COMP 546

Lecture 17

Linear Systems 2:

Fourier transform, filtering, convolution theorem

Tues. March 20, 2018

Recall last lecture

- convolution
- special behavior of sines and cosines
- complex numbers and Euler's formula

Today

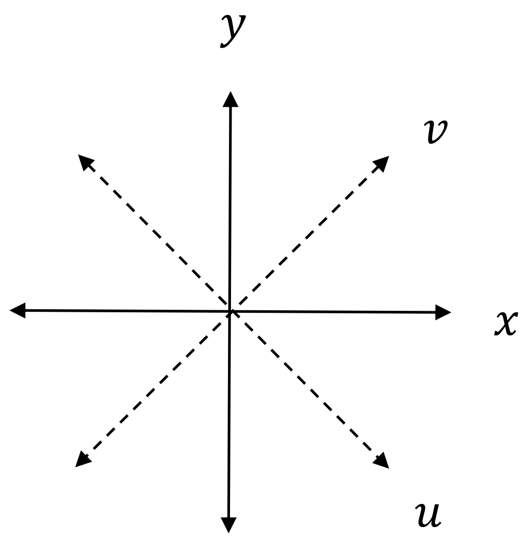
- Fourier transform
- convolution theorem
- filtering

Key idea from linear algebra: orthonormal basis vectors

Example:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

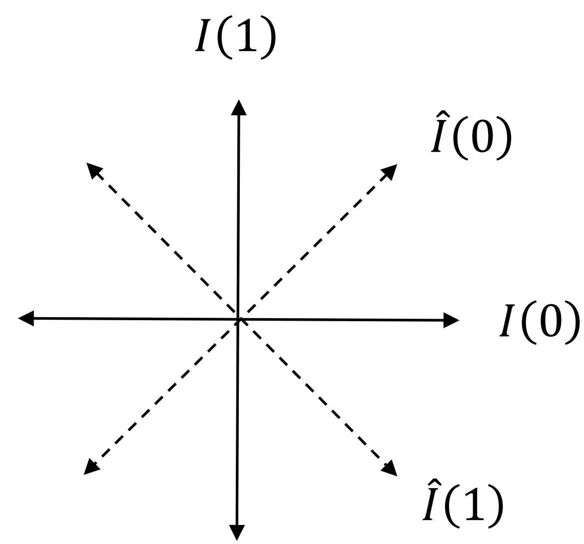


Fourier transform uses orthogonal basis vectors

Example:

$$\begin{bmatrix} \hat{I}(0) \\ \hat{I}(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} I(0) \\ I(1) \end{bmatrix}$$

$$\begin{bmatrix} I(0) \\ I(1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{I}(0) \\ \hat{I}(1) \end{bmatrix}$$



(1D) Fourier analysis

Fourier transform

map N-dimensional delta function (impulse function) basis
to an N-dimensional sinusoid function basis

$$\begin{tikzcd}[column sep=huge] \begin{bmatrix}\hat{I}(0)\\ \vdots \\ \vdots \\ \hat{I}(N-1)\end{bmatrix} \arrow[r, bend right, "\mathbf{F}^{-1}"'] & \begin{bmatrix}I(0)\\ \vdots \\ \vdots \\ I(N-1)\end{bmatrix} \arrow[l, bend right, "\mathbf{F}"'] \end{tikzcd}$$

