Christmas Holiday Aurora, their Relation to Earths Magnetic field and Solar Winds

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Abstract

Earths magnetic field is generated by the earths outer core. The core can be approximated as a magnetic dipole embedded in the centre of our planet. Solar winds which carry charged particles can react with this field, carrying them to earths atmosphere. This short study will use numerical methods, like RK4, to numerically approximate what paths positively unit charged particles (protons) will take inside the magnetic field. Notice how small errors accumulate, especially when rapid acceleration occurs, making very long term numerical calculations of this kind not advised. But the general trend of the particles coming closest to the earth at the poles are observed, and is what makes the oh so beautiful aurora appear.

1. Introduction

The motion of Earth's outer core, which is composed of liquid metals, generates a magnetic field strong enough to reach space, where it can interact with solar winds. Solar winds are a stream of charged particles released from the sun at high velocities. If earth is in the direction of one of these outbursts, particles can be bound to our magnetic field. There they will occasionally collide with the atmospheric particles, releasing energy in the form of light, which we know as Aurora. By numerical methods we will study these movements of the particles.

2. Theory

We assume Earth's magnetic field is a pure magnetic dipole, as the radius of the whole earth is much greater than the radius of its core. The general expression (coordinate free form) for a magnetic field of a dipole [1](page 255) is given by:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}] \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the center of the earth (our "perfect" dipole). To be able to solve this numerically, we must find the magnetic dipole moment $\vec{m}$ for the earth in (1). We consider such that $\vec{m}$ is along the magnetic dipole. This way using the property of a pure dipole [1](page 255):

$$\vec{B}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \tag{2}$$

at the equator, where $\theta = 90^\circ$, we receive:

$$\vec{B_0} = \frac{\mu_0}{4\pi}\frac{\vec{m}}{R_E^3} \tag{3}$$

Where $B_0 = 3.0\cdot 10^{-5}T$ [4], and $R_E = 6.371Mm$ (google). Now solving for $\vec{m}$ in (3), and substituting this in (1), we get:

$$\vec{B}(\vec{r}) = \frac{B_0 R_E^3}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}] \tag{4}$$

Setting the x-axis from the sun to the earth, and the z-axis perpendicular to the ecliptic of the earth, we can make a new coordinate system such that x and z fulfill these conditions, at a degree $\alpha = 9.6^\circ$ [4] from $\vec{m}$:

$$m_x = m\sin\alpha \tag{5}$$

$$m_y = 0 \tag{6}$$

$$m_z = m\cos\alpha \tag{7}$$

Then:

$$\vec{m} = \begin{bmatrix} m\sin\alpha \\ 0 \\ m\cos\alpha \end{bmatrix} \tag{8}$$

Where m is the magnitude of earths magnetic dipole moment. Using this, the property of a vector $\hat{r} = \frac{\vec{r}}{r}$ and dot product between $\vec{r}$ and $\vec{m}$, we can separate the Cartesian components of the magnetic field:

$$B_x = B_0\frac{R_E^3}{r^3}\left[\frac{3(xm_x + zm_z)x}{r^2} - m_x\right] \tag{9}$$

$$B_y = B_0\frac{R_E^3}{r^3}\left[\frac{3(xm_x + zm_z)y}{r^2}\right] \tag{10}$$

$$B_z = B_0\frac{R_E^3}{r^3}\left[\frac{3(xm_x + zm_z)z}{r^2} - m_z\right] \tag{11}$$

The force of a magnetic field $\vec{B}$ on a moving charge q with velocity $\vec{v}$ is [1](page 212):

$$\vec{F} = q\vec{v}\times\vec{B} \tag{12}$$

Using $F = ma$ we find the acceleration of the particle:

$$\vec{a} = \frac{q\vec{v}\times\vec{B}}{m} \tag{13}$$

From (12) and (Eq 5.11) [1](page 215) we see that the magnetic force is perpendicular to the magnetic field and velocity of the particle. Therefore the work done by magnetic field is always 0, and hence there is no change in the magnitude of velocity. In other words:

$$\Delta v = 0 \tag{14}$$

3. Methods

To track the trajectory of the particles we solve differential equations of motion. But these are not simple to solve and therefore we need numerical methods, which can approximate.

As the programming language i am using Python, with the help of numpy and matplotlib. Where the method implemented for solving the ODEs is RK4.

