

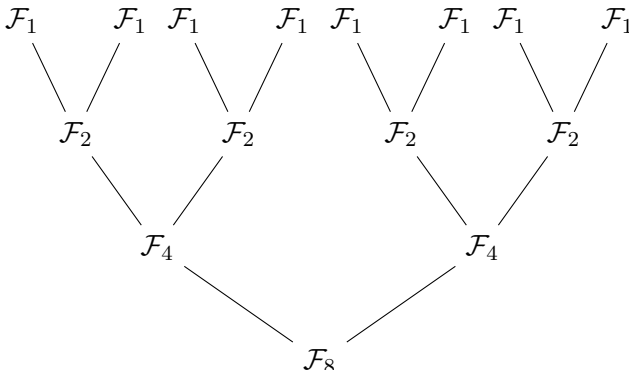
MATH 2160, Chapter 5 Summary & Exercises

Richard M. Slevinsky

A Conversation with Slevinsky

Problems	Solutions
How do I interpolate a periodic function and preserve the periodicity?	<p>There are two scenarios that are related by a variable transformation. If f is a function of a complex variable z, then the orthogonal polynomials with respect to $L^2(\mathbb{U}, dz/(iz))$ are the integer powers $\{z^k\}_{k=-\infty}^{+\infty}$. Both the function f and the basis z^k are periodic on the unit circle \mathbb{U}, because \mathbb{U} has no beginning and no end. This allows us to project f onto its <i>Laurent series</i>:</p> $f(z) = \sum_{k=-\infty}^{+\infty} f_k z^k,$ <p>where the coefficients are given by:</p> $f_k = \frac{1}{2\pi i} \int_{\mathbb{U}} \frac{f(z)}{z^{k+1}} dz.$ <p>If f is periodic on $[0, 2\pi)$, then we may represent it as a <i>Fourier series</i>:</p> $f(\theta) = \sum_{k=-\infty}^{+\infty} c_k e^{ik\theta},$ <p>where the coefficients are given by:</p> $c_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta.$
I thought a Fourier series was given in terms of sines and cosines.	<p>Indeed, yet another alternative representation gives:</p> $f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\theta + b_k \sin k\theta\},$ <p>where the coefficients are:</p> $a_k = c_k + c_{-k}, \quad b_k = i(c_k - c_{-k}).$

<p>OK, but all these coefficients are given by integrals. How should we approximate them <i>numerically</i>?</p>	<p>As we saw in our chapter on numerical differentiation and integration, the composite trapezoidal rule worked <i>exceptionally well</i> on integrands with periodicity. This is precisely because it <i>is the Gaussian quadrature rule</i> on $L^2(\mathbb{U}, dz/(iz))$. This leads us to the DFT matrix and its inverse.</p>
<p>What is the DFT matrix?</p>	<p>The <i>discrete Fourier transform</i> (DFT) matrix represents the transformation from N equispaced function samples on the unit circle to N approximate Maclaurin coefficients \hat{f}_k, for $k = 0, \dots, N - 1$. From a signal processor's point of view, function samples are a “signal” and Maclaurin coefficients represent the energy at “certain frequencies”:</p> $\hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f(e^{2\pi i j/N}) e^{-i 2\pi k j/N}.$ <p>To find the energy at a particular frequency, spin your signal around a circle at that frequency, and average over equispaced points.</p>
<p>Oh, I remember now!</p> $\mathcal{F}_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{bmatrix},$ <p>where $\omega = e^{-2\pi i/N}$.</p>	<p>Exactly! This matrix has <i>so much structure</i> that when N is a power of two, we may decompose its odd-even permutation into the product of a sparse matrix with $2N$ entries and a matrix containing two copies of the DFT matrix of half the size:</p> $\mathcal{F}_N P_N = \begin{bmatrix} I_{N/2} & \Omega_{N/2} \\ I_{N/2} & -\Omega_{N/2} \end{bmatrix} \begin{bmatrix} \mathcal{F}_{N/2} & 0 \\ 0 & \mathcal{F}_{N/2} \end{bmatrix}.$
<p>Let me see if I've got this right; if $N = 4$:</p> $\mathcal{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{bmatrix},$ <p>then since $\omega^4 = (e^{-2\pi i/4})^4 \equiv 1$:</p> $\mathcal{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{bmatrix}.$	<p>Yep, and if we swap the second and third columns:</p> $\mathcal{F}_4[:, [1; 3; 2; 4]] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & 1 & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^3 & \omega \end{bmatrix},$ <p>then we can write:</p> $\mathcal{F}_4[:, [1; 3; 2; 4]] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\omega \end{bmatrix} \begin{bmatrix} \mathcal{F}_2 & 0 \\ 0 & \mathcal{F}_2 \end{bmatrix}.$

