Algorithmic Challenges: From Suffix Array to Suffix Tree

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Higher School of Economics

Algorithms on Strings
Data Structures and Algorithms

Outline

1. Suffix Array and Suffix Tree
2. LCP Array Computation
3. Constructing Suffix Tree

Construct suffix Tree

Input: String S

Output: Suffix tree of S

- You already know how to construct suffix tree
- But $O(|S|^2)$ will only work for short strings
- You will learn to build it in $O(|S| \log |S|)$ which enables very long texts!

General Plan

- Construct suffix array in $O(|S| \log |S|)$
- Compute additional information in $O(|S|)$
- Construct suffix tree from suffix array and additional information in $O(|S|)$

Suffix array and suffix tree

$S = ababaa\$$

0 $\$$

1 $a\$$

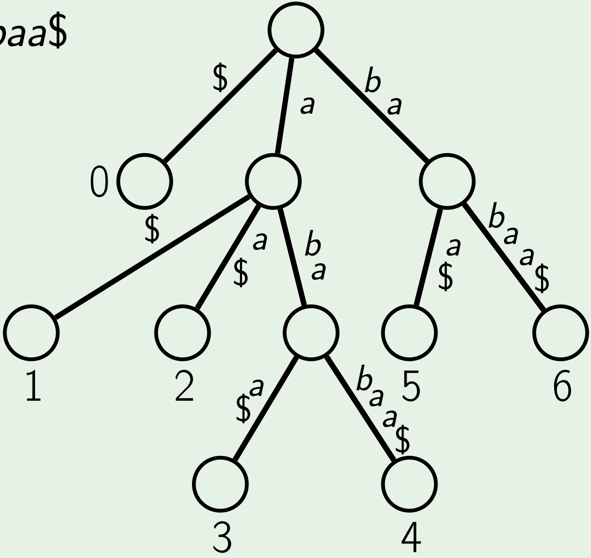
2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$



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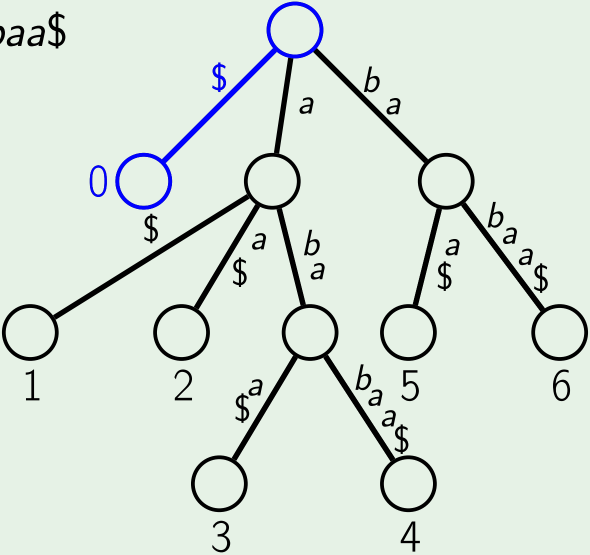

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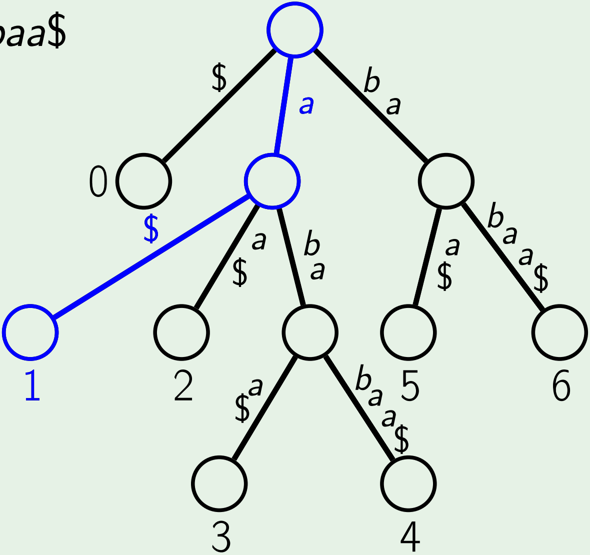

