \delta$.
Next we'll consider b and b. We note that $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \varepsilon$.
Finally, let's consider c and c^{-1}, and note $\beta \leftrightarrow \zeta$ and $\gamma \leftrightarrow \varepsilon$.
All together, we have

$$\alpha \leftrightarrow \gamma \leftrightarrow \varepsilon \leftrightarrow \beta \leftrightarrow \zeta \leftrightarrow \delta$$

Therefore, the surface has 1 vertex. $V = 1$ vertex.
Therefore, the surface has Euler characteristic, $\chi = V - E + F = 1 - 3 + 1 = -1$.

Therefore, a non-orientable surface with $\chi = -1 = 2 - g \iff g = 3$, is the connected sum of 3 projective planes, by the classification theorem.

Solution: (b)

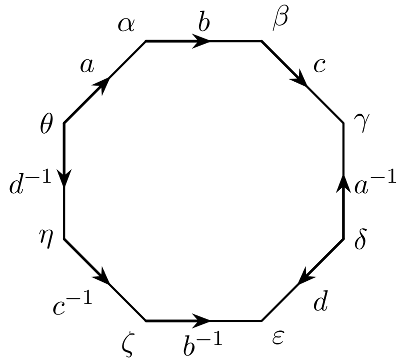

Figure 2: A drawing of the polygon represented by the word $abca^{-1}db^{-1}c^{-1}d^{-1}$

A word describes an orientable surface if and only if every symbol appears with both signs.
a appears with both signs.
b appears with both signs.
c appears with both signs.
d appears with both signs.
Therefore, this word describes a orientable surface.

$F = 1$ face, trivially.
The word has length $8 = 2 \cdot 4 \iff n = 4$.
$E = n = 4$ edges.
Calculating the vertices now,
Let's start with a and a^{-1}. We note that $\alpha \leftrightarrow \gamma$ and $\theta \leftrightarrow \delta$.
Next we'll consider b and b^{-1}. We note that $\beta \leftrightarrow \varepsilon$ and $\alpha \leftrightarrow \zeta$.
Next we'll consider c and c^{-1}. We note that $\gamma \leftrightarrow \zeta$ and $\beta \leftrightarrow \eta$.
Finally, let's consider d and d^{-1}, and note $\varepsilon \leftrightarrow \eta$ and $\delta \leftrightarrow \theta$.
All together, we have

$$\alpha \leftrightarrow \gamma \leftrightarrow \zeta, \qquad \theta \leftrightarrow \delta, \qquad \beta \leftrightarrow \varepsilon \leftrightarrow \eta$$

Therefore, the polygon has 3 vertices.
$V = 3$ vertices.
Therefore, the polygon has Euler characteristic, $\chi = V - E + F = 3 - 4 + 1 = 0$.

Therefore, a orientable surface with $\chi = 0 = 2 - 2g \iff g = 1$, is the connected sum of 1 torus, by the classification theorem, ie, a torus.

Question 2: 5 marks

Consider the closed surface represented by the following word:

$$a\,b\,c\,a^{-1}\,d\,b\,c^{-1}\,d^{-1}.$$

Use the word rules from lectures to identify the surface in the form stated in the classification theorem.

Solution:

Firstly, we'll note that b appears with only one sign, hence the surface the word represents is non-orientable.

We'll start transforming the word by applying rule 6, "moving outside xx pairs," to the two b's.

$$a\,b\,c\,a^{-1}\,d\,b\,c^{-1}\,d^{-1} \overset{r.6}{=} a\,b\,b\,d^{-1}\,a\,c^{-1}\,c^{-1}\,d^{-1}$$

We'll then apply rule 6 again to the two d^{-1}'s.

$$a\,b\,b\,d^{-1}\,a\,c^{-1}\,c^{-1}\,d^{-1} \overset{r.6}{=} a\,b\,b\,d^{-1}\,d^{-1}\,c\,c\,a^{-1}$$

Next, we'll apply rule 2, "cycling edges," to move the a^{-1} to the front of the word.

