Section 2 homework

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June 2, 2024

Exercises for Section 2.1

A. Write each of the following sets by listing their elements between braces.

1. $\{5x-1 : x \in \mathbb{Z}\} = \{\ldots, -11, -6, -1, 4, 9, \ldots\}$
2. $\{3x+2 : x \in \mathbb{Z}\} = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$
3. $\{x \in \mathbb{Z} : -2 \leq x < 7\} = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
4. $\{x \in \mathbb{N} : -2 < x \leq 7\} = \{1, 2, 3, 4, 5, 6, 7\}$
5. $\{x \in \mathbb{R} : x^2 = 3\} = \{-\sqrt{3}, \sqrt{3}\}$
6. $\{x \in \mathbb{R} : x^2 = 9\} = \{-3, 3\}$
7. $\{x \in \mathbb{R} : x^2 + 5x = -6\} = \{-3, -2\}$

11 $\{x \in \mathbb{Z} : |x| < 5\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
12 $\{x \in \mathbb{Z} : |2x| < 5\} = \{-2, -1, 0, 1, 2\}$
13 $\{x \in \mathbb{Z} : |6x| < 5\} = \{0\}$
14 $\{5x : x \in \mathbb{Z}, |2x| \leq 8\} = \{-20, -15, -10, -5, 0, 5, 10, 15, 20\}$

B. Write each of the following sets in set-builder notation.

17 $\{2, 4, 8, 16, 32, 64 \ldots\} = \{2^x : x > 0, x \in \mathbb{Z}\}$
19 $\{\ldots, -6, -3, 0, 3, 6, 9, 12, 15, \ldots\} = \{3x : x \in \mathbb{Z}\}$
24 $\{-4, -3, -2, -1, -0, 1, 2\} = \{x : -4 \leq x \leq 2, x \in \mathbb{Z}\}$
25 $\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\} = \{2^x : x \in \mathbb{Z}\}$
26 $\{\ldots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, \ldots\} = \{3^x : x \in \mathbb{Z}\}$

C. Find the following cardinalities of the following sets.

29 $|\{\{1\}, \{2, \{3, 4\}\}, \phi\}| = 3$
30 $|\{\{1, 4\}, a, b, \{\{3, 4\}\}, \{\phi\}\}| = 5$
31 $|\{\{\{1\}, \{2, \{3, 4\}\}, \phi\}\}| = 1$
32 $|\{\{\{1, 4\}, a, b, \{\{3, 4\}\}, \{\phi\}\}\}| = 1$
33 $|\{x \in \mathbb{Z} : |x| < 10\}| = 19$
34 $|\{x \in \mathbb{N} : |x| < 10\}| = 9$
35 $|\{x \in \mathbb{Z} : x^2 < 10\}| = 7$
36 $|\{x \in \mathbb{N} : x^2 < 10\}| = 3$
37 $|\{x \in \mathbb{N} : x^2 < 0\}| = 0$
38 $|\{x \in \mathbb{N} : 5x \leq 20\}| = 4$

Exercises for Section 2.2

A Write out the indicated sets by listing their elements between braces.

2 Suppose $A = \{\pi, e, 0\}$ and $B = \{0, 1\}$.

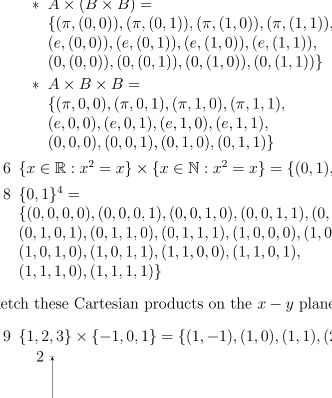
* $A \times B = \{(\pi, 0), (\pi, 1), (e, 0), (e, 1), (0, 0), (0, 1)\}$
* $B \times A = \{(0, \pi), (0, e), (0, 0), (1, \pi), (1, e), (1, 0)\}$
* $A \times A =$
  $\{(\pi, \pi), (\pi, e), (\pi, 0), (e, \pi), (e, e), (e, 0), (0, \pi), (0, e), (0, 0)\}$
* $B \times B = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$
* $A \times \phi = \{\}$
* $(A \times B) \times B =$
  $\{((\pi, 0), 0), ((\pi, 0), 1), ((\pi, 1), 0), ((\pi, 1), 1), ((e, 0), 0), ((e, 0), 1),$
  $((e, 1), 0), ((e, 1), 1), ((0, 0), 0), ((0, 0), 1), ((0, 1), 0), ((0, 1), 1)\}$
* $A \times (B \times B) =$
  $\{(\pi, (0, 0)), (\pi, (0, 1)), (\pi, (1, 0)), (\pi, (1, 1)),$
  $(e, (0, 0)), (e, (0, 1)), (e, (1, 0)), (e, (1, 1)),$
  $(0, (0, 0)), (0, (0, 1)), (0, (1, 0)), (0, (1, 1))\}$
* $A \times B \times B =$
  $\{(\pi, 0, 0), (\pi, 0, 1), (\pi, 1, 0), (\pi, 1, 1),$
  $(e, 0, 0), (e, 0, 1), (e, 1, 0), (e, 1, 1),$
  $(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1)\}$

