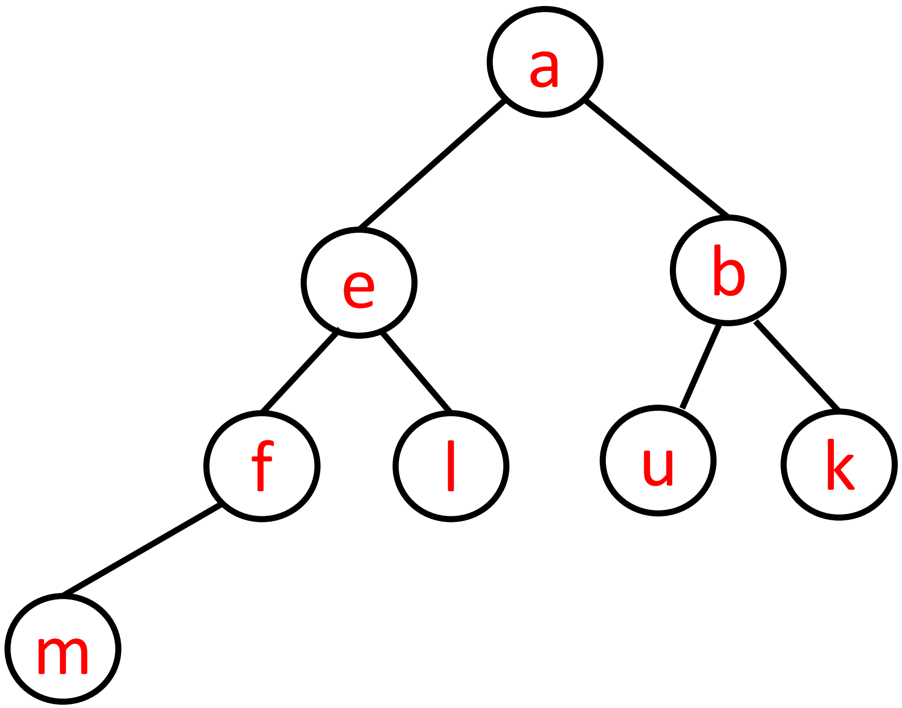
COMP 250

Lecture 24

heaps 3

Nov. 4, 2016

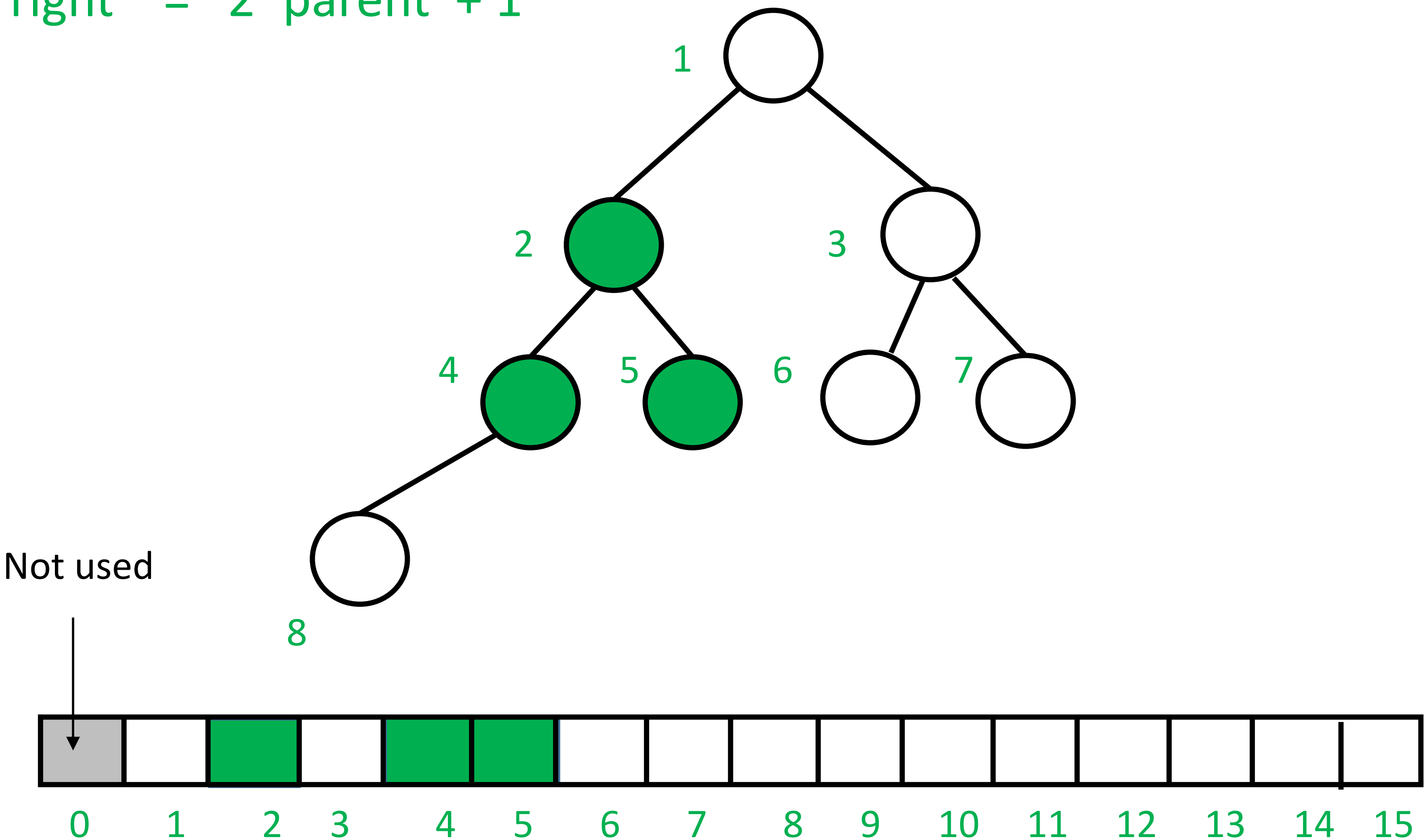
RECALL: min Heap (definition)



Complete binary tree with (unique) comparable elements, such that each node's element is less than its children's element(s).

