Redirection of Sound Waves

Mahsa Alikahi

January 24, 2025

Abstract

This document serves as a journal for the calculations performed. Two primary tasks are covered: first, the rederivation of necessary equations with appropriate references; and second, a discussion of the encountered challenges and the solutions implemented.

1 Introduction

We consider the scattering of a wave upon encountering a chain of cylinders. The goal is to compute the resulting behavior of the scattered wave.

2 Scattering of a Cylinder Object

The first problem we have to solve is the acoustic wave equation scattering from a cylinder.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) p = 0 \tag{1}$$

writing the equation in cylindrical coordinates and pertaining the variables to $p(r, \phi, z, t) = R(r)\Phi(\phi)Z(z)T(t)$, we rewrite the wave equation as

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Rr}\frac{\partial R}{\partial r} + \frac{1}{r^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} \tag{2}$$

in which;

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2$$
$$\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2$$
$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Rr}\frac{\partial R}{\partial r} + \frac{1}{r^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -k^2 + k_z^2 = -k_r^2$$

where $\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -n^2$, therefore for the radial differentiation equation we will have:

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + (k_r^2 - \frac{n^2}{r^2})R = 0$$

the solutions for each set of equations are:

$$R(r) = R_1 J_n(k_r r) + R_2 Y_n(k_r r)$$
$$T(t) = T_1 e^{-i\omega t}$$
$$Z(z) = Z_1 e^{ik_z z}$$
$$\Phi(\phi) = \Phi_1 e^{in\phi}$$

where $\omega = kc$, and n is an integer. Multiplying all the equations, for all possible nodes and all possible values for wave number in z axis, to have a solution in two dimensions, we will reach the most general solution:

$$p(r, \phi, z, t) = \sum_{n=-\infty}^{\infty} e^{in\phi}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk_z [A_n J_n(k_r r) + B_n Y_n(k_r r)] e^{ik_z z - i\omega t} \tag{3}$$

which is for the standing wave. The other form which is commonly used as well is:

$$p(r,\phi,z,t)=\sum_{n=-\infty}^{\infty} e^{in\phi}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk_z[C_n H_n^{(1)}(k_r r)+D_n H_n^{(2)}(k_r r)]e^{ik_z z-i\omega t} \tag{4}$$

by moving to the frequency domain and the boundary condition for a constant z, we reach that the incident and scattered wave are given as in the next section.

3 Equation of the scattered wave - Bozkho 2017

In this section, we are going to derive the equations of the scattering of a plane wave from a chain of perforated cylinders with impedance of:

$$Z_p=-\frac{i\omega\rho_0}{\sigma}\left[h\left(1-\frac{2}{s\sqrt{i}}\frac{J_1(s\sqrt{i})}{J_0(s\sqrt{i})}\right)^{-1}+4i\sqrt{2}\delta+\frac{16r}{3\pi}\left(1-2.5\sqrt{\frac{\sigma}{\pi}}\right)\right]$$

first we do it for a finite chain, since its easier, then we will move on to infinite chain. there a particular problem for a infinite chain, i calculating a series that converges very slowly.

3.1 Finite Chain of Cylinders

The incoming pressure plane wave $p(\mathbf{r},t)$ can be found by sovling the wave equation in a cylindrical coordinate. the solution for a plane wave will be found as:

$$\begin{aligned} p(\mathbf{r},t)&=p_0\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)=p\left(r_l,\varphi_l,t\right)\\ &=p_0 e^{ikr_l\cos(\varphi_l-\theta)-i\omega t}=p_0\sum_{n=-\infty}^{\infty} i^n J_n\left(kr_l\right)e^{in(\varphi_l-\theta)-i\omega t}\end{aligned} \tag{5}$$

The factor $e^{i\omega t}$ is omitted in further calculations. The field scattered by a system of shells is written as a superposition of outgoing cylindrical waves radiated by each scatterer. We write this field in polar coordinates centered at $x,y=0$:

$$p_{\mathrm{sc}}(r,\varphi)=\sum_{l'}\sum_{n=-\infty}^{+\infty} B_{l'n}H_n\left(kr_{l'}\right)e^{in\varphi_{l'}} \tag{6}$$

