HAND-IN ASSIGNMENT- MA1103
NTNU
SPRING 2022

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Topic 1 Curves:

Parametric functions, and curves:

Definition:

A vector-value function is defined to be a map $r: R^m \to R^n$ on an open interval $I = (a, b)$

To parametrize a function means to represent it in terms of one or more parameters. Parametric functions are vector-value functions that use parameters as their variables. A vector-value function is a function that for m input variables, outputs n vector coordinates. In other words, a parametric function is characterized as the solution space to a set of n functions in m variables. Graphically, the value of the vector function for a parameter value, represents the coordinate of a point.

If we define the vector-value function to be $r: R \to R^n$, we call it a parametrized curve. Meaning that geometrically all parametrized curves are one-dimensional objects in an n dimensional space. In $\mathbb{R}^3$ such a curve can be generalized to be:

$$r: \mathrm{R} \to \mathrm{R}^3, \qquad x = f(t), y = g(t), z = f(t); \qquad a \le t \le b$$

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

A useful interpretation of parametric curves, is looking at them as a particle moving through space, with respect to the time t. This interpretation leads to some useful applications as shown later in the text.

Derivatives of parametric functions, velocity, acceleration:

Definition:

For a differentiable vector-value functions $r: R \to R^n$, the derivative is defined to be:

$$r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$

Differentiability of parametric curves is very similar to differentiability of scalar functions in a single variable, $f: R \to R$, with it being the instantaneous rate of change with respect to the function variable. The actual process of finding the derivative is too, similar. A vector function in R^n being made up out of a set of n scalar functions, makes finding its derivative an easy process. We can look at each of the scalar functions by itself, and then differentiate them with respect to the curve parameter.

If we interpret a parametric curve as a particle moving through space with a time t, it is then common the denote the first and second derivatives as velocity and acceleration. It is important to not confuse the terms speed, and velocity, with velocity being the direction vector, while speed is the magnitude of velocity. For a curve $r: R \to R^3$ this can be denoted as follows:

$$r(t) = \langle f(t), g(t), h(t) \rangle\,,$$

$$r'(t) = v(t) = \langle f'(t), g'(t), h'(t) \rangle,$$

$$r''(t) = v'(t) = a(t) = \langle f''(t), g''(t), h''(t) \rangle$$

Arclength, and Arclength parametrizations:

The arclength of a curve is defined to be the distance between two points along the path of a curve, it can be derived as follows:

For a parametric curve $r: R \to R^n$ we define a set of points along its path to be $t_1, t_2, \dots, t_n$, spaced by the equal distance $\Delta t_i = t_i - t_{i-1}$. Then the segment between one point and the next can be defined to be a vector $r(t_2) - r(t_1)$, with a length equal $\|r(t_2) - r(t_1)\|$. We can then define the total sum of all the vector distances with a sigma sum to be:

$$\sum_{i}^{n} \|r(t_i) - r(t_{i-1})\|$$

Which can then be rewritten with the use of the relation $\Delta t_i = t_i - t_{i-1}$to be:

$$\sum_{i}^{n} \left\| \frac{r(t_{i-1} + \Delta t_i) - r(t_{i-1})}{\Delta t_i} \right\| \Delta t_i$$

We can now observe that if we let Δt_i approaches 0, the above term becomes an integral over the magnitude of a derivative of the curve, which we define to be the arclength of a curve:

$$L = \lim_{\Delta t_i \to 0} \sum_{i}^{n} \left\| \frac{r(t_{i-1} + \Delta t_i) - r(t_{i-1})}{\Delta t_i} \right\| \Delta t_i = \int_a^b ||r'(t)||dt$$

An application of the arclength is parametrizing a curve by it. A property of all parametrizations is that they are not unique; you can freely reparametrize all parametric functions; redefine their parameter variable. Reparametrizing a function to be parametrized by arclength changes the parameter value from t to s. From a geometric standpoint, it means that now the curve is not defined as a function of how much time has passed, but instead as a function of how much position of how much the object has travelled.

This is performed by finding the arclength function. For $a \leq t \leq b$ the arclength function is defined to be.

$$s = \int_a^t \|r'(\tau)\| \, d\tau$$

The actual process of finding the new parametrization consists simply of solving the above equation for t, and substituting the t of the original vector-value function.

