Proof Homework

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Exercises for Section 8

1. $x = 2m, m \in \mathbb{Z}$
   $x^2 = 4m^2 = 2(2m^2), n = 2m^2, n \in \mathbb{Z}$
   Therefore x^2 is a even integer.

3 $a = 2m + 1, m \in \mathbb{Z}$
$a^2 + 3a + 5 = (2m + 1)^2 + 6m + 3 + 5$
$= 4m^2 + 4m + 9 + 6m = 4m^2 + 10m + 9$
$= 4m^2 + 10m + 8 + 1 = 2(2m^2 + 5m + 4) + 1$
Where $(2m^2 + 5m + 4) = n \in \mathbb{Z}$, Therefore $a^2 + 3a + 5 = 2n + 1$, and is an odd integer.

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$$\begin{aligned} x &= 2m+1, m \in \mathbb{Z} \\ y &= 2n+1, n \in \mathbb{Z} \\ xy &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ 2mn + m + n &= b \in \mathbb{Z} \\ &= 2b + 1, \text{Therefore it's odd} \end{aligned}$$

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$$\begin{aligned} x &= 2m, m \in \mathbb{Z} \\ y &= 2n, n \in \mathbb{Z} \\ xy &= (2m)(2n) \\ &= 4mn \\ &= 2(2mn) \\ 2mn &= b \in \mathbb{Z} \\ &= 2b, \text{Therefore it's even} \end{aligned}$$

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$$\begin{aligned} b &= ac, c \in \mathbb{Z} \\ b^2 &= a^2c^2, c^2 = d \in \mathbb{Z} \\ b^2 &= a^2d, \text{Therefore} a^2 | b^2 \end{aligned}$$

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$$\begin{aligned} b &= am, m \in \mathbb{Z} \\ d &= cn, n \in \mathbb{Z} \\ bd &= (ac)(mn), \text{Therefore} ac | bd \end{aligned}$$

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$$\begin{aligned} &\text{Case 1, even n} \\ n &= 2a, a \in \mathbb{Z} \\ n^2 + 3n + 4 &= 4a^2 + 6a + 4 \\ = 2(2a^2 + 3a + 2), (2a^2 &+ 3a + 2) = c, c \in \mathbb{Z} \\ &= 2c, \text{Therefore, } n^2 + 3n + 4 \text{ is even} \end{aligned}$$

$$\begin{aligned} &\text{Case 2, odd n} \\ n &= 2b + 1, b \in \mathbb{Z} \\ n^2 + 3n + 4 &= 4b^2 + 4b + 1 + 6b + 3 + 4 \\ &= 4b^2 + 10b + 8 = 2(2b^2 + 5b + 4) \\ 2b^2 + 5b + 4 &= d \in \mathbb{Z}, = 2d \\ &\text{Therefore, } n^2 + 3n + 4 \text{ is even} \end{aligned}$$

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$$\begin{aligned} &\text{Case 1: Odd parity} \\ a &= 2x + 1, x \in \mathbb{Z} \\ b &= 2y + 1, y \in \mathbb{Z} \\ a + b &= (2x + 1) + (2y + 1) \\ &= 2x + 2y + 2 = 2(x + y + 1) \\ x + y + 1 &= z \in \mathbb{Z}, \text{Therefore, their sum is even} \end{aligned}$$

$$\begin{aligned} &\text{Case 2: Even parity} \\ a &= 2x, x \in \mathbb{Z} \\ b &= 2y, y \in \mathbb{Z} \\ a + b &= (2x) + (2y) \\ &= 2x + 2y = 2(x + y) \\ x + y &= z \in \mathbb{Z}, \text{Therefore, their sum is even} \end{aligned}$$

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$$\begin{aligned} a &= 2x, x \in \mathbb{Z} \\ b &= 2y + 1, y \in \mathbb{Z} \\ ab &= (2x)(2y + 1) \\ &= 4xy + 2x = 2(2xy + x) \\ 2xy + x &= z \in \mathbb{Z}, \text{Therefore, their product is even} \end{aligned}$$

Exercises for Section 9

1. By the contrapositive, suppose If n is odd then n^2 is odd

$$\begin{aligned} n &= 2a + 1, a \in \mathbb{Z} \\ n^2 &= (2a + 1)^2 \\ = 4a^2 + 4a + 1 &= 2(2a^2 + 2a) + 1 \\ &= 2a^2 + 2a \in \mathbb{Z} \\ \text{Therefore, by the contrapositive,If } n^2 &\text{ is even, then n is even} \end{aligned}$$