Here, the index l' numerates the shells in the chain, and H_n denotes the Hankel function of the first kind. Using Graf's theorem for $r_l<r_{ll'}$:

$$H_n\left(kr_{l'}\right)e^{in\varphi_{l'}}=\sum_{n'=-\infty}^{\infty} H_{n-n'}\left(kr_{ll'}\right)e^{i(n-n')\varphi_{ll'}}J_{n'}\left(kr_l\right)e^{in'\varphi_l}$$

Knowing it's a chain of cylinders, we can replace $r_{ll'}=|l-l'|d$ and $\varphi_{ll'}=\frac{\pi}{2}\operatorname{sgn}\left(l'-l\right)$. Also, by changing $n'\to -n'$:

$$H_n\left(kr_{l'}\right)e^{in\varphi_{l'}}=\sum_{n'=-\infty}^{\infty} i^{\left(n+n'\right)\operatorname{sgn}\left(l'-l\right)}H_{n+n'}\left(kr_{ll'}\right)J_{n'}\left(kr_l\right)e^{-in'\varphi_l}$$

Finally, by small adjustments:

$$H_n\left(kr_{l'}\right)e^{in\varphi_{l'}}=\sum_{n'=-\infty}^{\infty} i^{\left(n+n'\right)\operatorname{sgn}\left(l-l'\right)}H_{n+n'}\left(k\left|l-l'\right|d\right)J_{n'}\left(kr_l\right)e^{in'(\pi-\varphi_l)} \tag{7}$$

Replacing equation 7 in equation 6, we will have:

$$\begin{aligned} p_{\mathrm{sc}}\left(r_l,\varphi_l\right)=\sum_{n=-\infty}^{+\infty}\Bigg(&B_{ln}H_n\left(kr_l\right)e^{in\varphi_l}+\sum_{l'\neq l}B_{l'n}\\ &\times\sum_{n'=-\infty}^{+\infty} i^{\left(n+n'\right)\operatorname{sgn}\left(l-l'\right)}H_{n+n'}\left(k\left|l-l'\right|d\right)J_{n'}\left(kr_l\right)e^{in'(\pi-\varphi_l)}\Bigg)\end{aligned} \tag{8}$$

Pressure inside the lth cylinder is expanded over the Bessel functions:

$$p_{\text{in}}(r,\varphi) = \sum_{n=-\infty}^{+\infty} C_{ln} J_n\left(kr_l\right) e^{in\varphi_l} \tag{9}$$

3.1.1 Boundary Condition

We know that the impedance of a perforated shell is almost as much as a plane and is given by:

$$Z_p = -\frac{i\omega\rho_0}{\sigma}\left[h + \frac{16r}{3\pi}\left(1 - 2.5\sqrt{\frac{\sigma}{\pi}}\right)\right] \tag{10}$$

Also, from references, we know that:

$$Z_p = \frac{\Delta p}{v}$$

We are assuming the speed throughout each tube (perforated hole) is constant, so the speed at the beginning and the end is the same. Since impedance is given as the equation above, the boundary conditions are:

$$v_r|_{r=a} = v_r|_{r=b} = \frac{p|_{r=b} - p|_{r=a}}{Z_p} \tag{11}$$

Here, the radial velocity in the fluid is:

$$v_r = -\frac{i}{\omega\rho_0}\frac{\partial p}{\partial r}$$

So for the first boundary condition, we will have:

$$\left.\frac{\partial(p+p_{sc})}{\partial r}\right|_{r=a} = \left.\frac{\partial p_{in}}{\partial r}\right|_{r=b}$$

