COMP 250

Lecture 9

mathematical induction

Sept. 26, 2016

$$1 + 2 + 3 + \; \ldots . + (n - 1) + n = \frac{n(n + 1)}{2}$$

How to prove such a statement ?

(By "proof", we mean a formal logical argument that convincingly shows the statement is true.)

$$1 + 2 + \ldots + (n-1) + n$$

Write sum backwards:

$$n + (n-1) + \ldots + 2 + 1$$

Adding up n pairs gives $n * (n+1)$.

Dividing by 2 gives the result.

You should be 100% convinced by this proof.

Mathematical Induction

Consider statements of the form:

"For all n >= n0, P(n)"   where P(n) is either true or false for each n, and n0 is a constant.

Mathematical induction is a *general* technique for proving such statements.

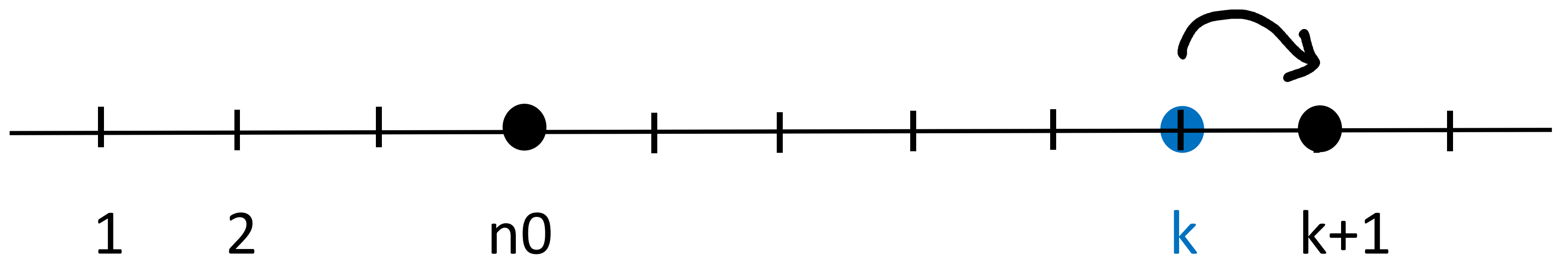
Mathematical induction requires proving two things:

<u>Base case:</u>

"P(n0) is true."

<u>Induction step:</u>

"For any k >= n0, if P(k) is true, then P(k+1) is also true."



Base case:

"P(n0) is true."

Induction step:

"For any k >= n0, if P(k) is true,   then P(k+1) is also true."

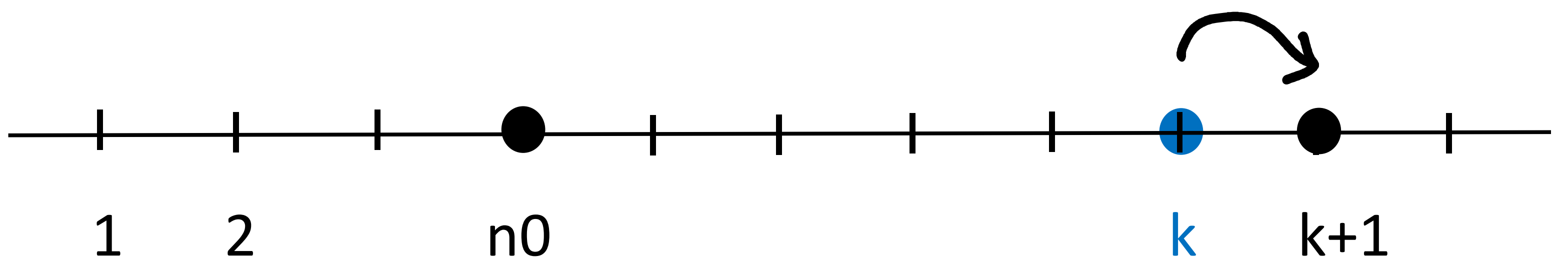
"P(k) is true" is called the "induction hypothesis".