<p>Whoa, that's a lot of omegas. I still don't understand how this gets us the $\mathcal{O}(N \log N)$ complexity, though.</p>	<p>If we continue the factorization <i>recursively</i>, then we can visualize it as a <i>tree</i>. For $N = 8$, for example:</p>  <p>There are four levels of the tree with eight 1-by-1 matrices at the leaves.</p>
<p>Very interesting. Wait, why are there hats on the coefficients? In which sense are they approximate?</p> <p>All of this technology transfers to Chebyshev polynomials of the first and second kinds, right?</p> <p>...making polynomial interpolation great again?</p> <p>We have two polynomial bases T_k and U_k. When should we use which one?</p>	<p>The phenomenon of <i>aliasing</i> shows us that the \hat{f}_k contain <i>all</i> the Laurent modes:</p> $\hat{f}_k = f_k + f_{k-N} + f_{k+N} + f_{k-2N} + f_{k+2N} + \cdots + .$ <p>Thus, if there is decay in the coefficients and if we use a lot of them, then we will converge to the true Laurent coefficients.</p> <p>Yep. The DCT and the DST allow us to create polynomial interpolants to $f \in C([-1, 1])$ in only $\mathcal{O}(N \log N)$ operations. Let me tell ya folks, this is “yuuuge.” Precisely.</p> <p>It depends what you would like to do, but there is a banded conversion between the bases, so we can change our representation very rapidly. Differentiation and integration is done term-by-term.</p>

Exercises

1. If we have the Laurent expansion of a function $f(z)$, then we may easily calculate its 2-norm only in terms of the coefficients. What is the expression for:

$$\|f\|_2^2 = \left| \int_{\mathbb{U}} |f(z)|^2 \frac{dz}{iz} \right|?$$

Alternatively, when $f(\theta)$ is a periodic function on $[0, 2\pi)$, what is the expression for:

$$\|f\|_2^2 = \int_0^{2\pi} |f(\theta)|^2 d\theta, \quad \text{in terms of } a_k \text{ and } b_k, \text{ or } c_k?$$

2. In chapter 5, we assume that the dimensions of the DFT matrix are powers of two. The Fundamental Theorem of Algebra states that every integer has a prime factorization. For example, $23,452 = 2^2 \cdot 11 \cdot 13 \cdot 41$ (check it!). Using this principle, optimal factorizations of the DFT matrix exist when the dimensions are *highly composite*, that is, a positive integer with more divisors than all smaller positive integers, such as $2^3 \cdot 3^2 \cdot 5$.

The first non-power-of-two highly composite number is $6 = 2 \cdot 3$. Carefully confirm the factorization of $\mathcal{F}_6 P_6$ in terms of I_3 , Ω_3 , and \mathcal{F}_3 . If you are feeling adventurous, can you derive a different factorization starting with a different permutation?

3. In problem 4 of chapter 3, we derived the $N(= N(a, \varepsilon))$ required to satisfy $\|f - p_N\|_\infty \leq \varepsilon \|f\|_\infty$ for $f(x) = (x^2 + a^2)^{-1}$. It was:

$$N(a, \varepsilon) \geq -\log \left(\varepsilon \frac{(a + \sqrt{a^2 + 1})^2 - 1}{a^2} \right) / \log(a + \sqrt{a^2 + 1}).$$

Use the DCT to examine how close the first N coefficients of the interpolant are to the first N coefficients of the expansion. In your opinion, is aliasing a concern for this particular function?

4. Use the conversion matrix:

$$\mathcal{C}_N = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & & & \\ & \frac{1}{2} & 0 & -\frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{2} & 0 & -\frac{1}{2} \\ & & & & \frac{1}{2} & 0 \\ & & & & & \frac{1}{2} \end{bmatrix},$$

between the T_k and U_k bases to confirm numerically the exact Chebyshev- U coefficients of e^x in Table 5.3 given the exact Chebyshev- T coefficients in Table 5.2 in terms of the modified Bessel functions of the first kind $I_k(1)$.