Modern Control Theory (EENG 720): Design Project 1

Mikhail Smirnov

Electrical and Computer Engineering
New York Institute of Technology
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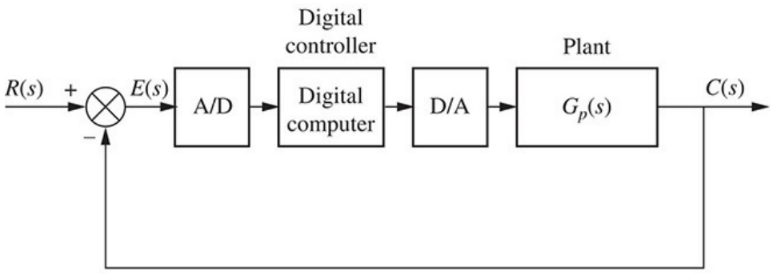

Figure 1: Full Digital Control System

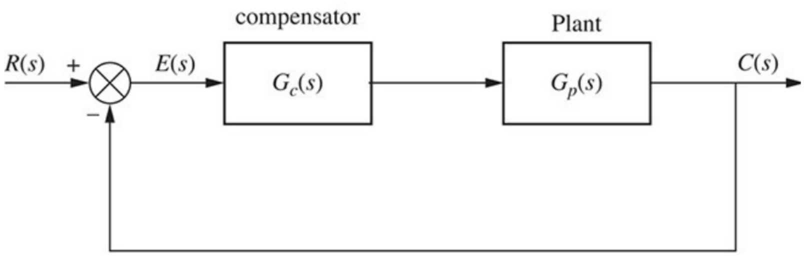

Figure 2: Analog s-domain Design

1 Introduction

In this project, a Proportional Derivative (PD) digital controller, $G_c(z)$ is designed based on the digital control system shown in Fig. 1. The controller is designed to meet two various sets of desired specifications in Section 2.2.1 and Section 2.2.2 based on the maximum percent overshoot ($MP\%$), settling time (t_s) and undamped natural frequency (ω_n). The design is initially modeled in the s-domain analytically (Fig. 2) and is then converted to the z-domain (Fig. 3) through both a zero-order hold (ZOH) and a bilinear (Tustin) transformation via MATLAB. Modeling will be done for three different cases: (i) an uncompensated system (Section 2.1), (ii) a PD Cascade Compensated System with the first set of specifications (Section 2.2.1), and (iii) a PD Cascade Compensated System with the second set of specifications (Section 2.2.2). MATLAB will first be used to obtain the discretized form of the controller and closed-loop transfer functions in Section 2.3. Afterwards, MATLAB will also be used to verify between the analytical and simulated results through simulation in Section 3. The simulation and modeling will be done assuming a unit-step input. While the steady-state error will be 0 for a unit-step input, the steady-state error for a ramp input will vary based on the design and will also be compared. In addition, the results (Section 3) will discuss the difference in the step responses, the difference between the uncompensated system and the two different sets of specifications for the PD controller design, and the difference in the varied parameters. An appendix (Section 4) will be provided to present the code of the MATLAB implementation.

2 Modeling and Analysis

Firstly, the transfer function of the analog plant is given as,

$$G_p(s) = \frac{1}{s(s+1)}. \tag{1}$$

The analysis must be done in the s-domain before designing the controller in the z-domain due to the system requiring analog to digital conversion. Sections 2.1 - 2.2 present the s-domain analysis and Section 2.3 later presents the z-domain analysis.

2.1 Uncompensated

In order to model the system as an uncompensated design, the controller, $G_c(s)$ is eliminated from the system in Fig. 2, leaving the closed loop transfer function in the s-domain to be,