Base case:

P(n0) is true.

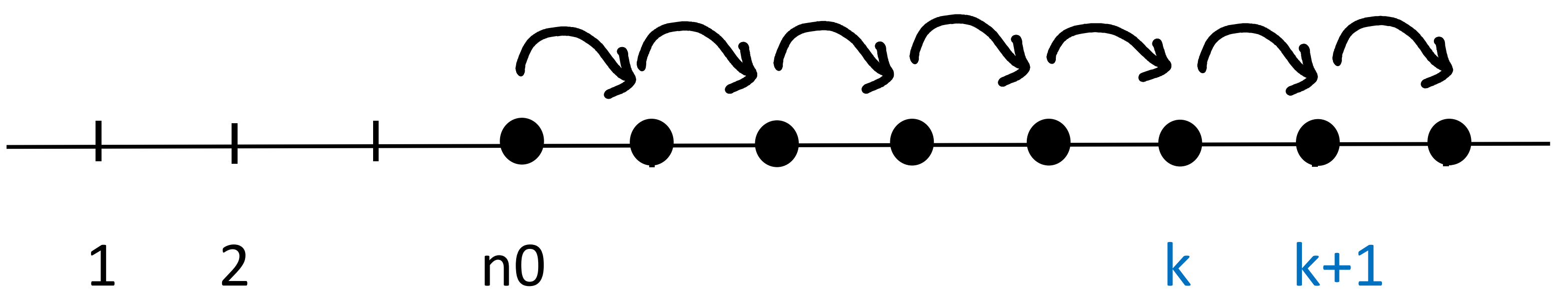
Induction step:

For any k >= n0, if P(k) then P(k+1).



Thus,

For any n >= n0, P(n) is true.



Statement: For all $n \geq 1$,

$$1 + 2 + 3 + \ldots . + (n - 1) + n = \frac{n(n + 1)}{2}$$

Proof (base case, n=1):

$$1 = \frac{1(1+1)}{2} \qquad \text{(true)}$$

Proof of Induction Step:

$$(1 + 2 + 3 + \ldots. + k) + k + 1$$

$$= \frac{k(k+1)}{2} + k + 1$$

Why? Because of the induction hypothesis

$$(1 + 2 + 3 + \ldots. + k) = \frac{k(k+1)}{2}$$

Proof of Induction Step:

$$(1 + 2 + 3 + \ldots. + k) + k + 1$$

by induction hypothesis

$$= \frac{k(k+1)}{2} + k + 1$$

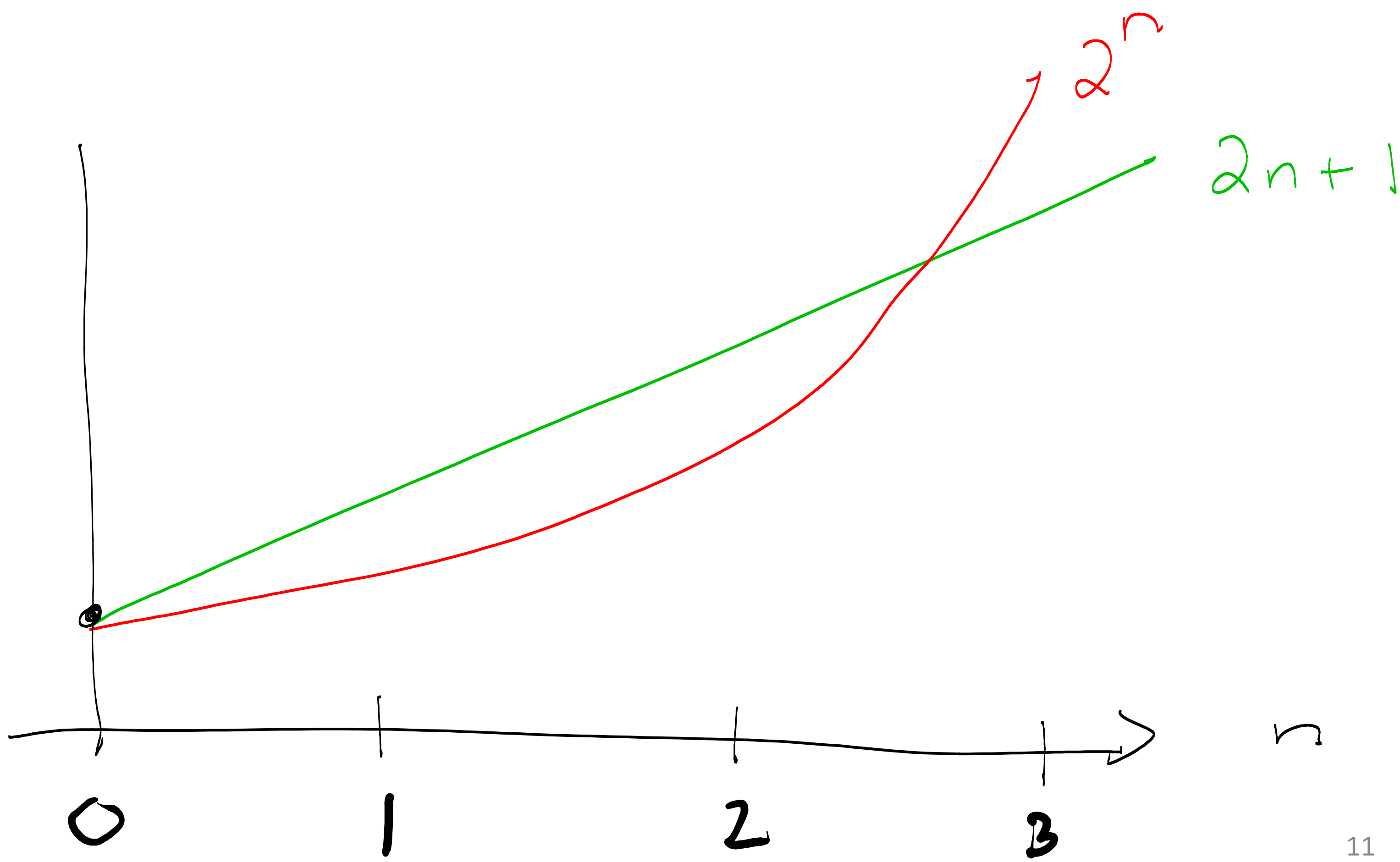
$$= \left(\frac{k}{2} + 1\right)(k + 1)$$

$$= \frac{1}{2}(k + 2)(k + 1)$$

Thus, $P(k)$ *is true* *implies* $P(k + 1)$ *is true*.

Example 2

Statement: For all $n \geq 3$, $2n + 1 < 2^n$.



Statement: For all $n \geq 3$, $2n + 1 < 2^n$.

Note: P(n) is false for n= 1, 2.

Proof (base case, n =3):

$$2*3 + 1 < 8$$ (true)

Proof of Induction Step:

We want to show that P(k) implies P(k+1).

