

# Complexity Theoretic Limitations on Learning Halfspaces

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## ABSTRACT

We study the problem of agnostically learning halfspaces which is defined by a fixed but unknown distribution  $\mathcal{D}$  on  $\mathbb{Q}^n \times \{\pm 1\}$ . We define  $\text{Err}_{\text{HALF}}(\mathcal{D})$  as the least error of a halfspace classifier for  $\mathcal{D}$ . A learner who can access  $\mathcal{D}$  has to return a hypothesis whose error is small compared to  $\text{Err}_{\text{HALF}}(\mathcal{D})$ .

Using the recently developed method of Daniely, Linial and Shalev-Shwartz we prove hardness of learning results assuming that random  $K$ -XOR formulas are hard to (strongly) refute. We show that no efficient learning algorithm has non-trivial worst-case performance even under the guarantees that  $\text{Err}_{\text{HALF}}(\mathcal{D}) \leq \eta$  for arbitrarily small constant  $\eta > 0$ , and that  $\mathcal{D}$  is supported in  $\{\pm 1\}^n \times \{\pm 1\}$ . Namely, even under these favorable conditions, and for every  $c > 0$ , it is hard to return a hypothesis with error  $\leq \frac{1}{2} - \frac{1}{n^c}$ . In particular, no efficient algorithm can achieve a constant approximation ratio. Under a stronger version of the assumption (where  $K$  can be poly-logarithmic in  $n$ ), we can take  $\eta = 2^{-\log^{1-\nu}(n)}$  for arbitrarily small  $\nu > 0$ . These results substantially improve on previously known results, that only show hardness of exact learning.

## Categories and Subject Descriptors

F.2 [Theory of Computation]: Analysis of algorithms and problem complexity

## General Terms

Hardness of Learning

## Keywords

Hardness of Learning, Halfspaces, Random XOR

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## 1. INTRODUCTION

In the problem of agnostically learning halfspaces, a learner is given an access to a distribution  $\mathcal{D}$  on  $\mathbb{Q}^n \times \{\pm 1\}$ . The goal is to output<sup>1</sup> (a description of) a classifier  $h : \mathbb{Q}^n \rightarrow \{\pm 1\}$  whose error,  $\text{Err}_{\mathcal{D}}(h) := \Pr_{(x,y) \sim \mathcal{D}}(h(x) \neq y)$ , is small comparing to  $\text{Err}_{\text{HALF}}(\mathcal{D})$  – the error of the best classifier of the form  $h_w(x) = \text{sign}(\langle w, x \rangle)$ . We say that a learning algorithm learns halfspaces if, given an accuracy parameter  $\epsilon > 0$ , it outputs a classifier with error at most  $\text{Err}_{\text{HALF}}(\mathcal{D}) + \epsilon$ . The learner is *efficient* if it runs in time  $\text{poly}(n, \frac{1}{\epsilon})$  and the output hypothesis can be evaluated in  $\text{poly}(n, \frac{1}{\epsilon})$  time given its description. The learner has an approximation ratio  $\alpha = \alpha(n)$  if it is guaranteed to return  $h$  with  $\text{Err}_{\mathcal{D}}(h) \leq \alpha \cdot \text{Err}_{\text{HALF}}(\mathcal{D}) + \epsilon$ . We emphasize that we consider the general, *improper*, setting where the learner has the freedom to return a hypothesis that is not a halfspace classifier.

The problem of learning halfspaces is as old as the field of machine learning, starting with the perceptron algorithm [56, 55], through the modern SVM [27, 61, 60, 28, 29] and AdaBoost [58, 57, 40]. Halfspaces are widely used in practice, have been extensively studied theoretically, and in fact motivated much of the existing theory, both the statistical and the computational.

Despite all that, the gap between the performance of best known algorithms and best known lower bounds is dramatic. Best known efficient algorithms [48] for the problem have a poor approximation ratio of  $\tilde{\Omega}(n)$ , and have performance better than trivial only when  $\text{Err}_{\text{HALF}}(\mathcal{D}) \leq \tilde{\Theta}(\frac{1}{n})$ . As for lower bounds, strong  $\mathcal{NP}$ -hardness results are known [8, 4, 41, 38] only for algorithms that are restricted to return a halfspace classifier (a.k.a. proper algorithms). For general algorithms, no  $\mathcal{NP}$ -hardness results are known, yet several results [34, 38, 50, 51, 44] show that it is hard to agnostically learn halfspaces under several (cryptographic and average case) complexity assumptions. However, these results are quantitatively very weak, as they only rule out exact learning (i.e., with approximation ratio 1). For example, they do not rule out algorithms that predict only 1.001 times worst than the best halfspace classifier.

The main result of this paper is a quantitatively strong hardness results, assuming that (strongly) refuting random  $K$ -XOR formulas is hard. Using the recently developed framework of the author with Linial and Shalev-Shwartz [33], we show that for arbitrarily small constant  $\eta > 0$  and every  $c > 0$ , no  $\text{poly}(n)$ -time algorithm can return a hypothe-

<sup>1</sup>Throughout, we require algorithms to succeed with a constant probability (that can be standardly amplified by repetition).

sis with error  $\leq \frac{1}{2} - \frac{1}{n^\epsilon}$ , even when it is guaranteed that  $\text{Err}_{\text{HALF}}(\mathcal{D}) \leq \eta$ , and that  $\mathcal{D}$  is supported in  $\{\pm 1\}^n \times \{\pm 1\}$ . This implies in particular, that there is no efficient learning algorithm with a constant approximation ratio. Under a stronger version of the assumption (where  $K$  is allowed to be poly-logarithmic in  $n$ ), we can take  $\eta = 2^{-\log^{1-\nu}(n)}$  for arbitrarily small  $\nu > 0$ . This implies hardness of approximation up to a factor of  $2^{\log^{1-\nu}(n)}$ . Interestingly, this is even stronger than the best known results [8, 41, 38] for proper algorithms.

## 1.1 The Random $K$ -XOR Assumption

Unless we face a dramatic breakthrough in complexity theory, it seems unlikely that hardness of learning can be established on standard complexity assumptions such as  $\mathbf{P} \neq \mathbf{NP}$  (see [7, 33]). Indeed, all currently known lower bounds are based on cryptographic and average-case assumptions. One type of such assumptions, on which we rely in this paper, concern the random  $K$ -XOR problem. This problem has been extensively studied, and assumptions about its intractability were used to prove hardness of approximation results [1], establish public-key cryptography [1, 6], and statistical-computational tradeoffs [12].

A  $K$ -tuple is a mapping  $C : \{\pm 1\}^n \rightarrow \{\pm 1\}^K$  in which each output coordinate is a literal and the  $K$  literals correspond to  $K$  different variables. The collection of  $K$ -tuples is denoted  $\mathcal{X}_{n,K}$ . A  $K$ -formula is a collection  $J = \{C_1, \dots, C_m\}$  of  $K$ -tuples. An instance to the  $K$ -XOR problem is a  $K$ -formula, and the goal is to find an assignment  $\psi \in \{\pm 1\}^n$  that maximizes  $\text{VAL}_{\psi, \text{XOR}}(J) := \frac{|\{j : \text{XOR}_K(C_i(\psi)) = 1\}|}{m}$ . We define the *value* of  $J$  as

$$\text{VAL}_{\text{XOR}}(J) := \max_{\psi \in \{\pm 1\}^n} \text{VAL}_{\psi, \text{XOR}}(J)$$

We will allow  $K$  to vary with  $n$  (but still be fixed for every  $n$ ). For example, we can look of the  $\lceil \log(n) \rceil$ -XOR problem.

We will consider the problem of distinguishing random formulas from formulas with high value. Concretely, for  $m = m(n)$ ,  $K = K(n)$  and  $\frac{1}{2} > \eta = \eta(n) > 0$ , we say that the problem  $\text{CSP}_m^{\text{rand}, 1-\eta}(\text{XOR}_K)$  is easy, if there exists an efficient randomized algorithm,  $\mathcal{A}$  with the following properties. Its input is a  $K$ -formula  $J$  with  $n$  variables and  $m$  constraints and its output satisfies:

- If  $\text{VAL}_{\text{XOR}}(J) \geq 1 - \eta$ , then

$$\Pr_{\text{coins of } \mathcal{A}} (\mathcal{A}(J) = \text{“non-random”}) \geq \frac{3}{4}$$

- If  $J$  is random<sup>2</sup>, then with probability  $1 - o_n(1)$  over the choice of  $J$ ,

$$\Pr_{\text{coins of } \mathcal{A}} (\mathcal{A}(J) = \text{“random”}) \geq \frac{3}{4}.$$

It is not hard to see that the problem gets easier as  $m$  gets larger, and as  $\eta$  gets smaller. For  $\eta = 0$ , the problem is actually easy, as it can be solved using Gaussian elimination. However, even for slightly larger  $\eta$ 's the problems seems to become hard. For example, for any constant  $\eta > 0$ , best known algorithms [37, 26, 25, 12, 3] only work with  $m =$

<sup>2</sup>To be precise, the  $K$ -tuples are chosen uniformly, and independently from one another.

$\Omega\left(n^{\frac{K}{2}}\right)$ . In light of that, we put forward the following two assumptions.

**ASSUMPTION 1.1.** *There are constants  $c > 0$  and  $\frac{1}{2} > \eta > 0$  such that for every  $K$  and  $m = n^{c \log(K) \sqrt{K}}$ , the problem  $\text{CSP}_m^{\text{rand}, 1-\eta}(\text{XOR}_K)$  is hard.*

**ASSUMPTION 1.2.** *There are constants  $c > 0$  and  $\frac{1}{2} > \eta > 0$  such that for every  $s$ ,  $K = \log^s(n)$  and  $m = n^{cK}$ , the problem  $\text{CSP}_m^{\text{rand}, 1-\eta}(\text{XOR}_K)$  is hard.*

We outline below some evidence to the assumptions, in addition to known algorithms' performance.

**Hardness of approximation.** Hastad's celebrated result [43] asserts that if  $\mathbf{P} \neq \mathbf{NP}$ , then for every  $\eta > 0$ , it is hard to distinguish  $K$ -XOR instances with value  $\geq 1 - \eta$  from instances with value  $\leq \frac{1}{2} + \eta$ . Since the value of a random formula is approximately  $\frac{1}{2}$ , we can interpret Hastad's result as claiming that it is hard to distinguish formulas with value  $1 - \eta$  from “semi-random”  $K$ -XOR formulas (i.e., formulas whose value is approximately the value of a random formula). Therefore, our assumptions can be seen as a strengthening of Hastad's result.

**Hierarchies and SOS lower bounds.** A family of algorithms whose performance has been analyzed are convex relaxations [24, 59, 2] that belong to certain *hierarchies* of convex relaxations. Among those hierarchies, the strongest is the Lasserre hierarchy (a.k.a. Sum of Squares). Algorithms from this family achieves state of the art results for the  $K$ -XOR and many similar problems. In [59] it is shown that relaxations in the Lasserre hierarchy that work in sub-exponential time cannot solve  $\text{CSP}_{n^{\frac{K}{2}-\epsilon}}^{\text{rand}, 1-\eta}(\text{XOR}_K)$  for any  $\eta, \epsilon > 0$ .

**Lower bounds on statistical algorithms.** Another family of algorithms whose performance has been analyzed are the so-called statistical algorithms. Similarly to hierarchies lower bounds, the results in [39] imply that statistical algorithms cannot solve  $\text{CSP}_{n^{\frac{K}{2}-\epsilon}}^{\text{rand}, 1-\eta}(\text{XOR}_K)$  for any  $\eta, \epsilon > 0$ .

**Resolution lower bounds.** The length of resolution refutations of random  $K$ -XOR formulas have been extensively studied (e.g. [42, 14, 13, 19]). It is known (Theorem 2.24 in [18]) that random formulas with  $n^{\frac{K}{2}-\epsilon}$  constraints only have exponentially long resolution refutations. This shows that yet another large family of algorithms (the so-called Davis-Putnam algorithms [35]) cannot efficiently solve  $\text{CSP}_{n^{\frac{K}{2}-\epsilon}}^{\text{rand}, 1-\eta}(\text{XOR}_K)$  for any  $\epsilon > 0$ .

