MATH 2160, Chapter 5 Summary & Exercises

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A Conversation with Slevinsky

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<tr><td>How do I interpolate a periodic function and preserve the periodicity?</td><td>There are two scenarios that are related by a variable transformation. If f is a function of a complex variable z, then the orthogonal polynomials with respect to $L^2(\mathbb{U}, \mathrm{d}z/(\mathrm{i}z))$ are the integer powers $\{z^k\}_{k=-\infty}^{+\infty}$. Both the function f and the basis z^k are periodic on the unit circle $\mathbb{U}$, because $\mathbb{U}$ has no beginning and no end. This allows us to project f onto its Laurent series:

$$f(z) = \sum_{k=-\infty}^{+\infty} f_k z^k,$$

where the coefficients are given by:

$$f_k = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{U}} \frac{f(z)}{z^{k+1}}\,\mathrm{d}z.$$

If f is periodic on $[0, 2\pi)$, then we may represent it as a Fourier series:

$$f(\theta) = \sum_{k=-\infty}^{+\infty} c_k e^{\mathrm{i}k\theta},$$

where the coefficients are given by:

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-\mathrm{i}k\theta}\,\mathrm{d}\theta.$$</td></tr>
<tr><td>I thought a Fourier series was given in terms of sines and cosines.</td><td>Indeed, yet another alternative representation gives:

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{a_k \cos k\theta + b_k \sin k\theta\right\},$$

where the coefficients are:

$$a_k = c_k + c_{-k}, \quad b_k = \mathrm{i}(c_k - c_{-k}).$$</td></tr>
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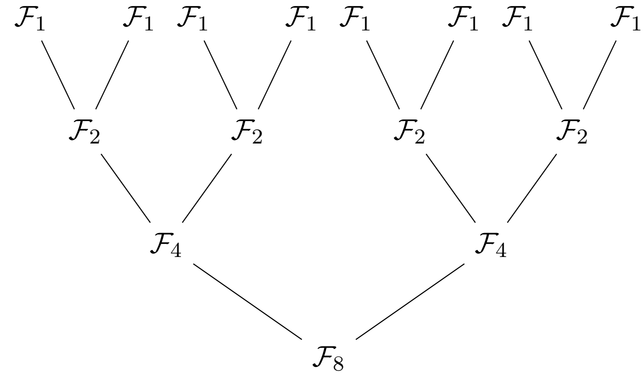
OK, but all these coefficients are given by integrals. How should we approximate them *numerically*?	As we saw in our chapter on numerical differentiation and integration, the composite trapezoidal rule worked *exceptionally well* on integrands with periodicity. This is precisely because it *is the Gaussian quadrature rule* on $L^2(\mathbb{U}, \mathrm{d}z/(\mathrm{i}z))$. This leads us to the DFT matrix and its inverse.
What is the DFT matrix?	The *discrete Fourier transform* (DFT) matrix represents the transformation from N equispaced function samples on the unit circle to N approximate Maclaurin coefficients $\hat{f}_k$, for $k = 0, \ldots, N-1$. From a signal processor's point of view, function samples are a "signal" and Maclaurin coefficients represent the energy at "certain frequencies": $$\hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f(e^{2\pi \mathrm{i} j/N}) e^{-\mathrm{i}2\pi k j/N}.$$ To find the energy at a particular frequency, spin your signal around a circle at that frequency, and average over equispaced points.
Oh, I remember now! $$\mathcal{F}_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{bmatrix},$$ where $\omega = e^{-2\pi \mathrm{i}/N}$.	Exactly! This matrix has *so much structure* that when N is a power of two, we may decompose its odd-even permutation into the product of a sparse matrix with $2N$ entries and a matrix containing two copies of the DFT matrix of half the size: $$\mathcal{F}_N P_N = \begin{bmatrix} I_{N/2} & \Omega_{N/2} \\ I_{N/2} & -\Omega_{N/2} \end{bmatrix} \begin{bmatrix} \mathcal{F}_{N/2} & 0 \\ 0 & \mathcal{F}_{N/2} \end{bmatrix}.$$
Let me see if I've got this right; if $N = 4$: $$\mathcal{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix},$$ then since $\omega^4 = (e^{-2\pi \mathrm{i}/4})^4 \equiv 1$: $$\mathcal{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{bmatrix}.$$	Yep, and if we swap the second and third columns: $$\mathcal{F}_4[:, [1;3;2;4]] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^3 & \omega \end{bmatrix},$$ then we can write: $$\mathcal{F}_4[:, [1;3;2;4]] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\omega \end{bmatrix} \begin{bmatrix} \mathcal{F}_2 & 0 \\ 0 & \mathcal{F}_2 \end{bmatrix}.$$