$$2(k+1)+1 = 2k+2+1$$

by induction hypothesis

$$< 2^k + 2$$

$$< 2^k + 2^k, \quad \text{for } k \geq 2$$

$$= 2^{k+1}$$

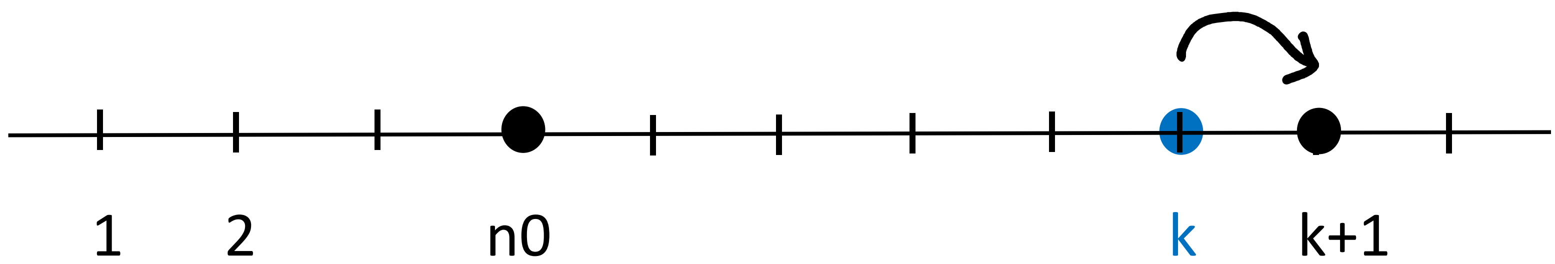
This condition is stronger than we need.

Base case:

P(n0) is true.

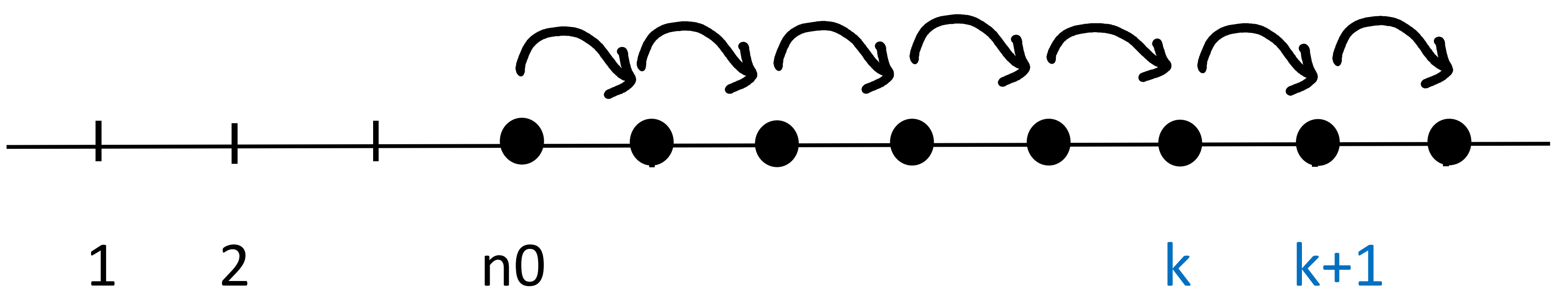
Induction step:

For any k >= n0, if P(k) then P(k+1).



