

Lecture 13

Hash Dictionaries

15-122: Principles of Imperative Computation (Fall 2020)
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In this lecture, we will discuss the data structure of hash tables further and use hash tables to implement a very basic interface of dictionaries. With this lecture, we will also begin to discuss a new separation of concerns. Previously, we have talked a great deal about the distinction between a library's interface (which the client can rely on) and a library's implementation (which should be able to change without affecting a correctly-designed client).

The interface defines not only the *types*, but also the available *operations* on them and the pre- and post-conditions for these operations. For general data structures it is also useful to note the asymptotic complexity of the operations so that potential clients can decide if the interface serves their purpose.

One wrinkle we have not yet encountered is that, in order for a library to provide its services, it may in turn require some operations supplied by the client. Hash tables provide an excellent example for this complication, so we will discuss the interface to hash tables in details before giving the hash table implementation.

For the purposes of this lecture we call the data structures and the operations on them provided by an implementation the *library* and code that uses the library the *client*.

Relating to our learning goals, we have

Computational Thinking: We discuss the separation of client interfaces and client implementations.

Algorithms and Data Structures: We discuss algorithms for hashing strings.

Programming: We revisit the **char** data type and use it to consider string hashing. We use this to implement a data structure based on a hash table.

1 Generic Data Structures — I

So far, all the data structures that we've considered, have always had particular type information that seemed irrelevant. In the implementation of queues, why is it important that we have a queue of *strings* in particular?

```
// typedef _____* queue_t;
bool queue_empty(queue_t Q)                /* 0(1) */
    /*@requires Q != NULL; @*/;
queue_t queue_new()                        /* 0(1) */
    /*@ensures \result != NULL; @*/;
void enq(queue_t Q, string x)              /* 0(1) */
    /*@requires Q != NULL; @*/;
string deq(queue_t S)                     /* 0(1) */
    /*@requires Q != NULL && !queue_empty(S); @*/ ;
```

It's both wasteful and a potential source of errors to have to rewrite our code if we want our program to use integers (or chars, or pointers to structs, or arrays of strings, ...) instead of strings. A way we deal with this is by creating a type, *elem*, that is used *by* the library but not defined *in* the library:

```
/** Client interface */
// typedef _____ elem;    // Supplied by client

/** Library interface */
// typedef _____* queue_t;
bool queue_empty(queue_t Q)                /* 0(1) */
    /*@requires Q != NULL; @*/;
queue_t queue_new()                        /* 0(1) */
    /*@ensures \result != NULL; @*/;
void enq(queue_t Q, elem x)                /* 0(1) */
    /*@requires Q != NULL; @*/;
elem deq(queue_t Q)                       /* 0(1) */
    /*@requires Q != NULL && !queue_empty(S); @*/ ;
```

The underscores in the library interface, before *queue_t*, mean that the client doesn't know how the abstract type *queue_t* is implemented beyond knowing that it is a pointer. The library is therefore free to change this implementation without breaking any (interface-respecting) client code. The underscores in the *client* interface mean that the *library* doesn't know how the abstract type *elem* is implemented, which means that the client is free to

change this implementation without breaking the library. The library's implementation just refers to the `elem` type, which it expects the client to have already defined, whenever it needs to refer to client data. Therefore, the client code must be split into (at least) two files: one, call it `queue-client.c0`, which defines `elem`, for example as

```
typedef string elem;
```

if we are interested in queues of strings, and the rest of the program, for example `main.c0`. Now, if the file containing the implementation of the queue library is called `queue.c0`, the overall program shall be compiled as

```
% cc0 queue-client.c0 queue.c0 main.c0
```

in order to respect the dependencies.

This approach is still not perfect, because any given program only supports a single type of queue element. We'll start working on that problem in the next lecture.

2 Generic Hash Dictionaries

When we implement the dictionary interface with a hash table, we'll call it a *hash dictionary* or `hdict`. Our hash dictionary implementation will be *generic*; it will work regardless of the type of entries to be stored in the table as well as of the type of their keys.

