Heat transfer simulations using discrete random processes

Stanislavs Dubrovskis, Toms Ozoliņš, Eduard Mrug, Markuss G. Ķēniņš

Team 13, supervisor Ena Salihovic

University of Groningen

Introduction

The experiment was conducted to see, whether it is possible to simulate heat flow with macroscopic processes. The inspiration for this experiment was taken from the 2016 International Physics Olympiad [1], in which phase transition was modelled using particles jumping over a wall between two bins on a speaker based on probability. This experiment used six instead of two compartments and looked at heat transfer along a 1D rod rather than binary phase transition.

Research questions:

1. How well do the discrete dynamics—distribution of oscillating particles into compartments—describe the continuous model of heat flow?
2. What is the relation between the equivalent thermal conductivity and the amplitude of the oscillating table?

Hypoteses:

1. There will be an analogy between the statistical mechanical model of heat transfer and the spread of particles used in the experiment.
2. Increasing the amplitude of the vibrating table will increase the rate at which the particles spread from the initial configuration. This is analogous to increasing the conductivity constant and observing quicker spread of heat quanta.

Theory

The predictions from continuous heat conduction are treated in the theory. Consider a rod of length L with hot and cold ends with temperature T_1 and T_2, respectively. The temperature $T(x,t)$ at point x with respect to time t is given by the heat equation [2] as follows

$$\frac{\partial T}{\partial t}(x,t) = \frac{k}{\sigma\delta}\frac{\partial^2 T}{\partial x^2}(x,t), \tag{1}$$

where k—coefficient of thermal conductivity, σ—specific heat, and δ—mass density. The coefficient of proportionality is

$$D = \frac{k}{\sigma\delta} \tag{2}$$

called the thermal diffusivity, which describes the rate of temperature distribution throughout the material.

The only time-independent solution was found to be (with $0 \le x \le L$)

$$T(x) = T_1 - \frac{T_1 - T_2}{L}x, \tag{3}$$

called the steady-state solution. Experimentally, the particles leaving the cold end—steady-state heat flux $\dot{q}$—may be measured, which is given by

$$\dot{q} = -k\frac{T_2 - T_1}{L}. \tag{4}$$

Note that the proportionality constant is the coefficient of thermal conductivity k $[\mathrm{W\,m^{-1}\,K^{-1}}]$.

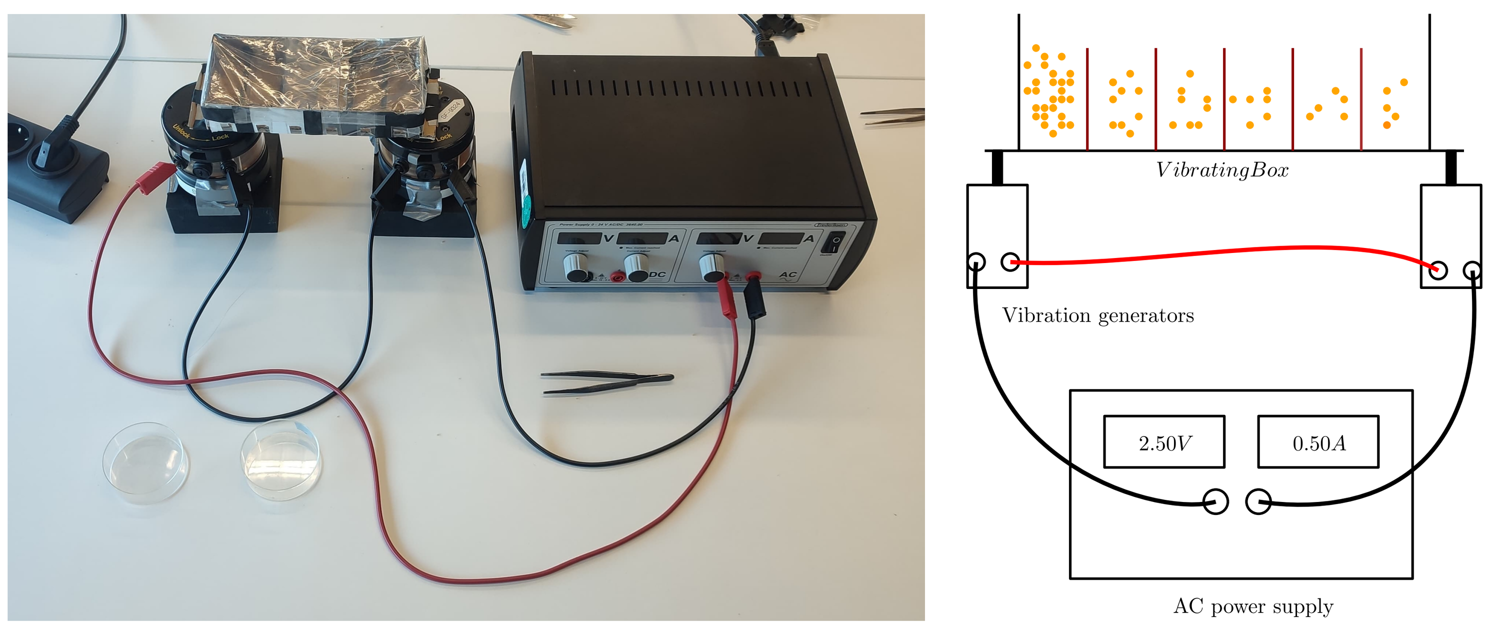
The time-dependent solution was computed (cf. [3, 4]), and then discretised over six bins, as in the experiment. To simplify calculations, arbitrary units were chosen, namely, $L = 1$, $T_2 = 1$, $T_1 = 0$ arb. units; then the temperature (0 to 1) in the j-th bin at time t was found to be given by