Inverse Fourier transform

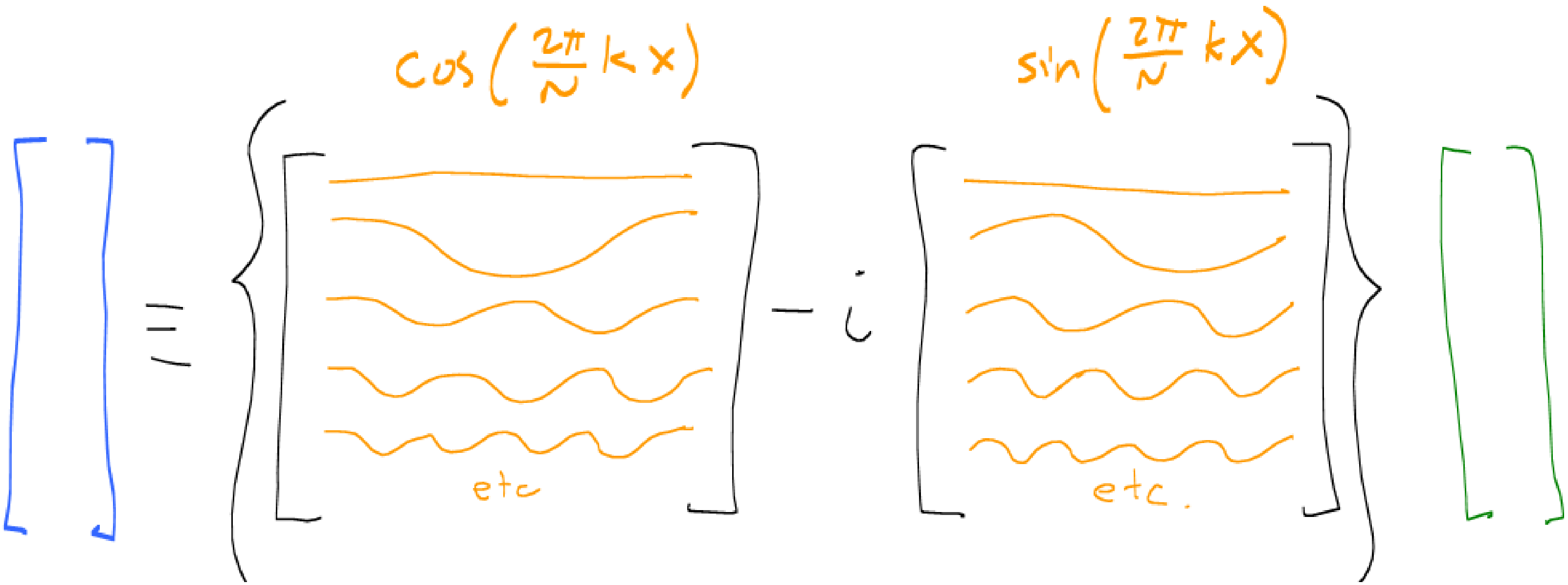
Fourier Transform

$$\hat{I}(k) = \sum_{x=0}^{N-1} \left(\cos\left(\frac{2\pi}{N} kx\right) - \; i \sin\left(\frac{2\pi}{N} kx\right) \right) I(x)$$

$$e^{-i \frac{2\pi}{N} k x}$$

$$\hat{I}(k) = \mathbf{F}\, I(x)$$

Fourier transform



$$\hat{I}(k) = \mathbf{F} \; I(x)$$

Define N x N Fourier transform matrix

$$\mathbf{F}_{k,x} \equiv e^{-i\frac{2\pi}{N}kx}$$

Claim: (see lecture notes for proof)

$$\mathbf{F}^{-1} = \frac{1}{N}\,\bar{\mathbf{F}}$$

where

$$\bar{\mathbf{F}}_{k,x} \equiv e^{\,i\frac{2\pi}{N}kx}$$

$$\hat{I}(k) = \left|\hat{I}(k)\right| e^{i\,\phi(k)}$$

amplitude
spectrum

phase
spectrum

Convolution Theorem

Let $I(x)$ and $h(x)$ be defined on $x \in \{0, 1, \ldots, N-1\}$.

$$\boldsymbol{F}\{\ I(x) * h(x)\ \} = \boldsymbol{F}\, I(x)\ \boldsymbol{F}\, h(x)$$

$$= \hat{I}(k) \quad \hat{h}(k)$$

See lecture notes for proof.

Convolution Theorem

Let $I(x)$ and $h(x)$ be defined on $x \in \{0, 1, \ldots, N-1\}$.