Consider: $A' = \frac{\partial A}{\partial r}$

$$\begin{aligned}p_0 \sum_{n=-\infty}^{\infty} i^n k J_n'\left(ka\right) e^{in(\varphi_l-\theta)} + \sum_{n=-\infty}^{+\infty} \Bigg[& B_{ln} k H_n'\left(ka\right) e^{in\varphi_l} + \sum_{l'\neq l} B_{l'n} \\ & \times \sum_{n'=-\infty}^{+\infty} i^{\left(n+n'\right)\operatorname{sgn}\left(l-l'\right)} H_{n+n'}\left(k\left|l-l'\right|d\right) k J_{n'}'\left(ka\right) e^{in'(\pi-\varphi_l)} \Bigg] \\ & = \sum_{n=-\infty}^{+\infty} C_{ln} k J_n'\left(kb\right) e^{in\varphi_l}\end{aligned}$$

to simplify the equation first by switching indices of n and n' in the second phrase and then change $n \to -n$ so the exponential go away.
Notice:

$$J_n(x) = (-1)^n J_{-n}(x)\ , \quad H_n(x) = (-1)^n H_{-n}(x)\ , \quad i^{-in\pi} = (-1)^n$$

$$(-1)^{(n-n')} i^{-(n-n')sgn(l-l')} = (-i^{-sgn(l-l')})^{(n-n')} = i^{(n-n')sgn(l-l')}$$

therefore:

$$p_0 i^n e^{-in\theta} + B_{ln}\frac{H_n'(ka)}{J_n'(ka)} + \sum_{l'\neq l}\sum_{n'=-\infty}^{\infty} B_{l'n'} i^{(n-n')\operatorname{sgn}(l-l')} H_{n-n'}(k|l-l'|d) e^{in\varphi_l} = C_{ln}\frac{J_n'(kb)}{J_n'(ka)} \tag{12}$$

The second boundary condition we will have:

$$\frac{p_{in}|_{r=b} - (p+p_{sc})|_{r=a}}{Z_p} = -\frac{-i}{\omega\rho_0}\left.\frac{\partial p_{in}}{\partial r}\right|_{r=b}$$

$$\sum_{n=-\infty}^{+\infty} C_{ln} J_n(kb)\, e^{in\varphi_l} - p_0 \sum_{n=-\infty}^{\infty} i^n J_n(ka)\, e^{in(\varphi_l-\theta)} - \sum_{n=-\infty}^{+\infty} \Bigg[B_{ln} H_n(ka)\, e^{in\varphi_l} + \sum_{l'\neq l} B_{l'n} \times \sum_{n'=-\infty}^{+\infty} i^{(n+n')\,\mathrm{sgn}(l-l')} H_{n+n'}(k\,|l-l'|\,d)\, J_{n'}(ka)\, e^{in'(\pi-\varphi_l)} \Bigg] = -\frac{ikZ_p}{\omega\rho_0} \sum_{n=-\infty}^{+\infty} C_{ln} J'_n(kb)\, e^{in\varphi_l}$$

Again by doing the same procedure in the last part we simplify the equation:

$$\begin{aligned} p_0 i^n e^{-in\theta} + B_{ln}\frac{H_n(ka)}{J_n(ka)} + \sum_{l'\neq l}\sum_{n'=-\infty}^{\infty} B_{l'n'} i^{(n+n')\,\mathrm{sgn}(l-l')} H_{n+n'}(k|l-l'|d) e^{in\varphi_l} \\ = C_{ln}\frac{J_n(kb)}{J_n(ka)} + \frac{iZ_p}{c_0\rho_0} C_{ln}\frac{J'_n(kb)}{J_n(ka)} \end{aligned} \tag{13}$$

dividing equations 13 and 12:

$$B_{ln}\left(\frac{H'_n(ka)}{J'_n(ka)} - \frac{H_n(ka)}{J_n(ka)}\right) = C_{ln}\left(\frac{J'_n(kb)}{J'_n(ka)} - \frac{J_n(kb)}{J_n(ka)} - \frac{iZ_p}{c_0\rho_0}\frac{J'_n(ka)}{J_n(ka)}\right)$$

by substituting the equation above in equation 12, we eliminate C_{ln} from the equations.