We need to think carefully about which types and functions are provided by the client of the hash dictionary, and which are provided by the library itself. Clearly, the library should determine the type of hash dictionaries:

```
/* library side types */
// typedef _____* hdict_t;
```

That is really the only type provided by the implementation. In addition, the library interface is supposed to provide a few functions:

```
/* library side functions */
hdict_t hdict_new(int capacity)          /* 0(1) */
/*@requires capacity > 0; @*/
/*@ensures \result != NULL; @*/ ;

entry hdict_lookup(hdict_t H, key k)     /* 0(1) avg. */
/*@requires H != NULL; @*/ ;
```

```
void hdict_insert(hdict_t H, entry x)    /* 0(1) avg. */
/*@requires H != NULL && x != NULL; @*/ ;
```

The function `hdict_new(int capacity)` takes the initial capacity of the hash table as an argument (which must be strictly positive) and returns a new hash dictionary without any entry in it.

The function `hdict_lookup(hdict_t H, key k)` answers the question of whether an entry with key `k` has been added to the dictionary already and, if the answer is positive, it returns this entry. We will see momentarily how to express these outcomes. This will allow us to add postconditions that the client can use to reason about his/her code.

The last function, `hdict_insert(hdict_t H, entry x)`, adds entry `x` to the dictionary. It too will be extended with a postcondition.

From these decisions we can see that the *client* must provide the type of entries and the type of their keys. Only the client can know what these might be in any particular use of the library. In this implementation, we don't need to know anything about the type key of keys. We will however require entries to have pointer type and be non-NULL. Doing so allows `hdict_lookup` to return NULL to signal that the dictionary does not contain any entry with the requested key. Thus, the client interface specifies that the following types be provided:

```
/* client-side types */
// typedef _____* entry;           // Supplied by client
// typedef _____ key;              // Supplied by client
```

Does the client also need to provide any functions? Yes! The hash table implementation needs functions that can operate on values of the types `key` and `entry` so that it can hash keys, determine whether keys are equal, and extract keys from entries. Since the library is supposed to be generic, the library implementer cannot write these functions; we require the client to provide them.

There are three of these “client-side” functions.

1. When looking up a key, it needs to match this key with the key of entries of interest in the dictionary. To do so, the library needs a function that recovers a key from an entry:

```
key entry_key(entry x)                // Supplied by client
/*@requires x != NULL; @*/ ;
```

Since `hdict_lookup` returns `NULL` to signal that an entry with the requested key is not present, we disallow `NULL` entries.

This function allows us to provide `hdict_insert` with a useful postcondition: after inserting an entry, we expect to be able to get it back when looking up its key.

```
void hdict_insert(hdict_t H, entry x)
/*@requires H != NULL && x != NULL; @*/
/*@ensures hdict_lookup(H, entry_key(x)) == x; @*/ ;
```

2. We also need a hash function which maps keys to integers.

```
int key_hash(key k);           // Supplied by client
```

The result, the *hash value*, can be any integer, so our hash table implementation will have to take both this arbitrary integer and m , the size of the hash table's table, into consideration when figuring out which index of the table the key hashes to. For the hash table implementation to achieve its advertised (average-case) asymptotic complexity, the resulting index should have the property that its results are evenly distributed between 0 and m . The hash set implementation will work correctly (albeit slowly) even if it maps every key to 0.

3. Hash table operations also need to check for the equality of keys in order to be able to tell whether two objects that collide are actually the same or not.

```
bool key_equiv(key k1, key k2);    // Supplied by client
```

With this function, we can add a postcondition to `hdict_lookup`: either it returns `NULL` or the returned entry has the key we were looking for:

```
entry hdict_lookup(hdict_t H, key k)
/*@requires H != NULL; @*/
/*@ensures \result == NULL
           || key_equiv(entry_key(\result), k); @*/ ;
```