$$G_{cl}(s) = \frac{G_p(s)}{1+G_p(s)}. \tag{2}$$

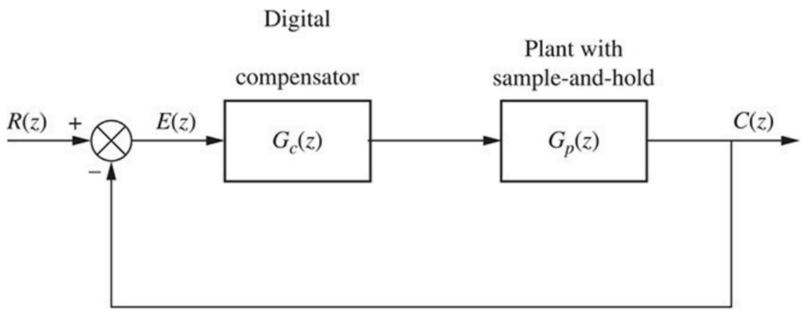

Figure 3: Digital z-domain Design

This results in the closed loop characteristic equation,

$$1 + G_p(s) = 0 \implies 1 + \frac{1}{s(s+1)} = 0. \tag{3}$$

Simplifying reduces the characteristic equation to,

$$s^2 + s + 1 = 0, \tag{4}$$

which is in the general form of the characteristic equation,

$$s^2 + 2\zeta\omega_n s + \omega_n^2, \tag{5}$$

where ζ is the damping ratio and ω_n is the undamped natural frequency. In order to solve for these two parameters, the coefficients of equation (4) and equation (5) are compared.

$$\begin{cases} \omega_n^2 = 1, & (6) \\ 2\zeta\omega_n = 1, & (7) \end{cases}$$

resulting in

$$\begin{cases} \omega_n = 1, & (8) \\ \zeta = \dfrac{1}{2\omega_n}. & (9) \end{cases}$$

Undamped Natural Frequency: $\omega_n = 1$ [rad/s]

Damping Ratio: $\zeta = 0.5$

Next, the remaining specifications of the uncompensated system are calculated,

Settling time: $t_s = \dfrac{4}{\zeta\omega_n} = \dfrac{4}{(0.5)(1)} = 8$ [s]

Rise time: $t_\zeta = \dfrac{1.8}{\omega_n} = \dfrac{1.8}{1} = 1.8$ [s]

Peak time: $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{1\sqrt{1-(0.5)^2}} = 3.63$ [s]

Maximum Percent Overshoot: $MP\% = e^{\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \cdot 100\% = e^{\left(\frac{-\pi(0.5)}{\sqrt{1-(0.5)^2}}\right)} \cdot 100\% = 16.3\%$

Lastly, it is important to mention that since the system described is a Type 1 system (one pole, $s = 0$, lies at the origin in the s-domain), the steady-state error due to a unit step input will always be 0. However for a unit ramp input, a Type 1 system's steady state error will vary based on the design. In the case of this uncompensated system, the steady state error due to a ramp input can be calculated as follows:

$$K_v = \lim_{s \to 0} sG(s) \quad \implies K_v = \lim_{s \to 0} \frac{s}{s(s+1)} = 1$$

Steady State Error (Ramp Input): $e_{ss} = \dfrac{1}{K_v} = 1 \implies e_{ss} = 100\%$

where K_v is the velocity error constant.
The following subsections will perform the same design procedure for a system that is compensated by a PD controller $G_c(s)$ instead, as represented by Fig. 2.

2.2 PD Design

A PD controller in the s-domain is generally represented as

$$G_c(s) = K_p + K_d s, \tag{10}$$

where K_p and K_d are proportional gain and derivative gain parameters, respectively.
The closed loop transfer function of the cascade-compensated analog s-domain design (Fig. 2) is modeled as,

$$G_{cl}(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} \tag{11}$$

This results in the closed loop characteristic equation,

$$1 + G_c(s)G_p(s) = 0 \implies 1 + (K_p + K_d s)\left(\frac{1}{s(s+1)}\right) = 0. \tag{12}$$

Simplifying reduces the characteristic equation to the following,

$$s(s+1) + (K_p + K_d s) = 0$$

$$\implies s^2 + (1 + K_d)s + K_p = 0 \tag{13}$$

Next, the coefficients are compared with the general form of the closed-loop characteristic equation:

$$s^2 + (1 + K_d)s + K_p = s^2 + 2\zeta\omega_n s + \omega_n^2$$

This leads to the following equations for calculating the proportional gain and derivative gain, K_p and K_d, respectively,

$$\begin{cases} K_p = \omega_n^2, & (14) \\ K_d = 2\zeta\omega_n - 1 & (15) \end{cases}$$

After obtaining the expressions for K_p and K_d, the controller, $G_c(s)$ can now be designed according to the specifications. Section 2.2.1 will design a PD controller, $G_{c1}(s)$, with one set of specifications and Section 2.2.2 will design a PD controller, $G_{c2}(s)$, with an alternative set of specifications that is meant to improve the system.

