COMP 250

Lecture 11

recursive algorithms 1

Oct. 2, 2017

Example 1: Factorial (iterative)

$$n! \;=\; 1 * 2 * 3 * \ldots * (n-1) * n$$

```
factorial( n ){    // assume n >= 1
     result = 1
     for (k = 2; k <= n; k++)
          result = result * k
    return result
 }
```

Factorial (recursive)

$$n! \;=\; (n - 1)! * n$$

```
factorial( n ){   // assume n >= 1
     if n == 1
        return 1
     else
        return factorial( n – 1 ) * n
}
```

Claim: the recursive factorial(n) algorithm returns $n!$.

Proof (by mathematical induction):

Base case: factorial(1) returns 1.

Induction step:

Take any k >= 1.

if factorial(k) returns $k!$

then factorial($k + 1$) returns ($k + 1$) !

Example 2: Fibonacci

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ….

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

Fibonacci (iterative)

```
fibonacci(n){
    if ((n == 0) | (n == 1))
      return n
    else{
      fib0 = 0
      fib1 = 1
      for k = 2 to n{
          fib2 = fib1 + fib0      //  Fib(n+2)
          fib0 = fib1             //  Fib(n)     in next pass
          fib1 = fib2             //  Fib(n+1)  in next pass
      }
      return fib2
    }
}
```

Fibonacci (recursive)

```
fibonacci(n){    //  assume n > 0
  if ((n == 0) || (n == 1))
    return n
  else
    return fibonacci(n-1) + fibonacci(n-2)
}
```

This is much simpler to express than the iterative version.

Claim: the recursive Fibonacci algorithm is correct.

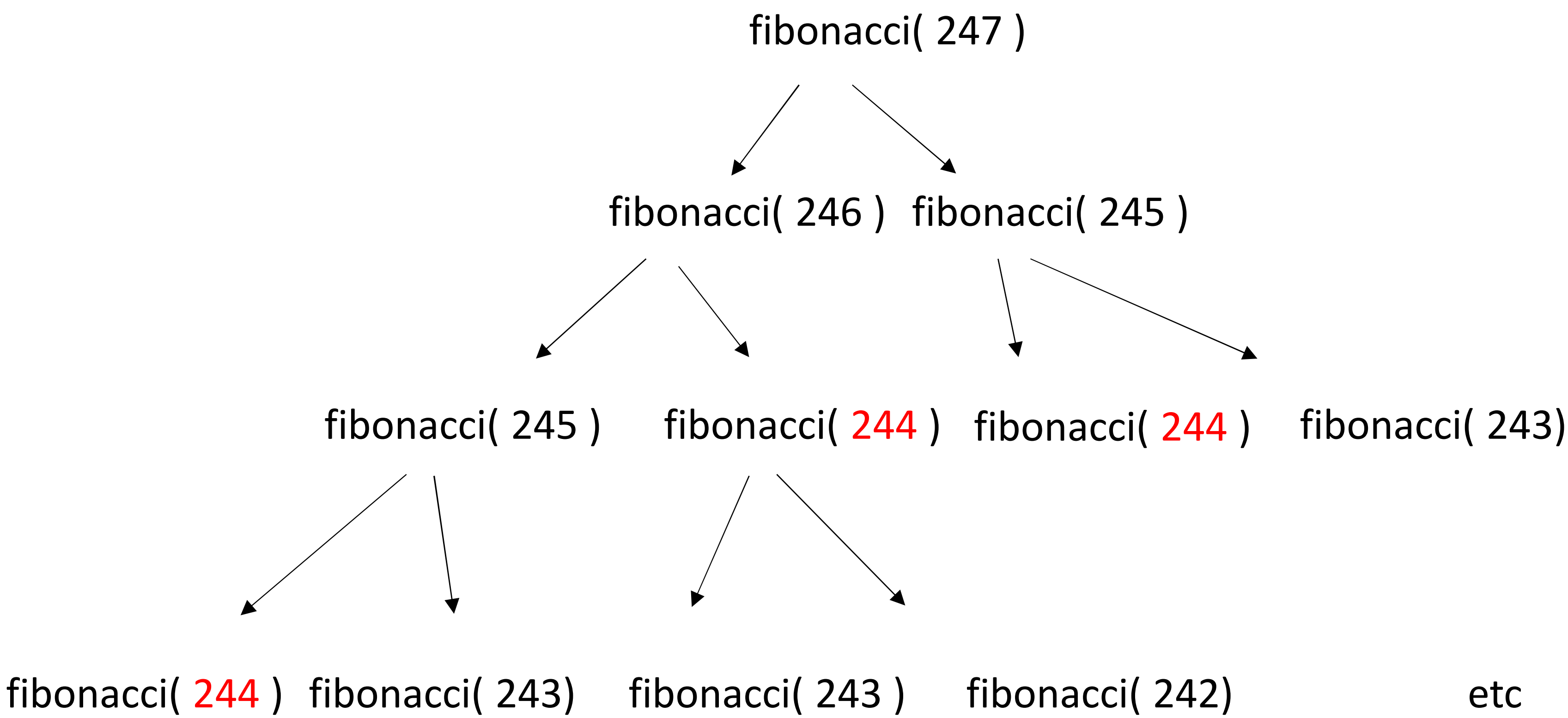
Proof:

Base case: Fib (0) returns 0. Fib(1) returns 1.

