The Hilbert matrix:

$$H_n = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}.$$

is a great example to illustrate ill-conditioning. For a symmetric ill-conditioned matrix, it is useful to examine sensitivities round eigenvectors as they all "point" in different directions, colloquially speaking.

```
H = #1 (generic function with 1 method)
```

```julia
H = n -> inv.((1:n) .+ (1:n)' .- 1)
```

```
A = 15×15 Array{Float64,2}:
 1.0                  0.5                  …  0.06666666666666667
 0.5                  0.3333333333333333      0.0625
 0.3333333333333333   0.25                    0.058823529411764705
 0.25                 0.2                     0.05555555555555555
 0.2                  0.16666666666666666     0.05263157894736842
 0.16666666666666666  0.14285714285714285  …  0.05
 0.14285714285714285  0.125                   0.047619047619047616
 ⋮                                         ⋱
 0.1                  0.09090909090909091     0.041666666666666664
 0.09090909090909091  0.08333333333333333  …  0.04
 0.08333333333333333  0.07692307692307693     0.038461538461538464
 0.07692307692307693  0.07142857142857142     0.037037037037037035
 0.07142857142857142  0.06666666666666667     0.03571428571428571
 0.06666666666666667  0.0625                  0.034482758620689655
```

```julia
A = H(15)
```

```julia
using LinearAlgebra
```

```
Eigen{Float64,Float64,Array{Float64,2},Array{Float64,1}}
values:
15-element Array{Float64,1}:
 -9.731674398306353e-18
  2.4471098684428205e-17
  8.417476172219603e-17
  1.3900578237791628e-14
  9.321996824788568e-13
  4.657781739919007e-11
  1.8029597497893784e-9
  ⋮
  2.710853922820439e-5
  0.0004364765944177553
  0.005639834755789189
  0.05721209253338421
  0.4266279570069751
  1.845927746153489
vectors:
15×15 Array{Float64,2}:
  4.027407113534054e-9    -3.8579037815039685e-8   …  -0.6643596446265618
 -6.434951197145829e-7     5.259822492602317e-6       -0.41379493314732985
  2.5683011205110927e-5   -0.00017821662657140948     -0.31263025106057074
 -0.00044963564925012576   0.0026140055386670736      -0.25497095328304914
  0.004312625558763815    -0.020536398884193487       -0.21683971746073358
 -0.02541271764022335      0.09560128949271934     …  -0.18940844991079595
  0.09820458417360232     -0.2748896282066456         -0.16856873009438833
  ⋮                                                ⋱
 -0.5937090230237423       0.028349623470173134       -0.12765836086544508
  0.5096716554877957       0.42890719110460424     …  -0.11827942474402993
 -0.2840155991095513      -0.4892558710585028         -0.11023685308104106
  0.09265662686030433      0.24070759428147576        -0.10325655189669361
 -0.013433180119699654    -0.04659932577569513        -0.09713593495171591
  0.0                      0.0                        -0.0917217396947767
```

```
Λ, V = eigen(A)
```

```
x = Float64[4.02741e-9, -6.43495e-7, 2.5683e-5, -0.000449636, 0.00431263, -0.0254
```

```
x = V[:,1]
```

```
Δx = Float64[-8.13282e-10, 2.73256e-8, -2.0499e-7, 5.05345e-7, -2.49295e-7, -3.69
```

```
Δx = 1e-6V[:, 8]
```

In the relative sense, the perturbation Δx is small in the 2-norm.

```
9.99999999985383e-7
```

```
norm((x+Δx)-x)/norm(x)
```

But the image of the perturbed $x + \Delta x$ under H_{15} is much farther away from the image of x.

```
y =  Float64[-8.67362e-19, -1.12757e-17, -1.38778e-17, -3.46945e-18, 5.20417e-18,
```

```
y = A*x
```

```
Δy =  Float64[-4.49661e-17, 1.51083e-15, -1.13339e-14, 2.79405e-14, -1.37835e-14,
```

```
Δy = A*Δx
```

```
1277.699978034397
```

```
norm(y+Δy-y)/norm(y)
```

It is always true that this relative perturbation is bounded above by the condition number of H_{15} multiplied by the relative perturbation in the domain.

```
true
```

```
norm(y+Δy-y)/norm(y) ≤ cond(A)*norm((x+Δx)-x)/norm(x)
```

This is because the condition number of the Hilbert matrix is indeed quite large.

```
2.5468136050354794e17
```

```
cond(A)
```