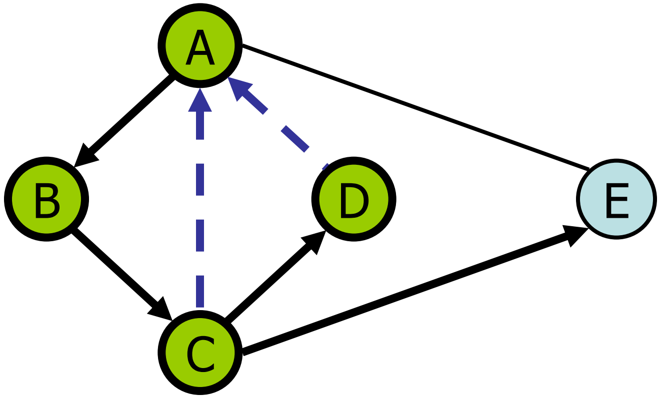
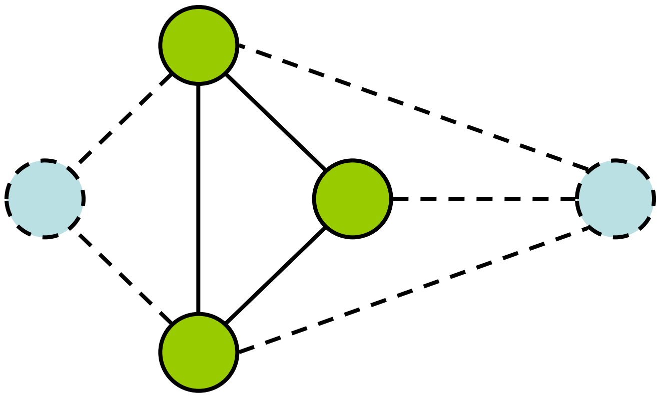
Graph Traversals

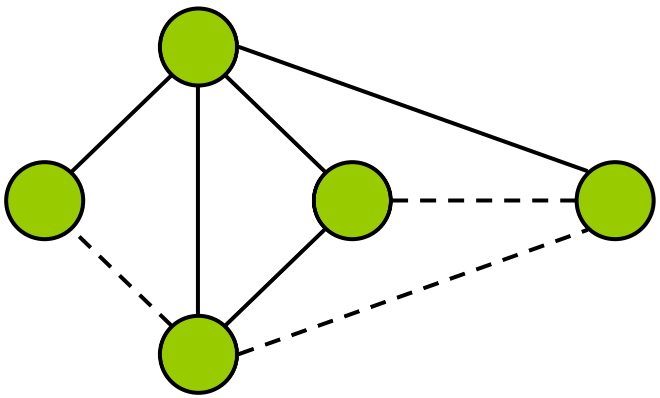


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



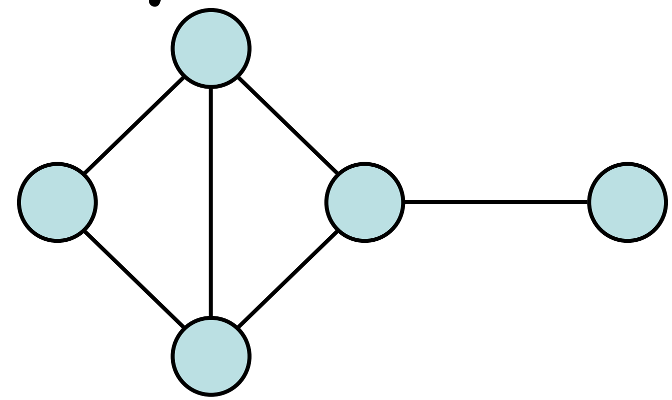
Subgraph



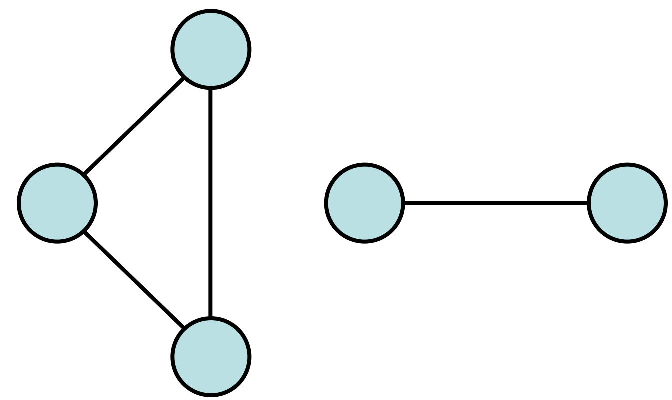
Spanning subgraph

Connectivity

- *A graph is connected if there is a path between every pair of vertices*
- *A connected component of a graph G is a maximal connected subgraph of G*



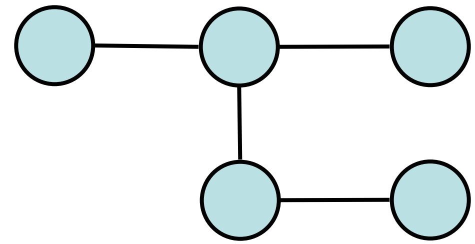
Connected graph



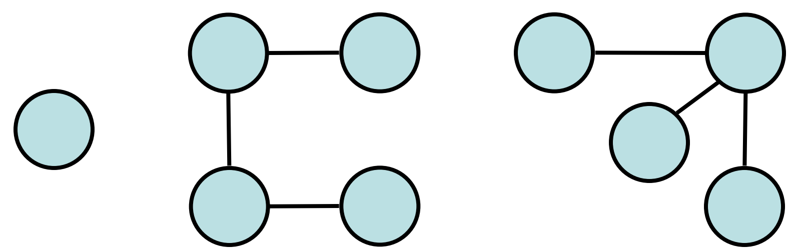
Non connected graph with two connected components

Trees and Forests

- A tree is an undirected graph T such that
  - T is connected
  - T has no cycles
- A forest is an undirected graph without cycles (a collection of trees).
- The connected components of a forest are trees



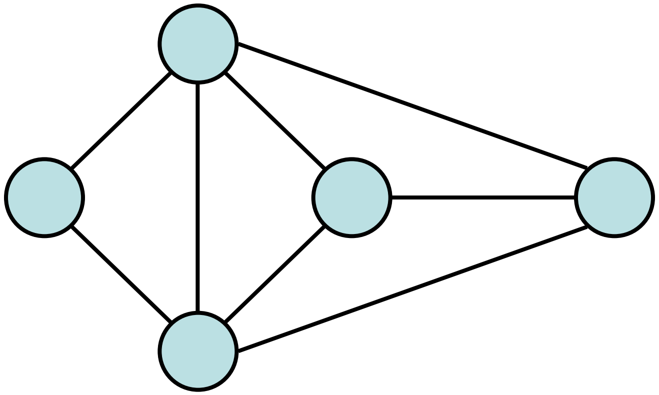
Tree



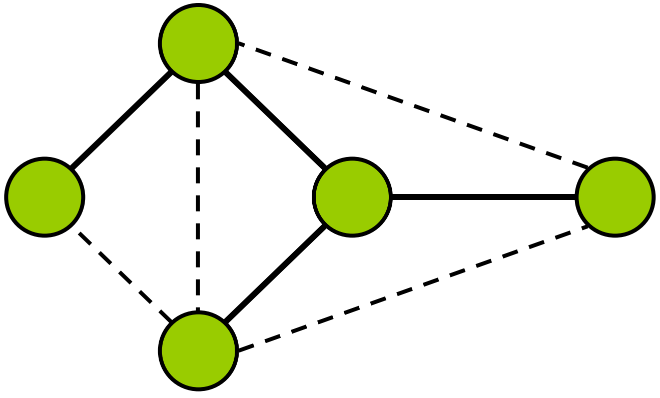
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



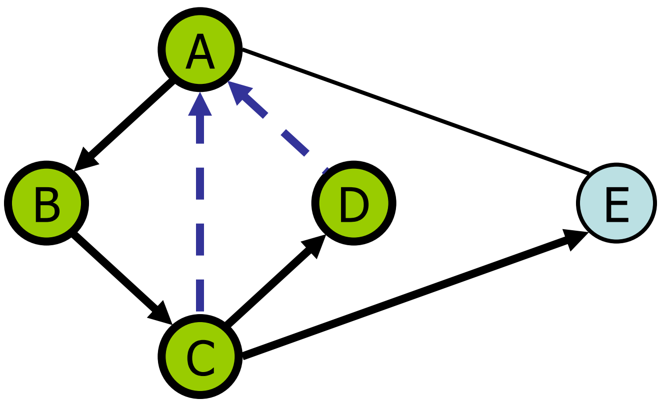
Spanning tree

Graph Traversals

A traversal of a graph G:

- Visits all the vertices and edges of G
- Determines whether G is connected
- Computes the connected components of G
- Computes a spanning forest of G
- Build a spanning tree in a connected graph

Depth-First Search



Outline and Reading

- Depth-first search
  - With a Stack
  - Recursive
  - Examples
  - Properties
  - Complexity
- Applications of DFS
  - Path finding
  - Cycle finding

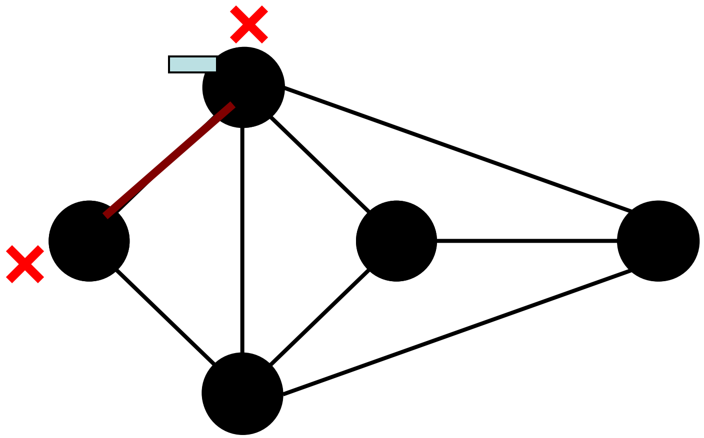
Depth-First Search

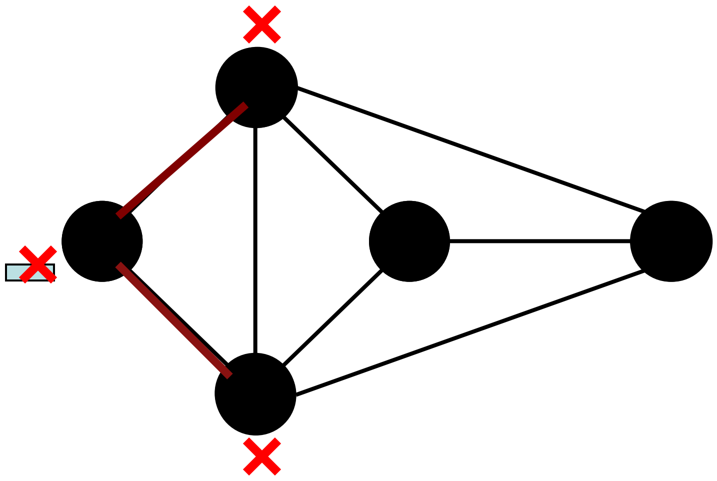
Depth-First Search is a graph traversal technique that:

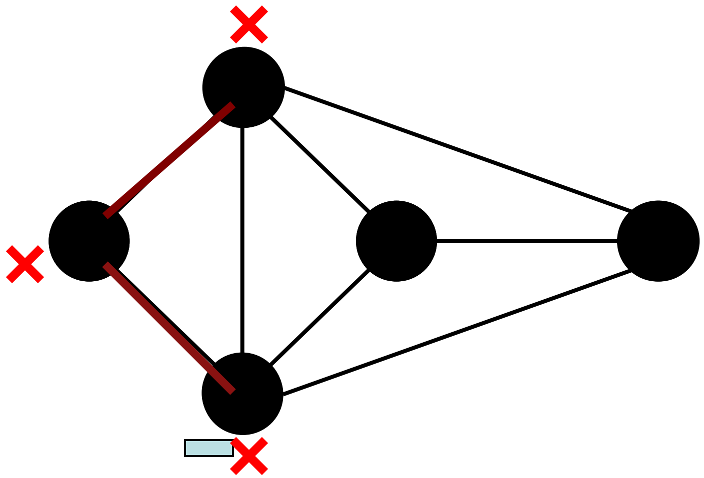
- on a graph with n vertices and m edges takes $O(n + m)$ time (which is O(m))
- can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph

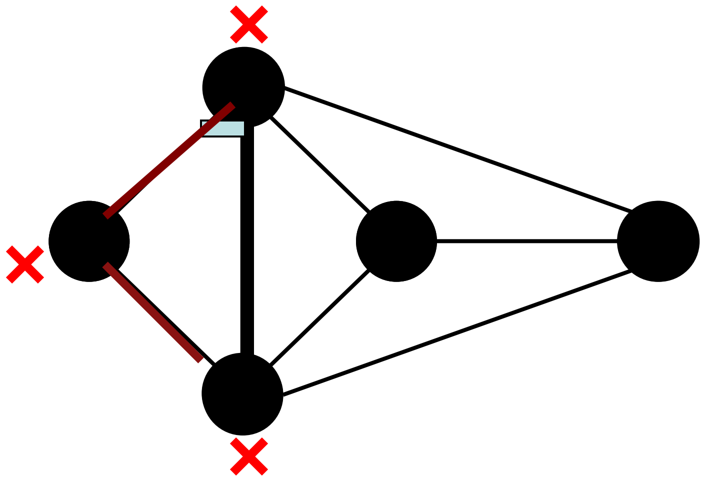
The idea:

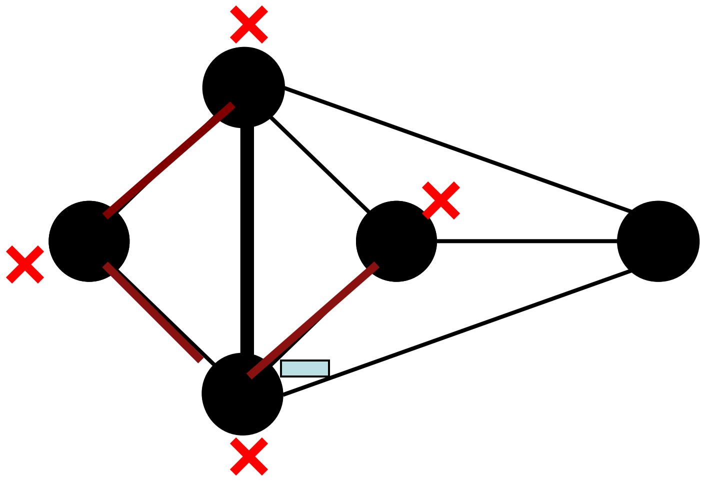
Starting at an arbitrary vertex, follow along a simple path until you get to a vertex which has no unvisited adjacent vertices.
Then start tracing back up the path, one vertex at a time, to find a vertex with unvisited adjacent vertices.