This completes the interface which we now summarize.

```
/*******/
/** Client interface ***/
/******/
```

```

// typedef _____* entry;           // Supplied by client
// typedef _____ key;             // Supplied by client

key entry_key(entry x)                  // Supplied by client
    /*@requires x != NULL; @*/ ;
int key_hash(key k);                    // Supplied by client
bool key_equiv(key k1, key k2);         // Supplied by client

/**** Library interface ****/

// typedef _____* hdict_t;

hdict_t hdict_new(int capacity)
    /*@requires capacity > 0; @*/
    /*@ensures \result != NULL; @*/ ;

entry hdict_lookup(hdict_t H, key k)
    /*@requires H != NULL; @*/
    /*@ensures \result == NULL
        || key_equiv(entry_key(\result), k); @*/ ;

void hdict_insert(hdict_t H, entry x)
    /*@requires H != NULL && x != NULL; @*/
    /*@ensures hdict_lookup(H, entry_key(x)) == x; @*/ ;

```

3 A Tiny Client

One sample application is to count word occurrences — say, in a corpus of Twitter data or in the collected works of Shakespeare. In this application, the keys are the words, represented as strings. Entries are pairs of words and word counts, the latter represented as integers.

```
/* **** */
/* client-side implementation */
/* **** */
struct wcount {
    string word;      // key
    int count;        // other data
};

// Fulfilling the client interface
typedef struct wcount* entry;
typedef string         key;

key entry_key(entry x)
//@requires x != NULL;
{
    return x->word;
}

int key_hash(key k) {
    return hash_string(k);           /* defined below */
}

bool key_equiv(key k1, key k2) {
    return string_equal(k1, k2);
}
```

4 A Universal Hash Function

One question we have to answer is how to hash strings, that is, how to map strings to integers so that the integers are evenly distributed no matter how the input strings are distributed.

We can get access to the individual characters in a string with the function `string_charat(s, i)`, and we can get the integer ASCII value of a

char with the function `char_ord(c)`; both of these are defined in the C0 **string** library. Therefore, our general picture of hashing strings looks like this:

```
int hash_string(string s) {
    int len = string_length(s);
    int h = 0;
    for (int i = 0; i < len; i++)
        //@loop_invariant 0 <= i;
    {
        int ch = char_ord(string_charat(s, i));
        // Do something to combine h and ch
    }
    return h;
}
```

Now, if we don't add anything to replace the comment, the function above will still allow the hash table to work correctly, it will just be very slow because the hash value of every string will be zero.

A slightly better idea is combining `h` and `ch` with addition or multiplication:

```
for (int i = 0; i < len; i++)
    //@loop_invariant 0 <= i;
{
    int ch = char_ord(string_charat(s, i));
    h = h + ch;
}
```

This is still pretty bad, however. We can see how bad by entering the $n = 45,600$ vocabulary words from the Collected Works of William Shakespeare, say, into a table with $m = 22,800$ chains (load factor is 2) and running `ht_stats`:

```
Hash table distribution: how many chains have size...
...0: 21217
...1: 239
...2: 132
...3: 78
...4: 73
...5: 55
...6: 60
...7: 46
```



```
...8: 42
...9: 23
...10+: 835
Longest chain: 176
```

Most of the chains are empty, and many of the chains are very, very long. One problem is that most strings are likely to have very small hash values when we use this hash function. An even bigger problem is that rearranging the letters in a string will always produce another string with the same hash value — so we know that "cab" and "abc" will always collide in a hash table. Hash collisions are inevitable, but when we can easily predict that two strings have the same hash value, we should be suspicious that something is wrong.

To address this, we can manipulate the value h in some way before we combine it with the current value. Some versions of Java use this as their default string hashing function.

```
for (int i = 0; i < len; i++)
    //@loop_invariant 0 <= i;
    {
        int ch = char_ord(string_charat(s, i));
        h = 31*h;
        h = h + ch;
    }
```

This does much better when we add all the vocabulary strings into the hash table:

```
Hash table distribution: how many chains have size...
...0: 3057
...1: 6210
...2: 6139
...3: 4084
...4: 2151
...5: 809
...6: 271
...7: 53
...8: 21
...9: 4
...10+: 1
Longest chain: 10
```

We can try adding a bit of randomness into this function in a number of different ways. For instance, instead of multiplying by 31, we could multiply by a number generated by the pseudo-random number generator from C0's library:

```
rand_t r = init_rand(0x1337BEEF);
for (int i = 0; i < len; i++)
    //@loop_invariant 0 <= i;
    {
        int ch = char_ord(string_charat(s, i));
        h = rand(r) * h;
        h = h + ch;
    }
```

If we look at the performance of this function, it is comparable to the Java hash function, though it is not actually quite as good — more of the chains are empty, and more are longer.