6 $\{x \in \mathbb{R} : x^2 = x\} \times \{x \in \mathbb{N} : x^2 = x\} = \{(0, 1), (1, 1)\}$

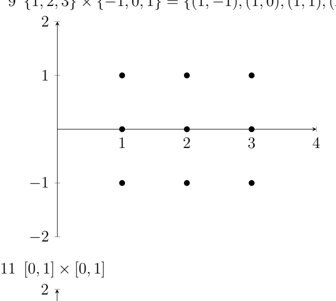
8 $\{0, 1\}^4 =$
$\{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 0),$
$(0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1), (1, 0, 0, 0), (1, 0, 0, 1),$
$(1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0), (1, 1, 0, 1),$
$(1, 1, 1, 0), (1, 1, 1, 1)\}$

B Sketch these Cartesian products on the $x-y$ plane $\mathbb{R}^2$ (or $\mathbb{R}^3$ for the last two.)

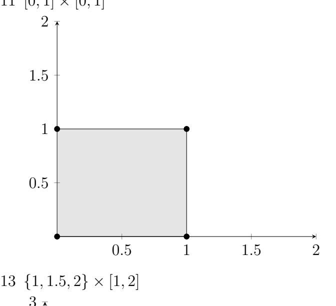
9 $\{1, 2, 3\} \times \{-1, 0, 1\} = \{(1, -1), (1, 0), (1, 1), (2, -1), (2, 0), (2, 1), (3, -1), (3, 0), (3, 1)\}$



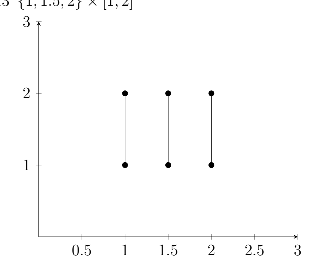
11 $[0, 1] \times [0, 1]$



13 $\{1, 1.5, 2\} \times [1, 2]$



15 $\{1\} \times [0, 1]$



Exercises for Section 2.3

A List all the subsets of the following sets.

1. $\{1, 2, 3, 4\} = \{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\},$
   $\{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$
2. $\{1, 2, \phi\} = \{\}, \{1\}, \{2\}, \{\phi\}, \{1, 2\}, \{1, \phi\}, \{2, \phi\}, \{1, 2, \phi\}$
3. $\{\{\mathbb{R}\}\} = \{\}, \{\{\mathbb{R}\}\}$
4. $\phi = \{\}$
5. $\{\phi\} = \{\}, \{\phi\}$
8. $\{\{0, 1\}, \{0, 1, \{2\}\}, \{0\}\} = \{\}, \{\{0, 1\}\}, \{\{0, 1, \{2\}\}\}, \{\{0\}\}, \{\{0, 1\}, \{0, 1, \{2\}\}\},$
   $\{\{0, 1\}, \{0\}\}, \{\{0, 1, \{2\}\}, \{0\}\}, \{\{0, 1\}, \{0, 1, \{2\}\}, \{0\}\}$

B Write out the following sets by listing their elements between braces.