2.2.1 Cascade Compensated Design With First Set of Specifications

The first set of desired design specifications entail:

- A **maximum percent overshoot** of 5% ($MP\% = 5\%$).
- An **undamped natural frequency** of 5 rad/s ($\omega_n = 5$ [rad/s]).

Next, the remaining specifications of the PD-compensated system are calculated. The damping ratio is calculated from the Maximum Percent Overshoot ($MP\%$):

$$MP\% = 0.05 = ^{\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \implies (\ln(0.05)^2 + \pi^2))\zeta^4 + (-2\ln(0.05)^2 - \pi^2)\zeta^2 + \ln(0.05)^2 = 0$$

$$\implies \zeta = 0.6901$$

$$\textbf{Damping Ratio:} \quad \zeta \approx 0.7$$

$$\textbf{Settling time:} \quad t_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.7)(5)} = 1.14 \text{ [s]}$$

$$\textbf{Rise time:} \quad t_\zeta = \frac{1.8}{\omega_n} = \frac{1.8}{5} = 0.36 \text{ [s]}$$

$$\textbf{Peak time:} \quad t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{5\sqrt{1-(0.7)^2}} = 0.88 \text{ [s]}$$

In addition, the proportional and derivative gain values, K_p and K_d, can be calculated from equations (14)-(15).

$$\begin{cases} K_p = \omega_n^2 = 5^2 = 25, & (16) \\ K_d = 2\zeta\omega_n - 1 = 2(0.7)(5) - 1 = 6 & (17) \end{cases}$$

$$\textbf{Proportional Gain:} \quad K_p = 25$$

$$\textbf{Derivative Gain:} \quad K_d = 6$$

Therefore, the analog controller in the s-domain is represented as,

$$G_{c1}(s) = 25 + 6s. \tag{18}$$

As stated in Section 2.1, the steady-state error due to a unit step for a Type 1 system will always be 0. However, in the case of this PD system, the steady state error due to a ramp input can be calculated as follows based on the loop gain, $G_{c1}(s)G_p(s)$:

$$K_v = \lim_{s \to 0} sG_{c1}(s)G_p(s) \quad \implies K_v = \lim_{s \to 0} \frac{s(25+6s)}{s(s+1)} = 25$$

$$\textbf{Steady State Error (Ramp Input):} \quad e_{ss} = \frac{1}{K_v} = 0.04 \implies e_{ss} = 4\%$$

2.2.2 Cascade Compensated Design With Second Set of Specifications

The second set of desired design specifications entail:

- A **settling time** of less than 0.5 seconds ($t_s < 0.5$ [s]).
- An **undamped natural frequency** of 10 rad/s ($\omega_n = 10$ [rad/s]).

Next, the remaining specifications of the PD-compensated system are calculated. The damping ratio is calculated from the desired settling time, (t_s):

$$t_s = \frac{4}{\zeta \omega_n} \implies \zeta = \frac{4}{t_s \omega_n}$$

$$\textbf{Damping Ratio:} \quad \zeta = \frac{4}{(0.5)(10)} = 0.8$$

$$\textbf{Rise time:} \quad t_\zeta = \frac{1.8}{\omega_n} = \frac{1.8}{10} = 0.18 \text{ [s]}$$

$$\textbf{Peak time:} \quad t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-(0.8)^2}} = 0.52 \text{ [s]}$$

$$\textbf{Maximum Percent Overshoot:} \quad MP\% = e^{\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \cdot 100\% = e^{\left(\frac{-\pi(0.8)}{\sqrt{1-(0.8)^2}}\right)} \cdot 100\% = 1.52\%$$

In addition, the proportional and derivative gain values, K_p and K_d, can be calculated from equations (14)-(15).