**Similar assumptions.** Several papers relied on similar assumptions. Alekhovich [1] assumed that

$$\text{CSP}_{Cn}^{\text{rand}, 1-\eta}(\text{XOR}_3)$$

is hard for some  $\eta < \frac{1}{2}$  and for every  $C > 0$ . Applebaum, Barak and Wigderson [6] assumed that

$$\text{CSP}_{n^{1.4}}^{\text{rand}, 1-n^{-0.2}}(\text{XOR}_3)$$

is hard. Barak and Moitra [12] made the assumption that

$$\text{CSP}_{n^{\frac{K}{2}-\epsilon}}^{\text{rand}, 1-\eta}(\text{XOR}_K)$$

is hard for every  $\epsilon > 0$  and  $\eta = \frac{1}{2} - o(1)$ . Assumptions on predicates different than  $K$ -XOR were made in [31, 36].

The assumption in [36] implies that  $\text{CSP}_{Cn}^{\text{rand}, 1-\eta}(\text{XOR}_3)$  is hard for every  $C > 0$  and  $\eta > 0$ . The assumption in [31] implies that  $\text{CSP}_{n^{1.5-\epsilon}}^{\text{rand}, 1-\eta}(\text{XOR}_3)$  is hard for every  $\epsilon > 0$  and  $\eta > 0$ . A much more general assumption was made in [11]. It implies in particular that  $\text{CSP}_{Cn}^{\text{rand}, 1-\eta}(\text{XOR}_K)$  is hard for every  $C > 0$  and  $\eta > 0$ .

## 1.2 Previous Results and Related Work

### *Upper bounds.*

When  $\text{Err}_{\text{HALF}}(\mathcal{D}) = 0$ , the problem of learning halfspaces can be solved efficiently using linear programming. However, even for slightly larger error values the problem seems to become much harder. Currently best known algorithms [48] have non trivial performance only when it is guaranteed that  $\text{Err}_{\text{HALF}}(\mathcal{D}) \leq \frac{\log(n)}{n}$ . This algorithm also achieves approximation ratio of  $n$ , which is currently the best known approximation ratio. Better guarantees are known under various assumptions on the marginal distribution [10, 20, 30, 44]. For example, a PTAS is known [30] when the marginal distribution is uniform.

### *Lower bounds for general algorithms.*

Several hardness assumptions imply that it is hard to agnostically learn halfspaces. Feldman, Gopalan, Khot and Ponnuswami [38] have shown that based on the security of the Ajtai-Dwork cryptosystem. Kalai, Klivans, Mansour and Servedio showed the same conclusion [44] based on the hardness of learning parity with noise. Daniely and Shalev-Shwartz [34] derived the same conclusion based on the hardness of refuting random  $K$ -SAT formulas. Klivans and Kothari [50] showed that assuming that learning sparse parity is hard, it is hard to learn halfspaces even when the marginal distribution is Gaussian. We note that all these results only rule out exact algorithms, but say nothing about approximation algorithms. We note however that by [9, 15], an algorithm with non trivial performance of  $\eta$ -realizable distributions will result with quasi-polynomial time algorithm for learning constant depth circuits.

### *Lower bounds for proper algorithms and hardness of approximation.*

If algorithms are restrict to return a halfspace, halfspaces learning is essentially equivalent [54] to the computational problem of *minimizing disagreements*. In this problem, given a sample  $S = \{(x_1, y_1), \dots, (x_m, y_m)\} \in \mathbb{Q}^n \times \{\pm 1\}$ , the goal is to find a vector  $w \in \mathbb{Q}^n$  that minimizes the fraction of pairs with  $\text{sign}(\langle w, x_i \rangle) \neq y_i$ . The optimal fraction is called the *error* of the sample. As a “standard” and basic computational problem, much is known about it. The problem have been shown  $\mathcal{NP}$ -hard already in Karp’s famous paper [46]. Soon after the discovery of the PCP theorem, Arora, Babai, Stern and Sweedyk [8] have shown that assuming that no quasi-polynomial time algorithm can solve  $\mathcal{NP}$ -hard problems, there is no efficient algorithm with an approximation ratio of  $2^{\log^{1-\epsilon}(n)}$ . Later on, the corresponding maximization problem was considered by several authors [5, 16, 23, 41, 38], culminating with Feldman, Gopalan, Khot and Ponnuswami [38] who showed that assuming that no quasi-polynomial time algorithm can solve  $\mathcal{NP}$  hard problems, no efficient algorithm can distinguish samples with error  $\geq \frac{1}{2} - 2^{-\sqrt{\log(n)}}$  from samples with error  $\leq 2^{-\sqrt{\log(n)}}$ .

### **Statistical-Queries Lower bounds.**

Statistical queries (SQ) algorithms [47] is a class of learning algorithms whose interaction with  $\mathcal{D}$  is done only via *statistical queries*. Concretely, an SQ-algorithm can specify any function  $Q : \{\pm 1\}^n \times \{\pm 1\} \rightarrow \{\pm 1\}$  and an error parameter  $\lambda > 0$ , and receive a number  $e$  satisfying  $|E_{(x,y) \sim \mathcal{D}}[Q(x, y)] - e| \leq \lambda$ . In this model, an algorithm is efficient if it makes polynomially many queries with error parameters satisfying  $\frac{1}{\lambda} \leq \text{poly}(n)$  (besides that, the algorithm is not restricted). While this class is strictly smaller than the class of all algorithms, most known algorithms admit an SQ version. In addition, as opposed to general algorithms, unconditional lower bounds are known for SQ algorithms [21] for several learning problems. In particular, it is known that it is hard to agnostically learn halfspaces using SQ-algorithm (e.g. [45]). We note that as with previously known lower bounds for general algorithms, these results only rule out exact algorithms.

### *Lower bounds on concrete algorithms.*

A few results [17, 53, 32] showing hardness of approximation results for several concrete families of algorithms (linear methods).

### *The methodology of [33].*

In light of the great success of complexity theory in establishing hardness of approximation results for standard computational problems, having such dramatic gaps between upper and lower bounds is perhaps surprising. The reason for the discrepancy between learning problems and computational problems is the fact that it is unclear how to reduce  $\mathcal{NP}$ -hard problems to learning problems (see [7, 33]). The main obstacle is the ability of a learning algorithm to return a hypothesis which does not belong to the learnt class (in our case, halfspaces). Until recently, there was only a single framework, due to Kearns and Valiant [49], to prove lower bounds on learning problems. The framework of [49] makes it possible to show that certain cryptographic assumptions imply hardness of certain learning problems. As indicated above, the lower bounds established by this method are quite far from the performance of best known algorithms.

In a recent paper [33] (see also [31]) we, together with Linial and Shalev-Shwartz, developed a new framework to prove hardness of learning based on hardness on average of CSP problems. Yet, in [33] we were not able to use our technique to establish hardness results that are based on a natural assumption on a well studied problem. Rather, we made a rather speculative hardness assumption, that is concerned with general CSP problems, most of which were never studied explicitly. We recognized it as the main weakness of our approach, and therefore the main direction for further research. About a year after, Allen, O’Donnell and Witmer [3] refuted the assumption of [33]. On the other hand we [34] were able to overcome the use of our speculative assumption, and prove hardness of learning of DNF formulas (and other problems) based on a natural assumption on the complexity of refuting random  $K$ -SAT instances, in the spirit of Feige’s assumption [36]. The current paper continues this line of work.

## 1.3 Results

We say that a distribution  $\mathcal{D}$  on  $\mathbb{Q}^n \times \{\pm 1\}$  is  $\eta$ -almost realizable if  $\text{Err}_{\text{HALF}}(\mathcal{D}) \leq \eta$ . An algorithm have *non-trivial*

performance w.r.t. a certain family of distributions if for some  $c > 0$ , its output hypothesis has error  $\leq \frac{1}{2} - \frac{1}{n^c}$  whenever the underlying distribution belongs to the family.

- THEOREM 1.3.** • *Under assumption 1.1, for all  $\eta > 0$ , there is no  $\text{poly}(n)$ -time algorithm with non-trivial performance on  $\eta$ -almost-realizable distributions on  $\{\pm 1\}^n \times \{\pm 1\}$ .*
- *Under assumption 1.2, for all  $\nu > 0$ , there is no  $\text{poly}(n)$ -time algorithm with non-trivial performance on  $2^{-\log^{1-\nu}(n)}$ -almost-realizable distributions on  $\{\pm 1\}^n \times \{\pm 1\}$ .*

These results imply in particular that under assumption 1.1 there is no efficient learning algorithm with a constant approximation ratio, and under assumption 1.2 there is no efficient learning algorithm with an approximation ratio of  $2^{\log^{1-\nu}(n)}$ . As mentioned above, this substantially improves on previously known results that only showed hardness of exact learning.

### Extension to large margin learning.

Large margin learning is a popular variant of halfspace learning (e.g. [60, 61]). Here, the learning algorithm faces an somewhat easier task, as it is not required to classify correctly examples that are very close to the separating hyperplane. Our basic theorem can be extended to the large margin case. Concretely, we say that a distribution  $\mathcal{D}$  on  $\{\pm 1\}^n \times \{\pm 1\}$  is  $\eta$ -almost-realizable-with-margin if there is  $w \in \mathbb{R}^d$  with  $\sum_{i=1}^n |w_i| \leq \text{poly}(n)$  such that

$$\Pr_{(x,y) \sim \mathcal{D}} (y \cdot \langle w, x \rangle \leq 1) \leq \eta$$

Theorem 1.3 can be extended to show that no efficient algorithm can perform better than trivial even when the distribution is almost realized with margin. Concretely, we have the following theorem.

- THEOREM 1.4.** • *Under assumption 1.1, for all  $\eta > 0$ , there is no  $\text{poly}(n)$ -time algorithm with non-trivial performance on distributions on  $\{\pm 1\}^n \times \{\pm 1\}$  that are  $\eta$ -almost-realizable-with-margin.*
- *Under assumption 1.2, for all  $\nu > 0$ , there is no  $\text{poly}(n)$ -time algorithm with non-trivial performance on distributions on  $\{\pm 1\}^n \times \{\pm 1\}$  that are  $2^{-\log^{1-\nu}(n)}$ -almost-realizable-with-margin.*

### Statistical queries version.

Our proof technique can be adapted to show unconditional lower bounds for SQ-algorithms. Concretely, we have the following.

**THEOREM 1.5.** *There is no efficient SQ-algorithm with non-trivial performance on distributions on  $\{\pm 1\}^n \times \{\pm 1\}$  that are  $2^{-\log^{1-\nu}(n)}$ -almost-realizable-with-margin.*

Again, this substantially improves on previously known results that only showed hardness of exact learning.

### Implications to hardness of approximation.

As explained in the previous section, the problem of proper learning is essentially equivalent to the problem of minimizing disagreements. As our results hold in particular for proper algorithms, we can conclude the following.

**THEOREM 1.6.** *Under assumption 1.2, for every  $\epsilon > 0$  and  $c > 0$ , no efficient algorithm can distinguish samples with error  $\geq \frac{1}{2} - n^{-c}$  from samples with error  $\leq 2^{-\log^{1-\epsilon}(n)}$ .*

We note that the conclusion of our theorem is stronger than the conclusions of the previously best known lower bounds for the problem (however, we rely on a stronger assumption). In particular, it implies that the problem is hard to approximate within a factor of  $2^{\log^{1-\epsilon}(n)}$ , implying the conclusion [8] of Arora et. al. It is also strengthened the conclusion [38] of Feldman et. al., improving the completeness parameter from  $2^{-\sqrt{\log(n)}}$  to  $2^{-\log^{1-\epsilon}(n)}$  and the soundness parameter from  $\frac{1}{2} - 2^{-\sqrt{\log(n)}}$  to  $\frac{1}{2} - n^{-c}$ . We remark that our result holds even if we assume that the input examples are binary, and in the large margin settings.

## 2. MAIN PROOF IDEAS

We next elaborate on the main ideas of our main theorem. The full proof is deferred to the appendix.

### 2.1 The Methodology of [33]

We first describe the methodology of [33] to prove hardness of learning. A *sample* is a collection

$$S = \{(x_1, y_1), \dots, (x_m, y_m)\} \subset X \times \{\pm 1\}$$

The *error* of  $h : X \rightarrow \{\pm 1\}$  is  $\text{Err}_h(S) = \frac{|\{j: h(x_j) \neq y_j\}|}{m}$ . The *error* of  $S$  w.r.t. a hypothesis class  $\mathcal{H} \subset \{\pm 1\}^X$  is  $\text{Err}_{\mathcal{H}}(S) = \min_{h \in \mathcal{H}} \text{Err}_h(S)$ . The basic idea behind [33] is that if it is hard to distinguish a sample with small error from a sample that is very random in a certain sense, then it is hard to (even approximately) learn  $\mathcal{H}$ .

