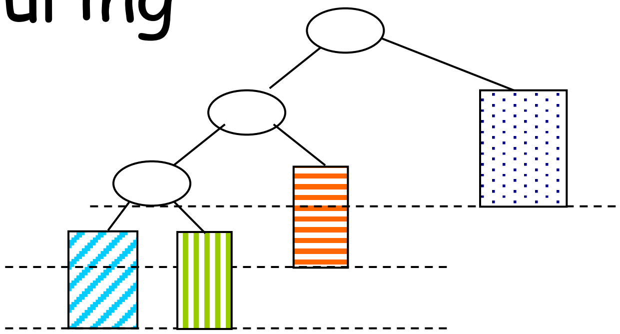
AVL Trees

Adel'son-Vel'skii and Landis

Data structure that implements MAP ADT

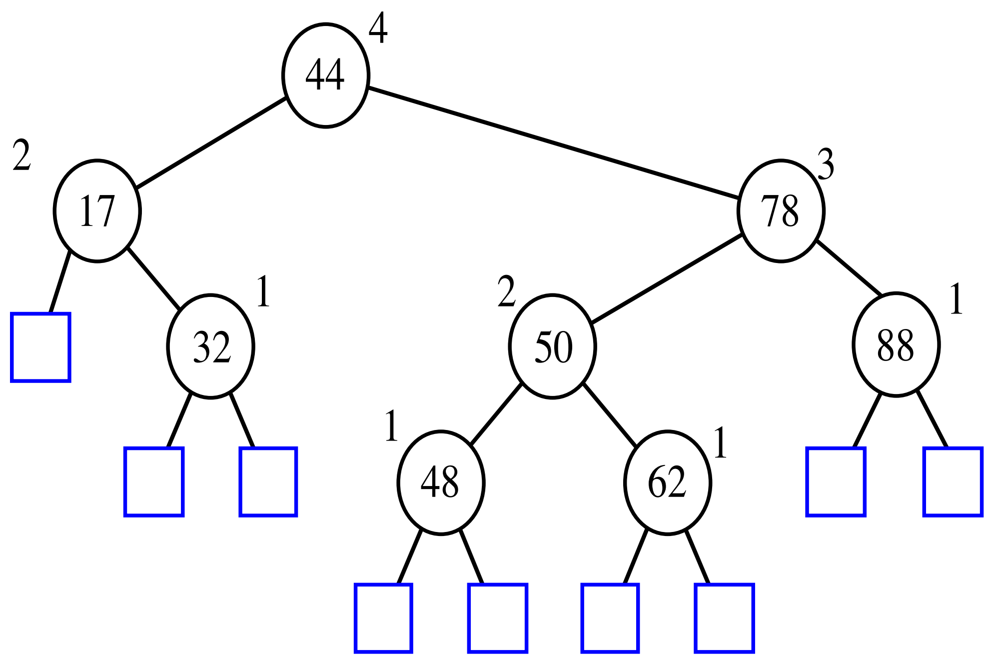
- Height of an AVL Tree
- Insertion and restructuring
- Removal and restructuring
- Costs



AVL Tree

- **AVL trees are balanced.**
- An AVL Tree is a ***binary search tree*** such that for every internal node v of T, the *heights of the trees rooted at the children of v can differ by at most 1.*

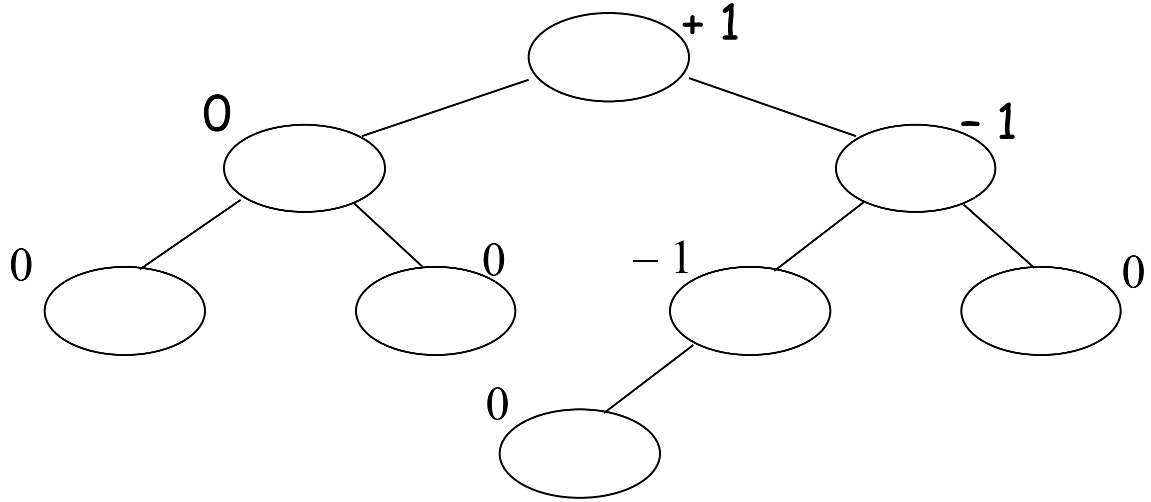
An example of an AVL tree where the heights are shown next to the nodes:



Balancing Factor

height(right subtree)– height(left subtree)

$\in\{-1, 0, 1\}$ for AVL tree



Height of an AVL Tree

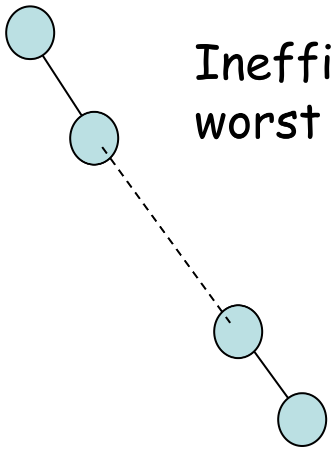
Note: "longest" possible heap with n nodes.

Always O(log n)

But cannot search efficiently

Note: "longest" possible binary tree with n nodes:

O(n)

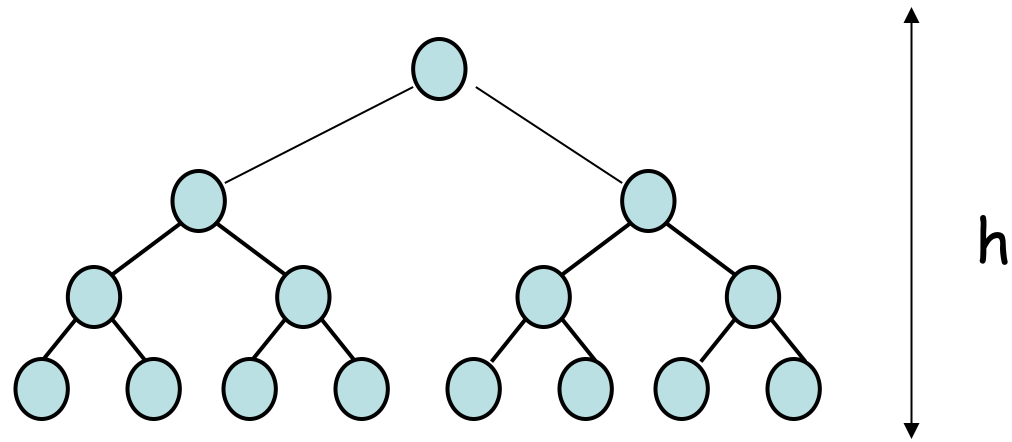


Inefficient to search in the worst case.