$$\begin{cases} K_p = \omega_n^2 = 10^2 = 100, & (19) \\ K_d = 2\zeta\omega_n - 1 = 2(0.8)(10) - 1 = 15 & (20) \end{cases}$$

$$\textbf{Proportional Gain:} \quad K_p = 100$$

$$\textbf{Derivative Gain:} \quad K_d = 15$$

Therefore, the analog controller in the s-domain is represented as,

$$G_{c2}(s) = 100 + 15s. \tag{21}$$

As stated in Section 2.1, the steady-state error due to a unit step for a Type 1 system will always be 0. However, in the case of this PD system, the steady state error due to a ramp input can be calculated as follows based on the loop gain, $G_{c2}(s)G_p(s)$:

$$K_v = \lim_{s \to 0} sG_{c2}(s)G_p(s) \quad \implies K_v = \lim_{s \to 0} \frac{s(100+15s)}{s(s+1)} = 100$$

$$\textbf{Steady State Error (Ramp Input):} \quad e_{ss} = \frac{1}{K_v} = 0.01 \implies e_{ss} = 1\%$$

	$G_p(s)$	$G_{c1}(s)G_p(s)$	$G_{c2}(s)G_p(s)$
ζ	0.5	0.7	0.8
ω_n [rad/s]	1	5	10
K_p	–	25	100
K_d	–	6	15
t_s [s]	8	1.14	0.5
t_p [s]	3.63	0.88	0.52
t_ζ [s]	1.8	0.36	0.18
$MP\%$	16.3%	5%	1.52%
e_{ss} (ramp)	100%	4%	1%

Table 1: Calculated Values of System Specifications

2.3 Discretization via MATLAB

After computing the closed loop transfer functions for the uncompensated system and for the cascade-compensated systems with the two different specifications in the s-domain, the transfer functions are discretized and converted to the z-domain via MATLAB. The full MATLAB code is provided in the appendix in Section 4. The sampling frequency was chosen to be $\omega_s = 100$ [rad/s], or a period of $T = 0.0628$s.

To begin, **the transfer function of the plant**, $G_p(s)$ is discretized using ZOH (zero-order hold):

$$G_p(s) = \frac{1}{s(s+1)} \xrightarrow{ZOH} G_p(z) = \frac{0.001933z + 0.001893}{z^2 - 1.939z + 0.9391}. \tag{22}$$

Firstly, the **closed-loop transfer function of the uncompensated system** was discretized by discretizing $G_p(s)$ using ZOH (zero-order-hold):

$$G_{cl}(s) = \frac{1}{s^2 + s + 1} \xrightarrow{ZOH} G_{cl}(z) = \frac{0.001933z + 0.001893}{z^2 - 1.937z + 0.941}. \tag{23}$$

Next, the **controller for the first set of specifications** ($K_p = 25$, $K_d = 6$) was discretized using the Bilinear (Tustin) method:

$$G_{c1}(s) = 25 + 6s \xrightarrow{Tustin} G_{c1}(z) = \frac{216z - 166}{z+1}. \tag{24}$$

The **discretized closed loop transfer function for the first set of specifications** then becomes:

$$G_{cl}(s) = \frac{6s + 25}{s^2 + 7s + 25} \to G_{cl}(z) = \frac{0.4175z^2 + 0.08801z - 0.3142}{z^3 - 0.5216z^2 - 0.912z + 0.6249}. \tag{25}$$

Lastly, the **controller for the second set of specifications** ($K_p = 100$, $K_d = 15$) was discretized using the Bilinear (Tustin) method. (Note: The sampling frequency was chosen to be $\omega_s = 200$ [rad/s], or a period of $T = 0.0314$s.)

$$G_{c2}(s) = 100 + 15s \xrightarrow{Tustin} G_{c2}(z) = \frac{1055z - 854.9}{z+1}. \tag{26}$$

The **discretized closed loop transfer function for the second set of specifications** then becomes:

$$G_{cl}(s) = \frac{15s + 100}{s^2 + 16s + 100} \to G_{cl}(z) = \frac{0.5152z^2 + 0.0923z - 0.4132}{z^3 - 0.4539z^2 - 0.9077z + 0.5559}. \tag{27}$$

3 Results and Discussion

After obtaining the equations for the digital controllers along with the discretized closed-loop transfer functions in the previous section, the closed loop system is simulated in MATLAB with a unit step input. The step responses are shown in the following figures, Figs. 4 and 5.

