A preface to Geometry

15-451
10/27/20

Cartesian Coordinates (René Descartes 1637)

d-dimensional space $\equiv$ vectors of d real numbers

the fundamental operation

Dot Product $p, q \in \mathbb{R}^d$

$$\langle p, q \rangle = p \cdot q = p^T q = \sum_{i=1}^{d} p_i q_i$$

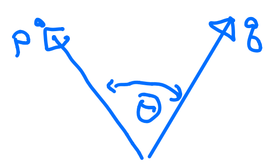
Properties

1) symmetric: $p^T q = q^T p$
2) bilinear: $(\alpha p)^T q = \alpha (p^T q)$
3) rigid motion invariant

$M \in \mathbb{R}^{d \times d}$ st $M^T M = I$ then

$$(Mp)^T (Mq) = p^T M^T M q = p^T q$$

4) Euclidean length $\equiv |p| = \sqrt{p^T p}$



5) Cosine $p^T q = |p| \cdot |q| \cdot \cos\theta$

p orthognal q iff $p^T q = 0$

6) Cauchy-Schwartz (1821)

$$(p^T q)^2 \le (p^T p)(q^T q)$$

Linear Programming in Fixed Dim

Goal: 1) Intro LP
2) Random Incremental 2D Alg

Def LP $\equiv \max c^T x$
subject $Ax \leq d$ where

$A^{n \times m} \in \mathbb{R}^{n \times m} \wedge x^{m \times 1} \wedge c^{m \times 1} \wedge d^{n \times 1}$

Note $x \leq y$ if $\forall i\ x_i \leq y_i$

Def The LP is feasible if
$\exists x\ Ax \leq d$

Note $\{x \mid Ax \leq d\}$ the feasible region
is convex $\equiv$ closed under convex comb. (Check)

2D case

$$\begin{pmatrix} a_1 & b_1 \\ \vdots & \\ a_n & b_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

Geometric View

Claim: $h_i \equiv \{(x,y) \mid a_i x + b_i y \le d_i\}$ is Half-plane or Half-Space

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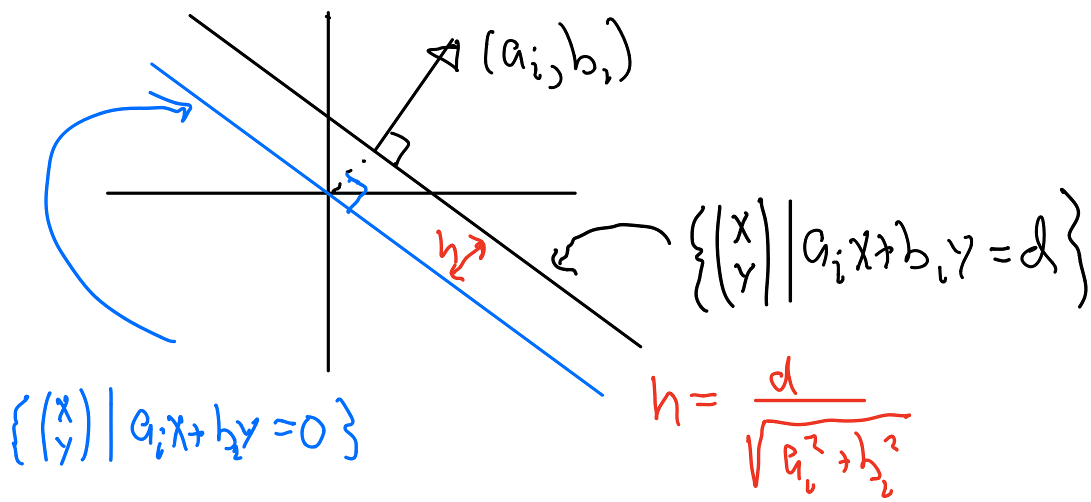
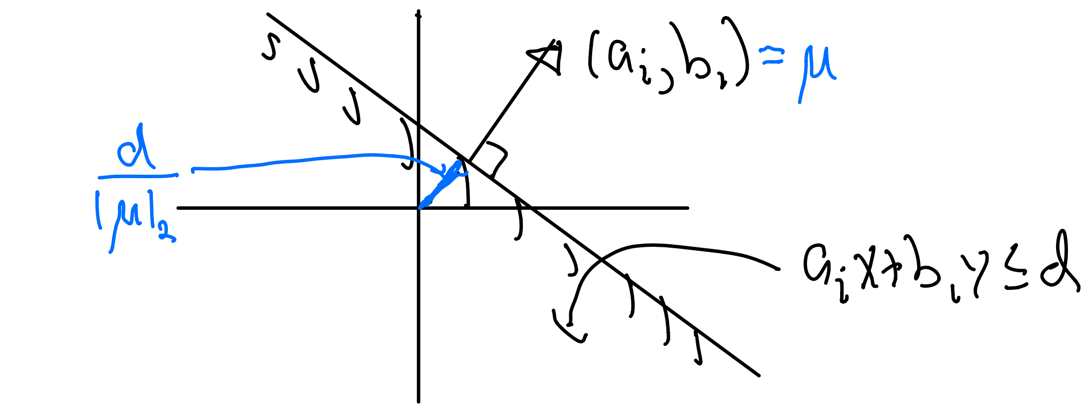
Consider vector (a_i, b_i)