We'll now see that the *height* of an AVL tree T storing n keys is O(log n).

Note: AVL tree with the highest possible number of internal nodes for a given height h:

Perfect:



$n = 2^h$

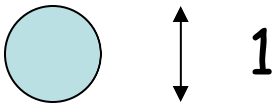
So $h = \log n$

To construct the "longest" possible AVL tree, we look for

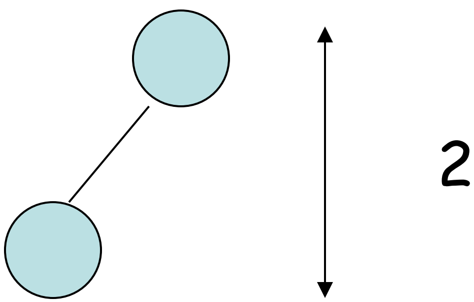
the *minimum number of nodes* of an AVL tree of height h.

$n(h)$

Easy to see that n(1) = 1 and n(2) = 2
(please note that dummy nodes are not shown but contributing to the height)



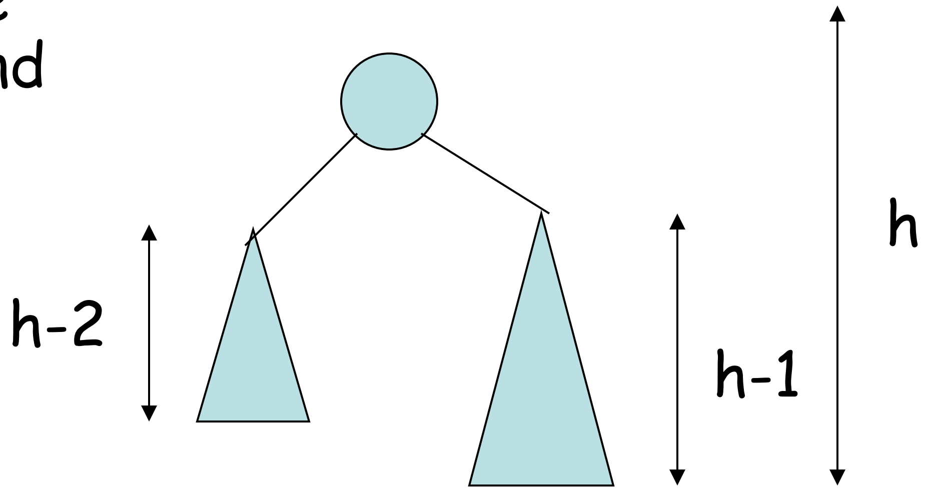
n(1) = 1



n(2) = 2

n(h): the *minimum number of internal nodes* of an AVL tree of height h.

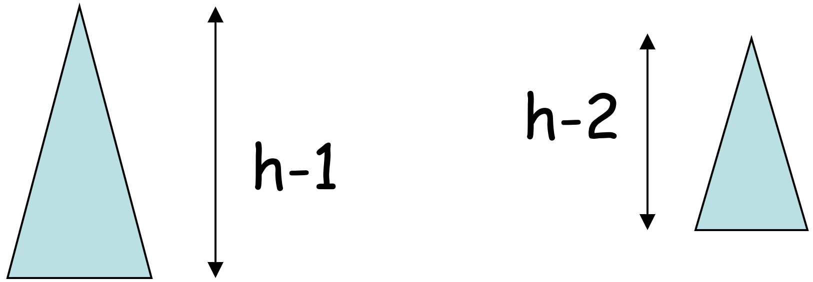
For n ≥ 3, an AVL tree of height h contains the root node, one AVL subtree of height h-1 and the other AVL subtree of height h-2.



$$n(h) = 1 + n(h-1) + n(h-2)$$

Height of an AVL Tree

$$n(h) = 1 + \boxed{n(h-1)} + n(h-2)$$



Clearly: $\boxed{n(h-1) > n(h-2)}$

so: $n(h) > 1 + n(h-2) + n(h-2)$

$n(h) > 1 + 2\,n(h-2) > 2\,n(h-2)$

$n(h) > 2n(h-2)$

for any h

Height of an AVL Tree

So, now we know: $n(h) > 2\ \boxed{n(h-2)}$

but then also: $\boxed{n(h-2)} > 2\ n(h-4)$

$n(h) > 4\ n(h-4)$

$n(h) > \boxed{4n(h-4)}$

but then also: $\boxed{n(h-4)} > 2n(h-6)$

$n(h) > 8n(h-6)$

We can continue:

$n(h) > 2n(h-2)$

$n(h) > 4n(h-4)$

$n(h) > 8n(h-6)$

...

$n(h) > 2^i n(h-2i)$

$n(h) > 2^i n(h-2i)$

$h-2i = 2$

And we know that

$n(1) = 1$

$n(2) = 2$

Now we pick I such that h is either 1 or 2

That is pick i= ceil(h/2)-1, and substitute:

for i = ceil(h/2) - 1

$$n(h) > 2^{ceil(h/2) - 1} n(1)$$

$$n(h) > 2^{h/2-1}$$

$$\log n(h) > \log 2^{(h/2)-1}$$

$$\log n(h) > h/2 - 1$$

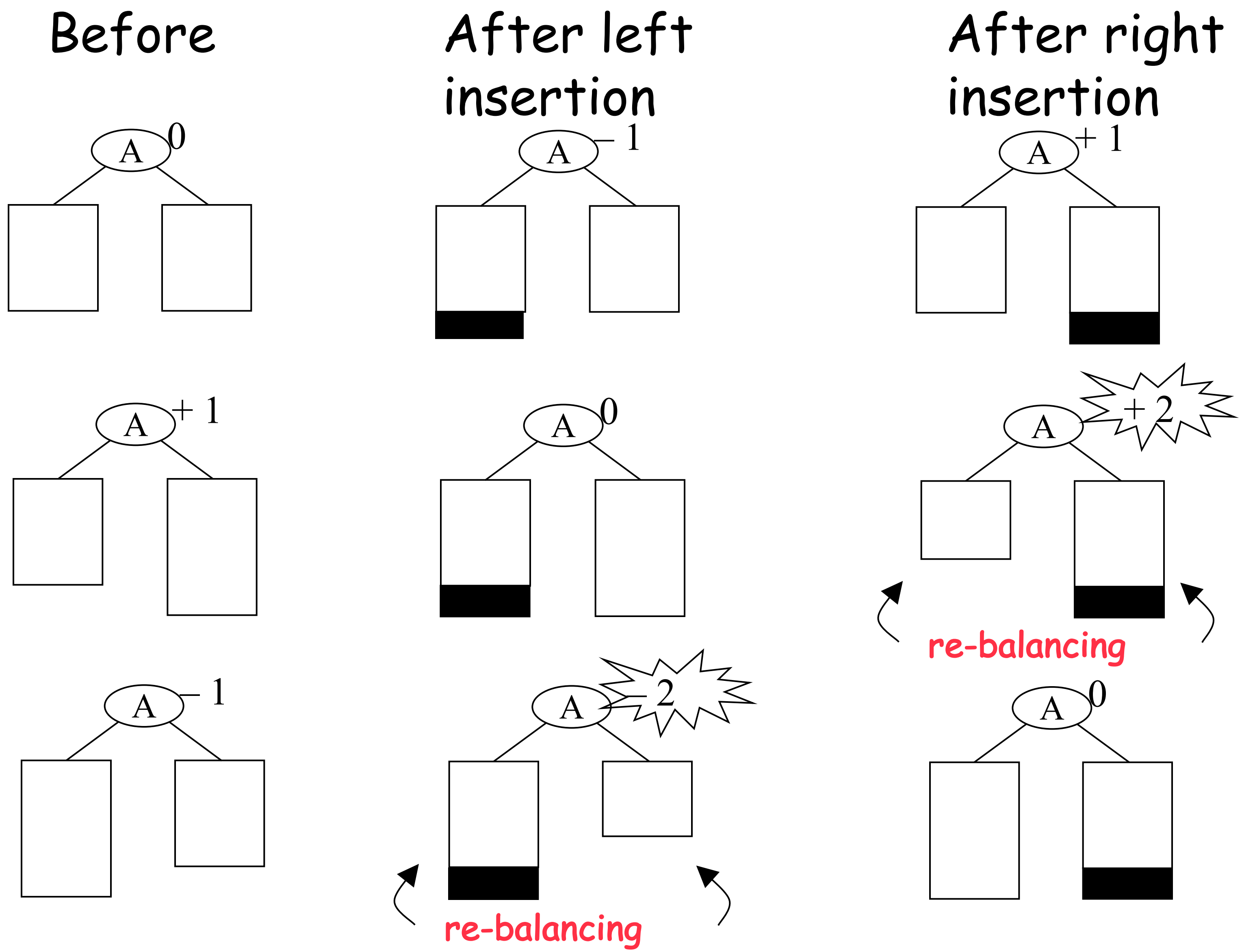
$$h < 2 \log n(h) + 2 < 2 \log n + 2$$

which means that h is O(log n)

Insertion

- A binary search tree T is called *balanced* if for every node v, the height of v's children differ by at most one.
- Inserting a node into an AVL tree involves performing an expandExternal(w) on T, which changes the heights of some of the nodes in T.
- If an insertion causes T to become unbalanced we have to rebalance...

Insertion


Before
After left insertion
After right insertion
A 0
A − 1
A + 1
A + 1
A 0
A + 2
A − 1
A − 2
A 0
re-balancing
re-balancing

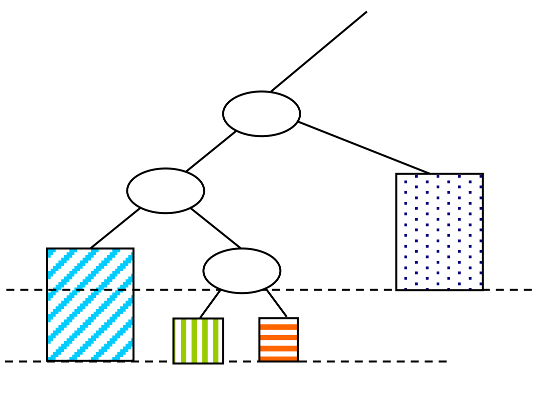
Rebalancing after insertion

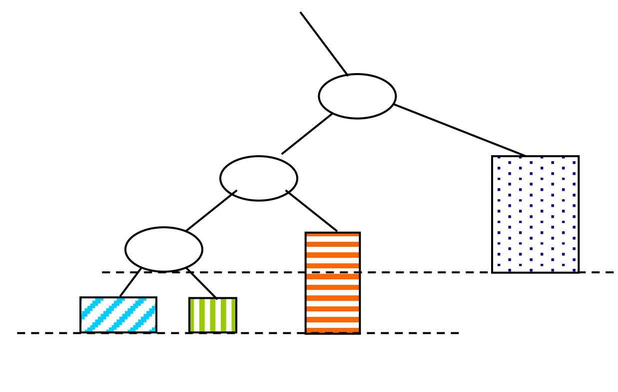
We are going to identify 3 nodes which form a grandparent, parent, child triplet and the 4 subtrees attached to them. We will rearrange these elements to create a new balanced tree.

Rebalancing

Step 1: Trace the path back from the point of insertion to the first node whose grandparent is unbalanced. Label this node x, its parent y, and grandparent z.

Examples .....





Rebalancing

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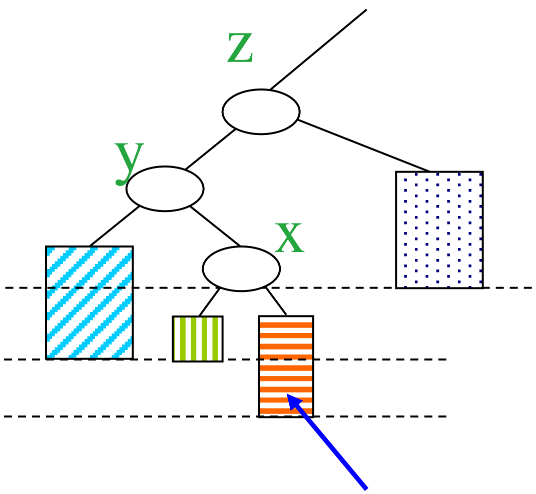
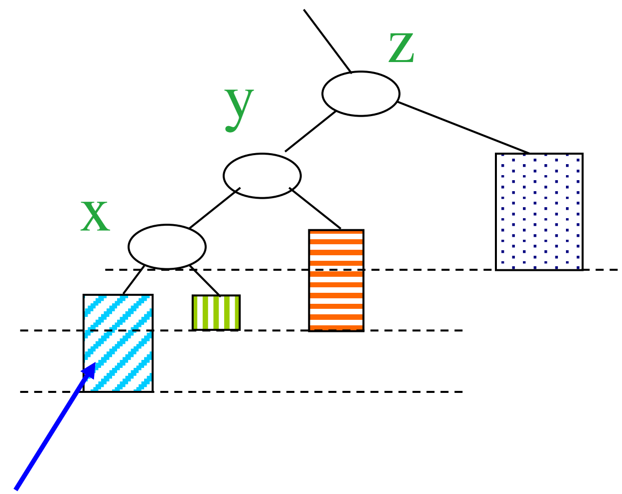
Examples .....