It is clear by all four plots that the PD compensated design is an improvement over the uncompensated system based on the peak time and rise time both being much shorter and the peak occuring much sooner. In addition, the step responses of the PD compensated designs begin approaching steady-state much sooner due to the increased damping ratio contributing to much more damped oscillations. When comparing the plots between the first PD system with the second PD system, it is clear that the second PD system has an improvement in its step response. Not only is the peak occuring sooner (due to the shorter peak and rise times), but the peak is much sharper and the response starts approaching steady-state much sooner. Lastly, it is clear that discretization produces small fluctuations and inaccuracies compared to a smooth continuous analog signal. However, with the sampling periods that were chosen, these types of results are to be expected in practice when designing a digital control system with hardware limitations and a limited maximum sampling rate.

Aside from the simulated step response plots, it is worth to also examine each of the calculated specifications of all three systems. Table 1 provides an overview of every calculated parameter for the uncompensated system and the two PD cascade-compensated systems. Since the values that are being varied in the design are the damping ratio, ζ, the undamped natural frequency, ω_n, the proportional gain, K_p, and the derivative gain, K_d, it is important to understand how they affect the overall system based on the settling time, t_s, peak time, t_p, rise time, t_ζ, the maximum percent overshoot, $MP\%$ and the steady-state error due a unit ramp input, e_{ss} (ramp). First, since the damping ratio affects the rate at which the response dampens, the settling time and peak time both greatly decrease compared to the uncompensated system. The maximum percent overshoot is also reduced due to an increasing damping ratio. The undamped natural frequency contributes to the rise time decreasing, as well as to the peak time decreasing. Lastly, the proportional and derivative gain contribute to producing a much more reliable system in which the maximum percent overshoot is decreased and the ramp steady-state error is greatly reduced. While having a higher value of K_p and K_d allows the values to provide more contribution towards reducing error and overshoot, implementing the controllers in practice can result in more difficulty. Overall, this analysis and simulation proves how much of a positive effect a controller can have on the system, both in an analog and in a digital setting.

