Lecture 4: Value Iteration and Policy Iteration

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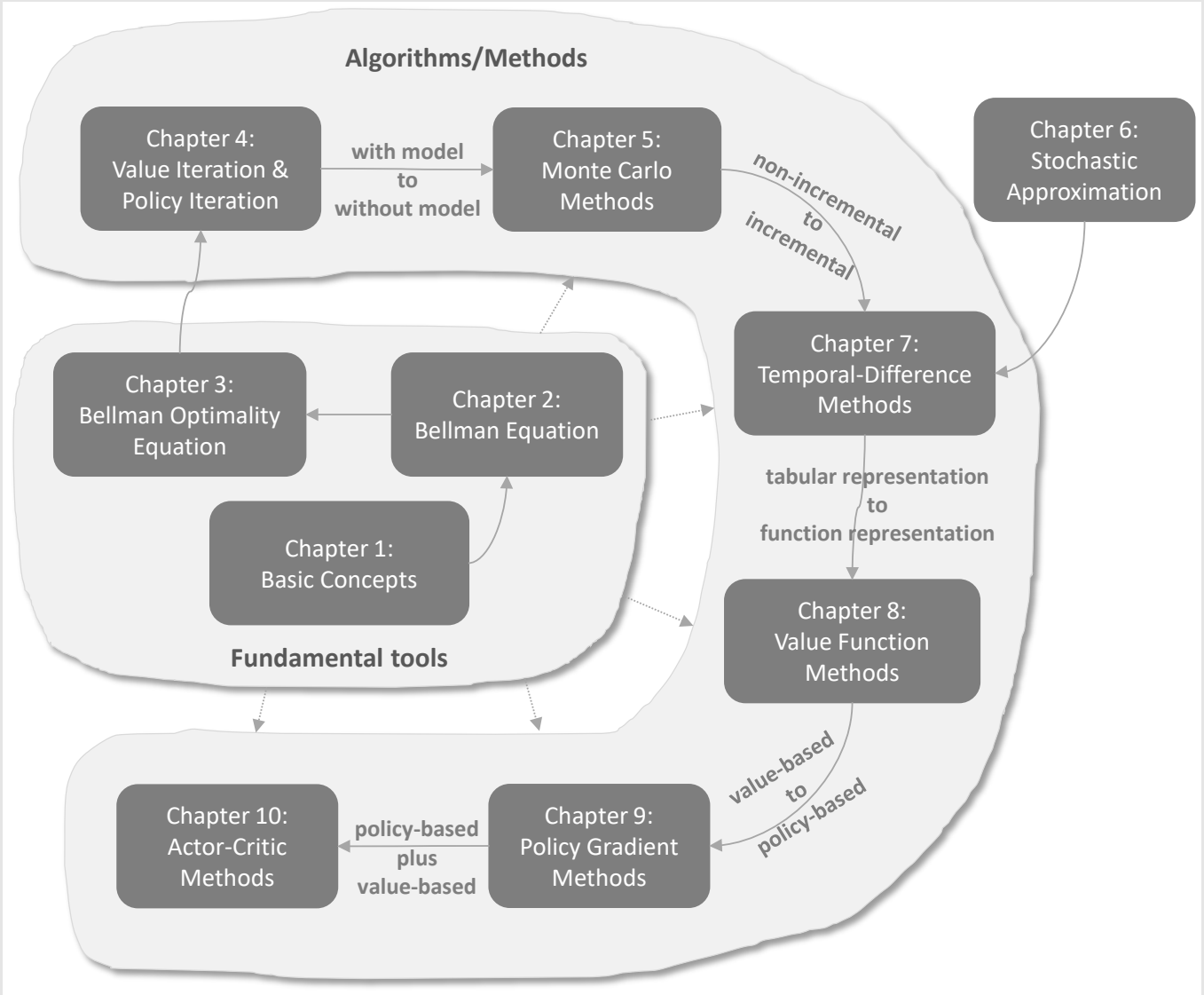
Outline


Algorithms/Methods
Chapter 4: Value Iteration & Policy Iteration
with model to without model
Chapter 5: Monte Carlo Methods
non-incremental to incremental
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Chapter 3: Bellman Optimality Equation
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Outline

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Value iteration algorithm

▷ How to solve the Bellman optimality equation?

$$v = f(v) = \max_{\pi}(r_\pi + \gamma P_\pi v)$$

▷ The **contraction mapping theorem** suggests an iterative algorithm:

$$v_{k+1} = f(v_k) = \max_{\pi}(r_\pi + \gamma P_\pi v_k), \quad k = 1, 2, 3 \ldots$$

where v_0 can be arbitrary. This algorithm can eventually find the optimal state value and an optimal policy.

▷ This algorithm is called value iteration!

▷ We next study the **implementation** of this algorithm.

Value iteration algorithm

The algorithm (matrix-vector form)

$$v_{k+1} = f(v_k) = \max_{\pi}(r_\pi + \gamma P_\pi v_k), \quad k = 1, 2, 3 \ldots$$

can be decomposed to two steps.

- Step 1: policy update. This step is to solve

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_k)$$

  where v_k is given.

- Step 2: value update.

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

Question: is v_k a state value? No, because it is not ensured that v_k satisfies a Bellman equation.

V_k不是 state value，但其最终可以收敛到 optimal state value.

Value iteration algorithm

▷ Next, we need to study the elementwise form in order to implement the algorithm.

