Lecture 18 Sherali-Adams Nov 3

Today: want to show a strong completeness theorem for SA

Throughout we're going to consider a system of linear inequalities

does also hold for higher degrees.

$$Q := \{ a_1 \cdot x \le b_1, \ \ldots, a_m \cdot x \le b_m \}$$

over n variables $x_1, \ldots, x_n \in \mathbb{R}$

Defn Let d be a positive integer. The d-th level Sherali-Adams polytope $SA_d(Q)$ is defined as follows.

(1) (Extend) For every inequality $q_i(x) \ge 0$ and every $S, T \subseteq [n]$ with $|S \cup T| \le d$, $S \cap T = \emptyset$, add the inequality

$$\prod_{i \in S} x_i \prod_{j \in T} (1 - x_j) \, q_i \ge 0$$

to $SA_d(Q)$

(2) (Linearize) For each $S \subseteq [n]$, $|S| \le d + \deg(Q)$, create a variable y_S.

Then, for each inequality $q_i \ge 0$ in Q, replace each monomial

$$\prod_{i \in S} x_i$$

occurring in q_i with the variable y_S, to make a new linear inequality $\tilde{q}_i \ge 0$.

- If Q has n variables then $SA_d(Q)$ has

$$\binom{n}{\leq d+\deg(Q)} := \sum_{i=0}^{d+\deg(Q)} \binom{n}{i}$$

- The original variables $x_1, \dots, x_n$ are naturally embedded in $SA_d(Q)$ in vars $y_{\{1\}}, y_{\{2\}}, \dots, y_{\{n\}}$.

Goal: $SA_n(Q)\restriction_{1,2,\dots,n} = \text{hull}_{\mathbb{Z}}(Q \cap \{0,1\}^n)$

Think **distributionally**

$\alpha \in \text{hull}_{\mathbb{Z}}(Q \cap \{0,1\}^n) \iff \alpha$ is represented by a probability distribution over $Q \cap \{0,1\}^n$

$$\alpha = \sum_{x \in Q \cap \{0,1\}^n} \lambda_x x \qquad \lambda_x \geq 0, \ \sum \lambda_x = 1$$

Today: Show $SA_d(Q)$ has a "distributional" interpretation for every d.

Main Lemma Let Q be a set of linear inequalities. For every $\alpha \in SA_d(Q)$ and every $S \subseteq [n]$ with $|S| = t$, $0 \le t \le d$

$$\alpha \text{ "}\in\text{" } \operatorname{conv}\left(\{\beta \in SA_{d-t}(Q) \mid \forall i \in S: \beta_i \in \{0,1\}\}\right)$$

<u>Remark 1</u> We mean the coordinates of α of size ≥ 1 and at most $\max\{1, d-t\}$.

<u>Remark 2</u> Set $t=d$: $(\alpha_{\{1\}}, \cdots, \alpha_{\{n\}})$ is a convex combination of points in $SA_0(Q)$ with d 0-1 coordinates!

<u>Cor</u> $\operatorname{hull}_{\mathbb{Z}}(Q \cap \{0,1\}^n) = SA_n(Q)\restriction_{1,2,\dots,n}$

<u>Pf</u> Immediate by setting $d=n$. $\square$

Proof of Main Lemma

Prove for $t=1$:

<u>Claim</u> For every $\alpha \in SA_d(Q)$ every $i \in [n]$ there exist points $\beta^{(0)}, \beta^{(1)} \in SA_{d-1}(Q)$ and $\lambda \in [0,1]$ s.t.

- $\beta^{(0)}_{\{i\}} = 0$, $\beta^{(1)}_{\{i\}} = 1$
- If $|S| \le d-1$ then $\alpha_S = \lambda \beta^{(1)}_S + (1-\lambda)\beta^{(0)}_S$

Pf. If $\alpha_{\{i\}} \in \{0,1\}$ then the vec. $(\alpha_S)_{|S| \le d-1}$ is in SA_{d-1}, and satisfies the claim. So, suppose

$$0 < \alpha_{\{i\}} < 1.$$

Let $\beta_S^{(1)} = \dfrac{\alpha_{S \cup \{i\}}}{\alpha_{\{i\}}}$, $\beta_S^{(0)} = \dfrac{\alpha_S - \alpha_{S \cup \{i\}}}{1 - \alpha_{\{i\}}}$, $\lambda = \alpha_{\{i\}}$

for all S, $|S| \le d-1$. Observe

- $\beta_{\{i\}}^{(1)} = 1$, $\beta_{\{i\}}^{(0)} = 0$ ✓
- $\alpha_S = \lambda \beta_S^{(1)} + (1-\lambda)\beta_S^{(0)}$

$$= \alpha_{S \cup \{i\}} + (\alpha_S - \alpha_{S \cup \{i\}}) = \alpha_S \checkmark$$

Prove $\beta^{(0)}, \beta^{(1)} \in SA_{d-1}(Q)$.

