Balanced Search Trees

- rotations
- AVL trees
- 2-3 trees

Background

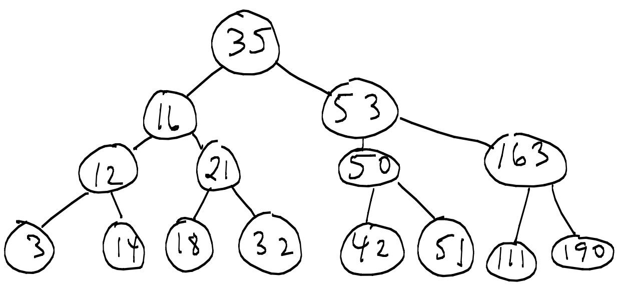
- COMP 250 binary search trees

Resources for this lecture

- rotations, AVL trees (slides only)
- 2-3 trees - Sedgewick 1

https://class.coursera.org/algs4partI-003/lecture/49

Binary Search Tree (Best Case)



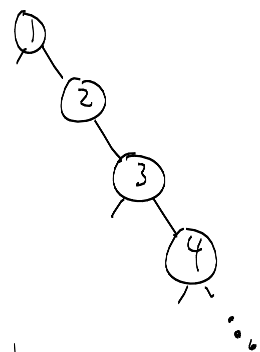
Tree is balanced.

Depths of all nodes are $O(\log n)$.

How to insert a key e.g 49?

Binary Search Tree (Worst Case)

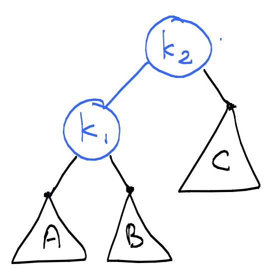
If keys are inserted in order then the BST behaves as a linked list and worst case searching for a key is $O(n)$.



We would like our BSTs to be "balanced". There are several ways to define "balanced".

The main goal is to keep the depths of all nodes to be $O(\log n)$.

Suppose we have

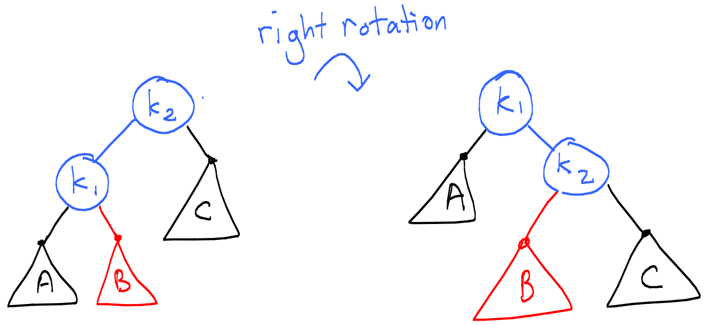


- k_1, k_2 are keys
- A, B, C are sub trees (of unspecified shape)

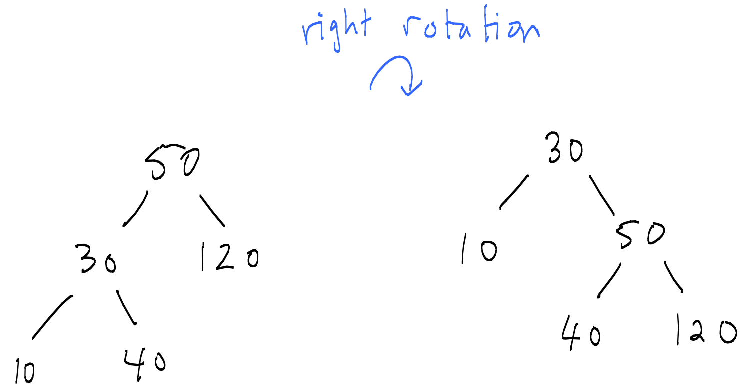
$$A < k_1 < B < k_2 < C$$

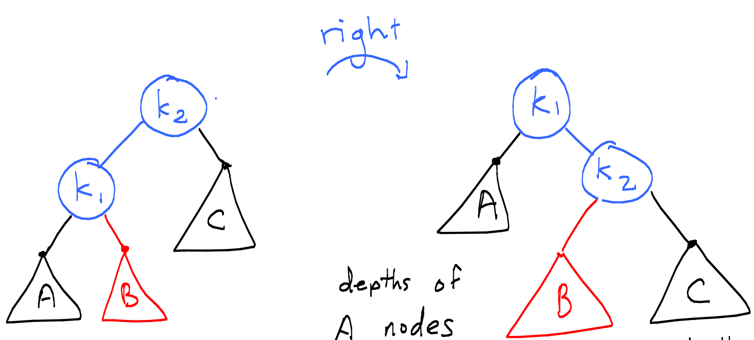
All keys in A are less than key k1. k1 is less than all keys in B, which are less than k2. k2 is less than all keys in C.

