COMP 250

Lecture 10

mathematical induction

Sept. 29, 2017

For all $n \geq 1$,

$$1 + 2 + 3 + \ \dots . + (n-1) + n = \frac{n(n+1)}{2}$$

How to prove such a statement ?

By “proof”, we mean a formal logical argument that convincely shows the statement is true.

Note that “convincely” is itself not well defined.

$$1 + 2 + \ldots + (n-1) + n$$

Rewrite by considering $n/2$ pairs :

$$1 + 2 + \cdots + \frac{n}{2} + (\frac{n}{2} + 1) \ldots + (n-1) + n$$

If n is even, then adding up the $n/2$ pairs gives $n/2$ *(n+ 1).

What if n is odd?

What if n is odd?

Then, n-1 is even. So,

$$1 + 2 + \ldots + (n-1) + n$$

$$= \left(\frac{n-1}{2} * n\right) + n$$

$$= \left(\frac{n-1}{2} + 1\right) * n$$

$= \frac{n+1}{2} * n$ which is the same formula as before.

Mathematical Induction

Consider a statement of the form:

"For all $n \geq n_0$, $P(n)$ is true" where n_0 is some constant and proposition $P(n)$ has value true or false for each n.

Mathematical induction is a general technique for proving such a statement.

"For all $n \geq n_0$, $P(n)$ is true"

For all $n \geq 1$,

$$1 + 2 + \ldots + (n-1) + n = \frac{n(n+1)}{2}$$

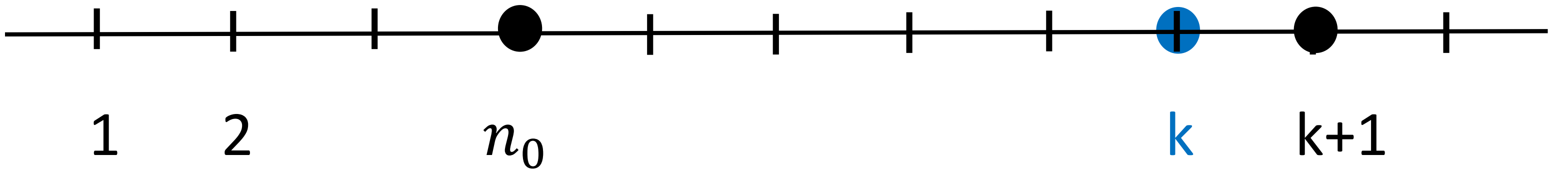
Mathematical induction requires proving two things:

Base case:

"P(n_0) is true."

Induction step:

"For any $k \geq n_0$, if $\mathrm{P}(k)$ is true, then $\mathrm{P}(k+1)$ is also true."



Base case:

"P(n_0) is true."

Induction step:

"For any k $\geq n_0$, if P(k) is true, then P(k+1) is also true."

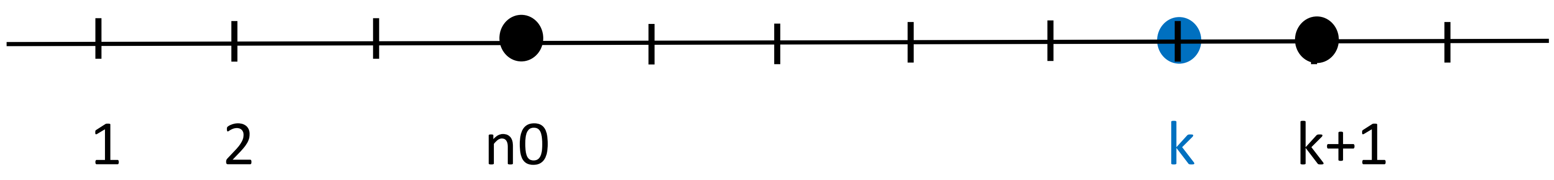
The statement "P(k) is true" is called the "induction hypothesis".

Base case:

P(n_0) is true.