$$Q_j(t) = \frac{1}{72}(13 - 2j) - \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left[1 + 11\cos\frac{n\pi}{6}\right]\sin\left[\frac{n\pi}{12}(2j-1)\right]\sin\frac{n\pi}{12}e^{-n^2\pi^2 Dt}, \tag{5}$$

where the only parameter is the coefficient of thermal diffusivity D $[\mathrm{m^2\,s^{-1}}]$. Fitting Eq. (5) to the experimentally obtained normalised distribution, the parameter D may be determined.

Methods

The actual set-up can be seen in Fig. 1a. Two vibration generators were connected to a box of six compartments and wired in series with an AC power supply, shown in Fig. 1b. The particles in use were couscous grains.



(a) (b)

Figure 1: The setup and scheme of the experiment. Besides the depicted materials, a scale with a precision of 0.01 g, and a brush were also used.

The measurement procedure was as follows: a certain amount of particles was added in the first compartment, the power supply was turned on for 5–20 s and stopped, the particles in each compartment were weighed/counted and returned. The last compartment was emptied and the first was refilled to the original amount in order to impose the boundary conditions required to reach the stable time-independent distribution. This was repeated until a steady flux—the number of particles removed from the system. The measurements were conducted for the following current amplitudes: $(0.50, 0.60, 0.70, 0.75, 0.80, 0.90, 0.98 \pm 0.01)$ A.

References

[1] C. Aegerter and A. Kish, "47th International Physics Olympiad E-2: Jumping beads—a model for phase transitions and instabilities," 2016.

[2] S. Robinson, "Derivation of the heat equation," 2009, lecture notes MATH/PHYS 4530, University of Michigan.

[3] K. R. Hiremath, "Partial differential equations," 2021, lecture notes, Indian Institute of Technology, Jodhpur.

[4] R. Daileda, "Partial differential equations," 2017, lecture notes, Trinity University.

Results

Where impossible by counting, the number of particles was found by weighing and using a conversion factor of (5.16 ± 0.03) particles per 0.01 g obtained from a larger sample. The following time evolution data was determined for different current amplitudes:

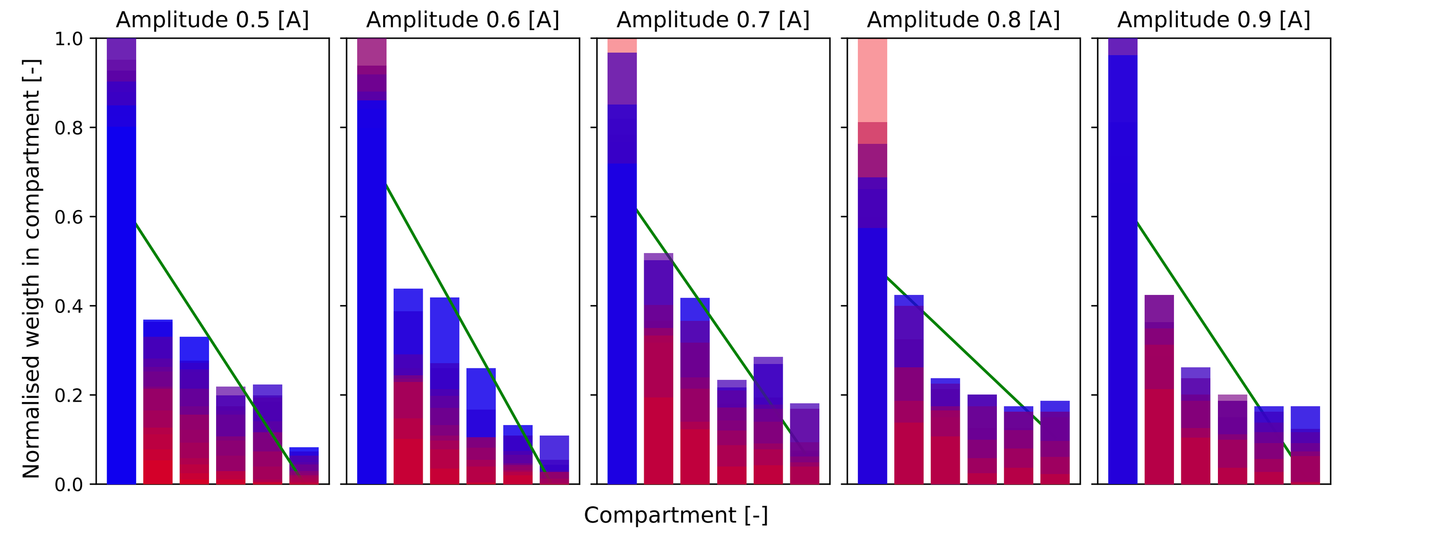


Figure 2: Some normalised weight time evolutions. Time increases from red to blue. Steady-state linear fits in green.

To determine the relation between k and the current amplitude, the steady state particle flux was used: $\dot{q} = \#\text{particles}/\text{time}$. The graph on the left shows the data with the first compartment always held at $m_1 = (0.80 \pm 0.01)$ g, while the one on the right depicts measurements with weight normalised by varying m_1 to avoid a liquid-like phase transition.

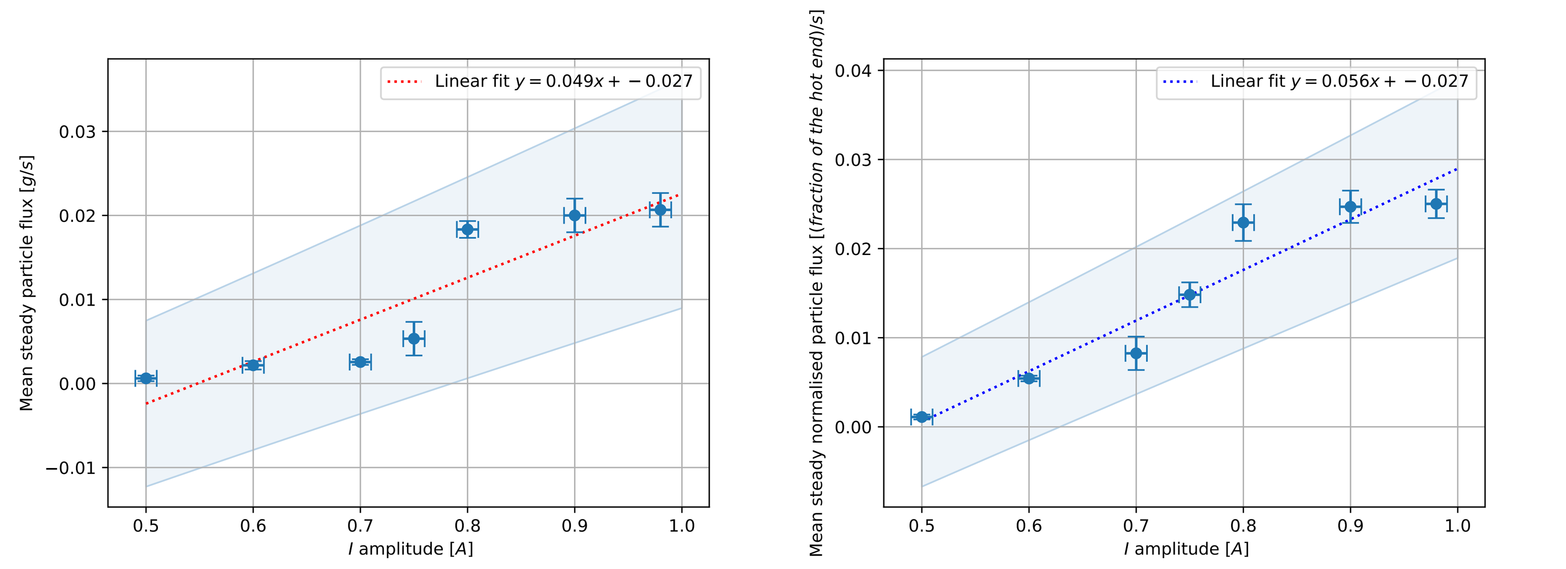


Figure 3: Steady particle flow versus current I. Original (left $a = (0.05 \pm 0.03)$ g/sA) Updated, normalised (right $a_{norm} = (0.06 \pm 0.01)$ 1/sA)