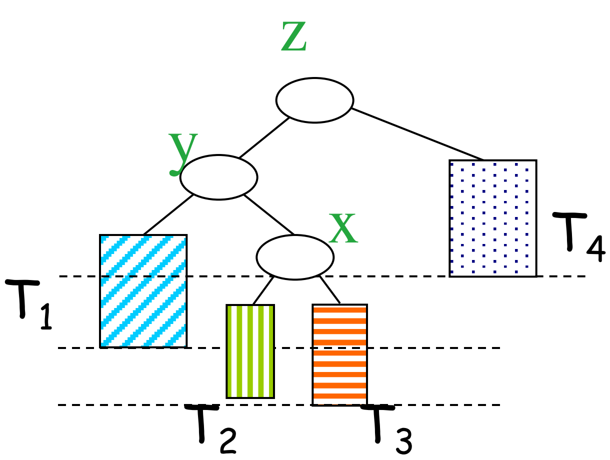


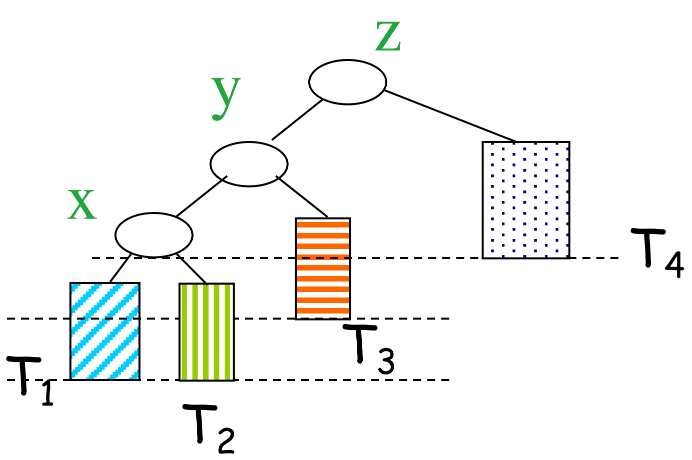


Rebalancing

Step 2: These nodes will have 4 subtrees connected to them. Label them T_1, T_2, T_3, T_4 from left to right.

Examples .....

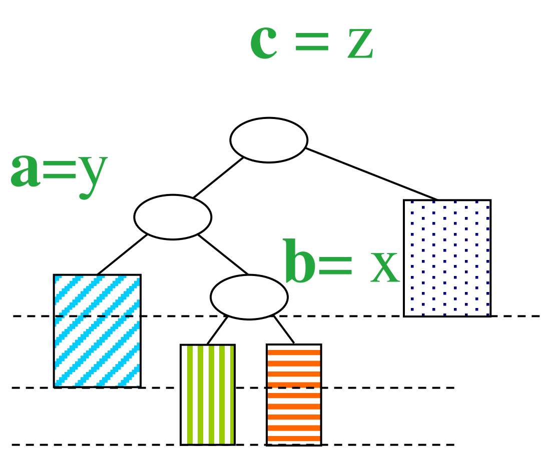




Rebalancing

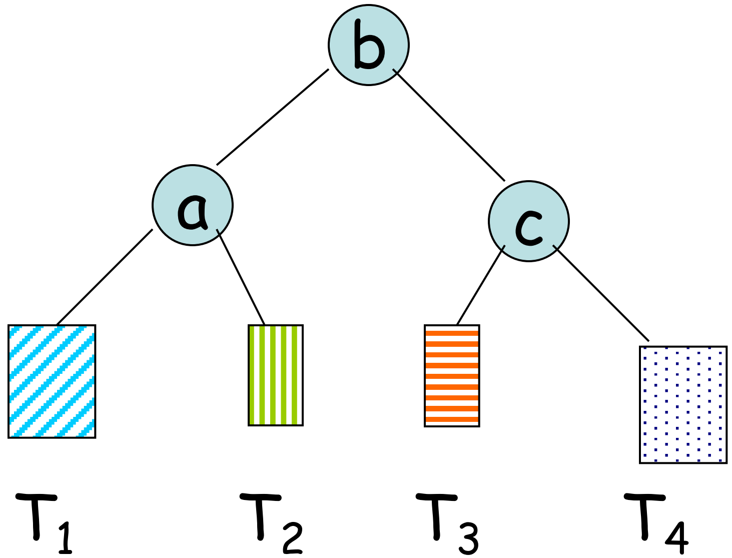
Step 3: Rename x, y, z to a, b, c according to their inorder traversal i.e. if y,x,z is the relative order of those nodes following the inorder traversal then label y 'a', x 'b' and z 'c'.