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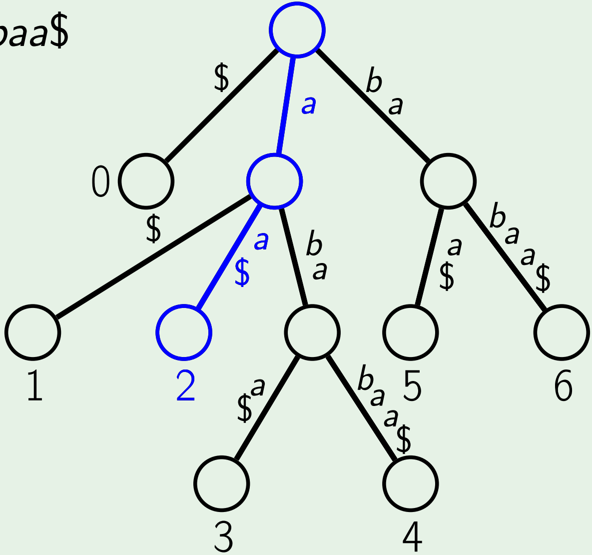

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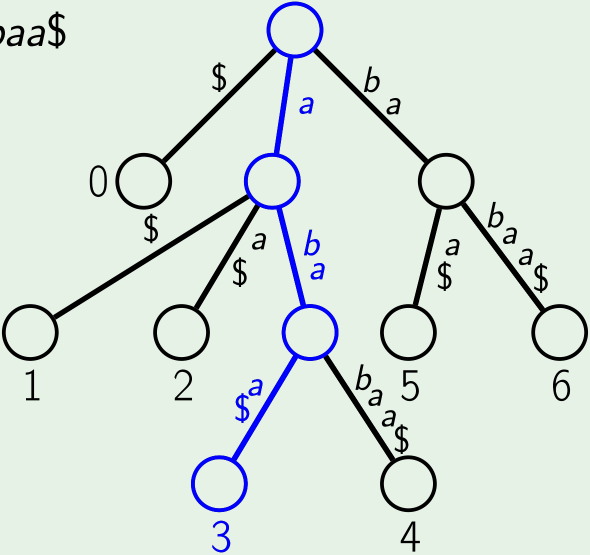

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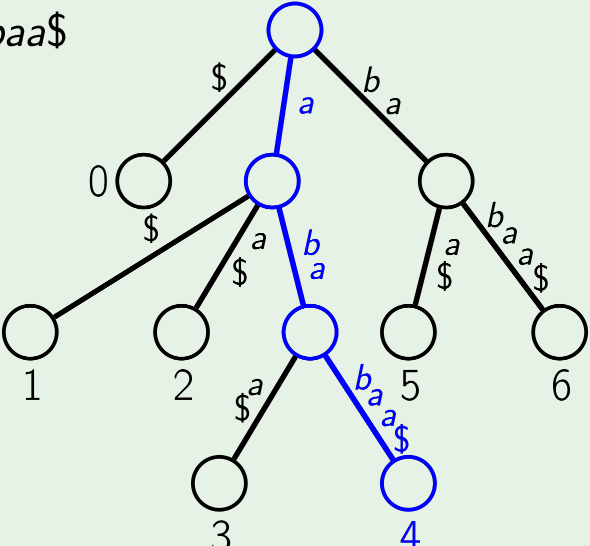

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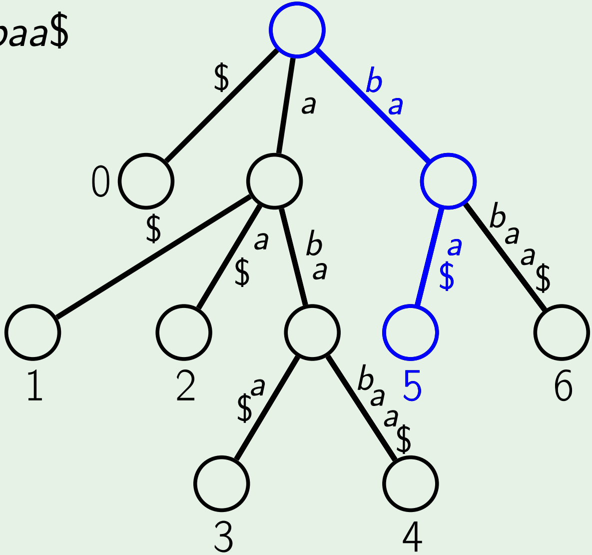