Induction step:

for k > 1
if fibonacci(k-1) and fibonacci(k) return F(k-1) and F(k)
then fibonacci(k+1) returns F(k+1).

However, the recursive Fibonacci algorithm is very inefficient. It computes the same quantity many times, for example:



Example 3: Reversing a list

input	(a b c d e f g h)
output	(h g f e d c b a)

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input (a b c d e f g h)

output (h g f e d c b a)

Idea of recursion:

a (b c d e f g h)

(h g f e d c b) a

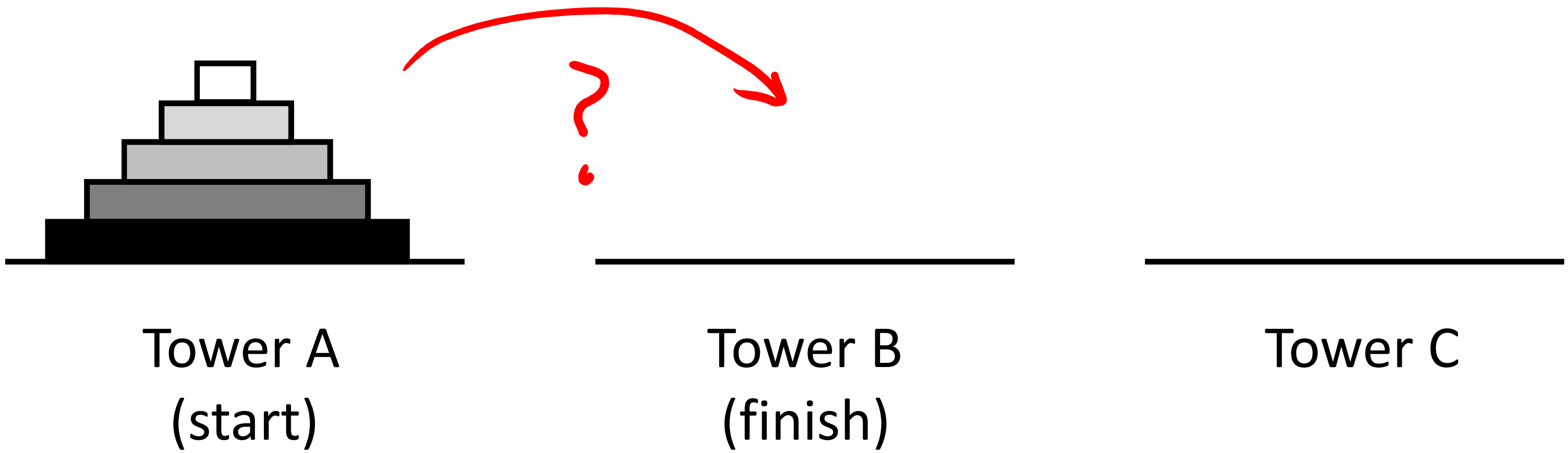
Example 3: Reversing a list (recursive)

```
reverse( list ){                  //   assume n > 0
  if list.size == 1               // base case
     return  list
  else{
     firstElement  =  removeFirst(list)
     list   =  reverse(list)    // list has only n-1 elements
     return  addLast(list,  firstElement )
     }
 }
```

Example 4: Sorting a list (recursive)

```
sort( list ) {                    //   assume size > 0
    if list.size == 1             //   base case
       return  list
    else{
        minElement = removeMin(list)
        list = sort( list )          //  has n-1 elements
        return  addFirst(list, minElement)
     }
}
//   reminiscent of selection sort
```