3

$$\begin{aligned} a &= 2x, x \in \mathbb{Z} \\ b &= 2y, y \in \mathbb{Z} \\ a^2(b^2 - 2b) &= 4x^2(4y^2 - 4y) \\ = 16x^2y^2 - 16x^2y &= 2(8x^2y^2 - 8x^2y) \\ &= 8x^2y^2 - 8x^2y \in \mathbb{Z} \\ \text{Therefore, by the contrapositive, if } a^2(b^2 - 2b) &\text{ is odd, then a and b are odd} \end{aligned}$$

5 By the contrapositive, If $x \geq 0$, then $x^2 + 5x \geq 0$

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$$\begin{aligned} &\text{Case 1: a odd, b even} \\ a &= 2x + 1, x \in \mathbb{Z} \\ b &= 2y, y \in \mathbb{Z} \\ a \cdot b &= 4xy + 2y \\ = 2(2xy + y), (2xy &+ y) \in \mathbb{Z} \\ &\text{a times b is even} \\ a + b &= 2x + 2y + 1 \\ &= 2(x + y) + 1 \\ &\text{a + b is odd} \end{aligned}$$

$$\begin{aligned} &\text{Case 2: b odd, a even} \\ a &= 2x, x \in \mathbb{Z} \\ b &= 2y + 1, y \in \mathbb{Z} \\ a \cdot b &= 4xy + 2x \\ = 2(2xy + x), (2xy &+ x) \in \mathbb{Z} \\ &\text{a times b is even} \\ a + b &= 2x + 2y + 1 \\ &= 2(x + y) + 1 \\ &\text{a + b is odd} \end{aligned}$$

$$\begin{aligned} &\text{Case 3: both odd} \\ a &= 2x + 1, x \in \mathbb{Z} \\ b &= 2y + 1, y \in \mathbb{Z} \\ a \cdot b &= 4xy + 2x + 2y + 1 \\ = 2(2xy + x + y) + 1, (2xy &+ x + y) \in \mathbb{Z} \\ &\text{a times b is odd} \\ a + b &= 2x + 2y + 1 + 1 \\ &= 2(x + y + 1) + 1 \\ &\text{a + b is even} \end{aligned}$$

Therefore, by the contrapositive, in all cases, there's at least one odd when at least a or b are odd

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$$\begin{aligned} n &= 3x, x \in \mathbb{Z} \\ n^2 &= 9x^2 = 3(3x^2) \qquad (1) \\ \text{Therefore, by the contrapositive, if 3 is not divisible by } n^2 &\text{, then 3 is not divisible by n} \end{aligned}$$

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$$\begin{aligned} x &= 2a + 1, a \in \mathbb{Z} \\ y &= 2b, b \in \mathbb{Z} \\ x^2(y + 3) &= (4a^2 + 4a + 1)(2b + 3) \\ &= 8a^2b + 8ab + 2b + 12a^2 + 12a + 2 + 1 \\ &= 2(4a^2b + 4ab + b + 6a^2 + 6a + 1) + 1 \\ &(4a^2b + 4ab + b + 6a^2 + 6a + 1) \in \mathbb{Z} \\ \text{Therefore, by the contrapositive, if } x^2(y + 3) &\text{ is even, then x is even or y is odd} \end{aligned}$$

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$$\begin{aligned} x &= 2a, a \in \mathbb{Z} \\ x^3 - 1 &= (2a)^3 - 1 \\ = 8a^3 - 1 &= 2(4a^3 - 1) + 1 \\ 4a^3 &- 1 \in \mathbb{Z} \\ \text{Therefore, by the contrapositive, if } x^3 - 1 &\text{ is even the x is odd} \end{aligned}$$

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$$\begin{aligned} n &= 2a + 1, a \in \mathbb{Z} \\ n^2 - 1 &= 8b, b \in \mathbb{Z} \\ n^2 - 1 &= 4a^2 + 4a = 4a(a + 1) \\ a(a + 1), \in \mathbb{Z} &= 2b \; even = 4(2b) = 8b \\ \text{Therefore, by direct proof, if n is odd, then 8 is divisible by } &n^2 - 1 \end{aligned}$$