Hash table distribution: how many chains have size...

```
...0: 3796
...1: 6214
...2: 5424
...3: 3589
...4: 2101
...5: 1006
...6: 455
...7: 145
...8: 48
...9: 15
...10+: 7
```

Longest chain: 11

Many other variants are possible; for instance, we could try directly applying the linear congruential generator to the hash value at every step:

```
for (int i = 0; i < len; i++)
    //@loop_invariant 0 <= i;
    {
        int ch = char_ord(string_charat(s, i));
        h = 1664525 * h + 1013904223;
        h = h + ch;
    }
```

The key goals are that we want a hash function that is very quick to compute and that nevertheless achieves good distribution across our hash table. Handwritten hash functions often do not work well, which can significantly affect the performance of the hash table. Whenever possible, the use of randomness can help to avoid any systematic bias.

5 Coherence

Recall the functions `key_hash` and `key_equiv` of our tiny client:

```
int key_hash(key k) {  
    return hash_string(k);           /* from hash-string.c0 */  
}  
  
bool key_equiv(key k1, key k2) {  
    return string_equal(k1, k2);  
}
```

In this example, a key is a string.

Consider replacing `key_equiv` with a function that checks that the input keys have the same length:

```
bool key_equiv(key k1, key k2) {  
    return string_length(k1) == string_length(k2);  
}
```

This does not feel right. Let's see what would happen when we use this `key_equiv` and our original `key_hash` in our hash dictionary. After populating it with a large corpus, for example the works of William Shakespeare, assume we lookup a misspelled word, for example "amlet". The function `key_hash` will convert it to a hash value which will be used to compute the index of a chain. If this chain contains an entry whose key has length 5 (the length of "amlet"), it will return this entry, although its key is not "amlet"! This result is incorrect.

In a well-designed hash table, keys that are considered equal should hash to the same value, a condition we call *coherence*. As we just saw, whenever the hashing and the equivalence functions are incoherent, there is the concrete risk of our hash dictionaries working incorrectly.

What if we weaken the hash function instead of the equality check? Consider now our original `key_equiv` and a hash function that returns

the ASCII code of the first letter of its input (or 0 when passed the empty string):

```
int key_hash(key k) {
    if (string_equal(k, "") return 0
    return char_ord(string_charat(k, 0));
}
```

Then every word starting with the same letter will have the same hash value, and therefore will end up at the same table index. Its chains will be very long for a large dictionary, while other indices will be empty. When looking up a word, once on the appropriate chain, `key_equiv` will correctly either find it or report that it is not in the dictionary. The issue here is performance, not correctness: this choice leads to long chains and therefore slow searches.

The problem with both of these setups is that `key_hash` and `key_equiv` used the information in the key in very different measures. Instead, our original choices were very much in harmony.

6 A Fixed-Size Implementation of Hash Tables

For simplicity, we will now write a non-resizing implementation of hash dictionaries. We leave it as an exercise to modify this code to use unbounded arrays to support on-demand resizing of the table.

A non-resizing implementation requires that we can a priori predict a good size, or we will not be able to get the advertised $O(1)$ average time complexity. Choose the size too large and it wastes space and slows the program down due to a lack of locality. Choose the size too small and the load factor will be high, leading to poor asymptotic (and practical) running time.