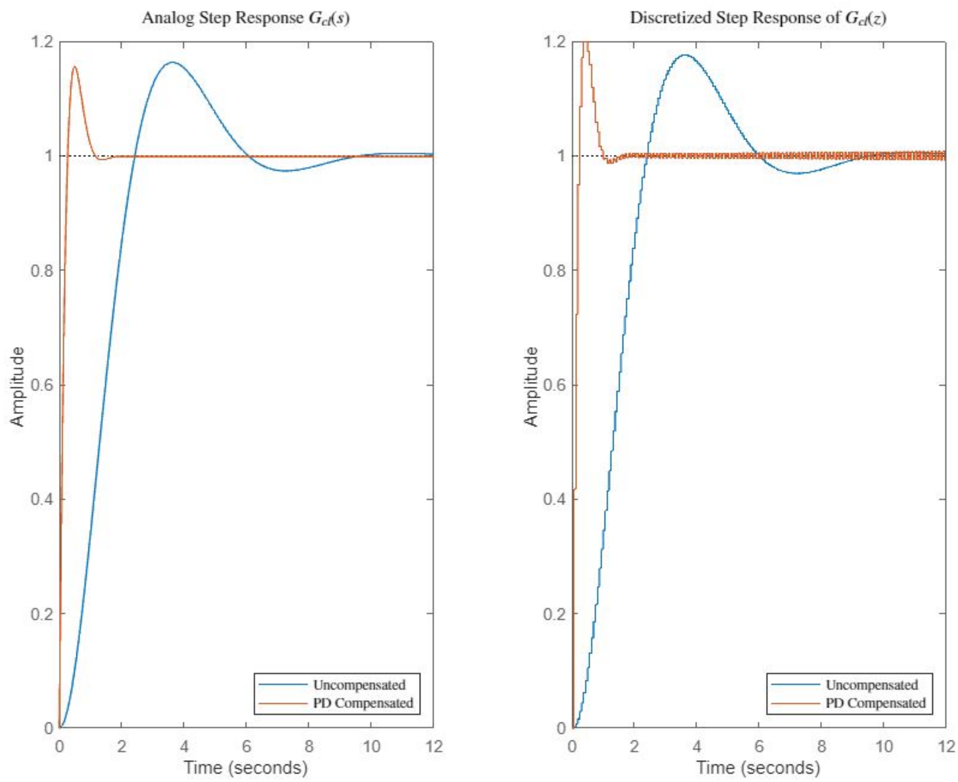

Figure 4: Analog and Discretized Step Responses of Uncompensated System and First PD System

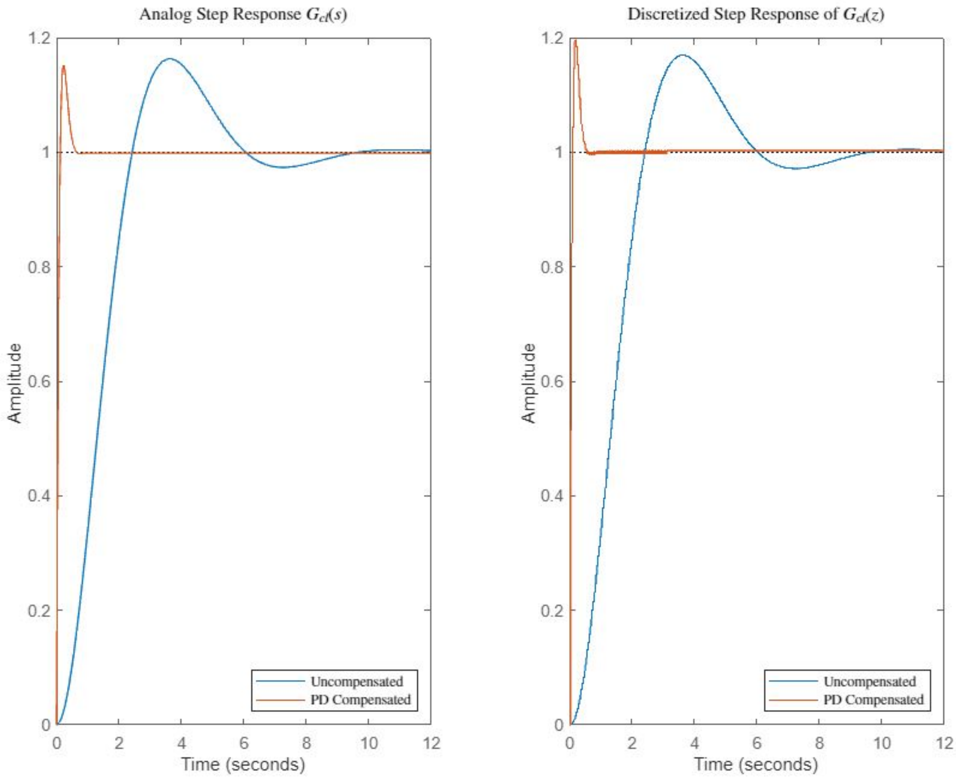

Figure 5: Analog and Discretized Step Responses of Uncompensated System and Second PD System