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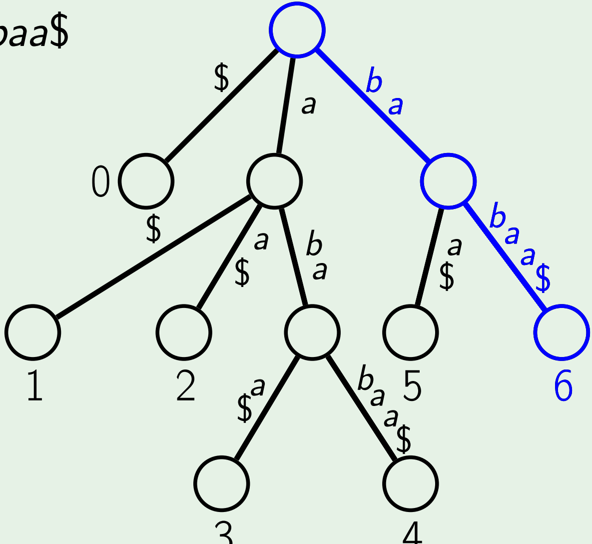

Definition

The longest common prefix (or just "lcp") of two strings S and T is the longest such string u that u is both a prefix of S and T. We denote by $LCP(S, T)$ the length of the "lcp" of S and T.

Example

$LCP(\text{"ababc"}, \text{"abc"}) = 2$
$LCP(\text{"a"}, \text{"b"}) = 0$

Suffix array, suffix tree and lcp

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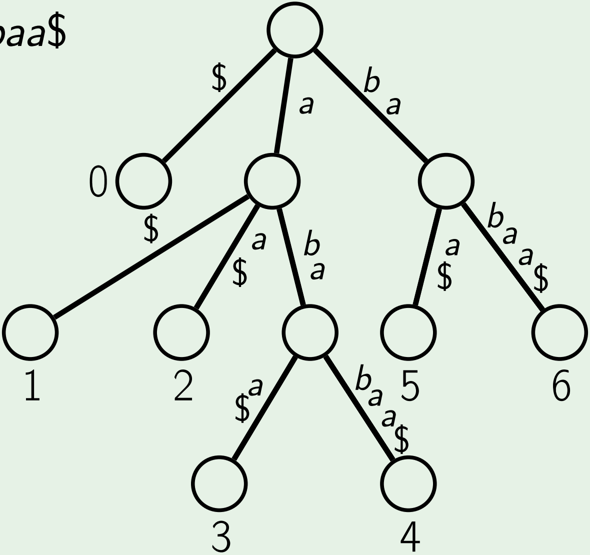

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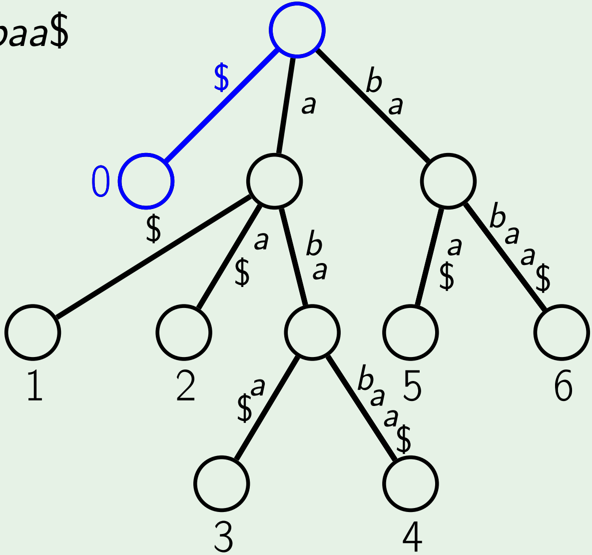

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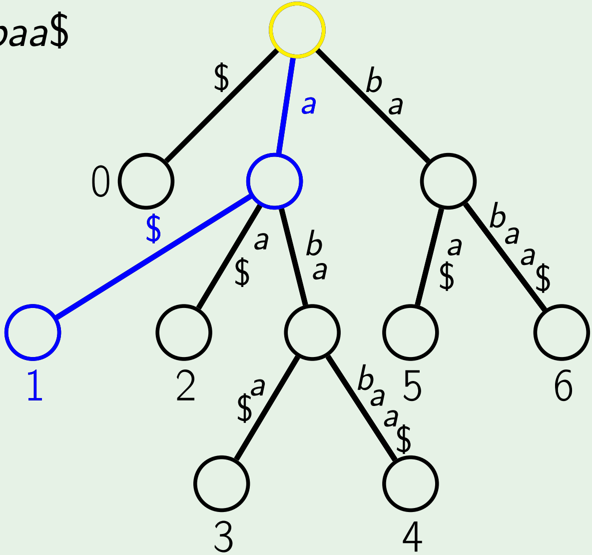