For the sake of concreteness, we restrict to the problem of learning halfspaces over the boolean cube. Namely, we take  $X = \{\pm 1\}^n$  and  $\mathcal{H} = \text{HALF} = \{h_w \mid w \in \mathbb{R}^n\}$ . We say that a sample is *strongly scattered*<sup>3</sup> if the labels (i.e., the  $y_i$ 's) are independent fair coins (in particular, they are independent from the  $x_i$ 's). For  $m = m(n)$  and  $\eta = \eta(m)$ , we denote by  $\text{HALF}_m^{\text{scat}, \eta}$  the problem of distinguishing a strongly-scattered sample from a sample with  $\text{Err}_{\text{HALF}}(S) \leq \eta$ . Concretely, we say that the problem is easy if there exists an efficient randomized algorithm  $\mathcal{A}$  with the following properties. Its input is a sample  $S = \{(x_1, y_1), \dots, (x_m, y_m)\} \subset \{\pm 1\}^n \times \{\pm 1\}$ , and its output satisfies:

- If  $\text{Err}_{\text{HALF}}(S) \leq \eta$ , then

$$\Pr_{\text{coins of } \mathcal{A}} (\mathcal{A}(S) = \text{"almost-realizable"}) \geq \frac{3}{4}$$

- If  $S$  is strongly scattered then, with probability  $1 - o_n(1)$  over the choice of the labels,

$$\Pr_{\text{coins of } \mathcal{A}} (\mathcal{A}(S) = \text{"scattered"}) \geq \frac{3}{4}.$$

**THEOREM 2.1.** [33] *If for every  $a > 0$  the problem*

$$\text{HALF}_{na}^{\text{scat}, \eta}$$

*is hard, then there is no efficient learning algorithm with non-trivial performance on  $\eta$ -almost realizable distributions.*

To be self contained, and since Theorem 2.1 is not identical to [33], we outline a proof.

<sup>3</sup>A weaker notion, called scattering, was used in [33].

PROOF. Assume toward a contradiction that the efficient algorithm  $\mathcal{L}$  is guaranteed to return a hypothesis with error  $\leq \frac{1}{2} - \frac{1}{n^c}$  on  $\eta$ -almost realizable distributions. Let  $M(n)$  be the maximal number of random bits used by  $\mathcal{L}$  when the examples lie in  $\{\pm 1\}^n$ . This includes both the bits describing the examples produced by the oracle and “standard” random bits. Since  $\mathcal{L}$  is efficient,  $M(n) < n^{c'}$  for some  $c' > 0$ . Define

$$q(n) = n^{2c' + 2c}.$$

We will derive a contradiction by showing how  $\mathcal{L}$  can be used to solve  $\text{HALF}_{q(n)}^{\text{s-scat}, \eta}$ . To this end, consider the algorithm  $\mathcal{A}$  defined below. On input  $S = \{(x_1, y_1), \dots, (x_m, y_m)\} \subset \{\pm 1\}^n \times \{\pm 1\}$ ,

1. Run  $\mathcal{L}$  on  $S$ , such that the examples’ oracle generates examples by choosing random examples from  $S$ .
2. Let  $h$  be the hypothesis that  $\mathcal{L}$  returns. If  $\text{Err}_S(h) \leq \frac{1}{2} - \frac{1}{n^c}$ , output “almost-realizable”. Otherwise, output “scattered”.

Next, we show that  $\mathcal{A}$  solves

$$\text{HALF}_{q(n)}^{\text{s-scat}, \eta}$$

Indeed, if  $\text{Err}_{\text{HALF}(S)} \leq \eta$ , then  $\mathcal{L}$  is guaranteed to return a hypothesis with  $\text{Err}_S(h) \leq \frac{1}{2} - \frac{1}{n^c}$ , and  $\mathcal{A}$  will output “almost-realizable”. What if  $S$  is strongly scattered? Let  $\mathcal{G} \subset \{\pm 1\}^{\{\pm 1\}^n}$  be the collection of functions that  $\mathcal{L}$  might return. We note that  $|\mathcal{G}| \leq 2^{n^{c'}}$ , since each hypothesis in  $\mathcal{G}$  can be described by  $n^{c'}$  bits. Namely, the random bits that  $\mathcal{L}$  uses and the description of the examples sampled by the oracle. Now, since  $\mathcal{D}$  is strongly-scattered, by Hoeffding’s bound, the probability that  $\text{Err}_S(h) \leq \frac{1}{2} - \frac{1}{n^c}$  for a single  $h : \{\pm 1\}^n \rightarrow \{\pm 1\}$  is  $\leq \exp\left(-2\frac{q(n)}{n^{2c}}\right)$ . By the union bound, the probability that  $\text{Err}_S(h) \leq \frac{1}{2} - \frac{1}{n^c}$  for some  $h \in \mathcal{G}$  is at most  $|\mathcal{G}| \exp\left(-2\frac{q(n)}{n^{2c}}\right) \leq 2^{n^{c'} - \frac{q(n)}{n^{2c}}} = 2^{n^{c'} - n^{2c'}}$ . It follows that the probability that  $\mathcal{A}$  responds “almost-realizable” is  $o(1)$ .  $\square$

## 2.2 An Overview

For the sake of simplicity, we will first explain how to prove a weaker version of Theorem 1.3. At the end of this section we will explain how to prove Theorem 1.3 in full.

**THEOREM 2.2.** *Suppose that for every  $K > 4$  the problem  $\text{CSP}_{n^{\frac{K}{4}}}^{\text{rand}, 1 - \frac{1}{100}}(\text{XOR}_K)$  is hard. Then, there is no efficient algorithm with non-trivial performance on  $\frac{2}{100}$ -almost-realizable distributions on  $\{-1, 1, 0\}^n \times \{\pm 1\}$ .*

The course of the proof is to reduce  $\text{CSP}_{n^{\frac{K}{4}}}^{\text{rand}, 1 - \frac{1}{100}}(\text{XOR}_K)$  to  $\text{HALF}_{n^{\frac{\sqrt{K}}{4 \log(K)}}}^{\text{s-scat}, \frac{2}{100}}$ . Given that reduction, and since  $K$  can be arbitrarily large, Theorem 2.2 follows from Theorem 2.1.

### The XOR Problem as a Learning Problems

The basic conceptual idea is to interpret the  $K$ -XOR problem as a learning problems. Every  $\psi \in \{\pm 1\}^n$  naturally defines  $h_\psi : \mathcal{X}_{n,K} \rightarrow \{\pm 1\}$ , by mapping each  $K$ -tuple  $C$  to the truth value of the corresponding constraint, given the

assignment  $\psi$ . Namely,  $h_\psi(C) = \text{XOR}(C(\psi))$ . Now, we can consider the hypothesis class  $\mathcal{H}_K = \{h_\psi \mid \psi \in \{\pm 1\}^n\}$ .

The  $K$ -XOR problem can be now formulated as follows. Given  $J = \{C_1, \dots, C_m\} \subset \mathcal{X}_{n,K}$ , find  $h_\psi \in \mathcal{H}_K$  with minimal error on the sample  $S = \{(C_1, 1), \dots, (C_m, 1)\}$ . Now, the problem  $\text{CSP}_m^{\text{rand}, 1 - \frac{1}{100}}(\text{XOR}_K)$  is the problem of distinguishing between the case the  $\text{Err}_{\mathcal{H}_K}(S) \leq \frac{1}{100}$  and the case where the different  $C_i$ ’s where chosen independently and uniformly from  $\mathcal{X}_{n,K}$ .

The mapping  $J \mapsto S$  is still not a reduction from

$$\text{CSP}_m^{\text{rand}, 1 - \frac{1}{100}}(\text{XOR}_K)$$

to the problem of distinguishing a strongly scattered sample from a sample with small error w.r.t. halfspaces. This is due to the following points:

- In the case that  $J$  is random,  $S$  is, in a sense, “very random”. Yet, it is not strongly-scattered.
- We need to measure the error w.r.t. halfspaces rather than  $\mathcal{H}_K$ .

Next, we explain how we address these two points.

### Making the Sample Scattered

Addressing the first point is relatively easy. Given a sample  $(C_1, 1), \dots, (C_m, 1)$ , we can produce a new sample

$$(C'_1, y'_1), \dots, (C'_m, y'_m)$$

as follows: for every  $i \in [m]$ , w.p.  $\frac{1}{2}$  we let  $(C'_i, y'_i) = (C_i, 1)$  and w.p.  $\frac{1}{2}$  we let  $(C'_i, y'_i) = (C'_i, -1)$ , where  $C'_i$  is obtained from  $C_i$  by flipping the sign of the first literal. It is not hard to see that this reduction maps random instances to strongly scattered instances. Also, it is not hard to see that the error of every  $h_\psi \in \mathcal{H}_K$  does not change when moving from the original sample to the new sample. Therefore, samples with error  $\leq \frac{1}{100}$  are mapped to samples with error  $\leq \frac{1}{100}$ .

Note that the reduction not only maps random instances to scattered instances, but in fact has a stronger property that will be useful for addressing the second point (Replacing  $\mathcal{H}_K$  by HALF). Concretely, if the original instance is random, then the  $(C'_i, y'_i)$ ’s are independent and uniform elements from  $\mathcal{X}_{n,K} \times \{\pm 1\}$ .

### Replacing $\mathcal{H}_K$ by HALF

To address the second point, we will show a reduction that maps a sample  $S = \{(C_1, y_1), \dots, (C_m, y_m)\} \subset \mathcal{X}_{n,K} \times \{\pm 1\}$  to a sample  $(x_1, y_1), \dots, (x_m, y_m) \in \{-1, 1, 0\}^{n^{\sqrt{K} \log(K)}} \times \{\pm 1\}$ . It will be convenient to give the reduction the option to output “not-random” instead of producing a new sample. For large enough  $K$ , if  $m = n^{\frac{K}{4}}$ , the reduction has the following properties:

- If the original sample is random (i.e., the examples  $(C_i, y_i)$  where chosen independently and uniformly from  $\mathcal{X}_{n,K} \times \{\pm 1\}$ ), then w.h.p. the reduction will produce a new sample, and given that a new sample was indeed produced, it will be strongly scattered.
- If the original sample has error  $\leq \frac{1}{100}$  w.r.t.  $\mathcal{H}_K$  then the reduction will either say that the original sample is not random, or it will produce a new sample with error  $\leq \frac{2}{100}$  w.r.t. halfspaces.

Putting  $n' = n^{\sqrt{K} \log(K)}$  and noting that  $m = (n')^{\frac{\sqrt{K}}{4 \log(K)}}$ , it is not hard to see that such a reduction, together with the previous reduction (the “scattering reduction”), indeed reduces  $\text{CSP}_{n \frac{K}{4}}^{\text{rand}, 1 - \frac{1}{100}}(\text{XOR}_K)$  to  $\text{HALF}_{n \frac{\sqrt{K}}{4 \log(K)}}^{\text{s-scat}, \frac{2}{100}}$ .

The reduction will work as follows. It will first test that  $J := \{C_1, \dots, C_m\}$  is *pseudo random*, in the sense that it satisfies certain (efficiently verifiable) properties (that will be specified later) that are possessed by random formulas w.h.p. If the test fails, the reduction will say that  $S$  is not random. Otherwise, the reduction will produce the sample  $\chi(S) := (\chi(C_1), y_1), \dots, (\chi(C_m), y_m) \in \{-1, 1, 0\}^{n^{\sqrt{K} \log(K)}} \times \{\pm 1\}$ , for a mapping  $\chi : \mathcal{X}_{n,K} \rightarrow \{-1, 1, 0\}^{n^{\sqrt{K} \log(K)}}$  that we will specify later.

Next, we explain why this reduction have the desired properties. We start with the case that  $S$  is random. In that case, the pseudo-randomness test will pass w.h.p. and the reduction will produce a new sample. Also, since the properties tested in this test do not depend on the labels, given that a new sample was indeed produced, the new sample is strongly scattered.