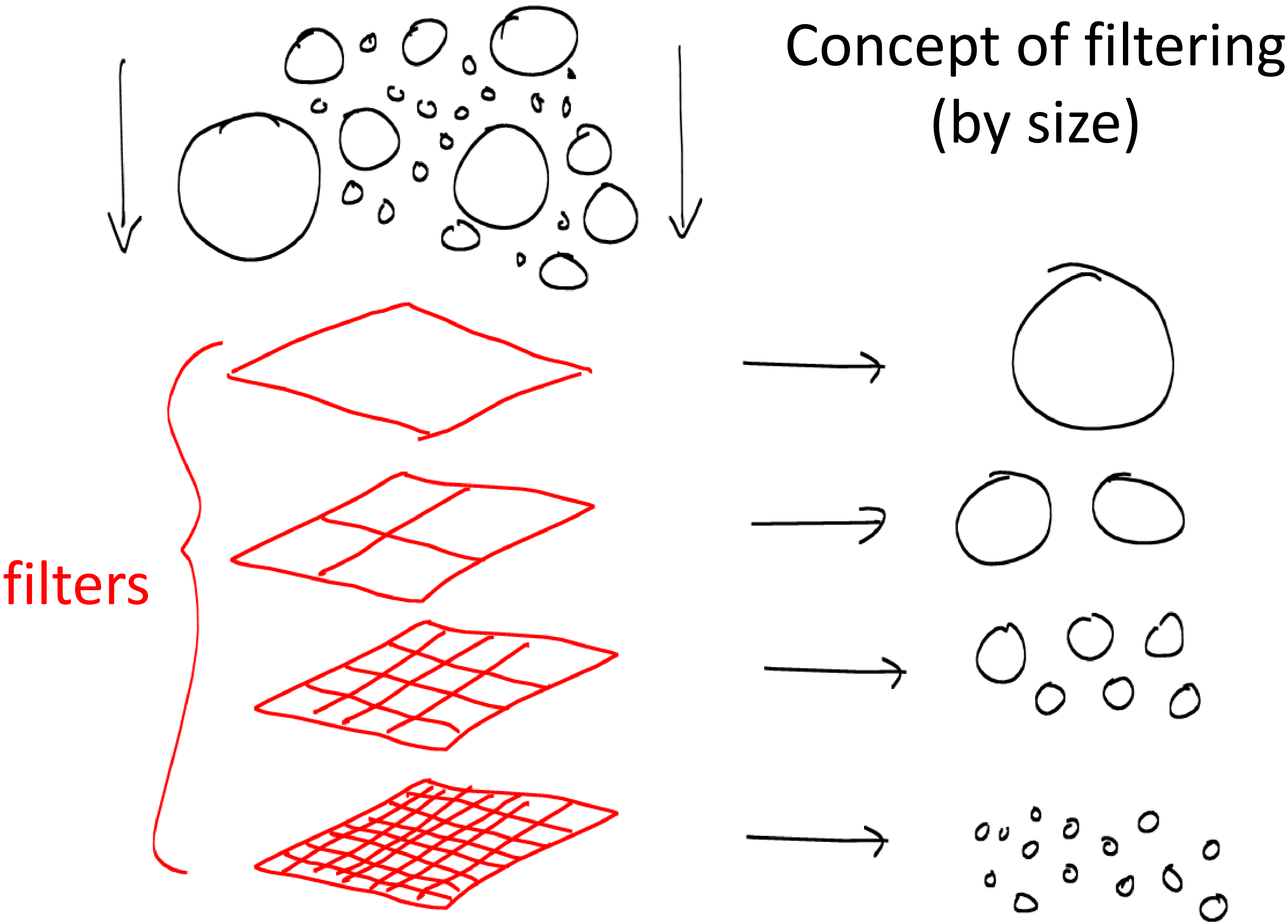
$$\boldsymbol{F} \{\ I(x) * h(x)\ \} \ = \ \boldsymbol{F}\, I(x)\ \ \boldsymbol{F}\, h(x)$$