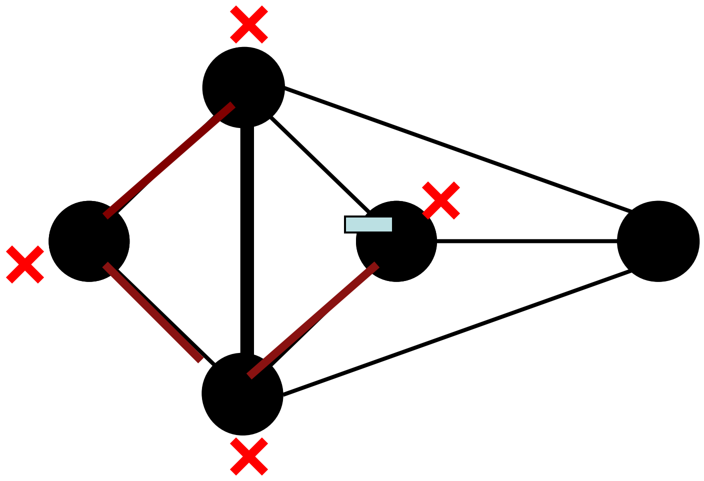


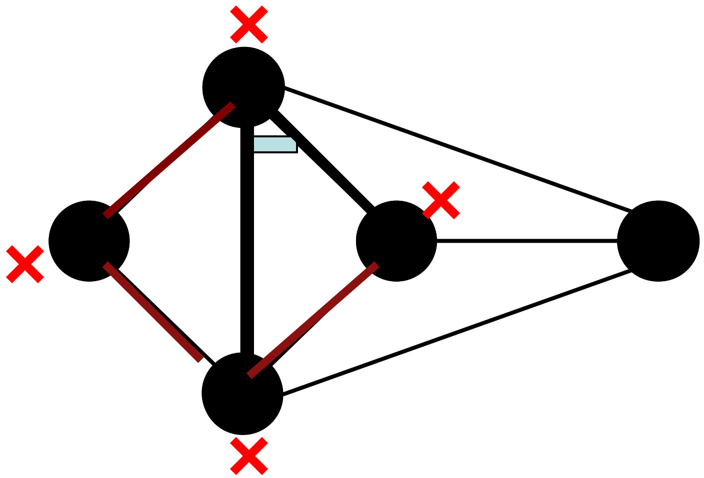


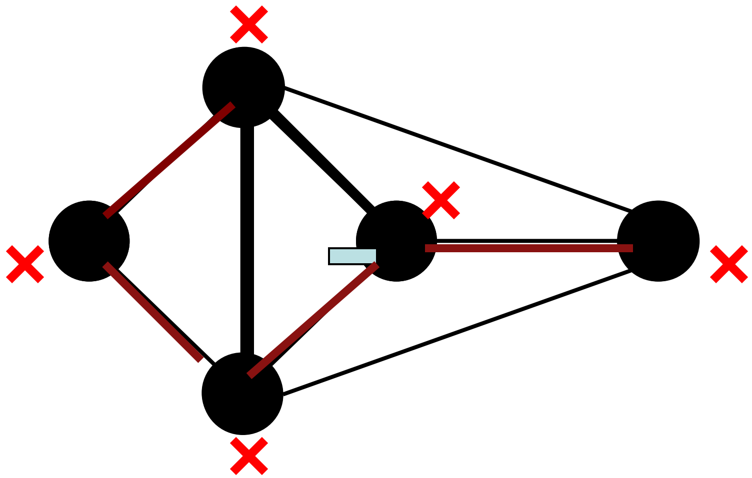


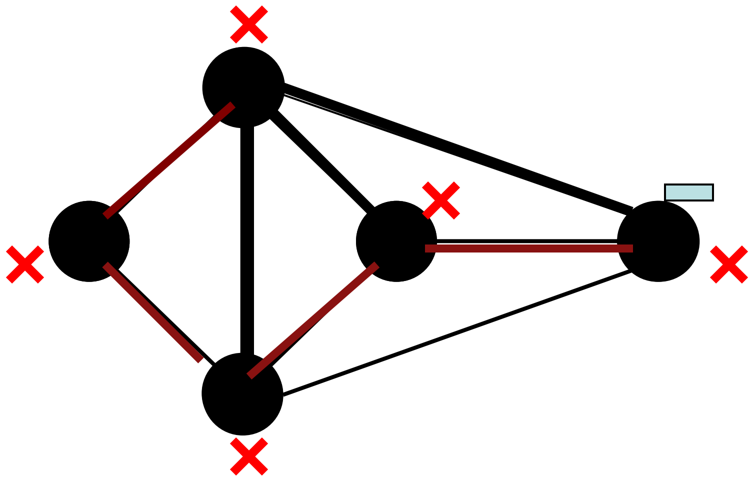


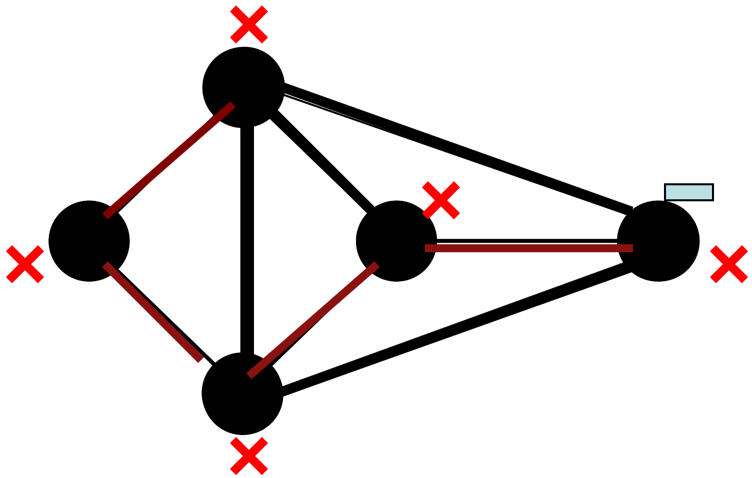


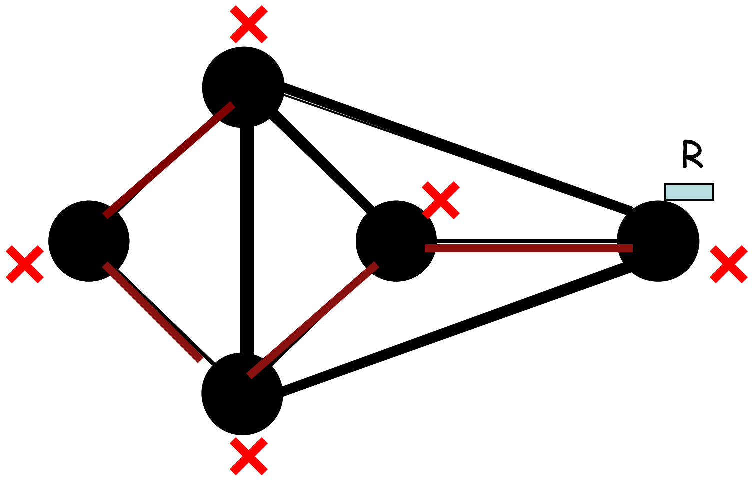








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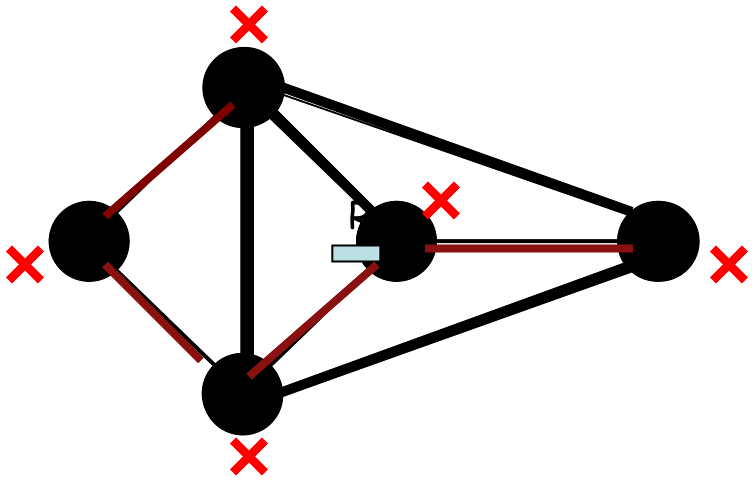
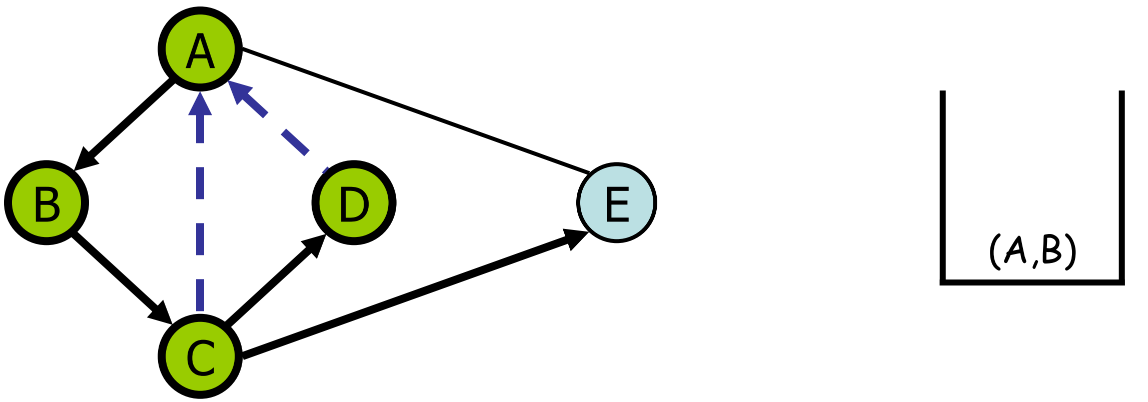
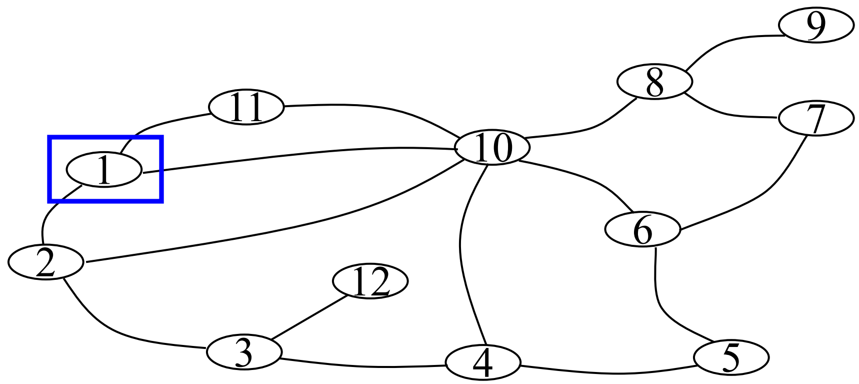
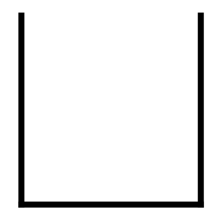


