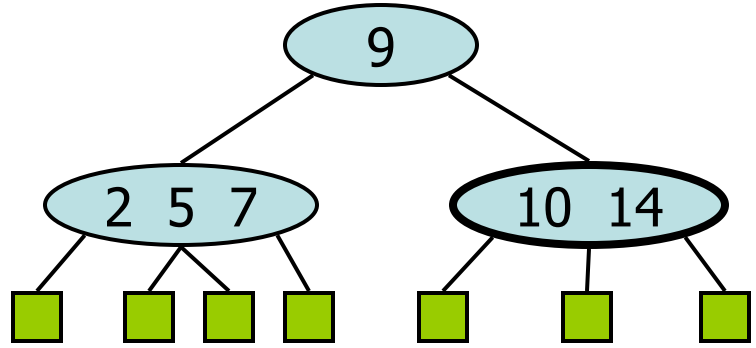
(2,4) Trees

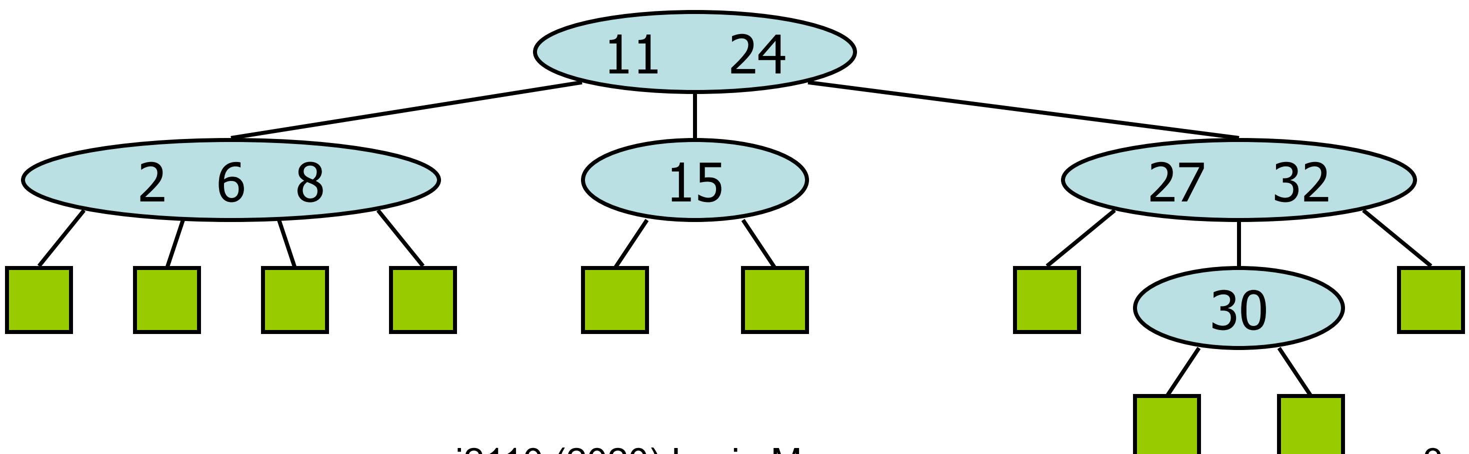


Outline and Reading

- Multi-way search tree
  - Definition
  - Search
- (2,4) tree
  - Definition
  - Search
  - Insertion
  - Deletion