$$\begin{aligned} p_0 i^n e^{-in\theta} + B_{ln}\frac{H'_n(ka)}{J'_n(ka)} + \sum_{l'\neq l}\sum_{n'=-\infty}^{\infty} B_{l'n'} i^{(n+n')\,\mathrm{sgn}(l-l')} H_{n-n'}(k|l-l'|d) e^{in\varphi_l} \\ = -B_{ln}\frac{\frac{H'_n(ka)}{J'_n(ka)} - \frac{H_n(ka)}{J_n(ka)}}{\frac{J'_n(kb)}{J'_n(ka)} - \frac{J_n(kb)}{J_n(ka)} - \frac{iZ_p}{c_0\rho_0}\frac{J'_n(ka)}{J_n(ka)}} \end{aligned}$$

which can be rewritten as:

$$\mathcal{S}_n B_{ln} + \sum_{l'\neq l}\sum_{n'=-\infty}^{\infty} i^{(n-n')\,\mathrm{sgn}(l-l')} H_{n-n'}(k\,|l-l'|\,d)\, B_{l'n'} \quad = -p_0 i^n e^{-in\theta} \tag{14}$$

where:

$$\mathcal{S}_n = \frac{H_n(ka) - H'_n(ka)\left(\frac{iZ_p}{\rho_0 c_0} + \frac{J_n(kb)}{J'_n(kb)}\right)}{J_n(ka) - J'_n(ka)\left(\frac{iZ_p}{\rho_0 c_0} + \frac{J_n(kb)}{J'_n(kb)}\right)} \tag{15}$$

3.2 Infinite chain of Cylinders

to solve for the infinite chain of cylinders, we are going to use Bloch Theorem: $B_{ln} = e^{ik_y ld} B_{0n}$ where $k_y = k\sin(\theta)$. so, rewriting equation 13 for $l = 0$:

$$\mathcal{S}_n B_{0n} + \sum_{l'\neq 0}\sum_{n'=-\infty}^{\infty} i^{(n-n')\,\mathrm{sgn}(-l')} H_{n-n'}(k\,|-l'|\,d)\, e^{ik_y l'd} B_{0n'} \quad = -p_0 i^n e^{-in\theta}$$

simplifying the equation with $b_n = i^{-n} B_{0n}$:

$$\begin{aligned} &\mathcal{S}_n b_n + \sum_{l'=1}^{\infty}\sum_{n'=-\infty}^{\infty} i^{-(n-n')-n} H_{n-n'}(kl'd)\, e^{ik_y l'd} B_{0n'} + \sum_{l'=-\infty}^{-1}\sum_{n'=-\infty}^{\infty} i^{-n'} H_{n-n'}(k-l'd)\, e^{ik_y l'd} B_{0n'} \\ &= \mathcal{S}_n b_n + \sum_{n'=-\infty}^{\infty}\sum_{l'=1}^{\infty} H_{n-n'}(kl'd)\left[(-1)^{n'-n} e^{ik_y l'd} + e^{-ik_y l'd}\right] \quad b_{n'} \\ &= -p_0 e^{-in\theta} \end{aligned}$$

defining $F(n)$ as:

$$F(n) = \sum_{l'=1}^{+\infty} H_n\left(kl'd\right)\left[e^{ik_y l'd} + (-1)^n e^{-ik_y l'd}\right] \tag{16}$$

Therefore, by solving the equation below, we can find the scattering constants.

$$\mathcal{S}_n b_n + \sum_{n'=-\infty}^{\infty} F(n'-n) b_{n'} = -p_0 e^{-in\theta} \tag{17}$$

however, equation 16 is not easy to calculate easily.

Remember, the lattice some is repeated in the pressure field as well, so by using the same tricks we have mentioned in the equation, we must calculate the scattered pressure field. in equation 8 we have used Graf theorem and there are some sense of the lattice sum in the equation. now by consider the center at $l = 0$ and the relation below:

$$B_{ln} = i^n e^{ik_y ld} b_n \tag{18}$$

we can resume the simplification.

$$\begin{aligned} p_{\text{sc}}(r,\varphi) = \sum_{n=-\infty}^{+\infty} \Bigg(& i^n b_n H_n(kr) e^{in\varphi} + \sum_{l'=1}^{\infty} i^n e^{ik_y l'd} b_n \sum_{n'=-\infty}^{+\infty} i^{-\left(n+n'\right)} H_{n+n'}\left(k\left|-l'\right|d\right) J_{n'}(kr) e^{in'(\pi-\varphi)} \\ & + \sum_{l'=-1}^{-\infty} i^n e^{ik_y l'd} b_n \sum_{n'=-\infty}^{+\infty} i^{\left(n+n'\right)} H_{n+n'}\left(k\left|-l'\right|d\right) J_{n'}(kr) e^{in'(\pi-\varphi)} \Bigg) \end{aligned}$$

by changing $l' \to -l'$ we can sum up the last two summations.