DFS Algorithm – With a Stack

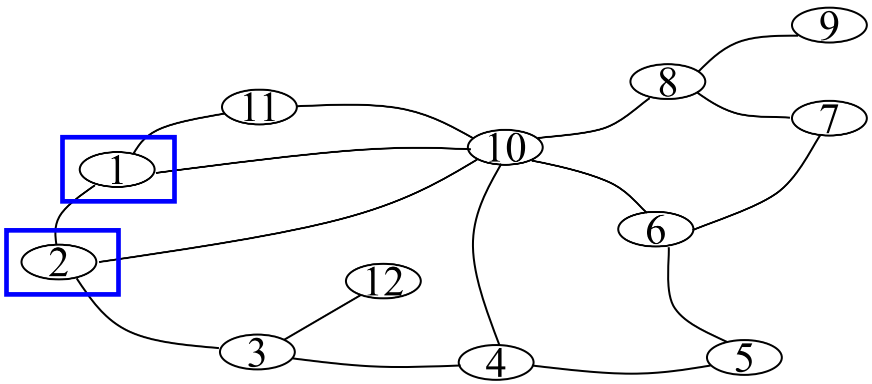




Visited : $\{1\}$

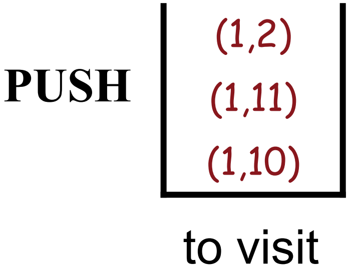


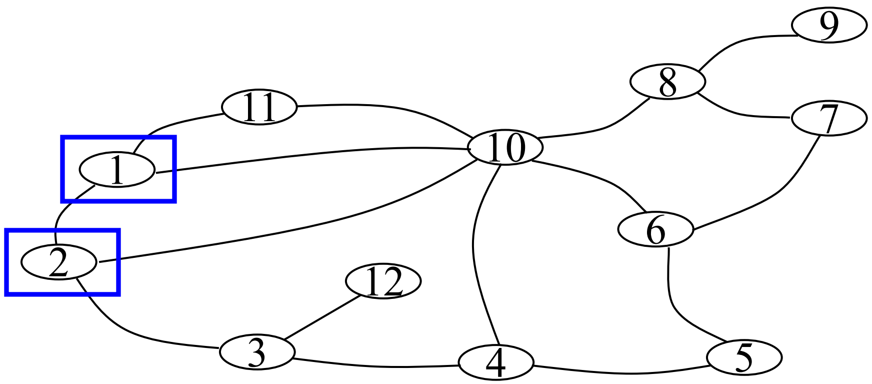
to visit



Visited : $\{1\}$

POP: (1,2)
Visited : $\{1,2\}$
T=$\{(1,2)\}$

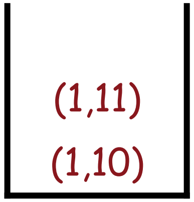




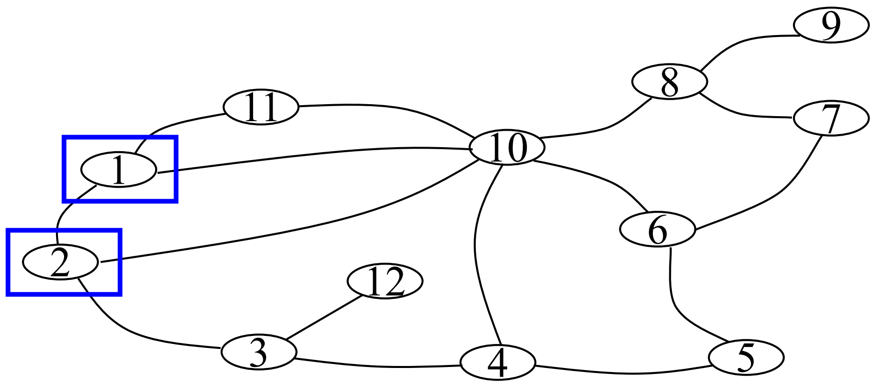
Visited : $\{1\}$

POP: (1,2)
Visited : $\{1,2\}$
$T=\{(1,2)\}$

PUSH



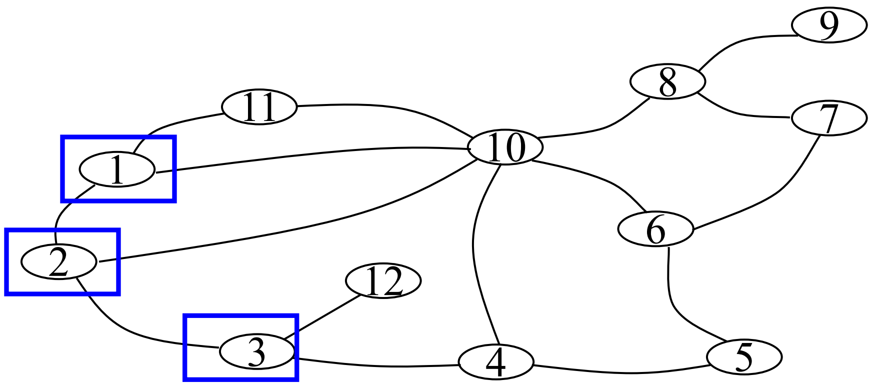
to visit



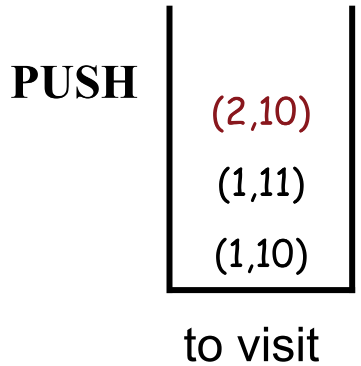
POP: (2,3)
Visited : {1,2,3}
T={(1,2), (2,3)}

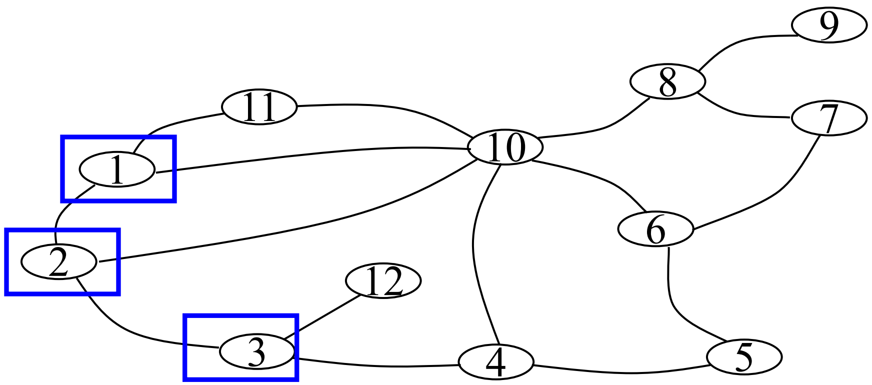
PUSH

to visit
(2,3)
(2,10)
(1,11)
(1,10)



POP: (2,3)
Visited : {1,2,3}
T={(1,2), (2,3)}



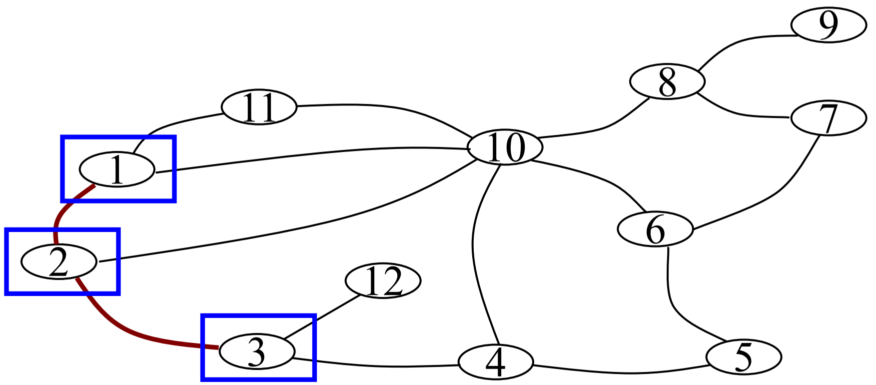


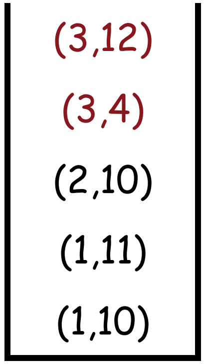
PUSH

(3,12)
(3,4)
(2,10)
(1,11)
(1,10)

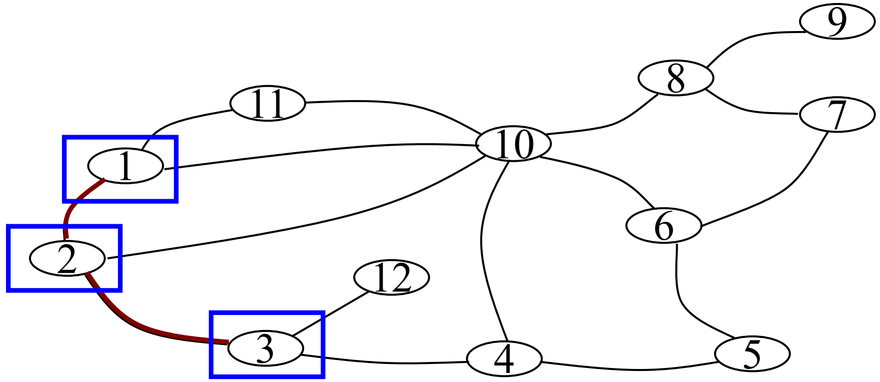
to visit

T={(1,2), (2,3)}





(3,12)
(3,4)
(2,10)
(1,11)
(1,10)



POP → (3,12)

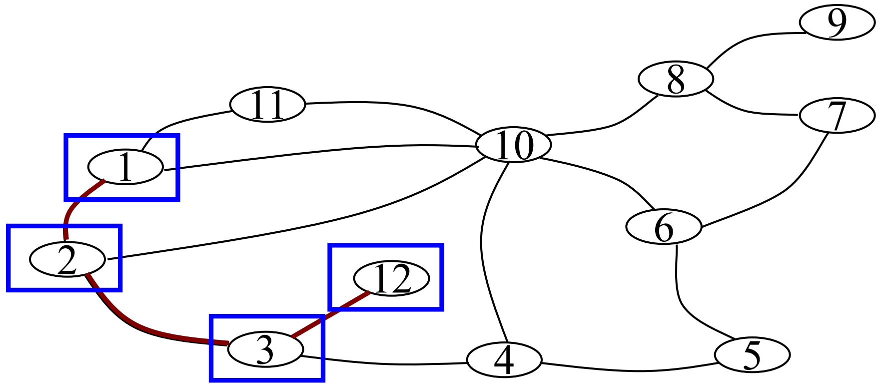
Visited : {1,2,3,12}

T={(1,2), (2,3), (3,12)}

PUSH
(nothing in this case)

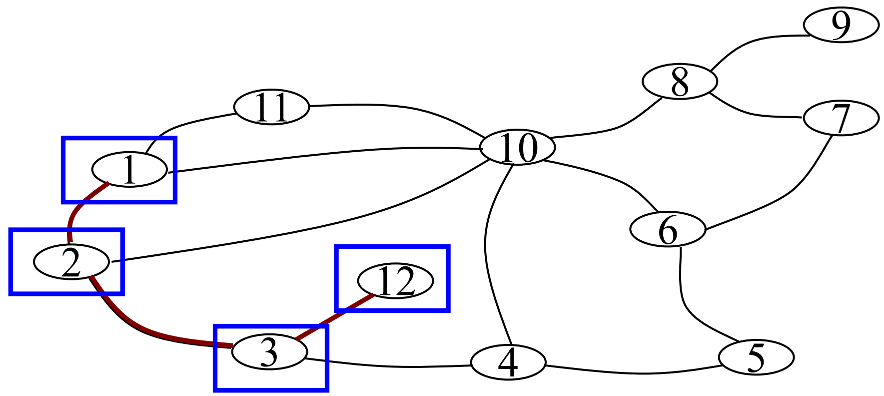
(3,4)
(2,10)
(1,11)
(1,10)

(3,4)
(2,10)
(1,11)
(1,10)



T={(1,2), (2,3), (3,12)}

(3,4)
(2,10)
(1,11)
(1,10)



POP → (3,4)

Visited : {1,2,3,12,4}

T={(1,2), (2,3), (3,12), (3,4)}

PUSH

(4,4) and (4,10)

(4,10)
(4,5)
(2,10)
(1,11)
(1,10)

POP → (4,10)

Visited : {1,2,3,12,4,10}

T={(1,2), (2,3), (3,12), (3,4), (4,10)}

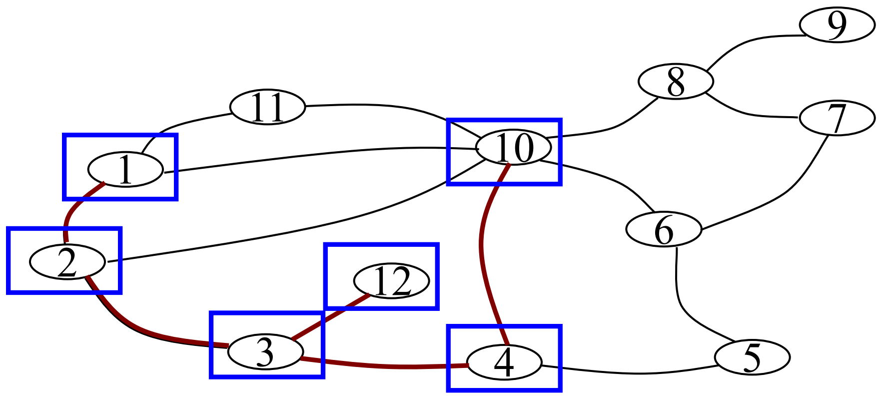
PUSH

(10,6) and (10,8)