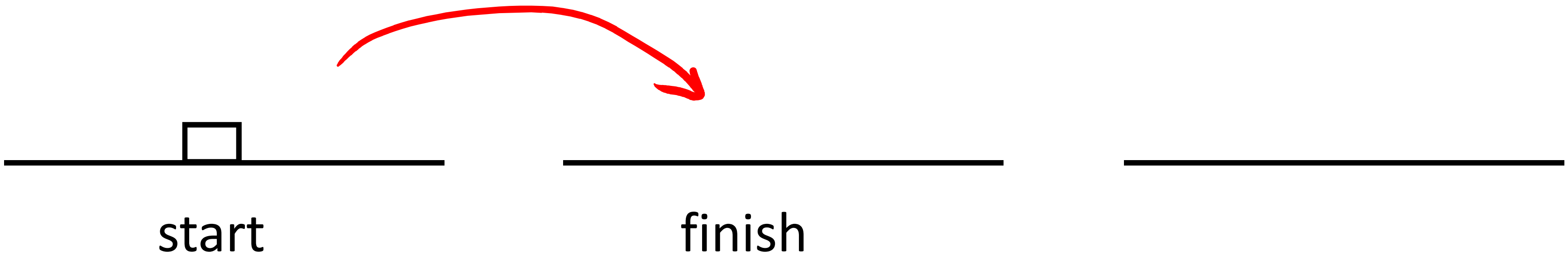
Example 5: Tower of Hanoi



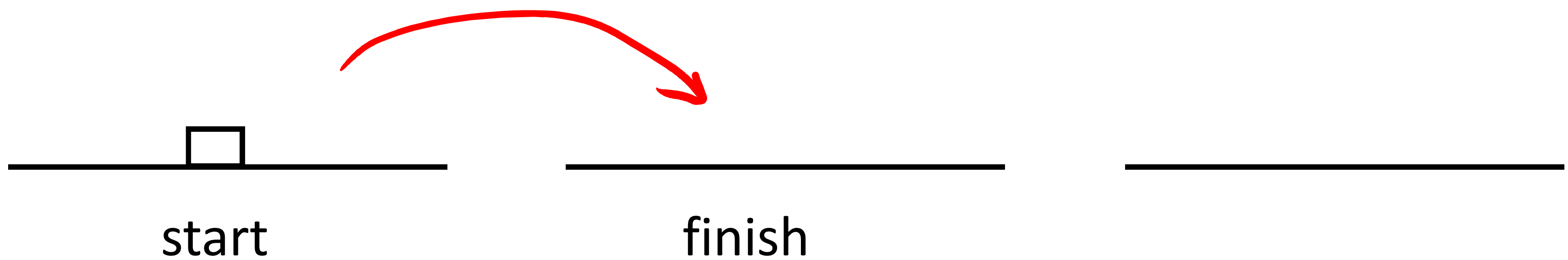
Problem: Move n disks from start tower to finish tower such that:

- move one disk at a time
- you can have a smaller disk on top of bigger disk (but you can't have a bigger disk onto a smaller disk)