First consider the SA inequalities from $1 \geq 0$. They look like:

$q_i \geq 0$

$\forall A, B \subseteq [n]$, $|A \cup B| = d$, $A \cap B = \emptyset$ add inequality

$$\sum_{S \subseteq B} (-1)^{|S|} y_{A \cup S} \geq 0$$

Remember ↗ this is from $\prod_{i \in A} x_i \prod_{j \in B} (1-x_j) \geq 0$ and linearizing.

Plug in $\beta^{(1)}$ first

$$\sum_{S \subseteq B} (-1)^{|S|} \beta_{A \cup S}^{(1)} = \frac{1}{\alpha_{\{i\}}} (q_i) \sum_{S \subseteq B} (-1)^{|S|} \alpha^{(1)}_{A \cup S \cup \{i\}} \geq 0$$

This is satisfied by α in $SA_d(Q)$, it corresponds to the inequality for $(A \cup \{i\}, B)$

Plug in $\beta^{(0)}$ then

$$\sum_{J \subseteq B} (-1)^{|J|} \beta^{(0)}_{A \cup J} = \frac{1}{1-\alpha_{\{i\}}} (q_i) \sum_{J \subseteq B} (-1)^{|J|} \alpha_{A \cup J} - \alpha_{A \cup J \cup \{i\}}$$

$$= \frac{1}{1-\alpha_{\{i\}}} (q_i) \sum_{J \subseteq B \cup \{i\}} (-1)^{|J|} \alpha_{A \cup J}$$

$$\geq 0$$

since the $(A, B \cup \{i\})$ inequality is satisfied in $SA_d(Q)$

$\square$ (Proof of Claim)

The lemma immediately follows by recursing on all sets of coordinates of size $\leq d$.

$\square$

<u>Remark</u> It's easy to prove (by reversing the previous argument) that the converse also holds, so this gives a characterization of points in $SA_d(Q)$.

Let's give a more "distributional" description of what happens in $SA_d(Q)$.

Defn A degree-d pseudo-distribution on $\{0,1\}^n$ is given by a family

$$\{\mu_S \mid \forall S \subseteq [n], |S| \leq d\}$$

of probability distributions, where μ_S is supported on $\{0,1\}^S$ such that

$\forall A \subseteq B$, $|B| \leq d$ then

$$\Pr_{x \sim \mu_A}\left[\prod_{i \in A} x_i = 1\right] = \Pr_{y \sim \mu_B}\left[\prod_{i \in A} y_i = 1\right].$$

Fact 1 (Exercise)

Let $\{\mu_S\}$ be a degree-d pseudo-distribution then define

$$z \in \mathbb{R}^{\binom{n}{\leq d}}$$

by $\quad z_S = \Pr_{x \sim \mu_S}\left[\prod_{i \in S} x_i\right]$ then $\quad z \in SA_d(Q)$.

ex PHP_n^{n+1} defined by the inequalities

- $x_{ik} + x_{jk} \leq 1 \quad \forall i \neq j \in [n+1], \forall k \in [n]$
- $\sum_{i=1}^{n+1} x_{ij} \geq 1 \quad \forall j \in [n]$
- $0 \leq x_{ij} \leq 1$

Consider the following distribution μ_S for $S \subseteq [n+1]$ defined as follows:

- Map all pigeons in S to $|S|$ holes u.a.r (i.e. with probability $1/\binom{n}{|S|}|S|!$)

  All other pigeons are unmapped.

<u>Fact 1</u> (Consistent Marginals)

If $S \subseteq T$, and $R = \{k_1, \cdots, k_{|S|}\}$ is a set of holes then

$$\Pr_{\pi \sim \mu_S}[\pi \text{ maps all pigeons in } S \text{ to } R]$$

$$=$$

$$\Pr_{\pi \sim \mu_T}[\pi \text{ maps all pigeons in } S \text{ to } R]$$

<u>Pf</u> Easy independence!

<u>Corollary</u> $SA_{n-1}(PHP_n^{n+1}) \neq \emptyset$

i.e. PHP_n^{n+1} requires degree $\geq n-1$ to refute in SA.

<u>Thm</u> The (B-W) size-degree tradeoff holds for SA!

$$S_{SA}(F) \geq 2^{\Omega\left(\frac{(\deg_{SA}(F) - \deg(F))^2}{n}\right)}$$