Section 2 homework

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Exercises for Section 2.1

A. Write each of the following sets by listing their elements between braces.

1. $\{5x-1 : x \in \mathbb{Z}\} = \{\dots, -11, -6, -1, 4, 9, \dots\}$
2. $\{3x+2 : x \in \mathbb{Z}\} = \{\dots, -4, -1, 2, 5, 8, \dots\}$
3. $\{x \in \mathbb{Z} : -2 \le x < 7\} = \{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
4. $\{x \in \mathbb{N} : -2 < x \le 7\} = \{1, 2, 3, 4, 5, 6, 7\}$
5. $\{x \in \mathbb{R} : x^2 = 3\} = \{-\sqrt{3}, \sqrt{3}\}$
6. $\{x \in \mathbb{R} : x^2 = 9\} = \{-3, 3\}$
7. $\{x \in \mathbb{R} : x^2 + 5x = -6\} = \{-3, -2\}$

11 $\{x \in \mathbb{Z} : |x| < 5\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

12 $\{x \in \mathbb{Z} : |2x| < 5\} = \{-2, -1, 0, 1, 2\}$

13 $\{x \in \mathbb{Z} : |6x| < 5\} = \{0\}$

14 $\{5x : x \in \mathbb{Z}, |2x| \le 8\} = \{-20, -15, -10, -5, 0, 5, 10, 15, 20\}$

B. Write each of the following sets in set-builder notation.

17 $\{2, 4, 8, 16, 32, 64 \dots\} = \{2 \cdot 2^x : x \ge 0, x \in \mathbb{Z}\}$

19 $\{\dots, -6, -3, 0, 3, 6, 9, 12, 15, \dots\} = \{3x : x \in \mathbb{Z}\}$

24 $\{-4, -3, -2, -1, -0, 1, 2\} = \{x : -4 \le x \le 2, x \in \mathbb{Z}\}$

25 $\{\dots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \dots\} = \{2^x : x \in \mathbb{Z}\}$

26 $\{\dots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, \dots\} = \{3^x : x \in \mathbb{Z}\}$

C. Find the following cardinalities of the following sets.

29 $\{\{1\}, \{2, \{3, 4\}\}, \phi\} = 3$

30 $\{\{1, 4\}, a, b, \{\{3, 4\}\}, \{\phi\}\} = 5$

31 $\{\{\{1\}, \{2, \{3, 4\}\}, \phi\}\} = 1$

32 $\{\{\{1, 4\}, a, b, \{\{3, 4\}\}, \{\phi\}\}\} = 1$

33 $\{x \in \mathbb{Z} : |x| < 10\} = 19$

34 $\{x \in \mathbb{N} : |x| < 10\} = 9$

35 $\{x \in \mathbb{Z} : x^2 < 10\} = 7$

36 $\{x \in \mathbb{N} : x^2 < 10\} = 3$

37 $\{x \in \mathbb{N} : x^2 < 0\} = 0$

38 $\{x \in \mathbb{N} : 5x \le 20\} = 4$

Exercises for Section 2.2

A Write out the indicated sets by listing their elements between braces.