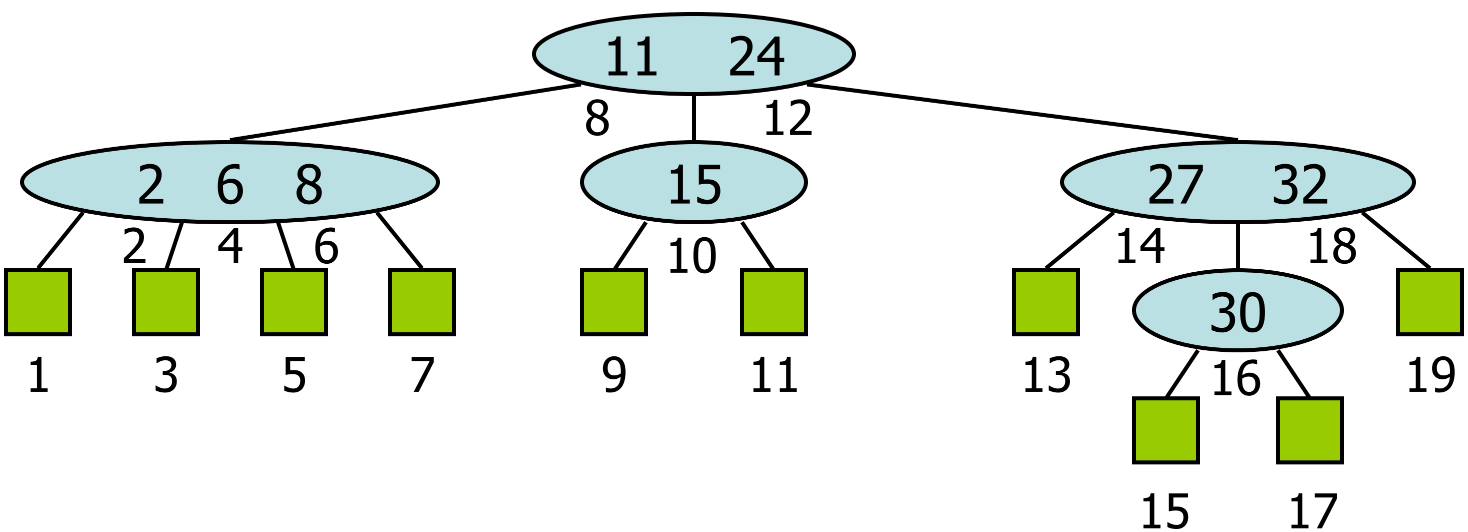
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores ***d-1*** key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\ v_2\ \ldots\ v_d$ storing keys $k_1\ k_2\ \ldots\ k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i ($i = 2, \ldots, d - 1$)
    - keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders



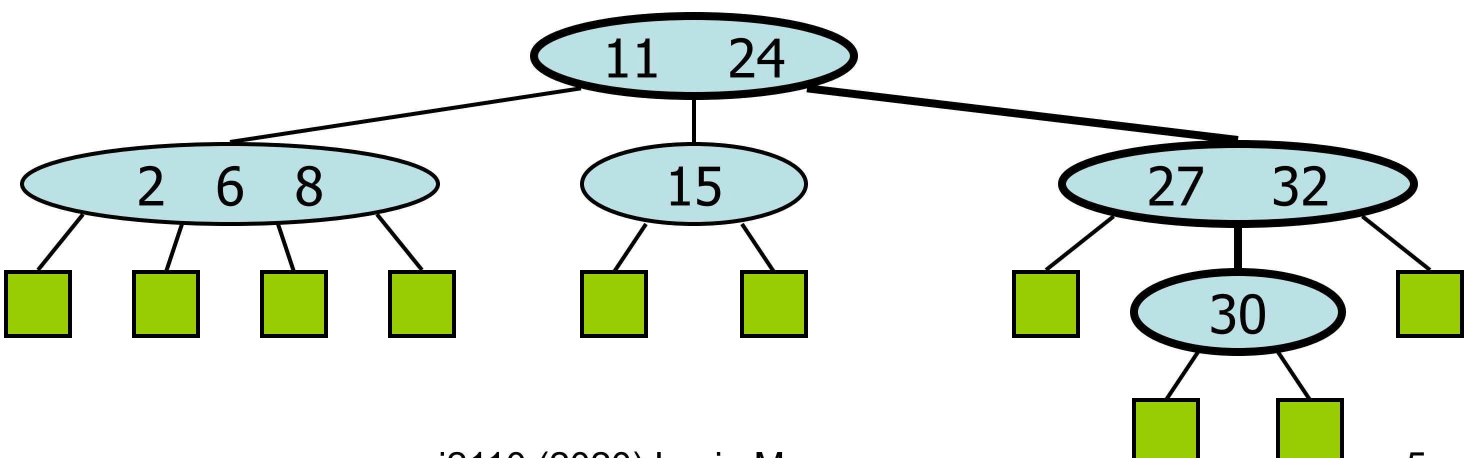
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order