Heap index relations

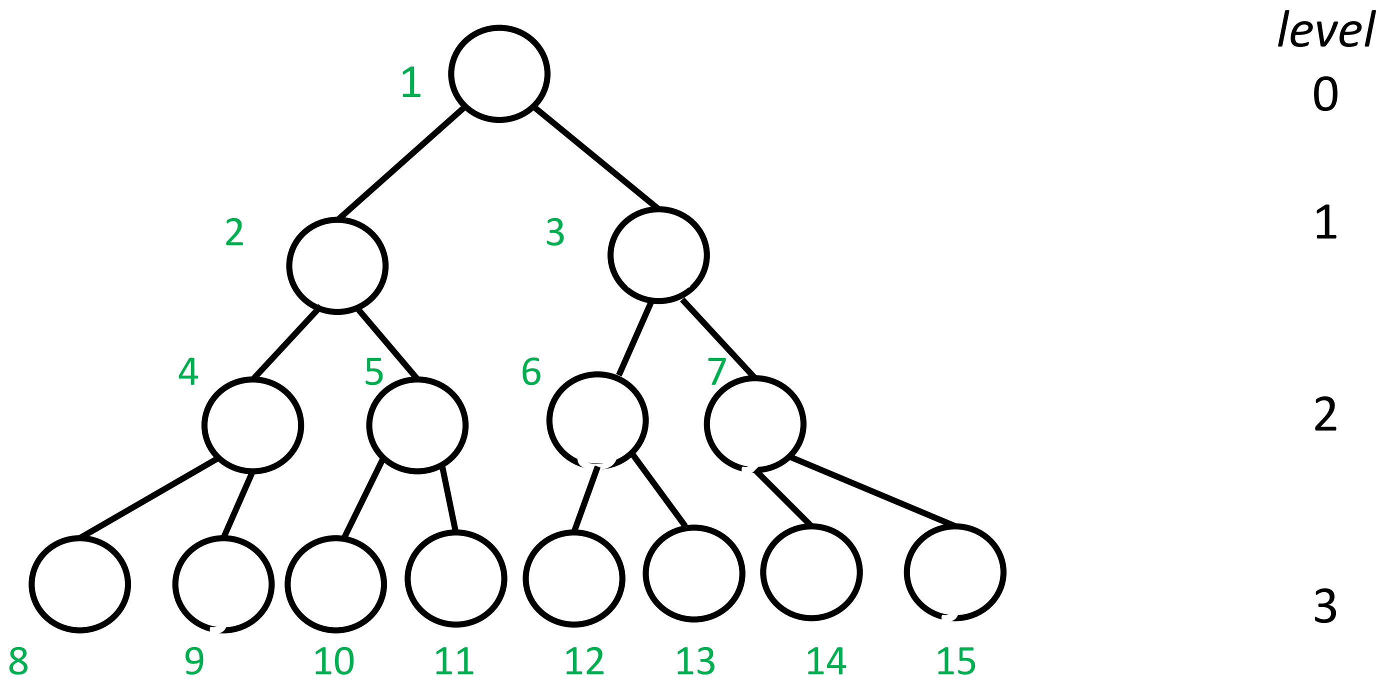
parent = child / 2
left = 2*parent
right = 2*parent + 1



RECALL: How to build a heap

Given a list with size elements:

```
buildHeap(list){
    create new heap array              // length > list.size
    for (k = 0; k < list.size; k++)
        add( list[k] )                 // add to heap[ ]
}
```



