Lecture 16 – Sherali-Adams

Oct 27

Recently seen
- Algebraic proof systems (NS, PC, Ideal Proof System)
- Semi-Algebraic proof systems (Cutting Planes)

Two more examples: Sherali-Adams and Sum-of-Squares

- Natural optimization algorithms that correspond to these systems

Let $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be an unsatisfiable CNF formula.

We can encode F either as polynomial equations (e.g. algebraic pfs) or linear inequalities (e.g. Cutting planes)

Defn Let P, Q be sets of polynomials over $\mathbb{R}$ in variables $x_1, x_2, \cdots, x_n$. We assume Q contains the polynomials

- $x_i^2 - x_i$
- $x_i - x_i^2$
- 1

For any $S, T \subseteq [n]$, $S \cap T = \emptyset$, let

$$J_{S,T} := \prod_{i \in S} x_i \prod_{j \in T} (1 - x_j)$$ ← "non-negative" junta

A Sherali-Adams proof of the polynomial q from P and Q is an expression of the form

$$\sum_{\substack{p=0 \\ p\in P}} h_p p + \sum_{\substack{r\geq 0 \\ r\in Q}} \left(\sum c_{ir} \mathfrak{Z}_{ir}\right) r = q$$

where each $c_{ir} \in \mathbb{R}_{\geq 0}$, $\mathfrak{Z}_{ir}$ is a non-negative junta, and each h_p is any polynomial.

i.e. if $P := \{p_1, \dots, p_m\}$ and $Q := \{r_1, \dots, r_\ell\}$ then an SA proof is a proof of the inequality

$$q \geq 0$$

from the system

$$\begin{matrix} p_1 = 0 & & r_1 \geq 0 \\ \vdots & \text{and} & \vdots \\ p_m = 0 & & r_\ell \geq 0 \end{matrix}.$$

A Sherali-Adams refutation is a proof of -1

- The degree of the refutation is
$$\max\{\deg(h_p p), \deg(\mathfrak{Z}_{ir} r)\}$$
- The size of the refutation is the number of monomials obtained by expanding out all polynomials before cancellations.

By definition, a Nullstellensatz refutation of P over $\mathbb{R}$ is also a Sherali-Adams refutation of $(P, \{x_i^2 - x_i, x_i - x_i^2\})$

Thm [Assignment 3]

Sherali-Adams can size- and degree- simulate Resolution!

If there is a Resolution proof of size s and width w, then there is a Sherali-Adams refutation with size $\text{poly}(s,w)$ and degree w.

ex] Let $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be an unsat CNF formula, for any clause

$$C = \bigvee_{i \in S} x_i \vee \bigvee_{j \in T} \overline{x_j}$$

let $\tilde{C} = \sum_{i \in S} x_i + \sum_{j \in T} (1 - x_j)$, and (over $\{0,1\}$ solns)

we can encode C as

$$\tilde{C} \geq 1.$$

$x_i^2 - x_i = 0$ encoded as

$x_i^2 - x_i \geq 0$

$x_i - x_i^2 \geq 0$

A SA proof would look like

$$\sum_{i=1}^{m} \Big(\sum_{k_i} c_{k_i} J_{k_i}\Big)(\tilde{C}_i - 1) + \sum_{i=1}^{n} y_{i,1}(x_i^2 - x_i) + y_{i,2}(x_i - x_i^2) + \underset{(1 \geq 0)}{y_0} = -1$$

where $y_0, y_{i,1}, y_{i,2}$ are non-negative combinations of non-negative juntas $J_{S,T}$.

ex) $F = x_1 \wedge (\bar{x}_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3) \wedge \bar{x}_3$

$$x_1 \geq 1 \qquad (1-x_1)+x_2 \geq 1 \qquad (1-x_2)+x_3 \geq 1 \qquad 1-x_3 \geq 1$$

along with $x_i^2 - x_i \geq 0$, $x_i - x_i^2 \geq 0$, $1 \geq 0$

$$\begin{aligned} & 1 \cdot (x_1 - 1) + 1 \cdot (1 - x_1 + x_2 - 1) \\ & + 1 \cdot (1 - x_2 + x_3 - 1) \\ & + 1 \cdot (1 - x_3 - 1) \\ & = -1 \end{aligned}$$

$P = \emptyset$

$Q = \{ x_1 - 1, (1-x_1)+x_2-1, (1-x_2)+x_3-1, (1-x_3)-1, x_i^2 - x_i, x_i - x_i^2, 1 \}$

Already stronger than NS! NS requires degree $\Theta(\log n)$ to refute over any field (for the formula with n variables).

What about other proof systems?

- SA is definitely stronger than NS over $\mathbb{R}$
- On the assignment, prove that SA can simulate Resolution
- SA vs. Polynomial Calculus over $\mathbb{R}$? Separations in both directions!
- SA vs. Cutting Planes?
  - There is a formula that requires $\Omega(n)$-degree SA with short CP

proofs. (PHP^{n+1}_n !)