10. $\{X \subseteq \mathbb{N} : |X| \leq 1\} = \{\phi, \{1\}\, \{2\}, \{3\}, \cdots, \{x\} : x \in \mathbb{N}\}$
11. $\{X : X \subseteq \{3, 2, a\}$ and $|X| = 4\} = \{\}$
12. $\{X : X \subseteq \{3, 2, a\}$ and $|X| = 1\} = \{\{3\}, \{2\}, \{a\}\}$

C Decide if the following statements are true or false. Explain.

13 $\mathbb{R}^3 \subseteq \mathbb{R}^3$. True, since it's a subset of the same set.
14 $\mathbb{R}^2 \subseteq \mathbb{R}^3$. False, since $\mathbb{R}^2$ is in the form of (x_1, x_2) and $\mathbb{R}^3$ is in the form of (x_1, x_2, x_3).
15 $\{(x, y) : x - 1 = 0\} \subseteq \{(x, y) : x^2 - x = 0\}$. True, since the right set has $(1, y)$ which is the set on the left.
16 $\{(x, y) : x^2 - x = 0\} \subseteq \{(x, y) : x - 1 = 0\}$. False, since the left one has 2 elements and the right one has only 1.

Exercises for Section 2.4

A Find the indicated sets.

1. $P(\{\{a, b\}, \{c\}\}) = \{\{\{a, b\}\}, \{\{c\}\}, \{\{a, b\}, \{c\}\}, \phi\}$

5 $P(P(\{2\})) = \{\phi, \{\phi\}, \{\{2\}\}, \{\phi, \{2\}\}\}$
7 $P(\{a, b\}) \times P(\{0, 1\}) =$
$\{(\phi, \phi), (\phi, \{0\}), (\phi, \{1\}), (\phi, \{0, 1\}), (\{a\}, \phi), (\{a\}, \{0\}), (\{a\}, \{1\}), (\{a\}, \{0, 1\}),$
$(\{b\}, \phi), (\{b\}, \{0\}), (\{b\}, \{1\}), (\{b\}, \{0, 1\}),$
$(\{a, b\}, \phi), (\{a, b\}, \{0\}), (\{a, b\}, \{1\}), (\{a, b\}, \{0, 1\})\}$
9 $P(\{a, b\} \times \{0\}) = \{\phi, \{(a, 0)\}, \{(b, 0)\}, \{(a, 0), (b, 0)\}\}$
10 $\{X \in P(\{1, 2, 3\}) : |X| \leq 1\} = \{\phi, \{1\}, \{2\}, \{3\}\}$
11 $\{X \subseteq P(\{1, 2, 3\}) : |X| \leq 1\} = \{\phi, \{\phi\}, \{\{1\}\}, \{\{2\}\}, \{\{3\}\},$
$\{\{1, 2\}\}, \{\{1, 3\}\}, \{\{2, 3\}\}, \{\{1, 2, 3\}\}\}$
12 $\{X \in P(\{1, 2, 3\}) : 2 \in X\} = \{\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$

B Suppose that $|A| = m$ and $|B| = n$. Find the following cardinalities.

13 $|P(P(A))| = 2^{2^m}$
14 $|P(A)| = 2^m$
15 $|P(A \times B)| = 2^{mn}$
16 $|P(A) \times P(B)| = 2^{m+n}$
17 $|\{X \in P(A) : |X| \leq 1\}| = m + 1$
18 $|P(A \times P(B))| = 2^{m2^n}$
19 $|P(P(P(A \times \phi)))| = 4$
20 $|\{X \subseteq P(A) : |X| \leq 1\}| = 1 + 2^m$

Exercises for Section 2.5

1. Suppose $A = \{4, 3, 6, 7, 1, 9\}$, $B = \{5, 6, 8, 4\}$, $C = \{5, 8, 4\}$. Find:
   * $A \cup B = \{4, 3, 5, 6, 7, 1, 9, 8\}$
   * $A \cap B = \{4, 6\}$
   * $A - B = \{3, 7, 1, 9\}$
   * $A - C = \{3, 6, 7, 1, 9\}$
   * $B - A = \{5, 8\}$
   * $A \cap C = \{4\}$
   * $B \cap C = \{5, 4, 8\}$
   * $B \cup C = \{5, 4, 8, 6\}$
   * $C - B = \{\}$