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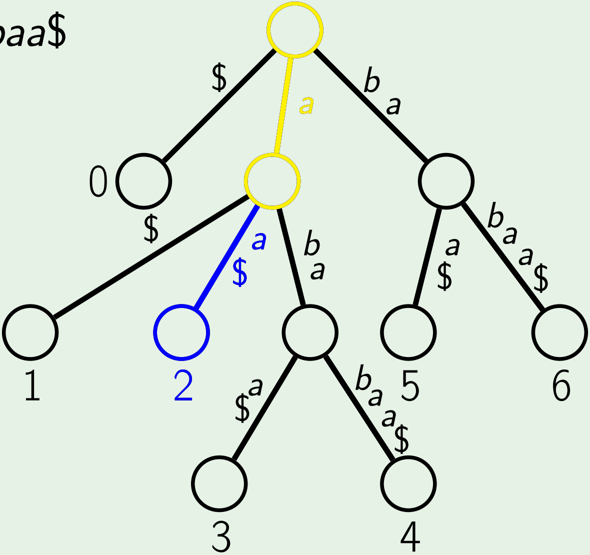

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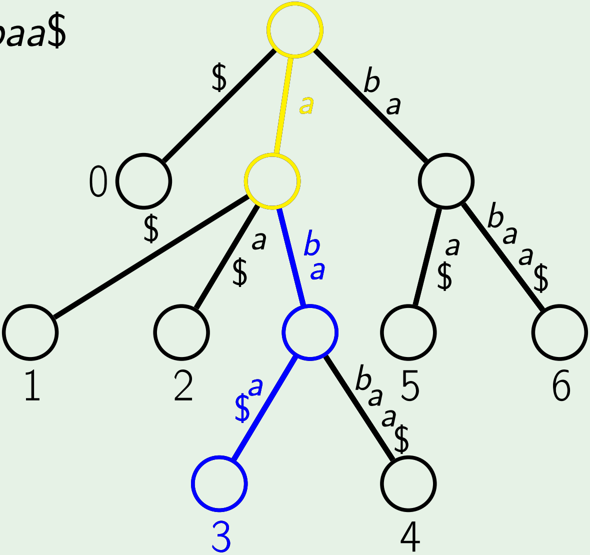

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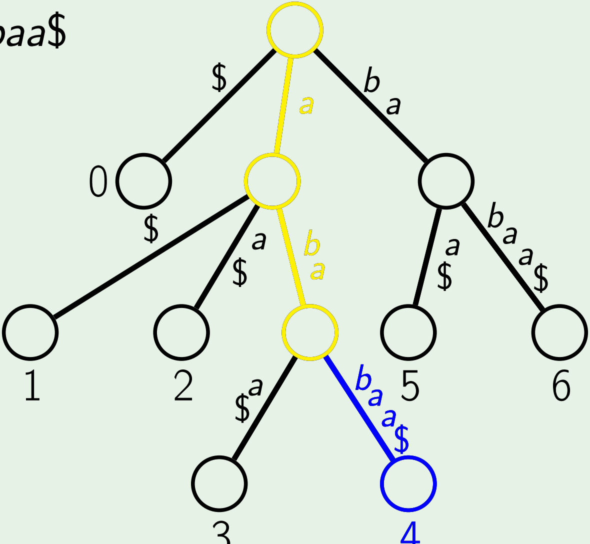

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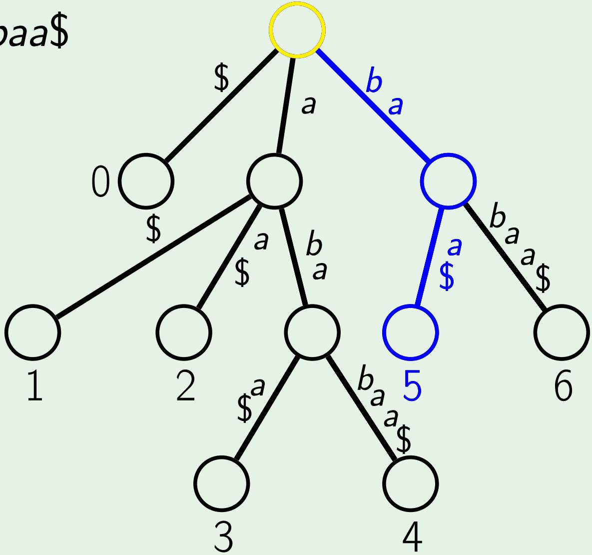