Thus: h_i is half-plane normal to (a_i, b_i)



A Geometric View of 2D-LP

Input: to 2D-LP

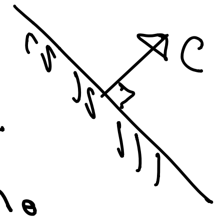
1) Half-planes $\{h_1, \dots, h_n\}$
2) vector $c \in \mathbb{R}^2$

Goal: find $x \in \bigcap_{i=1} h_i$ farthest in c direction.

Def: $\partial h_i \equiv \mathrm{bdry}(h_i)$

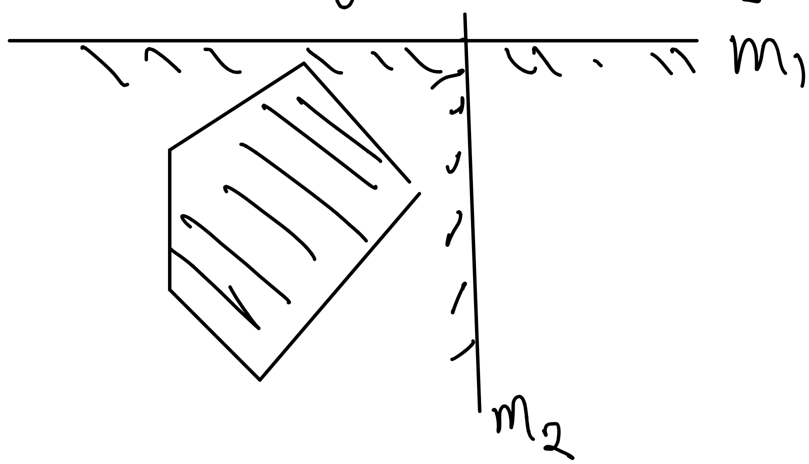
Simplification to 2D-LP

1) No h_i normal to c ie



2) Bounded feasible region.
3) Given a "bding" box m_1 & m_2

i.e.



$O(n \log n)$	$O(n)$
Sorting	Selection
Convex Hull	LP (fixed dim)
Half-Space Inter	?
Meshing	?

The 1D-LP Problem

Input: Constraints $a_i x \leq b_i$ $\quad a_i \neq 0$

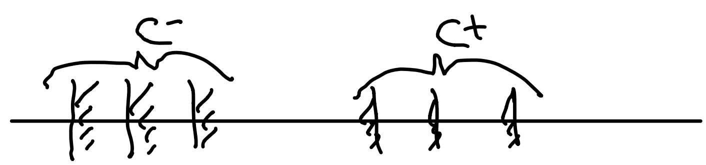
Goal: Max x satisfing constraints. $\quad$ Sign$(c) = \pm 1$

Note WLOG $a_i = \pm 1$

2-Types of contraints.

