# Probability 101

15-451
9/29/20

**Random Variable:** Samples $\to \mathbb{R}$

For discrete Random Variable the definition is straight forward.

Ex: Sample space $= \{H, T\}$; RV $X$

$$Prob[X = H] = 1/2$$

$$Prob[X = T] = 1/2$$

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In the continuous case one needs some care in general:

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We will assume they a given by

1) Prob Density Fcn

2) Cumulative Dist

# The Exponential Distribution

## Prob Density Fcn

**Def** The Exponential RV $X_\beta$.

$$\text{Prob}[X_\beta = \mu] = \begin{cases} \beta e^{-\beta\mu} & \mu \geq 0 \\ 0 & \text{o.w} \end{cases}$$

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## More Formally:

$$\text{Prob}\left[X_\beta \in [u, u+dx]\right] = \begin{cases} \beta e^{-\beta u}\, dx & u \geq 0 \\ 0 & \text{o.w} \end{cases}$$

## Cumulative Dist

**Def** $F_\beta(Y) \equiv \text{Prob}[X_\beta \leq Y]$

## For the Exponential

$$F_\beta(Y) = \int_0^Y \beta e^{-\beta X}\, dX \equiv \left[-e^{-\beta X}\right]_0^Y = 1 - e^{-\beta Y}$$

**Def** <u>Expected Value</u>

$$\mathbb{E}_X[X] \equiv \int_{-\infty}^{\infty} y \, P_r[X = y] \, dy$$

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Two ways to compute $\mathbb{E}[X_\beta]$.

1) <u>Def</u> $\mathbb{E}[X_\beta] = \int_0^\infty y \beta e^{-\beta y} dy = 1/\beta$

Using integration by parts.

2) $\mathbb{E}[X_\beta] = \int_0^\infty \text{Prob}[X_\beta \geq y] \, dy$

## For the Exponential

Now $\text{Prob}[X_\beta \geq y] = e^{-\beta y}$

thus $\mathbb{E}[X_\beta] = \int_0^\infty e^{-\beta y} = -\frac{1}{\beta} e^{-\beta y} \Big]_0^\infty = 1/\beta$

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## Order Statistics

Suppose $X_1, \dots, X_n$ random variables

**Def** $X_{(k)} \equiv \text{Select}_k(X_1, \dots, X_n)$

ie $X_{(1)} \le X_{(2)} \le \dots \le X_{(n)}$

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Suppose $X_1, \dots, X_n$ are iid st

1) $f(u) = \text{Prob}[X_i = u]$ (PDF)

2) $F(u) = \text{Prob}[0 \le X_i \le u]$

Then

$$\text{Prob}[X_{(1)} = u] = n(1 - F(u))^{n-1} f(u)$$

If $X_1, \dots, X_n$ are iid exponentials.

$$\text{Prob}[X_{(1)} = u] = n\left(e^{-\beta u}\right)^{n-1} \beta e^{-\beta u} = n\beta\, e^{-n\beta u}$$

$$X_{(1)} \equiv \text{Exp}(n\beta) \quad \text{Thus } \mathbb{E}\text{xp}[X_{(1)}] = \frac{1}{n\beta}$$

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Let $S_i = X_{(i+1)} - X_{(i)}$ for $i \geq 0$

By Memoryless property $S_i = \text{Exp}((n-i)\beta)$

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Pf: Consider $S_1 = X_{(2)} - X_{(1)}$

Suppose $X_1 = X_{(1)}$ thus

$X_2, \dots, X_n \geq X_{(1)}$

Thus $X_i \equiv X_i' + X_{(1)}$ $\quad X_i'$ new exponential

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Thus $\mathbb{E}[S_i] = \frac{1}{(n-i)\beta}$

By Linearity of expectation.

$$\mathbb{E}[X_{(n)}] = \sum_{i=0}^{n-1} \mathbb{E}[S_i] = \frac{1}{\beta}\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)$$

$$= \frac{H_n}{\beta}$$

$$\approx \frac{\ln n}{\beta}$$

Thus $\mathbb{E}[X_{(1)}] = \frac{1}{n\beta}$

$$\mathbb{E}[X_{(n)}] = \frac{\ln n + O(1)}{\beta}$$

## Concentration for $X_{(n)}$?

$$\Pr\left[X_i \geq \frac{c\ln n}{\beta}\right] = e^{-\frac{\beta c \ln n}{\beta}} = e^{-c\ln n} = n^{-c}$$

by union bound we get.

$$\Pr\left[X_{(n)} \geq \frac{c\ln n}{\beta}\right] \leq n \cdot n^{-c} = \frac{1}{n^{(c-1)}}$$

thus

$$\Pr\left[X_{(n)} \geq \frac{2\ln n}{\beta}\right] \leq \frac{1}{n}$$

# Generating Dist or Random Variables

Let $X_f$ be the random variable of PDF $f : \mathbb{R} \to \mathbb{R}^+$ where $\int_{-\infty}^{\infty} f(x)\,dx = 1$

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Note Not clear that RV exist.