Suffix array, suffix tree and lcp

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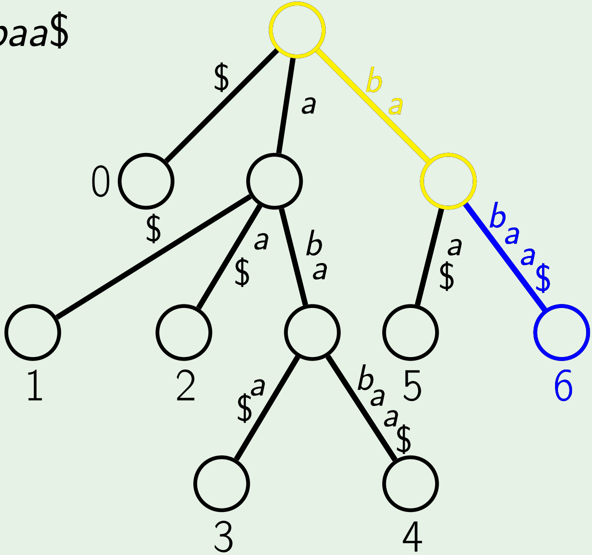

LCP array

Definition

Consider suffix array A of string S in the raw form, that is $A[0] < A[1] < A[2] < \cdots < A[|S| - 1]$ are all the suffixes of S in lexicographic order. LCP array of string S is the array lcp of size $|S| - 1$ such that for each i such that $0 \leq i \leq |S| - 2$,

$$lcp[i] = LCP(A[i], A[i + 1])$$

LCP array

$S = ababaa\$$

0 $\$$

1 $a\$$

2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$

$$lcp = [\ ,\ ,\ ,\ ,\ ,\ ]$$

LCP array

$S = ababaa\$$

0 $\$$

1 $a\$$

2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$

$lcp = [\ ,\ ,\ ,\ ,\ ,\ ]$

LCP array

$S = ababaa\$$

0 $\$$

1 $a\$$

2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$

$lcp = [0, \ , \ , \ , \ , \ ]$

LCP array

$S = ababaa\$$

0 $\$$

1 $a\$$

2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$

$lcp = [0, 1, \ , \ , \ , \ ]$

LCP array

$S = ababaa\$$

0 $\$$

1 $a\$$ $\qquad lcp = [0, 1, 1, \ , \ , \ ]$

2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$

LCP array

$S = ababaa\$$

0 $\$$

1 $a\$$

2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$

$$lcp = [0, 1, 1, 3, \quad , \quad]$$

LCP array

$S = ababaa\$$

0 $\$$

1 $a\$$ $lcp = [0, 1, 1, 3, 0, \ \ ]$

2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$

LCP array

$S = ababaa\$$

0 $\$$

1 $a\$$

2 $aa\$$

3 $abaa\$$

4 $ababaa\$$

5 $baa\$$

6 $babaa\$$

$lcp = [0, 1, 1, 3, 0, 2]$

LCP array property

Lemma

For any $i < j$, $LCP(A[i], A[j]) \leq lcp[i]$ and $LCP(A[i], A[j]) \leq lcp[j-1]$.

Proof

$$\begin{array}{ll} & \dots \\ i & ababababa \\ i+1 & abababc \\ & \dots \\ j & abbcabab \end{array}$$

Proof

	$\ldots$
i	$ababababa$
$i+1$	$abababc$
	$\ldots$
j	$abbcabab$

Proof

	$\ldots$
i	*ababababa*
$i+1$	*xxxxxxxxx*
	$\ldots$
j	*abbcabab*

If $LCP(A[i], A[j]) > LCP(A[i], A[i+1])$

Proof

$$
\begin{array}{ll}
 & \ldots \\
i & ababababa \\
i+1 & xxxxxxxxx \quad k = 1 \\
 & \ldots \\
j & abbcabab
\end{array}
$$

If $LCP(A[i], A[j]) > LCP(A[i], A[i+1])$
Consider $k = LCP(A[i], A[i+1])$

Proof

$\dots$

i $\quad ababababa$

$i+1$ $\quad a\_$ $\qquad k = 1$

$\dots$

j $\quad abbcabab$

If $k = |A[i+1]|$, then $A[i+1] < A[i]$ – contradiction

Proof

$\ldots$

i $\quad abababababa$

$i+1$ $\quad axxxxxxxx \quad k = 1$

$\ldots$

j $\quad abbcabab$

Otherwise $A[j][k] = A[i][k] \neq A[i+1][k]$

Proof

$$\begin{array}{ll}
 & \dots \\
i & ababababa \\
i+1 & acxxxxxxx \quad k = 1 \\
 & \dots \bigvee \\
j & abbcabab
\end{array}$$