(10,8)
(10,6)
(4,5)
(2,10)
(1,11)
(1,10)

• • •

(10,8)
(10,6)
(4,5)
(2,10)
(1,11)
(1,10)



T={(1,2), (2,3), (3,12), (3,4), (4,10)}

• • •

Complexity

Elementary operations: Pop, Push, and visits

Number of PUSH:

$$\sum_{v \in V} d(v) = 2m$$

Number of POP:

$$\sum_{v \in V} d(v) = 2m$$

Visit of a node: n

O(n+m) = O(m)

DFS Algorithm – Recursive version

```
DFS(v)
Mark v visited
 ∀w ∈ Adjacent(v)
        if w not visited
                visit w
                DFS(w)
```

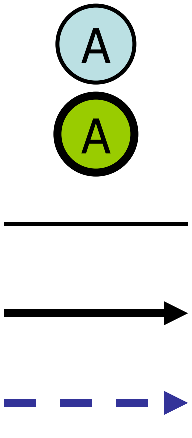
DFS Again - More Detail…

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

Algorithm *DFS*(*G*)
  Input graph *G*
  Output labeling of the edges of *G* as discovery edges and back edges
 for all $u \in G.vertices()$
  setLabel(*u, UNEXPLORED*)
 for all $e \in G.edges()$
  setLabel(*e, UNEXPLORED*)
 for all $v \in G.vertices()$
  if *getLabel*(*v*) = *UNEXPLORED*
   DFS(*G, v*)

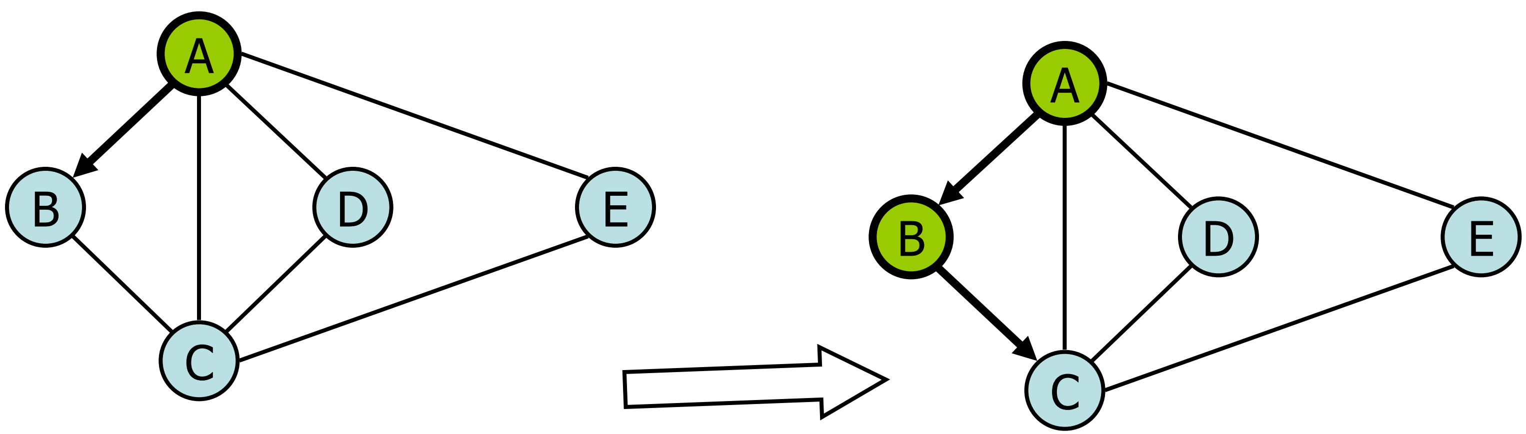
Algorithm *DFS*(*G, v*)
  Input graph *G* and a start vertex *v* of *G*
  Output labeling of the edges of *G* in the connected component of *v* as discovery edges and back edges
 setLabel(*v, VISITED*)
 for all $e \in G.incidentEdges(v)$
  if *getLabel*(*e*) = *UNEXPLORED*
   $w \leftarrow opposite(v,e)$
   if *getLabel*(*w*) = *UNEXPLORED*
    setLabel(*e, DISCOVERY*)
    DFS(*G, w*)
   else
    setLabel(*e, BACK*)

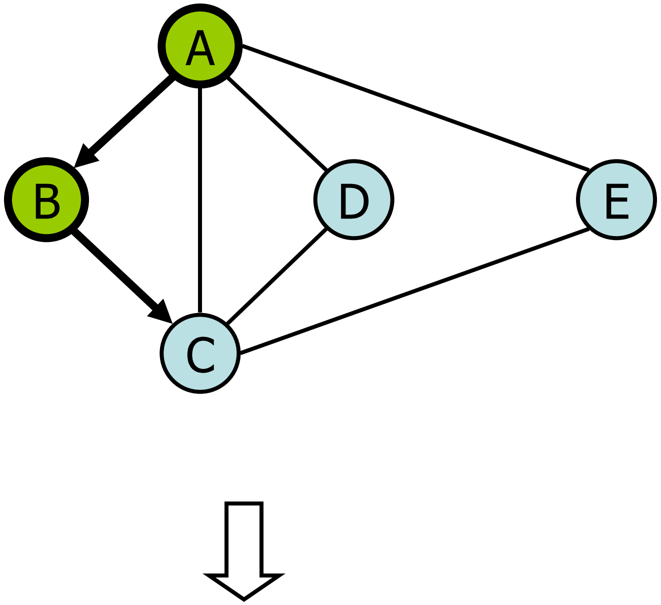
Example



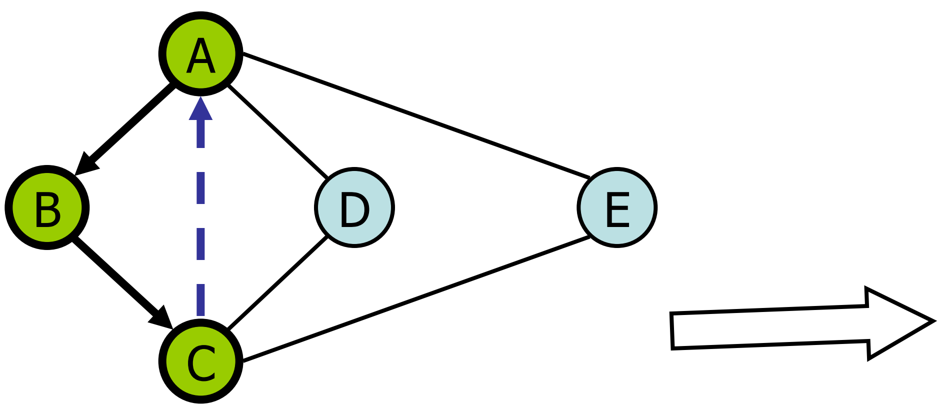
unexplored vertex
visited vertex
unexplored edge
discovery edge
back edge

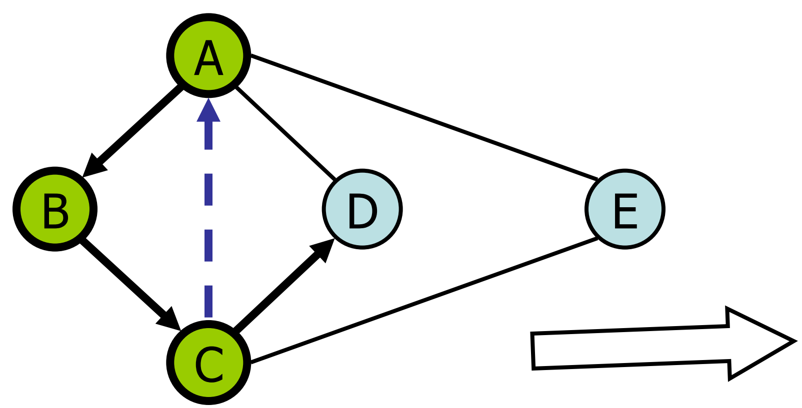
```
Algorithm DFS(G, v)
    setLabel(v, VISITED)
    for all  e ∈ G.incidentEdges(v)
        if  getLabel(e) = UNEXPLORED
            w ← opposite(v,e)
            if getLabel(w) = UNEXPLORED
                setLabel(e, DISCOVERY)
                DFS(G, w)
            else
                setLabel(e, BACK)
```

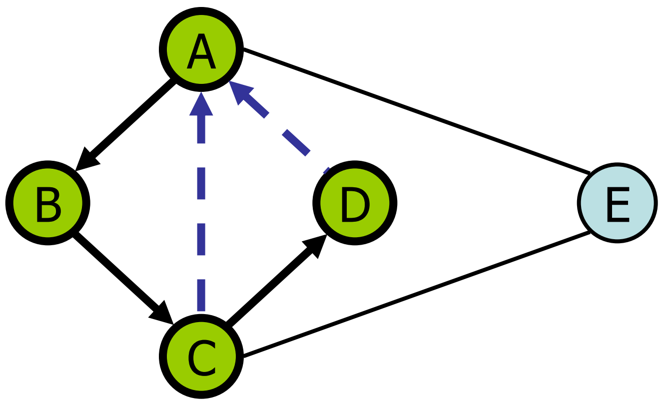




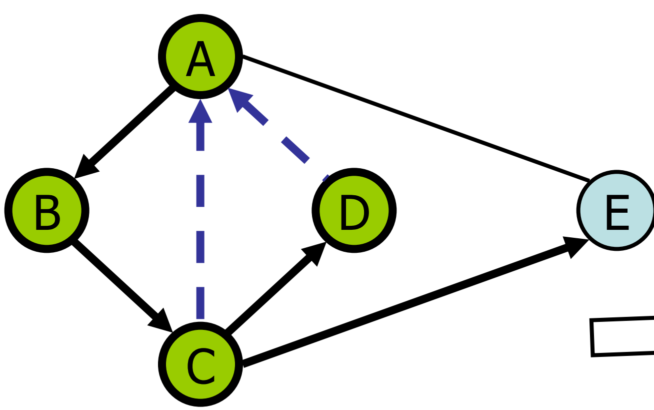

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  for all  e ∈ G.incidentEdges(v)
    if  getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        DFS(G, w)
      else
        setLabel(e, BACK)
```

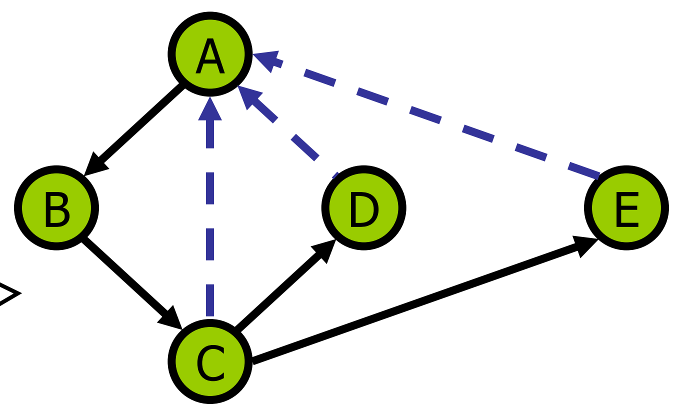






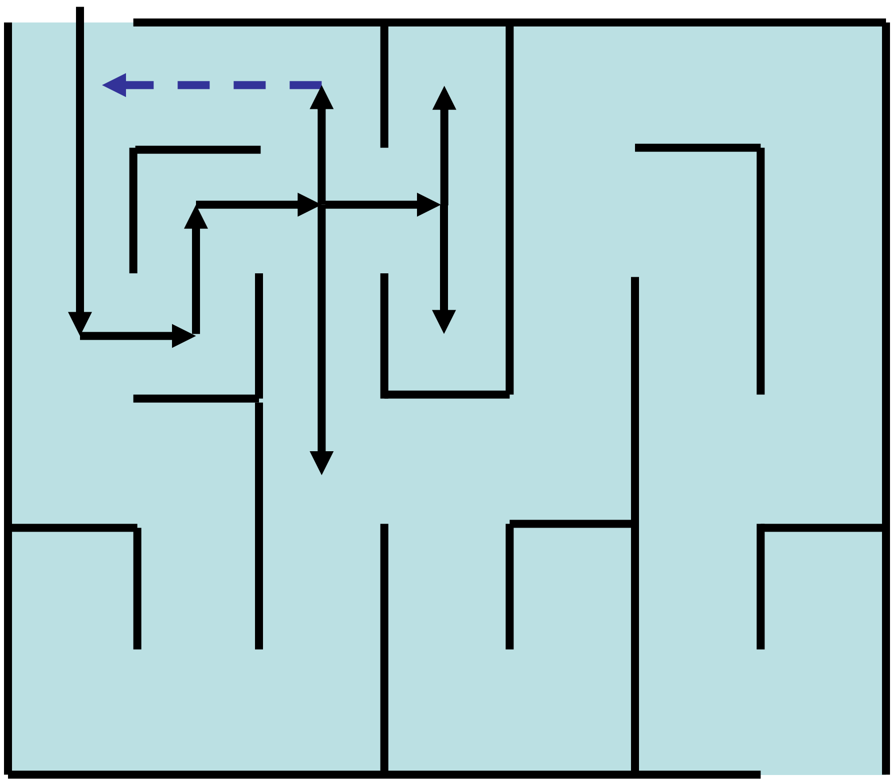