Frenet–Serret frame

When interpreting a curve in R^3 as the position of an object moving through space, we can use a something called a Frenet- Serret frame, or TNB frame, to better describe the properties of that object. The TNB frame consists of three-unit vectors moving together with the imagined object, and together formulating a moving orthonormal basis for R^3.

The first vector that makes up the TNB frame is the unit tangent vector, $\vec{T}(t)$. Like the name suggests it is just the tangent vector normalized. Therefor it will always point in the direction of the objects movement, or rate of change of position. The formula for it is:

$$\vec{T}(t) = \frac{\vec{r'}(t)}{\|\vec{r}'(t)\|}$$

The next vector in the frame is the unit normal vector, $\vec{N}(t)$. It is the rate of change, or the derivative of $\vec{T}(t)$. The formula for it is:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

The unit normal vector is always perpendicular to the unit tangent vector. It indicates the change of the direction of the unit tangent vector, or the direction in which it bends or curves. $\vec{T}(t)$ and $\vec{N}(t)$ always being perpendicular can be shown as follows:

The dot product between the unit tangent vector and itself can be written as, with the angle between them obviously being $\theta = 0$:

$$\vec{T} \cdot \vec{T} = \|\vec{T}\|\|\vec{T}\| \cos\theta = \|\vec{T}\|^2 = 1$$

We can then take the derivative with respect to t of both sides:

$$\frac{d}{dt}\left(\vec{T} \cdot \vec{T}\right) = \frac{d}{dt}(1)$$

Following the chain rule, we get:

$$\vec{T} \cdot \vec{T}' + \vec{T} \cdot \vec{T}' = 2\left(\vec{T} \cdot \vec{T}'\right) = 0$$

Which leads to:

$$\vec{T} \cdot \vec{T}' = 0$$

$\vec{T}'$ is clearly just the normal vector that is not normalized, with normalizing a vector not changing its direction, we can conclude by saying that unit normal vector, and unit tangent vector, are orthogonal.

$$\vec{T} \cdot \vec{N} = 0$$

The last vector that makes up the TNB frame, is the binormal vector, with it being defined as being the cross product between the unit tangent, and unit normal vectors.

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The unit binormal can be interpreted as a representation of a normal vector (not to be confused with the unit normal vector), that defines a plane that best fits the curve at a given instant of time. Therefore, if we found the TNB frame for a curve in R^2, the binormal vector would always give the normal vector defining the xy-plane.

A property of curves related to the TNB frame, is its curvature. The curvature of a curve defines how much a curve is bending, it can be related to the unit normal vector, where the unit normal vector defines in which direction a curve is bending, the curvature defines how much a curve would bend. Where R is the radius of the circle created if the curvature kept on turning with a constant curvature.

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{1}{R}$$

Above stated formula would require the reparameterization of a curve as a function of the arclength. As this process can be difficult or time consuming, there exist an alternative formulation of the curvature:

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

In R^3 the curvature at a given point can be interpreted as the radius of a circle shaped curve in R^2 of the above stated radius that is on the plane created from the binormal vector.

Application of theory:

In this section there will be given examples of applications of the above state theory, on the example function.

$\gamma(t) = \langle t, \cos t \sin t, \sin^2 t \rangle$

This function can be visualized as a helix line in R^3 (Figure 1.1).

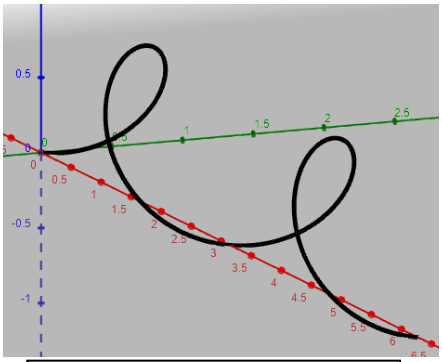

Figure 1.1: Graph of the function $\gamma(t)$

Curve Derivatives:

The derivatives of $\gamma(t)$ can be found with accordance to the process explained in the theoretical part to be:

$$\frac{d}{dt}\gamma(t) = \gamma'(t) = \langle \frac{d}{dt}(t), \frac{d}{dt}(\cos t \sin t), \frac{d}{dt}(\sin^2 t) \rangle = \langle 1, \cos 2t, \sin 2t \rangle,$$

$$\gamma''(t) = \langle \frac{d}{dt}(1), \frac{d}{dt}(\cos 2t), \frac{d}{dt}(\sin 2t) \rangle = \langle 0, -2\sin 2t, 2\cos 2t \rangle$$

