Tutorial: Induction

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October 7, 2010

In this tutorial, we cover some more examples of proofs by induction. Recall that in proofs by induction, we have two main steps. The *base case* just checks if the statement is true for a first value of the argument, n_0. The second step, the *induction step*, assumes that the statement that we want to prove is true for any arguments up to and including n, and then shows that it is also true for $n+1$. The trick is (almost) always to isolate in the statement for $n+1$ a fragment which can use the statement for n (or for some other $k < n$), for which we can use the induction hypothesis.

1. Prove that $2^n < n!, \forall n \geq 4$.
   Proof:
   Base case: For $n = 4$, we have $2^4 = 16$ and $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$
   Induction step: Suppose the $2^n = n!$. Is it true that $2^{n+1} < (n+1)!$? We have:

   $$\begin{aligned} 2^n &< n! \text{ (by the induction hypothesis)} \\ 2 &< (n+1) \text{ (because } n \geq 4) \end{aligned}$$

   By multiplying the two together, we get the result. This concludes the proof.

2. Consider the Fibonacci numbers, defined as follows: $f_1 = f_2 = 1, f_n = f_{n-1} + f_{n-2}$. Prove that:

   (a) f_{3n} is even, $\forall n \geq 1$
   Proof:
   Base case: $f_3 = f_1 + f_2 = 1 + 1 = 2$ which is even
   Induction step: $f_{3(n+1)} = f_{3n+2} + f_{3n+1} = f_{3n+1} + f_{3n} + f_{3n+1} = 2f_{3n+1} + f_{3n}$. The first term is obviously even and the second is even form the induction hypothesis - q.e.d.

   (b) $f_{n-1}f_{n+1} = f_n^2 + (-1)^n, \forall n \geq 2$
   Base case: $n = 2$: we have $f_1 f_3 = 2$ and $f_2^2 + (-1)^2 = 1 + 1 = 2$ which concludes the proof
   Induction step: Suppose the result is true up to n. We have:

   $$\begin{aligned} f_n f_{n+2} &= f_n(f_{n+1} + f_n), \text{ by definiton} \\ &= f_n^2 + f_n f_{n+1} \end{aligned}$$

$$
\begin{aligned}
&= f_{n-1}f_{n+1} - (-1)^n + f_n f_{n+1}, \text{ by induction hypothesis} \\
&= f_{n+1}(f_{n-1} + f_n) + (-1)^{n+1}, \text{ by simple algebra} \\
&= f_{n+1}^2 + (-1)^{n+1}
\end{aligned}
$$

$mbox, by definition$

which concludes the proof

3. Prove that $1^3 + 2^3 + \ldots n^3 = (1 + 2 + \ldots + n)^2$
   Proof:
   Base case: $n = 1$ - obvious **Induction step:**

$$
\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= (1+2+\ldots+n)^2 + (n+1)^2(n+1), \text{ using the induction hypothesis} \\
&= (1+2+\ldots n)^2 + (n+1)^2 + 2\frac{n(n+1)}{2}(n+1) \\
&= (1+2+\ldots n)^2 + (n+1)^2 + 2(1+2+\ldots+n)(n+1), \text{ using identity we showed in class} \\
&= (1+2+\ldots(n+1))^2
\end{aligned}
$$