$$2^{level} \leq \quad i \quad < 2^{level+1}$$

$$level \leq \log_2 i < level + 1$$

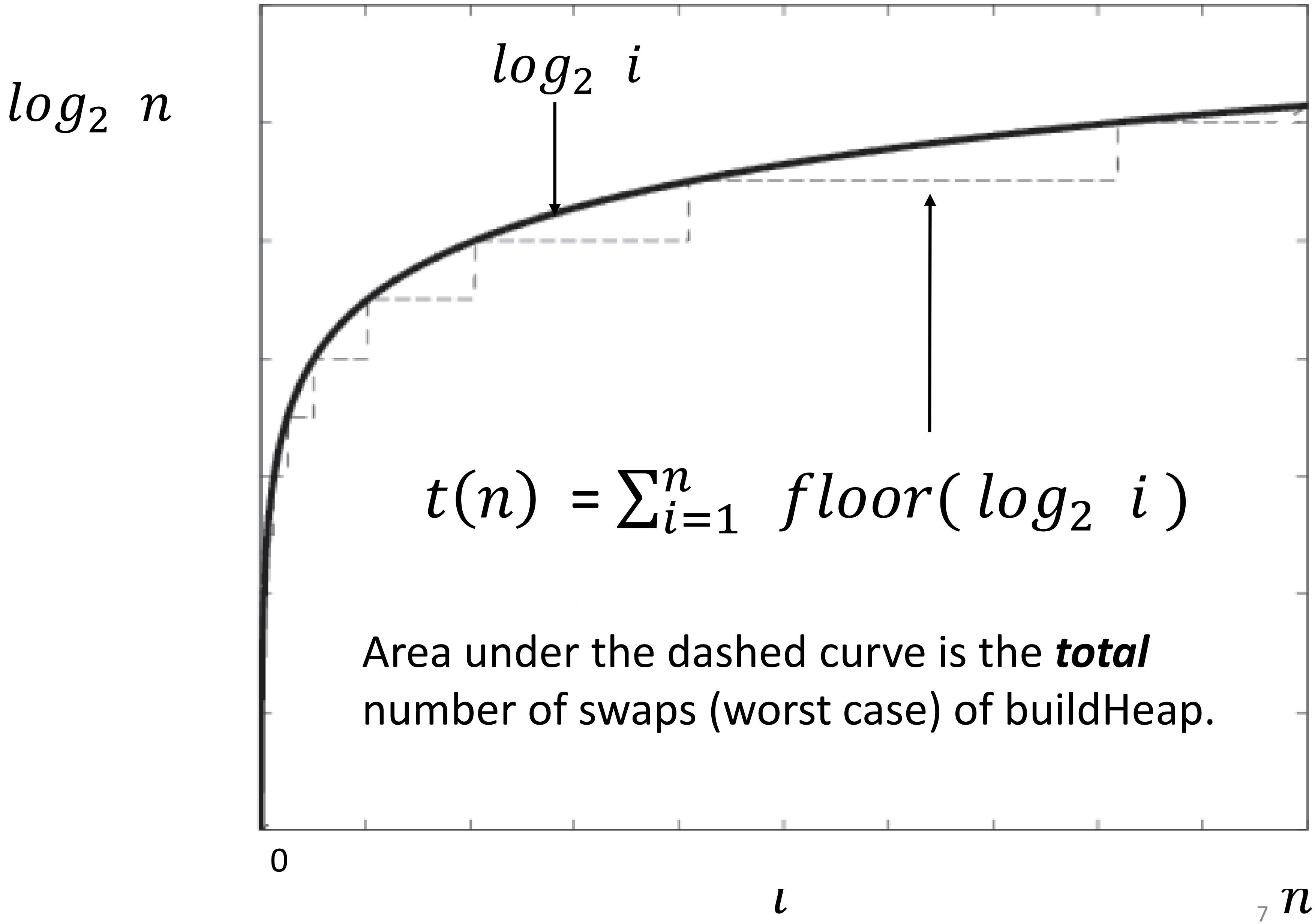
Thus, $level = floor(\log_2 i\ )$

Worse case of buildHeap

$$t(n) = \sum_{i=1}^{n} number\ of\ swaps\ for\ node\ i$$

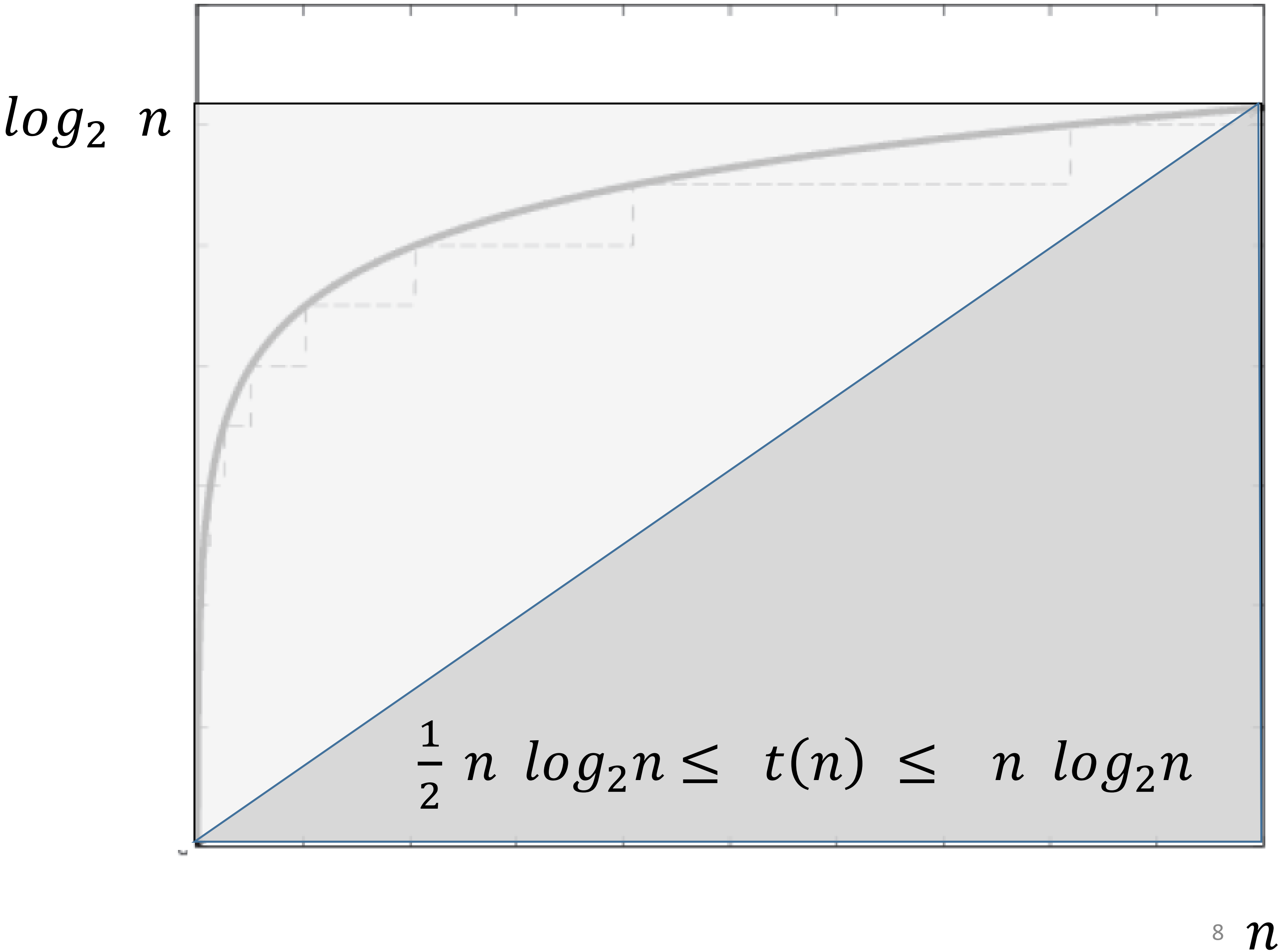
$$= \sum_{i=1}^{n} level\ of\ node\ i$$

$$= \sum_{i=1}^{n} floor(\ log_2\ i\ )$$



$$t(n) = \sum_{i=1}^{n} floor(\,log_2\ i\,)$$

Area under the dashed curve is the ***total*** number of swaps (worst case) of buildHeap.


$log_2\ n$