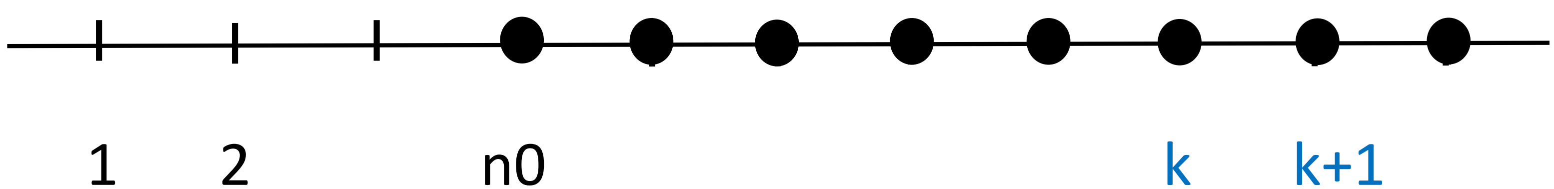
Induction step:

For any k >= n_0, if P(k) is true
then P(k+1) is true.



Thus we have proved:

For any n >= n_0, P(n) is true.



Statement: For all $n \geq 1$,

$$1 + 2 + 3 + \ \ldots . + (n-1) + n = \frac{n(n+1)}{2}$$

Proof (base case, n=1):

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$$1 + 2 + 3 + \ \ldots . + (n-1) + n = \frac{n(n+1)}{2}$$

Proof (base case, n=1):

$$1 \ = \ \frac{1(1+1)}{2} \qquad \text{(true)}$$

induction *hypothesis* is that $\mathrm{P}(k)$ is true:

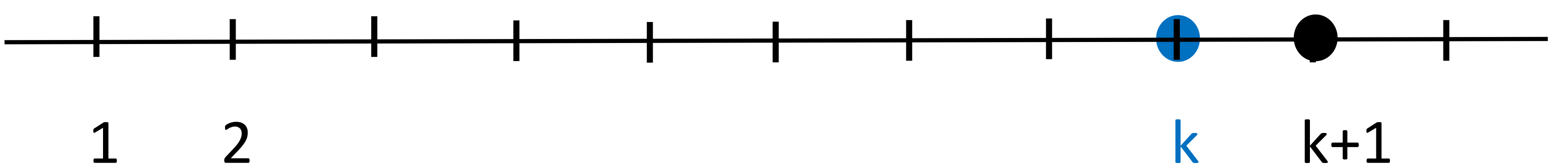
$$1 + 2 + \ \ldots + \ k \ = \frac{k(k+1)}{2}$$

Base case:

P(1) is true.

Induction step:

For any k >= n_0, if P(k) is true
then P(k+1) is true.



Proof of Induction Step:

$$(1 + 2 + 3 + \ldots + k) + k + 1$$

by induction hypothesis

$$= \frac{k(k+1)}{2} + k + 1$$

$$=$$

Proof of Induction Step:

$$(1 + 2 + 3 + \ \ldots . + \ k) \ + k + 1$$

by induction hypothesis

$$= \frac{k(k+1)}{2} + k + 1$$

$$= \left(\frac{k}{2} + 1\right)(k+1)$$

$$= \frac{1}{2}(k+2)(k+1)$$

Thus, $P(k)$ *is true* *implies* $P(k+1)$ *is true*.

Possible confusion

$P(k)$ has value true or false (Boolean).

So, $P(k)$ $is\ true$ means what?

Examples

“$3 = 2 + 1$” is true.

“$3 = 2 + 2$” is false.

Examples

"$3 = 2 + 1$" is true.