```
Algorithm DFS(G, v)
  setLabel(v, VISITED)
  for all  e ∈ G.incidentEdges(v)
    if  getLabel(e) = UNEXPLORED
      w ← opposite(v,e)
      if getLabel(w) = UNEXPLORED
        setLabel(e, DISCOVERY)
        DFS(G, w)
      else
        setLabel(e, BACK)
```





DFS and Maze Traversal

- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)



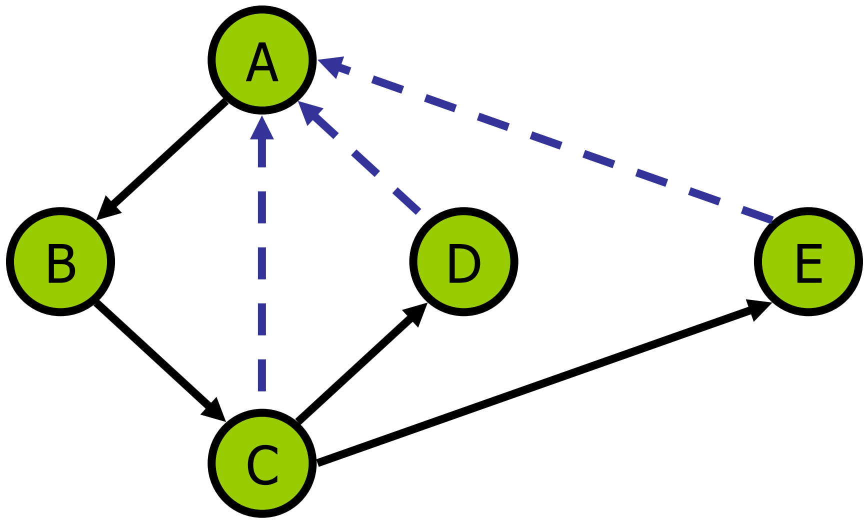
Properties of DFS

Property 1

$DFS(G, v)$ visits all the vertices and edges in the connected component of v

Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of v



Analysis of DFS + labeling

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED

2n

- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK

2m

- Method incidentEdges is called once for each vertex

$$\sum_{v \in V} d(v) = 2m$$

If the graph is implemented with adjacency list

- DFS runs in $O(n+m)$ time provided the graph is represented by the adjacency list structu

- If the graph is connected (m >= n-1) then

$$O(n+m) = O(m)$$

Conclusion

If we represent the graph with an adjacency list

Complexity of DFS is O(n+m)

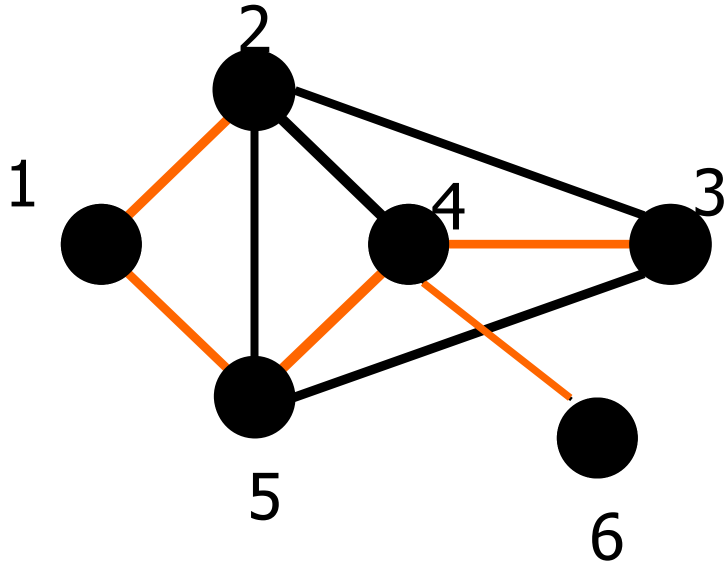
WORST CASE: $m = O(n^2)$.

Question: With adjacency matrix ?

With adjacency matrix DFS is always $O(n^2)$, even if m is much smaller than n^2.

Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $DFS(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack



2 -- 6

$2_{(2,1)}$ $1_{(1,5)}$ $5_{(5,4)}$ 4 ~~(4,3)~~ ~~3~~ (4,6) 6

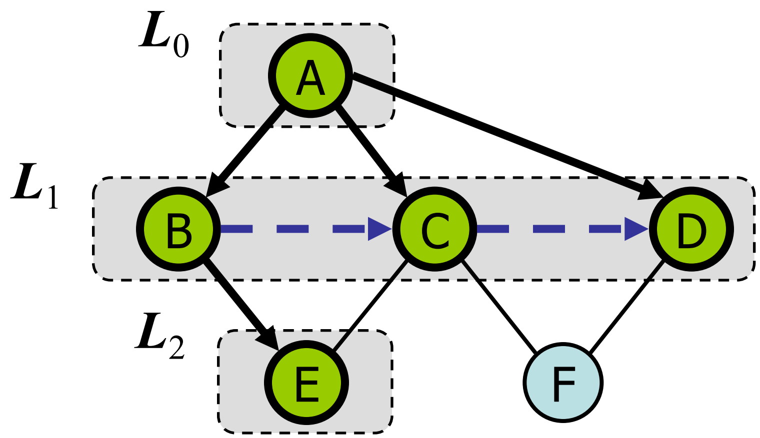
Algorithm *pathDFS*(*G, v, z*)
  setLabel(*v, VISITED*)
  S.push(*v*)
  if *v* = *z*
    return *S.elements*()
  for all *e* ∈ *G.incidentEdges*(*v*)
    if *getLabel*(*e*) = *UNEXPLORED*
      w ← *opposite*(*v,e*)
      if *getLabel*(*w*) = *UNEXPLORED*
        setLabel(*e, DISCOVERY*)
        S.push(*e*)
        pathDFS(*G, w, z*)
        S.pop()
      else
        setLabel(*e, BACK*)
  S.pop()

Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex w

