15-451
10/22/20

Quick Sort & Backward Analysis

We consider yet another analysis of QS.

Recall: QS(M) (distinct keys)

1) Pick random $a \in M$
2) Split M into: $S < a < L$ $\left(\begin{array}{c}|M|-1 \\ \text{comparisons}\end{array}\right)$
3) return QS(S) * a * QS(L)

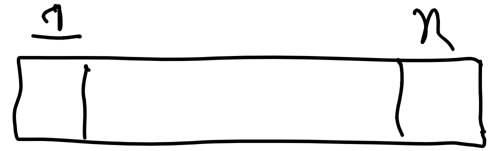
Note: The two calls can be done in parallel or interleaved.

Goal: Bound Expect # of comparisons.

Claim: Expect cost of dart game is the same as QS.

Game: DG ≡ Dart Game

Init: empty board

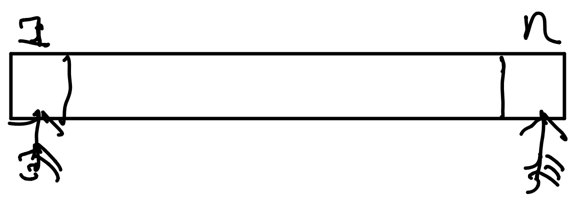


While ∃ an empty square
  Pick random empty sq.
Cost ≡ # empty sqs to left and right of dart.

Claim: Expect cost of dart game
  is the same as QS.

Game: BW-DG Backward Dart Game

Init: Full board of darts

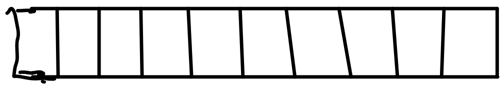


While ∃ dart remove a random one.
Cost: # empty sqs to left & right
  of dart.

Claim: $\mathbb{E}[\text{Cost}(DG)] = \mathbb{E}[\text{Cost}(BW\text{-}DG)]$

Forward Game

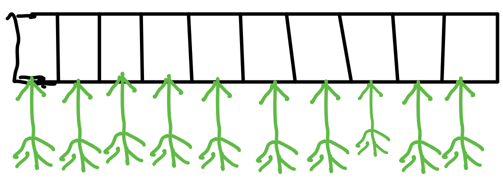
Darts #



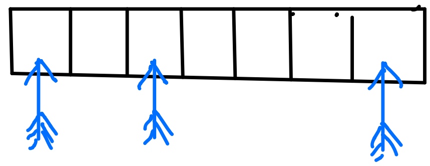
Cost

Backwards Game

Darts #



Cost



Cost = 1 4 3

Total Cost = 8 $\quad \mathbb{E}(\text{cost}) = 8/3$

Analysis of BW-DG

Assume there are i darts on board.

Note: Total Cost of i darts $\le 2(n-i) \le 2n$

Expect cost $\le 2n/i$

Proof using random variables:

Consider Random variables:

$RV_i \equiv$ cost of removing ith dart.

$$S_j^i \equiv \begin{cases} 1 & \text{if 1) } Sq_j \text{ is empty \& } \\ & \text{2) closest dart to left or right} \\ & \text{is removed.} \\ 0 & \text{o.w} \end{cases}$$

Now $Pr\left[S_j^i = 1\right] \leq \frac{2}{i}$

$$T_i = \sum_{j=1}^{n} S_j^i$$

Note $T_i = RV_i$

Now:

$$\mathbb{E}(T_i) = \sum \mathbb{E}(S_j^i) = \sum Pr\left[S_j^i = 1\right] \leq \frac{2n}{i}$$

Consider $T = \sum_{i=1}^{n} RV_i = \sum_{i=1}^{n} T_i$

$$\mathbb{E}[T] = \sum_{i=1}^{n} \mathbb{E}(T_i) \leq \sum_{i=1}^{n} \frac{2n}{i}$$

$$= 2n H_n \approx 2n \log n$$