Arclength of a curve:

The arclength of $\gamma(t)$ $function$, for one full period; $0 \leq t \leq 2\pi$ is found as follows in accordance to the formula defined in the theory:

$$L = \int_a^b ||r'(t)||dt = \int_0^{2\pi} \sqrt{1^2 + (\cos(2t))^2 + (\sin(2t))^2}\, dt$$

We can observe the trigonometric identity $\cos^2 x + \sin^2 x = 1$ and simplify the expression to

$$L = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$$

Parametrizing a curve by its arclength:

Parametrizing $\gamma(t)$ $function$, for one full period; $0 \leq t \leq 2\pi$ by arclength requires it to be defined an arclength function, in accordance to theory, and the previously found result for the arclength integral:

$$s = \int_a^t \|r'(\tau)\| \, d\tau = \int_0^t \sqrt{2} \, dt = t\sqrt{2} \Rightarrow t = \frac{s}{\sqrt{2}}$$

We can then substitute t in the previous function to parametrize it by the arclength.

$$\gamma(s) = \langle \frac{s}{\sqrt{2}}, \cos\left(\frac{s}{\sqrt{2}}\right) \sin\left(\frac{s}{\sqrt{2}}\right), \sin^2\left(\frac{s}{\sqrt{2}}\right) \rangle \ ; 0 \leq s \leq 2\pi\sqrt{2}$$

Frenet-Serret frame components:

Finding the unit tangent vector, unit normal vector, and unit binormal vector, is the process of following the formulas stated in the theory, and using the previously found results of the derivative:

$$\vec{T}(t) = \frac{\vec{r'}(t)}{\|\vec{r}'(t)\|} = \frac{\langle 1, \cos 2t, \sin 2t \rangle}{\sqrt{2}} = \langle \frac{1}{\sqrt{2}}, \frac{\cos 2t}{\sqrt{2}}, \frac{\sin 2t}{\sqrt{2}} \rangle$$

Taking the derivative of the unit tangent vector, and dividing it by its length gives:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|} = \frac{\langle 0, -\frac{2\sin 2t}{\sqrt{2}}, \frac{2\cos 2t}{\sqrt{2}} \rangle}{\sqrt{2}} = \langle 0, -\sin 2t, \cos 2t \rangle$$

Finding the binormal vector requires taking the cross product between the unit tangent vector, and the unit normal vector, following the rules of cross product for vectors in R^3, the process is as follows:

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \langle \frac{1}{\sqrt{2}}, \frac{\cos 2t}{\sqrt{2}}, \frac{\sin 2t}{\sqrt{2}} \rangle \times \langle 0, -\sin 2t, \cos 2t \rangle$$

$$\vec{B}(t) = \langle \begin{vmatrix} \frac{\cos 2t}{\sqrt{2}} & , \frac{\sin 2t}{\sqrt{2}} \\ -\sin 2t & \cos 2t \end{vmatrix}, -\begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{\sin 2t}{\sqrt{2}} \\ 0 & \cos 2t \end{vmatrix}, \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{\cos 2t}{\sqrt{2}} \\ 0 & -\sin 2t \end{vmatrix} \rangle$$

$$\vec{B}(t) = \langle \frac{1}{\sqrt{2}}, -\frac{\cos 2t}{\sqrt{2}}, -\frac{\sin 2t}{\sqrt{2}} \rangle$$

Calculating the curvature is again using one of the formulas given in the theory, here the process is:

$$\kappa(t) = \frac{\left\|\vec{T}'(t)\right\|}{\|\vec{r}'(t)\|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Meaning that for every point at that curve, the curvature remains constant, and represents a circle with a radius of 1.