2 Suppose $A = \{\pi, e, 0\}$ and $B = \{0, 1\}$.

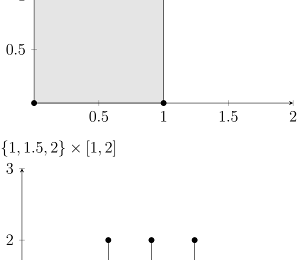
- $A \times B = \{(\pi,0), (\pi,1), (e,0), (e,1), (0,0), (0,1)\}$
- $B \times A = \{(0,\pi), (0,e), (0,0), (1,\pi), (1,e), (1,0)\}$
- $A \times A =$
  $\{(\pi,\pi), (\pi,e), (\pi,0), (e,\pi), (e,e), (e,0), (0,\pi), (0,e), (0,0)\}$
- $B \times B = \{(0,0), (0,1), (1,0), (1,1)\}$
- $A \times \phi = \{(\pi), (e), (0)\}$
- $(A \times B) \times B =$
  $\{((\pi,0),0), ((\pi,0),1), ((\pi,1),0), ((\pi,1),1), ((e,0),0), ((e,0),1),$
  $((e,1),0), ((e,1),1), ((0,0),0), ((0,0),1), ((0,1),0), ((0,1),1)\}$
- $A \times (B \times B) =$
  $\{(\pi,(0,0)), (\pi,(0,1)), (\pi,(1,0)), (\pi,(1,1)),$
  $(e,(0,0)), (e,(0,1)), (e,(1,0)), (e,(1,1)),$
  $(0,(0,0)), (0,(0,1)), (0,(1,0)), (0,(1,1))\}$
- $A \times B \times B =$
  $\{(\pi,0,0), (\pi,0,1), (\pi,1,0), (\pi,1,1),$
  $(e,0,0), (e,0,1), (e,1,0), (e,1,1),$
  $(0,0,0), (0,0,1), (0,1,0), (0,1,1)\}$

6 $\{x \in \mathbb{R} : x^2 = x\} \times \{x \in \mathbb{N} : x^2 = x\} = \{(0,1), (1,1)\}$

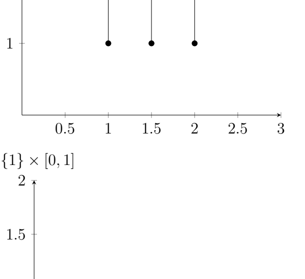
8 $\{0,1\}^4 =$
$\{(((0,0),0),0), (((0,0),0),1), (((0,0),1),0), (((0,0),1),1), (((0,1),0),0),$
$(((0,1),0),1), (((0,1),1),0), (((0,1),1),1), (((1,0),0),0), (((1,0),0),1),$
$(((1,0),1),0), (((1,0),1),1), (((1,1),0),0), (((1,1),0),1),$
$(((1,1),1),0), (((1,1),1),1)\}$

B Sketch these Cartesian products on the $x-y$ plane $\mathbb{R}^2$ (or $\mathbb{R}^3$for the last two.)

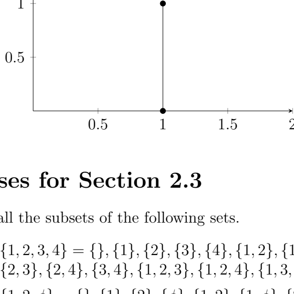
9 $\{1,2,3\} \times \{-1,0,1\} = \{(1,-1), (1,0), (1,1), (2,-1), (2,0), (2,1), (3,-1), (3,0), (3,1)\}$



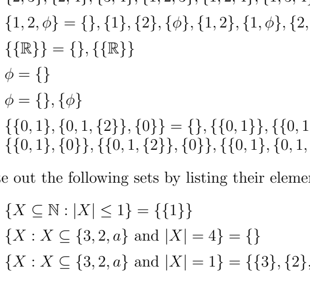
11 $[0,1] \times [0,1]$



13 $\{1, 1.5, 2\} \times [1,2]$



15 $\{1\} \times [0,1]$



Exercises for Section 2.3

A List all the subsets of the following sets.

1. $\{1,2,3,4\} = \{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\},$
   $\{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}$
2. $\{1, 2, \phi\} = \{\}, \{1\}, \{2\}, \{\phi\}, \{1,2\}, \{1,\phi\}, \{2,\phi\}, \{1,2,\phi\}$
3. $\{\{\mathbb{R}\}\} = \{\}, \{\{\mathbb{R}\}\}$
4. $\phi = \{\}$
5. $\phi = \{\}, \{\phi\}$
8. $\{\{0,1\}, \{0,1,\{2\}\}, \{0\}\} = \{\}, \{\{0,1\}\}, \{\{0,1,\{2\}\}\}, \{\{0\}\}, \{\{0,1\}, \{0,1,\{2\}\}\},$
   $\{\{0,1\}, \{0\}\}, \{\{0,1,\{2\}\}, \{0\}\}, \{\{0,1\}, \{0,1,\{2\}\}, \{0\}\}$

B Write out the following sets by listing their elements between braces.