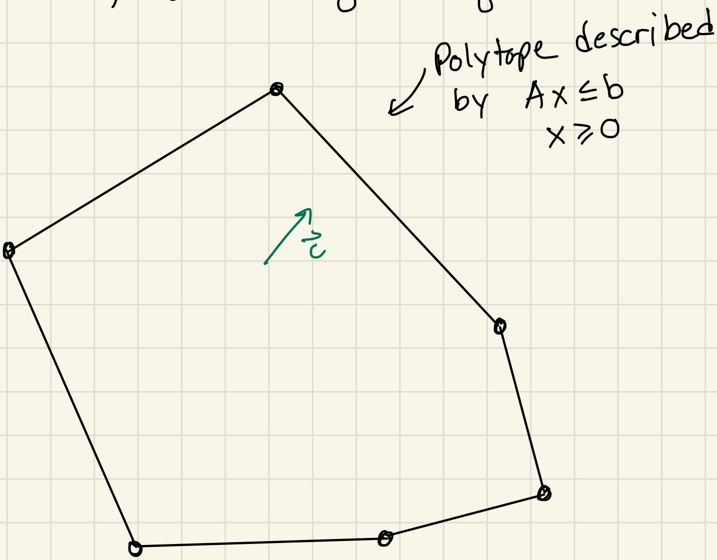
Open Problem: Find a CNF F s.t. SA has efficient proofs (size/degree) while CP requires long proofs.

Thm Sherali-Adams requires degree $\Omega(\text{expansion}(G))$ to refute $Tsei_G$.

The Linear Programming Perspective

Recall the definitions of Linear Programming:

$$\max \quad c \cdot x$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0$$



Linear Programs are often used to give approximation algorithms for NP-Hard problems.

ex] Max-SAT: Given $F = C_1 \wedge \cdots \wedge C_m$ CNF, find x maximizes the # of satisfied clauses.

Integer LP for Max-SAT

$$\max \sum_{i=1}^{m} c_i$$

$$\text{s.t.} \quad \tilde{C}_i \geq c_i \quad \text{for all } i = 1 \cdots m$$

$$0 \leq x_i \leq 1 \quad \text{for all } i = 1 \cdots n$$

$$0 \leq c_i \leq 1 \quad \text{for all } i = 1 \cdots n$$

$$x_i, c_i \in \mathbb{Z}$$

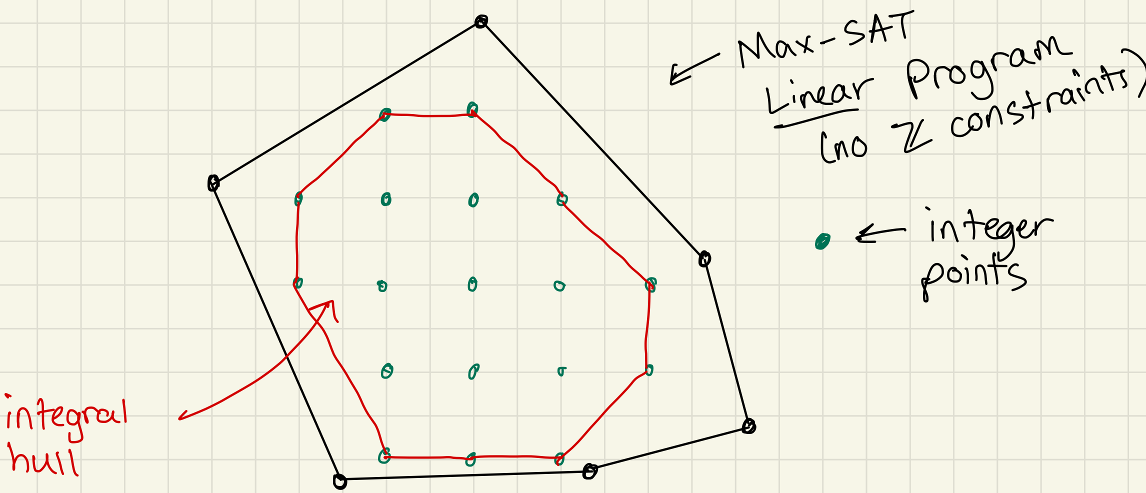
$S, T \subseteq [n]$, $S \cap T = \emptyset$

$$C = \bigvee_{i \in S} x_i \vee \bigvee_{j \in T} \overline{x_j}$$

$$\Rightarrow \tilde{C} := \sum_{i \in S} x_i + \sum_{j \in T} (1 - x_j)$$

C sat iff $\tilde{C} \geq 1$

Obtain the LP relaxation by removing $\mathbb{Z}$ constraint.



Sherali-Adams can be used to systematically add new constraints