$\frac{1}{2}\, n\ log_2 n \leq\ t(n)\ \leq\ n\ log_2 n$
n

Thus, worst case: buildHeap is $O(n\, log_2\ n)$

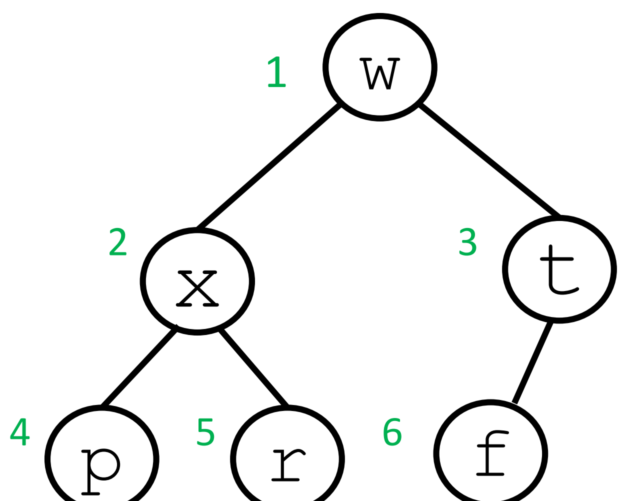
This is a tight $O(\ \ )$ bound.

Today, I will show you a $O(n)$ algorithm.

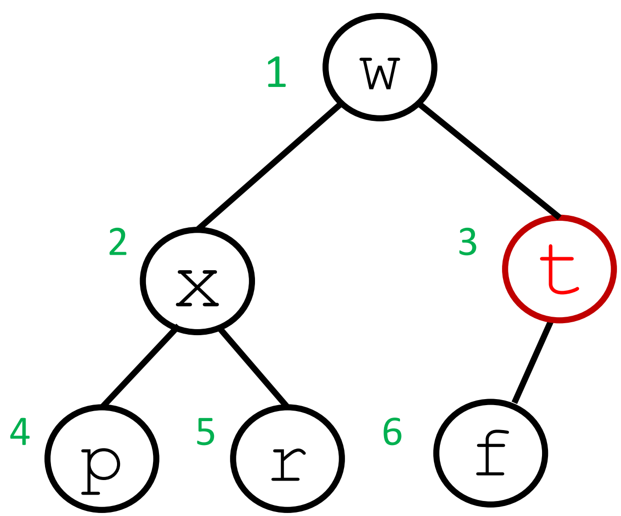
Given a list with size elements:

```
buildHeapFast(list){
    copy list into a heap array
    for (k = size/2;   k >= 1;    k--)
        downHeap( k, size )
}
```

1	2	3	4	5	6
w	x	t	p	r	f

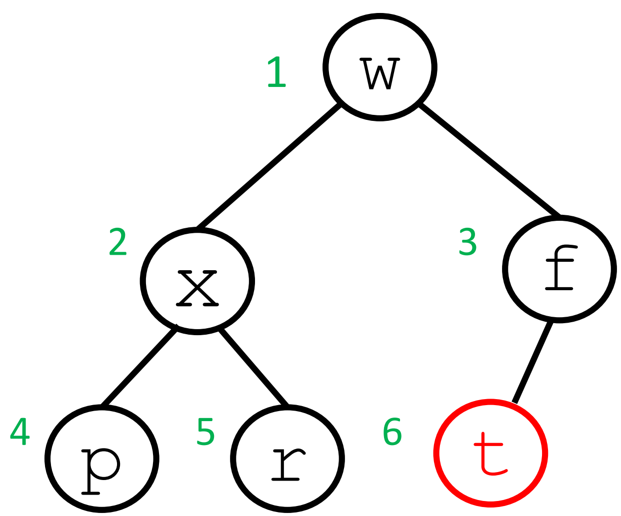


1	2	3	4	5	6
w	x	t	p	r	f



downHeap(3, 6)