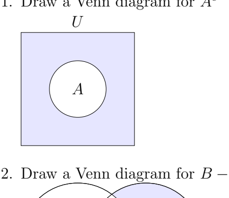
3. Suppose $A = \{0, 1\}$, $B = \{1, 2\}$. Find:
   * $(A \times B) \cap (B \times B) = \{(1, 1), (1, 2)\}$
   * $(A \times B) \cup (B \times B) = \{(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2)\}$
   * $(A \times B) - (B \times B) = \{(0, 1), (0, 2)\}$
   * $(A \cap B) \times A = \{(1, 0), (1, 1)\}$
   * $(A \times B) \cap B = \{\}$
   * $P(A) \cap P(B) = \{\phi, \{1\}\}$
   * $P(A) - P(B) = \{\{0\}, \{0, 1\}\}$
   * $P(A \cap B) = \{\phi, \{1\}\}$
   * $P(A \times B) = \{\phi, \{(0, 1)\}, \{(0, 2)\}, \{(1, 1)\}, \{(1, 2)\}, \{(0, 1), (0, 2)\}, \{(0, 1), (1, 1)\}, \{(0, 1), (1, 2)\},$
     $\{(0, 2), (1, 1)\}, \{(0, 2), (1, 2)\}, \{(1, 1), (1, 2)\}, \{(0, 1), (0, 2), (1, 1)\}, \{(0, 1), (0, 2), (1, 2)\},$
     $\{(0, 1), (1, 1), (1, 2)\}, \{(0, 2), (1, 1), (1, 2)\}, \{(0, 1), (0, 2), (1, 1), (1, 2)\}\}$

Exercises for Section 2.6

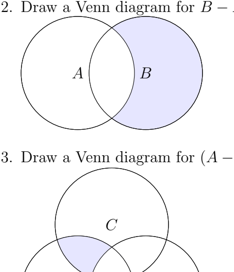
1. Let $A = \{4, 3, 6, 7, 1, 9\}$, $B = \{5, 6, 8, 4\}$ have universal set $U = \{0, 1, 2, \cdots, 10\}$. Find:
   * $\bar{A} = \{0, 2, 5, 8, 10\}$
   * $\bar{B} = \{0, 1, 2, 3, 7, 9, 10\}$
   * $A \cap \bar{A} = \{\}$
   * $A \cup \bar{A} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
   * $A - \bar{A} = \{1, 3, 4, 6, 7, 9\}$
   * $A - \bar{B} = \{4, 6\}$
   * $A - \bar{B} = \{5, 8\}$
   * $\bar{A} \cap B = \{5, 8\}$
   * $\overline{A \cap B} = \{0, 1, 2, 3, 5, 7, 8, 9, 10\}$

Exercises for Section 2.7

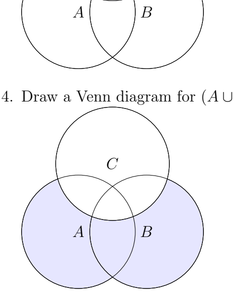
1. Draw a Venn diagram for $\bar{A}$.



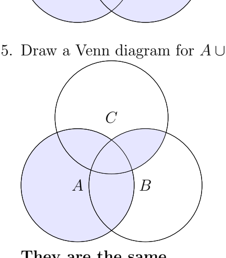
2. Draw a Venn diagram for $B - A$.



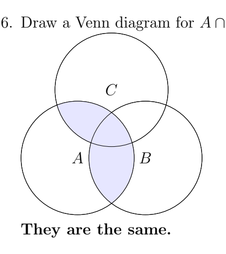
3. Draw a Venn diagram for $(A - B) \cap C$.



4. Draw a Venn diagram for $(A \cup B) - C$.

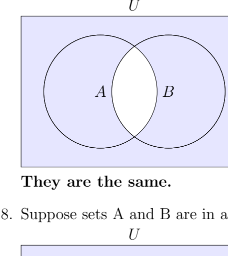


5. Draw a Venn diagram for $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.



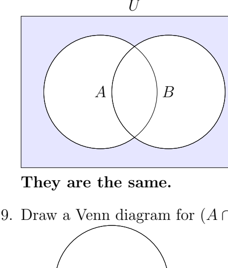
They are the same.