$$= \ \ \hat{I}(k) \quad \hat{h}(k)$$

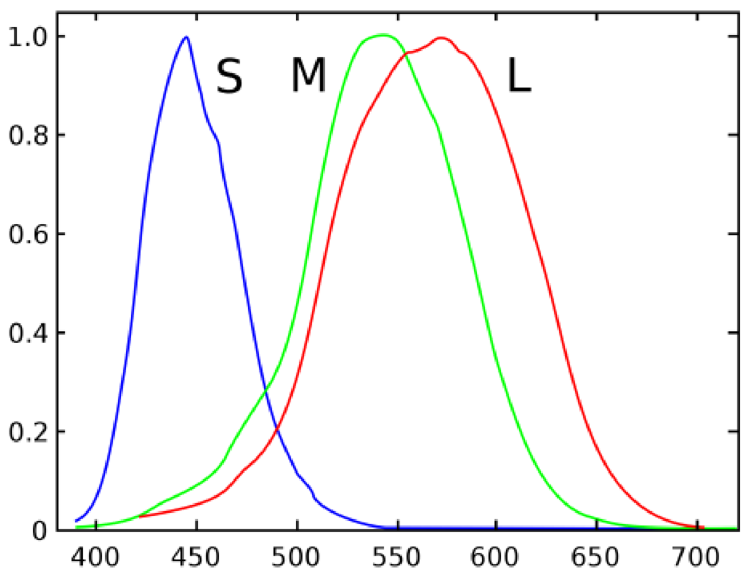
$$= \ \left|\hat{I}(k)\right| \ \left|\hat{h}(k)\right| \quad e^{-i\,\phi_I(k)} \ e^{-i\,\phi_h(k)}$$

Convolving an image $I(x)$ with a filter $h(x)$ changes the amplitude and phase of each frequency component.

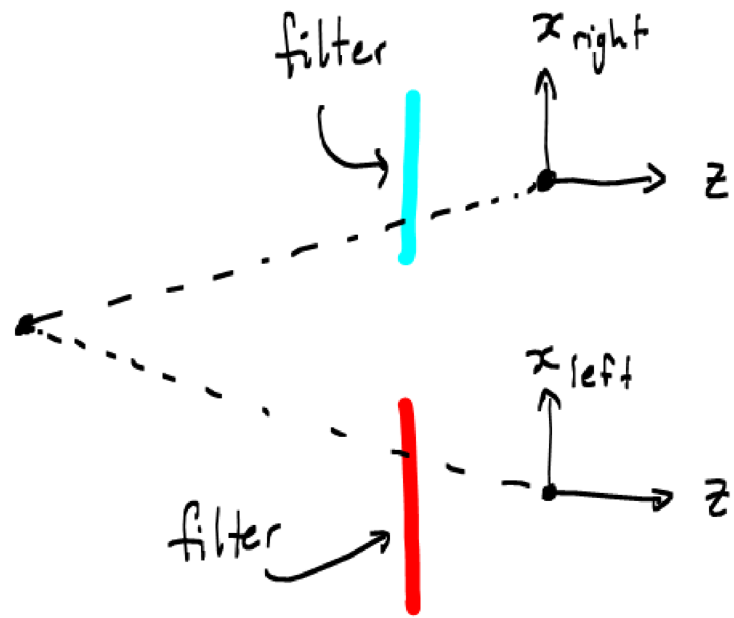
Concept of filtering (by size)



