

FastTransforms Documentation

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1 Introduction

FastTransforms is a C library based on the solutions of the two-dimensional harmonic polynomial connection problems in [1, 2] that have an $\mathcal{O}(n^3)$ run-time, where n is the polynomial degree, and are 2-normwise backward stable.

The transforms are separated into computational kernels that offer SSE, AVX, and AVX-512 vectorization on applicable Intel processors, and driver routines that are easily parallelized by OpenMP.

2 What FastTransforms actually computes

For every subsection below, the title of the subsection, of the form **a2b**, refers conceptually to the transform and the available functions are as follows:

- **ft_plan_a2b**, is a pre-computation,
- **ft_execute_a2b**, is a forward execution,
- **ft_execute_b2a**, is a backward execution,
- **ft_execute_a_hi2lo**, is a conversion to a tensor-product basis,
- **ft_execute_a_lo2hi**, is a conversion from a tensor-product basis,
- **ft_kernel_a_hi2lo**, is an orthonormal conversion from high to low order,
- **ft_kernel_a_lo2hi**, is an orthonormal conversion from low to high order.

The **ft_execute_*** functions are drivers that perform transforms as defined below. They are composed of computational kernels, **ft_kernel_*** functions, that may be assembled differently.

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2.1 sph2fourier

Spherical harmonics are:

$$Y_\ell^m(\theta, \varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} (-1)^{|m|} \sqrt{(\ell + \frac{1}{2}) \frac{(\ell - |m|)!}{(\ell + |m|)!}} P_\ell^{|m|}(\cos \theta), \quad (1)$$

where $P_\ell^m(\cos \theta)$ are the associated Legendre functions. A degree- n expansion in spherical harmonics is given by:

$$f_n(\theta, \varphi) = \sum_{\ell=0}^n \sum_{m=-\ell}^{+\ell} f_\ell^m Y_\ell^m(\theta, \varphi). \quad (2)$$

If spherical harmonic expansion coefficients are organized into the array:

$$F = \begin{pmatrix} f_0^0 & f_1^{-1} & f_1^1 & f_2^{-2} & f_2^2 & \cdots & f_n^{-n} & f_n^n \\ f_1^0 & f_2^{-1} & f_2^1 & f_3^{-2} & f_3^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n-2}^0 & f_{n-1}^{-1} & f_{n-1}^1 & f_n^{-2} & f_n^2 & & \vdots & \vdots \\ f_{n-1}^0 & f_n^{-1} & f_n^1 & 0 & 0 & \cdots & 0 & 0 \\ f_n^0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (3)$$

then **sph2fourier** returns the bivariate Fourier coefficients:

$$G = \begin{pmatrix} g_0^0 & g_0^{-1} & g_0^1 & \cdots & g_0^{-n} & g_0^n \\ g_1^0 & g_1^{-1} & g_1^1 & \cdots & g_1^{-n} & g_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{n-1}^0 & g_{n-1}^{-1} & g_{n-1}^1 & \cdots & g_{n-1}^{-n} & g_{n-1}^n \\ g_n^0 & 0 & 0 & \cdots & g_n^{-n} & g_n^n \end{pmatrix}. \quad (4)$$

That is:

$$f_n(\theta, \varphi) = \sum_{\ell=0}^n \sum_{m=-n}^{+n} g_\ell^m \frac{e^{im\varphi}}{\sqrt{2\pi}} \begin{cases} \cos(\ell\theta) & m \text{ even,} \\ \sin((\ell+1)\theta) & m \text{ odd.} \end{cases} \quad (5)$$

Since **sph2fourier** only transforms columns of the arrays, the routine is indifferent to the choice of longitudinal basis; it may be complex exponentials or sines and cosines, with no particular normalization.

2.2 spinsph2fourier

Spin-weighted spherical harmonics are:

$$Y_{\ell,m}^s(\theta, \varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}} \sqrt{(\ell + \frac{1}{2}) \frac{(\ell + \ell_0)! (\ell - \ell_0)!}{(\ell + \ell_1)! (\ell - \ell_1)!}} \\ \times \sin^{|m+s|}(\frac{\theta}{2}) \cos^{|m-s|}(\frac{\theta}{2}) P_{\ell-\ell_0}^{(|m+s|, |m-s|)}(\cos \theta). \quad (6)$$

where $P_n^{(\alpha,\beta)}(\cos\theta)$ are the Jacobi polynomials and $\ell_0 = \max\{|m|, |s|\}$ and $\ell_1 = \min\{|m|, |s|\}$. A degree- n expansion in spin-weighted spherical harmonics is given by:

$$f_n^s(\theta, \varphi) = \sum_{\ell=\ell_0}^n \sum_{m=-\ell}^{+\ell} f_\ell^m Y_{\ell,m}^s(\theta, \varphi). \quad (7)$$