```
Algorithm cycleDFS(G, v)
    setLabel(v, VISITED)
    S.push(v)
    for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
            w ← opposite(v,e)
            S.push(e)
            if getLabel(w) = UNEXPLORED
                setLabel(e, DISCOVERY)
                cycleDFS(G, w)
                S.pop(e)
            else
                T ← new empty stack
                repeat
                    o ← S.pop()
                    T.push(o)
                until o = w
                return T.elements()
    S.pop(v)
```

Breadth-First Search



Outline and Reading

- Breadth-first search
  - Algorithm
  - Example
  - Properties
  - Analysis
  - Applications
- DFS vs. BFS
  - Comparison of applications
  - Comparison of edge labels

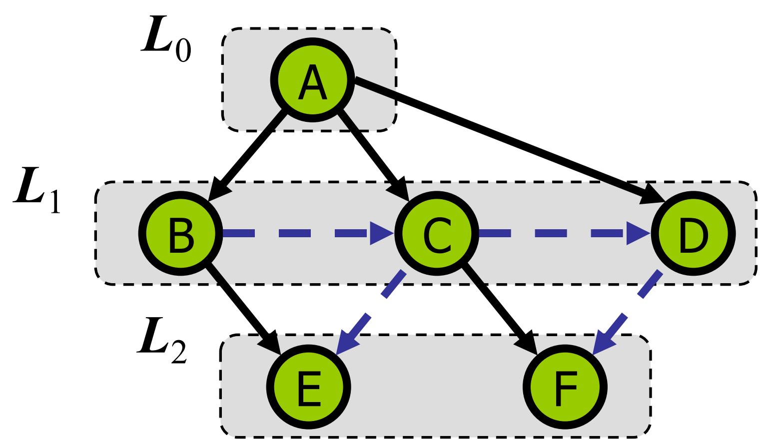
Breadth-First Search

Breadth-First Search (BFS) is a graph traversal technique that:

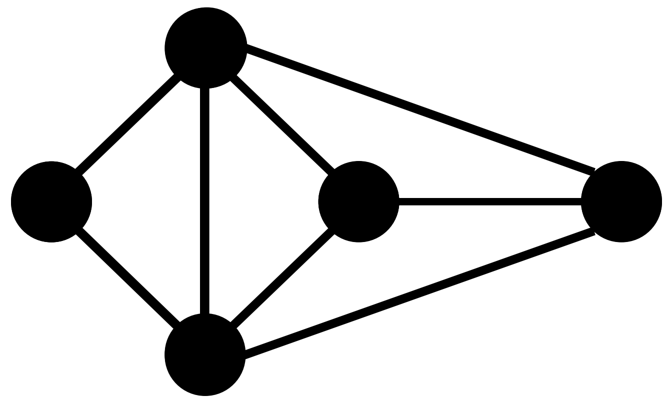
- on a graph with n vertices and m edges, takes $O(n + m)$ time
- can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

The idea:

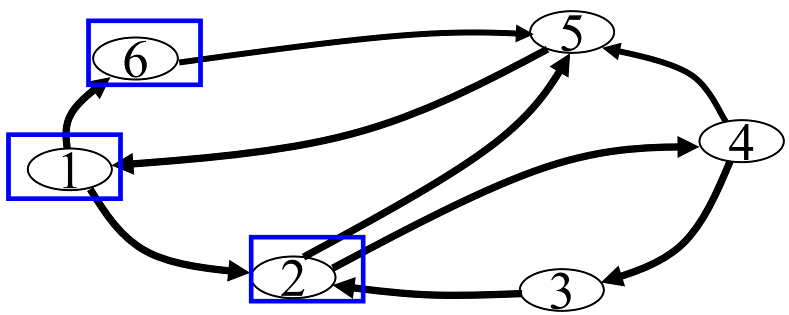
Visit a vertex and then visit all unvisited vertices that are adjacent to it before visiting a vertex which is 2 away from it.



level by level



Breath First Search with a Queue



visited : $\{1\}$
$T = \phi$

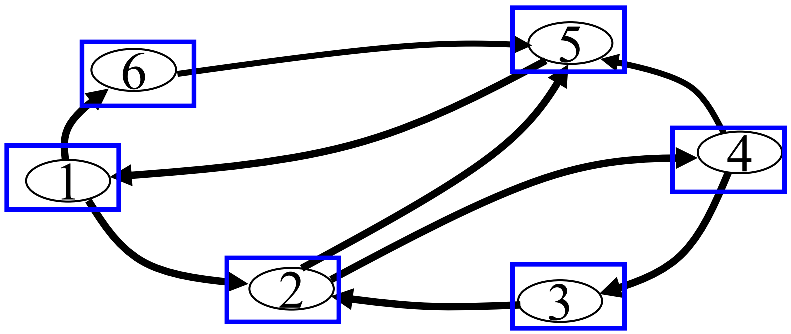
to visit: $\{(1,2), (1,6)\}$

$(1,2) - 2$ visited?
visited : $\{1,2\}$
$T = \{(1,2)\}$

to visit: $\{(1,6), (2,4), (2,5)\}$

$(1,6)$ - 6 visited ?
visited : $\{1,2,6\}$
$T = \{(1,2), (1,6)\}$

to visit: $\{(2,4), (2,5), (6,5)\}$



(2,4) - 4 visited?

visited: {1,2,6,4}

T = {(1,2), (1,6), (2,4)}

to visit: {(2,5), (6,5), (4,5), (4,3) }

(2,5) -5 visited ?

visited: {1,2,6,4,5}

T = {(1,2), (1,6), (2,4), (2,5)}

to visit: {(6,5), (4,5), (4,3)}

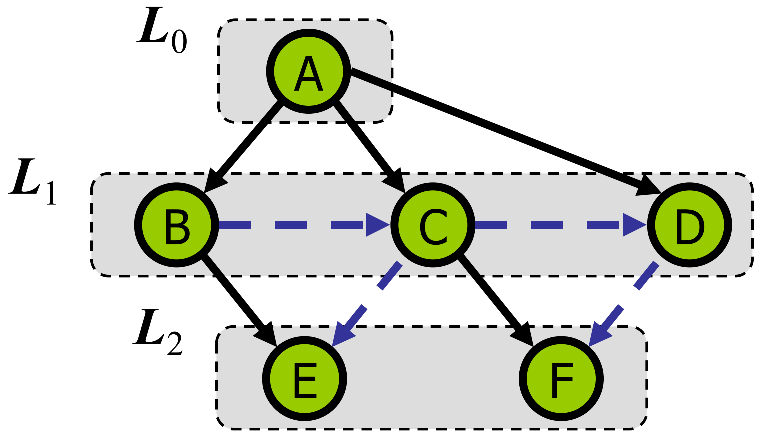
(6,5) - 5? already visited!

(4,5) - 5? already visited!

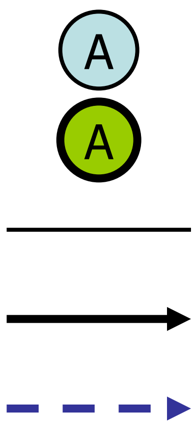
(4,3) - 3?

BFS with labeling

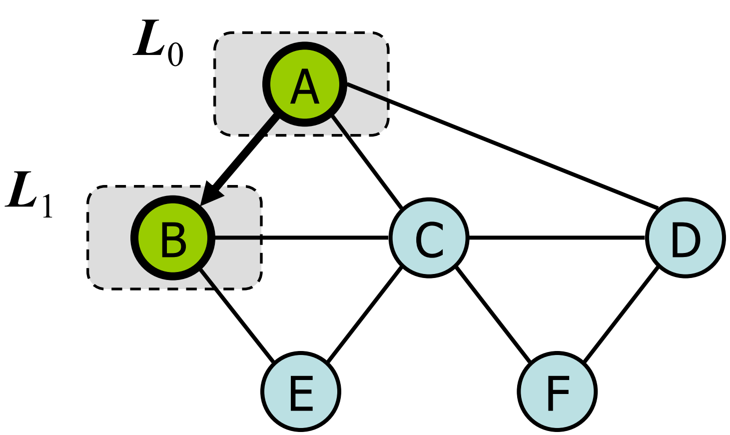
Using a sequence for each level

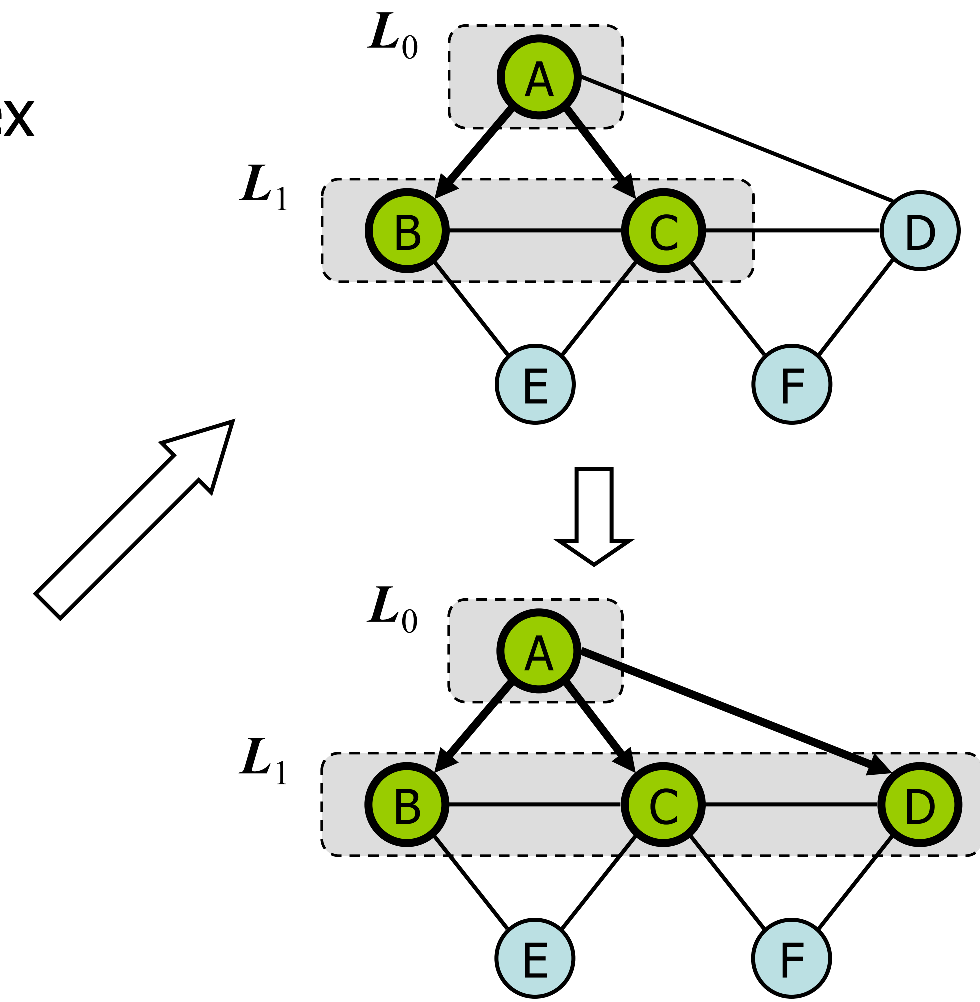


Example

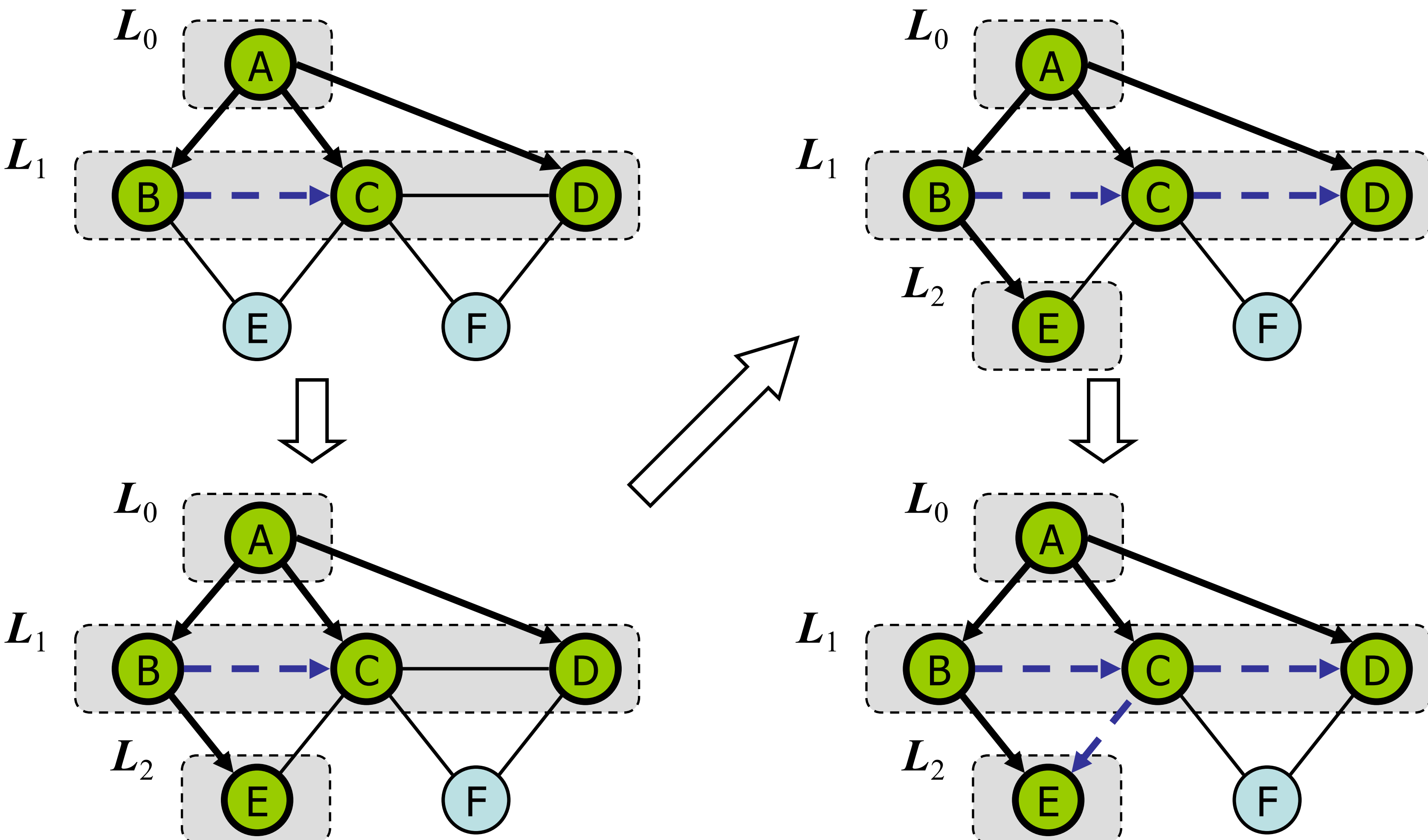


unexplored vertex
visited vertex
unexplored edge
discovery edge
cross edge

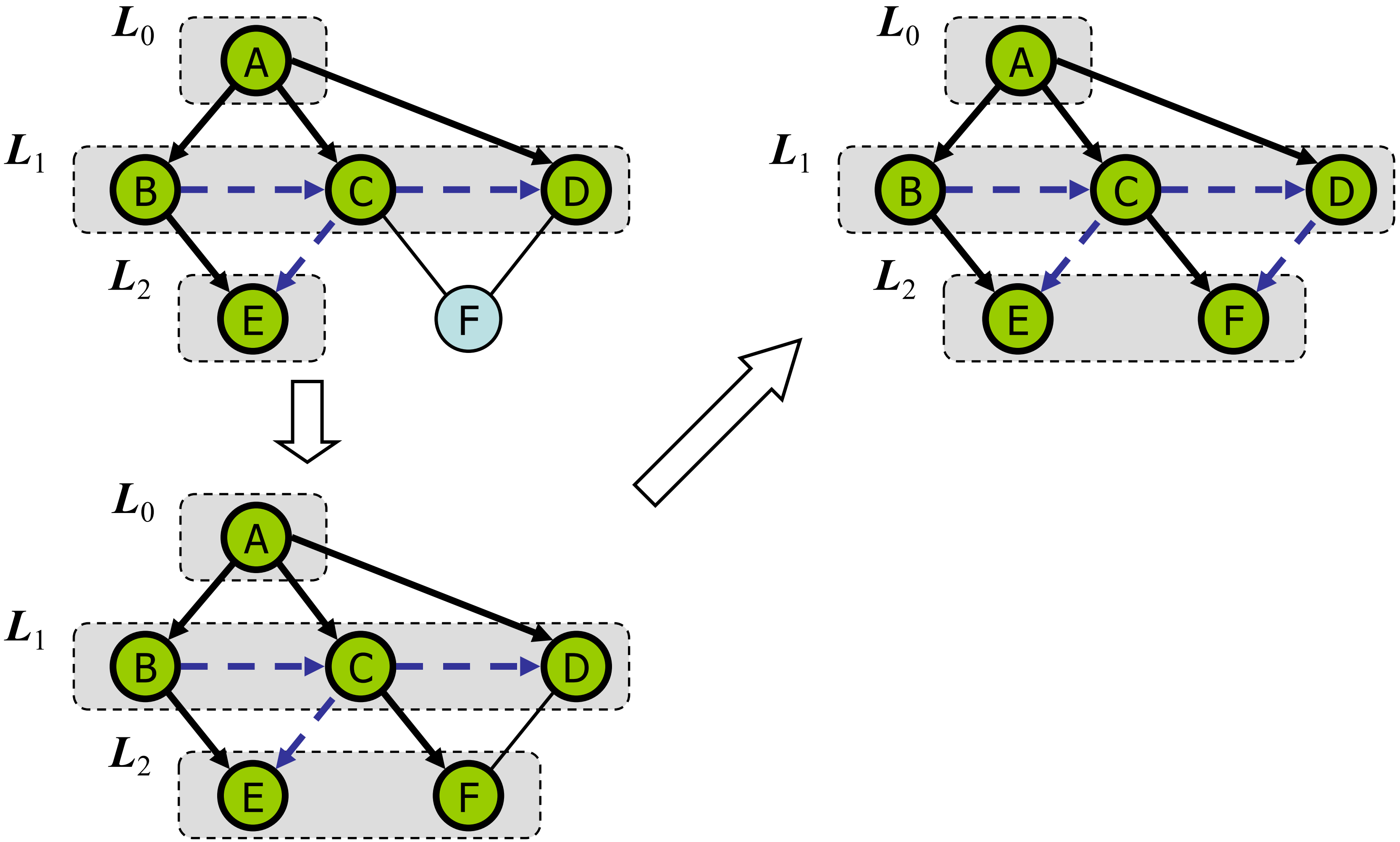




Example (cont.)



Example (cont.)



This version uses several queues instead of just one

BFS Again - more details

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm BFS(G)
    Input graph G
    Output labeling of the edges
        and partition of the
        vertices of G
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
        BFS(G, v)
```

```
Algorithm BFS(G, s)
  L0 ← new empty sequence
  L0.insertLast(s)
  setLabel(s, VISITED)
  i ← 0
  while ! Li.isEmpty()
    Li+1 ← new empty sequence
    for all v ∈ Li.elements()
      for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
          w ← opposite(v,e)
          if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            setLabel(w, VISITED)
            Li+1.insertLast(w)
          else
            setLabel(e, CROSS)
    i ← i +1
```

Properties

Notation

G_s: connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

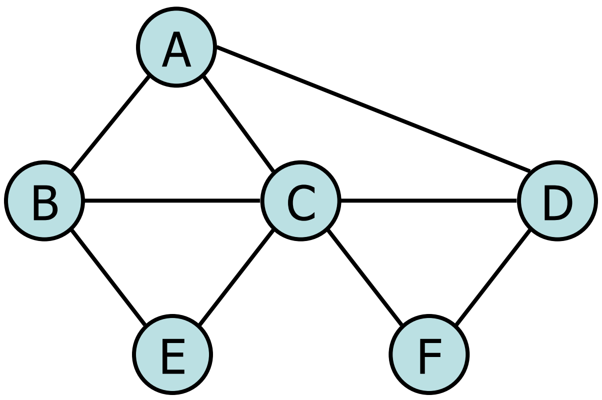
Property 2

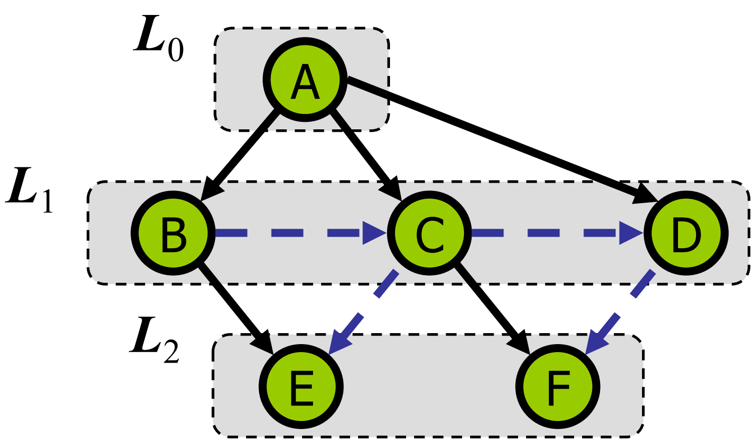
The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- The path of T_s from s to v has i edges
- Every path from s to v in G_s has at least i edges





Analysis

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

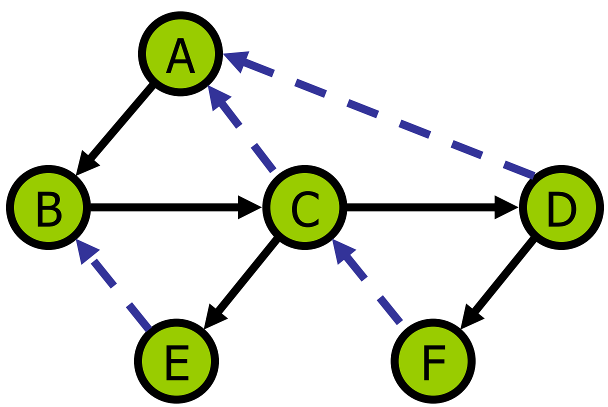
Question: what if your graph is represented with adjacency matrix? See DFS.