- Matrix-vector form is useful for **theoretical analysis**.
- Elementwise form is useful for **implementation**.

Value iteration algorithm - Elementwise form

▷ Step 1: Policy update

给每个状态 s 一个初始价值 $v_0(s)=0$

The elementwise form of

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_k)$$

is

$$\pi_{k+1}(s) = \arg\max_{\pi} \sum_a \pi(a|s) \underbrace{\left(\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_k(s') \right)}_{q_k(s,a)}, \quad s \in \mathcal{S}$$

已知 未知 对每个 s

The optimal policy solving the above optimization problem is

$$\pi_{k+1}(a|s) = \begin{cases} 1 & a = a_k^*(s) \\ 0 & a \neq a_k^*(s) \end{cases}$$

只选择 action value 最大的 action (确定性策略)

where $a_k^*(s) = \arg\max_a q_k(a,s)$. π_{k+1} is called a **greedy policy**, since it simply selects the greatest q-value.

▷ Step 2: Value update

由 Policy update，已求出所有的 π.

The elementwise form of

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

is

因为是确定性的，该值为1

对每个 S

$$v_{k+1}(s) = \sum_a \pi_{k+1}(a|s) \underbrace{\left(\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_k(s') \right)}_{q_k(s,a)}, \quad s \in \mathcal{S}$$

Since π_{k+1} is greedy, the above equation is simply

$$v_{k+1}(s) = \max_a q_k(a, s)$$

Value iteration algorithm - Pseudocode

▷ Procedure summary:

$v_k(s) \rightarrow q_k(s,a) \rightarrow$ greedy policy $\pi_{k+1}(a|s) \rightarrow$ new value $v_{k+1} = \max_a q_k(s,a)$

$v_0 = 0$ $q_0(s,a)$ $\pi_1(a,s)$ $v_1 = \max_a q_0(s,a)$

v_1 $q_1(s,a)$ $\pi_2(a,s)$ $v_2 = \max_a q_1(s,a)$

Pseudocode: Value iteration algorithm

Initialization: The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all (s,a) are known. Initial guess v_0.

Aim: Search the optimal state value and an optimal policy solving the Bellman optimality equation.

While v_k has not converged in the sense that $\|v_k - v_{k-1}\|$ is greater than a predefined small threshold, for the kth iteration, do

For every state $s \in \mathcal{S}$, do

For every action $a \in \mathcal{A}(s)$, do

q-value: $q_k(s,a) = \sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a)v_k(s')$

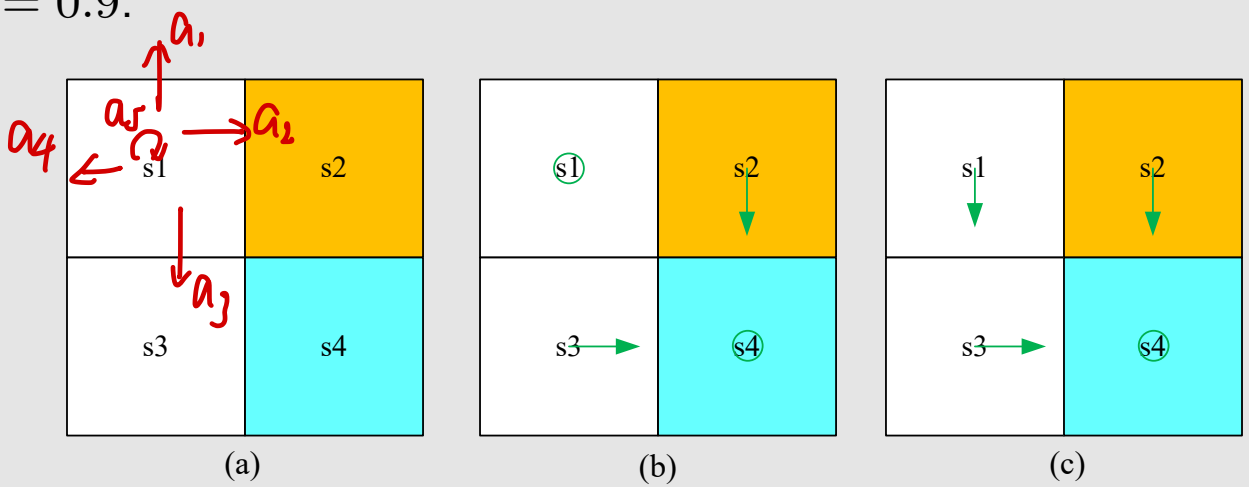
Maximum action value: $a_k^*(s) = \arg\max_a q_k(a,s)$

Policy update: $\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise

Value update: $v_{k+1}(s) = \max_a q_k(a,s)$

Value iteration algorithm - Example

▷ The reward setting is $r_{\text{boundary}} = r_{\text{forbidden}} = -1$, $r_{\text{target}} = 1$. The discount rate is $\gamma = 0.9$.



q-table: The expression of $q(s, a)$.

q-value	a_1	a_2	a_3	a_4	a_5
s_1	$-1 + \gamma v(s_1)$	$-1 + \gamma v(s_2)$	$0 + \gamma v(s_3)$	$-1 + \gamma v(s_1)$	$0 + \gamma v(s_1)$
s_2	$-1 + \gamma v(s_2)$	$-1 + \gamma v(s_2)$	$1 + \gamma v(s_4)$	$0 + \gamma v(s_1)$	$-1 + \gamma v(s_2)$
s_3	$0 + \gamma v(s_1)$	$1 + \gamma v(s_4)$	$-1 + \gamma v(s_3)$	$-1 + \gamma v(s_3)$	$0 + \gamma v(s_3)$
s_4	$-1 + \gamma v(s_2)$	$-1 + \gamma v(s_4)$	$-1 + \gamma v(s_4)$	$0 + \gamma v(s_3)$	$1 + \gamma v(s_4)$