Rotation



$A < k_1 < B < k_2 < C$

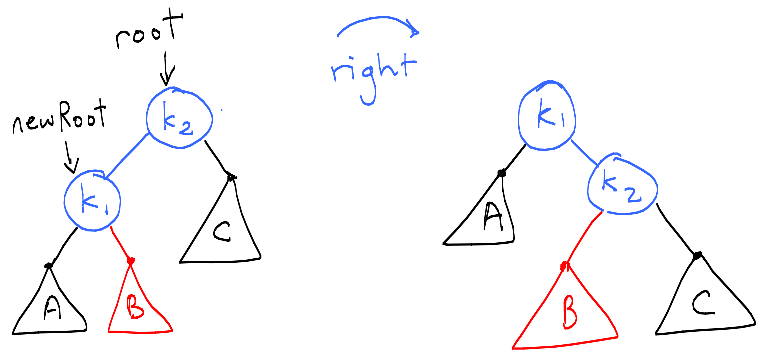




depths of A nodes decrease by 1

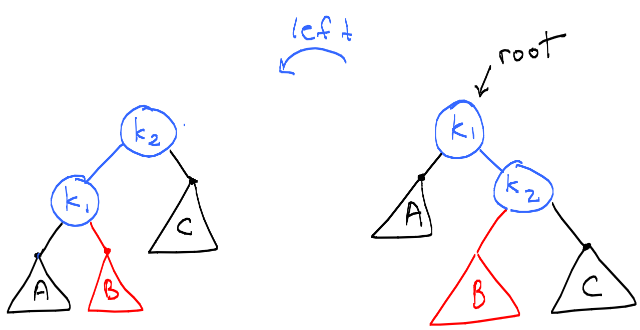
no change in depth of B nodes

depths of C nodes increase by 1



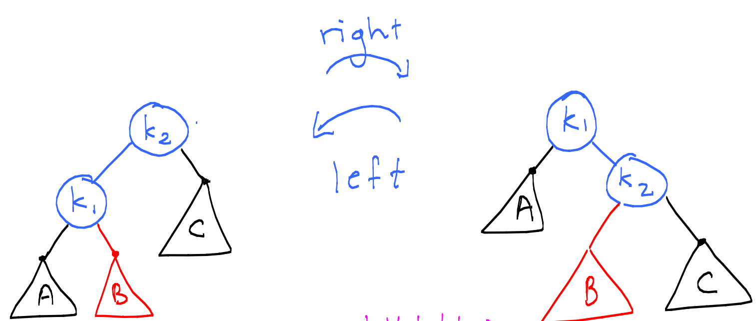

```
rotateRight( root ){
    newRoot =  root.left
    root.left  = newRoot.right
    newRoot.right = root
    return newRoot
}
```

argument and returned value are nodes, not keys



Exercise:

```
rotateLeft( root ){
      :
}
```

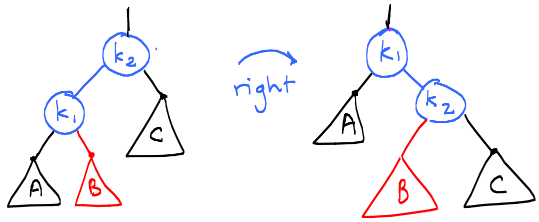


slight butchering of notation here (key argument vs node)