Whoa, that's a lot of omegas. I still don't understand how this gets us the $\mathcal{O}(N \log N)$ complexity, though.	If we continue the factorization *recursively*, then we can visualize it as a *tree*. For $N = 8$, for example:  There are four levels of the tree with eight 1-by-1 matrices at the leaves.
Very interesting. Wait, why are there hats on the coefficients? In which sense are they approximate?	The phenomenon of *aliasing* shows us that the $\hat{f}_k$ contain *all* the Laurent modes: $$\hat{f}_k = f_k + f_{k-N} + f_{k+N} + f_{k-2N} + f_{k+2N} + \cdots + .$$ Thus, if there is decay in the coefficients and if we use a lot of them, then we will converge to the true Laurent coefficients.
All of this technology transfers to Chebyshev polynomials of the first and second kinds, right?	Yep. The DCT and the DST allow us to create polynomial interpolants to $f \in C([-1,1])$ in only $\mathcal{O}(N \log N)$ operations. Let me tell ya folks, this is "yuuuuuuuuuuuuuuuuuuuuuuuuuuuuuuuuuuuuuge."
...making polynomial interpolation great again?	Precisely.
We have two polynomial bases T_k and U_k. When should we use which one?	It depends what you would like to do, but there is a banded conversion between the bases, so we can change our representation very rapidly. Differentiation and integration is done term-by-term.

Exercises

1. If we have the Laurent expansion of a function $f(z)$, then we may easily calculate its 2-norm only in terms of the coefficients. What is the expression for:

$$\|f\|_2^2 = \left| \int_{\mathbb{U}} |f(z)|^2 \, \frac{\mathrm{d}z}{\mathrm{i}z} \right| ?$$

   Alternatively, when $f(\theta)$ is a periodic function on $[0, 2\pi)$, what is the expression for:

$$\|f\|_2^2 = \int_0^{2\pi} |f(\theta)|^2 \, \mathrm{d}\theta, \quad \text{in terms of } a_k \text{ and } b_k, \text{ or } c_k?$$

2. In chapter 5, we assume that the dimensions of the DFT matrix are powers of two. The Fundamental Theorem of Algebra states that every integer has a prime factorization. For example, $23,452 = 2^2 \cdot 11 \cdot 13 \cdot 41$ (check it!). Using this principle, optimal factorizations of the DFT matrix exist when the dimensions are *highly composite*, that is, a positive integer with more divisors than all smaller positive integers, such as $2^3 \cdot 3^2 \cdot 5$.

The first non-power-of-two highly composite number is $6 = 2\cdot 3$. Carefully confirm the factorization of $\mathcal{F}_6 P_6$ in terms of I_3, Ω_3, and $\mathcal{F}_3$. If you are feeling adventurous, can you derive a different factorization starting with a different permutation?

3. In problem 4 of chapter 3, we derived the $N(= N(a,\varepsilon))$ required to satisfy $\|f - p_N\|_\infty \le \varepsilon \|f\|_\infty$ for $f(x) = (x^2+a^2)^{-1}$. It was:

$$N(a,\varepsilon) \ge -\log\left(\varepsilon \frac{(a+\sqrt{a^2+1})^2 - 1}{a^2}\right) / \log(a+\sqrt{a^2+1}).$$

Use the DCT to examine how close the first N coefficients of the interpolant are to the first N coefficients of the expansion. In your opinion, is aliasing a concern for this particular function?

4. Use the conversion matrix:

$$\mathcal{C}_N = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & & & \\ & \frac{1}{2} & 0 & -\frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{2} & 0 & -\frac{1}{2} \\ & & & & \frac{1}{2} & 0 \\ & & & & & \frac{1}{2} \end{bmatrix},$$

between the T_k and U_k bases to confirm numerically the exact Chebyshev-U coefficients of e^x in Table 5.3 given the exact Chebyshev-T coefficients in Table 5.2 in terms of the modified Bessel functions of the first kind $I_k(1)$.