But we can ask if we have one can we generate more.

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Def: $f, g \in$ PDF's with RV Variables $X_f$ & $X_g$

we say $f \leq g$ if $\exists$ det process $D$ s.t. $X_f = D(X_g)$

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Let $\mathcal{U}$ be uniform RV with PDF $u$.

ie $u(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$

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Let $\mathcal{U}_2$ be uniform $[0,2]$ then

$X_{u_2} = 2X_u$ thus $\mathcal{U}_2 \leq \mathcal{U}$

& $u \leq \mathcal{U}_2$ ?

## Generating Exponential Dist from Uniform

PDF $\equiv f(x) = \beta e^{-\beta x}$ for $0 < \beta$ & $x \geq 0$

$$F(x) = \int_0^x f(x)\,dx = 1 - e^{-\beta x}$$

Thus $F: [0,\infty] \longrightarrow [0,1]$ 1-1 & onto

We get that $F(X_f)$ is uniform $[0,1]$

Thus: $u \leq f$ but we want $f \leq u$

Lets find $F^{-1}$ ie

Solve for $x$ in $y = F(x) = 1 - e^{-\beta x}$

iff $e^{-\beta x} = 1 - y$

iff $-\beta x = \ln(1-y)$

iff $x = -\frac{1}{\beta}\ln(1-y)$ but $1-y$ is uniform $[0,1]$

$$x = \frac{-\ln(X_u)}{\beta} = X_{Exp}$$

Thus $X_{Exp} \leq X_u$

**Alg** Given RV $U$ uniform $[0,1]$

Return $\frac{-\ln U}{\beta}$

# Generating Normal Dist

The PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

Setting $\sigma = 1$ we get Gauss's Unit Normal

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

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What if we try computing CDF!

ie $F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$

Thm $F(x)$ is not an elementary fcn!

Question: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx \stackrel{?}{=} 1$

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<u>note</u> $f(x) = x e^{-x^2/2}$ is OK since

$$\frac{d}{dx}\left(-e^{-x^2/2}\right) = x e^{-x^2/2}$$

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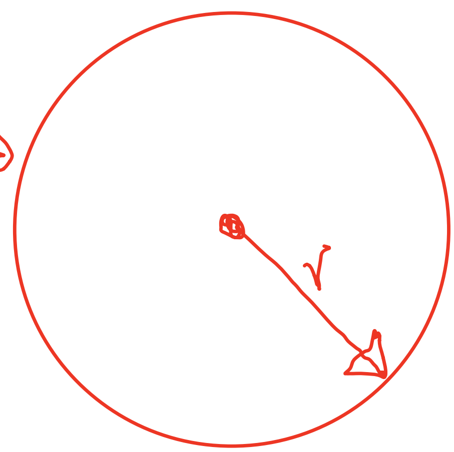
Trick compute 2D normal.

Let $f(x,y) = \frac{1}{2\pi} e^{-x^2/2} \cdot e^{-y^2/2}$

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$= \frac{1}{2\pi} e^{-r^2/2} \text{ (in polar)}$$

PDF $= \frac{1}{2\pi} e^{-r^2/2}$



In Polar: $f(r,\theta) = \frac{1}{2\pi} e^{-r^2/2}$ (Symmetric)

Lets compute the cumulative that is Prob $f(r,\theta)$ is in a disk of radius $r$.

$$D(R) = \int_0^R \frac{2\pi r}{2\pi} e^{-r^2/2}\, dr = \int_0^R r e^{-r^2/2}\, dr$$

$$= -e^{-r^2/2}\Big]_0^R = 1 - e^{-R^2/2}$$

$$Y = 1 - e^{-R^2/2}$$

$$e^{-R^2/2} = 1 - Y$$

$$-R^2/2 = \ln(1-Y)$$

$$R = \sqrt{-2\ln(1-Y)}$$

$$R = \sqrt{-2\ln(Y)}$$

Alg Pick $u, v$ uniform $[0,1]$ (In polar) return $(\theta, r)$ where

$$\theta = 2\pi u$$

$$r = \sqrt{-2\ln(v)}$$

Or return: $x = r\cos\theta$

$y = r\sin\theta$

Do we need to compute $\cos\theta$ & $\sin\theta$?