Example



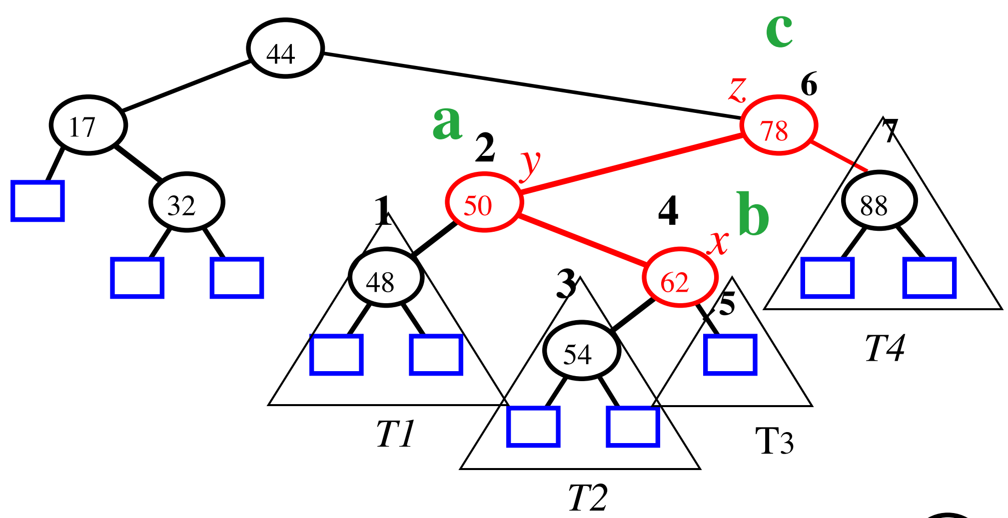
Rebalancing

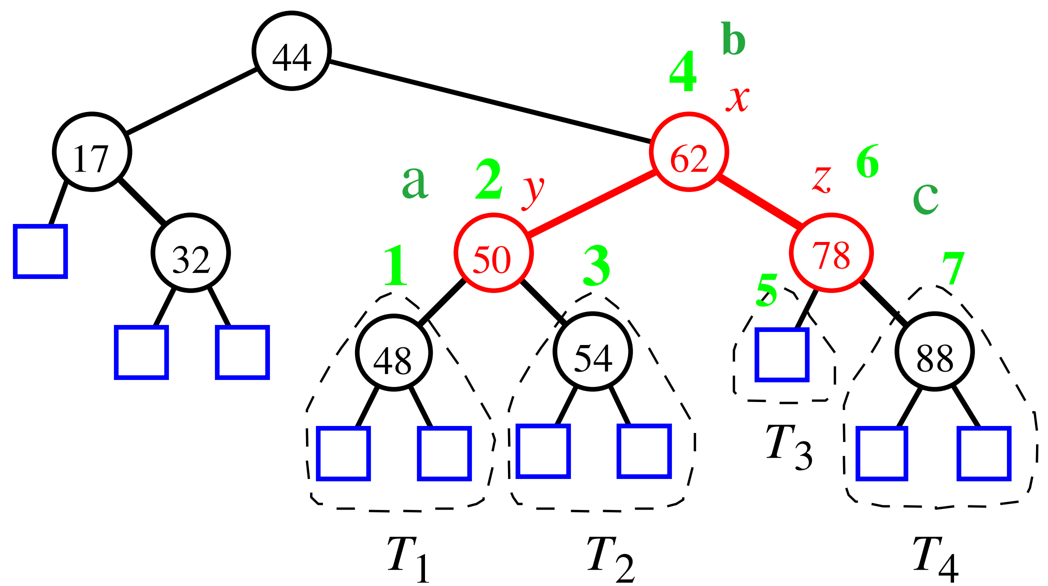
Step 4: Replace the tree rooted at z with the following tree:



Rebalance done!

Example: after inserting 54





Does this really work?

We need to see that the new tree is :

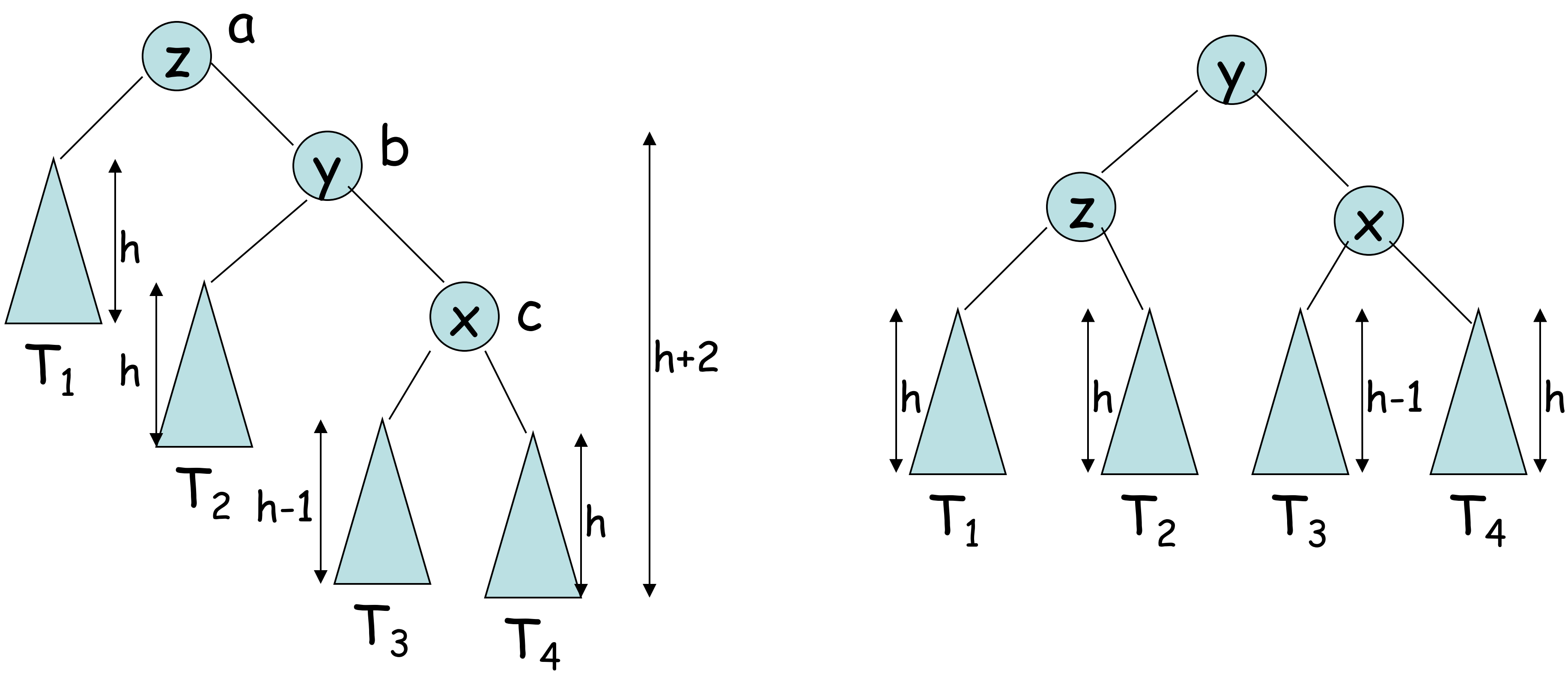
a) A Binary search tree - the inorder traversal of our new tree should be the same as that of the old tree

Inorder traversal: by definition is T1 a T2 b T3 c T4

b) Balanced: have we fixed the problem?