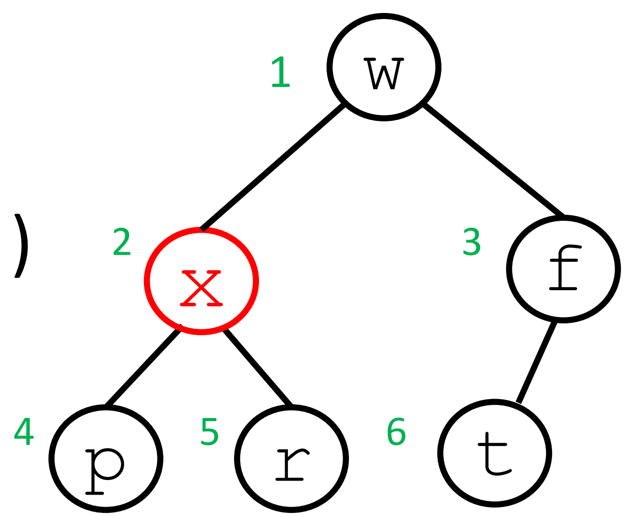
1	2	3	4	5	6
w	x	f	p	r	t



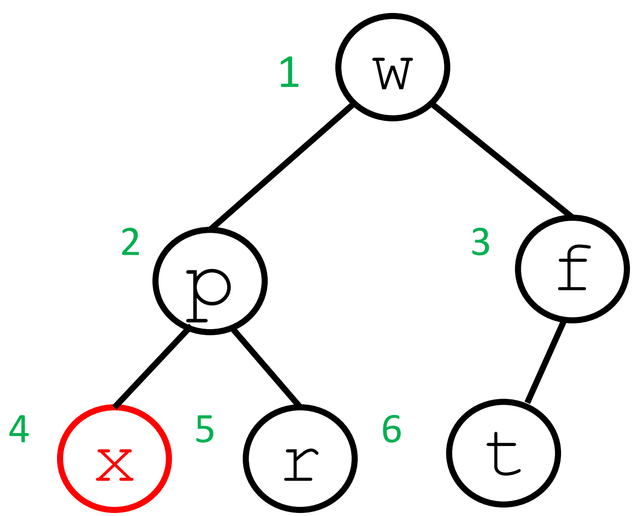
downHeap(3, 6)

1	2	3	4	5	6
w	x	f	p	r	t

downHeap(2, 6)

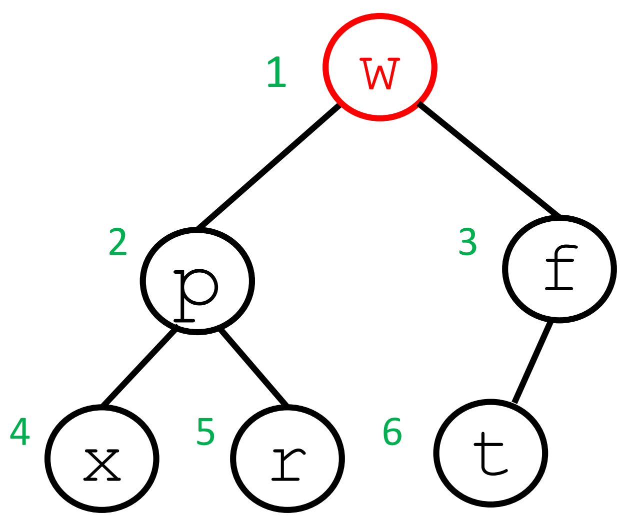


1	2	3	4	5	6
w	p	f	x	r	t

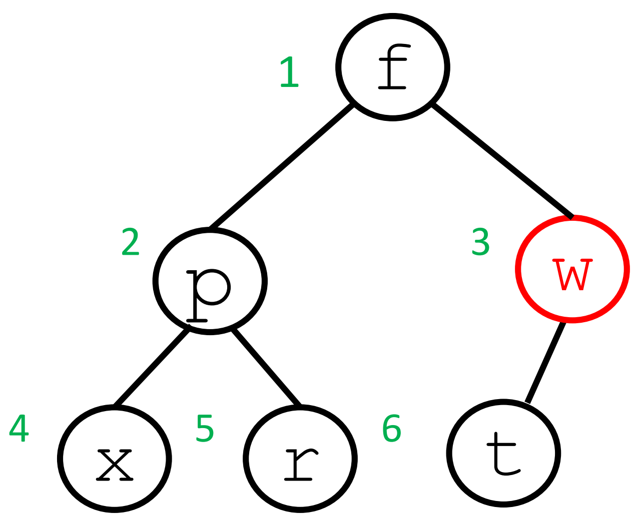


1	2	3	4	5	6
w	p	f	x	r	t

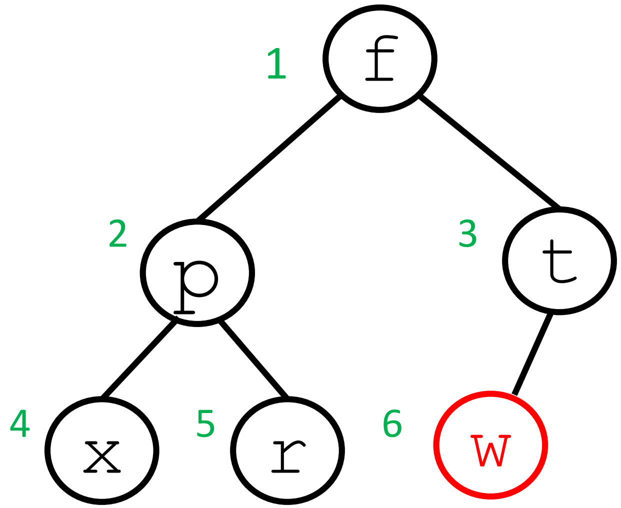
downHeap(1, 6)