$C^+ = \{ i \mid x \leq b_i \}$

$C^- = \{ i \mid -x \leq b_i \}$ or $\{ i \mid -b_i \leq x \}$



Let $\alpha = \max \{ -b_i \mid i \in C^- \}$

$\beta = \min \{ b_i \mid i \in C^+ \}$

Note: feasible iff $\alpha \leq \beta$

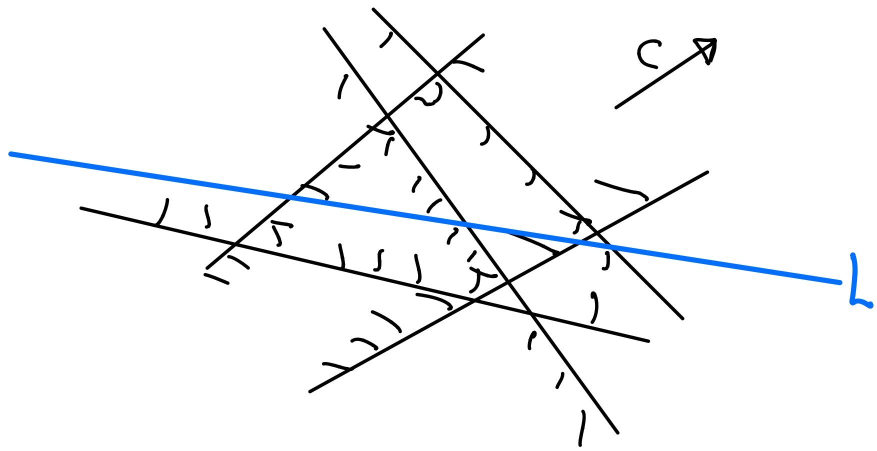
Return β if feasible

Thus

Thm 1D-LP is $O(n)$ time.

Solving a 1D-LP as a Restriction of a 2D-LP

Given 2D-LP $Ax \le d$ & $c \in \mathbb{R}^2$



Random Incremental 2D-LP

Proc: 2D-LP($m_1, m_2, h_1, \ldots, h_n, c$) ($m_1$ & m_2 form bdin box)

1) $V_0 \leftarrow$ 2D-LP(m_1, m_2, c)

   ie $V_0 = \partial m_1 \wedge \partial m_2$

2) Randomly order $h_1, \ldots, h_n$

3) for $i=1$ to n do

   if $V_{i-1} \in h_i$ then $V_i \leftarrow V_{i-1}$ (*)

   else (Make & solve 1D-LP prob)

   Let $L = \partial h_i$

   $L_1' = L \cap h_1, \ldots L_{i-1}' = L \cap h_{i-1}$

   $C' =$ projection(c, L) note $c' \neq 0$

   $V_i =$ 1D-LP$(L_1', \cdots, L_{i-1}', c')$ (**)

   if V_i is "undef" report "no solution" & halt.

Correctness

Claim: At time (*) or (**)

$$V_i = LP(m_1, m_2, \cdots h_i, c)$$

pf Induction on i

Base case OK

Suppose V_{i-1} is correct.

Case a: $V_{i-1} \in h_i$ then

$V_{i-1} \in$ feasible region and thus opt.

Case b: $V_{i-1} \notin h_i$

Claim $V_i \in \partial h_i \equiv L.$

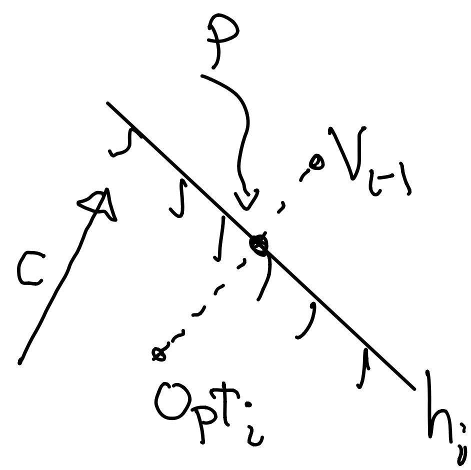
Picture Proof

$c^T V_{i-1} \geq c^T Opt_i$ (Objective is nonincreasing)