Applications

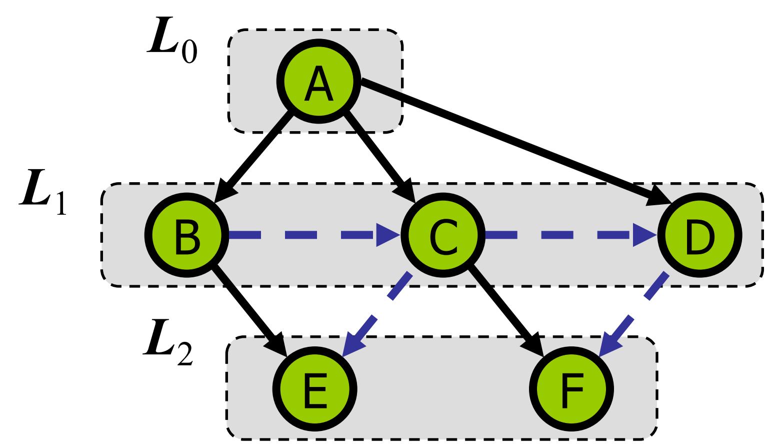
- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n+m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS vs. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
*)Biconnected components	√	



DFS



BFS

*) "A connected graph is *biconnected* if the removal of any single vertex (and all edges incident on that vertex) can not disconnect the graph."

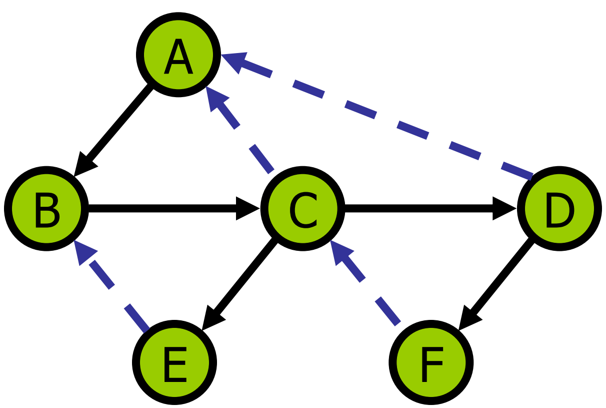
DFS vs. BFS (cont.)

Back edge (v,w)

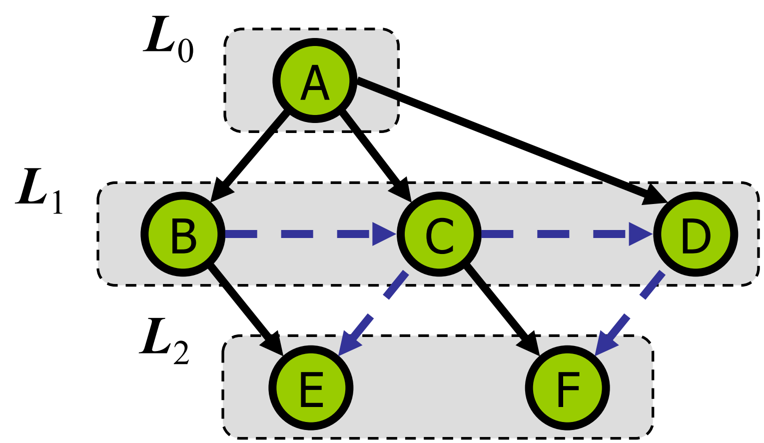
– w is an ancestor of v in the tree of discovery edges

Cross edge (v,w)

– w is in the same level as v or in the next level in the tree of discovery edges