4 Appendix

4.1 Part 1 MATLAB Code (PD Design With First Set of Specifications)

```
% Mikhail Smirnov EENG720: Modern Control Theory
% Design Project #1: PD Controller

% Simulate closed loop: Gcl(s) = Gc(s) Gp(s) / (1 + Gc(s) Gp(s))
% Gp(s) = 1 / (s^2 + s)
Gp = tf(1, [1 1 0]); % Plant transfer function in s-domain

% Uncompensated closed-loop transfer function in s-domain
Gcl_uncomp = feedback(Gp,1)

% Part 1: Proportional and derivative gain values
Kp1 = 25;
Kd1 = 6;

% PD Controller transfer function in s-domain ( = Kp + Kd s)
Gc = tf([Kd1 Kp1], 1);
% Loop gain: L(s) = Gc(s) * Gp(s)
L = Gc * Gp;
% Closed loop transfer function with PD in s-domain
Gcl = feedback(L,1)

% Discretized
omega_s = 100; % Sampling frequency
T = (2 * pi)/omega_s % Sampling period
% G(s) ----ZOH----> G(z)
Gz_zoh = c2d(Gp, T, 'zoh') % Discretized plant
% C(s) ----tustin----> C(z)
Cz_tustin = c2d(Gc, T, 'tustin') % Discretized controller
Lz = Gz_zoh * Cz_tustin; % Discretized loop gain

% Discretized Closed Loop Transfer Function of Uncompensated
Gcl_uncomp_z = feedback(Gz_zoh, 1)
% Discretized Closed Loop Transfer Function with PD
Gcl_z = feedback(Lz, 1)
% PLOTTING ----------------------------------------------------------
% Simulate Gcl(s)
figure(1)
subplot(1,2,1)
hold on
step(Gcl_uncomp) % Step response of uncompensated
step(Gcl) % Step response of Part 1 PD
title("Analog Step Response $G_{cl}(s)$", 'Interpreter', 'latex')
legend({"Uncompensated", "PD Compensated"}, 'Interpreter' ,'latex', ...
    'Location', 'southeast')
hold off

% Simulate Gcl(z)
subplot(1,2,2)
hold on
step(Gcl_uncomp_z)
step(Gcl_z)
title("Discretized Step Response of $G_{cl}(z)$", 'Interpreter','latex')
legend({"Uncompensated", "PD Compensated"}, 'Interpreter' ,'latex', ...
    'Location', 'southeast')
hold off
xlim([0 12])
ylim([0 1.2])
```

4.2 Part 2 MATLAB Code (PD Design With Second Set of Specifications)

```
% Mikhail Smirnov EENG720: Modern Control Theory
% Design Project #1: PD Controller

% Simulate closed loop: Gcl(s) = Gc(s) Gp(s) / (1 + Gc(s) Gp(s))
% Gp(s) = 1 / (s^2 + s)
Gp = tf(1, [1 1 0]); % Plant transfer function in s-domain

% Uncompensated closed-loop transfer function in s-domain
Gcl_uncomp = feedback(Gp,1)

% Part 1: Proportional and derivative gain values
Kp1 = 25;
Kd1 = 6;

% PD Controller transfer function in s-domain ( = Kp + Kd s)
Gc = tf([Kd1 Kp1], 1);
% Loop gain: L(s) = Gc(s) * Gp(s)
L = Gc * Gp;
% Closed loop transfer function with PD in s-domain
Gcl = feedback(L,1)

% Discretized
omega_s = 100; % Sampling frequency
T = (2 * pi)/omega_s % Sampling period
% G(s) ----ZOH----> G(z)
Gz_zoh = c2d(Gp, T, 'zoh') % Discretized plant
% C(s) ----tustin----> C(z)
Cz_tustin = c2d(Gc, T, 'tustin') % Discretized controller
Lz = Gz_zoh * Cz_tustin; % Discretized loop gain

% Discretized Closed Loop Transfer Function of Uncompensated
Gcl_uncomp_z = feedback(Gz_zoh, 1)
% Discretized Closed Loop Transfer Function with PD
Gcl_z = feedback(Lz, 1)
% PLOTTING ---------------------------------------------------------------
% Simulate Gcl(s)
figure(1)
subplot(1,2,1)
hold on
step(Gcl_uncomp) % Step response of uncompensated
step(Gcl) % Step response of Part 1 PD
title("Analog Step Response $G_{cl}(s)$", 'Interpreter', 'latex')
legend({"Uncompensated", "PD Compensated"}, 'Interpreter' ,'latex', ...
    'Location', 'southeast')
hold off

% Simulate Gcl(z)
subplot(1,2,2)
hold on
step(Gcl_uncomp_z)
step(Gcl_z)
title("Discretized Step Response of $G_{cl}(z)$", 'Interpreter','latex')
legend({"Uncompensated", "PD Compensated"}, 'Interpreter' ,'latex', ...
    'Location', 'southeast')
hold off
xlim([0 12])
ylim([0 1.2])
```