

# MATH 2160, Chapter 2 Summary & Exercises

Richard M. Slevinsky

## A Conversation with Slevinsky

| Problems  | Solutions  |
|---|--|
| $Ax = b$ , $A$ square.  | A good algorithm is Gaussian elimination. It is a direct algorithm that terminates after $\mathcal{O}(n^3)$ operations. Partial pivoting ensures that it is stable; however, there are corner cases where the rounding errors can accumulate <i>geometrically</i> with the problem dimension.  |
| $Ax = b$ , $A$ square and multiple RHS.                             | First, compute a matrix factorization, such as $LUP$ or $QR$ . Each of these costs $\mathcal{O}(n^3)$ operations, but solution of linear systems with factorizations consisting of triangular or orthogonal matrices costs only $\mathcal{O}(n^2)$ operations.   |
| $Ax = b$ , $A$ rectangular with more rows than columns.             | This is a least-squares problem. DO NOT SOLVE $A^*Ax = A^*b$ . Instead, use a reduced $QR$ factorization, where $Q$ is now a rectangular matrix with orthonormal columns and $R$ is still square and upper triangular.   |
| $Ax = b$ , $A$ rectangular with more columns than rows.             | Focus! We didn't study this! This is an ill-posed problem, but it is useful in image compression.  |
| $A = V\Lambda V^{-1}$ ?   | Generically, a matrix is not guaranteed to have a spectral decomposition.  |
| Fine, what about when $A \in \mathbb{R}^{n \times n}$ is symmetric? | Yes! Even better, the eigenvectors can be chosen to be orthonormal: $A = Q\Lambda Q^T$ .   |
| $A = U\Sigma V^*$ ?   | Yes! Every matrix $A \in \mathbb{C}^{m \times n}$ has a singular value decomposition.  |
| Cool, but why is this useful?                                       | For one, we now know how to calculate the matrix 2-norm, since $\ A\ _2 = \sigma_1$ , its largest singular value. For another, if we just take the first $r$ columns of $U$ and $V$ and the $r \times r$ principal submatrix of $\Sigma$ , we have the best rank- $r$ approximation to $A$ , which is another useful matrix compression technique. |
| What happens when $A$ is large?                                     | The main lessons of this chapter are to take advantage of structure of your linear system; structure usually transpires from the problem you are trying to solve. Important structures are symmetry, sparsity patterns, and definiteness, which are all useful when solving linear systems iteratively.  |

## Exercises

1. What can we say about the eigenvalues of a unitary matrix?
2. Determine the SVDs of the following matrices (by hand):

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

3. Let  $\rho(A)$  denote the *spectral radius* of  $A \in \mathbb{C}^{n \times n}$ , i.e. the largest eigenvalue in absolute value  $|\lambda|$ . Let  $\|\cdot\|_p$  denote the  $p$ -norm on  $\mathbb{C}^n$  and the induced matrix norm on  $\mathbb{C}^{n \times n}$ . Show that  $\rho(A) \leq \|A\|_p$  for every  $1 \leq p \leq \infty$ .
4. Suppose  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times m}$  is the matrix obtained by rotating  $A$  clockwise  $90^\circ$ . Do  $A$  and  $B$  have the same singular values? Prove that the answer is yes or give a counterexample.
5. The matrix:

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & & & 1 \end{bmatrix} = \frac{1}{2}(I_n + S_n) \in \mathbb{R}^{n \times n}$$

represents the averaging of the coordinates of an  $n$ -sided polygon in a plane. Here,  $I_n$  is the identity matrix and  $S_n$  is the right-circular shift matrix. Although  $A$  is not symmetric, it does have a spectral decomposition. Can you find it?

6. Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & & \\ & \ddots & \ddots & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

- (a) What are the eigenvalues and determinant of  $A$ ?
- (b) What is  $A^{-1}$ ?
- (c) Give nontrivial bounds on  $\sigma_1$  and  $\sigma_n$ , the first and last singular values of  $A$ . Use JULIA to build your intuition on the problem, but the bounds should be derived analytically.