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$$\begin{aligned} a - b &= nx, x \in \mathbb{Z} \\ a - c &= ny, y \in \mathbb{Z} \\ c - b &= n(x - y) \\ \text{Therefore, by direct proof, c} &\equiv b \text{ (mod n)} \end{aligned}$$

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$$\begin{aligned} a - 1 &= 5x, x \in \mathbb{Z} \\ a &= 5x + 1 \\ a^2 &= 25x^2 + 10x + 1 \\ a^2 - 1 &= 25x^2 + 10x \\ a^2 - 1 &= 5(5x^2 + 2x) \\ 5x^2 &+ 2x \in \mathbb{Z} \\ \text{Therefore, if a congruence 1 mod 5, then } a^2 &\equiv 1 \text{ (mod 5)} \end{aligned}$$

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$$\begin{aligned} a - b &= nx, x \in \mathbb{Z} \\ ac - bc &= nxc, xc = y, y \in \mathbb{Z} \\ \text{Therefore, if } a &\equiv b(mod\ n)\text{,then } ca \equiv cb(mod\ n) \end{aligned}$$

Exercises for Section 10

1.

$$\begin{aligned} n &= 2a + 1, a \in \mathbb{Z} \\ n^2 &= 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1 \\ 2b &+ 1, b = (2a^2 + 2a) \in \mathbb{Z} \\ \text{Therefore, by the contradiction, } n^2 &\text{ is even and odd, which is a contradiction.} \end{aligned}$$

3

$$\begin{aligned} &\text{By contradiction, } \sqrt[3]{2} \text{ is rational} \\ \sqrt[3]{2} &= a/b, \text{In their simpliest form} \\ 2 &= a^3/b^3 = 2b^3 = a^3, a\ is\ even \\ 2b^3 &= (2c)^3 = 8c^3 = 2(4c^3), b\ is\ even \\ &\text{Therefore, by contradiction, a, b are both even and odd} \end{aligned}$$

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$$\begin{aligned} &\text{By contradiction, if a is rational and ab is irrational, then b is rational} \\ a &= n/m, n, m \in \mathbb{Z} \\ b &= x/y, x, y \in \mathbb{Z} \\ ab &= nx/my \\ &\text{Therefore, by contradiction, ab is both rational and irrational} \end{aligned}$$

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$$\begin{aligned} &\text{By contradiction, integers a and b exist, for which 18a+6b} = 1 \\ 1 &= 18a + 6b \\ 1 &= 2(9a + 3b) \\ &\text{Therefore, by contradiction, 1 is even and odd} \end{aligned}$$

- Prove that the sum of a rational number and an irrational number is always irrational.

$$\begin{aligned} &\text{By contradiction, the sum of a rational number and an irrational number is always rational} \\ a &= x/y, x, y \in \mathbb{Z} \\ b &= irrational \\ a + b &= n/m, n, m \in \mathbb{Z} \\ b &= n/m - x/y \\ n/m\ and\ x/y, &\text{ are rational, therefore, by contradiction, b is rational and irrational} \end{aligned}$$

- Prove that the product of a nonzero rational number and an irrational number is always an irrational number. (Why "nonzero"?)

$$\begin{aligned} &\text{By contradiction, a product of a nonzero rational number} \\ &\text{and an irrational number is always a rational number} \\ a &= x/y, x, y \in \mathbb{Z} \\ ab &= n/m, n, m \in \mathbb{Z} \\ b &= (n/m)/(x/y) \\ b &= ny/mx \\ &\text{Therefore, by contradiction, b is both rational and irrational} \end{aligned}$$

Exercises for Section 12

1.

$$\begin{aligned} x &= 2a, a \in \mathbb{Z} \\ 3x + 5 &= 6a + 5 = 6a + 4 + 1 \\ &= 2(3a + 2) + 1 \\ &\text{Therefore it's odd} \end{aligned}$$

3

$$\begin{aligned} a &= 2n + 1 \\ a^3 + a^2 + a &= (4n^2 + 4n + 1)(2n + 1) + (4n^2 + 4n + 1) + 2n + 1 \\ &= 8n^3 + 8n^2 + 2n + 4n^2 + 4n + 1 + 4n^2 + 4n + 1 + 2n + 1 \\ &= 8n^3 + 16n^2 + 12n + 3 = 2(4n^3 + 8n^2 + 6n + 1) + 1 \\ a &= 2n \\ &\text{By the contrapositive, it's odd} \\ a^3 + a^2 + a &= 8n^3 + 4n^2 + 2n = 2(4n^3 + 2n^2 + n) \\ &\text{Therefore, by direct proof it's even} \end{aligned}$$

