COMP-423B: Quiz 2 Solutions

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<u>**General Comments**</u>

- For any questions/comments regarding the correction of the quiz: if I have made a mistake adding up your points, or you feel that your arguments were not understood, please contact me (Matt) at wahab@cim.mcgill.ca

1. **(3 points)**

   Marking:
   2 marks for part (a.), 1 mark for part (b.).

   (a) What is the "inverted file" of the following: `abccbaabc` ?

   Solution:

symbol	n_i	t_i								
a	3	1	0	0	0	0	1	1	0	0
b	3	0	1	0	0	1	0	0	1	0
c	3	0	0	1	1	0	0	0	0	1

   We can also encode the gaps between instances of a symbol.

symbol	n_i	offset		
a	3	1	5	1
b	3	2	3	3
c	3	3	1	4

   (b) How can "header" information be used to compress inverted files?

   Solution:

   We use gap or run-length encoding. Define a prefix code for the gaps of each list (say use a Golomb code). The header file needs to send n and n_i for each list. CHECK THIS IN NOTES.

2. **(4 points)**

   (a) Encode the following sequence using LZ2 (sliding window):

   `baabaaaabb`

   Assume the window size is $n_w = 4$. Your answer should include a parsing of the above sequence into phrases.

   Solution:

   The phrase is parsed as follows: b, a, a, baa, aa, b, b.

length	symbol	offset
0	'b'	-
0	'a'	-
1	-	1
3	-	3
2	-	2
0	'b'	-
1	-	1

   Let C_0 be the code for phrase length, C_1 be the code for offset, and C_2 be the code for symbols. This yields the following code:

$$C_0(0)C_2(b)C_0(0)C_2(a)C_0(1)C_1(1)C_0(3)C_1(3)C_0(2)C_1(2)C_0(0)C_2(b)C_0(1)C_1(1)$$

   (b) For the same sequence as in (a), show the phrase table built by an LZ3 encoder.

   Solution:

position	symbol	parent
1	'b'	0
2	'a'	0
3	'b'	2
4	'a'	2
5	'b'	4
6	'b'	0

3. **(3 points)**

   (a) If the alphabet is $\{0, 1\}$, then a Lempel Ziv algorithm takes any finite sequence of bits (a sequence) and maps it to another finite sequence of bits (a codeword for the sequence). Thus, the LZ algorithm defines a code on the set of finite bit strings. This code is not a prefix code. Why not? Give a counter example (and specify the LZ method).

Solution:

In general the Lempel Ziv algorithm does not define a prefix code because if we have two strings s_1 and s_2, such that s_1 is a prefix of s_2, then the code for s_1, $LZ(s_1)$, is a prefix of the code for s_2, $LZ(s_2)$.

For a specific counter example let's use LZII to encode strings $s_1 = 0$ and $s_2 = 00$. Let C_0 be the code for phrase length, C_1 be the code for offset, and C_2 be the code for symbols.

$\text{LZII}(s_1) = C_0(0)C_2(0)$
$\text{LZII}(s_2) = C_0(0)C_2(0)C_0(1)C_1(0)$

Notice that the code for s_1 is a prefix for the code for s_2.

(b) How could one modify the LZ algorithm/code so that it does define a prefix code?

Solution:

There are several ways to do this:

- Have the algorithm encode the length of the sequence and send it at the end (or beginning) of the encoded sequence.
- Introduce a 'null' character that is encoded at the end of the encoded sequence. Note that the codeword for this 'null' character must not be a prefix of any codeword in C_0.

4. **(2 points)**

Consider the following conditional probability matrix for a first order Markov model, where the number of symbols in the alphabet is $N = 3$. Assume that these conditional probabilities are the same for all j.

$$P(X_{j+1} \mid X_j) = \frac{1}{8}\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 1 \\ 6 & 1 & 5 \end{bmatrix}$$

If the marginal probability $p(X_1)$ is uniform, then what are the marginal probabilities $p(X_2)$ and $p(X_3)$?