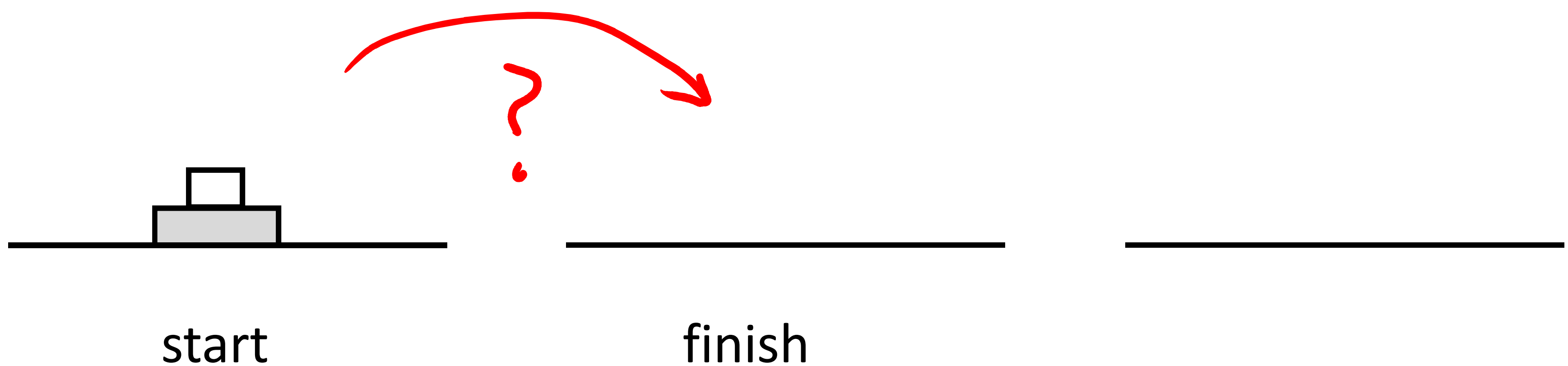
Example: n = 1



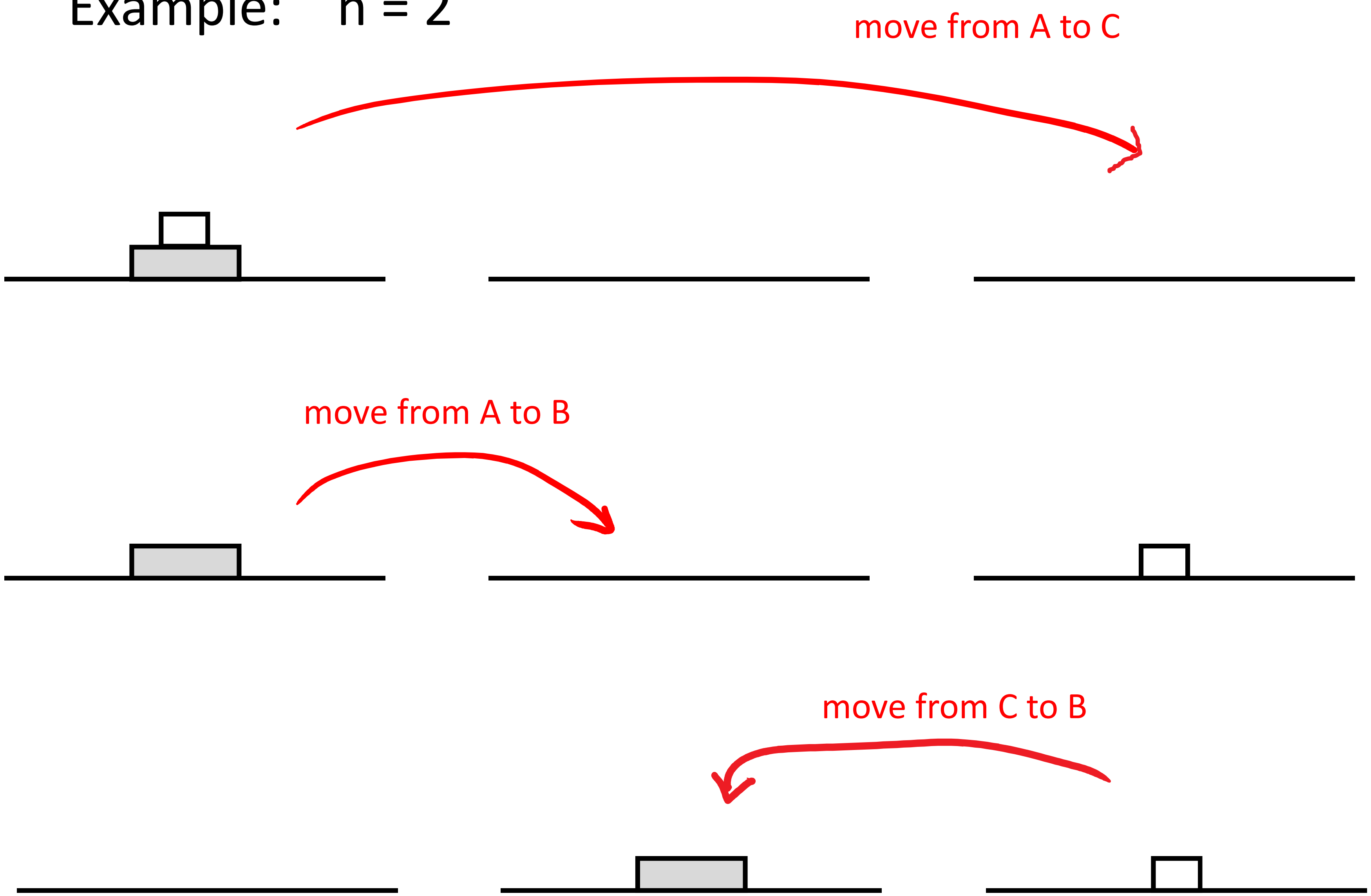
Example: n = 1



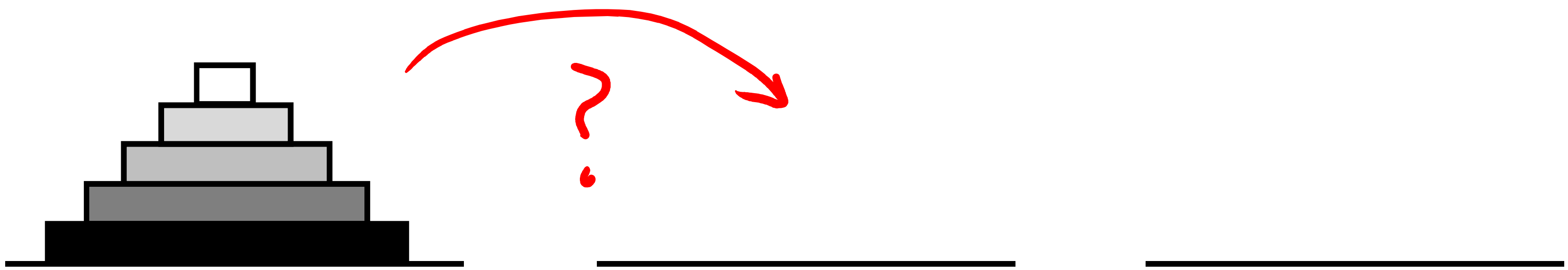
Example: n = 2



Example: n = 2



Q: How to move 5 disks from tower 1 to 2 ?

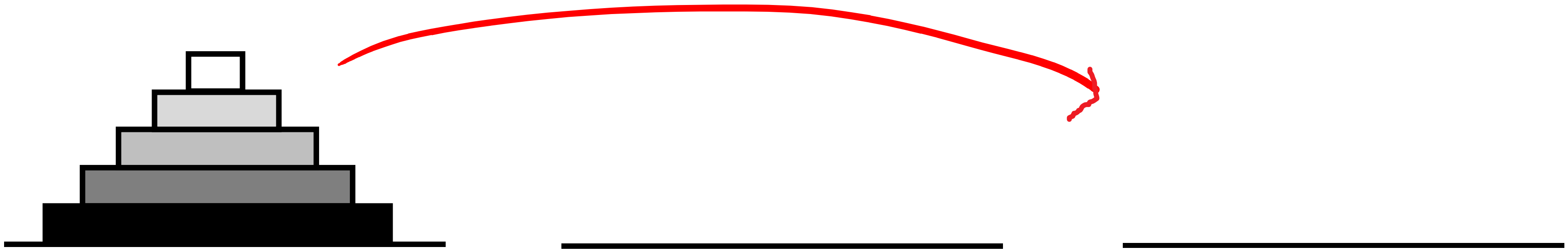


start finish

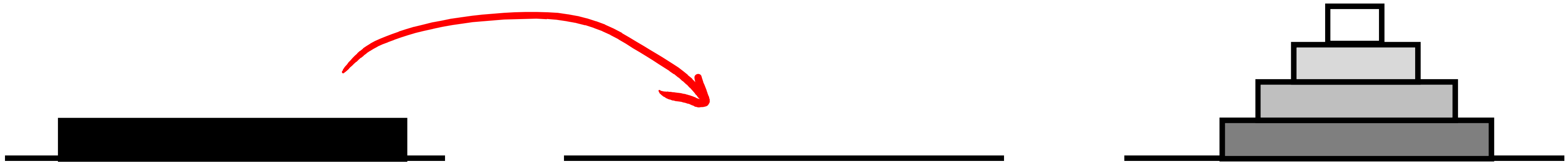
A: Think recursively.