rotateRight (k_2)
rotateLeft (k_1) } gives back the original tree

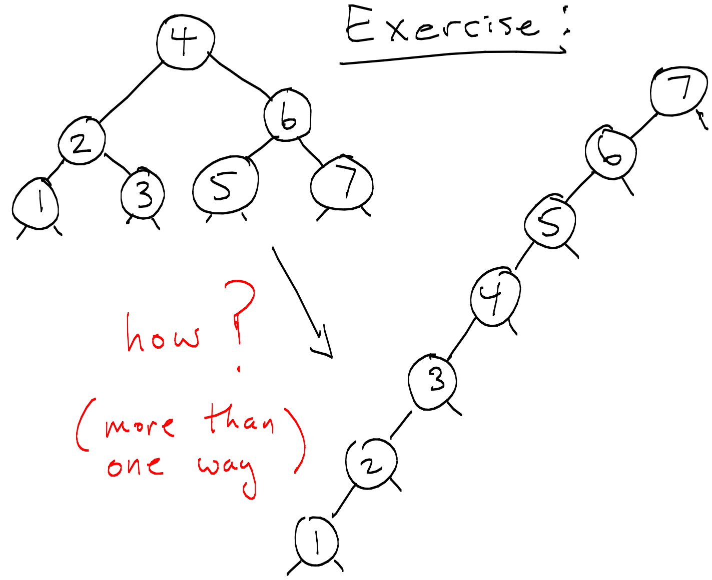
notice keys differ

What if k_2 is not the root of the tree i.e. what if k_2 has a parent?



Exercise: rotateRight(k_2) maintains the BST property for the whole tree. Why?

Exercise:

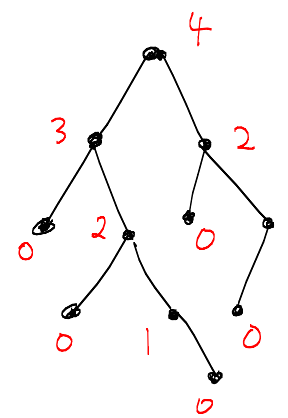


Balanced search trees

- AVL trees
- red-black trees used in Java's TreeMap class. If you are interested, see https://class.coursera.org/algs4partI-003/lecture/50
- 2-3 trees

Recall from COMP 250:

the height of a node in a binary tree is the length of longest path from that node to a leaf.



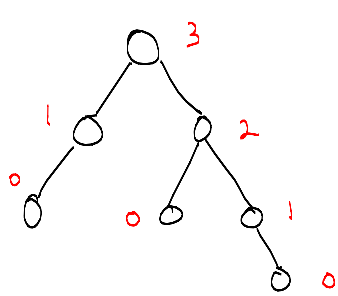
AVL tree

[Adelson-Velskii, Landis 1962]

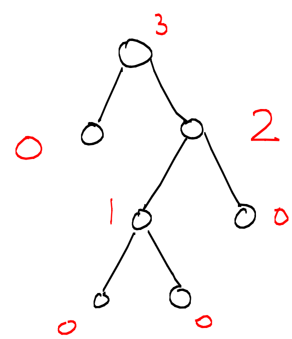
Balance Condition:

For any node in an AVL tree, the height of its left subtree differs by at most 1 from the height of its right subtree.

Examples: valid AVL or not?

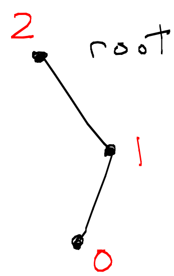


yes



no

Weird but common example



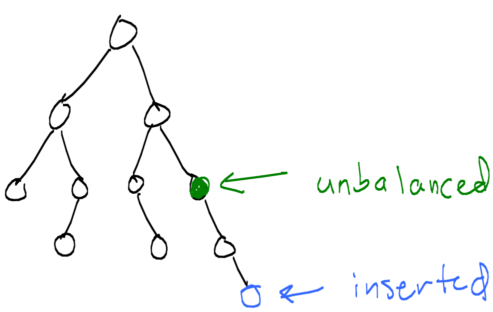
root's left child is null

Define the height of an empty tree to be -1.

Because of time constraints, we will cover only insertion into AVL trees, but we ignore deletion.

(Recall from COMP 250 that deletion from a BST is trickier than insertion. This is the case for AVL trees too. See wikipedia for details if you are interested.)
http://en.wikipedia.org/wiki/AVL_tree

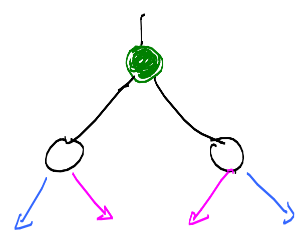
Suppose we have an AVL tree and we do an **insertion** that causes the heights of the left and right subtrees of some node to differ by 2. How can we rebalance the tree?