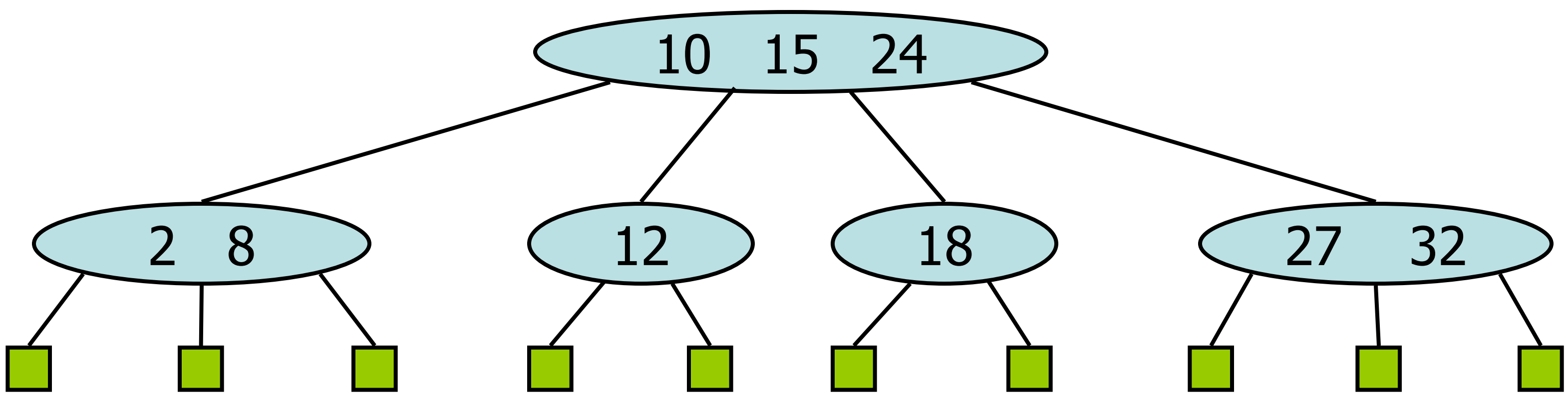
Multi-Way Searching

- Similar to search in a binary search tree
- At each internal node with children v_1 v_2 … v_d and keys k_1 k_2 … k_{d-1}
  - $k = k_i$ ($i = 1, \ldots, d - 1$): the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ ($i = 2, \ldots, d - 1$): we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



(2,4) Tree

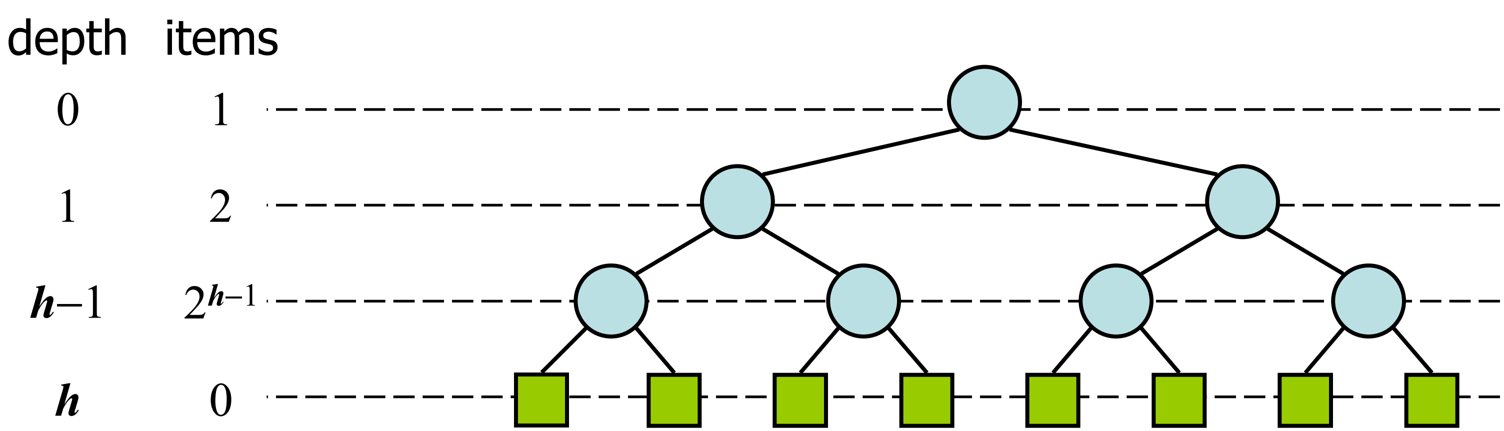
- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