Example: n = 5

Somehow move 4 disks from A to C



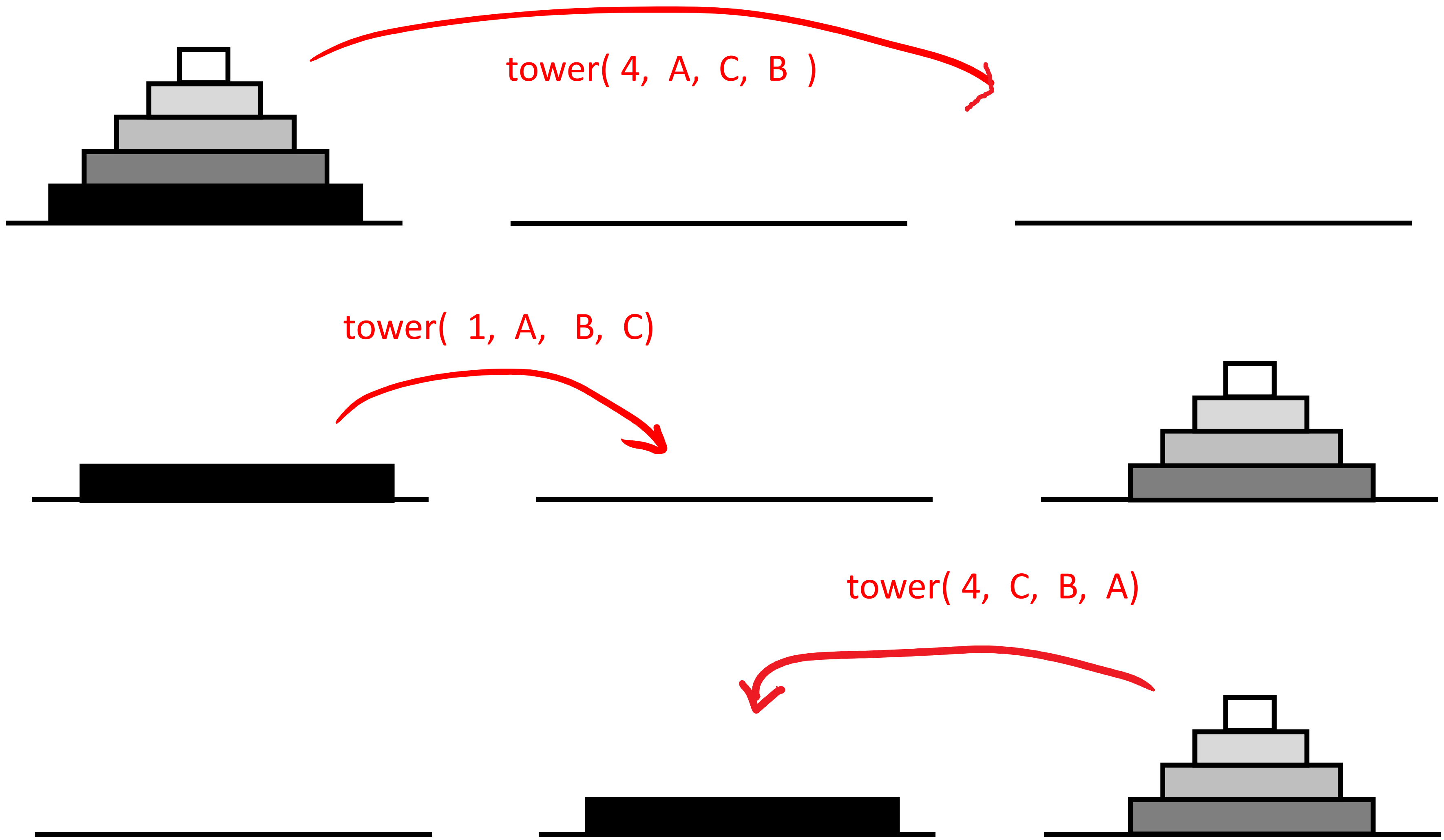
move 1 disk from A to B



Somehow move 4 disks from C to B

```
tower(n, start, finish, other){  //  e.g.  tower(5,  A, B, C)

    if  n > 0 {

            tower( n-1, start, other,  finish)

            move from start to finish

            tower( n-1, other, finish, start)
     }
}
```

Example: n = 5 tower(5, A, B, C)



Claim: the tower( ) algorithm is correct, namely it moves the blocks from start to finish without breaking the two rules (one at a time, and can't put bigger one onto smaller one).

Proof: (**sketch**)

Base case: tower(0, *, *, *) is correct.

Induction step:

for any k > 0, if tower(k, *, *, *) is correct
then tower(k + 1, *, *, *) is correct.