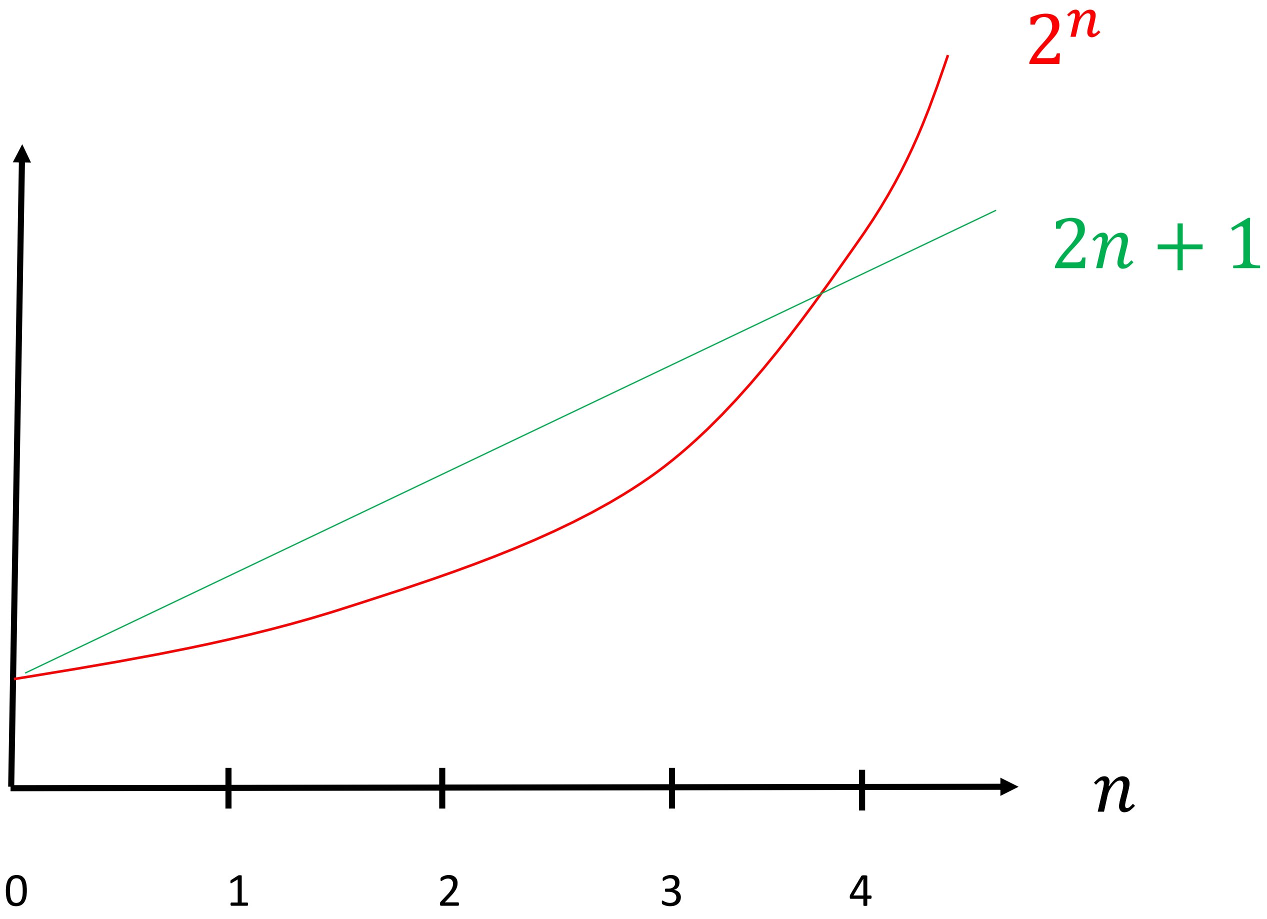
"$3 = 2 + 2$" is false.

"If $3 = 2 + 2$ then $5 > 7$" is true.

If this is a mystery to you, then I strongly advise you to take MATH 240 or MATH 318 (logic).

Mathematical Induction: Example 2

Prove the following statement:
For all $n \geq 3$, $2n + 1 < 2^n$.



Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Note: P(n) is false for n= 1, 2.

But that has nothing to do with what we need to prove.

Proof (base case, n =3):

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Note: P(n) is false for n= 1, 2.

But that has nothing to do with what we need to prove.

Proof (base case, n =3):

2*3 + 1 < 8 (true)

Proof of Induction Step:

We want to show that P(k) implies P(k+1).

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Proof of Induction Step:

We want to show that P(k) implies P(k+1).

$$2(k+1) + 1 \quad = \quad ?$$

(This is $2n + 1$ where $n = k + 1$.)