10. $\{X \subseteq \mathbb{N} : |X| \le 1\} = \{\{1\}\}$
11. $\{X : X \subseteq \{3, 2, a\}$ and $|X| = 4\} = \{\}$
12. $\{X : X \subseteq \{3, 2, a\}$ and $|X| = 1\} = \{\{3\}, \{2\}, \{a\}\}$

C Decide if the following statements are true or false. Explain.

13 $\mathbb{R}^3 \subseteq \mathbb{R}^3$. True, since it's a subset of the same set.

14 $\mathbb{R}^2 \subseteq \mathbb{R}^3$

15 $\{(x,y) : x - 1 = 0\} \subseteq \{(x,y) : x^2 - x = 0\}$

16 $\{(x,y) : x^2 - x = 0\} \subseteq \{(x,y) : x - 1 = 0\}$

Exercises for Section 2.4

A Find the indicated sets.

1. $P(\{\{a,b\}, \{c\}\}) = \{\{\{a,b\}\}, \{\{c\}\}, \{\{a,b\}, \{c\}\}, \phi\}$

5 $P(P(\{2\})) = \{\phi, \{\phi\}, \{\{2\}\}, \{\phi, \{2\}\}\}$

7 $P(\{a,b\}) \times P(\{0,1\}) =$
$\{(\phi,\phi), (\phi,\{0\}), (\phi,\{1\}), (\phi,\{0,1\}), (\{a\},\phi), (\{a\},\{0\}), (\{a\},\{1\}), (\{a\},\{0,1\}),$
$(\{b\},\phi), (\{b\},\{0\}), (\{b\},\{1\}), (\{b\},\{0,1\}),$
$(\{a,b\},\phi), (\{a,b\},\{0\}), (\{a,b\},\{1\}), (\{a,b\},\{0,1\})\}$

9 $P(\{a,b\} \times \{0\}) = \{\phi, \{(a,0)\}, \{(b,0)\}, \{(a,0),(b,0)\}\}$

10 $\{X \in P(\{1,2,3\}) : |X| \le 1\} = \{\phi, \{1\}, \{2\}, \{3\}\}$

11 $\{X \subseteq P(\{1,2,3\}) : |X| \le 1\} = \{\phi, \{\phi\}, \{\{1\}\}, \{\{2\}\}, \{\{3\}\},$
$\{\{1,2\}\}, \{\{1,3\}\}, \{\{2,3\}\}, \{\{1,2,3\}\}\}$

12 $\{X \in P(\{1,2,3\}) : 2 \in X\} = \{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

B Suppose that $|A| = m$ and $|B| = n$. Find the following cardinalities.

13 $|P(P(P(A)))| = 2^{2^{2^m}}$

14 $|P(P(A))| = 2^{2^m}$

15 $|P(A \times B)| = 2^{mn}$

16 $|P(A) \times P(B)| = 2^{m+n}$

17 $|X \in P(A) : |X| \le 1| = m + 1$

18 $|P(A \times P(B))| = 2^{m \cdot 2^n}$

19 $|P(P(P(A \times \phi)))| = 4$

20 $|X \subseteq P(A) : |X| \le 1| = 1 + 2^m$

Exercises for Section 2.5

1. Suppose $A = \{4,3,6,7,1,9\}, B = \{5,6,8,4\}, C = \{5,8,4\}$. Find:

- $A \cup B = \{4,3,5,6,7,1,9,8\}$
- $A \cap B = \{4,6\}$
- $A - B = \{3,7,1,9\}$
- $A - C = \{3,6,7,1,9\}$
- $B - A = \{5,8\}$
- $A \cap C = \{4\}$
- $B \cap C = \{5,4,8\}$
- $B \cup C = \{5,4,8,6\}$
- $C - B = \{\}$