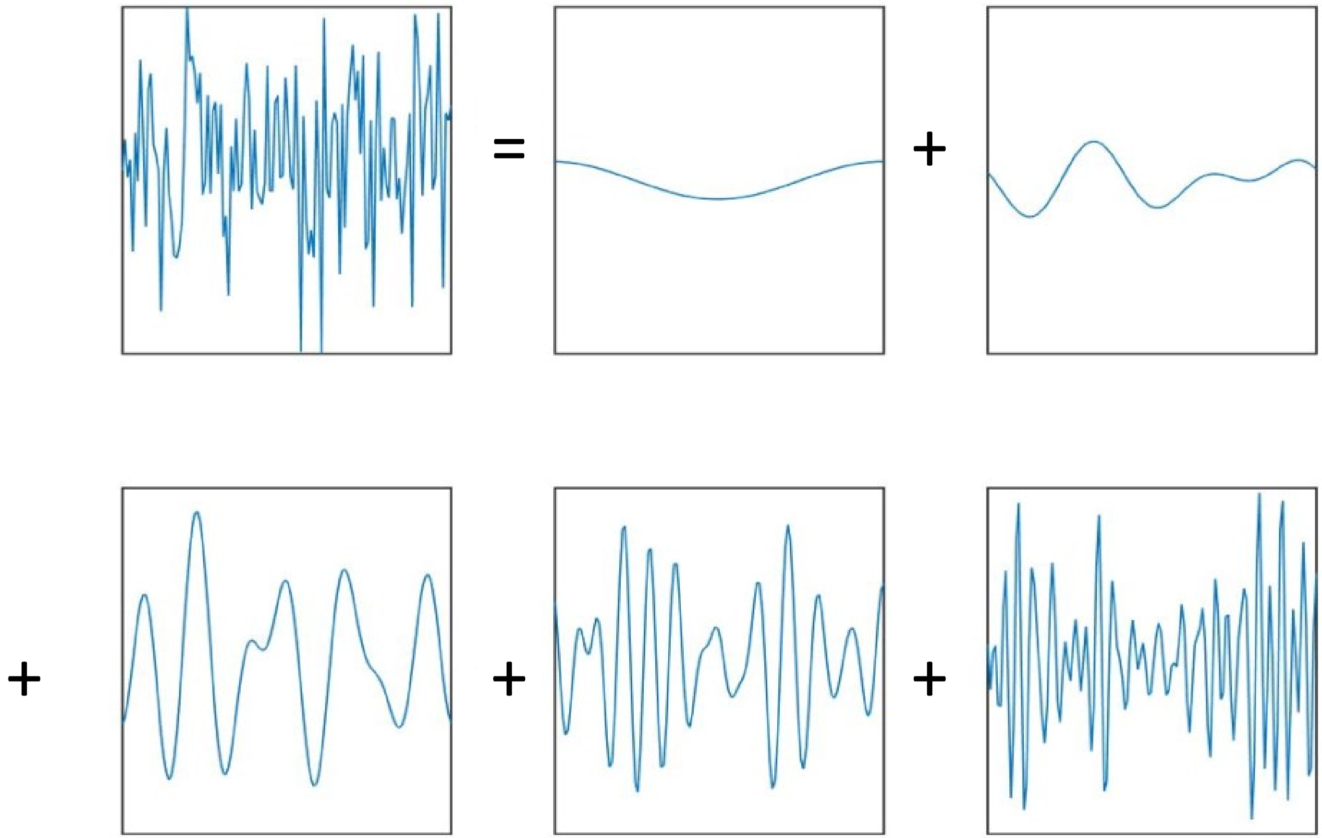
Color filtering
(by frequency or wavelength)



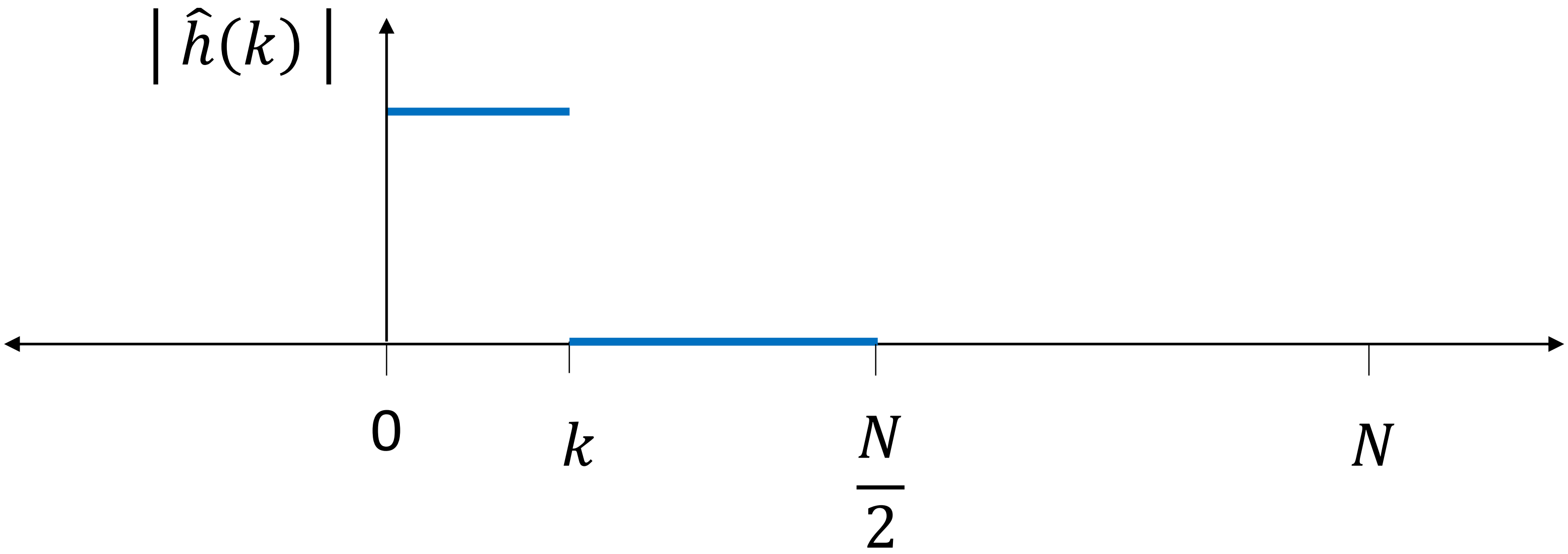
wavelength (1/frequency)