Value iteration algorithm - Example

$q(s,a_n) = r(s,a_n) + \gamma V(s')$

- $k = 0$: let $v_0(s_1) = v_0(s_2) = v_0(s_3) = v_0(s_4) = 0$ 初始化

q-value	a_1	a_2	a_3	a_4	a_5
s_1	-1	-1	0	-1	0
s_2	-1	-1	1	0	-1
s_3	0	1	-1	-1	0
s_4	-1	-1	-1	0	1

Step 1: Policy update:

$$\pi_1(a_5|s_1) = 1,\ \pi_1(a_3|s_2) = 1,\ \pi_1(a_2|s_3) = 1,\ \pi_1(a_5|s_4) = 1$$

Step 2: Value update:

$$v_1(s_1) = 0,\ v_1(s_2) = 1,\ v_1(s_3) = 1,\ v_1(s_4) = 1.$$

This policy is visualized in Figure (b).

Value iteration algorithm - Example

- $k = 1$: since $v_1(s_1) = 0, v_1(s_2) = 1, v_1(s_3) = 1, v_1(s_4) = 1$, we have

q-table	a_1	a_2	a_3	a_4	a_5
s_1	$-1+\gamma 0$	$-1+\gamma 1$	$0+\gamma 1$	$-1+\gamma 0$	$0+\gamma 0$
s_2	$-1+\gamma 1$	$-1+\gamma 1$	$1+\gamma 1$	$0+\gamma 0$	$-1+\gamma 1$
s_3	$0+\gamma 0$	$1+\gamma 1$	$-1+\gamma 1$	$-1+\gamma 1$	$0+\gamma 1$
s_4	$-1+\gamma 1$	$-1+\gamma 1$	$-1+\gamma 1$	$0+\gamma 1$	$1+\gamma 1$

Step 1: Policy update:

$$\pi_2(a_3|s_1) = 1,\ \pi_2(a_3|s_2) = 1,\ \pi_2(a_2|s_3) = 1,\ \pi_2(a_5|s_4) = 1.$$

Step 2: Value update:

$$v_2(s_1) = \gamma 1,\ v_2(s_2) = 1+\gamma 1,\ v_2(s_3) = 1+\gamma 1,\ v_2(s_4) = 1+\gamma 1.$$

This policy is visualized in Figure (c).

$\|v_k - v_{k+1}\| < \theta$

The policy is already optimal!!

- $k = 2, 3, \ldots$. Stop when $\|v_k - v_{k+1}\|$ is smaller than a predefined threshold.

Outline

Policy iteration algorithm

$V_{\pi_0} = r_{\pi_0} + \gamma P_{\pi_0} V_{\pi_0}$

↓

$V_{\pi_0} = ?$

↓

所有的 action value

↓

$\pi_1 = \arg\max_{\pi} q(s,a)$

↓

V_{π_1}

↓

π_2

↓

...

▷ Algorithm description:

Given a random **initial policy** π_0,

- Step 1: policy evaluation (PE)

  This step is to calculate the state value of π_k:

$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

  Note that v_{π_k} is a state value function.

- Step 2: policy improvement (PI)

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_{\pi_k})$$

  The maximization is componentwise!

Similar to the value iteration algorithm? Be patient. We will compare them later.

Policy iteration algorithm

▷ The algorithm leads to a sequence

$$\pi_0 \xrightarrow{PE} v_{\pi_0} \xrightarrow{PI} \pi_1 \xrightarrow{PE} v_{\pi_1} \xrightarrow{PI} \pi_2 \xrightarrow{PE} v_{\pi_2} \xrightarrow{PI} \dots$$

PE=policy evaluation, PI=policy improvement

▷ Questions:

- Q1: In the policy evaluation step, how to get the state value v_{π_k} by solving the Bellman equation?
- Q2: In the policy improvement step, why is the new policy π_{k+1} better than π_k?
- Q3: Why such an iterative algorithm can finally reach an optimal policy?
- Q4: What is the relationship between this policy iteration algorithm and the previous value iteration algorithm?

Policy iteration algorithm

▷ **Q1: In the policy evaluation step, how to get the state value v_{π_k} by solving the Bellman equation?**

$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

- Closed-form solution:

$$v_{\pi_k} = (I - \gamma P_{\pi_k})^{-1} r_{\pi_k}$$

- Iterative solution:

$$v_{\pi_k}^{(j+1)} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}^{(j)}, \quad j = 0, 1, 2, ...$$

Already studied in the lecture about Bellman equation.

▷ Policy iteration is an iterative algorithm with another iterative algorithm embedded in the policy evaluation step!

Policy iteration algorithm

▷ **Q2: In the policy improvement step, why is the new policy π_{k+1} better than π_k?**

Lemma (Policy Improvement)

If $\pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_k})$, then $v_{\pi_{k+1}} \geq v_{\pi_k}$ for any k.

See the proof in the book.