If $A[j][k] = A[i][k] < A[i+1][k]$, then $A[j] < A[i+1]$ — contradiction

Proof

$$\begin{array}{ll}
 & \cdots \\
i & ababababa \\
 & \quad\;\; \vee \\
i+1 & aaxxxxxxx \qquad k = 1 \\
 & \cdots \\
j & abbcabab
\end{array}$$

If $A[i][k] > A[i+1][k]$, then $A[i] > A[i+1]$ — contradiction $\square$

Computing LCP array

- For each i, compute $LCP(A[i], A[i+1])$ via comparing $A[i]$ and $A[i+1]$ character-by-character
- $O(|S|)$ for each i, $O(|S|)$ different i — total time $O(|S|^2)$
- How to do this faster?

Outline

1. Suffix Array and Suffix Tree
2. LCP Array Computation
3. Constructing Suffix Tree

Idea

Lemma

Let h be the longest common prefix between S_{i-1} and its adjacent (next) suffix in the suffix array of string S. Then the longest common prefix between S_i and its adjacent (next) suffix in the suffix array is at least $h - 1$.

$S = abracadabra\$$

index	sorted suffix	LCP
$\dots$	$\dots$	$\dots$
$i = 10$	`a$`	
7	`abra$`	
$\dots$	$\dots$	$\dots$
$j = 3$	`acadabra$`	
$\dots$	$\dots$	$\dots$
$i - 1 = 9$	`ra$`	
$j - 1 = 2$	`racadabra$`	

$$S = abracadabra\$$$

index	sorted suffix	LCP
...	...	...
$i = 10$	a$	
7	abra$	
...	...	...
$j = 3$	acadabra$	
...	...	...
$i - 1 = 9$	ra$	$h = 2$
$j - 1 = 2$	racadabra$	

$S = abracadabra\$$

index	sorted suffix	LCP
$\ldots$	$\ldots$	$\ldots$
$i = 10$	`a$`	
7	`abra$`	
$\ldots$	$\ldots$	$\ldots$
$j = 3$	`acadabra$`	
$\ldots$	$\ldots$	$\ldots$
$i - 1 = 9$	`ra$`	$h = 2$
$j - 1 = 2$	`racadabra$`	

$S = abracadabra\$$

index	sorted suffix	LCP
$\dots$	$\dots$	$\dots$
$i = 10$	a$	$1 \geq h - 1$
7	abra$	
$\dots$	$\dots$	$\dots$
$j = 3$	acadabra$	
$\dots$	$\dots$	$\dots$
$i - 1 = 9$	ra$	$h = 2$
$j - 1 = 2$	racadabra$	

Idea

- Start by computing $LCP(A[0], A[1])$ directly
- Instead of computing to $LCP(A[1], A[2])$, move $A[0]$ one position to the right **in the string**, get some $A[k]$ and compute $LCP(A[k], A[k+1])$
- Repeat this until LCP array is fully computed
- Length of the LCP never decreases by more than one each iteration

Notation

- Let $A_{n(i)}$ be the suffix starting in the next position **in the string** after $A[i]$

Example

a	b	a	b	d	a	b	c

- $A[0] =$ "ababdabc", $A[1] =$ "abc"
- Compute $LCP(A[0], A[1]) = 2$ directly
- $LCP(A_{n(0)}, A_{n(1)}) \geq LCP(A[0], A[1]) - 1$
- $A[0] < A[1] \Rightarrow A_{n(0)} < A_{n(1)}$
- LCP of $A_{n(0)}$ with the next **in order** $A[j]$ is also at least $LCP(A[0], A[1]) - 1$

Example

a	b	a	b	d	a	b	c

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- LCP of $A_{n(0)}$ with the next **in order** $A[j]$ is also at least $LCP(A[0], A[1]) - 1$

Example

a	b	a	b	d	a	b	c

- $LCP(A_{n(0)}, A_{n(1)}) \geq LCP(A[0], A[1]) - 1$
- $A[0] < A[1] \Rightarrow A_{n(0)} < A_{n(1)}$
- LCP of $A_{n(0)}$ with the next **in order** $A[j]$ is also at least $LCP(A[0], A[1]) - 1$
- Compute $LCP(A_{n(0)}, A[j])$ directly, but don't compare first $LCP(A[0], A[1]) - 1$ characters: they are equal

Algorithm

- Compute $LCP(A[0], A[1])$ directly, save as *lcp*
- First suffix goes to the next **in the string**
- Second suffix is the next **in the order**
- Compute LCP knowing that first $lcp - 1$ characters are equal, save *lcp*
- Repeat