Topic 2 Differentiability of functions of two variables

Partial derivative:

Definition:

For a function of two variables $f: R^2 \to R$, at a point $p = (x_o, y_o)$ the partial derivatives with respect to x, and y, are defined to be:

$$\frac{\partial f}{\partial x}_{(x_o,y_o)} = f_{x(x_o,y_o)} = \lim_{h\to 0}\frac{f(x_o+h,\ y_o) - f(x_o,y_o)}{h}$$

$$\frac{\partial f}{\partial y}_{(x_o,y_o)} = f_{y(x_o,y_o)} = \lim_{h\to 0}\frac{f(x_o,y_o+h) - f(x_o,y_o)}{h}$$

In single variable calculus, the derivative of a function $f: R \to R$ at a point tells the rate of change of the function with respect to the function variable at that point. Graphically this would represent the linear approximation of the slope of the tangent line at that point. This interpretation of the derivative can be translated into multivariable functions, $f: R^2 \to R$, using the partial derivative.

The partial derivative behaves similarly to the single variable derivative. Instead of giving the rate of change of the whole function, it only tells the rate of change in the x, or y direction. The only difference in the process of finding the actual derivative lies in the fact that function variables that are not being differentiated with respect to, are seen as constants. Otherwise, the process follows the usual rules of differentiation. Graphically its application shares some similarities. Where in $f: R \to R$ the derivative tells us the slope of the tangent line, in $f: R^2 \to R$ the partials tell us the slopes of a plane in their respective directions. The plane $T_p(x, y)$ at a point $p = (x_o, y_o)$, can then be given by:

$$T_p(x,y) = f_{(x_o,y_o)} + f_{x(x_o,y_o)}(x - x_o) + f_{y(x_o,y_o)}(y - y_o)$$

In addition to those basic properties, the partial derivatives have other important applications that will come to light later in the text.

Differentiability, and continuous differentiability

Definition:

A function of two variables $f: R^2 \to R$ is defined to be differentiable at a point $p = (x_o, y_o)$ if:

$$\lim_{(h,k)\to(0,0)}\frac{f(x_o+h, y_o+k) - f(x_o,y_o) - hf_x(x_o,y_o) - kf_y(x_o,y_o)}{\sqrt{h^2+k^2}} = 0$$

In functions of single variables, a function being differentiable means that its derivative exists at every point in the domain, and therefore the tangent line can be found for any point on the function.

Conversely, for functions of two variables $f: R^2 \to R$ a function being differentiable implies the existence of the directional derivative in any direction, and existence of a tangent plane, at any point in its domain. Graphically this means that the graph has no sharp edges, turns, points, etc, in other words that it is smooth.

A way to check if a function is differentiable, is checking if it is continuously differentiable. A function is continuously differentiable, if all its partial derivatives are continuous. Continuous differentiability implies differentiability; however, this implication does not hold the other way around, this means that a function can still be differentiable, even where its partial derivatives are not continuous. In cases where there are points in the function where the partial derivatives are not continuous, one might apply the actual limit definition to those points to determine if a function is differentiable at its entire domain.

Multivariable chain rule:

Definition:

For a function of two variables $f: R^2 \to R, z = f(x, y)$ Let the function variables be functions of one or more variables, then the chain rule is defined as:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

The multivariable chain rule considers compositions of functions, where the function variables of a multivariable functions, themselves, are functions. In both single, and multivariable calculus, the rule considering compositions of functions is called the chain rule, however, the rule itself clearly differs vastly between those two.

The directional derivatives, and the gradient:

Definition:

$The\ direction\ derivative\ of\ a\ \ function\ of\ n\ variables\ f: R^2 \to R, at\ a\ point$

$p = (x_o, y_o), in\ the\ direction\ of\ a\ unit\ vector\ v = (v_x, v_y)\ is\ defined\ as:$

$$\partial_v(p) = \lim_{h \to 0} \frac{f(p + hv) - f(p)}{h}$$

An extension of the partial derivatives is the directional derivative. Where the partial derivatives tell the rate of change in the direction of the axis, the directional derivative tells the rate of change in the direction of any vector at a point. Actually, if you choose the vector direction to be $v = (1,0)$, or $v = (0,1)$ the directional derivative will be equal to the partial derivative in the according directions at a point. It is important to note that because we only use the vector to indicate a direction, it must be normalised. It is also then possible to interpret the direction vector as an angle, by calculating either $\theta = \cos^{-1} v_x$, or $\theta = \sin^{-1} v_y$.