We next deal with the case that  $\text{Err}_{\mathcal{H}_K}(S) \leq \frac{1}{100}$ . We will show that in this case either the reduction will say that  $S$  is not random, or the new sample will have error  $\leq \frac{2}{50}$  w.r.t. halfspaces. To this end we will finally specify  $\chi$  and describe the list of pseudo-random properties.

It will be convenient to define  $\chi$  as a composition  $\chi = \rho \circ \pi$  where  $\pi : \mathcal{X}_{n,K} \rightarrow \{-1, 1, 0\}^{n^K}$  and  $\rho : \{-1, 1, 0\}^{n^K} \rightarrow \{-1, 1, 0\}^{n^{\sqrt{K} \log(K)}}$ . We first define  $\pi$ . The indicator vector of a literal is the vector in  $\{0, -1, 1\}^n$  whose all coordinates are zero except the coordinate corresponding to the literal, that is 1 (−1) if the literal is un-negated (negated). We define  $\pi(C)$  as a concatenation of  $K$  vectors, where the  $i$ ’s vector is the indicator vector of the  $i$ ’th literal in  $C$ . As for  $\rho$ , we let  $\rho(x) \in \{-1, 1, 0\}^{n^{\sqrt{K} \log(K)}}$  be a vector consisting of all products of the form  $x_{i_1} \dots x_{i_r}$  for  $r \leq \frac{1}{2} \sqrt{K} \log(K)$ , and padded with zeros in the remaining coordinates. We note that for large enough  $n$ , the number of such products is  $\leq (nK + 1)^{\frac{1}{2} \sqrt{K} \log(K)} \leq n^{\sqrt{K} \log(K)}$ .

Now, it is not hard to see that the error of  $\chi(S)$  w.r.t. halfspaces is exactly the error of  $\pi(S)$  w.r.t.  $\text{POL}_d$  – the hypothesis class consisting of threshold functions of polynomials of degree at most  $d = \frac{1}{2} \sqrt{K} \log(K)$ . Therefore, we will want to show that if  $\text{Err}_{\mathcal{H}_K}(S) \leq \frac{1}{100}$  then  $\text{Err}_{\text{POL}_d}(\pi(S)) \leq \frac{2}{100}$ .

Suppose that  $\text{Err}_{\mathcal{H}_K}(S) \leq \frac{1}{100}$ , and let  $\psi \in \{\pm 1\}^n$  such that  $\text{Err}_{h_\psi}(S) \leq \frac{1}{100}$ . To show that  $\text{Err}_{\text{POL}_d}(\pi(S)) \leq \frac{2}{100}$  it is enough to construct a degree  $\leq d$  polynomial  $p : \{-1, 1, 0\}^{n^K} \rightarrow \mathbb{R}$  such that  $\Pr_{j \sim [m]}(h_\psi(C_j) \neq \text{sign}(p(\pi(C_j)))) \leq \frac{1}{100}$ . Indeed, in that case,

$$\begin{aligned} \text{Err}_{\text{POL}_d}(\pi(S)) &\leq \Pr_{j \sim [m]}(y_j \neq \text{sign}(p(\pi(C_j)))) \\ &\leq \Pr_{j \sim [m]}(y_j \neq h_\psi(C_j)) + \Pr_{j \sim [m]}(h_\psi(C_j) \neq \text{sign}(p(\pi(C_j)))) \\ &\leq \text{Err}_{h_\psi}(S) + \frac{1}{100} \leq \frac{1}{100} + \frac{1}{100} = \frac{2}{100} \end{aligned}$$

We note that we will actually find  $p$  with a stronger property – namely, that  $\Pr_{j \sim [m]}(h_\psi(C_j) \neq p(\pi(C_j))) \leq \frac{1}{100}$ . This stronger property is not needed for proving the simplified version (Theorem 2.2), but will be needed for proving Theorem 1.3.

Unfortunately, it is not the case that we can always find such a polynomial. To overcome it, as explained above, the reduction will first check a set of pseudo-random properties. Concretely, for all small enough sets of literals, the reduction will check that the number of  $K$ -tuples that contain all the literals in the set is close to what is expected for a random sample. For example, it will check that the fraction of  $K$ -tuples containing the literal  $x_7$  is approximately  $\frac{K}{2n}$ .

It therefore remains to show that for pseudo-random  $S$  there is degree  $\leq d$  polynomial  $p : \{-1, 1, 0\}^{n^K} \rightarrow \mathbb{R}$  such that  $\Pr_{j \sim [m]}(h_\psi(C_j) \neq p(\pi(C_j))) \leq \frac{1}{100}$ . To this end, consider the linear map  $T_\psi : \mathbb{R}^{n^K} \rightarrow \mathbb{R}^K$  that is defined as

$$T_\psi(v_1 | v_2 | \dots | v_K) = (\langle v_1, \psi \rangle, \langle v_2, \psi \rangle, \dots, \langle v_K, \psi \rangle)$$

We note that  $\forall C \in \mathcal{X}_{n,K}$ ,  $T_\psi(\pi(C)) = C(\psi)$ . We will consider polynomials of the form  $p = p' \circ T_\psi$  where  $p' : \{\pm 1\}^K \rightarrow \mathbb{R}$  is a degree  $\leq d$  polynomial. We note that for such  $p$  we have

$$\Pr_{j \sim [m]}(h_\psi(C_j) \neq p(\pi(C_j))) = \Pr_{j \sim [m]}(\text{XOR}(C_j(\psi)) \neq p'(C_j(\psi)))$$

It is therefore enough to find a degree  $\leq d$  polynomial  $p'$  for which  $\Pr_{z \sim \mathcal{D}(\psi)}(\text{XOR}(z) = p'(z)) \geq 0.99$ . Here,  $\mathcal{D}(\psi)$  is the distribution of the random variable  $C_j(\psi)$  where  $j \sim [m]$ . Now, even though it is not possible to do that for general distributions, the fact that the sample is pseudo-random implies that  $\mathcal{D}(\psi)$  is close, in a certain sense, to the uniform distribution on  $\{\pm 1\}^K$ . Now, for uniform  $z \in \{\pm 1\}^K$  we have that  $\left| \sum_{i=1}^K z_i \right| \leq \frac{1}{2} \sqrt{K} \log(K)$  w.p.  $1 - o_K(1)$ , and the same holds for  $z \sim \mathcal{D}(\psi)$ . Therefore, since  $\text{XOR}(z)$  is fully determined given  $\sum_{i=1}^K z_i$ , and since  $\sum_{i=1}^K z_i$  takes  $\leq d$  values in the interval  $[-d, d]$ , we can take  $p'(z) = p''(\sum_{i=1}^K z_i)$  where  $p'' : \mathbb{R} \rightarrow \mathbb{R}$  is a degree  $\leq \frac{1}{2} \sqrt{K} \log(K)$  polynomial that satisfies  $p''(\sum_{i=1}^K z_i) = \text{XOR}(z)$  whenever  $\left| \sum_{i=1}^K z_i \right| \leq \sqrt{K} \log(K)$ .

### Proving Theorem 1.3 in Full

Theorem 2.2 differs from Theorem 1.3 in two aspects. The first is that in Theorem 1.3 the completeness parameter in the assumption can be arbitrarily close to  $\frac{1}{2}$  (rather than 0.99 in Theorem 2.2) and at the same time, the conclusion holds for  $\eta$ -almost realizable distributions for arbitrarily small  $\eta$  (rather than  $\eta = 0.02$  in Theorem 2.2). The second aspect, that is much more minor, is that in Theorem 2.2 the hard distribution can be chosen to be supported in  $\{-1, 1, 0\}^n \times \{\pm 1\}$ , while Theorem 1.3 is slightly more restrictive and requires that the distribution will be supported in  $\{\pm 1\}^n \times \{\pm 1\}$ .

To address the first aspect, we will first reduce the random XOR problem to the random majority-of- $q$ -XORs problem, and then we will follow a similar (but slightly more involved) argument as the one described above. The reduction will work as follows. Given a XOR-formula with  $m$  XOR-constraints, the reduction will produce a MXOR-formula with  $\frac{m}{q}$  constraints, each of which is a majority of  $q$  random XOR-constraints from the original formula. This reduction will amplify the completeness parameter toward 1.

To address the second aspect, we note that halfspaces on  $\{-1, 1, 0\}^n$  can be realized by halfspaces on  $\{\pm 1\}^{2n}$  using the map  $\Psi : \{-1, 1, 0\}^n \rightarrow \{\pm 1\}^{2n}$  that is defined as follows:

$$\Psi(x) = (\Psi(x_1), \dots, \Psi(x_n)),$$

where for  $x \in \{0, -1, 1\}$ ,  $\Psi(x) = \begin{cases} (1, 1) & x = 1 \\ (-1, -1) & x = -1 \\ (-1, 1) & x = 0 \end{cases}$ . It

is not hard to see that for every  $w \in \mathbb{R}^n$  we have  $h_w = h_{w'} \circ \Psi$  where  $w' = \frac{1}{2}(w_1, w_1, \dots, w_n, w_n)$ . This observation shows that if there is no efficient algorithm with non-trivial performance on  $\eta(n)$ -almost-realizable distributions on  $\{-1, 1, 0\}^n \times \{\pm 1\}$ , then if there is no efficient algorithm with non-trivial performance on  $\eta(\frac{n}{2})$ -almost-realizable distributions on  $\{\pm 1\}^n \times \{\pm 1\}$ .

### A Road Map

In section 3.1 we analyze elementary properties of random formulas, and define accordingly a notion of a pseudo-random formula. We also show that if  $J = \{C_1, \dots, C_m\} \subset \mathcal{X}_{n,K}$  is a pseudo-random formula, then for  $j \sim [m]$  and every assignment  $\psi \in \{\pm 1\}^n$ , the distribution of the random variable  $C_j(\psi) \in \{\pm 1\}^K$  is close to the uniform distribution in a certain sense. In section 3.2 we show that for pseudo-random formulas, for every every assignment  $\psi \in \{\pm 1\}^n$ , the mapping  $C_j \mapsto C_j(\psi)$  can be approximated by a low-degree polynomial. Finally, the full reduction is outlined in section 3.3. In section 3.4 we briefly explain how our argument can be extended to prove Theorems 1.4 and 1.5.

## 3. PROOF OF THEOREM 1.3

### 3.1 Pseudo-random Formulas

A *partial  $K$ -tuple* supported in  $A \subset [K]$  is a mapping  $C : \{\pm 1\}^n \rightarrow \{-1, 1, *\}^K$  such that the output coordinates corresponding to  $A$  are literals corresponding to  $|A|$  different variables, and the remaining coordinates are the constant function  $*$ . The *size* of  $C$  is  $|A|$ . We denote by  $\mathcal{X}_{n,K,A}$  the collection of partial  $K$ -tuples that are supported in  $A$ . We note that

$$|\mathcal{X}_{n,K,A}| = (2n)(2n-2) \dots (2n+2-2|A|) \quad (1)$$

For  $A \subset [K]$  we denote by  $\Pi_A : \{\pm 1\}^K \rightarrow \{-1, 1, *\}^K$  the function that maps all coordinates in  $[K] \setminus A$  to  $*$  and leaves the remaining coordinates unchanged. For  $C \in \mathcal{X}_{n,K}$  we denote by  $C_A : \{\pm 1\}^n \rightarrow \{-1, 1, *\}^K$  the partial  $K$ -tuple  $\Pi_A \circ C$ . For a  $K$ -formula  $J$ , and a partial  $K$ -tuple  $C$  we define the *frequency* of  $C$  in  $J$  as  $\text{Fr}_J(C) = \frac{|\{C' \in J : C'_A = C\}|}{m}$ . We note that the for random  $J$ , if  $C$  is a partial  $K$ -tuple that is supported in a set  $A$  of size  $t$ , then  $\text{Fr}_J(C)$  is an average of  $m$  independent Bernulli variables with parameter  $p_{n,t} := \frac{1}{|\mathcal{X}_{n,K,A}|} = \frac{1}{(2n)(2n-2) \dots (2n-t+1)}$ . By Hoeffding's bound we have