`LCPOfSuffixes`$(S, i, j, equal)$

```
lcp ← equal
while i + lcp < |S| and j + lcp < |S|:
    if S[i + lcp] == S[j + lcp]:
        lcp ← lcp + 1
    else:
        break
return lcp
```

```
InvertSuffixArray(order)
pos ← array of size |order|
for i from 0 to |pos| − 1:
    pos[order[i]] ← i
return pos
```

ComputeLCPArray(S, $order$)

```
lcpArray ← array of size |S| − 1
lcp ← 0
posInOrder ← InvertSuffixArray(order)
suffix ← order[0]
for i from 0 to |S| − 1:
   orderIndex ← posInOrder[suffix]
   if orderIndex == |S| − 1:
      lcp ← 0
      suffix ← (suffix + 1) mod |S|
      continue
   nextSuffix ← order[orderIndex + 1]
   lcp ← LCPOfSuffixes(S, suffix, nextSuffix, lcp − 1)
   lcpArray[orderIndex] ← lcp
   suffix ← (suffix + 1) mod |S|
return lcpArray
```

Analysis

Lemma

This algorithm computes LCP array in $O(|S|)$

Proof

- Each comparison increases *lcp*
- $lcp \leq |S|$
- Each iteration *lcp* decreases by at most 1
- Number of comparisons is $O(|S|)$ $\square$

Outline

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Building suffix tree

$S = ababaa\$$

6 $\$$

5 $a\$$

4 $aa\$$

2 $abaa\$$

0 $ababaa\$$

3 $baa\$$

1 $babaa\$$

Building suffix tree

$S = ababaa\$$

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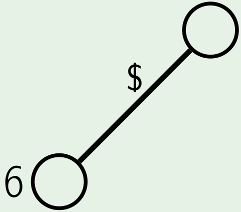

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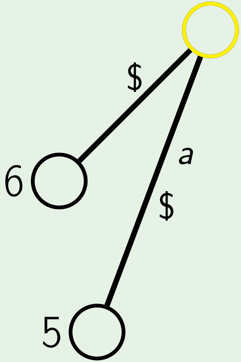

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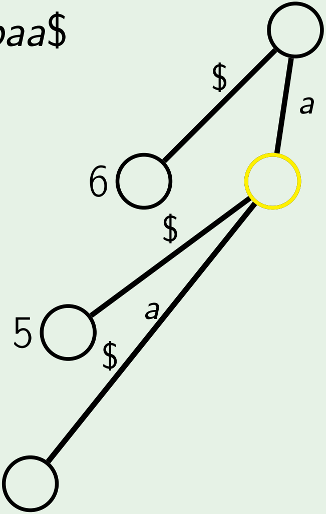

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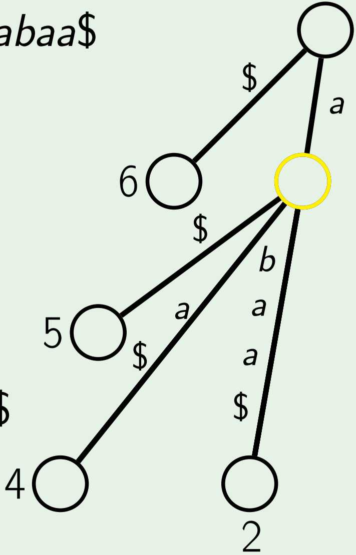

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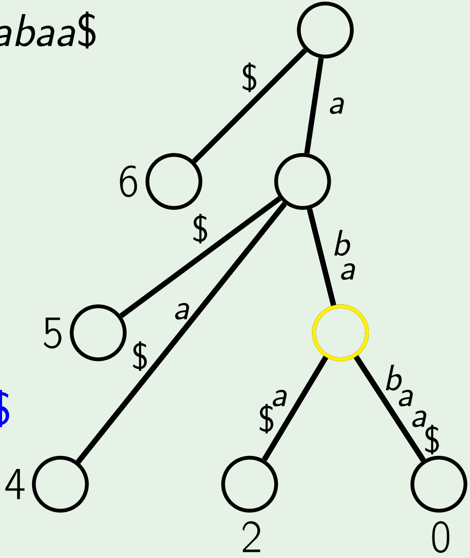