We start with the type of lists to represent the chains of entries, and the hash table type itself.

```
/* ***** */
/* library-side implementation */
/* ***** */
typedef struct chain_node chain;
struct chain_node {
    entry data;           // != NULL
    chain* next;
};
```

```

typedef struct hdict_header hdict;
struct hdict_header {
    int size;                // 0 <= size
    int capacity;           // 0 < capacity
    chain*[] table;         // \length(table) == capacity
};

```

The first thing after the definition of a data structure is a function to verify its invariants. Besides the invariants noted above we should check the hash index of the key of every entry in each chain stored in $A[i]$ is indeed i . (This `is_hdict` function is incomplete.)

```

bool is_hdict(hdict* H) {
    return H != NULL
        && 0 <= H->size
        && 0 < H->capacity
        && is_array_expected_length(H->table, H->capacity);
    /* && there are no NULL entries */
    /* && each entry satisfies its own representation invariants */
    /* && there aren't entries with equal key */
    /* && the number of entries matches the size */
    /* && every entry in H->table[i] hashes to i */
    /* && ... */
}

```

Recall that the test on the length of the array must be inside an annotation, because the `\length` function is not available when the code is compiled without dynamic checking enabled.

In order to check that the keys in a hash dictionary hash to the correct index, we need a way of mapping the hash value returned by `key_hash` to an index of the table. This is a common enough operation that we'll write a helper function:

```

int index_of_key(hdict* H, key k)
    //@requires is_hdict(H);
    //@ensures 0 <= \result && \result < H->capacity;
{
    return abs(key_hash(k) % H->capacity);
}

```

Allocating a hash table is straightforward.

```

hdict* hdict_new(int capacity)
//@requires capacity > 0;
//@ensures is_hdict(\result);
{
    hdict* H = alloc(hdict);
    H->size = 0;
    H->capacity = capacity;
    H->table = alloc_array(chain*, capacity);
    return H;
}

```

Equally straightforward is searching for an entry with a given key. We omit the standard loop invariant.

```

entry hdict_lookup(hdict* H, key k)
//@requires is_hdict(H);
//@ensures \result == NULL || key_equiv(entry_key(\result), k);
{
    int i = index_of_key(H, k);
    for (chain* p = H->table[i]; p != NULL; p = p->next) {
        if (key_equiv(entry_key(p->data), k))
            return p->data;
    }
    return NULL;
}

```

Inserting an entry follows generally the same structure as search. If we find an entry in the correct chain with the same key we replace it. If we find none, we insert a new node at the beginning of the chain.

```

void hdict_insert(hdict* H, entry x)
//@requires is_hdict(H);
//@requires x != NULL;
//@ensures is_hdict(H);
//@ensures x == hdict_lookup(H, entry_key(x));
{
    key k = entry_key(x);
    int i = index_of_key(H, k);
    for (chain* p = H->table[i]; p != NULL; p = p->next) {
        //@assert p->data != NULL; // Not given by loop invariant!
        if (key_equiv(entry_key(p->data), k)) {
            p->data = x;
        }
    }
}

```

```

        return;
    }
}

// prepend new entry
chain* p = alloc(chain);
p->data = x;
p->next = H->table[i];
H->table[i] = p;
(H->size)++;
}

```

7 Hash Sets

A *set* is a collection of elements without duplicates. Elementary operations we will be interested in are inserting an element into a set and asking whether an element is a member of a set.

The basic technology of hash table can also be leveraged to give an efficient implementation of sets. Specifically, we can think of a set as a dictionary where both keys and entries are the elements. This allows us to simplify and specialize the hash dictionary interface into a hash set interface which provides the abstract type `hset` and the above operations.

- We collapse the types `key` and `entry` into a single type that we call `elem`.
- Extracting a key from an entry becomes vacuous. Consequently we can do without an `entry_key` function.
- The lookup function of dictionaries underlies a membership test: it returns `NULL` if the element is not in the set, and the element itself if it is. We can streamline this behavior by having the hash set version, which we call `hset_contains`, return a boolean. This also lets us do without the requirement that the data contained in the hash table be a pointer type: `elem` can be any type.