LEMMA 3.1. *Let  $J = \{C_1, \dots, C_m\}$  be a random formula. Then, for every partial  $K$ -tuple of size  $t$  we have*

$$\Pr_j (|\text{Fr}_J(C) - p_{n,t}| \geq \tau) \leq 2 \exp(-2m\tau^2)$$

We say that  $J$  is  $(t, \tau)$ -pseudo-random if  $|\text{Fr}_J(C) - p_{n,t'}| < \tau$  for every partial  $K$ -tuple  $C$  of size  $t' \leq t$ . We say that  $J$  is  $\tau$ -pseudo-random if it is  $(K, \tau)$ -pseudo-random. By lemma 3.1, the fact that the number of partial  $K$ -tuples is

$$1 + \sum_{j=1}^K \binom{K}{j} (2n)(2n-2) \dots (2n+2-2j) \leq 2^K (2n)^K \leq (2n)^{2K},$$

and the union bound we have

LEMMA 3.2. *Let  $J = \{C_1, \dots, C_m\}$  be a random formula. Then the probability that  $J$  is not  $\tau$ -pseudo-random is at most  $(2n)^{2K} 2 \exp(-2m\tau^2)$*

For a formula  $J = \{C_1, \dots, C_m\} \subset \mathcal{X}_{n,K}$  and  $\psi \in \{\pm 1\}^n$ , we denote by  $\mathcal{D}(J, \psi)$  the distribution of the random variable  $C_j(\psi) \in \{\pm 1\}^K$  where  $j \sim \text{Uni}([m])$ . A vector  $z \in \{-1, 1, *\}^K$  is *supported* in  $A \subset [K]$  if  $A = \{i \in [K] \mid z_i \neq *\}$ . We say that a distribution  $\mathcal{D}$  on  $\{\pm 1\}^K$  is  $(t, \mu)$ -close to the uniform distribution if for every  $z \in \{-1, 1, *\}^K$  that is supported in a set  $A \in \binom{[K]}{\leq t}$  we have that

$$\left| \Pr_{z' \sim \mathcal{D}} (\Pi_A(z') = z) - 2^{-|A|} \right| \leq \mu$$

LEMMA 3.3. *If  $J$  is  $(t, \tau)$ -pseudo-random then for every  $\psi \in \{\pm 1\}^n$ ,  $\mathcal{D}(J, \psi)$  is  $(t, n^t \tau)$ -close to the uniform distribution.*

PROOF. Fix a vector  $z \in \{-1, 1, *\}^K$  supported in  $A \in \binom{[K]}{\leq t}$ . We have

$$\begin{aligned} \Pr_{z' \sim \mathcal{D}(J, \psi)} (\Pi_A(z') = z) &= \Pr_{j \sim \text{Uni}([m])} ((C_j)_A(\psi) = z) \\ &= \frac{1}{m} |\{j \in [m] \mid (C_j)_A(\psi) = z\}| \\ &= \frac{1}{m} \sum_{C \in \mathcal{X}_{n,K,A} \mid C(\psi) = z} |\{j \mid (C_j)_A = C\}| \\ &= \sum_{C \in \mathcal{X}_{n,K,A} \mid C(\psi) = z} \text{Fr}_J(C) \end{aligned}$$

Denote  $U = \{C \in \mathcal{X}_{n,K,A} \mid C(\psi) = z\}$  and note that  $|U| = n(n-1) \dots (n-|A|+1) = \frac{|\mathcal{X}_{n,K,A}|}{2^{|A|}}$ . By the  $(t, \tau)$ -pseudo-randomness of  $J$  we have

$$\begin{aligned} \left| \Pr_{j \sim \text{Uni}([m])} ((C_j)_A(\psi) = z) - 2^{-|A|} \right| &= \left| \sum_{C \in U} \text{Fr}_J(C) - 2^{-|A|} \right| \\ &\leq \left| \sum_{C \in U} p_{n,|A|} - 2^{-|A|} \right| + |U| \tau \\ &= |U| p_{n,|A|} - 2^{-|A|} + |U| \tau \\ &= |U| \tau \leq n^{|A|} \tau \leq n^t \tau \end{aligned}$$

□

### 3.2 Realizing Assignments by Polynomials

In this section we will show that when a formula  $J = \{C_1, \dots, C_m\}$  is pseudo-random then for every assignment  $\psi \in \{\pm 1\}^n$  the mapping  $C \mapsto \text{XOR}(C(\psi))$  can be approximately realized by a low-degree polynomial on  $J$ . Namely, there is a low degree polynomial  $p : \mathcal{X}_{n,K} \rightarrow \{\pm 1\}$  such that on most  $K$ -tuples  $C \in J$ , we have  $p(C) = \text{XOR}(C(\psi))$ . To this end, we must represent  $K$ -tuples as vectors. The way we will do this is the following. Recall that the indicator vector of a literal is the vector in  $\{0, -1, 1\}^n$  whose all coordinates are zero except the coordinate corresponding to the literal, that is 1  $(-1)$  if the literal is un-negated (negated). Also, we defined  $\pi : \mathcal{X}_{n,K} \rightarrow \{0, -1, 1\}^{nK}$  such that  $\pi(C)$  is a concatenation of  $K$  vectors, where the  $i$ 's vector is the indicator vector of the  $i$ 'th literal in  $C$ .

We will use the following version of Chernoff's bound from [52].

LEMMA 3.4. *Let  $\mathcal{D}$  be a distribution on  $\{\pm 1\}^K$ ,  $1 \geq \beta > 0$  and  $\frac{1}{2} \leq \alpha < \frac{1+\beta}{2}$ . Assume that for every  $z \in \{-1, 1, *\}^K$  that is supported in a set  $A$  of size  $r = \lceil \beta K \rceil$  we have*

$\Pr_{z' \sim \mathcal{D}} (\Pi_A(z') = z) \leq \alpha^r$ . Then

$$\Pr_{z \sim \mathcal{D}} \left( \left| \sum_{i=1}^K z_i \right| \geq \beta K \right) \leq 2 \exp \left( -D \left( \frac{1+\beta}{2}, \alpha \right) K \right)$$

LEMMA 3.5. Let  $\mathcal{D}$  be a distribution on  $\{\pm 1\}^K$  that is  $(t, \mu)$ -close to the uniform distribution, and let  $d$  such that  $\frac{K2^d\mu}{d} < d \leq t$ . Then, there exists a degree  $\leq d$  polynomial  $p : \{\pm 1\}^K \rightarrow \mathbb{R}$  such that

$$\Pr_{z \sim \mathcal{D}} (\text{XOR}(z) \neq p(z)) \leq 2 \exp \left( -D \left( \frac{1}{2} + \frac{d}{2K}, \frac{1}{2} + \frac{2^{d-1}\mu}{d} \right) K \right)$$

PROOF. Let  $\Lambda : \{\pm 1\}^K \rightarrow \mathbb{R}$  be the linear mapping  $\Lambda(z) = \sum_{i=1}^K z_i$ . We note that  $\text{XOR}(z)$  is fully determined given  $\Lambda(z)$ . Namely there is  $f : \mathbb{R} \rightarrow \{\pm 1\}$  such that  $\text{XOR} = f \circ \Lambda$ . We note that the image of  $\Lambda$  has at most  $d+1$  values in the interval  $[-d, d]$ . Therefore, there is a degree  $\leq d$  polynomial  $q : \mathbb{R} \rightarrow \mathbb{R}$  that coincides with  $f$  on  $\Lambda(\{\pm 1\}^K) \cap [-d, d]$ . Consider now the degree  $\leq d$  polynomial  $p = q \circ \Lambda$ . We have that  $p$  coincides with  $\text{XOR}$  whenever  $|\sum_{i=1}^K z_i| \leq d$ . Hence, we have

$$\Pr_{z \sim \mathcal{D}} (\text{XOR}(z) \neq p(z)) \leq \Pr_{z \sim \mathcal{D}} \left( \left| \sum_{i=1}^K z_i \right| \geq d \right).$$

The lemma will now follow from Lemma 3.4 when  $\beta = \frac{d}{K}$  and  $\alpha = \frac{1}{2} + \frac{2^{d-1}\mu}{d}$ . It remains to show that the conditions of the Lemmas hold, namely, that  $\alpha < \frac{1+\beta}{2}$  and that for every  $z \in \{-1, 1, *\}$  that is supported in a set  $A$  of size  $r = \lceil \beta K \rceil = d$  we have  $\Pr_{z' \sim \mathcal{D}} (\Pi_A(z') = z) \leq \alpha^d$ . The first conditions follows from the requirement that  $\frac{K2^d\mu}{d} < d$ . For the second condition, since  $\mathcal{D}$  is  $(t, \mu)$ -close to the uniform distribution and  $d \leq t$  we have

$$\begin{aligned} \alpha^d &= \sum_{i=0}^d \binom{d}{i} \left( \frac{1}{2} \right)^{d-i} \left( \frac{2^{d-1}\mu}{d} \right)^i \\ &\geq \left( \frac{1}{2} \right)^d + d \left( \frac{1}{2} \right)^{d-1} \left( \frac{2^{d-1}\mu}{d} \right) \\ &= \left( \frac{1}{2} \right)^d + \mu \\ &\geq \Pr_{z' \sim \mathcal{D}} (\Pi_A(z') = z) \end{aligned}$$

□

LEMMA 3.6. Let  $J = \{C_1, \dots, C_m\} \in \mathcal{X}_{n,K}$  be  $(t, \tau)$ -pseudo-random,  $\psi \in \{\pm 1\}^n$  and  $d$  such that  $\frac{K2^d n^d \tau}{d} < d \leq t$ . Then, there exists a degree  $\leq d$  polynomial  $p : \{0, -1, 1\}^{nK} \rightarrow \mathbb{R}$  such that

$$\Pr_{j \sim [m]} (\text{XOR}(C_j(\psi)) \neq p(\pi(C_j))) \leq 2 \exp \left( -D \left( \frac{1}{2} + \frac{d}{2K}, \frac{1}{2} + \frac{2^{d-1} n^d \tau}{d} \right) K \right)$$

PROOF. Since  $J$  is  $(t, \tau)$ -pseudo-random and  $d \leq t$ , it is also  $(d, \tau)$ -pseudo-random. By Lemma 3.3  $\mathcal{D}(J, \psi)$  is  $(d, n^d \tau)$ -close to the uniform distribution. Therefore, by Lemma 3.5 there is degree  $\leq d$  polynomial  $p'$  for which

$$\Pr_{j \sim [m]} (\text{XOR}(z) \neq p'(z)) \leq 2 \exp \left( -D \left( \frac{1}{2} + \frac{d}{2K}, \frac{1}{2} + \frac{2^{d-1} n^d \tau}{d} \right) K \right).$$

Now, let  $T : \mathbb{R}^{nK} \rightarrow \mathbb{R}^K$  be the linear map

$$T(v_1 | v_2 | \dots | v_K) = (\langle v_1, \psi \rangle, \langle v_2, \psi \rangle, \dots, \langle v_K, \psi \rangle)$$

Note that for all  $C \in \mathcal{X}_{n,K}$ ,  $T(\pi(C)) = C(\psi)$ . Consider the degree  $\leq d$  polynomial  $p = p' \circ T$ . We have

$$\begin{aligned} \Pr_{j \sim [m]} (\text{XOR}(C_j(\psi)) \neq p(\pi(C_j))) &= \Pr_{j \sim [m]} (\text{XOR}(C_j(\psi)) \neq p'(T(\pi(C_j)))) \\ &= \Pr_{j \sim [m]} (\text{XOR}(C_j(\psi)) \neq p'(C_j(\psi))) \\ &= \Pr_{z \sim \mathcal{D}(J, \psi)} (\text{XOR}(z) \neq p'(z)) \\ &\leq 2e \left( -D \left( \frac{1}{2} + \frac{d}{2K}, \frac{1}{2} + \frac{2^{d-1} n^d \tau}{d} \right) K \right) \end{aligned}$$

□

### 3.3 The Eeduction

#### Step I: Amplifying the Gap – From XOR to Majority of XORs

For odd  $q$ , we define the predicate  $\text{MXOR}_{q,K} : \{\pm 1\}^{qK} \rightarrow \{\pm 1\}$  by

$$\text{MXOR}_{q,K}(z) = \text{MAJ}(\text{XOR}(z_1, \dots, z_k), \dots, \text{XOR}(z_{(q-1)K+1}, \dots, z_{qK}))$$