1	2	3	4	5	6
f	p	w	x	r	t



1	2	3	4	5	6
f	p	t	x	r	w



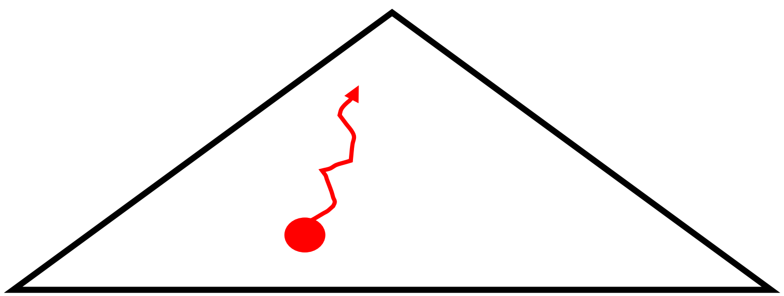
```
buildHeapFast(list){
    copy list into a heap array
    for (k = size/2;   k >= 1;    k--)
          downHeap( k, size )
}
```

Claim: this algorithm is O(n).

Intuition for why this algorithm is so fast?

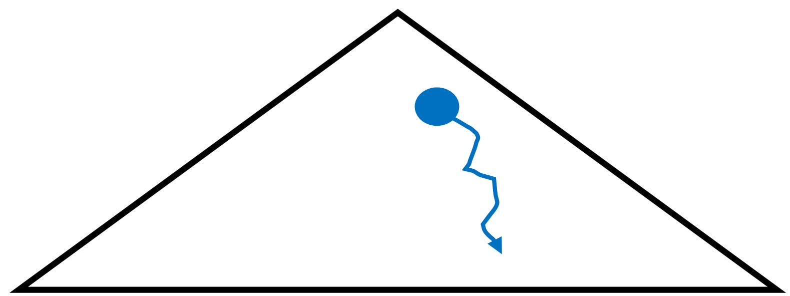
buildheap algorithms

last lecture



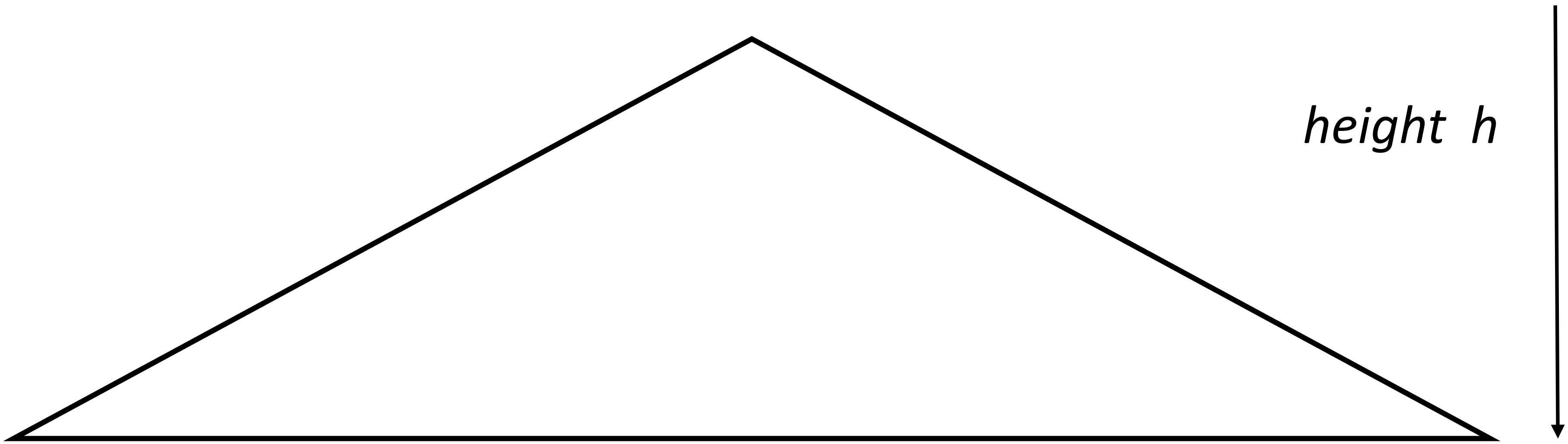