Policy iteration algorithm

▷ **Q3: Why can such an iterative algorithm finally reach an optimal policy?**

Since every iteration would improve the policy, we know

$$v_{\pi_0} \leq v_{\pi_1} \leq v_{\pi_2} \leq \cdots \leq v_{\pi_k} \leq \cdots \leq v^*.$$

As a result, v_{π_k} keeps **increasing** and will converge.

Still need to prove what value it converges to.

Theorem (Convergence of Policy Iteration)

The state value sequence $\{v_{\pi_k}\}_{k=0}^{\infty}$ generated by the policy iteration algorithm converges to the optimal state value v^*. As a result, the policy sequence $\{\pi_k\}_{k=0}^{\infty}$ converges to an optimal policy.

The proof is given in my book.

Policy iteration algorithm

▷ **Q4: What is the relationship between policy iteration and value iteration?**

Will be explained in detail later.

Policy iteration algorithm - Elementwise form

Step 1: Policy evaluation 通过 π_k 求出 V_{π_k} (用 iteration 求)

- ▷ Matrix-vector form: $v_{\pi_k}^{(j+1)} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}^{(j)}, \quad j = 0, 1, 2, \ldots$
- ▷ Elementwise form:

$$v_{\pi_k}^{(j+1)}(s) = \sum_a \pi_k(a|s) \left(\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}^{(j)}(s') \right), \quad s \in \mathcal{S}$$

Stop when j is sufficiently large or $\|v_{\pi_k}^{(j+1)} - v_{\pi_k}^{(j)}\|$ is sufficiently small.

Step 2: Policy improvement

通过 V_{π_k} 和 π_k ⟶ 每个 s 的 optimal action value ↓ π_{k+1}

- ▷ Matrix-vector form: $\pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_k})$
- ▷ Elementwise form

$$\pi_{k+1}(s) = \arg\max_\pi \sum_a \pi(a|s) \underbrace{\left(\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}(s') \right)}_{q_{\pi_k}(s,a)}, \quad s \in \mathcal{S}.$$

Here, $q_{\pi_k}(s,a)$ is the action value under policy π_k. Let

$$a_k^*(s) = \arg\max_a q_{\pi_k}(a,s)$$

Then, the greedy policy is

$$\pi_{k+1}(a|s) = \begin{cases} 1 & a = a_k^*(s), \\ 0 & a \neq a_k^*(s). \end{cases}$$

Policy iteration algorithm - Implementation

Pseudocode: Policy iteration algorithm

Initialization: The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all (s,a) are known. Initial guess π_0.

Aim: Search for the optimal state value and an optimal policy.

While v_{π_k} has not converged, for the kth iteration, do

Policy evaluation:

Initialization: an arbitrary initial guess $v_{\pi_k}^{(0)}$

While $v_{\pi_k}^{(j)}$ has not converged, for the jth iteration, do

For every state $s \in \mathcal{S}$, do

$$v_{\pi_k}^{(j+1)}(s) = \sum_a \pi_k(a|s) \left[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}^{(j)}(s') \right]$$

Policy improvement:

For every state $s \in \mathcal{S}$, do

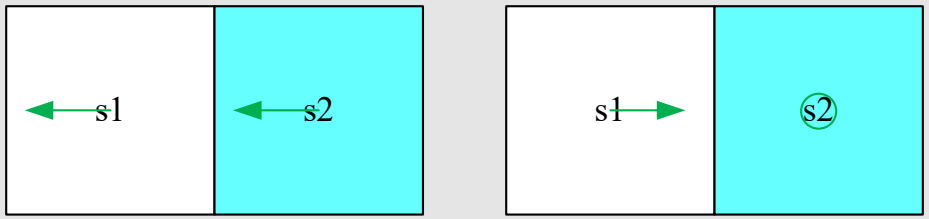
For every action $a \in \mathcal{A}$, do

$$q_{\pi_k}(s,a) = \sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}(s')$$

$a_k^*(s) = \arg\max_a q_{\pi_k}(s,a)$

$\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise

Policy iteration algorithm - Simple example



(a) initial policy (b) optimal policy

- The reward setting is $r_{\text{boundary}} = -1$ and $r_{\text{target}} = 1$. The discount rate is $\gamma = 0.9$.
- Actions: a_ℓ, a_0, a_r represent go left, stay unchanged, and go right.
- Aim: use policy iteration to find out the optimal policy.

Policy iteration algorithm - Simple example

▷ Iteration $k = 0$: Step 1: policy evaluation

π_0 is selected as the policy in Figure (a). The Bellman equation is

$\pi_0 = [a_1, a_1]$ 贝尔曼公式

$$\begin{aligned} v_{\pi_0}(s_1) &= -1 + \gamma v_{\pi_0}(s_1), \\ v_{\pi_0}(s_2) &= 0 + \gamma v_{\pi_0}(s_1). \end{aligned}$$

方法①：iteration

- Solve the equations directly: 方法①：直接求解

$$v_{\pi_0}(s_1) = -10, \quad v_{\pi_0}(s_2) = -9.$$

$V_{\pi_0} = r_{\pi_0} + \gamma P_{\pi_0} V_{\pi_0}$

$$\begin{bmatrix} V_{\pi_0}(s_1) \\ V_{\pi_0}(s_2) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_{\pi_0}(s_1) \\ V_{\pi_0}(s_2) \end{bmatrix}$$