Height of a (2,4) Tree

- Theorem: A (2,4) tree storing n items has height $O(\log n)$

  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \dots, h-1$ and no items at depth h, we have
  $$n \geq 1 + 2 + 4 + \dots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log(n + 1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



Height of a (2,4) Tree

Min # of items n: When all internal nodes have 1 key and 2 children

$n = 2^{h+1}-1$ $h = O(\log n)$

"perfect" binary tree

Max # of items n: When all internal nodes have 3 keys and 4 children

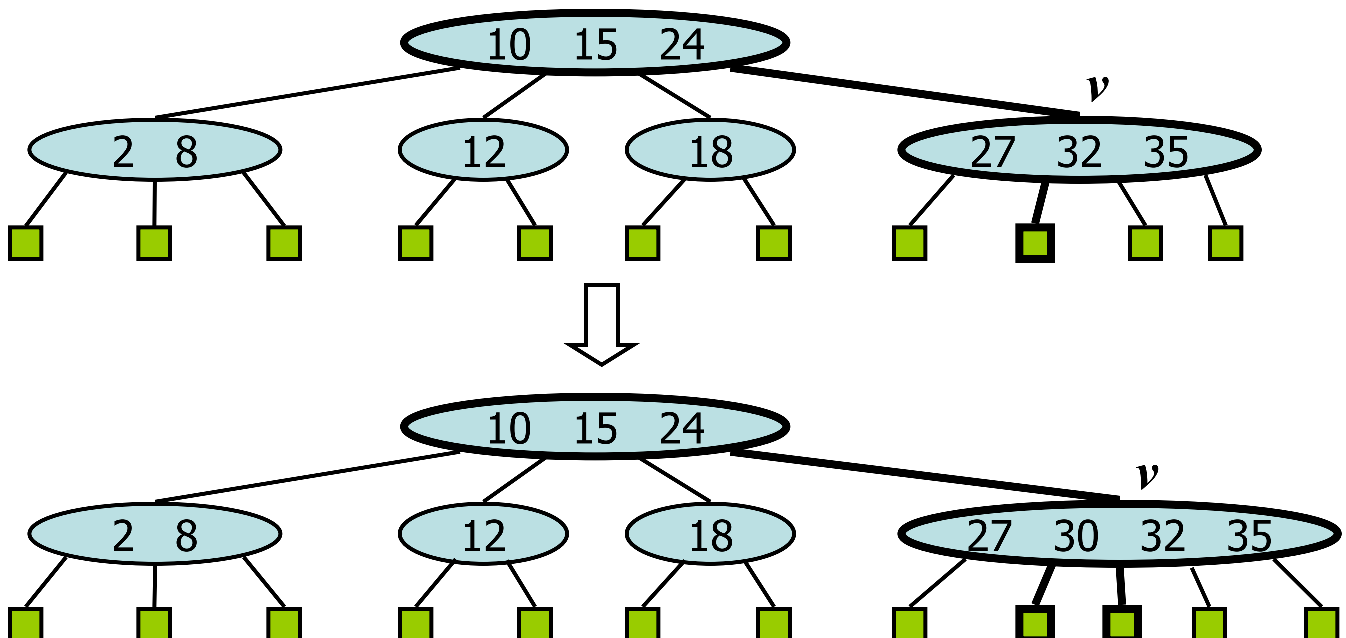
$$n = \sum_{i=0}^{h} 4^i = (4^{h+1}-1)/3$$