Proof of Induction Step:

We want to show that P(k) implies P(k+1).

$$2(k+1) + 1 = 2k + 2 + 1$$

by induction hypothesis

$$< 2^k + 2$$

Proof of Induction Step:

We want to show that P(k) implies P(k+1).

$$2(k+1)+1 \;=\; 2k+2+1$$

by induction hypothesis

$$< \; 2^k + 2$$

$$< \; 2^k + 2^k, \quad \text{for } k \geq 3$$

This inequality is also true for k >=2 but we don't care because we are trying to prove for k>=3.

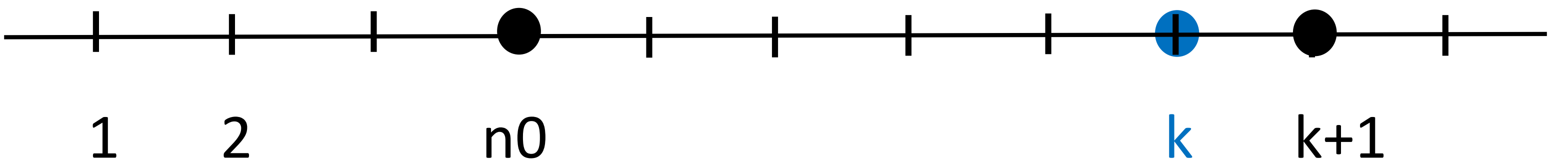
$$= \; 2^{k+1}$$

Base case:

$P(n_0)$ is true.

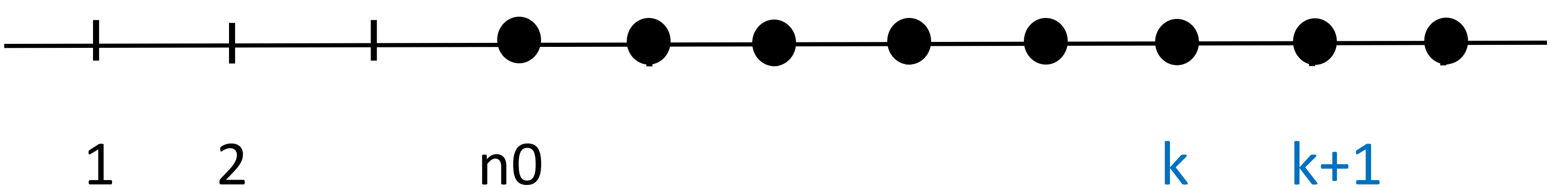
Induction step:

For any k >= n_0, if P(k) then P(k+1).