- Solve the equations iteratively. Select the initial guess as $v_{\pi_0}^{(0)}(s_1) = v_{\pi_0}^{(0)}(s_2) = 0$:

$$\begin{cases} v_{\pi_0}^{(1)}(s_1) = -1 + \gamma v_{\pi_0}^{(0)}(s_1) = -1, \\ v_{\pi_0}^{(1)}(s_2) = 0 + \gamma v_{\pi_0}^{(0)}(s_1) = 0, \end{cases}$$

$$\begin{cases} v_{\pi_0}^{(2)}(s_1) = -1 + \gamma v_{\pi_0}^{(1)}(s_1) = -1.9, \\ v_{\pi_0}^{(2)}(s_2) = 0 + \gamma v_{\pi_0}^{(1)}(s_1) = -0.9, \end{cases}$$

$$\begin{cases} v_{\pi_0}^{(3)}(s_1) = -1 + \gamma v_{\pi_0}^{(2)}(s_1) = -2.71, \\ v_{\pi_0}^{(3)}(s_2) = 0 + \gamma v_{\pi_0}^{(2)}(s_1) = -1.71, \end{cases}$$

$\ldots$

Policy iteration algorithm - Simple example

▷ Iteration $k = 0$: Step 2: policy improvement

The expression of $q_{\pi_k}(s, a)$:

已知 V_{π_0} ⟶ 求 q_{π_0} of each action

$q_{\pi_k}(s,a)$	a_ℓ	a_0	a_r
s_1	$-1+\gamma v_{\pi_k}(s_1)$	$0+\gamma v_{\pi_k}(s_1)$	$1+\gamma v_{\pi_k}(s_2)$
s_2	$0+\gamma v_{\pi_k}(s_1)$	$1+\gamma v_{\pi_k}(s_2)$	$-1+\gamma v_{\pi_k}(s_2)$

Substituting $v_{\pi_0}(s_1) = -10, v_{\pi_0}(s_2) = -9$ and $\gamma = 0.9$ gives

$q_{\pi_0}(s,a)$	a_ℓ	a_0	a_r
s_1	-10	-9	-7.1
s_2	-9	-7.1	-9.1

选最大的

By seeking the greatest value of q_{π_0}, the improved policy is:

$$\pi_1(a_r|s_1) = 1, \quad \pi_1(a_0|s_2) = 1.$$

This policy is optimal after one iteration! In your programming, should continue until the stopping criterion is satisfied.

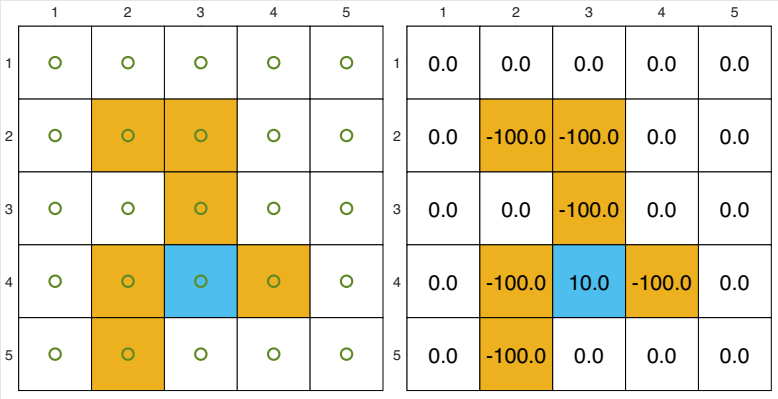
Policy iteration algorithm - Simple example

Excise! Set the left cell as the target area.

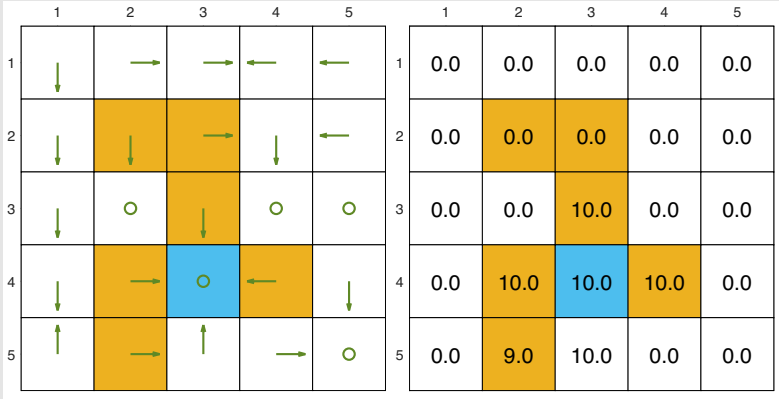
Now you know another powerful algorithm searching for optimal policies! Now let's apply it and see what we can find.

Policy iteration algorithm - Complicated example

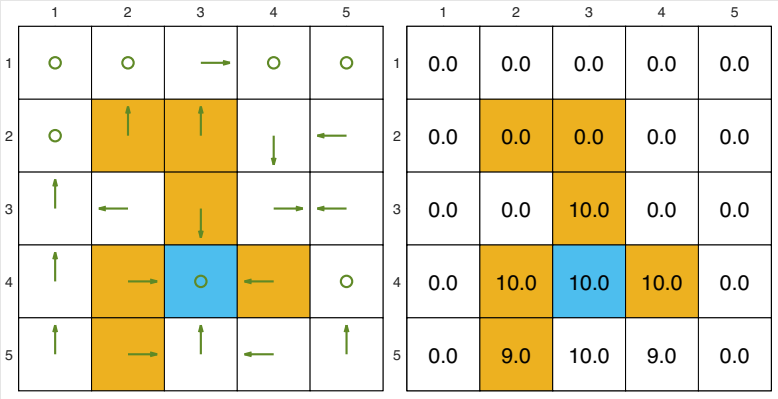
- Setting: $r_{\text{boundary}} = -1$, $r_{\text{forbidden}} = -10$, $r_{\text{target}} = 1$, $\gamma = 0.9$.
- Let's check out the intermediate policies and state values.