$$a\,b\,b\,d^{-1}\,d^{-1}\,c\,c\,a^{-1} \overset{r.2}{=} a^{-1}\,a\,b\,b\,d^{-1}\,d^{-1}\,c\,c$$

For the sake of sematics (not-shortcutting anything), we'll apply rule 1, "relabelling edges," to swap around a and a^{-1}.

$$a^{-1}\,a\,b\,b\,d^{-1}\,d^{-1}\,c\,c \overset{r.1}{=} a\,a^{-1}\,b\,b\,d^{-1}\,d^{-1}\,c\,c$$

Now, we can apply rule 5, "cancelling xx^{-1} pairs" to cancel out the aa^{-1} at the front of the word.

$$a\,a^{-1}\,b\,b\,d^{-1}\,d^{-1}\,c\,c \overset{r.5}{=} b\,b\,d^{-1}\,d^{-1}\,c\,c$$

Finally, we'll clean this up by applying rule 1 twice: to relabel b as a; and to relabel d^{-1} as b.

$$b\,b\,d^{-1}\,d^{-1}\,c\,c \overset{r.1}{=} a\,a\,b\,b\,c\,c$$

In accordance with the classification theorem, the word $abca^{-1}dbc^{-1}d^{-1}$, therefore, represents a surface which is non-orientable and the sum of 3 projective planes with Euler characteristic -1.

Question 3: 8 marks

Consider the sequence $S = (5, 3, 3, 3, 1, 1)$.

(a) Show that S is graphical, and draw a graph with the degree sequence S.

(b) Prove that the graph with the degree sequence S is unique up to graph isomorphism.

Solution: (a)

To show S is graphical, we'll use Theorem 25.76. First, we'll note that S is monotone decreasing ($d_1 = 5 \geq d_2 = 3 \geq ... \geq d_p = d_6 = 1$), with $p = 6 \geq 2$ and $\Delta = 5 \geq 1$. Therefore, we can apply the theorem; the first reduction:

$$(5, 3, 3, 3, 1, 1) \xrightarrow{Thm.25.76} (3-1, 3-1, 3-1, 1-1, 1-1) = (2, 2, 2, 0, 0)$$

The sequence is monotone decreasing, with $p = 5 \geq 2$ and $\Delta = 2 \geq 1$. The second reduction:

$$(2, 2, 2, 0, 0) \xrightarrow{Thm.25.76} (2-1, 2-1, 0, 0) = (1, 1, 0, 0)$$

The sequence is monotone decreasing, with $p = 4 \geq 2$ and $\Delta = 1 \geq 1$. The third reduction:

$$(1, 1, 0, 0) \xrightarrow{Thm.25.76} (1-1, 0, 0) = (0, 0, 0)$$

Because the Δ for this sequence is $0 < 1$, the recurrsion ceases, but this reduced graph is trivially graphical because it represents the degree sequence of some graph (namely, three vertices with no edges). Therefore, S is graphical.

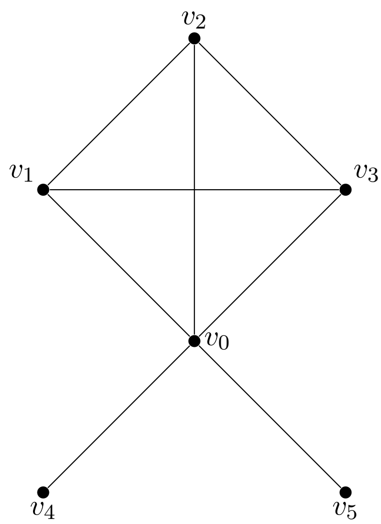

Figure 3: A drawing of a simple graph with degree sequence $S = (5, 3, 3, 3, 1, 1)$.

Solution: (b)

Claim. The graph with degree sequence $S = (5,3,3,3,1,1)$ is unique up to graph isomorphism.
Proof. By construction.

The graph, G, with degree sequence S is graphical.

