FastTransforms Documentation

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1 Introduction

`FastTransforms` is a C library based on the solutions of the two-dimensional harmonic polynomial connection problems in [1, 2] that have an $\mathcal{O}(n^3)$ run-time, where n is the polynomial degree, and are 2-normwise backward stable.

The transforms are separated into computational kernels that offer SSE, AVX, and AVX-512 vectorization on applicable Intel processors, and driver routines that are easily parallelized by OpenMP.

2 What `FastTransforms` actually computes

For every subsection below, the title of the subsection, of the form `a2b`, refers conceptually to the transform and the available functions are as follows:

- `ft_plan_a2b`, is a pre-computation,
- `ft_execute_a2b`, is a forward execution,
- `ft_execute_b2a`, is a backward execution,
- `ft_execute_a_hi2lo`, is a conversion to a tensor-product basis,
- `ft_execute_a_lo2hi`, is a conversion from a tensor-product basis,
- `ft_kernel_a_hi2lo`, is an orthonormal conversion from high to low order,
- `ft_kernel_a_lo2hi`, is an orthonormal conversion from low to high order.

The `ft_execute_*` functions are drivers that perform transforms as defined below. They are composed of computational kernels, `ft_kernel_*` functions, that may be assembled differently.

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2.1 `sph2fourier`

Spherical harmonics are:

$$Y_\ell^m(\theta,\varphi) = \frac{e^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}(-1)^{|m|}\sqrt{(\ell+\tfrac{1}{2})\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_\ell^{|m|}(\cos\theta), \tag{1}$$

where $P_\ell^m(\cos\theta)$ are the associated Legendre functions. A degree-n expansion in spherical harmonics is given by:

$$f_n(\theta,\varphi) = \sum_{\ell=0}^{n}\sum_{m=-\ell}^{+\ell} f_\ell^m Y_\ell^m(\theta,\varphi). \tag{2}$$

If spherical harmonic expansion coefficients are organized into the array:

$$F = \begin{pmatrix} f_0^0 & f_1^{-1} & f_1^1 & f_2^{-2} & f_2^2 & \cdots & f_n^{-n} & f_n^n \\ f_1^0 & f_2^{-1} & f_2^1 & f_3^{-2} & f_3^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ f_{n-2}^0 & f_{n-1}^{-1} & f_{n-1}^1 & f_n^{-2} & f_n^2 & & \vdots & \vdots \\ f_{n-1}^0 & f_n^{-1} & f_n^1 & 0 & 0 & \cdots & 0 & 0 \\ f_n^0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{3}$$

then `sph2fourier` returns the bivariate Fourier coefficients:

$$G = \begin{pmatrix} g_0^0 & g_0^{-1} & g_0^1 & \cdots & g_0^{-n} & g_0^n \\ g_1^0 & g_1^{-1} & g_1^1 & \cdots & g_1^{-n} & g_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{n-1}^0 & g_{n-1}^{-1} & g_{n-1}^1 & \cdots & g_{n-1}^{-n} & g_{n-1}^n \\ g_n^0 & 0 & 0 & \cdots & g_n^{-n} & g_n^n \end{pmatrix}. \tag{4}$$

That is:

$$f_n(\theta,\varphi) = \sum_{\ell=0}^{n}\sum_{m=-n}^{+n} g_\ell^m \frac{e^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\begin{cases}\cos(\ell\theta) & m \text{ even},\\ \sin((\ell+1)\theta) & m \text{ odd}.\end{cases} \tag{5}$$

Since `sph2fourier` only transforms columns of the arrays, the routine is indifferent to the choice of longitudinal basis; it may be complex exponentials or sines and cosines, with no particular normalization.

2.2 `spinsph2fourier`

Spin-weighted spherical harmonics are:

$$Y_{\ell,m}^s(\theta,\varphi) = \frac{e^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\sqrt{(\ell+\tfrac{1}{2})\frac{(\ell+\ell_0)!(\ell-\ell_0)!}{(\ell+\ell_1)!(\ell-\ell_1)!}} \\ \times \sin^{|m+s|}(\tfrac{\theta}{2})\cos^{|m-s|}(\tfrac{\theta}{2})P_{\ell-\ell_0}^{(|m+s|,|m-s|)}(\cos\theta). \tag{6}$$

where $P_n^{(\alpha,\beta)}(\cos\theta)$ are the Jacobi polynomials and $\ell_0 = \max\{|m|, |s|\}$ and $\ell_1 = \min\{|m|, |s|\}$. A degree-$n$ expansion in spin-weighted spherical harmonics is given by:

$$f_n^s(\theta,\varphi) = \sum_{\ell=\ell_0}^{n} \sum_{m=-\ell}^{+\ell} f_\ell^m Y_{\ell,m}^s(\theta,\varphi). \tag{7}$$