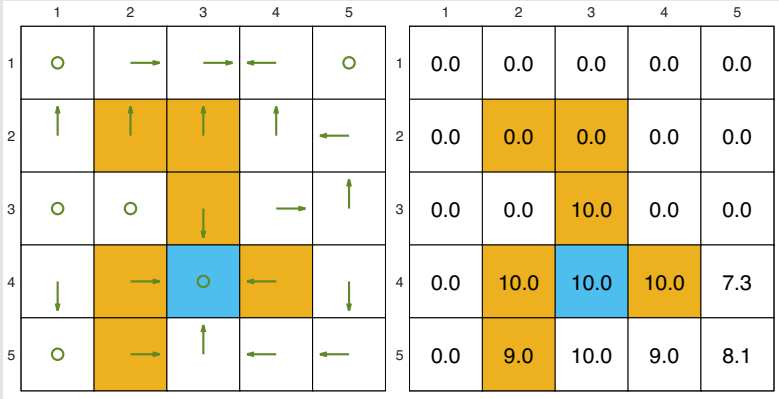
π_0 and v_{π_0}



π_1 and v_{π_1}



π_2 and v_{π_2}

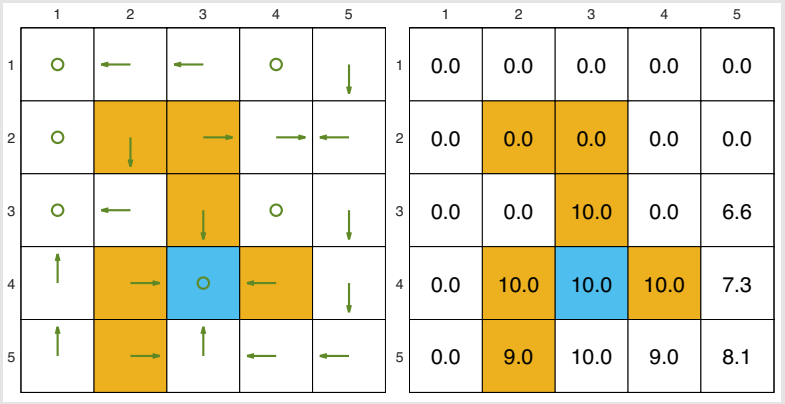


π_3 and v_{π_3}

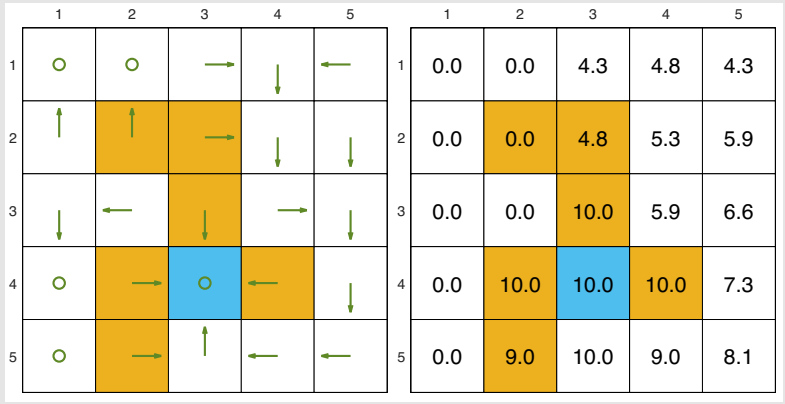
Policy iteration algorithm - Complicated example

一种现象：接近目标的 states 会先变好，远的后变好

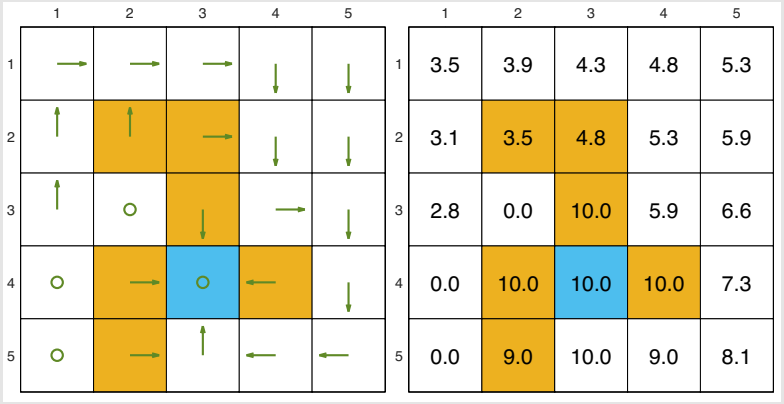
▷ Interesting pattern of the policies and state values