$$\begin{aligned} p_{\text{sc}}(r,\varphi) &= \sum_{n=-\infty}^{+\infty} \Bigg(i^n b_n H_n(kr) e^{in\varphi} + \sum_{l'=1}^{\infty} i^n e^{ik_y l'd} b_n \sum_{n'=-\infty}^{+\infty} i^{-\left(n+n'\right)} H_{n+n'}\left(kl'd\right) J_{n'}(kr) e^{in'(\pi-\varphi)} \\ &\quad + \sum_{l'=1}^{\infty} i^n e^{-ik_y l'd} b_n \sum_{n'=-\infty}^{+\infty} i^{\left(n+n'\right)} H_{n+n'}\left(kl'd\right) J_{n'}(kr) e^{in'(\pi-\varphi)} \Bigg) \\ &= \sum_{n=-\infty}^{+\infty} i^n b_n H_n(kr) e^{in\varphi} \\ &\quad + \sum_{n=-\infty}^{+\infty} \sum_{l'=1}^{\infty} b_n \sum_{n'=-\infty}^{+\infty} i^{-n'} \left(e^{ik_y l'd} + (-1)^{(n+n')} e^{-ik_y l'd}\right) H_{n+n'}\left(kl'd\right) J_{n'}(kr) e^{in'(\pi-\varphi)} \\ &= \sum_{n=-\infty}^{+\infty} i^n b_n H_n(kr) e^{in\varphi} + \sum_{n=-\infty}^{+\infty} \sum_{n'=-\infty}^{+\infty} i^{-n'} b_n F(n+n') J_{n'}(kr) e^{in'(\pi-\varphi)} \end{aligned}$$

now by$n' \to -n'$ we see that:

$$p_{\text{sc}}(r,\varphi) = \sum_{n=-\infty}^{+\infty} i^n b_n H_n(kr) e^{in\varphi} + \sum_{n=-\infty}^{+\infty} \sum_{n'=-\infty}^{+\infty} i^{n'} b_n F(n-n') J_{n'}(kr) e^{in'\varphi} \tag{19}$$

4 Far Field Calculation for Infinite chain of Cylinders

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - n\pi/2 - \pi/4)$$

$$H_n^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-n\pi/2-\pi/4)}$$

by using approximations of Bessel functions in far field we can find the pattern of intensity of the scattered wave:

$$p_{\text{sc}}(r,\varphi) = \sqrt{\frac{2}{\pi kr}} \left[\sum_{n=-\infty}^{+\infty} i^n b_n e^{i(n\varphi+kr-n\pi/2-\pi/4)} + \sum_{n=-\infty}^{+\infty} \sum_{n'=-\infty}^{+\infty} i^{n'} b_n F(n-n') \cos(kr-n\pi/2-\pi/4) e^{in'\varphi} \right] \tag{20}$$

the velocity of the scattered wave:

$$v_{r,\mathrm{sc}}\left(r,\varphi\right)=\frac{-i}{\omega\rho_0}\sqrt{\frac{2k}{\pi r}}\left[\sum_{n=-\infty}^{+\infty}i^n b_n e^{i(n\varphi+kr-n\pi/2-\pi/4)}-\sum_{n=-\infty}^{+\infty}\sum_{n'=-\infty}^{+\infty}i^{n'}b_n F(n-n')\sin(kr-n\pi/2-\pi/4)e^{in'\varphi}\right] \tag{21}$$

so the intensity in far field normalized to $I_0=\frac{p_0 v_0}{2}=\frac{p_0^2}{2\rho_0 c_0}$:

$$\frac{I_{sc}}{I_0}=\frac{p_{\mathrm{sc}}\left(r,\varphi\right)v^*_{r,\mathrm{sc}}\left(r,\varphi\right)}{p_0 v_0} \tag{22}$$

using the final equations in the Mathematica code has given the needed results of the Bozkho article.