RK4 or Runge-Kutta 4 method, is a technique used to analyze Differential equations by linear approximation, where small tangents lines over very short distances approximate a solution to the ODE [2]. The 4 refers to it being a fourth order RK method, and how for every "step" it takes, it takes 3 smaller steps, as corrections for the original calculation. RK4 is defined as:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{15}$$
$$t_{n+1} = t_n + h \tag{16}$$
$$k_1 = f(t_n, y_n) \tag{17}$$
$$k_2 = f(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}) \tag{18}$$
$$k_3 = f(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}) \tag{19}$$
$$k_4 = f(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}) \tag{20}$$

where $f = y'(y(t), t)$ is the rate of change of y and h is the step size.

The numerical error using RK4 and Runge-Kutta in general, accumulates with each step and is proportional to the step size [2]:

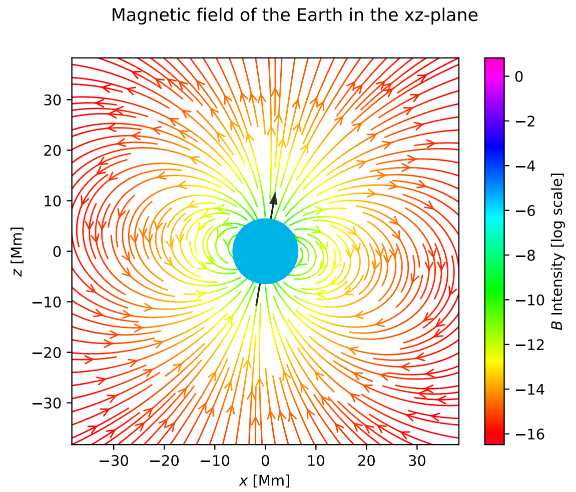
$$\|e_n\| = O(h^4) \tag{21}$$

where e_N is the error in the n-th step and O (big O) can be understood as a constant.

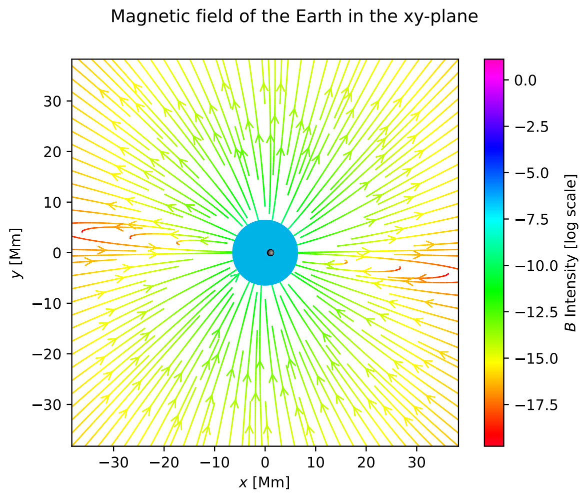
As RK4 is a fourth order method, it is important choosing a suitable h that is not too large to prevent inaccurate results. But big changes over small time periods will produce more inaccurate results.

4. Results

Using numpys meshgrid functionality, we can calculate earths magnetic field (see code).



The blue circle is the Earth with the magnetic dipole moment represented by the arrow. The x-axis is from the sun toward the Earth, and the z-axis is perpendicular to the ecliptic. Notice that the strength of the field decreases rapidly as we go further away from the center, much like our real earth.



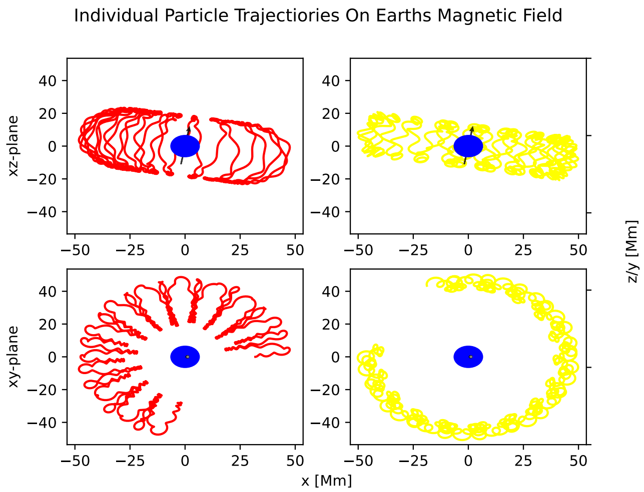
Notice the small arrow still indicating the magnetic dipole moment. As the dipole is not aligned with the z axis, the streamlines do not point radially out as we expect from a point charge. Also notice how the field curves in the x-axis because of this slight tilt.