upHeap based

today

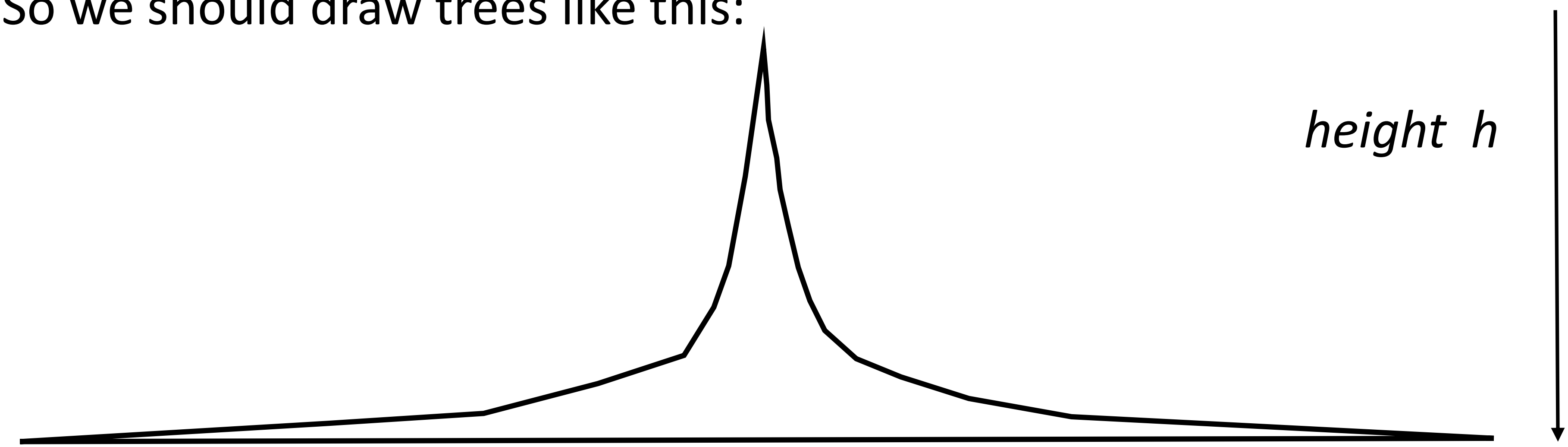


downHeap based

We tends to draw binary trees like this:



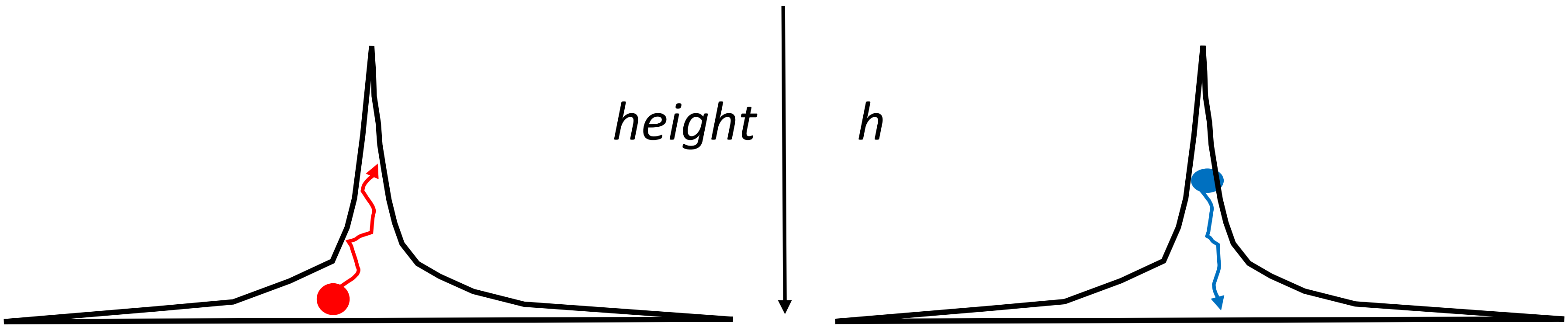
But the number of nodes doubles at each level.
So we should draw trees like this:



buildheap algorithms

last lecture

today



Most nodes swap *h* times in worst case.

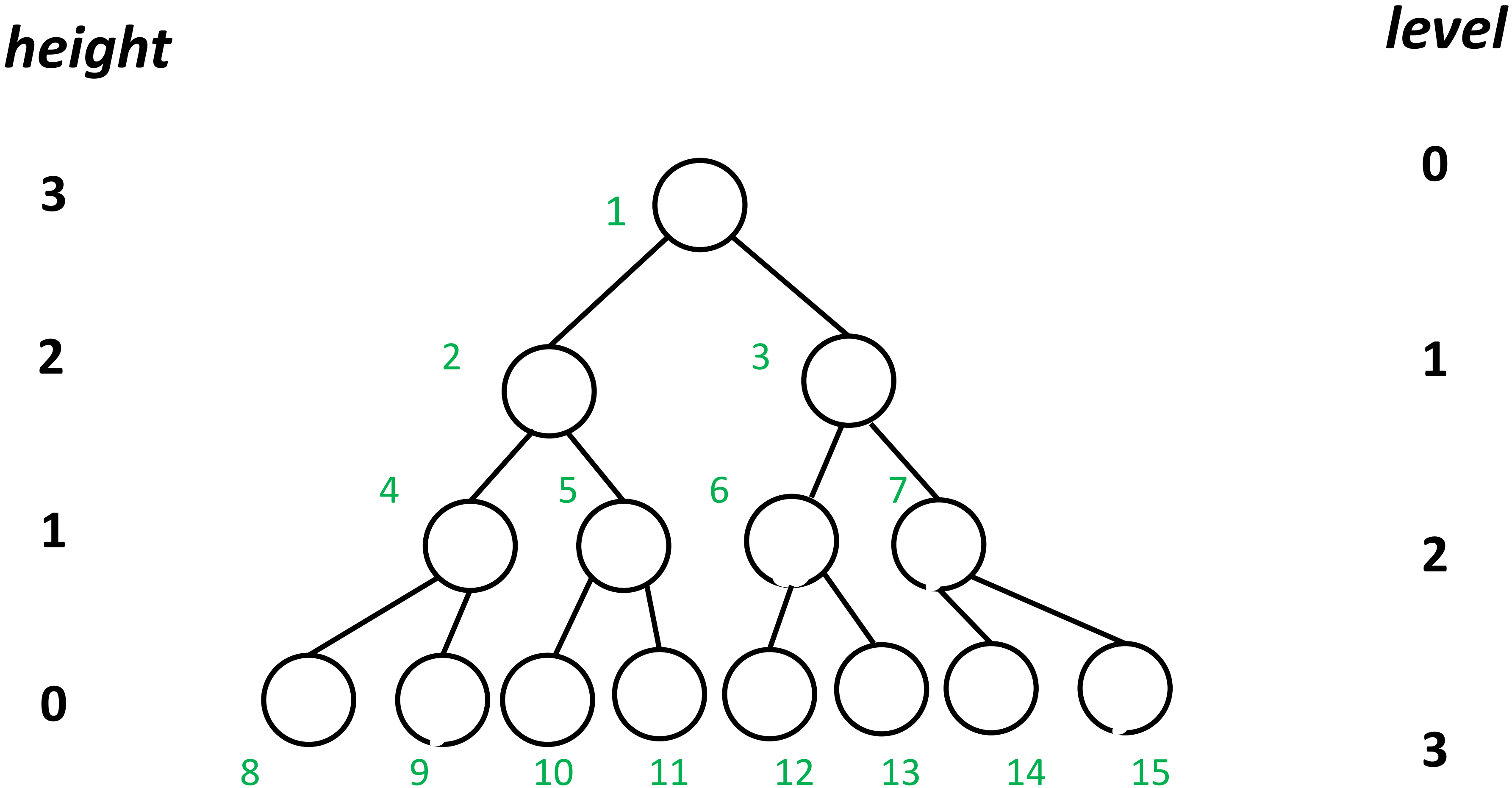
Few nodes swap h times in worst case.

