MATH 2160, Chapter 2 Summary & Exercises

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A Conversation with Slevinsky

Problems	Solutions
$Ax = b$, A square.	A good algorithm is Gaussian elimination. It is a direct algorithm that terminates after $\mathcal{O}(n^3)$ operations. Partial pivoting ensures that it is stable; however, there are corner cases where the rounding errors can accumulate *geometrically* with the problem dimension.
$Ax = b$, A square and multiple RHS.	First, compute a matrix factorization, such as LUP or QR. Each of these costs $\mathcal{O}(n^3)$ operations, but solution of linear systems with factorizations consisting of triangular or orthogonal matrices costs only $\mathcal{O}(n^2)$ operations.
$Ax = b$, A rectangular with more rows than columns.	This is a least-squares problem. DO NOT SOLVE $A^*Ax = A^*b$. Instead, use a reduced QR factorization, where Q is now a rectangular matrix with orthonormal columns and R is still square and upper triangular.
$Ax = b$, A rectangular with more columns than rows.	Focus! We didn't study this! This is an ill-posed problem, but it is useful in image compression.
$A = V\Lambda V^{-1}$?	Generically, a matrix is not guaranteed to have a spectral decomposition.
Fine, what about when $A \in \mathbb{R}^{n\times n}$ is symmetric?	Yes! Even better, the eigenvectors can be chosen to be orthonormal: $A = Q\Lambda Q^\top$.
$A = U\Sigma V^*$?	Yes! Every matrix $A \in \mathbb{C}^{m\times n}$ has a singular value decomposition.
Cool, but why is this useful?	For one, we now know how to calculate the matrix 2-norm, since $\|A\|_2 = \sigma_1$, its largest singular value. For another, if we just take the first r columns of U and V and the $r \times r$ principal submatrix of Σ, we have the best rank-r approximation to A, which is another useful matrix compression technique.
What happens when A is large?	The main lessons of this chapter are to take advantage of structure of your linear system; structure usually transpires from the problem you are trying to solve. Important structures are symmetry, sparsity patterns, and definiteness, which are all useful when solving linear systems iteratively.

Exercises

1. What can we say about the eigenvalues of a unitary matrix?

2. Determine the SVDs of the following matrices (by hand):

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

3. Let $\rho(A)$ denote the *spectral radius* of $A \in \mathbb{C}^{n\times n}$, i.e. the largest eigenvalue in absolute value $|\lambda|$. Let $\|\cdot\|_p$ denote the p-norm on $\mathbb{C}^n$ and the induced matrix norm on $\mathbb{C}^{n\times n}$. Show that $\rho(A) \leq \|A\|_p$ for every $1 \leq p \leq \infty$.

4. Suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ is the matrix obtained by rotating A clockwise 90°. Do A and B have the same singular values? Prove that the answer is yes or give a counterexample.

5. The matrix:

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & & & 1 \end{bmatrix} = \frac{1}{2}(I_n + S_n) \in \mathbb{R}^{n\times n}$$

   represents the averaging of the coordinates of an n-sided polygon in a plane. Here, I_n is the identity matrix and S_n is the right-circular shift matrix. Although A is not symmetric, it does have a spectral decomposition. Can you find it?

6. Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & & \\ & \ddots & \ddots & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

   (a) What are the eigenvalues and determinant of A?

   (b) What is A^{-1}?

   (c) Give a nontrivial bounds on σ_1 and σ_n, the first and last singular values of A. Use JULIA to build your intuition on the problem, but the bounds should be derived analytically.