# Yet another Normal Dist Alg

Let $B \equiv \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

The unit ball

Note: $\text{Area}(B) = \pi$

Consider RV $D$

$\text{Prob}[D = (x,y) \in B] = 1/\pi$

$\text{Prob}[D = (x,y) \notin B] = 0$

Alg $UB[u,v]$ where $u$ & $v$ are uniform $[0,1]$

1) Set $U = 2U-1$ & $V = 2V-1$
"$U$ & $V$ are uniform $[-1,1]$"

2) Set $W = U^2+V^2$

3) Return $(u,v)$ if $W \leq 1$
else restart (Try again)

Observe that

$(u,v)$ is uniform over $B$.

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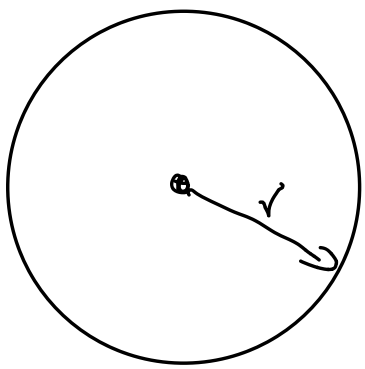
What can we say about $w$?

Lets start with the RV $\sqrt{w}$.

<u>Claim</u> PDF of $\sqrt{w}$ is $2r$.

<u>pf</u>



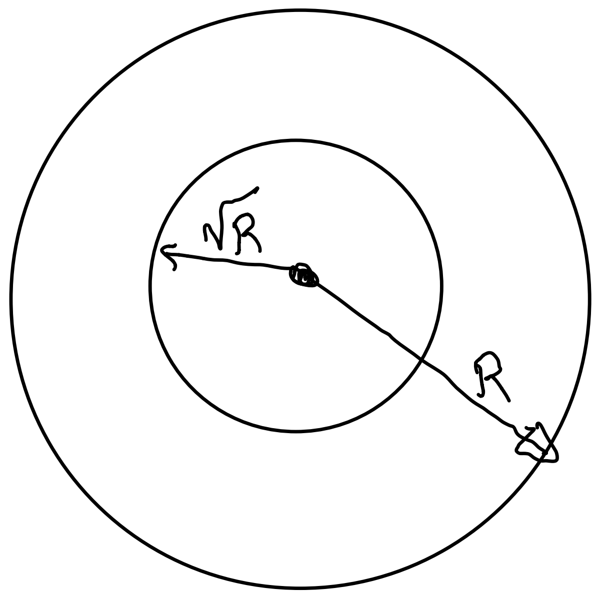

$$PDF(w) = \frac{2\pi r}{\pi}$$

Thus $CDF(\sqrt{w}) \equiv \int_0^R 2r\,dr = R^2$

<u>Consider</u>: RV $W = (\sqrt{w})^2$

Question How do we get CDF of $W$ from $\sqrt{w}$

We look at the preimage




$$CDF_W(R) = CDF_{\sqrt{W}}(\sqrt{R}) = R$$

Thus $W$ is uniform $[0,1]$

Note $(U,V)/\sqrt{W}$ is uniform on unit circle

## The Box-Muller Alg

Alg BM(u,v) where u & v are uniform [0,1].

1) Set $U = 2u-1$ & $V = 2v-1$ (uniform [-1,1])

2) Set $W = U^2 + V^2$

   if $W > 1$ then restart.

Note: (U,V) is uniform over unit disk.

3) Set $A = \sqrt{\frac{-2 \ln W}{W}}$ (scaling)

4) Return $T_1 = AU$ & $T_2 = AV$

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Claim: BM generates 2D unit Gaussian.

After step 2

1) $(U,V)$ is uniform over $B$.

2) $\left(U/\sqrt{W}, V/\sqrt{W}\right)$ is uniform over unit circle.

3) $W$ is uniform [0,1]

Since $W$ is $[0,1]$ uniform

The Radial RV $R$ is just

$$R = \sqrt{-2\ln W}$$

For the angle RV $\Theta$

we need $\cos\Theta$, $\sin\Theta$

But these are

$$U/\sqrt{W} \;\&\; V/\sqrt{W}$$

Thus we return

$$\left(\frac{R}{\sqrt{W}}U, \frac{R}{\sqrt{W}}V\right) \qquad \square$$