$n = (4^{h+1}-1)/3$ $h = O(\log_4 n)$

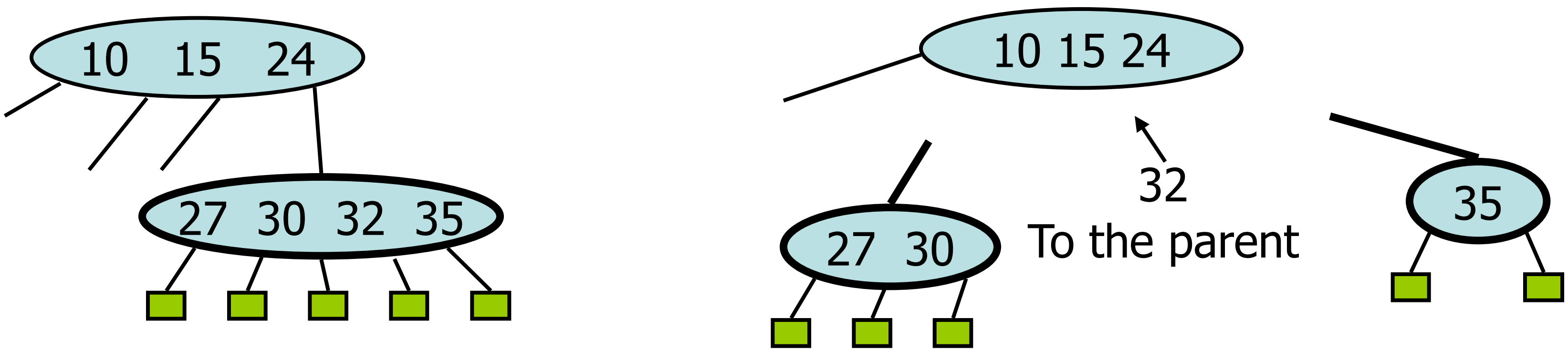
→ Search O(log n)

Insertion

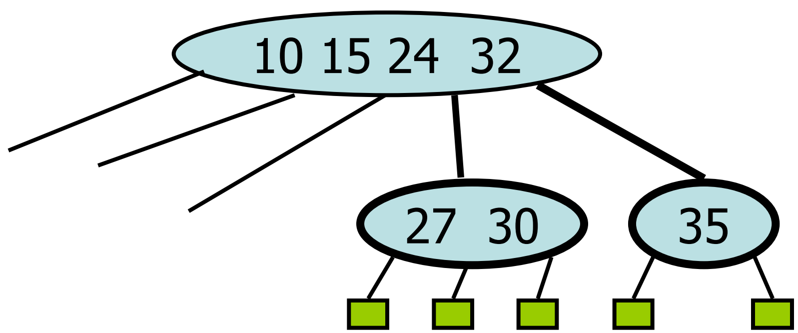
- We insert a new item (*k*, *o*) at the parent *v* of the leaf reached by searching for *k*
  - We preserve the depth property but
  - We may cause an overflow (i.e., node *v* may become a 5-node)
- Example: inserting key 30 causes an overflow