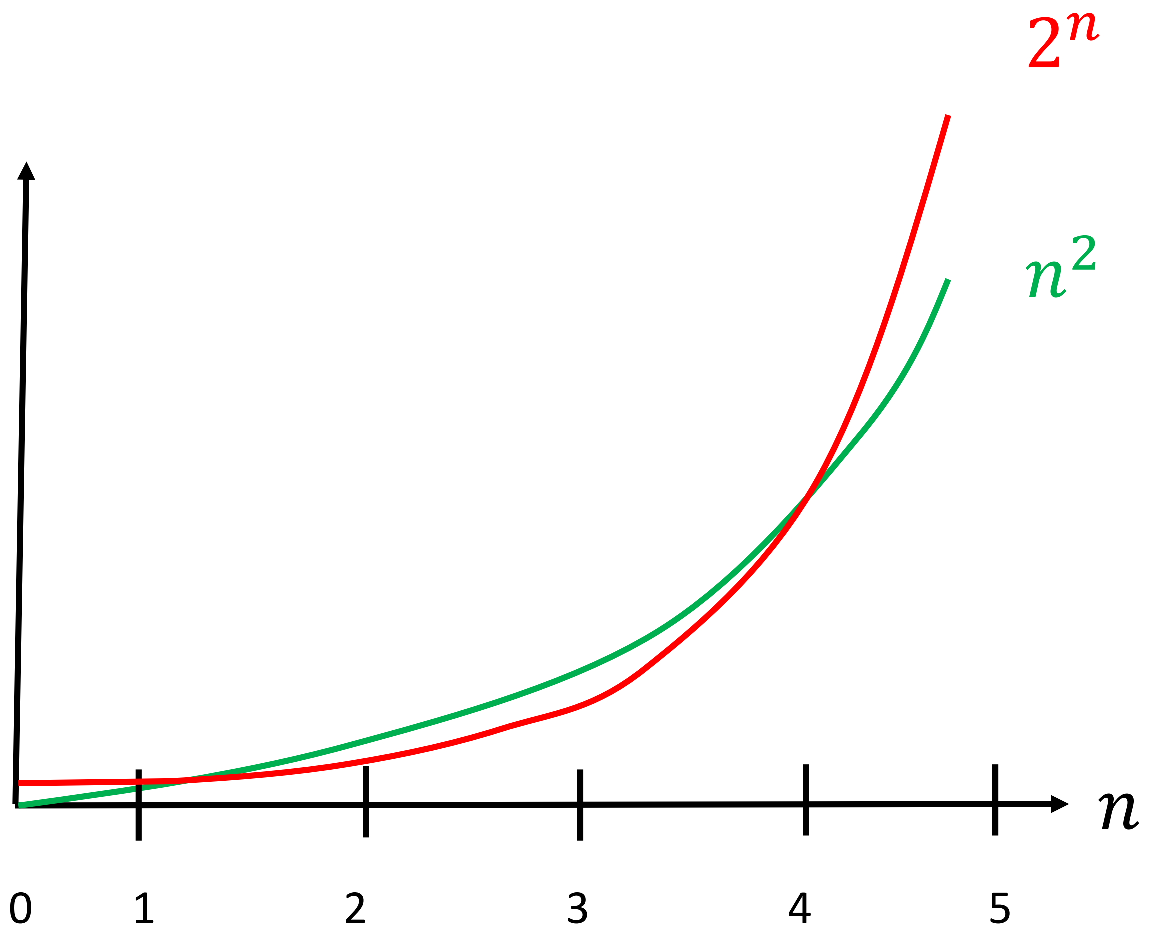
Thus,

For any n >= n_0, P(n) is true.



Example 3

Statement: For all $n \geq 5$, $n^2 < 2^n$.



Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case (n = 5):

Induction step:

Statement: For all $n \geq 5, \quad n^2 < 2^n$.

Base case (n = 5):

25 < 32

Induction step:

What do we assume ?

What do we want to prove ?

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case (n = 5):

25 < 32

Induction step:

What do we assume ?

$$k^2 < 2^k, \quad k \geq 5$$

What do we want to show ?

$$(k+1)^2 < 2^{k+1}$$

Statement: For all $n \geq 5, \quad n^2 < 2^n$.

Base case (n = 5):

25 < 32

Induction step:

$$(k+1)^2 = k^2 + 2k + 1$$

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case (n = 5):

25 < 32

Induction step:

$$(k+1)^2 = k^2 + 2k + 1$$

by induction hypothesis

$$< 2^k + 2k + 1$$

Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case (n = 5):

25 < 32

Induction step:

$$(k+1)^2 = k^2 + 2k + 1$$

by induction hypothesis

$$< 2^k + 2k + 1$$

by Example 2

$$< 2^k + 2^k$$

$$= 2^{k+1}$$

Example 4 : Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ….

$F(0)$ = 0
$F(1)$ = 1
$F(n+2)$ = $F(n+1) + F(n)$, for $n \geq 0$.

Statement: For all $n \geq 0$, $F(n) < 2^n$

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

Base case(s):

$n = 0$: $0 < 2^0$ is true.

$n = 1$: $1 < 2^1$ is true.

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

Induction step:

$$F(k+1) = F(k) + F(k-1)$$

by induction hypothesis

$$< 2^k + 2^{k-1}$$

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

Induction step:

$$F(k+1) = F(k) + F(k-1)$$

by induction hypothesis

$$< 2^k + 2^{k-1}$$

$$< 2^k + 2^k$$

$$= 2^{k+1}$$