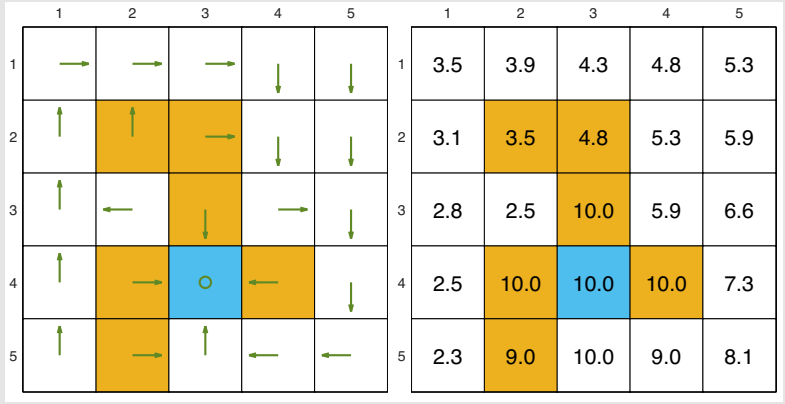
π_4 and v_{π_4}



π_5 and v_{π_5}

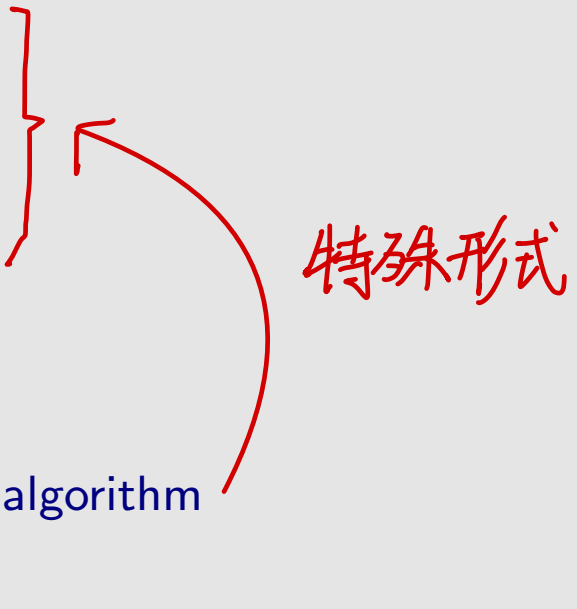


π_9 and v_{π_9}



π_{10} and $v_{\pi_{10}}$

Outline



Compare value iteration and policy iteration

Policy iteration: start from π_0

- Policy evaluation (PE):

$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

- Policy improvement (PI):

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_{\pi_k})$$

Value iteration: start from v_0

- Policy update (PU):

$$\pi_{k+1} = \arg\max_{\pi}(r_\pi + \gamma P_\pi v_k)$$

- Value update (VU):

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

Compare value iteration and policy iteration

▷ The two algorithms are very similar:

Policy iteration: $\pi_0 \xrightarrow{PE} v_{\pi_0} \xrightarrow{PI} \pi_1 \xrightarrow{PE} v_{\pi_1} \xrightarrow{PI} \pi_2 \xrightarrow{PE} v_{\pi_2} \xrightarrow{PI} \dots$

Value iteration: $u_0 \xrightarrow{PU} \pi_1' \xrightarrow{VU} u_1 \xrightarrow{PU} \pi_2' \xrightarrow{VU} u_2 \xrightarrow{PU} \dots$

PE=policy evaluation. PI=policy improvement.

PU=policy update. VU=value update.

Compare value iteration and policy iteration

▷ Let's compare the steps carefully:

	Policy iteration algorithm	Value iteration algorithm	Comments
1) Policy:	π_0	N/A	
2) Value:	$v_{\pi_0} = r_{\pi_0} + \gamma P_{\pi_0} v_{\pi_0}$	$v_0 \doteq v_{\pi_0}$	
3) Policy:	$\pi_1 = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_0})$	$\pi_1 = \arg\max_\pi (r_\pi + \gamma P_\pi v_0)$	The two policies are the same
4) Value:	$v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$	$v_1 = r_{\pi_1} + \gamma P_{\pi_1} v_0$	$v_{\pi_1} \geq v_1$ since $v_{\pi_1} \geq v_{\pi_0}$
5) Policy:	$\pi_2 = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_1})$	$\pi_2' = \arg\max_\pi (r_\pi + \gamma P_\pi v_1)$	
⋮	⋮	⋮	⋮

已知policy，用贝尔曼公式 和 iteration，求 state value

用贝尔曼最优公式，由旧的 v_k 推出新的 v_{k+1} 和 π_{k+1}

- They start from the same initial condition.
- The first three steps are the same.
- The fourth step becomes different:
  - In policy iteration, solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$ requires an iterative algorithm (an infinite number of iterations)
  - In value iteration, $v_1 = r_{\pi_1} + \gamma P_{\pi_1} v_0$ is a one-step iteration

Compare value iteration and policy iteration

这是Policy iteration中，由π_1求V_{π_1}的过程

Consider the step of solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$:

value iteration 的初始值

不同点!!!
原本设 $V_{\pi_1}^{(0)} = \vec{0}$
现在：$V_{\pi_1}^{(0)} = V_0$

$$v_{\pi_1}^{(0)} = v_0$$

value iteration $\leftarrow v_1 \longleftarrow v_{\pi_1}^{(1)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(0)}$

在第j步的时候截断!!!
将 $V_{\pi_1}^{(j)}$ 放入下一步求 π_2

$$v_{\pi_1}^{(2)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(1)}$$

$$\vdots$$

truncated policy iteration $\leftarrow \bar{v}_1 \longleftarrow v_{\pi_1}^{(j)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(j-1)}$

j=1 ⟶ value iteration

$$\vdots$$

j=∞ ⟶ policy iteration

policy iteration $\leftarrow v_{\pi_1} \longleftarrow v_{\pi_1}^{(\infty)} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(\infty)}$

- The value iteration algorithm computes *once*.
- The policy iteration algorithm computes *an infinite number of iterations*.
- The truncated policy iteration algorithm computes *a finite number of iterations* (say j). The rest iterations from j to ∞ are truncated.

policy iteration是不实际的，因为要计算∞步.
(可以设计一个 $\|V_{k+1} - V_k\| < \theta$)

Truncated policy iteration - Pseudocode

Pseudocode: Truncated policy iteration algorithm

Initialization: The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all (s,a) are known. Initial guess π_0.