The resulting interface for hash sets is as follows:

```

/*****
*** Client interface ***
*****/

```

```
// typedef _____ elem;           // Supplied by client
bool elem_equiv(elem x, elem y);      // Supplied by client
int elem_hash(elem x);                // Supplied by client

/***** Library interface *****/
/***** Library interface *****/

// typedef _____* hset_t;

hset_t hset_new(int capacity)           /* 0(1) */
/*@requires capacity > 0; @*/
/*@ensures \result != NULL; @*/ ;

bool hset_contains(hset_t H, elem x)    /* 0(1) avg. */
/*@requires H != NULL; @*/ ;

void hset_add(hset_t H, elem x)         /* 0(1) avg. */
/*@requires H != NULL; @*/
/*@ensures hset_contains(H, x); @*/ ;
```


8 Exercises

Exercise 1 (sample solution on page 19). We want to use the hash dictionaries developed in this chapter to keep track of grades and other student data, and access them by student id. A student id is an integer and the associated grades are stored in a character array of some fixed length tracked outside the hash dictionary. A student record also includes notes about the student (a string) and whether this student is auditing the course (a boolean).

Implement the client interface of hash dictionaries by defining the types `entry` and `key` and the functions `entry_key`, `key_hash` and `key_equiv`.

Exercise 2 (sample solution on page 19). We are extending the hash table interface with a new function

```
entry[] hdict_tabulate(hdict* H)
/*@requires H != NULL; @*/ ;
```

that returns an array with all the entries in the hash table, in some order of your choice. Include contracts and annotations as appropriate.

Exercise 3. Extend the hash table implementation so it dynamically resizes itself when the load factor exceeds a certain threshold. When doubling the size of the hash table you will need to explicitly insert every entry from the old hash table into the new one, because the result of hashing depends on the size of the hash table.

Exercise 4. Redo the library implementation for a different client interface that has a function `key_hash(key k, int m)` that returns a result between 0 (inclusive) and `m` (exclusive).

Exercise 5. Extend the hash table interface with a new function `hdict_tabulate` that returns an array with the entries in the hash table, in some order of your choice.

Exercise 6. Extend the hash table interface with a new function to delete the entry with a given key if present in the table. To be extra ambitious, shrink the size of the hash table once the load factor drops below some minimum, similarly to the way we could grow and shrink unbounded arrays.

Exercise 7. The hash dictionaries in this lecture relied on entries embedding their keys (which we retrieved with the function `entry_key`). A different design is for the hash table to implement a mapping from keys to values, where values do not embed the key. The interface of such hash dictionaries is

```
/*******/
/* Client interface */
```

```

/*****/

// typedef _____ key;           // Supplied by client
// typedef _____ *value;        // Supplied by client
bool key_equiv(key x, key y);        // Supplied by client
int key_hash(key x);                 // Supplied by client

/*****/
/** Library interface */
/*****/

// typedef _____* hdict_t;

hdict_t hdict_new(int capacity)
/*@requires capacity > 0; @*/
/*@ensures \result != NULL; @*/ ;

value hdict_lookup(hdict_t H, key k)
/*@requires H != NULL; @*/ ;

void hdict_insert(hdict_t H, key k, value v)
/*@requires H != NULL; @*/
/*@requires v != NULL; @*/
/*@ensures hdict_lookup(H, k) == v; @*/ ;

```

In particular, `hdict_insert` takes both a key and a value as parameters.
Implement this interface.

Exercise 8. Write an implementation of hash sets based on the interface provided in this chapter.

Sample Solutions

Solution of exercise 1 This solution assumes that we have written a function `hash_int` that maps integers to values uniformly distributed over the range of all C0 integers.

```
struct student_record {
    int student_id;
    char[] grades;
    string notes;
    bool auditor;
};

typedef struct student_record* entry;
typedef int key;

key entry_key(entry x)
//@requires x != NULL;
{
    return x->student_id;
}

int key_hash(key k) {
    return hash_int(k);
}

bool key_equiv(key k1, key k2) {
    return k1 == k2;
}
```

Solution of exercise 2

```
entry[] hdict_tabulate(hdict* H)
//@requires is_hdict(H);
//@ensures is_hdict(H) && \length(\result) == H->size;
{
    entry[] result = alloc_array(entry, H->size);
    int j = 0; // indexes result

    for (int i = 0; i < H->capacity; i++)
        //@loop_invariant 0 <= i && i <= H->capacity;
        {
            for (chain* p = H->table[i]; p != NULL; p = p->next)
                //@loop_invariant 0 <= j && j <= H->size;
                {
                    result[j] = p->data;
                    j++;
                }
        }
    return result;
}
```