A  $(q, K)$ -tuple is an element in  $\mathcal{X}_{n,q,K} := (\mathcal{X}_{n,K})^q$ . For a  $(q, K)$ -tuple  $C = (C^1, \dots, C^q)$  and an assignment  $\psi \in \{\pm 1\}^n$  we denote  $C(\psi) = (C^1(\psi), \dots, C^q(\psi)) \in \{\pm 1\}^{qK}$ . A  $(q, K)$ -formula is a collection  $J = \{C_1, \dots, C_m\}$  of  $(q, K)$ -tuples. An instance to the  $(q, K)$ -MXOR problem is a  $(q, K)$ -formula, and the goal is to find an assignment  $\psi \in \{\pm 1\}^n$  that maximizes  $\text{VAL}_{\psi, \text{MXOR}}(J) := \frac{|\{j : \text{MXOR}_{q,K}(C_j(\psi)) = 1\}|}{m}$ . We define the *value* of  $J$  as

$$\text{VAL}_{\text{MXOR}}(J) := \max_{\psi \in \{\pm 1\}^n} \text{VAL}_{\psi, \text{MXOR}}(J)$$

For  $m = m(n)$ ,  $(q, K) = (q(n), K(n))$  and  $\frac{1}{2} > \eta = \eta(n) > 0$ , we say that the problem  $\text{CSP}_m^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K})$  is easy, if there exists an efficient randomized algorithm,  $\mathcal{A}$  with the following properties. Its input is a  $(q, K)$ -formula  $J$  with  $n$  variables and  $m$  constraints and its output satisfies:

- If  $\text{VAL}_{\text{MXOR}}(J) \geq 1 - \eta$ , then

$$\Pr_{\text{coins of } \mathcal{A}} (\mathcal{A}(J) = \text{"non-random"}) \geq \frac{3}{4}$$

- If  $J$  is random<sup>4</sup> then, with probability  $1 - o_n(1)$  over the choice of  $J$ ,

$$\Pr_{\text{coins of } \mathcal{A}} (\mathcal{A}(J) = \text{"random"}) \geq \frac{3}{4}.$$

LEMMA 3.7. The problem  $\text{CSP}_m^{\text{rand}, \eta}(\text{XOR}_K)$  can be efficiently reduced to  $\text{CSP}_{\lfloor \frac{m}{q} \rfloor}^{\text{rand}, 1-4 \exp(-2(\eta - \frac{1}{2})^2 q)}(\text{MXOR}_{q,K})$ .

We will use the following version of Chernoff's bound from [52].

LEMMA 3.8. Let  $\mathcal{D}$  be a distribution on  $\{0, 1\}^q$  and  $\eta < \frac{1}{2}$ . Assume that for every  $A \subset [q]$  we have  $\Pr_{z \sim \mathcal{D}} (\forall i \in A, z_i = 1) \leq \eta^{|A|}$ . Then

$$\Pr_{z \sim \mathcal{D}} \left( \sum_{i=1}^q z_i \geq \frac{q}{2} \right) \leq \exp \left( -D \left( \frac{1}{2}, \eta \right) q \right) \leq \exp \left( -2 \left( \frac{1}{2} - \eta \right)^2 q \right)$$

<sup>4</sup>To be precise, the  $(q, K)$ -tuples are chosen uniformly, and independently from one another.



PROOF. (of Lemma 3.7) Given a  $K$ -formula  $J = \{C_1, \dots, C_m\} \in \mathcal{X}_{n,K}$  we will produce a  $(q, K)$ -formula  $J'$  as follows. We first randomly throw  $m - m \lfloor \frac{m}{q} \rfloor$  of  $J$ 's  $K$ -tuples. Then, we randomly partition the remaining tuples into  $\lfloor \frac{m}{q} \rfloor$  equally sized ordered bundles. For each such bundle  $\{C_{j_1}, \dots, C_{j_q}\}$ , we add to  $J'$  the  $(q, K)$ -tuple  $C = (C_{j_1}, \dots, C_{j_q})$ . To see that the reduction works note that:

- If  $J$  is random then so is  $J'$ .
- Assume now that  $\text{VAL}_{\text{XOR}}(J) \geq 1 - \eta$ , and let  $\psi \in \{\pm 1\}^n$  be an assignment with  $\text{VAL}_{\psi, \text{XOR}}(J) \geq 1 - \eta$ . Consider a single random bundle  $\{C_{j_1}, \dots, C_{j_q}\}$ . By Lemma 3.8 we have that the probability that for most  $K$ -tuples in the bundle we have  $\text{XOR}_K(C(\psi)) = -1$  is  $\leq \exp\left(-2\left(\eta - \frac{1}{2}\right)^2 q\right)$ . Hence,  $E[1 - \text{VAL}_{\text{MXOR}}(J')] \leq \exp\left(-2\left(\eta - \frac{1}{2}\right)^2 q\right)$ . By Markov's inequality we have that  $\text{VAL}_{\text{MXOR}}(J') \geq 1 - 4\exp\left(-2\left(\eta - \frac{1}{2}\right)^2 q\right)$  w.p.  $\geq \frac{3}{4}$ .

□

### Step II: Making the Sample Scattered – From (MXOR) to (MXOR, $\neg$ MXOR)

A labeled  $(q, K)$ -formula is a collection  $J = \{(C_1, y_1), \dots, (C_m, y_m)\} \in \mathcal{X}_{n,q,K}$ . An instance to the  $(q, K)$ -(MXOR,  $\neg$ MXOR) problem is a labeled  $(q, K)$ -formula, and the goal is to find an assignment  $\psi \in \{\pm 1\}^n$  that maximizes  $\text{VAL}_{\psi, \text{MXOR}}(J) := \frac{|\{j: \text{MXOR}_{q,K}(C_j(\psi)) = y_j\}|}{m}$ . We define the value of  $J$  as

$$\text{VAL}_{\text{MXOR}}(J) := \max_{\psi \in \{\pm 1\}^n} \text{VAL}_{\psi, \text{MXOR}}(J)$$

For  $m = m(n)$ ,  $(q, K) = (q(n), K(n))$  and  $\frac{1}{2} > \eta = \eta(n) > 0$ , we say that the problem

$$\text{CSP}_m^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$$

is easy, if there exists an efficient randomized algorithm,  $\mathcal{A}$  with the following properties. Its input is a labeled  $(q, K)$ -formula  $J$  with  $n$  variables and  $m$  constraints and its output satisfies:

- If  $\text{VAL}_{\text{MXOR}}(J) \geq 1 - \eta$ , then

$$\Pr_{\text{coins of } \mathcal{A}}(\mathcal{A}(J) = \text{"non-random"}) \geq \frac{3}{4}$$

- If  $J$  is random<sup>5</sup> then, with probability  $1 - o_n(1)$  over the choice of  $J$ ,

$$\Pr_{\text{coins of } \mathcal{A}}(\mathcal{A}(J) = \text{"random"}) \geq \frac{3}{4}.$$

LEMMA 3.9. The problem  $\text{CSP}_m^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K})$  can be efficiently reduced to  $\text{CSP}_m^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$ .

PROOF. Given an instance  $J = \{C_1, \dots, C_m\}$  to

$$\text{CSP}_m^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K})$$

, we will produce an instance  $J'$  to

$$\text{CSP}_m^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$$

<sup>5</sup>To be precise, the  $(q, K)$ -tuples and the labels are chosen uniformly, and independently from one another.

as follows. For each  $C_j$ , w.p.  $\frac{1}{2}$  we will add to  $J'$  the pair  $(C_j, 1)$ , and w.p.  $\frac{1}{2}$  we will add the pair  $(C_j', -1)$  where  $C_j' = (C_j^1, \dots, C_j^q)$  is obtained from  $C_j = (C_j^1, \dots, C_j^q)$  by flipping, for each  $C_j^i$ , the sign of the first literal. It is not hard to see that if  $J$  is random then so is  $J'$ . Also, for every  $\psi \in \{\pm 1\}^n$ ,  $\text{VAL}_{\psi, \text{MXOR}}(J) = \text{VAL}_{\psi, \text{MXOR}}(J')$ , and therefore, if  $\text{VAL}_{\text{MXOR}}(J) \geq 1 - \eta$  then  $\text{VAL}_{\text{MXOR}}(J') \geq 1 - \eta$  as well. □

### Step III: Enforcing Pseudo-randomness

We say that a labeled  $(q, K)$ -formula  $J$  is  $(t, \tau)$  pseudo-random if the  $K$ -formula consisting of all the  $K$ -tuples that appear in  $J$  is  $(t, \tau)$ -pseudo-random. For  $m = m(n)$ ,  $(q, K) = (q(n), K(n))$ ,  $(t, \tau) = (t(n), \tau(n))$  and  $\frac{1}{2} > \eta = \eta(n) > 0$ , we say that the problem  $\text{CSP}_{m, (t, \tau)}^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$  is easy, if there exists an efficient randomized algorithm,  $\mathcal{A}$  with the following properties. Its input is a labeled  $(q, K)$ -formula  $J$  with  $n$  variables and  $m$  constraints that is  $(t, \tau)$ -pseudo-random. Its output satisfies:

- If  $\text{VAL}_{\text{MXOR}}(J) \geq 1 - \eta$ , then

$$\Pr_{\text{coins of } \mathcal{A}}(\mathcal{A}(J) = \text{"non-random"}) \geq \frac{3}{4}$$

- If  $J$  is random<sup>6</sup> then, with probability  $1 - o_n(1)$  over the choice of  $J$ ,

$$\Pr_{\text{coins of } \mathcal{A}}(\mathcal{A}(J) = \text{"random"}) \geq \frac{3}{4}.$$

LEMMA 3.10. For  $r = r(n) \geq 4$ , the problem

$$\text{CSP}_{n^r}^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$$

can be efficiently reduced to

$$\text{CSP}_{n^r, (r, n^{-\frac{r}{4}})}^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$$

PROOF. Given an instance  $J$  to

$$\text{CSP}_{n^r}^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$$

we will simply check whether it is  $(r, n^{-\frac{r}{4}})$ -pseudo-random or not. If it not, we will say that  $J$  is not random. Otherwise, we will leave it as is as an instance to

$$\text{CSP}_{n^r, (r, n^{-\frac{r}{4}})}^{\text{rand}, 1-\eta}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$$

To see that this reduction works, note that

- If  $\text{VAL}_{\text{MXOR}}(J) \geq \eta$ , we will either say that it is not random or produce an  $(r, n^{-\frac{r}{4}})$ -pseudo-random instance with value  $\geq \eta$ .

- If  $J$  is random, by lemma 3.2, it is  $(r, n^{-\frac{r}{4}})$ -pseudo-random with probability at least

$$1 - (2n)^K 2 \exp\left(-2n^r n^{-\frac{r}{2}}\right) = 1 - 2 \exp\left(K \log(2n) - 2n^r n^{-\frac{r}{2}}\right) \geq 1 - o_n(1).$$

Hence, w.p.  $\geq 1 - o_n(1)$  the reduction will produce an instance. Now, conditioning on this event, the produced formulas is a random labeled  $(q, K)$ -formula that is  $(r, n^{-\frac{r}{4}})$ -pseudo-random. □

<sup>6</sup>To be precise,  $J$  is chosen uniformly at random from all  $(t, \tau)$ -pseudo-random labeled  $(q, K)$ -formulas.