How many moves does tower(1, …) make ?

tower(1, start, finish, other)

tower(0, start, other, finish)

move start
to other

tower(0, other, finish, start)

Answer: 1

How many moves does tower(2, ...) make ?

tower(2, start, finish, other)

tower(1, start, other, finish)

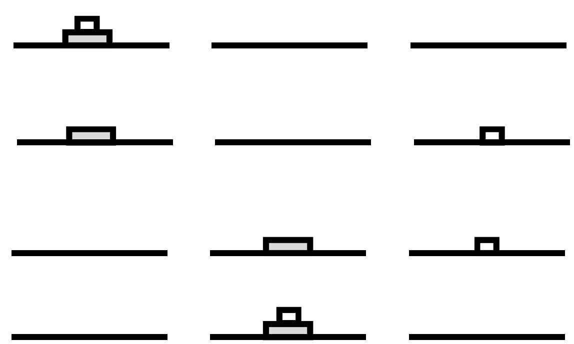
move

tower(1, other, finish, start)

move

move

Answer: 1 + 2

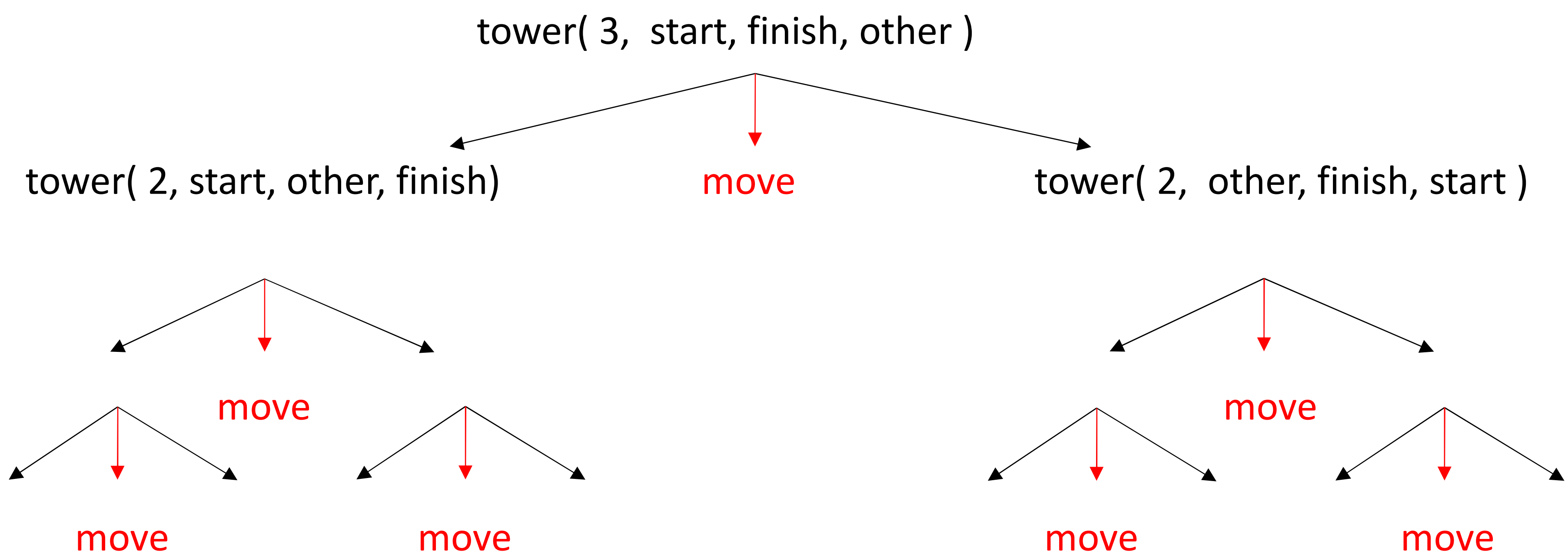


move from A to C

