Lecture 19 Sherali-Adams Nov 5

Goal: Link the "polytope view" with the "proof view"

- Let $Q = \{q_1 \geq 0, \cdots, q_m \geq 0\}$ be a system of polynomial inequalities over vars $x_1 \cdots x_n$.
- $B = \{x_i^2 - x_i \geq 0, x_i - x_i^2 \geq 0\}_{i=1}^{n}$
- $\forall S, T \subseteq [n]$, $S \cap T$, let $J_{S,T} := \prod_{i \in S} x_i \prod_{j \in T} (1 - x_j)$
- A conical junta is a non-negative linear combination of $J_{S,T}$s:

$$y := \sum_{i=1}^{\ell} c_i J_{S_i, T_i} \qquad c_i \in \mathbb{R}$$

- An SA-refutation of Q is given by a list of conical juntas

$$y_1, \cdots, y_m, \mathcal{H}_1, \cdots, \mathcal{H}_{2n}$$

such that $\displaystyle\sum_{i=1}^{m} y_i q_i + \sum_{j=1}^{2n} \mathcal{H}_j p_j = -1$

where $p_i \in B$, $q_i \in Q$.

Thm There is a degree-d Sherali-Adams refutation of Q iff $SA_d(Q) = \emptyset$.

Need a link between "polynomial" representations and the "linear" representations, use pseudo-expectation

Defn Let q be a polynomial over $x_1 \cdots x_n$. The multilinearization of q is the polynomial obtained by replacing all terms x_i^c with x_i in q.

A degree-d pseudo-expectation for Q is a linear function

$$\tilde{\mathbb{E}} : \{\text{polynomials over } x_1 \cdots x_n\} \to \mathbb{R} \quad \text{s.t.}$$

(1) $\tilde{\mathbb{E}}[1] = 1$

(2) $\tilde{\mathbb{E}}[J_{S,T}] \geq 0$ for all degree-d non-neg juntas $J_{S,T}$

(3) $\tilde{\mathbb{E}}[q J_{S,T}] \geq 0$ for any $q \in Q$, all $J_{S,T}$ with $\deg(J_{S,T}) \leq d$.

(4) $\tilde{\mathbb{E}}[p] = \tilde{\mathbb{E}}[q]$ if p and q have the same multilinearization.

Lemma Let $\alpha \in \mathbb{R}^{\binom{n}{\leq d}}$, consider any function

$$\tilde{\mathbb{E}} : \{\text{polys over } x_1 \cdots x_n\} \to \mathbb{R}$$

by

$$\tilde{\mathbb{E}}\left[\prod_{i \in S} x_i\right] = \alpha_S$$

and then extended by (multi)linearity.

Then

$$\alpha \in SA_d(Q) \iff \tilde{\mathbb{E}} \text{ is a degree-}d\text{ pseudo-expectation}$$

Pf Sketch Let $S, T \subseteq [n]$ satisfy $|S \cup T| \leq d$, $S \cap T = \emptyset$. Consider the inequality in $SA_d(Q)$:

$$\sum_{R \subseteq T} (-1)^{|R|} \alpha_{S \cup R} \geq 0 \qquad \begin{array}{c} SA_d(Q) \\ \rotatebox{90}{$\in$} \\ \alpha_\emptyset = 1 \end{array}$$

$$\Longleftrightarrow \quad \sum_{R \subseteq T} (-1)^{|R|} \tilde{\mathbb{E}}\Big[\prod_{i \in S \cup R} x_i\Big] \geq 0$$

$$\Longleftrightarrow \quad \tilde{\mathbb{E}}\Big[\sum_{R \subseteq T} (-1)^{|R|} \prod_{i \in S \cup R} x_i\Big] \geq 0$$

$$\Longleftrightarrow \quad \tilde{\mathbb{E}}\big[\mathcal{J}_{S,T}\big] \geq 0$$

other inequalities follow similarly. □

<u>Thm</u> There is a degree-d Sherali-Adams refutation of Q iff $SA_d(Q) = \emptyset$.

<u>Pf</u> ($\Rightarrow$) Suppose by contradiction $\alpha \in SA_d(Q)$, and let $\tilde{\mathbb{E}}_\alpha$ be the pseudo-expec. for α.

Let's consider a degree-d SA ref. of Q:

$$\sum_{i=1}^{m} \mathcal{J}_i q_i + \sum_{j=1}^{2n} \mathcal{H}_j p_j = -1$$

$q_i \in Q$, $p_i \in B$, $\mathcal{J}_i, \mathcal{H}_j$ are conical juntas.