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$$\begin{aligned} a &= 2n + 1 \\ a^3 &= 8n^3 + 12n^2 + 6n + 1 \\ &= 2(4n^3 + 6n^2 + 3n) + 1 \\ &\text{It's odd if a is odd} \\ a &= 2n \\ a^3 &= 8n^3 = 2(4n^3) \\ \text{Therefore, by the contrapositive, a is odd if and only if } &a^3 \text{ is odd.} \end{aligned}$$

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$$\begin{aligned} a &= 14m, m \in \mathbb{Z} \\ a &= 7(2m) = 2(7m) \\ &\text{Therefore, if 14—a then 7—a and 2—a} \\ a &= 2x, x \in \mathbb{Z} \\ a &= 7y, y \in \mathbb{Z} \\ y &= 2z, z \in \mathbb{Z} \\ a &= 14z \\ &\text{Therefore, 14 — a if and only if 7—a and 2—a} \end{aligned}$$

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$$\begin{aligned} &\text{By the contrapositive, if a is even and b is odd then } (a - 3)b^2 \text{ is odd} \\ a &= 2n, n \in \mathbb{Z} \\ b &= 2m + 1, m \in \mathbb{Z} \\ (a - 3)b^2 = (2n - 3)(4m^2 + 4m + 1) &= 8nm^2 + 8nm + 2n - 12m^2 - 12m - 4 + 1 \\ &= 2(4nm^2 + 4nm + n - 6m^2 - 6m - 2) + 1 \\ &\text{Therefore, it's odd} \end{aligned}$$

$$\begin{aligned} &\text{Case 1: a is odd} \\ a &= 2x + 1, x \in \mathbb{Z} \\ (a - 3)b^2 &= (2x + 1 - 3)b^2 = 2(x - 1)b^2 \\ &\text{Therefore, it's even} \end{aligned}$$

$$\begin{aligned} &\text{Case 2: b is even} \\ b &= 2x, x \in \mathbb{Z} \\ (a - 3)b^2 &= (a - 3)(2x)^2 = 2(a - 3)2x^2 \\ &\text{Therefore, it's even} \end{aligned}$$

Therefore, $(a - 3)b^2$ is always even

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$$\begin{aligned} a &= 2m, m \in \mathbb{Z} \\ b &= 2n + 1, n \in \mathbb{Z} \\ a + b &= 2m + 2n + 1 = 2(m + n) + 1 \\ &\text{Therefore, a + b is odd} \end{aligned}$$

$$\begin{aligned} &\text{Case 1: a, b are even} \\ a &= 2m, m \in \mathbb{Z} \\ b &= 2n, n \in \mathbb{Z} \\ a + b &= 2m + 2n = 2(m + n) \\ &\text{It's even} \end{aligned}$$

$$\begin{aligned} &\text{Case 2: a, b are odd} \\ a &= 2m + 1, m \in \mathbb{Z} \\ b &= 2n + 1, n \in \mathbb{Z} \\ a + b &= 2m + 2n + 2 = 2(m + n + 1) \\ &\text{It's even} \end{aligned}$$

Therefore, a + b is even

17 By the existence method, 97 is that number.

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$$\text{By the existence method, the set is:} X = \{\mathbb{N}, 1, 2, 3, ...\}$$

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$$\text{By the existence method, the number is: 10}$$

Exercises for Section 13

1. By disproof, when x = -1 and y = -1, $|x + y| = 0, |x| + |y| = 2$

7 By disproof, when C = ϕ, A = 1,2,3, B = 4,5,6, A$\neq$ B

9 By disproof, when A = 1,2, B = 1, P(A) - P(B) is not a subset of P(A-B)

11 By disproof, when a = 1, b = 2 $a + b = 3, ab = 2$, so a + b ¿ ab

13 By existence, the set is $X = \mathbb{R} \cup \{\phi\}$

29 By disproof, y = 2, x = 0, $|x + y| = |x - y|$, but y is 2