3 Suppose $A = \{0,1\}, B = \{1,2\}$. Find:

- $(A \times B) \cap (B \times B) = \{(1,1), (1,2)\}$
- $(A \times B) \cup (B \times B) = \{(0,1), (0,2), (1,1), (1,2), (2,1), (2,2)\}$
- $(A \times B) - (B \times B) = \{(0,1), (0,2)\}$
- $(A \cap B) \times A = \{(1,0), (1,1)\}$
- $(A \times B) \cap B = \{\}$
- $P(A) \cap P(B) = \{\phi, \{1\}\}$
- $P(A) - P(B) = \{\{0\}, \{0,1\}\}$
- $P(A \cap B) = \{\phi, \{1\}\}$
- $P(A \times B) = \{\phi, \{(0,1)\}, \{(0,2)\}, \{(1,1)\}, \{(1,2)\},$
  $\{(0,1),(0,2)\}, \{(0,1),(1,1)\}, \{(0,1),(1.2)\},$
  $\{(0,2),(1,1)\}, \{(0,2),(1,2)\}, \{(1,1),(1,2)\}$
  $\{(0,1),(0,2),(1,1)\}, \{(0,1),(0,2),(1,2)\}, \{(0,1),(1,1),(1,2)\}, \{(0,2),(1,1),(1,2)\}$
  $\{(0,1),(0,2),(1,1),(1,2)\}\}$

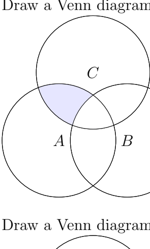
Exercises for Section 2.6

1. Let $A = \{4,3,6,7,1,9\}, B = \{5,6,8,4\}$ have universal set $U = \{0,1,2,\cdots,10\}$. Find:

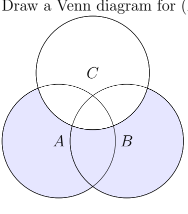
- $A^c = \{0,2,5,8,10\}$
- $B^c = \{0,1,2,3,7,9,10\}$
- $A \cap A^c = \{\}$
- $A \cup A^c = \{0,1,2,3,4,5,6,7,8,9,10\}$
- $A - A^c = \{1,3,4,6,7,9\}$
- $A - B^c = \{4,6\}$
- $A^c - B^c = \{5,8\}$
- $A^c \cap B = \{5,8\}$
- $A^c \cap B^c = \{0,1,2,3,4,6,7,9,10\}$

Exercises for Section 2.7

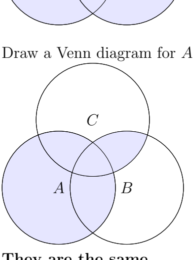
1. Draw a Venn diagram for A^c



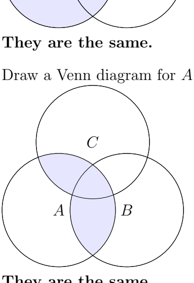
2. Draw a Venn diagram for $B - A$



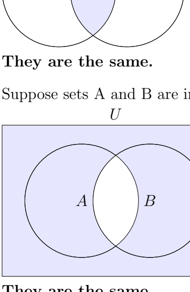
3. Draw a Venn diagram for $(A - B) \cap C$



4. Draw a Venn diagram for $(A \cup B) - C$

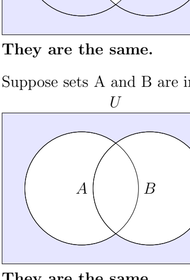


5. Draw a Venn diagram for $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.



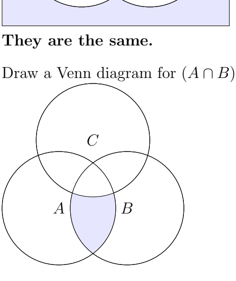
They are the same.

6. Draw a Venn diagram for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.



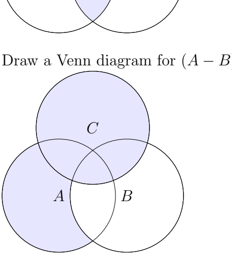
They are the same.