Aim: Search for the optimal state value and an optimal policy.

While v_k has not converged, for the kth iteration, do

Policy evaluation:

Initialization: select the initial guess as $v_k^{(0)} = v_{k-1}$. The maximum iteration is set to be j_{truncate}.

While $j < j_{\text{truncate}}$, do

For every state $s \in \mathcal{S}$, do

$$v_k^{(j+1)}(s) = \sum_a \pi_k(a|s) \left[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_k^{(j)}(s') \right]$$

Set $v_k = v_k^{(j_{\text{truncate}})}$

Policy improvement:

For every state $s \in \mathcal{S}$, do

For every action $a \in \mathcal{A}(s)$, do

$$q_k(s,a) = \sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_k(s')$$

$a_k^*(s) = \arg\max_a q_k(s,a)$

$\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise

Truncated policy iteration - Convergence

▷ Will the truncation undermine convergence?

Proposition (Value Improvement)

Consider the iterative algorithm for solving the policy evaluation step:

$$v_{\pi_k}^{(j+1)} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}^{(j)}, \quad j = 0, 1, 2, ...$$

If the initial guess is selected as $v_{\pi_k}^{(0)} = v_{\pi_{k-1}}$*, it holds that*

$$v_{\pi_k}^{(j+1)} \geq v_{\pi_k}^{(j)}$$

for every $j = 0, 1, 2, \ldots$.

在iteration中，$V_{\pi_k}^{(j+1)} \geq V_{\pi_k}^{(j)}$

所以在j处截断，也一定会收敛。

For the proof, see the book.

Truncated policy iteration - Convergence

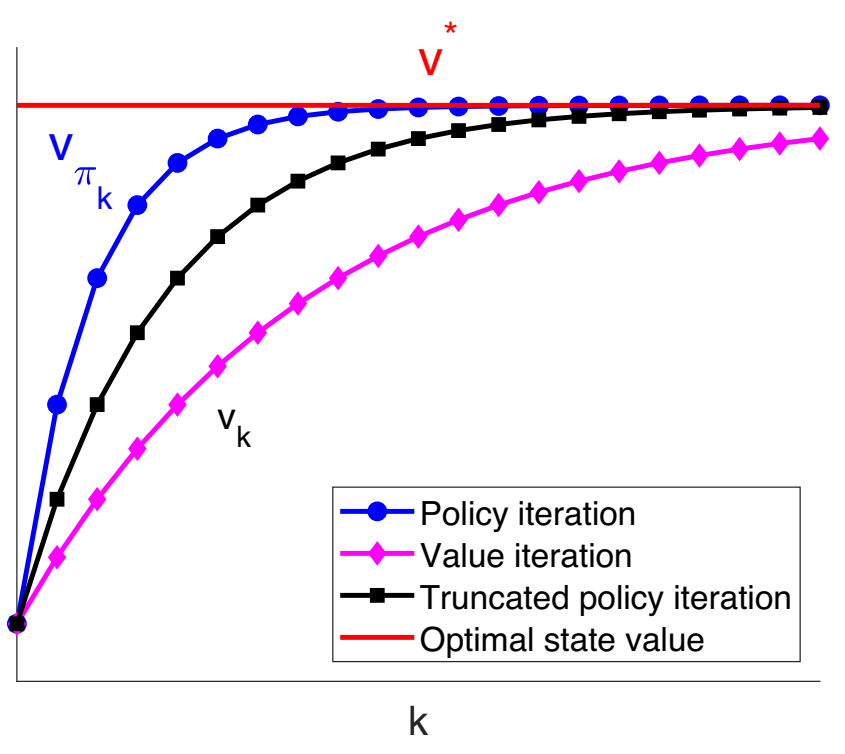


Figure: Illustration of the relationship among value iteration, policy iteration, and truncated policy iteration.

The convergence proof of PI is based on that of VI. Since VI converges, we know PI converges.

Summary

▷ Value iteration: it is the iterative algorithm solving the Bellman optimality equation: given an initial value v_0,

$$v_{k+1} = \max_\pi(r_\pi + \gamma P_\pi v_k)$$
$$\Updownarrow$$
$$\begin{cases} \text{Policy update: } \pi_{k+1} = \arg\max_\pi(r_\pi + \gamma P_\pi v_k) \\ \text{Value update: } v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k \end{cases}$$

▷ Policy iteration: given an initial policy π_0,

$$\begin{cases} \text{Policy evaluation: } v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k} \\ \text{Policy improvement: } \pi_{k+1} = \arg\max_\pi(r_\pi + \gamma P_\pi v_{\pi_k}) \end{cases}$$

▷ Truncated policy iteration