A concept closely related to the directional derivative is the gradient. Gradient of a function is a vector-value function, consisting of the partial derivatives. It can be derived from the formula for the directional derivative as follows:

The directional derivative can be restated as:

$$D_v f(x_o, y_o) = \frac{d}{dt} f\left(x_o + tv_x, y_o + tv_y\right)$$

Which, following the multivariable chain rule can be rewritten to:

$$D_v f(x_o, y_o) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Redefine the right side of the equation as a dot product:

$$D_v f(x_o, y_o) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

We can now define gradient as the left side of the dot product, and denote it as $\nabla f(x, y)$:

$$D_v f(x_o, y_o) = \nabla f \cdot v = \|\nabla f\|\|v\| \cos\theta = \|\nabla f\| \cos\theta$$

Then, for a function of two variables $f: R^2 \to R$, the gradient is defined as:

$$\nabla f(x, y) = \langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \rangle$$

Probably one of the clearest applications of the gradient, looking at the above stated formula, is its use to find the partial derivative without having to use the limit definition, making it a much simpler process. However, the gradient itself has its own additional applications.

Let $r(t) = \langle x(t), y(t) \rangle$ be a level curve to a graph, such that $f\left(x(t), y(t)\right) = C$, where C is a constant. Then, differentiating f with respect to t, is given by the multivariable chain rule as:

$$\frac{d}{dt} f\left(x(t), y(t)\right) = \frac{d}{dt} C \Longrightarrow \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

Redefining the left side of the equation as a dot product, where $\frac{dr}{dt}$ is the tangent vector of the level curve:

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0 \Longrightarrow \nabla f \cdot \frac{dr}{dt} = 0$$

The dot product between the gradient and the tangent curve being 0, means that those vectors are perpendicular to each other. The gradient being perpendicular to the tangent curve means that the tangent is always normal to any level surface. Because of this property, the gradient can be used to find the normal line to the graph at any point, given as a parametric function:

$$r(t) = \langle x_o, y_o, z_o \rangle + t\nabla f(x_o, y_o, z_o)$$

An important property of the gradient is that it is always pointing in the direction of the largest increase of the function. When looking at the cosine definition of the dot product of the directional derivative, we can observe that the value will always be largest when $\theta = 0$. At that angle the direction vector, and the gradient are parallel, therefor the direction of the highest increase is always the direction in which the gradient is pointing.

This can be visualized by interpreting the gradient as a vector-value function that represents a vector field, where at all points the vectors are in the direction of the largest increase, and their magnitude represents how large that increase is. Graphing such vector function could like Figure 2.1, where the vector field is shown on the xy-plane, with the corresponding function above it.

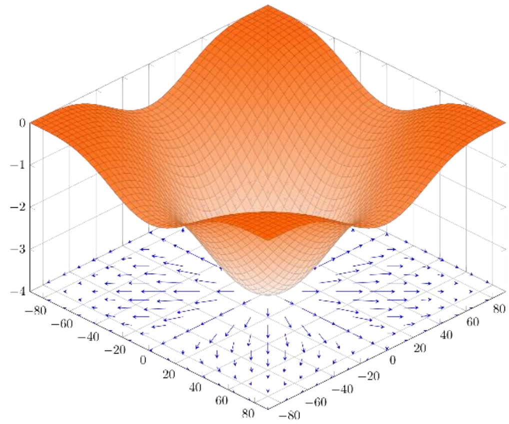

Figure 2.1: Graph and vector field of the function $f(x,y) = -(\cos^2 x + \cos^2 y)^2$

Source: https://en.wikipedia.org/wiki/Gradient#/media/File:3d-gradient-cos.svg

Applications of theory:

In this section there will be given examples of applications of the above state theory, on the example function:

$$f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

Partial Derivatives, and the Gradient:

In accordance with theory, finding the partial derivatives requires considering the variables that are not being derived, as constants, therefore the process of finding the partial derivatives is a as follows:

Using the chain rule:

$$\frac{\partial f}{\partial x}(x,y) = \left(-\frac{y^3}{(x^2+y^2)^2}\right)\left(\frac{\partial}{\partial x}(x^2+y^2)\right) = -\frac{2y^3x}{(x^2+y^2)^2}$$

Using the quotient rule:

$$\frac{\partial f}{\partial y}(x,y) = \frac{\left(\frac{\partial}{\partial y}(y^3)\right)(x^2+y^2) - \left(\frac{\partial}{\partial y}(x^2+y^2)\right)(y^3)}{(x^2+y^2)^2} = \frac{3y^2x^2+y^4}{(x^2+y^2)^2}$$