$\Rightarrow$ 1) $c^T p \geq c^T Opt_i$

2) p feasible

$\therefore p = Opt_i$



2D-LP Timing

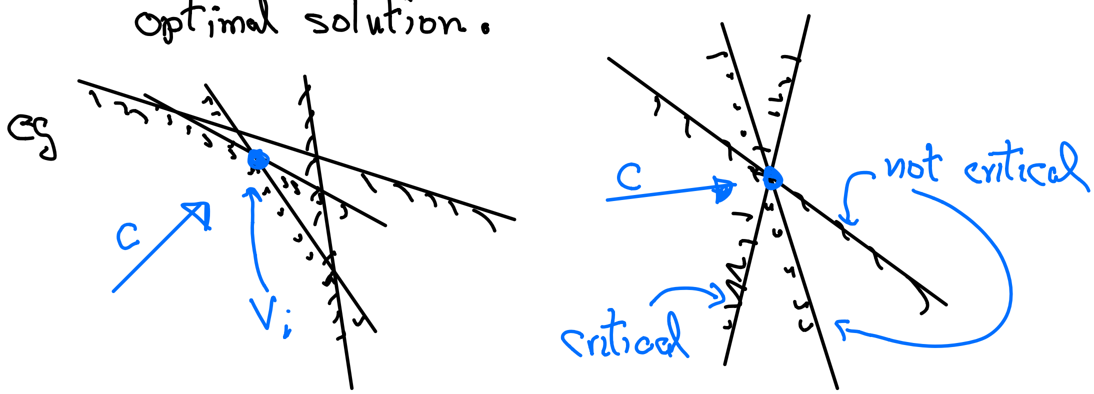
Claim 2D-LP is $O(n)$ expected time.

pf We use backwards analysis.
We start with all the constraints and randomly remove one at a time.

Suppose we remove h_j from $h_1, \ldots, h_i$.

Def h_j is critical if removing it changes optimal solution.

eg



Note: At most 2 critical contraints

Backward cost:

$$BW\text{-}Cost(h_j) = \begin{cases} k & \text{if } h_j \text{ not critical constraint.} \\ k \cdot i & \text{o.w.} \end{cases}$$

$$Forward\text{-}Cost = \begin{cases} K & \text{if } (*) \\ K \cdot i & \text{if } (**) \end{cases}$$

Thus worst case for step 3) is exactly 2 critical constraints.

Let $E_i \equiv$ Expect cost of step 3) at time i

$$E_i \leq \left(\left(\frac{2}{i}\right) k \cdot i + \left(\frac{i-2}{i}\right) k \right) \quad \text{some constant } k$$

$$\leq 2k + k = 3k$$

Thus total expected work is

$$\sum_{i=1}^{n} 3k = O(n).$$

Determining Unbdness or Finding the bding box

LP $\equiv$ Max $C^T X$ subject $AX \leq b$

<u>Lemma</u> LP is unbounded iff

1) it is feasible
2) $\exists d$ s.t. $C^T d > 0$ & $Ad \leq 0$

<u>pf</u> ($\Leftarrow$) By 1) $\exists \bar{x}$ $A\bar{x} \leq b$

pick any $\alpha \geq 0$ & d by 2).

Now: $A(\alpha d + \bar{x}) \leq \alpha Ad + A\bar{x} \leq \alpha Ad + b \leq b$

thus $\alpha d + \bar{x}$ is feasible $\forall \alpha \geq 0$.

$C^T(\alpha d + \bar{x}) = \alpha C^T d + C^T \bar{x}$

objective goes to ∞ with α.