move from A to B

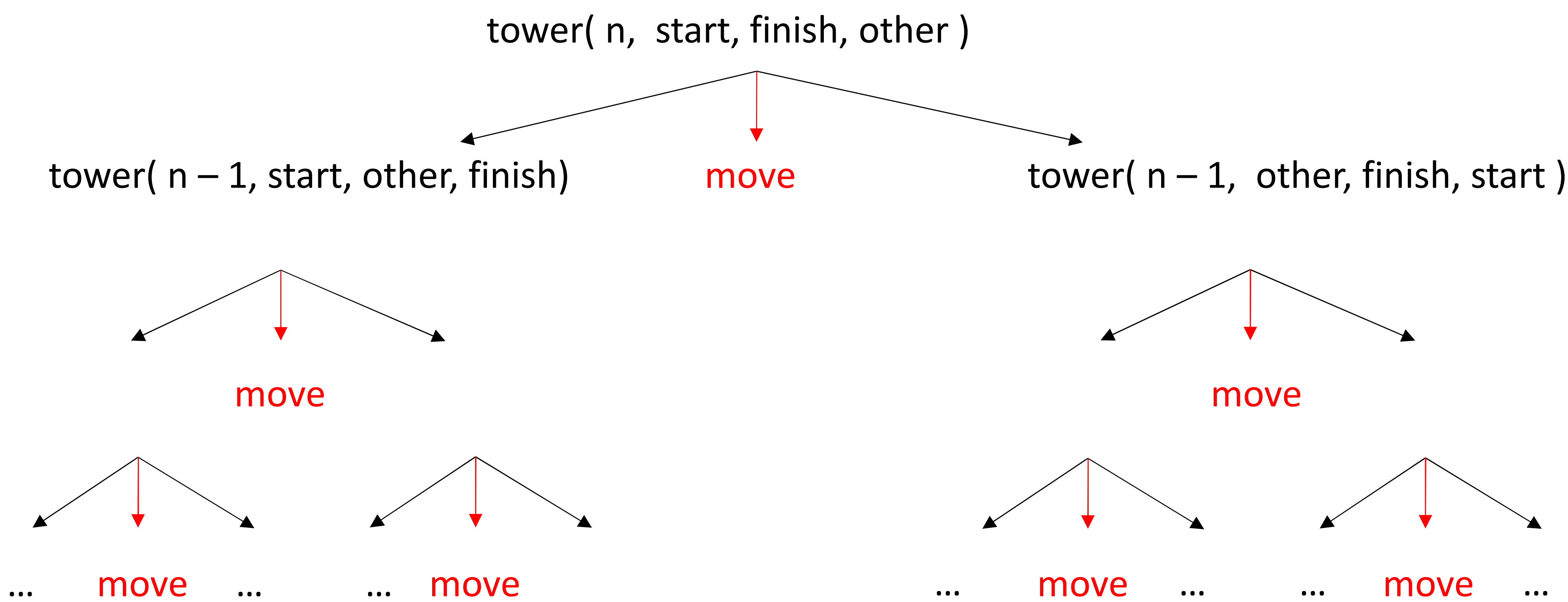
move from C to B

How many moves does tower(3, ...) make ?



Answer: $1 + 2 + 4 = 2^0 + 2^1 + 2^2$

How many moves does tower(n, ...) make ?



Answer: $1 + 2 + 4 + \ ... + \ 2^{n-1} = 2^{n} - 1$

Recall (lecture 7): "call stack"

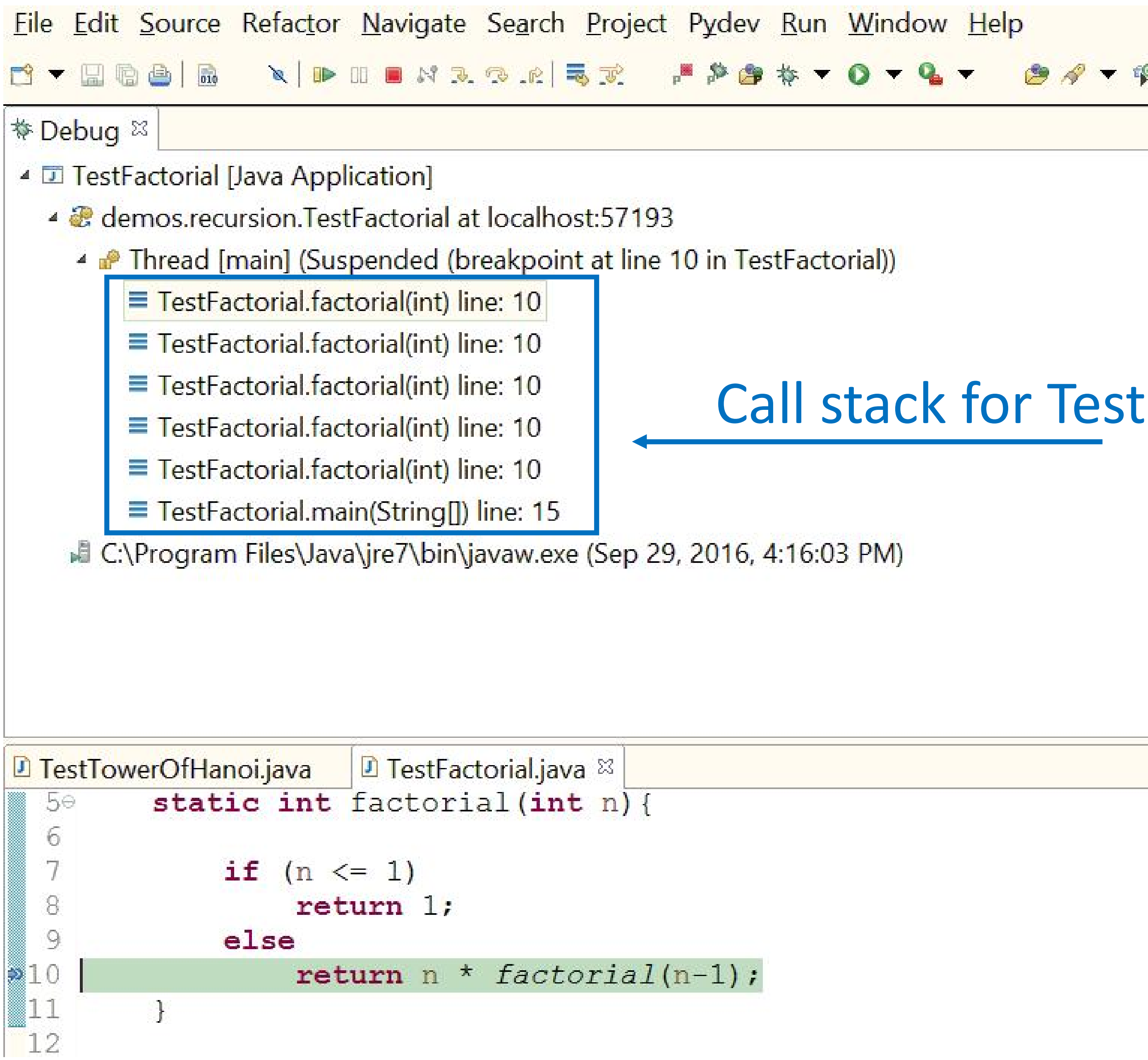
```
void   mA( ) {
            mB( );
            mC( );
        }

void  main( ){
            mA(  );
 }
```