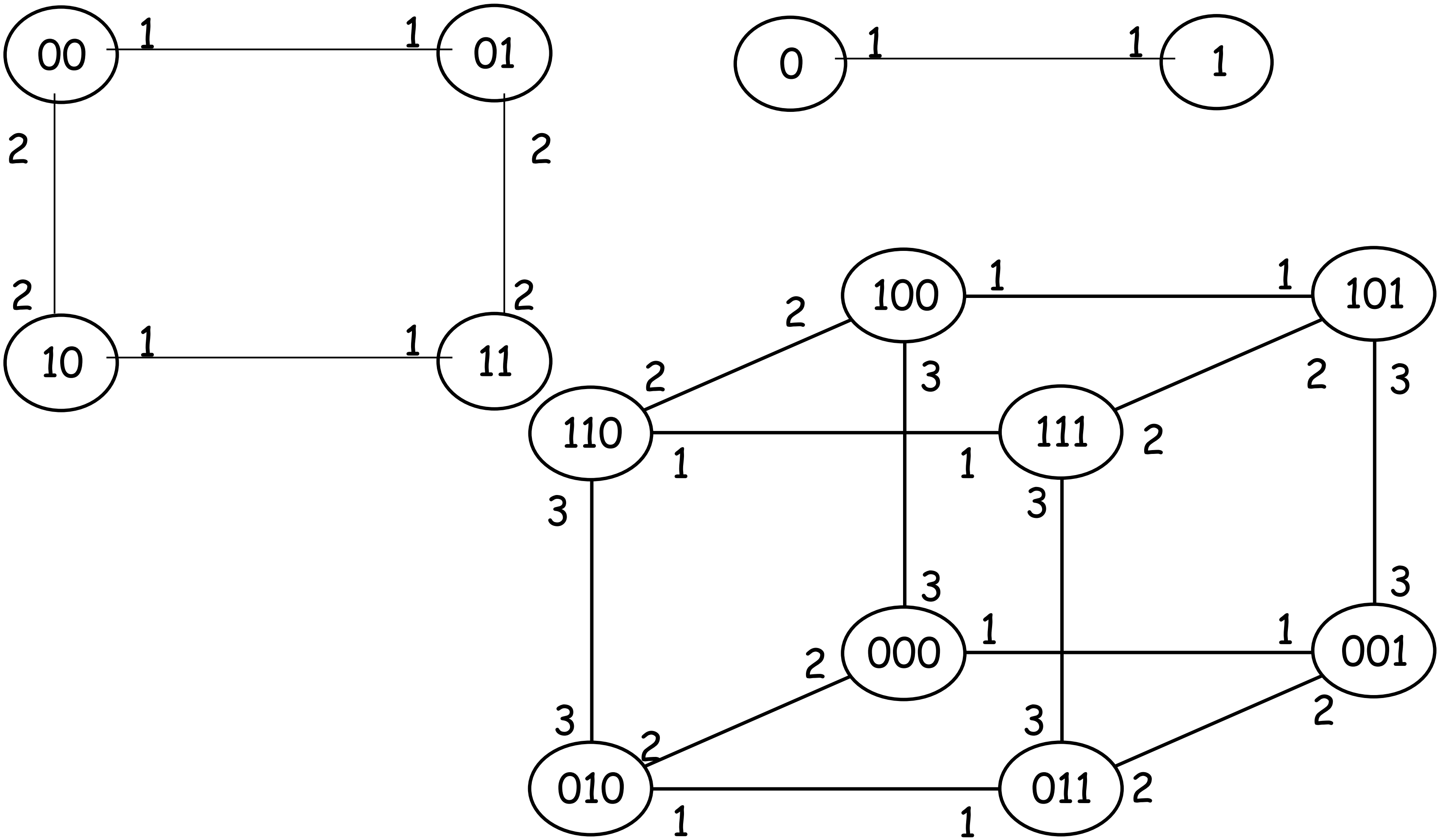


DFS

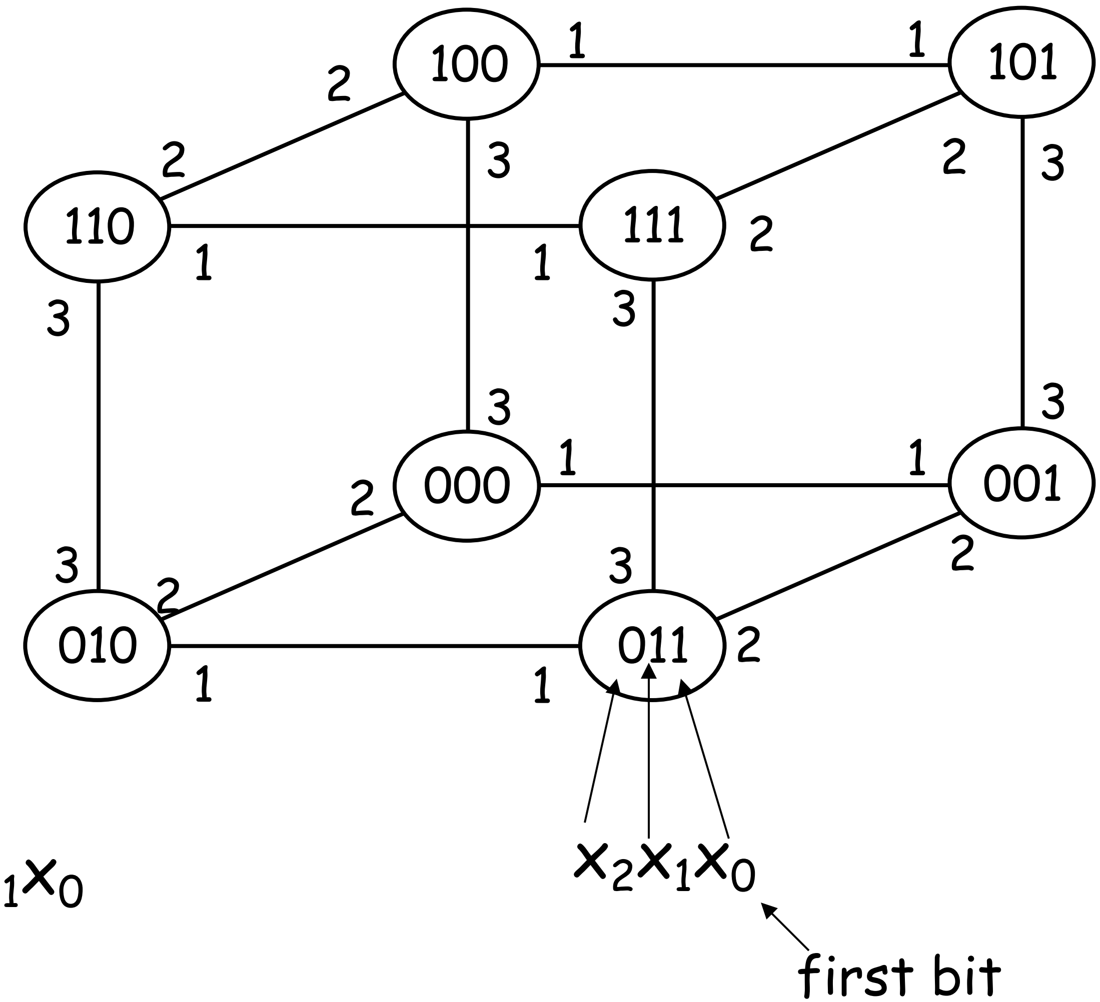


BFS

Example: DFS/BFS in the hypercube

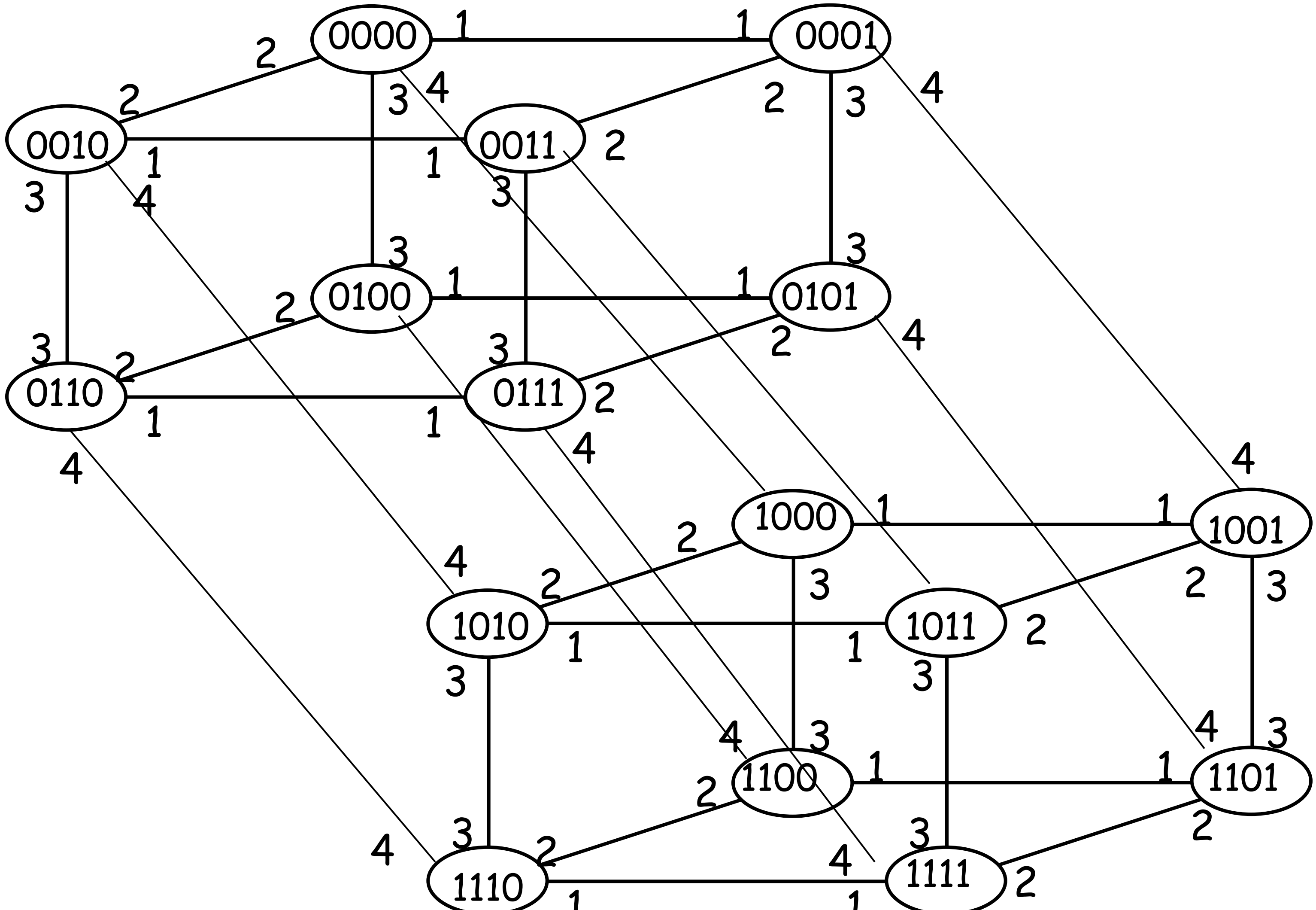


Each link between two nodes is labeled by the dimension of the bit by which the nodes' name differ.



$X = x_k x_{k-1} \ldots\ldots x_1 x_0$

K-bit name


0000
0001
0010
0011
0100
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0110
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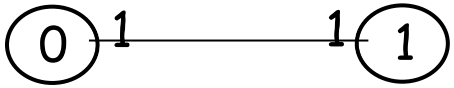
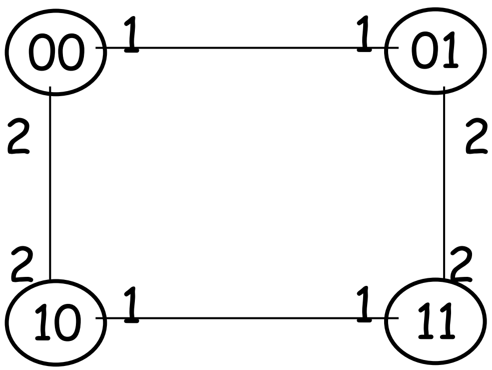
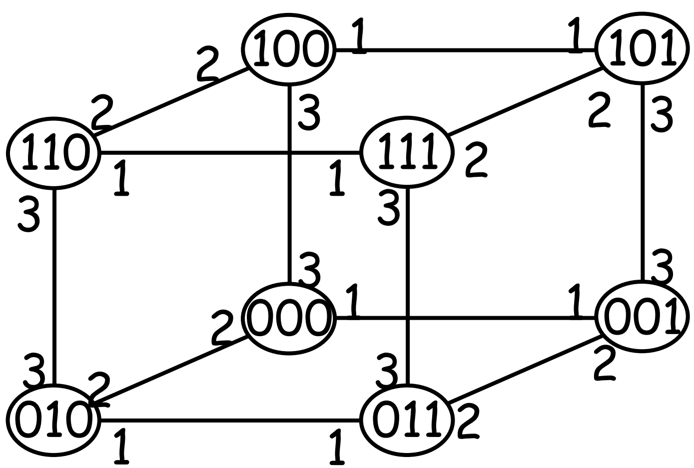
1001
1010
1011
1100
1101
1110
1111



n=2 dimensions=1



n=4 dimensions=2



n=8 dimensions=3

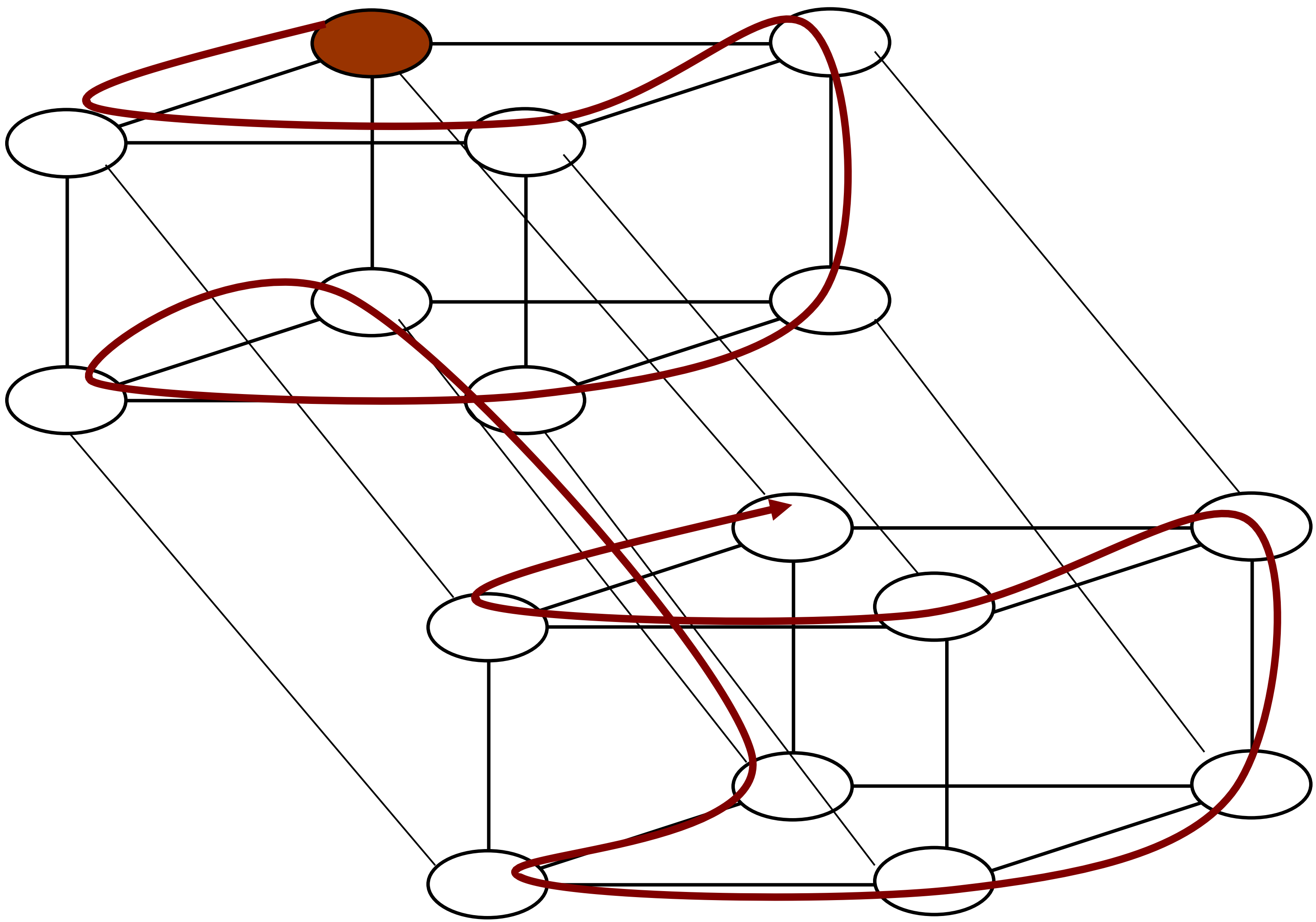
dimensions = log n

A hypercube of dimension d has $n = 2^d$ nodes

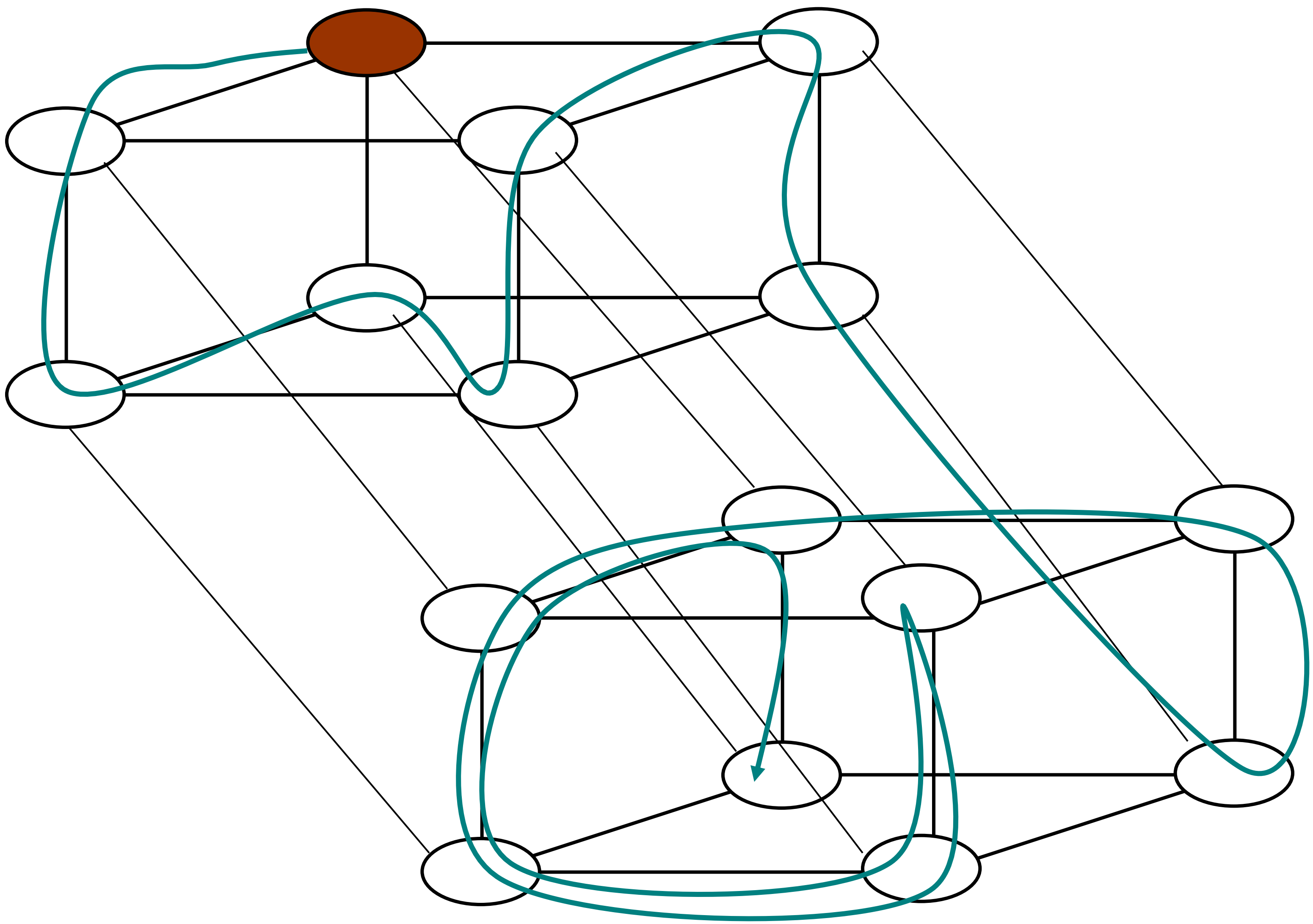
Each node has d links

➔ $m = n\, d/2 = O(n \log n)$

A Depth-first traversal



Another Depth-first traversal



A Breadth-first traversal