6. Draw a Venn diagram for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.



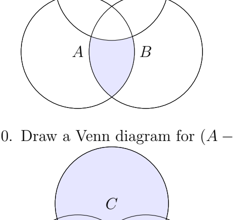
They are the same.

7. Suppose sets A and B are in a universal set U. Draw Venn diagrams for $\overline{A \cap B}$ and $\bar{A} \cup \bar{B}$.



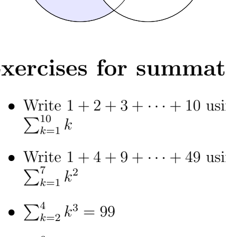
They are the same.

8. Suppose sets A and B are in a universal set U. Draw Venn diagrams for $\overline{A \cup B}$ and $\bar{A} \cap \bar{B}$.

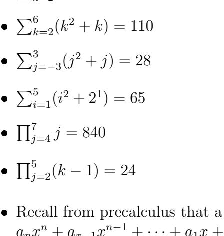


They are the same.

9. Draw a Venn diagram for $(A \cap B) - C$.



10. Draw a Venn diagram for $(A - B) \cup C$.



Exercises for summation

* Write $1 + 2 + 3 + \cdots + 10$ using sigma notation.
  $\sum_{k=1}^{10} k$
* Write $1 + 4 + 9 + \cdots + 49$ using sigma notation.
  $\sum_{k=1}^{7} k^2$
* $\sum_{k=2}^{4} k^3 = 99$
* $\sum_{k=2}^{6} (k^2 + k) = 110$
* $\sum_{j=-3}^{3} (j^2 + j) = 28$
* $\sum_{i=1}^{5} (i^2 + 2^1) = 65$
* $\prod_{j=4}^{7} j = 840$
* $\prod_{j=2}^{5} (k - 1) = 24$
* $\prod_{i=1}^{3} \frac{i+1}{i} = 4$
* $\sum_{i=1}^{3} \sum_{j=1}^{2} (j) = 9$
* $\sum_{i=1}^{3} \sum_{j=1}^{2} (i) = 12$
* $\prod_{i=1}^{2} \prod_{j=4}^{6} (i - j) = 1440$
* $\sum_{i=1}^{2} \prod_{j=4}^{6} (i - j) = -84$
* $\prod_{i=2}^{4} \sum_{j=1}^{3} (i + j) = 3240$
* Recall from precalculus that a polynomial of degree n has the form
  $a_n x^n + a_{x-1} x^{n-1} + \cdots + a_1 x + a_0$,
  where $a_n \neq 0$. Express the form of a polynomial using summation notation.
  $= \sum_{i=0}^{n} (a_{n-i} x^{k-i})$

Exercises for section 2.8

1. Suppose $A_1 = \{a, b, d, e, g, f\}$, $A_2 = \{a, b, c, d\}$, $A_3 = \{b, d, a\}$, $A_4 = \{a, b, h\}$.
   * $\cup_{i=1}^{4} A_i = \{a, b, c, d, e, g, f, h\}$
   * $\cap_{i=1}^{4} A_i = \{a, b\}$

3 For each $n \in \mathbb{N}$, let $A_n = \{0, 1, 2, 3, \cdots, n\}$.

* $\cup_{i \in \mathbb{N}} A_i = \{0\} \cup \mathbb{N}$
* $\cap_{i \in \mathbb{N}} A_i = \{0, 1\}$

4 For each $n \in \mathbb{N}$, let $A_n = \{-2n, 0, 2n\}$.

* $\cup_{i \in \mathbb{N}} A_i = \{0, -2, 2, -4, 4\} \cup \{2n : n \in \mathbb{N}\} \cup \{-2n : n \in \mathbb{N}\}$
* $\cap_{i \in \mathbb{N}} A_i = \{0\}$

5
* $\cup_{i \in \mathbb{N}} [i, i+1] = [1, \infty)$
* $\cap_{i \in \mathbb{N}} [i, i+1] = \phi$

6
* $\cup_{i \in \mathbb{N}} [0, i+1] = [0, \infty)$
* $\cap_{i \in \mathbb{N}} [0, i+1] = [0, 2]$

9
* $\cup_{X \in P(\mathbb{N})} X = \mathbb{N}$
* $\cap_{X \in P(\mathbb{N})} X = \phi$