If spin-weighted spherical harmonic expansion coefficients with $s = 2$, for example, are organized into the array:

$$F = \begin{pmatrix} f_2^0 & f_2^{-1} & f_2^1 & f_2^{-2} & f_2^2 & \cdots & f_n^{-n} & f_n^n \\ f_3^0 & f_3^{-1} & f_3^1 & f_3^{-2} & f_3^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_n^0 & f_n^{-1} & f_n^1 & f_n^{-2} & f_n^2 & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (8)$$

then `spinsph2fourier` returns the bivariate Fourier coefficients:

$$G = \begin{pmatrix} g_0^0 & g_0^{-1} & g_0^1 & \cdots & g_0^{-n} & g_0^n \\ g_1^0 & g_1^{-1} & g_1^1 & \cdots & g_1^{-n} & g_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{n-1}^0 & g_{n-1}^{-1} & g_{n-1}^1 & \cdots & g_{n-1}^{-n} & g_{n-1}^n \\ g_n^0 & 0 & 0 & \cdots & g_n^{-n} & g_n^n \end{pmatrix}. \quad (9)$$

That is:

$$f_n(\theta, \varphi) = \sum_{\ell=0}^n \sum_{m=-n}^{+n} g_\ell^m \frac{e^{im\varphi}}{\sqrt{2\pi}} \begin{cases} \cos(\ell\theta) & m+s \text{ even,} \\ \sin((\ell+1)\theta) & m+s \text{ odd.} \end{cases} \quad (10)$$

Since `spinsph2fourier` only transforms columns of the arrays, the routine is indifferent to the choice of longitudinal basis; it may be complex exponentials or sines and cosines, with no particular normalization.

2.3 tri2cheb

Triangular harmonics are:

$$\tilde{P}_{\ell,m}^{(\alpha,\beta,\gamma)}(x, y) = (2-2x)^m \tilde{P}_{\ell-m}^{(2m+\beta+\gamma+1,\alpha)}(2x-1) \tilde{P}_m^{(\gamma,\beta)}\left(\frac{2y}{1-x} - 1\right), \quad (11)$$

where the tilde implies that the univariate Jacobi polynomials are orthonormal. A degree- n expansion in triangular harmonics is given by:

$$f_n(x, y) = \sum_{\ell=0}^n \sum_{m=0}^{\ell} f_\ell^m \tilde{P}_{\ell,m}^{(\alpha,\beta,\gamma)}(x, y). \quad (12)$$

If triangular harmonic expansion coefficients are organized into the array:

$$F = \begin{pmatrix} f_0^0 & f_1^1 & f_2^2 & \cdots & f_n^n \\ \vdots & \vdots & \vdots & \ddots & 0 \\ f_{n-2}^0 & f_{n-1}^1 & f_n^2 & \ddots & \vdots \\ f_{n-1}^0 & f_n^1 & 0 & \cdots & 0 \\ f_n^0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (13)$$

then `tri2cheb` returns the bivariate Chebyshev coefficients:

$$G = \begin{pmatrix} g_0^0 & g_0^1 & \cdots & g_0^n \\ g_1^0 & g_1^1 & \cdots & g_1^n \\ \vdots & \vdots & \ddots & \vdots \\ g_n^0 & g_n^1 & \cdots & g_n^n \end{pmatrix}. \quad (14)$$

That is:

$$f_n(x, y) = \sum_{\ell=0}^n \sum_{m=0}^n g_\ell^m T_\ell(2x-1) T_m\left(\frac{2y}{1-x} - 1\right). \quad (15)$$

2.4 disk2cxf

Disk harmonics are Zernike polynomials:

$$Z_\ell^m(r, \theta) = \sqrt{2\ell+2r^{|m|}} P_{\frac{\ell-|m|}{2}}^{(0, |m|)}(2r^2-1) \frac{e^{im\theta}}{\sqrt{2\pi}}. \quad (16)$$

A degree- $2n$ expansion in disk harmonics is given by:

$$f_{2n}(r, \theta) = \sum_{\ell=0}^{2n} \sum_{m=-\ell, 2}^{+\ell} f_\ell^m Z_\ell^m(r, \theta), \quad (17)$$

where the $, 2$ in the inner summation index implies that the inner summation runs from $m = -\ell$ in steps of 2 up to $+\ell$. If disk harmonic expansion coefficients are organized into the array:

$$F = \begin{pmatrix} f_0^0 & f_1^{-1} & f_1^1 & f_2^{-2} & f_2^2 & \cdots & f_{2n}^{-2n} & f_{2n}^{2n} \\ f_2^0 & f_3^{-1} & f_3^1 & f_4^{-2} & f_4^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{2n-4}^0 & f_{2n-3}^{-1} & f_{2n-3}^1 & f_{2n-2}^{-2} & f_{2n-2}^2 & & \vdots & \vdots \\ f_{2n-2}^0 & f_{2n-1}^{-1} & f_{2n-1}^1 & f_{2n}^{-2} & f_{2n}^2 & \cdots & 0 & 0 \\ f_{2n}^0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (18)$$

then `disk2cxf` returns the even Chebyshev–Fourier coefficients:

$$G = \begin{pmatrix} g_0^0 & g_0^{-1} & g_0^1 & g_0^{-2} & g_0^2 & \cdots & g_0^{-2n} & g_0^{2n} \\ g_2^0 & g_2^{-1} & g_2^1 & g_2^{-2} & g_2^2 & \cdots & g_2^{-2n} & g_2^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-2}^0 & g_{2n-2}^{-1} & g_{2n-2}^1 & g_{2n-2}^{-2} & g_{2n-2}^2 & \cdots & g_{2n-2}^{-2n} & g_{2n-2}^{2n} \\ g_{2n}^0 & 0 & 0 & g_{2n}^{-2} & g_{2n}^2 & \cdots & g_{2n}^{-2n} & g_{2n}^{2n} \end{pmatrix}. \quad (19)$$

That is:

$$f_{2n}(r, \theta) = \sum_{\ell=0}^n \sum_{m=-2n}^{+2n} g_{2\ell}^m \frac{e^{im\theta}}{\sqrt{2\pi}} \begin{cases} T_{2\ell}(r) & m \text{ even}, \\ T_{2\ell+1}(r) & m \text{ odd}. \end{cases} \quad (20)$$

Since `disk2cxf` only transforms columns of the arrays, the routine is indifferent to the choice of azimuthal basis; it may be complex exponentials or sines and cosines, with no particular normalization.

2.5 tet2cheb

Tetrahedral harmonics are:

$$\begin{aligned} \tilde{P}_{k,\ell,m}^{(\alpha,\beta,\gamma,\delta)}(x, y, z) &= (2-2x)^{\ell+m} \tilde{P}_{k-\ell-m}^{(2\ell+2m+\beta+\gamma+\delta+2,\alpha)}(2x-1) \\ &\times \left(2 - \frac{2y}{1-x}\right)^m \tilde{P}_{\ell}^{(2m+\gamma+\delta+1,\beta)}\left(\frac{2y}{1-x} - 1\right) \\ &\times \tilde{P}_m^{(\delta,\gamma)}\left(\frac{2z}{1-x-y} - 1\right), \end{aligned} \quad (21)$$

where the tilde implies that the univariate Jacobi polynomials are orthonormal. A degree- n expansion in tetrahedral harmonics is given by:

$$f_n(x, y, z) = \sum_{k=0}^n \sum_{\ell=0}^k \sum_{m=0}^{k-\ell} f_{k,\ell,m}^m \tilde{P}_{k,\ell,m}^{(\alpha,\beta,\gamma,\delta)}(x, y, z), \quad (22)$$

$$= \sum_{m=0}^n \sum_{\ell=0}^{n-m} \sum_{k=\ell+m}^n f_{k,\ell,m}^m \tilde{P}_{k,\ell,m}^{(\alpha,\beta,\gamma,\delta)}(x, y, z). \quad (23)$$

If tetrahedral harmonic expansion coefficients are organized into the rank-3 array whose m^{th} slice is:

$$F[0 : n-m, 0 : n-m, m] = \begin{pmatrix} f_{m,0}^m & f_{1+m,1}^m & f_{2+m,2}^m & \cdots & f_{n,n-m}^m \\ \vdots & \vdots & \vdots & \ddots & 0 \\ f_{n-2,0}^m & f_{n-1,1}^m & f_{n,2}^m & \ddots & \vdots \\ f_{n-1,0}^m & f_{n,1}^m & 0 & \cdots & 0 \\ f_{n,0}^m & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (24)$$

then `tet2cheb` returns the trivariate Chebyshev coefficients stored in the rank-3 array G . That is:

$$f_n(x, y, z) = \sum_{k=0}^n \sum_{\ell=0}^n \sum_{m=0}^n g_{k,\ell}^m T_k(2x-1) T_\ell \left(\frac{2y}{1-x} - 1 \right) T_m \left(\frac{2z}{1-x-y} - 1 \right). \quad (25)$$

References

- [1] R. M. Slevinsky. Fast and backward stable transforms between spherical harmonic expansions and bivariate Fourier series. *Appl. Comput. Harmon. Anal.*, 2017.
- [2] R. M. Slevinsky. Conquering the pre-computation in two-dimensional harmonic polynomial transforms. arXiv:1711.07866, 2017.