$|S| = 6$, hence $|V(G)| = 6$.
$\Delta(S) = 5 = |V(G)| - 1$, so the vertex with the greatest degree, call it v_0 has $d(v_0) = 5$, and must be connected to all other vertices in the graph.
Therefore, $\{\{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}, \{v_0, v_4\} \{v_0, v_5\}\} = M \subseteq E(G)$.

There are two vertices, call them v_4 and v_5 with $d(v_4) = d(v_5) = 1$.
The only choice for the other endpoint of these edges is v_0.
These edges, namely $\{v_4, v_0\}$ and $\{v_5, v_0\}$, are already in the constructed set.

The remaining three vertices, $v_1,\ v_2,\ v_3$ have $d(v_1) = d(v_2) = d(v_3) = 3$.
Each of these has one edge joining it to v_0, namely $\{v_1, v_0\},\ \{v_2, v_0\},\ \{v_3, v_0\}$, all of which are already included in our constructed set.
v_1 has 2 incident edges left to account for and only 2 choices: v_2 and v_3.
v_2 has 2 incident edges left to account for and only 2 choices: v_1 and v_3.
v_3 has 2 incident edges left to account for and only 2 choices: v_1 and v_2.
Therefore, $\{\{v_1, v_2\}, \{v_1, v_3\}\} \cup \{\{v_2, v_1\}, \{v_2, v_3\}\} \cup \{\{v_3, v_1\}, \{v_3, v_1\}\}$
$= \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\} = T \subseteq E(G)$

$M \cup T = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}, \{v_0, v_4\}, \{v_0, v_5\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\} \subseteq E(G)$

By the Handshake Theorem, $\displaystyle\sum_{i=0}^{p-1} d(v_i) = d(v_0) + d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$
$= 5 + 3 + 3 + 3 + 1 + 1 = 16 = 2q \iff q = 8$, therefore, $|E(G)| = 8$.

$|M \cup T| = 8$.

Therefore, $M \cup T = E(G)$.

By our construction, this is the only graph with degree sequence S (a consequence of the various forced decisions made throughout the process).

Therefore, the graph with degree sequence S is unique up to graph isomorphism. □

Question 4: 7 marks

Suppose that G is a graph on $4k+3$ vertices, where each vertex has degree at least $2k+1$, for some integer $k \geq 1$.

(a) Can G be $(2k+1)$-regular? Justify your answer.

(b) Prove that G is connected.

Solution: (a)

Suppose G were $(2k+1)$-regular.
Then it would be a graph with $4k+3$ vertices, where each vertex has degree $2k+1$.

Then, we could calculate this graph's degree total:

$$(4k+3)(2k+1) = 8k^2 + 10k + 3 = 2(4k^2 + 5k + 1) + 1 = 2z + 1,\ z \in \mathbb{Z},$$

which has odd parity.

We can also calculate this graphs degree total with the Handshake Theorem, namely:

$$\sum_{\forall v \in V(G)} d(v) = 2q,\ q \in \mathbb{Z},$$

which has even parity.

Therefore, no $(2k+1)$-regular graph with $4k+3$ vertices exists, because of this clear contradiction.
Therefore, G cannot be $(2k+1)$-regular.

Solution: (b)

Suppose G is a graph on $4k+3$ vertices such that each vertex has degree at least $2k+1$, for some integer $k \geq 1$.
Claim. G is connected.

Proof. By contradiction.

For the sake of contradiction, suppose that G is disconected.
Then G can be decomposed into at least 2 connected subgraphs.

Let one of these components have t vertices.
Then each vertex in this component has degree at most $t-1$.

Since, $\delta(G) \geq 2k+1$, we know $t-1 \geq 2k+1$, $t \geq 2k+2$.

Now, since G can be decomposed into *at least* 2 components,

$$|V(G)| \geq 2(2k+2) = 4k+4 > 4k+3 = |V(G)| \qquad ※$$

This is a clear contradiction because $V(G)$ must simultaneously be equal to $4k+3$ and greater than or equal to $4k+4$. Absurd.
Therefore, G is not disconected.

Therefore, G is connected. □