To study the particles from the solar wind in this magnetic field, we simulate them with the proton mass $m_p = 1.65 \times 10^{-27} kg$, charge $q = 1.6 \times 10^{-19} C$ and constant velocity $24 \times 10^6 m/min$ [3]. To study different initial conditions and how they effect the paths they take, we define:

	Proton 1	Proton 2
x_0	$5R_E$	$-7R_E$
y_0	0	$0.5R_E$
z_0	$2R_E$	$2R_E$

We set the time step as $\Delta t = 0.0001$ min for an interval of

100 minutes, as this gave good results, without having to wait a long time.



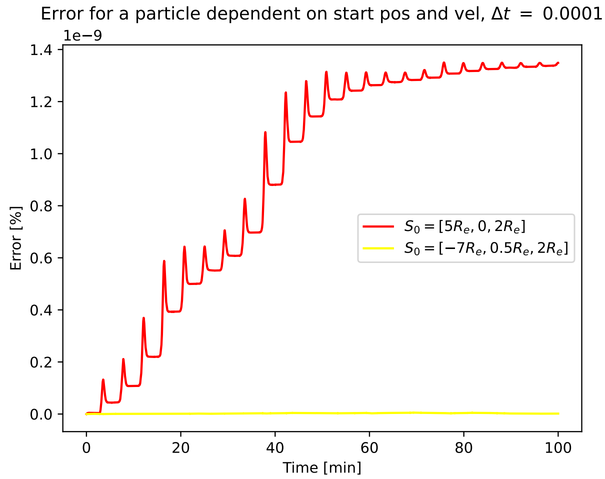
Trajectories of 2 protons around the earth, with different initial conditions, following earths magnetic field. Notice how the proton "fall" into the poles.

The two particles behave a little differently, as they have different initial conditions. The red one had a more direct trajectory towards the earth, while the yellow one started more perpendicular to the earth. Therefore the yellow proton is more "stable" as it can hang further away. But notice the similar path they take. They are following the field lines, as expected of charged particles in this type of field. Most importantly, they are the closest to the earth at the poles (where the magnetic field is the strongest). When they come too close to the earth they will collide with the atmosphere and react creating different colors of light, and thus creating the Aurora. Since they are the closest at the magnetic poles, here we expect the Aurora to be strongest, which of course is true!

5. Discussion

Using (14), we can check the relative error in our simulation using:

$$e_n = \frac{\Delta v_n}{v_0} = \frac{v_n - v_0}{v_0} \tag{22}$$



Notice the jumps in the red, and how they almost perfectly align with both the frequency and amount of times the red charge comes towards earths magnetic pole, where the magnetic field is the strongest. This supports the idea of strong acceleration being bad for these methods of numerical analysis.

As expected the error accumulates with each step.

6. Summary and conclusions

By approximating the earths magnetic field as a perfect magnetic dipole in the center of the earth, and using RK4 as a numerical method, we have simulated the movement of some trapped charged particles across earths field. We see they follow the magnetic field lines of the earth while orbiting it as well, much like we expect of a charge particle in a dipole magnetic field. They get the closest to earth at the magnetic poles, where they interact and create the famous (Christmas holiday) Aurora.

Numerical methods can be unreliable, and as we have seen especially at points of high acceleration. But they are good enough at showing the general, and wonderful trend, of how the solar wind in earths magnetic field make Aurora.

References

[1] Introduction to electrodynamics (Fourth Edition). David J. Griffiths and Cambridge University Press (2013).
[2] Wikipedia Contributors (2017). Runge–Kutta methods. Acquired at: https://en.wikipedia.org/wiki/Runge
[3] Dr. David H. Hathaway. Solar Physics. Acquired at: https://solarscience.msfc.nasa.gov/SolarWind.shtml
[4] Wkipedia Contributors (2019). Earth's magnetic field. Acquired at: https://en.wikipedia.org/wiki/Earth