($\Rightarrow$) compactness argument

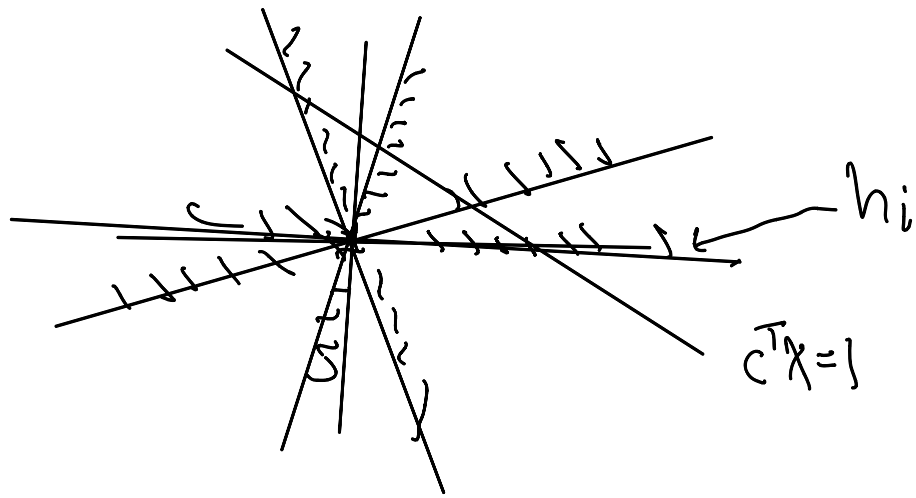
<u>Lemma</u> $\exists d$ $Ad < 0$ then $Ax \leq b$ is feasible

<u>pf</u> $V = Ad$ then $V < 0$ pick $\alpha > 0$ st $\alpha V \leq b$ thus $A\alpha d \leq b$.

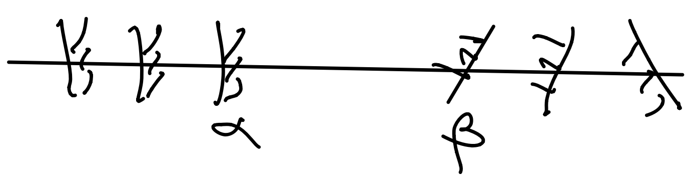
Finding d for unbdness

d exists iff $\exists d$ s.t. $c^T d = 1$ & $Ad \leq 0$

Our feasible region is $Ax \leq 0$
we intersect it with line $c^T x = 1$



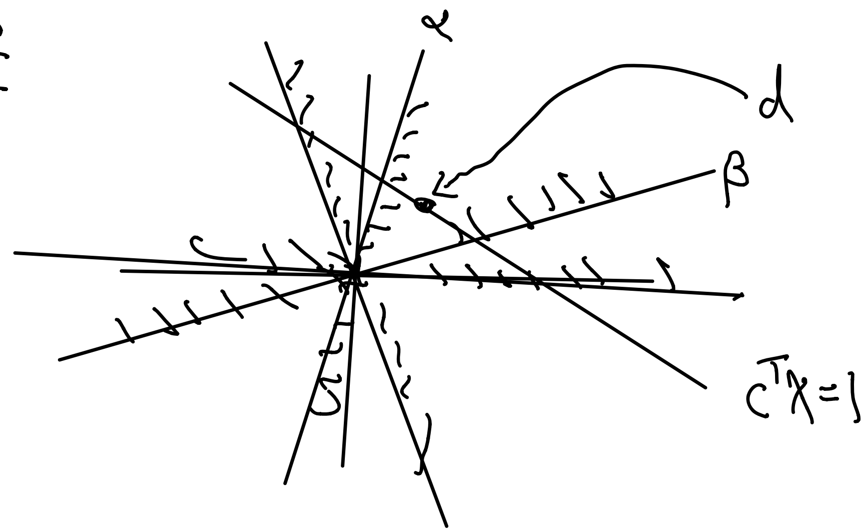
this is a 1D-LP



Note $\exists d$ iff $\beta \geq \alpha$

<u>Claim</u> If $\beta > \alpha$ then $Ax \le b$ is feasible and thus $c^T x$, $Ax \le b$ unbded.

<u>Pf</u>



Pick d as in figure then $Ad < 0$

Thus $Ax \le b$ is feasible.

There are 3 cases

1) $\alpha < \beta$ 2) $\alpha > \beta$ 3) $\alpha = \beta$

Case 1) $\alpha < \beta$ return "unbded"

Case 2) $\alpha > \beta$ return bounding box:

Assume one halfspace corresponding to α
" " β

Say h_α and h_β
Return BB $\{h_\alpha, h_\beta\}$

Case 3) $\alpha = \beta$

<u>Subcase a</u> $h_\alpha \cap h_\beta = \emptyset$

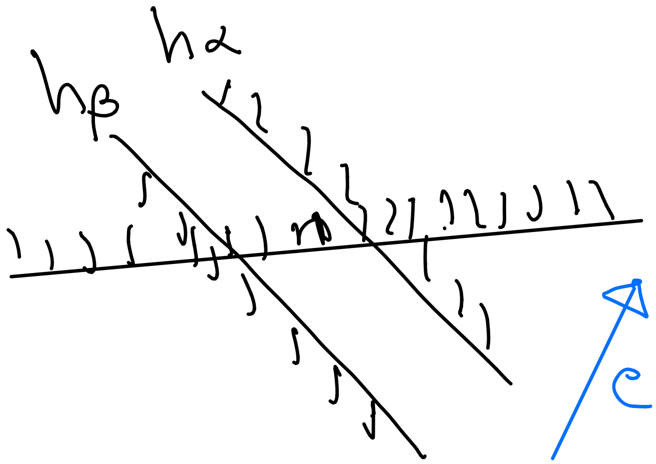
$Ax \leq b$ not feasible

<u>Subcase b</u> $h_\alpha \cap h_\beta \neq \emptyset$

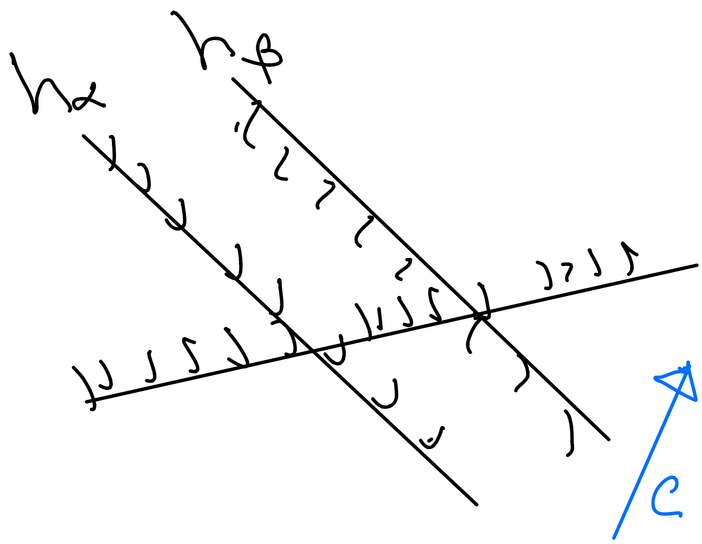
<u>Claim</u> $Ax \leq b$ feasible thus

$Ax \leq b$ $c^T x$ unbded.

Case a



Case b



LP for fixed dim

Proc: $d\text{-}LP(h_1, \dots, h_n, c)$

0) Run Boundedness$(h_1, \dots, h_n, c)$

if unbdd then quit.

else pick Bounding Box $\{g_1, \dots, g_d\} \subseteq \{h_1, \dots, h_n\}$

Set $V_0 = \bigcap \partial g_i$

1) Let $h_{d+1}, \dots h_n$ be random order of $H - G$

2) For $i = d+1$ to n do

if $V_{i-1} \in h_i$ then $V_i \leftarrow V_{i-1}$

else (Make & Solve $(d-1)$-LP prob)

if "undef" report "no solution" & halt

else $V_i \leftarrow (d-1)$-LP

<u>Claim</u> d-LP is expected runtime
$C_d \cdot n$ where $C_d = O(2^d \cdot d!)$

We know $C_2 < \infty$

Suppose $C_{d-1} < \infty$ i.e.
$C_{d-1} \cdot n$ bds expected cost to
make & solve $(d-1)$-LP.

<u>Note</u> At most d critical constraints.
Thus by BA we get

$$\text{Cost}(h_i) = \begin{cases} d K_d & \text{if } h_i \text{ not critical} \\ C_{d-1}\, i & \text{o.w.} \end{cases}$$

$$E_i = \frac{\left(d\, C_{d-1} \cdot i + (i-d) K_d\right)}{i} \leq d\, C_{d-1} + d K_d$$

Total Expected $\leq d \cdot n \left(C_{d-1} + K_d \right)$

Assuming $K_d < C_{d-1}$

$C_d \leq d \cdot 2\, C_{d-1}$

$C_d = O(2^d \cdot d!) = O(d!)$