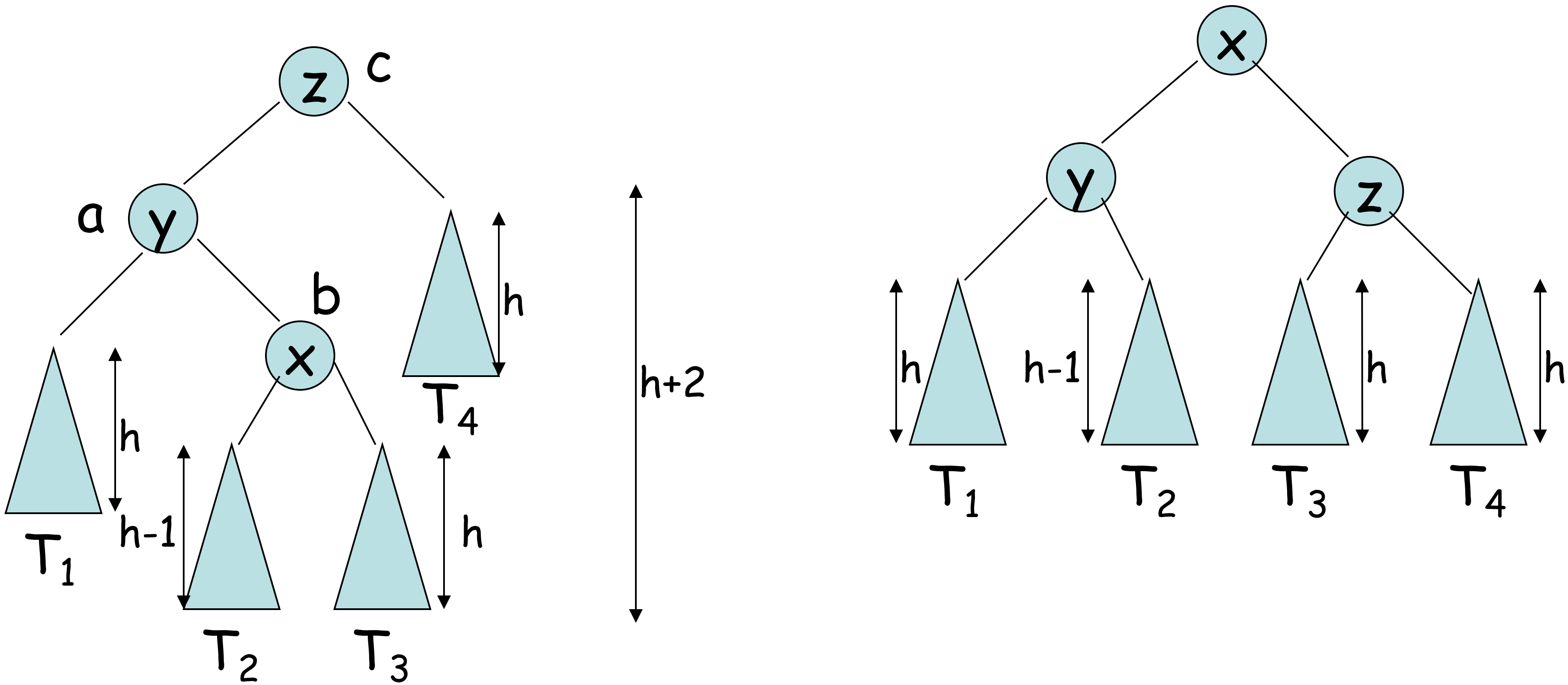
We consider 2 types of examples

Example 1



Inorder: T1 z T2 y T3 x T4

Example 2



Inorder: T1 y T2 x T3 z T4

An Observation…

Notice that in both cases, the new tree rooted at b has the same height that the old tree rooted at z had before insertion.

So.. once we have done one rebalancing act, we are done.

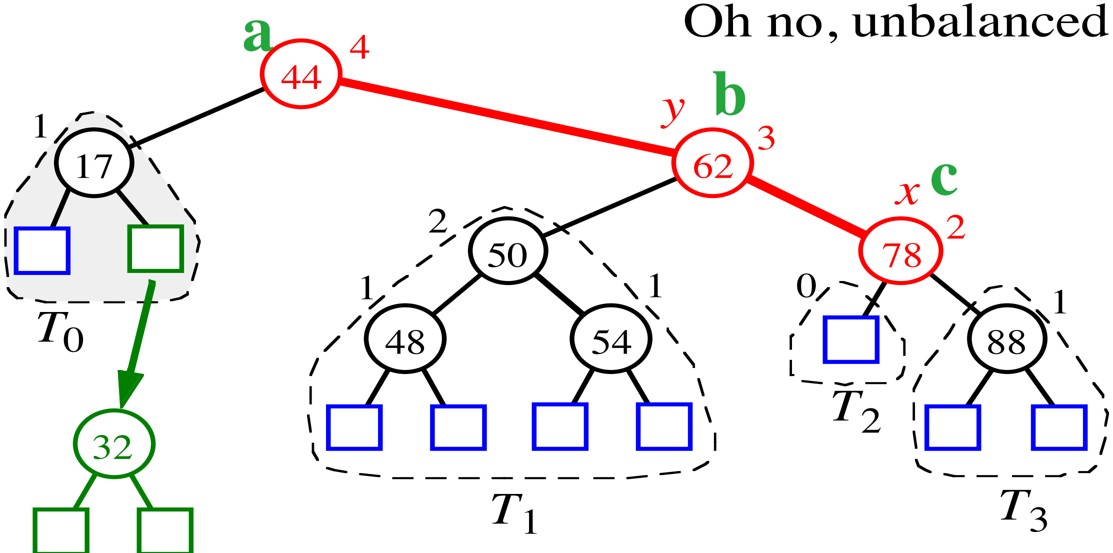
Removal

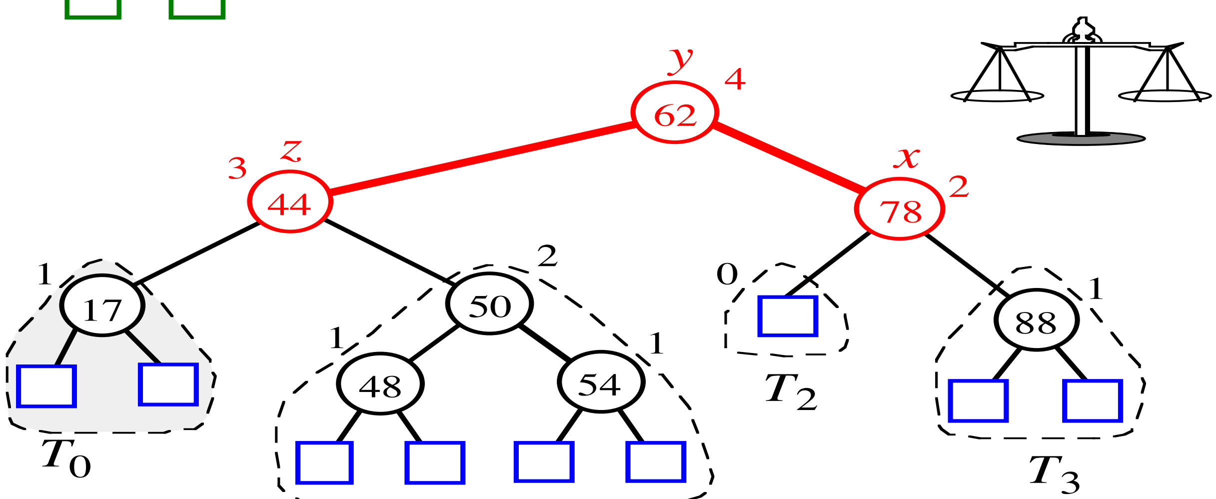
- We can easily see that performing a removeAboveExternal(w) can cause T to become unbalanced.
- Let *z* be the first unbalanced node encountered while travelling up the tree from w. Also, let y be **the child of z with the larger height, and let x be the child of y with the larger height.**
- We can perform operation restructure(x) to restore balance at the subtree rooted at z.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached

Removal (contd.)

the choice of x is not unique !!!