		mB		mC		
	mA	mA	mA	mA	mA	
main	main	main	main	main	main	main

Recursive methods & Call stack

```
factorial(  n  ){
     if  n == 1
         return  1
     else
         return  factorial( n – 1 )  * n
 }
```

			factorial(1)			
		factorial(2)	factorial(2)	factorial(2)		
	factorial(3)	factorial(3)	factorial(3)	factorial(3)	factorial(3)	
main	main	main	main	main	main	main



Call stack for TestFactorial

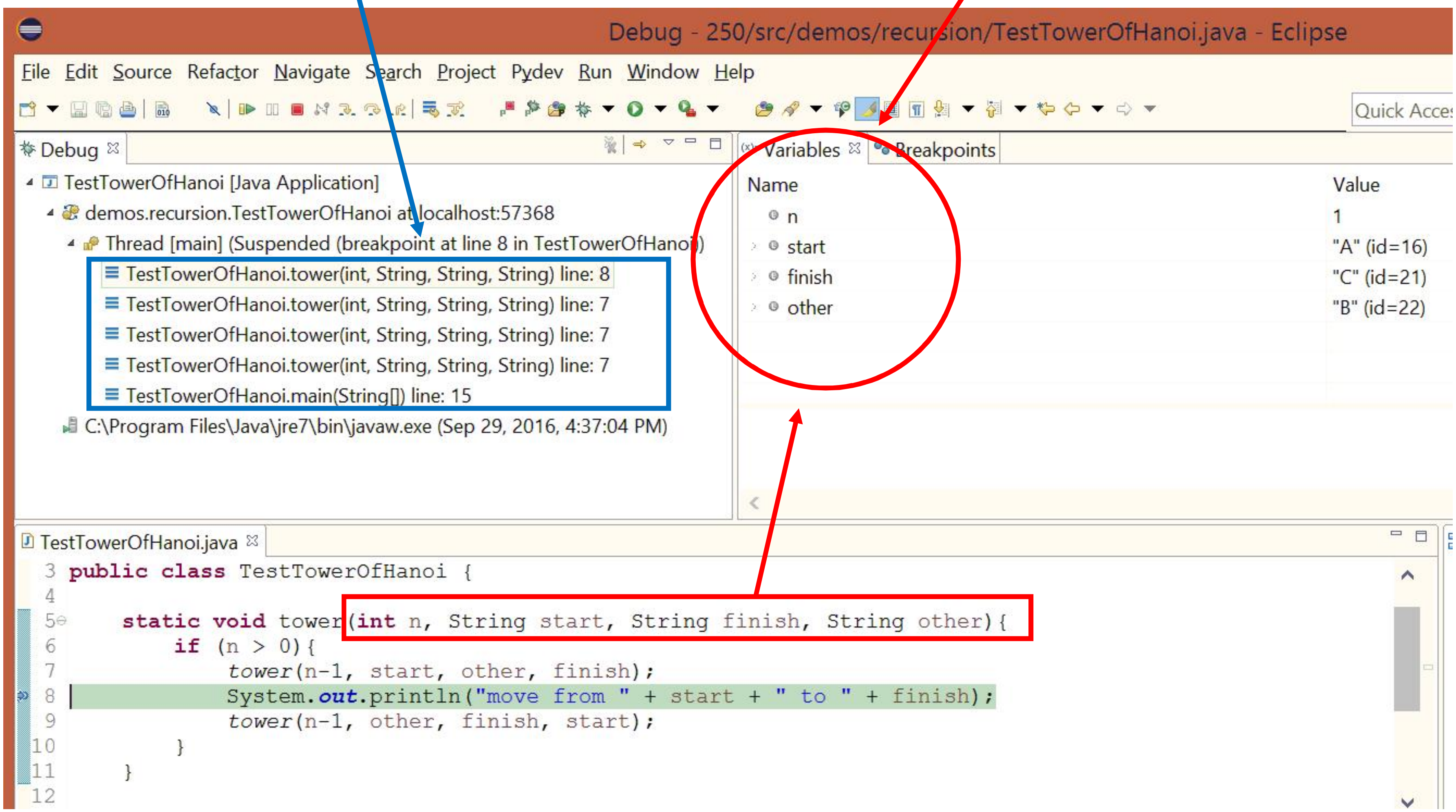
Stack frame (details in COMP 273)

The call stack consists of “frames” that contain:

- the parameters passed to the method
- local variables of a method
- information about where to return (“which line number in which method in which class?”)

Call stack for TestTowerOfHanoi

parameters in current stack frame



Call stack

```
void   mA( ) {
          mB( );
          mA( );
          mC( );
       }
```

A method can make both recursive and non-recursive calls.

There is a single call stack for all methods.