Therefore, the gradient will be:

$$\nabla f(x,y) = \langle -\frac{2y^3x}{(x^2+y^2)^2}, \frac{3y^2x^2+y^4}{(x^2+y^2)^2} \rangle$$

Differentiability, and continuous Differentiability:

We can recall from theory that a function is continuously differentiable, if all the partial derivatives exist, and are continuous. For the function $f(x, y)$, we can observe that both partial derivatives are continuous everywhere where the denominator is not equal to zero, and that both partial derivatives have the same denominator.

$$(x^2 + y^2)^2 \neq 0$$

Clearly, because of the square terms, the only case in which the denominator can be equal to zero, is at the point $(x, y) = (0,0)$, therefore it is not necessary to check the continuity at the point (1,1). Therefore, in this case it is required to check the limit to determine if the partial derivatives are continuous at the point (0,0).

$$\lim_{(x,y)\to(0,0)} -\frac{2xy^3}{(x^2 + y^2)^2}$$

Approaching the limit from x=y:

$$\lim_{y\to 0} -\frac{2y^4}{(2y^2)^2} = -\frac{1}{2}$$

Approaching the limit from x=0:

$$\lim_{y\to 0} \frac{0}{y^4} = 0$$

From two approaches to the limit being different, we can conclude that thef_x, is not continuous at (0,0). The requirement for continuous differentiability is that all the partial derivatives are continuous, therefore it is not necessary to check if f_y is continuous. Because of continuous differentiability implying differentiability, we can conclude that $f(x, y)$ is continuously differentiable everywhere but at (0,0), we have not yet determined if the function is differentiable at (0,0), to which we will be required to use the limit definition of differentiability to determine if it is.

$$\lim_{(h,k)\to(0,0)} \frac{f(x_o + h, y_o + k) - f(x_o, y_o) - hf_x(x_o, y_o) - kf_y(x_o, y_o)}{\sqrt{h^2 + k^2}} = 0$$

Because of the partials not being continuous at (0,0), it will be required to use the limit definition to calculate the partial derivatives at (0,0).

$$f_x = \lim_{h\to 0} \frac{(f(h, 0) - f(0,0))}{h} = \lim_{h\to 0} \left(\frac{f(h, 0)}{h}\right) = \frac{0}{h^3} = 0$$

$$f_y = \lim_{k\to 0} \frac{(f(0, k) - f(0,0))}{h} = \lim_{h\to 0} \left(\frac{f(0, k)}{k}\right) = \frac{k^3}{k^3} = 1$$

Therefore, the limit definition of differentiability becomes:

$$\lim_{(h,k)\to(0,0)} \frac{f(h, k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{\frac{k^3}{h^2 + k^2} - k}{\sqrt{h^2 + k^2}}$$

There requirement for differentiability is that at from any approach the limit would be equal to zero. We can then approach the limit from $h = k$:

$$\lim_{k\to 0}\frac{\frac{k^3}{k^2+k^2}-k}{\sqrt{k^2+k^2}} = \lim_{k\to 0} -\frac{k^3}{2k^2\sqrt{2k^2}} = \lim_{k\to 0}\frac{k^3}{2^{\frac{3}{2}}k^3} = \frac{1}{2\sqrt{2}}$$

$\frac{1}{2\sqrt{2}} \neq 0$ from which we can conclude that $f(x,y)$ is not differentiable at (0,0).

Directional Derivative:

Because of the function not being differentiable at (0,0), we can not determine any directional derivatives and therefore the direction of greatest increase, at that point.

Determining the direction of greatest increase at (1,1) in accordance to theory, is simply a matter of checking the direction in which the gradient is pointing at that point, the process is as follows:

$$\nabla f(1,1) = \langle -\frac{2(1)^3(1)}{((1)^2+(1)^2)^2}, \frac{3(1)^2(1)^2+(1)^4}{((1)^2+(1)^2)^2}\rangle = \langle -\frac{1}{2}, 1\rangle$$

Which means that at the point (1,1) the function is increasing fastest in the direction of the vector $\langle -\frac{1}{2}, 1\rangle$, with a magnitude of $\frac{\sqrt{5}}{2}$.

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