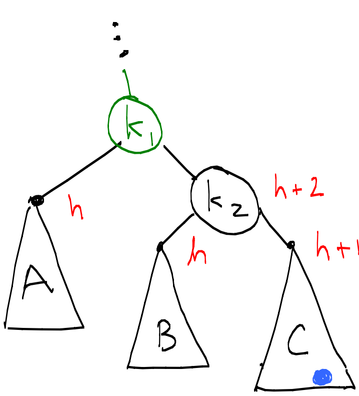
There are four ways (two pairs of ways) that the imbalance could have occured, namely the insertion was:

- to the left subtree of the left child (outside)
- to the right subtree of the right child (outside)

- to the right subtree of the left child (inside).
- to the left subtree of the right child (inside)



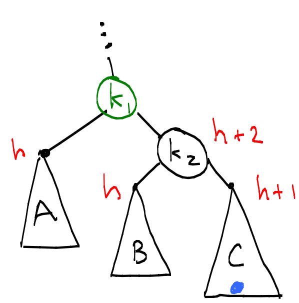
"outside"



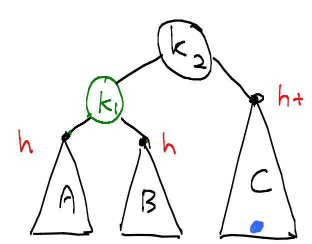
e.g. Insertion was in tree C and extended C's height from h to h+1, creating imbalance at subtree rooted at k_1 (but not k_2).

Question: How to rebalance k_1?

Answer: rotateLeft(k_1)

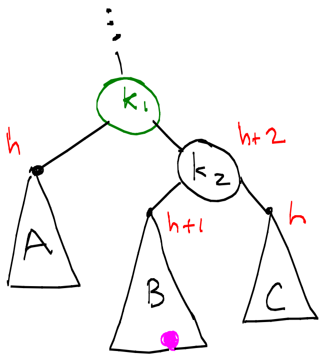


BEFORE



AFTER (balanced)

"inside"



e.g. Insertion was into tree B, extending its height from h to h+1, and creating imbalance at subtree rooted at k_1

Exercise: How to rebalance k_1 ?

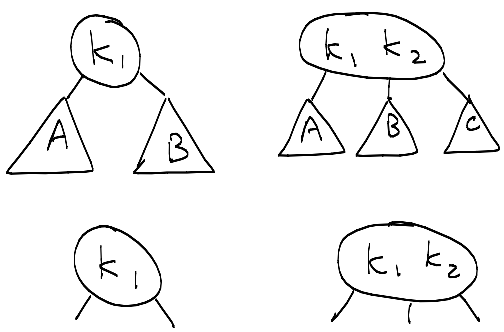
2-3 trees

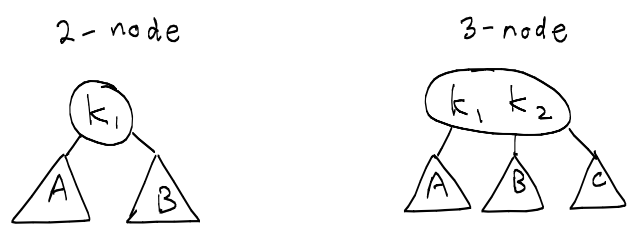
(balanced search trees that don't use rotations)

Each node of a 2-3 tree either has 1 or 2 keys (called a "2 node" or "3 node", respectively, i.e. number of children)

internal

leaf



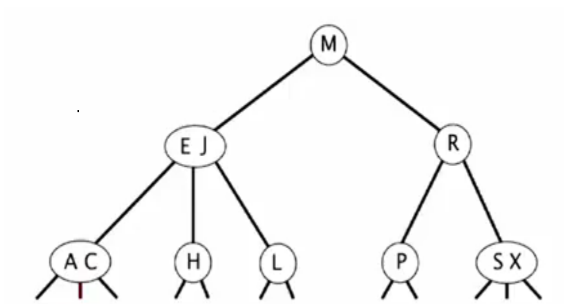


search tree order condition: $A < k_1 < B < k_2 < C$

balance condition: height(A) = height(B) = height(C)

Searching (find) in a 2-3 tree uses same idea as BST.