Another parameter is the time to reach the steady state. In the following graph, the range of times declared steady is plotted against the current amplitude of the oscillators. The best-fit curve is an exponential decay with $a = (40 \pm 10)$ s:

$$t_{\text{steady}} = a + Ce^{b \cdot x} \tag{6}$$

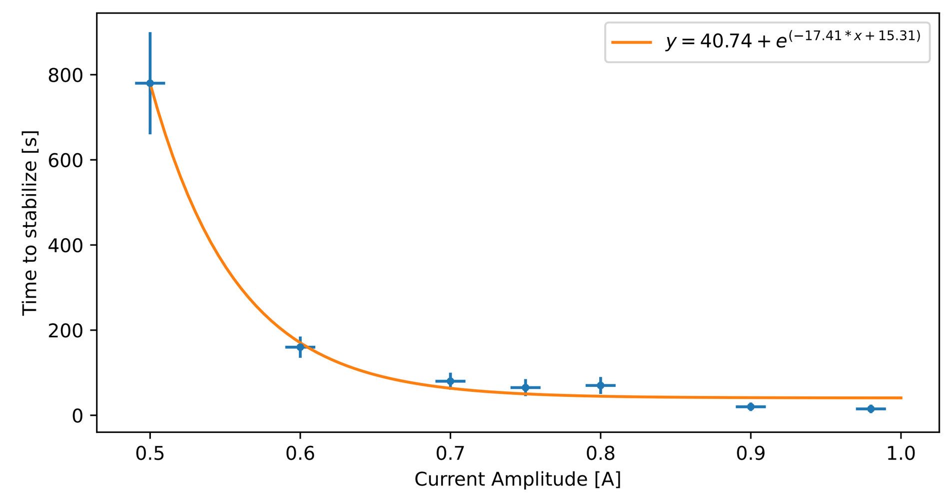


Figure 4: The time it takes to reach the steady state as a function of the current amplitude.

Finally, the system's coefficient of thermal diffusivity for each amplitude was determined by fitting the entire time evolution of the particle distribution. The following graph once again shows that particles dissipate faster with higher amplitudes.

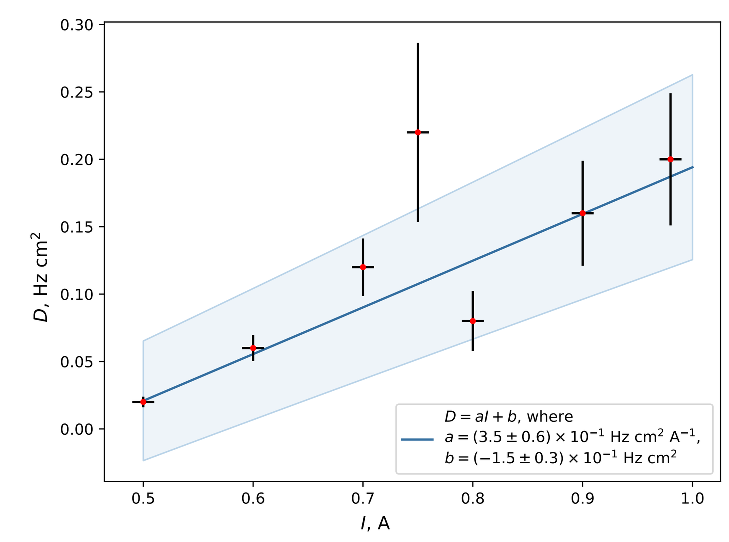


Figure 5: Best-fitted thermal diffusivity D depending on the current I. $D = 0\,\mathrm{cm^2/s}$ at $I_{\text{crit}} = (0.4 \pm 0.2)$ A.

Discussion and conclusions

Heating. Inability to fit the theoretical prediction $Q_k(t)$ is described as homogeneous heating—explained by the energy barrier of the first wall: once crossed, the bins are empty and thus equivalent, effects from interaction (as predicted) are only observed with enough particles. The x-intercept ($D = 0$) current $I_{\text{crit}} = (0.4 \pm 0.2)$ A is explained as the critical current to cross the first wall.

Linearity. The system approached steady-state linearity. The time required to reach it is exponentially decaying with respect to current with (40 ± 20) s offset—time required even for very large amplitudes; it is explained as the vibrations primarily change the z-direction velocity and particles must cross each wall separately.

Conductivity. The effective conductivity was linear with the current after normalisation.

Errors. Errors mostly consisted of liquefaction, despite trying to avoid it by changing the number of particles followed by normalisation. Other errors arose from imperfectly maintained boundary conditions and equipment precision.

Conclusions. Research questions were answered, and hypotheses were confirmed:

1. (1st RQ) The distribution describes continuous heat conduction with a homogeneous heating term. Thus an analogy exists, confirming the hypothesis.
2. (2nd RQ) The effective coefficient of thermal conductivity is proportional to the amplitude of vibrations. The spread is thus faster, confirming the second hypothesis.
3. A lower bound exists for the time required to reach an approximate steady state for any amplitude.
4. Liquefaction was observed with too high a number of particles.