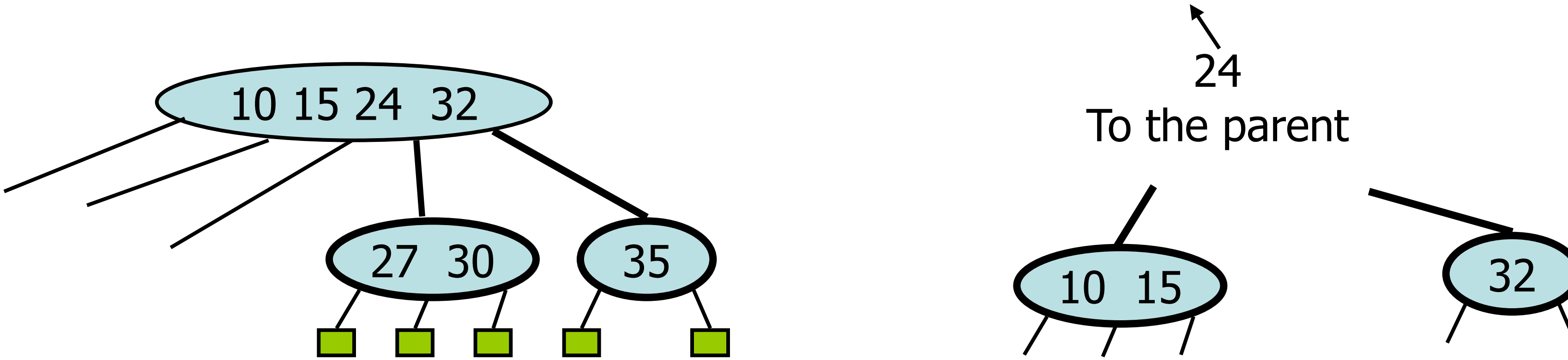
Overflow and Split



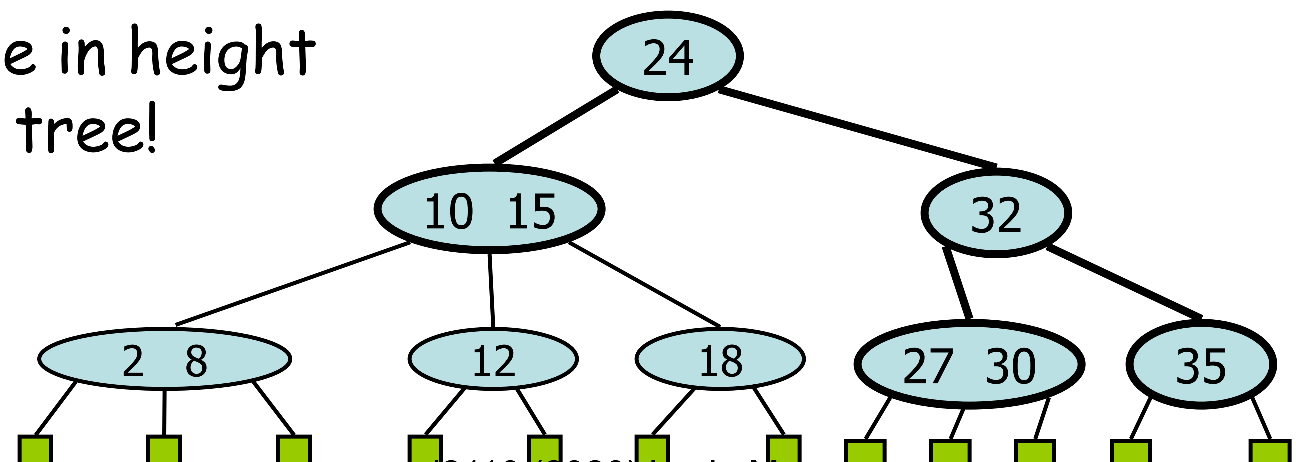
Overflow again!



Overflow and Split Again

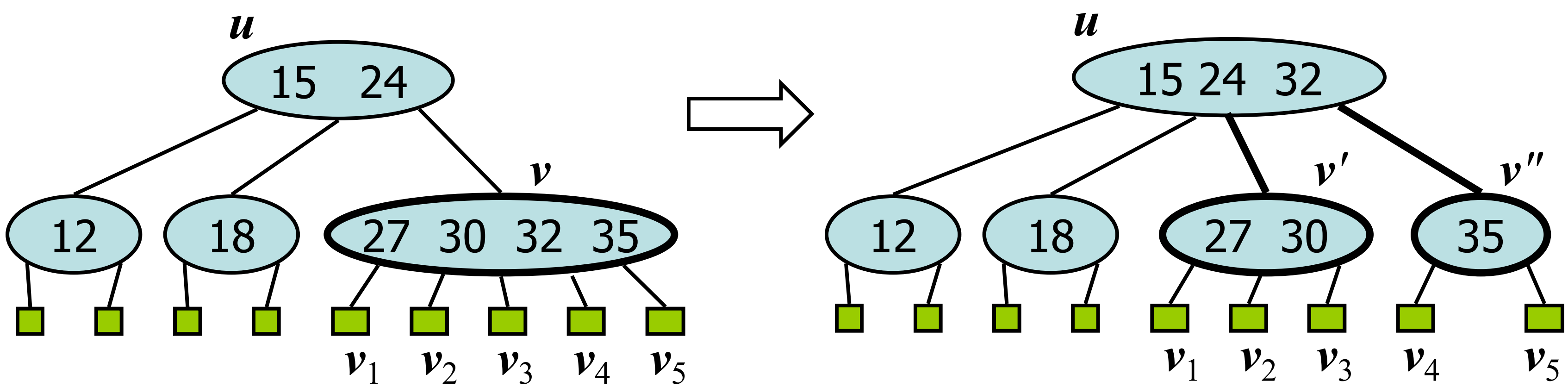


Increase in height of tree!



Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced by nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent of node u



Analysis of Insertion

Algorithm ***insertItem***(k, o)

1. We search for key k to locate the insertion node v
2. We add the new item (k, o) at node v
3. **while** ***overflow***(v)

    if ***isRoot***(v)

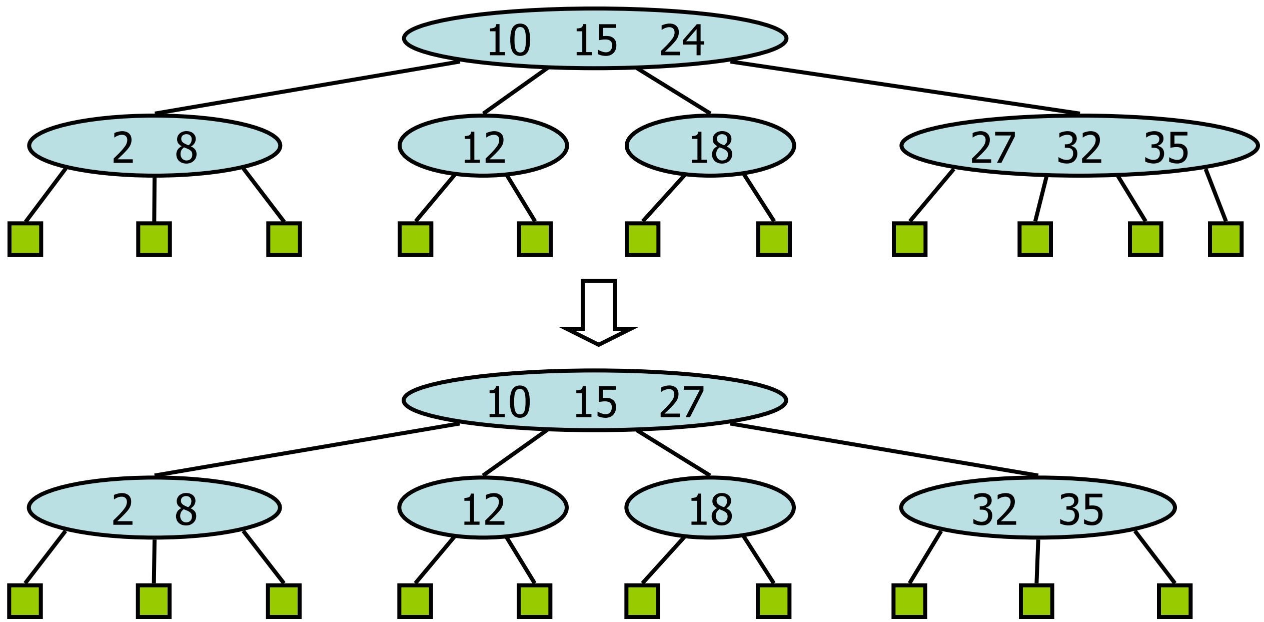
        create a new empty root above v

    $v \leftarrow$ ***split***(v)

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Step 2 takes $O(1)$ time
  - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits
- Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