Oh no, unbalanced!

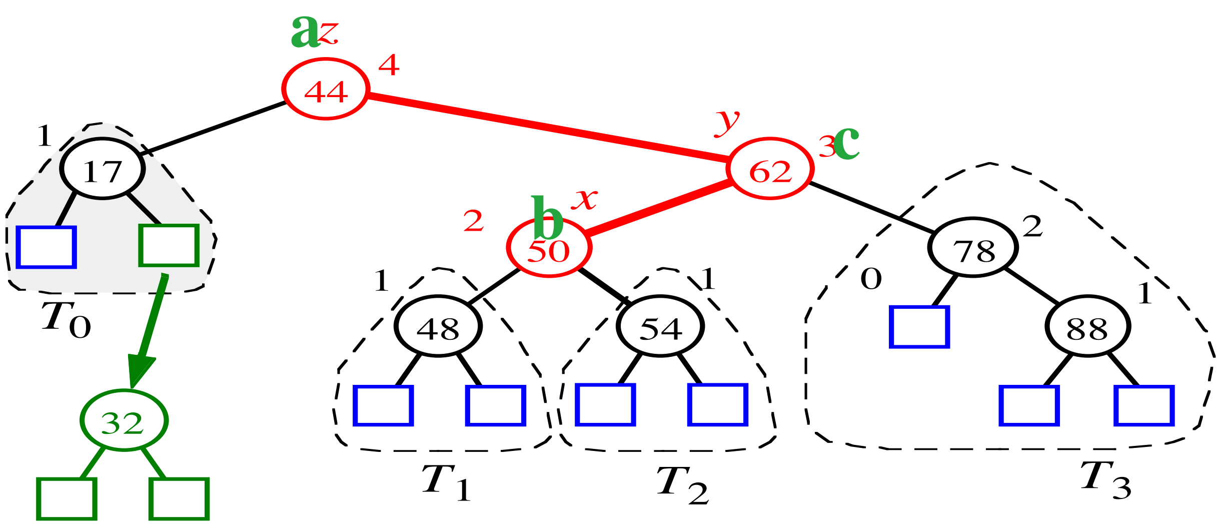




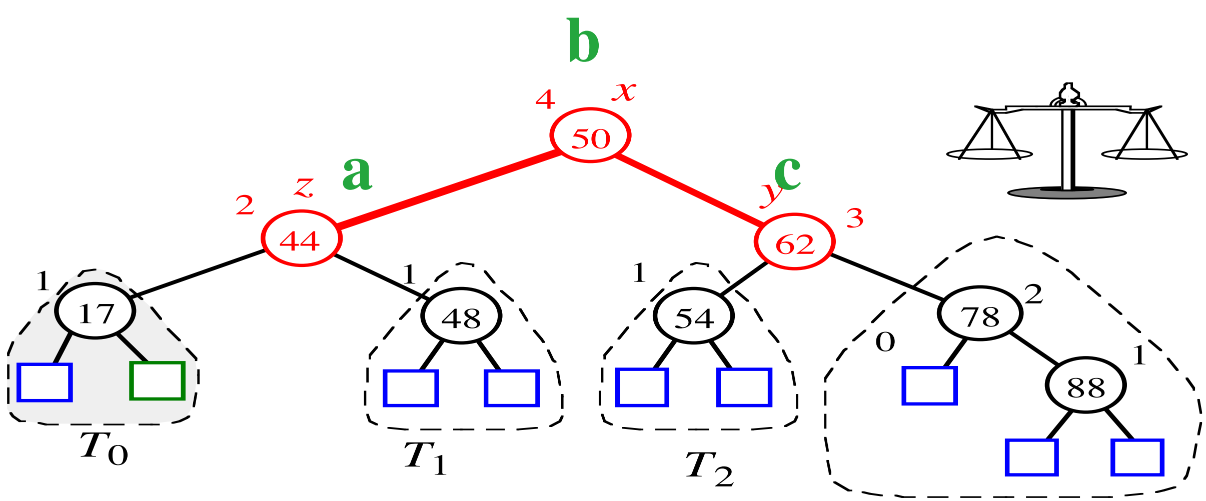
Whew, balanced!

Removal (contd.)

- we could choose a different x:



Oh no, unbalanced!



Whew, balanced!

COMPLEXITY

Searching: findElement(k):

Inserting: insertItem(k, o):

Removing: removeElement(k):

$O(\log n)$

Some implementation details are very important:

The trinode restructure is accomplished using the rotation operation:

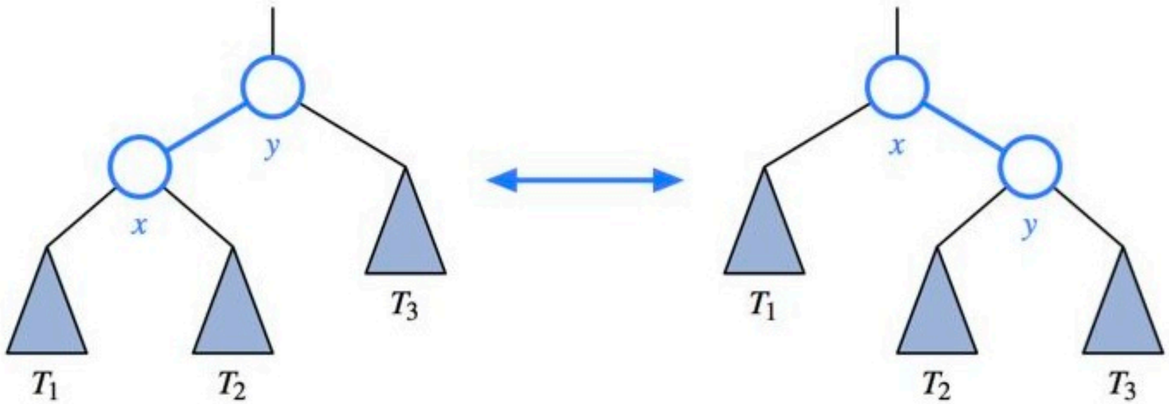


Figure 11.8: A rotation operation in a binary search tree. A rotation can be performed to transform the left formation into the right, or the right formation into the left. Note that all keys in subtree T_1 have keys less than that of position x, all keys in subtree T_2 have keys that are between those of positions x and y, and all keys in subtree T_3 have keys that are greater than that of position y.