Solution:

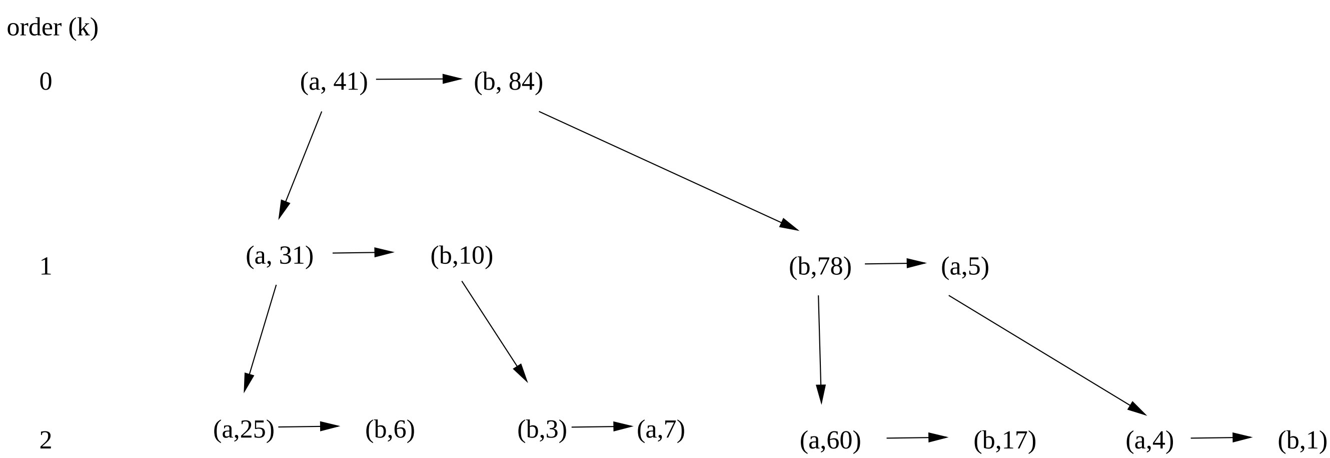
For this we use the formula for marginal probability: $p(X_{j+1}) = p(X_{j+1}|X_j)p(X_j)$

$$p(X_2) = p(X_2 \mid X_1)p(X_1) = \frac{1}{8}\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 1 \\ 6 & 1 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$$

$$p(X_3) = p(X_3 \mid X_2)p(X_2) = \frac{1}{8}\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 1 \\ 6 & 1 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{7}{32} \\ \frac{17}{32} \end{bmatrix}$$

5. **(3 points)**

The following trie shows the frequency counts of a sequence, after j symbols have been seen by the encoder.



(a) Estimate the conditional probability of X_{j+1}, assuming:

- a 0^{th} order Markov model;
- a 1^{st} order Markov model;

You may assume that the alphabet is $\{\mathtt{a}, \mathtt{b}\}$ only. Be sure to state any other assumptions you make.

Hint: the count at a node at level $k+1$ is the number of times that the symbol at that node is the next symbol, given the previous k symbols.

Solution:

The 0^{th} order Markov model is:

$$P(X_{j+1}) = \frac{1}{125}\begin{bmatrix} 41 \\ \\ 84 \end{bmatrix}$$

The 1^{st} order Markov model is:

$$P(X_{j+1} \mid X_j) = \begin{bmatrix} \frac{31}{41} & \frac{5}{84} \\ \\ \frac{10}{41} & \frac{78}{84} \end{bmatrix}$$

(b) What were the symbols X_j and X_{j-1}?

Solution:

This was a trick question. To find out X_j and X_{j-1} you simply needed to see where the counts did not add up. For level $k = 1$ we can see that $78 + 5 \neq 84$ ($X_j = b$), and at level $k = 2$ that $60 + 17 \neq 78$ ($X_{j-1} = b$).