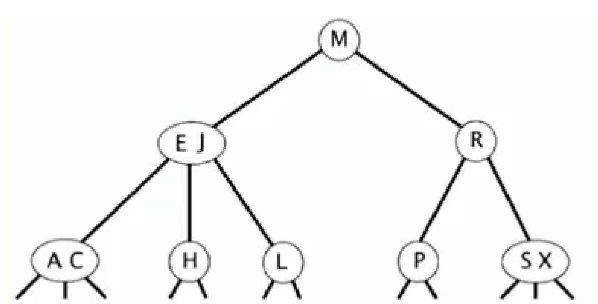
https://class.coursera.org/algs4partI-003/lecture/49



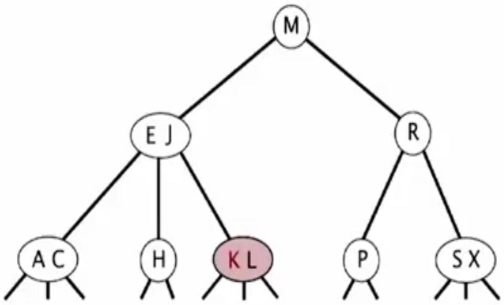
e.g. find(H)

find(B) ... fail

Insertion of a key always occurs into a leaf node



insert(k)

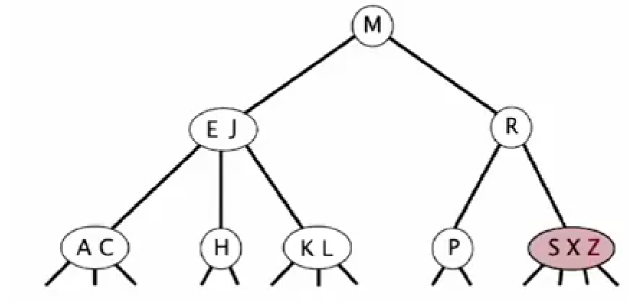


The problem comes when the leaf node already is a 3-node (i.e. two keys present already)

e.g. insert (z)

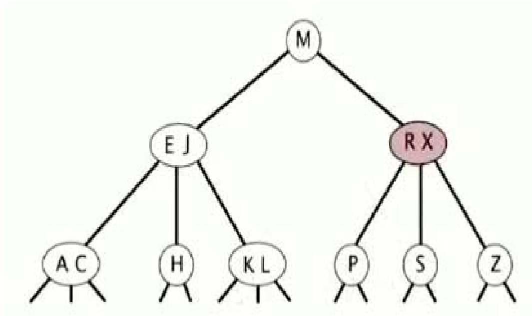
Inserting a key into 3 node makes a temporary 4- node

(3 keys, 4 empty child refs)



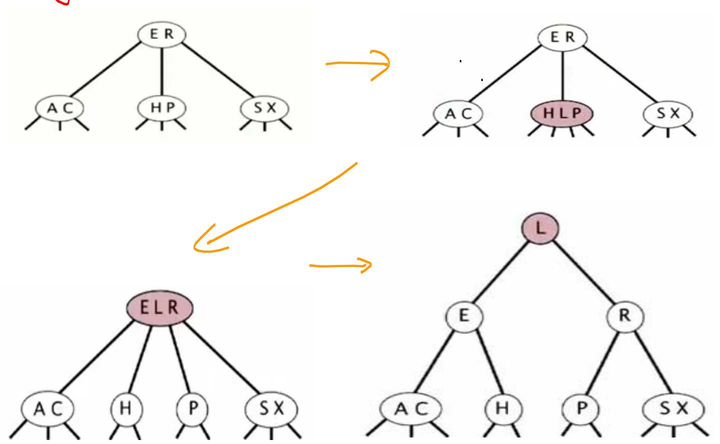
What to do ?

Split 4-node into two 2-nodes and move middle key (X) to parent.



Now we are balanced again !

e.g. insert (L)



Height of tree has increased by 1.

Summary Insertion in 2-3 tree

two phases :

- downward (search for leaf node where key is inserted)
- upward (split if necessary)

Next lecture :

hash tables

Prepare by reviewing my COMP 250 notes on this topic.
(or M. Blanchette's slides)