Trinode restructuring using rotation operation:

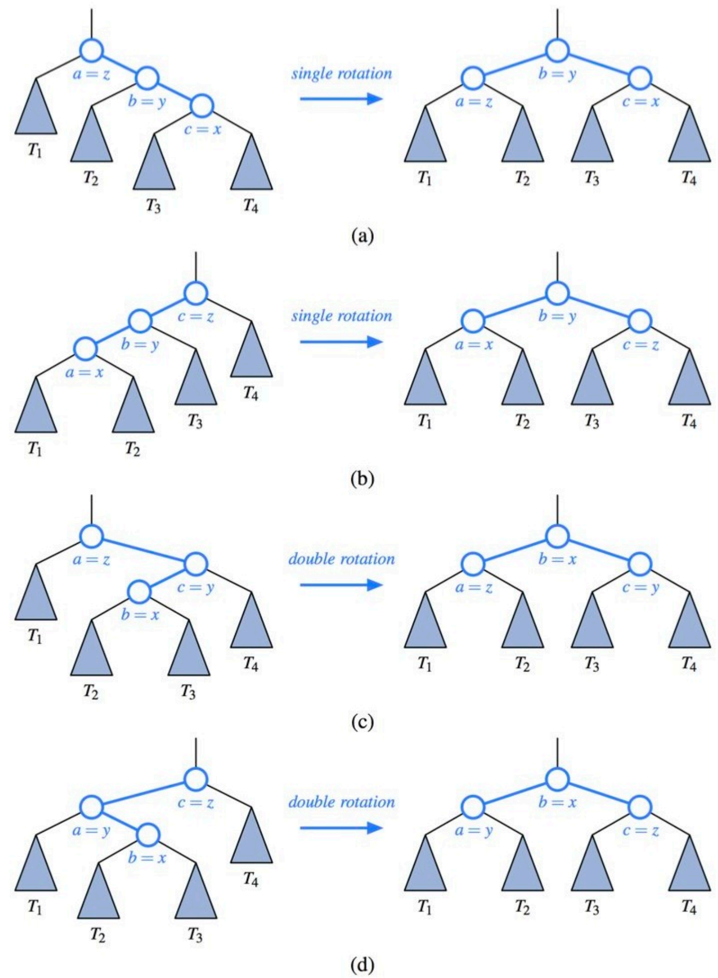


Figure 11.9: Schematic illustration of a trinode restructuring operation: (a and b) require a single rotation; (c and d) require a double rotation.

```
  /** Relinks a parent node with its oriented child node. */
  private void relink(Node<Entry<K,V>> parent, Node<Entry<K,V>> child,
                      boolean makeLeftChild) {
    child.setParent(parent);
    if (makeLeftChild)
      parent.setLeft(child);
    else
      parent.setRight(child);
  }
```

Rotate:

```
  /** Rotates Position p above its parent. */
  public void rotate(Position<Entry<K,V>> p) {
    Node<Entry<K,V>> x = validate(p);
    Node<Entry<K,V>> y = x.getParent();        // we assume this exists
    Node<Entry<K,V>> z = y.getParent();        // grandparent (possibly null)
    if (z == null) {
      root = x;                                // x becomes root of the tree
      x.setParent(null);
    } else
      relink(z, x, y == z.getLeft());          // x becomes direct child of z
    // now rotate x and y, including transfer of middle subtree
    if (x == y.getLeft()) {
      relink(y, x.getRight(), true);           // x's right child becomes y's left
      relink(x, y, false);                     // y becomes x's right child
    } else {
      relink(y, x.getLeft(), false);           // x's left child becomes y's right
      relink(x, y, true);                      // y becomes left child of x
    }
  }
```

Restructure:

```
  /** Performs a trinode restructuring of Position x with its parent/grandparent. */
  public Position<Entry<K,V>> restructure(Position<Entry<K,V>> x) {
    Position<Entry<K,V>> y = parent(x);
    Position<Entry<K,V>> z = parent(y);
    if ((x == right(y)) == (y == right(z))) {  // matching alignments
      rotate(y);                               // single rotation (of y)
      return y;                                // y is new subtree root
    } else {                                   // opposite alignments
      rotate(x);                               // double rotation (of x)
      rotate(x);
      return x;                                // x is new subtree root
    }
  }
}
```

Code Fragment 11.10: The BalanceableBinaryTree class, which is nested within the TreeMap class definition (continued from Code Fragment 11.9).

Rebalancing operation for AVL insertions and deletions:

```
/**
 * Utility used to rebalance after an insert or removal operation. This traverses the
 * path upward from p, performing a trinode restructuring when imbalance is found,
 * continuing until balance is restored.
 */
protected void rebalance(Position<Entry<K,V>> p) {
  int oldHeight, newHeight;
  do {
    oldHeight = height(p);                    // not yet recalculated if internal
    if (!isBalanced(p)) {                     // imbalance detected
      // perform trinode restructuring, setting p to resulting root,
      // and recompute new local heights after the restructuring
      p = restructure(tallerChild(tallerChild(p)));
      recomputeHeight(left(p));
      recomputeHeight(right(p));
    }
    recomputeHeight(p);
    newHeight = height(p);
    p = parent(p);
  } while (oldHeight != newHeight && p != null);
}
/** Overrides the TreeMap rebalancing hook that is called after an insertion. */
protected void rebalanceInsert(Position<Entry<K,V>> p) {
  rebalance(p);
}
/** Overrides the TreeMap rebalancing hook that is called after a deletion. */
protected void rebalanceDelete(Position<Entry<K,V>> p) {
  if (!isRoot(p))
    rebalance(parent(p));
}
}
```

At this point in the class I discuss how AVL trees are implemented in the 6th edition of the textbook by Goodrich, Tamassia and Goldwasser.

Please, refer to pages:

466-470 class TreeMap<K,V>
Note methods: put(K key, V value), remove(K key)

475-478 class BalancedBinaryTree<K,V>
Note: hooks for rebalancing present in TreeMap,
Methods: rotate, restructure

486-487 class AVLTreeMap<K,V>
Note methods: rebalanceInsert, rebalanceDelete, rebalance