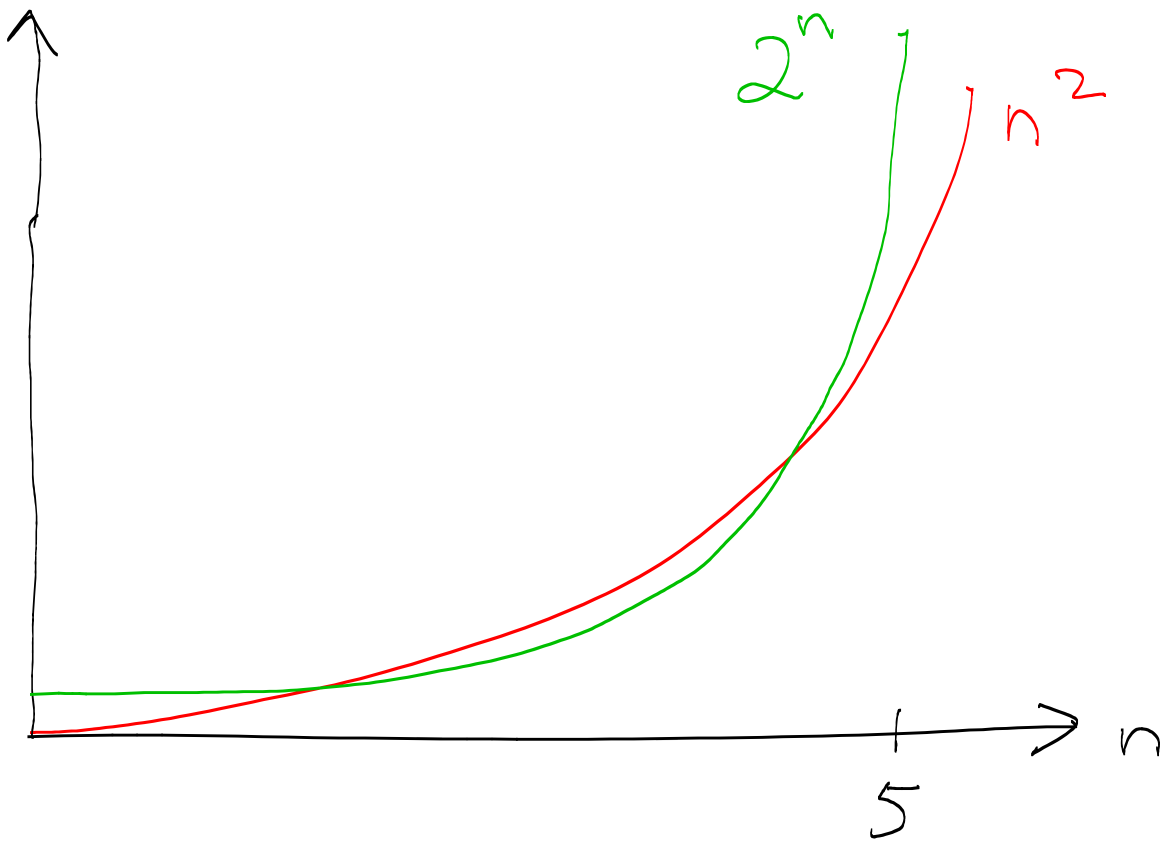
Thus,

For any n >= n0, P(n) is true.



Example 3

Statement: For all $n \geq 5$, $n^2 < 2^n$.



Statement: For all $n \geq 5$, $n^2 < 2^n$.

Base case (n = 5):

25 < 32

Induction step:

$$(k+1)^2 = k^2 + 2k + 1$$

by induction hypothesis

$$< 2^k + 2k + 1$$

by Example 2

$$< 2^k + 2^k$$

$$= 2^{k+1}$$

Example 4 : Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ….

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

Statement: For all $n \geq 0$, $F(n) < 2^n$

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

Base case(s):

$n = 0$: $0 < 2^0$ is true.

$n = 1$: $1 < 2^1$ is true.

$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$, for $n \geq 0$.

Induction step:

$$F(k+1) = F(k) + F(k-1)$$

by induction hypothesis

$$< 2^k + 2^{k-1}$$

$$< 2^k + 2^k$$

$$= 2^{k+1}$$

How to use mathematical induction to prove an algorithm is correct ? e.g. insertion sort

```
insertion_sort( list ){
  for k = 1 to n - 1 {
    elementK =  list[k]
    i = k
    while (i > 0) and (list[i - 1] > elementK ){
      list[i] =  list[i - 1]
      i  = i -1
    }
    list[i] = elementK
  }
}
```

`P(k) =`

"At the start of pass `k` through the `for` loop, `list[0,... k-1]` contains the same elements as the original list, and these `k` elements are now sorted."

Q: What is the base case?

Q: What is the induction step?

Base case:

`P(1) =`

"At the start of pass `1` through the `for` loop, `list[0]` contains the same element as the original list, and these elements are now sorted."

Induction step

if

"At the start of pass `k` through the `for` loop, `list[0,... k-1]` contains the same `k` elements as the original list, and these elements are now sorted."

then

"At the start of pass `k+1` through the `for` loop
(i.e. at the *end* of pass `k` through the `for` loop)
`list[0,... k]` contains the same elements as the original list, and these elements are now sorted."

You may have already been convinced that this algorithm was correct.

I would argue, however, that your proof (or intuition) uses mathematical induction.

If not, then what *does* it use?