#### Step IV: From MXOR to Polynomials

Let  $\text{POL}_{u,d}$  be the hypothesis class of all functions

$$h : \{0, -1, 1\}^u \rightarrow \{\pm 1\}$$

that are thresholds of degree  $\leq d$  polynomials. For  $u = u(n)$ ,  $d = d(n)$ ,  $m = m(n)$  and  $\eta = \eta(n)$  we consider the problem  $\text{POL}(u, d)_m^{\text{s-scat}, \eta}$  of distinguishing a strongly-scattered sample from a sample with  $\text{Err}_{\text{POL}_d}(S) \leq \eta$ . Concretely, the input is a sample  $S = \{(x_1, y_1), \dots, (x_m, y_m)\} \subset \{-1, 1, 0\}^u \times \{\pm 1\}$ , and we say that the problem is easy if there exists an efficient randomized algorithm,  $\mathcal{A}$  with the following properties. Its input is such a sample  $S$ , and its output satisfies:

- If  $\text{Err}_{\text{POL}_d}(S) \leq \eta$ , then

$$\Pr_{\text{coins of } \mathcal{A}} (\mathcal{A}(S) = \text{"almost-realizable"}) \geq \frac{3}{4}$$

- If  $S$  is strongly scattered then, with probability  $1 - o_n(1)$  over the choice of the labels,

$$\Pr_{\text{coins of } \mathcal{A}} (\mathcal{A}(J) = \text{"scattered"}) \geq \frac{3}{4}.$$

LEMMA 3.11. For  $d$  such that

$$\frac{K2^d n^d \tau}{d} < d \leq t$$

$\text{CSP}_{m, (t, \tau)}^{\text{rand}, 1-\eta}(\text{MXOR}_{q, K}, \neg \text{MXOR}_{q, K})$  can efficiently reduced to  $\text{POL}(nqK, d)_m^{\text{s-scat}, \eta'}$  where

$$\eta' = \eta + 2q \exp \left( -D \left( \frac{1}{2} + \frac{d}{2K}, \frac{1}{2} + \frac{2^{d-1} n^d \tau}{d} \right) K \right)$$

PROOF. Given a labeled  $(q, K)$ -formula

$$J = \{(C_1, y_1), \dots, (C_m, y_m)\} \subset \mathcal{X}_{n, q, K} \times \{\pm 1\}$$

we will simply produce the sample

$$S = \{(\pi(C_1), y_1), \dots, (\pi(C_m), y_m)\} \subset \{-1, 1, 0\}^{nqK} \times \{\pm 1\}$$

Here,  $\Pi : \mathcal{X}_{n, q, K} \rightarrow \{-1, 1, 0\}^{nqK}$  is the mapping

$$\pi(C^1, \dots, C^q) = (\pi(C^1), \dots, \pi(C^q))$$

, where  $\pi : \mathcal{X}_{n, K} \rightarrow \{-1, 1, 0\}^{nK}$  is as defined in section 3.2.

Clearly if  $J$  is random then  $S$  is scattered. It remains to show that if  $\text{VAL}_{\text{MXOR}}(J) \geq 1 - \eta$  and  $J$  is  $(t, \tau)$ -pseudo-random then there is a degree  $\leq d$  polynomial that errs on  $\leq \eta'$  fraction of the examples.

Indeed, let  $\psi \in \{\pm 1\}^n$  be an assignment that satisfies  $\geq 1 - \eta$  fraction of  $J$ 'th  $(q, K)$ -tuples. By lemma 3.6, there is a polynomial  $p : \{-1, 1, 0\}^{nK} \rightarrow \mathbb{R}$  of degree  $\leq d$  that satisfies  $p(\pi(C)) = \text{XOR}(C(\psi))$  on

$$1 - 2 \exp \left( -D \left( \frac{1}{2} + \frac{d}{2K}, \frac{1}{2} + \frac{2^{d-1} n^d \tau}{d} \right) K \right)$$

fraction of  $J$ 's  $K$ -tuples (here,  $J$ 's  $K$ -tuples are the  $K$ -tuples that appear in one of  $J$ 's  $(q, K)$ -tuples).

Let  $p' : \{-1, 1, 0\}^{nqK} \rightarrow \mathbb{R}$  be the degree  $\leq d$  polynomial  $p'(x) = \sum_{j=1}^q p(z_{(j-1)K+1}, \dots, z_{jK})$ . It is not hard to check that  $\text{sign}(p'(\pi(C))) = \text{MXOR}(C(\psi))$  on

$$1 - 2q \exp \left( -D \left( \frac{1}{2} + \frac{d}{2K}, \frac{1}{2} + \frac{2^{d-1} n^d \tau}{d} \right) K \right)$$

fraction of  $J$ 's  $(q, K)$ -tuples.

Therefore, we have that  $\text{sign}(p'(\pi(C_i))) \neq y_i$  for at most  $\eta + 2q \exp \left( -D \left( \frac{1}{2} + \frac{d}{2K}, \frac{1}{2} + \frac{2^{d-1} n^d \tau}{d} \right) K \right)$  fraction of the samples in  $S$ .  $\square$

#### Step V: From Polynomials to Halfspaces

Let  $u = u(n)$ ,  $m = m(n)$  and  $\eta = \eta(n)$ . Similarly to  $\text{POL}(u, d)_m^{\text{s-scat}, \eta}$ , we define  $\text{HALF}(u)_m^{\text{s-scat}, \eta}$  as the problem of distinguishing a strongly-scattered sample consisting of  $m$  examples in  $\{\pm 1\}^u \times \{\pm 1\}$ , from a sample with  $\text{Err}_{\text{HALF}}(S) \leq \eta$ .

LEMMA 3.12. The problem  $\text{POL}(u, d)_m^{\text{s-scat}, \eta}$  can efficiently reduced to  $\text{HALF}(2(u+1)^d)_m^{\text{s-scat}, \eta}$

PROOF. It will be convenient to decompose the reduction into two steps, where the second only deals with the issue of replacing  $\{-1, 1, 0\}^{(u+1)^d}$  by  $\{\pm 1\}^{2(u+1)^d}$ . Given a sample

$$S = \{(x_1, y_1), \dots, (x_m, y_m)\} \subset \{-1, 1, 0\}^u \times \{\pm 1\}$$

the reduction will first produce the sample

$$\rho(S) = \{(\rho(x_1), y_1), \dots, (\rho(x_m), y_m)\} \subset \{-1, 1, 0\}^{(u+1)^d} \times \{\pm 1\},$$

where  $\rho : \{-1, 1, 0\}^u \rightarrow \{-1, 1, 0\}^{(u+1)^d}$  is defined as follows.

We index the coordinates in  $\{-1, 1, 0\}^{(u+1)^d}$  by the functions in  $([u] \cup \{*\})^{[d]}$  and we let

$$\forall f \in ([u] \cup \{*\})^{[d]}, \quad \rho_f(x) = \prod_{j=1}^d x_{f(j)}.$$

(where  $x_* := 1$ ). It is not hard to see that the collection of degree  $\leq d$  polynomial functions from  $\{-1, 1, 0\}^u$  to  $\mathbb{R}$  equals to

$$\{x \mapsto \langle w, \rho(x) \rangle \mid w \in \mathbb{R}^{(u+1)^d}\}.$$

Hence,  $\text{Err}_{\text{POL}_{u,d}}(S) = \text{Err}_{\text{HALF}}(\rho(S))$ .

In the second step the reduction will produce the sample

$$\Psi(\rho(S)) = \{(\Psi(\rho(x_1)), y_1), \dots, (\Psi(\rho(x_m)), y_m)\} \subset \{\pm 1\}^{2(u+1)^d} \times \{\pm 1\}$$

$\Psi : \{-1, 1, 0\}^{(u+1)^d} \rightarrow \{\pm 1\}^{2(u+1)^d}$  that is defined as follows:

$$\Psi(x) = (\Psi(x_1), \dots, \Psi(x_n)),$$

$$\text{where for } x \in \{0, -1, 1\}, \quad \Psi(x) = \begin{cases} (1, 1) & x = 1 \\ (-1, -1) & x = -1 \\ (-1, 1) & x = 0 \end{cases}$$

is not hard to see that for every  $w \in \mathbb{R}^{(u+1)^d}$  we have  $h_w = h_{w'} \circ \Psi$  where  $w' = \frac{1}{2}(w_1, w_1, \dots, w_{(u+1)^d}, w_{(u+1)^d})$ . Therefore we have

$$\text{Err}_{\text{HALF}}(\Psi(\rho(S))) \leq \text{Err}_{\text{HALF}}(\rho(S)) = \text{Err}_{\text{POL}_{u,d}}(S)$$

Also, it is clear that if  $S$  is strongly scattered then so is  $\Psi(\rho(S))$ . To summarize, the reduction  $S \mapsto \Psi(\rho(S))$  forms a reduction from  $\text{POL}(u, d)_m^{\text{s-scat}, \eta}$  to  $\text{HALF}(2(u+1)^d)_m^{\text{s-scat}, \eta}$ .  $\square$

#### Connecting the Dots

We start with the hard problem  $\text{CSP}_{n, r}^{\text{rand}, \eta}(\text{XOR}_K)$ . Using Lemma 3.7 and Lemma 3.9, we reduce it to

$$\text{CSP}_{\lfloor \frac{n}{q} \rfloor}^{\text{rand}, 2^{-K}}(\text{MXOR}_{q, K}, \neg \text{MXOR}_{q, K})$$

Since  $\eta$  is bounded away from  $\frac{1}{2}$ , this can be done with  $q = C \cdot K$  for a constant  $C = C(\eta)$ . We can reduce farther to  $\text{CSP}_{n^{r-1}}^{\text{rand}, 2^{-K}}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$  (by simply throwing away random  $\lfloor \frac{n^r}{q} \rfloor - n^{r-1}$  tuples from the input formula)

Now, we use Lemma 3.10 to reduce to

$$\text{CSP}_{n^{r-1}, (r-1, n - \frac{r-1}{4})}^{\text{rand}, 2^{-K}}(\text{MXOR}_{q,K}, \neg \text{MXOR}_{q,K})$$

Using Lemma 3.11, for every  $d$  such that  $\frac{K2^d n^{d-r+1}}{d} < d \leq r$  we can reduce to

$$\text{POL}(nCK^2, d)^{\text{s-scat}, \eta'}$$

for

$$\eta' \leq 2^{-K} + 2CK \exp \left( -2 \left( \frac{d}{2K} - \frac{2^{d-1} n^{d-r+1}}{d} \right)^2 K \right).$$

We will choose  $d$  such that  $d = o(r)$  and  $d = \omega(\sqrt{\log(K)K})$  (the exact choice depend on the assumption we start with and will be specified later). It is not hard to check that for such a choice, it holds that for large enough  $r$  and  $K$ ,  $\frac{K2^d n^{d-r+1}}{d} < d \leq r-1$  and  $\eta' \leq 2^{-C_1 \frac{d^2}{K}}$  for a constant  $C_1 > 0$ .

Now, by Lemma 3.12 we can reduce farther to

$$\text{HALF}(n^{4d})^{\text{s-scat}, \eta'}$$

Putting  $n' = n^{4d}$ , we conclude that  $\text{HALF}(n')^{\text{s-scat}, \eta'}$ . Since  $d = o(r)$ , this can be reduced farther (by simply randomly throwing examples from the input sample) to  $\text{HALF}(n')_{n'^a}^{\text{s-scat}, \eta'}$  for any constant  $a > 0$ . By Theorem 2.1, we conclude that there is no efficient learning algorithm that can return a hypothesis with non-trivial error on a distribution  $\mathcal{D}$  on  $\{\pm 1\}^{n'} \times \{\pm 1\}$  that is  $\eta'$ -almost realizable by halfspaces.

For the choice of  $d$  and the calculation of  $\eta'$  in terms of  $n'$ , we split to two cases, according to the assumption we started with.

**Case 1 (Assumption 1.1).** Here,  $r = c \log(K) \sqrt{K}$  and  $K$  is constant. We will choose  $d = \log^{\frac{2}{3}}(K) \sqrt{K}$ . We will have  $\eta' \leq 2^{-\Omega(\log^{\frac{4}{3}}(K))}$ . Since  $K$  can be arbitrarily large,  $\eta'$  can be arbitrarily small.

**Case 2 (Assumption 1.2).** Here,  $r = cK$  and  $K = \log^s(n)$ . Here, we will choose  $d = \frac{K}{\log \log K}$ . Note that

$$\begin{aligned} \eta' &\leq 2^{-C_3 \frac{K}{(\log \log K)^2}} \\ &= 2^{-C_3 \frac{\log^s(n)}{(\log \log K)^2}} \\ &\leq 2^{-\log^{s-1}(n)} \end{aligned}$$

Now,  $\log(n') = 4d \log(n) = \frac{4}{\log \log K} \log^{s+1}(n) \leq \log^{s+1}(n)$ . Hence,

$$\eta' \leq 2^{-\log^{\frac{s-1}{s+1}}(n')}$$

Since  $s$  can be arbitrarily large, we can get  $\eta' \leq 2^{-\log^{1-\nu}(n')}$

### 3.4 How to Prove Theorems 1.4 and 1.5?

We next briefly explain how our argument can be extended to prove Theorems 1.4 and 1.5.

Theorem 1.4 is proved analogously to Theorem 1.3. The only difference is that we have to verify certain properties

of the vector defining the halfspace that almost realizes the sample. Namely, we have to make sure that (i) the sum of its coefficients is polynomial in the dimension and (ii) that whenever we guarantee that its prediction is correct, its inner product with the instance is  $\geq 1$  in absolute value. These two facts can be straight forwardly verified, by carefully going over the proof.