Apply $\tilde{\mathbb{E}}_\alpha$ to both sides:

$x_i^2 - x_i$, so $\tilde{\mathbb{E}}_\alpha[x_i^2 - x_i] = \tilde{\mathbb{E}}[x_i] - \tilde{\mathbb{E}}[x_i] = 0$

$$\tilde{\mathbb{E}}_\alpha\Big[\underbrace{\sum_{i=1}^{m} \mathcal{J}_i q_i}_{\geq 0} + \underbrace{\sum_{j=1}^{2n} \mathcal{H}_j p_j}_{=0}\Big] = \tilde{\mathbb{E}}_\alpha[-1] = -1$$

Contradiction! So $SA_d(Q) = \emptyset$!

($\Leftarrow$) Suppose $SA_d(Q) = \emptyset$. Since $SA_d(Q) = \emptyset$,

there is a non-negative linear combination of the defining inequalities that yields -1: "Farkas Lemma"

$$\sum_i c_i \underline{L_i} = -1$$

where $c_i \in \mathbb{R}_{\geq 0}$, each L_i is a linear inequality from $SA_d(Q)$.

Replace each L_i with it's corresponding polynomial term:

$$\sum_i c_i \, \mathfrak{I}_{S_i, T_i} \, q_{j_i}$$

We might not get -1 right away because this isn't the linearization!

So: use boolean inequalities to linearize!

ex) $\mathfrak{I}_{S_i, T_i} = x_1 x_2 \qquad q_{j_i} = x_1 + x_2 - 1 \leftarrow x_1 + x_2 \geq 1$

$\hookrightarrow \quad x_1^2 x_2 + x_1 x_2^2 - x_1 x_2$

Add $x_2(x_1^2 - x_1) + x_1(x_2^2 - x_2)$ to get

$$x_1 x_2 + x_1 x_2 - x_1 x_2 = x_1 x_2.$$ Multilinear!

After multilinearizing we get

$$\sum_i c_i \, \mathfrak{I}_{S_i, T_i} \, q_{j_i} + \sum_{j=1}^{2n} H_j p_j = -1 \qquad \square$$

By working a little bit harder we can prove derivational completeness

<u>Thm</u> Let Q be a system of polynomial inequalities. Consider another inequality

$$p(x) \geq c$$

Then there is a SA-proof of $p(x)-c$ from Q

$$\Longleftrightarrow$$

$p(x) \geq c$ holds for all $x \in \{0,1\}^n$ that satisfy all inequalities in Q.

Applications

Because of this strong completeness, SA can reason about optimization problems!

ex) Vertex Cover → A vertex cover of a graph $G=(V,E)$ is a subset of vertices $U \subseteq V$ that touch every edge in G

<u>LP Relaxation</u>

$$\min \sum_{u \in V} x_u$$

$$\text{s.t.} \quad x_u + x_v \geq 1 \quad \forall uv \in E$$

$$0 \leq x_u \leq 1$$

This LP relaxation achieves a 2-approximation in the sense that

$$\frac{1}{2} vc(G) \leq val_{LP}(G) \leq vc(G)$$

Equivalently: if $\sum_{u \in V} x_u \geq s$ is a valid inequality for the smallest vc of G, then

$$\sum_u x_u \geq \frac{1}{2} s$$

from the inequalities in the LP.

Achieve 2-approx. provable by degree-0 Sherali-Adams!

Question. We know $\sum_u x_u \geq VC(G)$ is a valid inequality over the VC inequalities for G.

What degree of SA is needed to prove

$$\sum_u x_u \geq VC(G) \; ?$$

or $$\sum_u x_u \geq \left(\frac{1}{2}+\varepsilon\right) VC(G) \text{ for } \varepsilon > 0 \, ?$$

[Charikar-Makarychev-Makarychev 09]

There are infinite families of graphs $\{G_n\}$ s.t. any SA proof of

$$\sum_u x_u \geq \left(\frac{1}{2}+\varepsilon\right) VC(G)$$

requires degree n^{δ} for $\delta(\varepsilon) > 0$.

Def Let $P:\{0,1\}^K \to \{0,1\}$ be any predicate. An instance $\mathcal{I}$ of P-CSP is given by e.g. k-SAT

$$P(T_1) \wedge P(T_2) \wedge \cdots \wedge P(T_m) \qquad P := \mathrm{OR}$$

where each T_i is an ordered list of boolean literals x_j^o or $\overline{x_j^o}$.

The goal is to find an input assignment x such that

$$\mathcal{I}(x) := \sum_{i=1}^{m} P(T_i(x))$$

i.e. maximize the # of sat constraints.

Thm [Chan et al 13, Kothari et al 17]

Suppose that degree-d SA cannot achieve an α-approximation for the P-CSP problem. Then **no** "structured" linear programming relaxation with at most $n^{O(d)}$ constraints achieving an α-approximation for P-CSP.

Defn [Linear Extended Formulation]

A polytope Q is a "structured" LP relaxation of P-CSP if for every instance $\mathcal{I}$ and every $x \in \{0,1\}^n$ there are vectors $w_{\mathcal{I}} \in \mathbb{R}^D$, $v_x \in Q$ s.t.

$$\mathcal{I}(x) = w_{\mathcal{I}} \cdot v_x .$$