How to show buildHeapFast is $O(n)$?

Worst case number of swaps needed to add node i is the the height of that node.

(Recall the height of a node is the length of the longest path from that node to a leaf.)

$$t(n) = \sum_{i=1}^{n} \; height\ of\ node\ i$$



height
3
2
1
0
level
0
1
2
3
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

Worse case of buildHeap**Fast** ?

How many elements at $level\ \ l$? $(l\ \in\ 0, \ldots, \ h)$

What is the height of each $level\ \ l$ node?

Worse case of buildHeap**Fast** ?

$level\ \ l$ has 2^l elements, $l \in 0, \ldots, h$

$level\ \ l$ nodes have height $h - l$.

$t(n) = \sum_{i=1}^{n} \quad height\ of\ node\ i$

$= \quad ?$

Worse case of buildHeap**Fast** ?

$level\ \ l$ has 2^l elements, $l \ \in \ 0, \ldots, \ h$

$level\ \ l$ nodes have height $h - l$.

$$t(n) = \sum_{i=1}^{n} \ height\ of\ node\ i$$

$$= \sum_{l=0}^{h} \ (h\ - l)\ 2^l$$

$$
\begin{aligned}
t_{worstcase}(h) &= \sum_{l=0}^{h} (h-l)\, 2^l \\
&= h \sum_{l=0}^{h} 2^l - \sum_{l=0}^{h} l\, 2^l
\end{aligned}
$$

Easy

(number of nodes)

Difficult

(sum of node depths)

I have removed the next two slides which derived a closed for expression for the second summation (the difficult one). Please see lecture notes for a slightly different derivation, which is simpler.

$$
\begin{aligned}
t_{worstcase}(h) &= \sum_{l=0}^{h} (h-l)\, 2^l \\
&= h \sum_{l=0}^{h} 2^l - \sum_{l=0}^{h} l\, 2^l \\
&= h(2^{h+1}-1) - (h-1)2^{h+1} - 2 \\
&= 2^{h+1} - h - 2
\end{aligned}
$$

(See lecture notes)

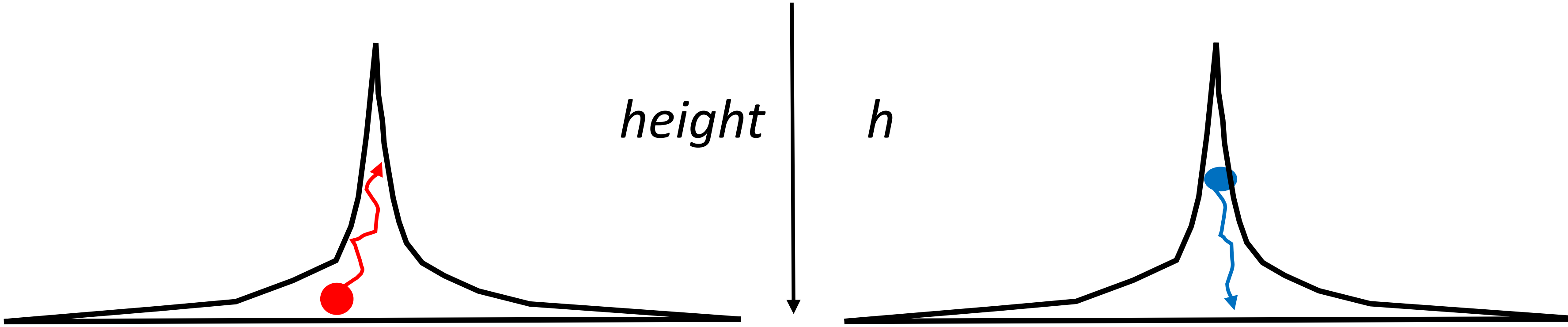
In terms of n, we have

$$t(n) = n - \log(n+1)$$

Summary: buildheap algorithms

last lecture

today



$O(n\ \ log_2\ n)$

$O(n)$