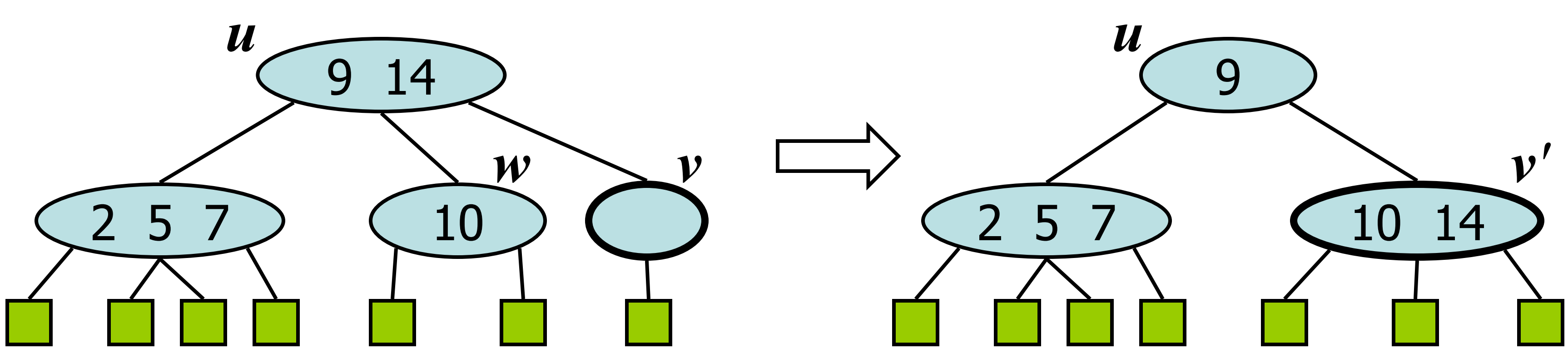
Deletion

- We reduce deletion of an item to the case where the item is at the node with leaf children
- Otherwise, we replace the item with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter item
- Example: to delete key 24, we replace it with 27 (inorder successor)



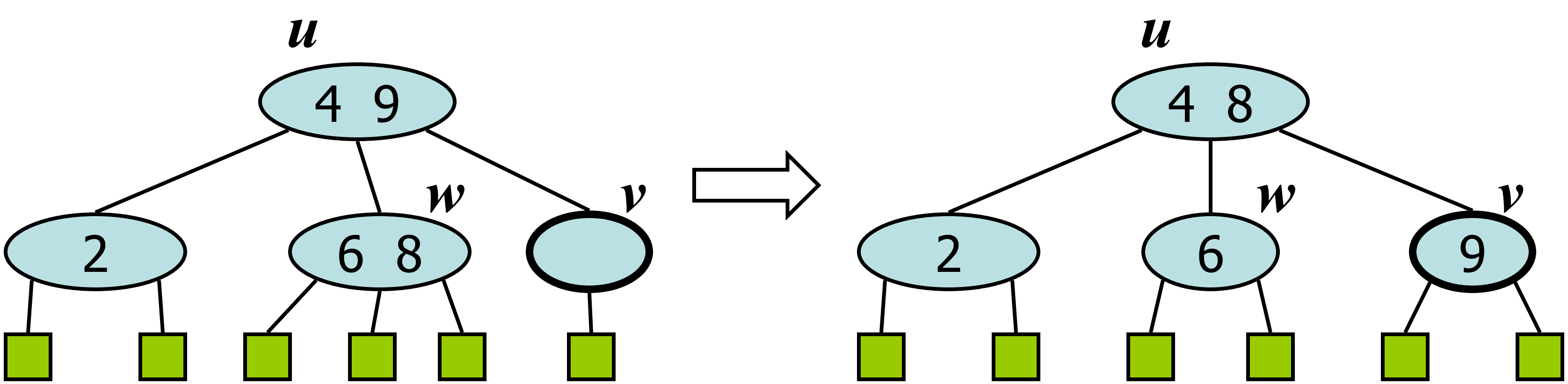
Underflow and Fusion

- Deleting an item from a node *v* may cause an underflow, where node *v* becomes a 1-node with one child and no keys
- To handle an underflow at node *v* with parent *u*, we consider two cases
- Case 1: the adjacent siblings of *v* are 2-nodes
  - Fusion operation: we merge *v* with an adjacent sibling *w* and move an item from *u* to the merged node *v*′
  - After a fusion, the underflow may propagate to the parent *u*



Underflow and Transfer

- To handle an underflow at node *v* with parent *u*, we consider two cases
- Case 2: an adjacent sibling *w* of *v* is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of *w* to *v*
    2. we move an item from *u* to *v*
    3. we move an item from *w* to *u*
  - After a transfer, no underflow occurs



Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the item
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time