Theorem 1.5 can also be proved analogously to Theorem 1.3. The only difference is that we have to verify that (i) the problem we start with cannot be solved efficiently using statistical queries and that (ii) the reduction steps can be done using statistical queries. This strategy can indeed be carried out, if one is using the result [39] of Feldman, Perkins and Vempala, that shows the SQ-hardness of the initial problem. However, a simpler strategy can be applied. Concretely, we can use our argument to reduce the uniform  $K$ -sparse-parity learning problem to the problems of learning halfspaces. In this parity problem, the learning algorithm is given an access to examples  $(x, h(x))$  where  $x \in \{\pm 1\}^n$  is uniformly distributed and  $h$  computes the XOR of  $K$  unknown variables. It is known [22, 21] that no SQ-algorithm for the problem can return a classifier with error  $\leq \frac{1}{2} - n^{-o(K)}$  using  $n^{o(K)}$  queries with error parameters  $n^{-o(K)}$ . Lemma 3.5 shows that for any XOR of  $K$  variables, there is a polynomial threshold function of degree  $d$  that agree with  $h$  on all but  $2 \exp\left(-\frac{d^2}{2K}\right)$ -fraction of the examples (w.r.t. the uniform distribution). As in the proof of Theorem 1.3, this fact establishes a reduction to the problem of agnostically learning halfspaces. All is left to show is that this reduction can be implemented using statistical queries.

Indeed, the reduction is of the following form. It introduces a mapping  $\Psi : \{\pm 1\}^n \rightarrow \{\pm 1\}^{n'}$  and reduce the original learning problem to the problem of learning  $(\Psi(x), h(x))$ , where  $x$  is sampled from the original distribution (the uniform distribution in our case). Now, given a statistical query  $Q : \{\pm 1\}^{n'} \times \{\pm 1\} \rightarrow \{\pm 1\}$ , in order to evaluate  $E_x[Q(\Psi(x), h(x))]$ , we can simply query the oracle of the original problem with the function  $\tilde{Q}(x, y) := Q(\Psi(x), y)$ .

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## 5. REFERENCES

- [1] M. Alekhnovich. More on average case vs approximation complexity. In *Foundations of Computer Science, 2003. Proceedings. 44th Annual IEEE Symposium on*, pages 298–307. IEEE, 2003.
- [2] M. Alekhnovich, S. Arora, and I. Tzourakis. Towards strong nonapproximability results in the lovász-schrijver hierarchy. In *Proceedings of the thirty-seventh annual ACM symposium on Theory of computing*, pages 294–303. ACM, 2005.
- [3] S. Allen, R. O'Donnell, and D. Witmer. How to refute a random csp? In *FOCS*, 2015.

- [4] E. Amaldi and V. Kann. On the approximability of minimizing nonzero variables or unsatisfied relations in linear systems. *Theoretical Computer Science*, 209(1):237–260, 1998.
- [5] E. Amaldi and V. Kann. On the approximability of minimizing nonzero variables or unsatisfied relations in linear systems. In *Theoretical Computer Science*, 1998.
- [6] B. Applebaum, B. Barak, and A. Wigderson. Public key cryptography from different assumptions. In *STOC*, 2010.
- [7] B. Applebaum, B. Barak, and D. Xiao. On basing lower-bounds for learning on worst-case assumptions. In *Foundations of Computer Science, 2008. FOCS'08. IEEE 49th Annual IEEE Symposium on*, pages 211–220. IEEE, 2008.
- [8] S. Arora, L. Babai, J. Stern, and Z. Sweedyk. The hardness of approximate optima in lattices, codes, and systems of linear equations. In *Foundations of Computer Science, 1993. Proceedings., 34th Annual Symposium on*, pages 724–733. IEEE, 1993.
- [9] J. Aspnes, R. Beigel, M. Furst, and S. Rudich. The expressive power of voting polynomials. *Combinatorica*, 14(2):135–148, 1994.
- [10] P. Awasthi, M.-F. Balcan, and P. Long. The power of localization for efficiently learning linear separators with noise. In *STOC*, 2014.
- [11] B. Barak, G. Kindler, and D. Steurer. On the optimality of semidefinite relaxations for average-case and generalized constraint satisfaction. In R. D. Kleinberg, editor, *ITCS*, pages 197–214, 2013.
- [12] B. Barak and A. Moitra. Tensor prediction, rademacher complexity and random 3-xor. In *Arxiv preprint arXiv:1501.06521 v2*, 2015.
- [13] P. Beame, R. Karp, T. Pitassi, and M. Saks. On the complexity of unsatisfiability proofs for random k-cnf formulas. In *Proceedings of the thirtieth annual ACM symposium on Theory of computing*, pages 561–571. ACM, 1998.
- [14] P. Beame and T. Pitassi. Simplified and improved resolution lower bounds. In *Foundations of Computer Science, 1996. Proceedings., 37th Annual Symposium on*, pages 274–282. IEEE, 1996.
- [15] R. Beigel, N. Reingold, and D. Spielman. The perceptron strikes back. In *Structure in Complexity Theory Conference, 1991., Proceedings of the Sixth Annual*, pages 286–291. IEEE.
- [16] S. Ben-David, N. Eiron, and P. Long. On the difficulty of approximately maximizing agreements. In *JCSS*, 2003.
- [17] S. Ben-David, D. Loker, N. Srebro, and K. Sridharan. Minimizing the misclassification error rate using a surrogate convex loss. In *ICML*, 2012.
- [18] E. Ben-Sasson. Expansion in proof complexity. In *Hebrew University*. Citeseer, 2001.
- [19] E. Ben-Sasson and A. Wigderson. Short proofs are narrow—resolution made simple. In *Proceedings of the thirty-first annual ACM symposium on Theory of computing*, pages 517–526. ACM, 1999.
- [20] E. Blais, R. O’Donnell, and K. Wimmer. Polynomial regression under arbitrary product distributions. In *COLT*, 2008.
- [21] A. Blum, M. Furst, J. Jackson, M. Kearns, Y. Mansour, and S. Rudich. Weakly learning dnf and characterizing statistical query learning using fourier analysis. In *Proceedings of the twenty-sixth annual ACM symposium on Theory of computing*, pages 253–262. ACM, 1994.
- [22] A. Blum, A. Kalai, and H. Wasserman. Noise-tolerant learning, the parity problem, and the statistical query model. *Journal of the ACM (JACM)*, 50(4):506–519, 2003.
- [23] N. Bshouty and L. Burroughs. Maximizing agreements and coagnostic learning. In *Theoretical Computer Science*, 2006.
- [24] J. Buresh-Oppenheim, N. Galesi, S. Hoory, A. Magen, and T. Pitassi. Rank bounds and integrality gaps for cutting planes procedures. In *Foundations of Computer Science, 2003. Proceedings. 44th Annual IEEE Symposium on*, pages 318–327. IEEE, 2003.
- [25] A. Coja-Oghlan, C. Cooper, and A. Frieze. An efficient sparse regularity concept. *SIAM Journal on Discrete Mathematics*, 23(4):2000–2034, 2010.
- [26] A. Coja-Oghlan, A. Goerdt, and A. Lanka. Strong refutation heuristics for random k-sat. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 310–321. Springer, 2004.
- [27] C. Cortes and V. Vapnik. Support-vector networks. *Machine Learning*, 20(3):273–297, September 1995.
- [28] N. Cristianini and J. Shawe-Taylor. *An Introduction to Support Vector Machines*. Cambridge University Press, 2000.
- [29] N. Cristianini and J. Shawe-Taylor. *Kernel Methods for Pattern Analysis*. Cambridge University Press, 2004.
- [30] A. Daniely. A ptas for agnostically learning halfspaces. In *COLT*, 2015.
- [31] A. Daniely, N. Linial, and S. Shalev-Shwartz. More data speeds up training time in learning halfspaces over sparse vectors. In *NIPS*, 2013.
- [32] A. Daniely, N. Linial, and S. Shalev-Shwartz. The complexity of learning halfspaces using generalized linear methods. *COLT*, 2014.
- [33] A. Daniely, N. Linial, and S. Shalev-Shwartz. From average case complexity to improper learning complexity. *STOC*, 2014.
- [34] A. Daniely and S. Shalev-Shwartz. Complexity theoretic limitations on learning dnf’s. In *Arxiv preprint arXiv:1404.3378 v1*, 2014.
- [35] M. Davis, G. Logemann, and D. Loveland. A machine program for theorem-proving. *Communications of the ACM*, 5(7):394–397, 1962.
- [36] U. Feige. Relations between average case complexity and approximation complexity. In *Proceedings of the thirty-fourth annual ACM symposium on Theory of computing*, pages 534–543. ACM, 2002.
- [37] U. Feige and E. Ofek. Easily refutable subformulas of large random 3cnf formulas. In *Automata, languages and programming*, pages 519–530. Springer, 2004.
- [38] V. Feldman, P. Gopalan, S. Khot, and A. Ponnuswami. New results for learning noisy parities and halfspaces. In *In Proceedings of the 47th*

- Annual IEEE Symposium on Foundations of Computer Science*, 2006.
- [39] V. Feldman, W. Perkins, and S. Vempala. On the complexity of random satisfiability problems with planted solutions. In *STOC*, 2015.
  - [40] Y. Freund and R. Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. In *EUROCOLT*, pages 23–37. Springer-Verlag, 1995.
  - [41] V. Guruswami and P. Raghavendra. Hardness of learning halfspaces with noise. In *Proceedings of the 47th Foundations of Computer Science (FOCS)*, 2006.
  - [42] A. Haken. The intractability of resolution. *Theoretical Computer Science*, 39:297–308, 1985.
  - [43] J. Håstad. Some optimal inapproximability results. *Journal of the ACM (JACM)*, 48(4):798–859, 2001.
  - [44] A. Kalai, A. Klivans, Y. Mansour, and R. Servedio. Agnostically learning halfspaces. In *Proceedings of the 46th Foundations of Computer Science (FOCS)*, 2005.
  - [45] A. T. Kalai, A. R. Klivans, Y. Mansour, and R. A. Servedio. Agnostically learning halfspaces. *SIAM Journal on Computing*, 37(6):1777–1805, 2008.
  - [46] R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, 1972.
  - [47] M. Kearns. Efficient noise-tolerant learning from statistical queries. In *STOC*, pages 392–401, 1993.
  - [48] M. Kearns and M. Li. Learning in the presence of malicious errors. *SIAM Journal on Computing*, 22(4):807–837, Aug. 1993.
  - [49] M. Kearns and L. G. Valiant. Cryptographic limitations on learning Boolean formulae and finite automata. In *STOC*, pages 433–444, May 1989.
  - [50] A. Klivan and P. Kothari. Embedding hard learning problems in gaussian space. In *RANDOM*, 2014.
  - [51] A. R. Klivans and A. A. Sherstov. Cryptographic hardness for learning intersections of halfspaces. In *FOCS*, 2006.
  - [52] N. Linial and Z. Luria. Chernoff’s Inequality - A very elementary proof. *Arxiv preprint arXiv:1403.7739 v2*, 2014.
  - [53] P. Long and R. Servedio. Learning large-margin halfspaces with more malicious noise. In *NIPS*, 2011.
  - [54] L. Pitt and L. Valiant. Computational limitations on learning from examples. *Journal of the Association for Computing Machinery*, 35(4):965–984, Oct. 1988.
  - [55] F. Rosenblatt. The perceptron: A probabilistic model for information storage and organization in the brain. *Psychological Review*, 65:386–407, 1958. (Reprinted in *Neurocomputing* (MIT Press, 1988).).
  - [56] F. Rosenblatt. *Principles of Neurodynamics*. Spartan, New York, 1962.
  - [57] R. Schapire. The strength of weak learnability. In *FOCS*, pages 28–33, Oct. 1989.
  - [58] R. E. Schapire and Y. Freund. *Boosting: Foundations and algorithms*. MIT press, 2012.
  - [59] G. Schoenebeck. Linear level lasserre lower bounds for certain k-csps. In *Foundations of Computer Science, 2008. FOCS’08. IEEE 49th Annual IEEE Symposium on*, pages 593–602. IEEE, 2008.
  - [60] B. Schölkopf and A. J. Smola. *Learning with Kernels: Support Vector Machines, Regularization, Optimization and Beyond*. MIT Press, 2002.
  - [61] V. N. Vapnik. *Statistical Learning Theory*. Wiley, 1998.