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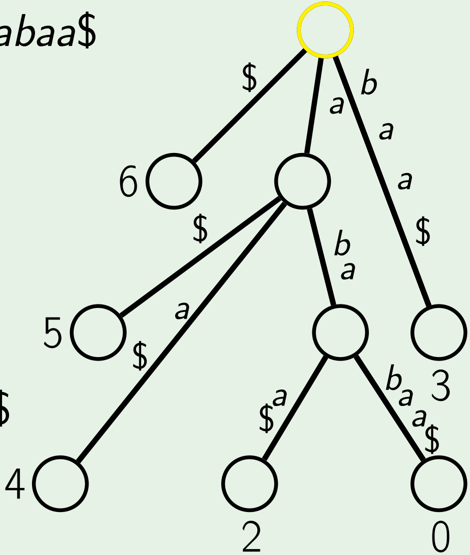

Building suffix tree

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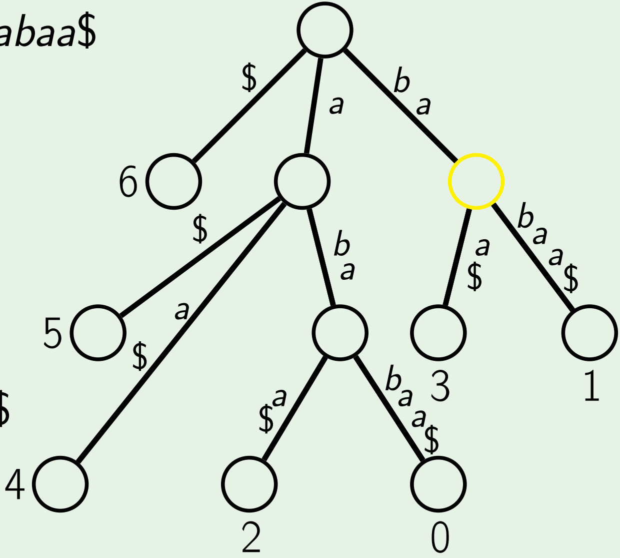

Algorithm

- Build suffix array and LCP array
- Start from only root vertex
- Grow first edge for the first suffix
- For each next suffix, go up from the leaf until *LCP* with previous is below
- Build a new edge for the new suffix

```
class SuffixTreeNode:
SuffixTreeNode parent
Map<char, SuffixTreeNode> children
integer stringDepth
integer edgeStart
integer edgeEnd
```

$\mathtt{STFromSA}(S, order, lcpArray)$

```
root ← new SuffixTreeNode(
   children = {}, parent = nil, stringDepth = 0,
   edgeStart = −1, edgeEnd = −1)
lcpPrev ← 0
curNode ← root
for i from 0 to |S| − 1:
   suffix ← order[i]
   while curNode.stringDepth > lcpPrev:
      curNode ← curNode.parent
   if curNode.stringDepth == lcpPrev:
      curNode ← CreateNewLeaf(curNode, S, suffix)
   else:
      edgeStart ← order[i − 1] + curNode.stringDepth
      offset ← lcpPrev − curNode.stringDepth
      midNode ← BreakEdge(curNode, S, edgeStart, offset)
      curNode ← CreateNewLeaf(midNode, S, suffix)
   if i < |S| − 1:
      lcpPrev ← lcpArray[i]
return root
```

`CreateNewLeaf`($node, S, suffix$)

$leaf \leftarrow$ `new` $SuffixTreeNode($
$\quad children = \{\}$,
$\quad parent = node$,
$\quad stringDepth = |S| - suffix$,
$\quad edgeStart = suffix + node.stringDepth$,
$\quad edgeEnd = |S| - 1)$
$node.children[S[node.edgeStart]] \leftarrow leaf$
`return` $leaf$

BreakEdge($node$, S, $start$, $offset$)

$startChar \leftarrow S[start]$
$midChar \leftarrow S[start + offset]$
$midNode \leftarrow$ `new` $SuffixTreeNode($
  $children = \{\}$,
  $parent = node$,
  $stringDepth = node.stringDepth + offset$,
  $edgeStart = start$,
  $edgeEnd = start + offset - 1)$
$midNode.children[midChar] \leftarrow node.children[startChar]$
$node.children[startChar].parent \leftarrow midNode$
$node.children[startChar].edgeStart \leftarrow start + offset$
$node.children[startChar] \leftarrow midNode$
`return` $midNode$

Analysis

Lemma

This algorithm runs in $O(|S|)$

Proof

- Total number of edges in suffix tree is $O(|S|)$
- For each edge, we go at most once down and at most once up
- Constant time to create a new edge and possibly a new node

Conclusion

- Can build suffix tree from suffix array in linear time
- Can build suffix tree from scratch in time $O(|S| \log |S|)$