7. Suppose sets A and B are in a universal set U. Draw Venn diagrams for $A \cap B^c$ and $A^c \cup B^c$.



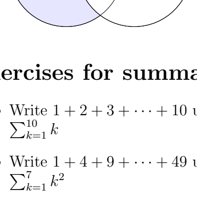
They are the same.

8. Suppose sets A and B are in a universal set U. Draw Venn diagrams for $A \cup B^c$ and $A^c \cap B^c$.

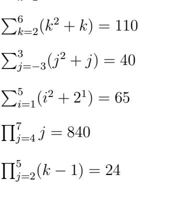


They are the same.

9. Draw a Venn diagram for $(A \cap B) - C$.



10. Draw a Venn diagram for $(A - B) \cup C$.



Exercises for summation

- Write $1 + 2 + 3 + \cdots + 10$ using sigma notation.
  $\sum_{k=1}^{10} k$
- Write $1 + 4 + 9 + \cdots + 49$ using sigma notation.
  $\sum_{k=1}^{7} k^2$
- $\sum_{k=2}^{4} k^3 = 99$
- $\sum_{k=2}^{6} (k^2 + k) = 110$
- $\sum_{j=-3}^{3} (j^2 + j) = 40$
- $\sum_{i=1}^{5} (i^2 + 2^1) = 65$
- $\prod_{j=4}^{7} j = 840$
- $\prod_{j=2}^{5} (k - 1) = 24$
- $\prod_{i=1}^{3} \frac{i+1}{i} = 4$
- $\sum_{i=1}^{3} \sum_{j=1}^{2} (j) = 9$
- $\sum_{i=1}^{3} \sum_{j=1}^{2} (i) = 12$
- $\prod_{i=1}^{2} \prod_{j=4}^{6} (i - j) = 1440$
- $\sum_{i=1}^{2} \prod_{j=4}^{6} (i - j) = -84$
- $\prod_{i=2}^{4} \sum_{j=1}^{3} (i + j) = 3240$
- Recall from precalculus that a polynomial of degree n has the form
  $a_n x^n + a_{x-1} x^{n-1} + \cdots + a_1 x + a_0$,
  where $a_n \neq 0$. Express the form of a polynomial using summation notation.
  $= \sum_{i=0}^{n} (a_{n-i} x^{k-i})$

Exercises for section 2.8

1. Suppose $A_1 = \{a,b,d,e,g,f\}, A_2 = \{a,b,c,d\}, A_3 = \{b,d,a\}, A_4 = \{a,b,h\}$.

- $\cup_{i=1}^{4} A_i = \{a,b,c,d,e,g,f,h\}$
- $\cap_{i=1}^{4} A_i = \{a,b\}$

3 For each $n \in \mathbb{N}$, let $A_n = \{0,1,2,3,\cdots,n\}$.

- $\cup_{i \in \mathbb{N}} A_i = \{0,1,2,3,\cdots,i\}$
- $\cap_{i \in \mathbb{N}} A_i = \{0,1\}$

4 For each $n \in \mathbb{N}$, let $A_n = \{-2n, 0, 2n\}$.

- $\cup_{i \in \mathbb{N}} A_i = \{0,-2,2,-4,-4,\cdots,-2i,2i\}$
- $\cap_{i \in \mathbb{N}} A_i = \{0\}$

5
- $\cup_{i \in \mathbb{N}} [i, i+1] = [1, \infty)$
- $\cap_{i \in \mathbb{N}} [i, i+1] = \phi$

6
- $\cup_{i \in \mathbb{N}} [0, i+1] = [0, \infty)$
- $\cap_{i \in \mathbb{N}} [0, i+1] = [0, 2]$

9
- $\cup_{X \in P(\mathbb{N})} X = \mathbb{N}$
- $\cap_{X \in P(\mathbb{N})} X = \phi$