If spin-weighted spherical harmonic expansion coefficients with $s = 2$, for example, are organized into the array:

$$F = \begin{pmatrix} f_2^0 & f_2^{-1} & f_2^1 & f_2^{-2} & f_2^2 & \cdots & f_n^{-n} & f_n^n \\ f_3^0 & f_3^{-1} & f_3^1 & f_3^{-2} & f_3^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ f_n^0 & f_n^{-1} & f_n^1 & f_n^{-2} & f_n^2 & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{8}$$

then `spinsph2fourier` returns the bivariate Fourier coefficients:

$$G = \begin{pmatrix} g_0^0 & g_0^{-1} & g_0^1 & \cdots & g_0^{-n} & g_0^n \\ g_1^0 & g_1^{-1} & g_1^1 & \cdots & g_1^{-n} & g_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{n-1}^0 & g_{n-1}^{-1} & g_{n-1}^1 & \cdots & g_{n-1}^{-n} & g_{n-1}^n \\ g_n^0 & 0 & 0 & \cdots & g_n^{-n} & g_n^n \end{pmatrix}. \tag{9}$$

That is:

$$f_n(\theta,\varphi) = \sum_{\ell=0}^{n} \sum_{m=-n}^{+n} g_\ell^m \frac{e^{\mathrm{i}m\varphi}}{\sqrt{2\pi}} \begin{cases} \cos(\ell\theta) & m+s \text{ even}, \\ \sin((\ell+1)\theta) & m+s \text{ odd}. \end{cases} \tag{10}$$

Since `spinsph2fourier` only transforms columns of the arrays, the routine is indifferent to the choice of longitudinal basis; it may be complex exponentials or sines and cosines, with no particular normalization.

2.3 `tri2cheb`

Triangular harmonics are:

$$\tilde{P}_{\ell,m}^{(\alpha,\beta,\gamma)}(x,y) = (2-2x)^m \tilde{P}_{\ell-m}^{(2m+\beta+\gamma+1,\alpha)}(2x-1)\tilde{P}_m^{(\gamma,\beta)}\left(\frac{2y}{1-x} - 1\right), \tag{11}$$

where the tilde implies that the univariate Jacobi polynomials are orthonormal. A degree-n expansion in triangular harmonics is given by:

$$f_n(x,y) = \sum_{\ell=0}^{n} \sum_{m=0}^{\ell} f_\ell^m \tilde{P}_{\ell,m}^{(\alpha,\beta,\gamma)}(x,y). \tag{12}$$

If triangular harmonic expansion coefficients are organized into the array:

$$F = \begin{pmatrix} f_0^0 & f_1^1 & f_2^2 & \cdots & f_n^n \\ \vdots & \vdots & \vdots & \iddots & 0 \\ f_{n-2}^0 & f_{n-1}^1 & f_n^2 & \iddots & \vdots \\ f_{n-1}^0 & f_n^1 & 0 & \cdots & 0 \\ f_n^0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{13}$$

then `tri2cheb` returns the bivariate Chebyshev coefficients:

$$G = \begin{pmatrix} g_0^0 & g_0^1 & \cdots & g_0^n \\ g_1^0 & g_1^1 & \cdots & g_1^n \\ \vdots & \vdots & \ddots & \vdots \\ g_n^0 & g_n^1 & \cdots & g_n^n \end{pmatrix}. \tag{14}$$

That is:

$$f_n(x,y) = \sum_{\ell=0}^{n} \sum_{m=0}^{n} g_\ell^m T_\ell(2x-1) T_m\left(\frac{2y}{1-x} - 1\right). \tag{15}$$

2.4 `disk2cxf`

Disk harmonics are Zernike polynomials:

$$Z_\ell^m(r,\theta) = \sqrt{2\ell+2}\, r^{|m|} P_{\frac{\ell-|m|}{2}}^{(0,|m|)}(2r^2-1) \frac{e^{\mathrm{i}m\theta}}{\sqrt{2\pi}}. \tag{16}$$

A degree-$2n$ expansion in disk harmonics is given by:

$$f_{2n}(r,\theta) = \sum_{\ell=0}^{2n} \sum_{m=-\ell,2}^{+\ell} f_\ell^m Z_\ell^m(r,\theta), \tag{17}$$

where the $,2$ in the inner summation index implies that the inner summation runs from $m=-\ell$ in steps of 2 up to $+\ell$. If disk harmonic expansion coefficients are organized into the array:

$$F = \begin{pmatrix} f_0^0 & f_1^{-1} & f_1^1 & f_2^{-2} & f_2^2 & \cdots & f_{2n}^{-2n} & f_{2n}^{2n} \\ f_2^0 & f_3^{-1} & f_3^1 & f_4^{-2} & f_4^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ f_{2n-4}^0 & f_{2n-3}^{-1} & f_{2n-3}^1 & f_{2n-2}^{-2} & f_{2n-2}^2 & & \vdots & \vdots \\ f_{2n-2}^0 & f_{2n-1}^{-1} & f_{2n-1}^1 & f_{2n}^{-2} & f_{2n}^2 & \cdots & 0 & 0 \\ f_{2n}^0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{18}$$

then `disk2cxf` returns the even Chebyshev–Fourier coefficients:

$$G = \begin{pmatrix} g_0^0 & g_0^{-1} & g_0^1 & g_0^{-2} & g_0^2 & \cdots & g_0^{-2n} & g_0^{2n} \\ g_2^0 & g_2^{-1} & g_2^1 & g_2^{-2} & g_2^2 & \cdots & g_2^{-2n} & g_2^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-2}^0 & g_{2n-2}^{-1} & g_{2n-2}^1 & g_{2n-2}^{-2} & g_{2n-2}^2 & \cdots & g_{2n-2}^{-2n} & g_{2n-2}^{2n} \\ g_{2n}^0 & 0 & 0 & g_{2n}^{-2} & g_{2n}^2 & \cdots & g_{2n}^{-2n} & g_{2n}^{2n} \end{pmatrix}. \tag{19}$$

That is:

$$f_{2n}(r,\theta) = \sum_{\ell=0}^{n} \sum_{m=-2n}^{+2n} g_{2\ell}^m \frac{e^{im\theta}}{\sqrt{2\pi}} \begin{cases} T_{2\ell}(r) & m \text{ even}, \\ T_{2\ell+1}(r) & m \text{ odd}. \end{cases} \tag{20}$$

Since `disk2cxf` only transforms columns of the arrays, the routine is indifferent to the choice of azimuthal basis; it may be complex exponentials or sines and cosines, with no particular normalization.

2.5 `tet2cheb`

Tetrahedral harmonics are:

$$\begin{aligned} \tilde{P}_{k,\ell,m}^{(\alpha,\beta,\gamma,\delta)}(x,y,z) = {} & (2-2x)^{\ell+m} \tilde{P}_{k-\ell-m}^{(2\ell+2m+\beta+\gamma+\delta+2,\alpha)}(2x-1) \\ & \times \left(2 - \frac{2y}{1-x}\right)^m \tilde{P}_m^{(2m+\gamma+\delta+1,\beta)}\left(\frac{2y}{1-x} - 1\right) \\ & \times \tilde{P}_m^{(\delta,\gamma)}\left(\frac{2z}{1-x-y} - 1\right), \end{aligned} \tag{21}$$

where the tilde implies that the univariate Jacobi polynomials are orthonormal. A degree-n expansion in tetrahedral harmonics is given by:

$$f_n(x,y,z) = \sum_{k=0}^{n} \sum_{\ell=0}^{k} \sum_{m=0}^{k-\ell} f_{k,\ell}^m \tilde{P}_{k,\ell,m}^{(\alpha,\beta,\gamma,\delta)}(x,y,z). \tag{22}$$

$$f_n(x,y,z) = \sum_{m=0}^{n} \sum_{\ell=0}^{n-m} \sum_{k=\ell+m}^{n} f_{k,\ell}^m \tilde{P}_{k,\ell,m}^{(\alpha,\beta,\gamma,\delta)}(x,y,z). \tag{23}$$

If tetrahedral harmonic expansion coefficients are organized into the rank-3 array whose m^{th} slice is:

$$F[0:n-m, 0:n-m, m] = \begin{pmatrix} f_{m,0}^m & f_{1+m,1}^m & f_{2+m,2}^m & \cdots & f_{n,n-m}^m \\ \vdots & \vdots & \vdots & \iddots & 0 \\ f_{n-2,0}^m & f_{n-1,1}^m & f_{n,2}^m & \iddots & \vdots \\ f_{n-1,0}^m & f_{n,1}^m & 0 & \cdots & 0 \\ f_{n,0}^m & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{24}$$

then `tet2cheb` returns the trivariate Chebyshev coefficients stored in the rank-3 array G. That is:

$$f_n(x,y,z) = \sum_{k=0}^{n}\sum_{\ell=0}^{n}\sum_{m=0}^{n} g_{k,\ell}^{m} T_k(2x-1) T_\ell\left(\frac{2y}{1-x}-1\right) T_m\left(\frac{2z}{1-x-y}-1\right). \tag{25}$$

References

[1] R. M. Slevinsky. Fast and backward stable transforms between spherical harmonic expansions and bivariate Fourier series. *Appl. Comput. Harmon. Anal.*, 2017.

[2] R. M. Slevinsky. Conquering the pre-computation in two-dimensional harmonic polynomial transforms. arXiv:1711.07866, 2017.