Linear Filtering (by frequency “band”)

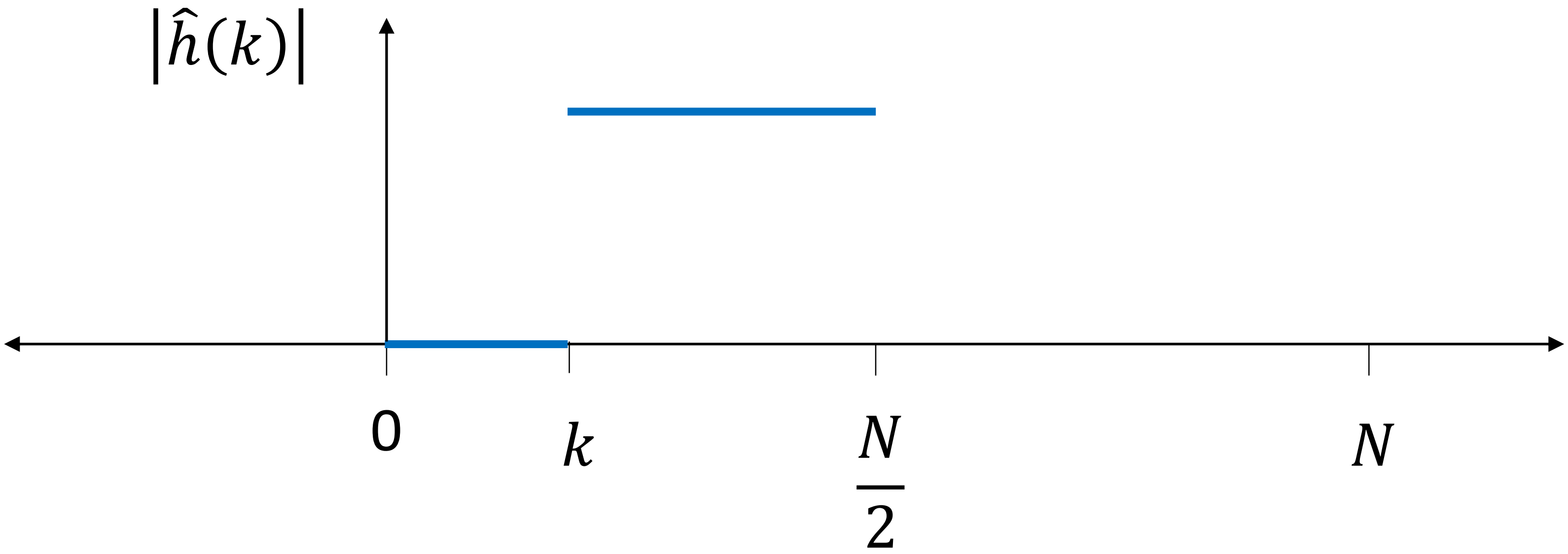


Ideal Low Pass Filter $h(x)$

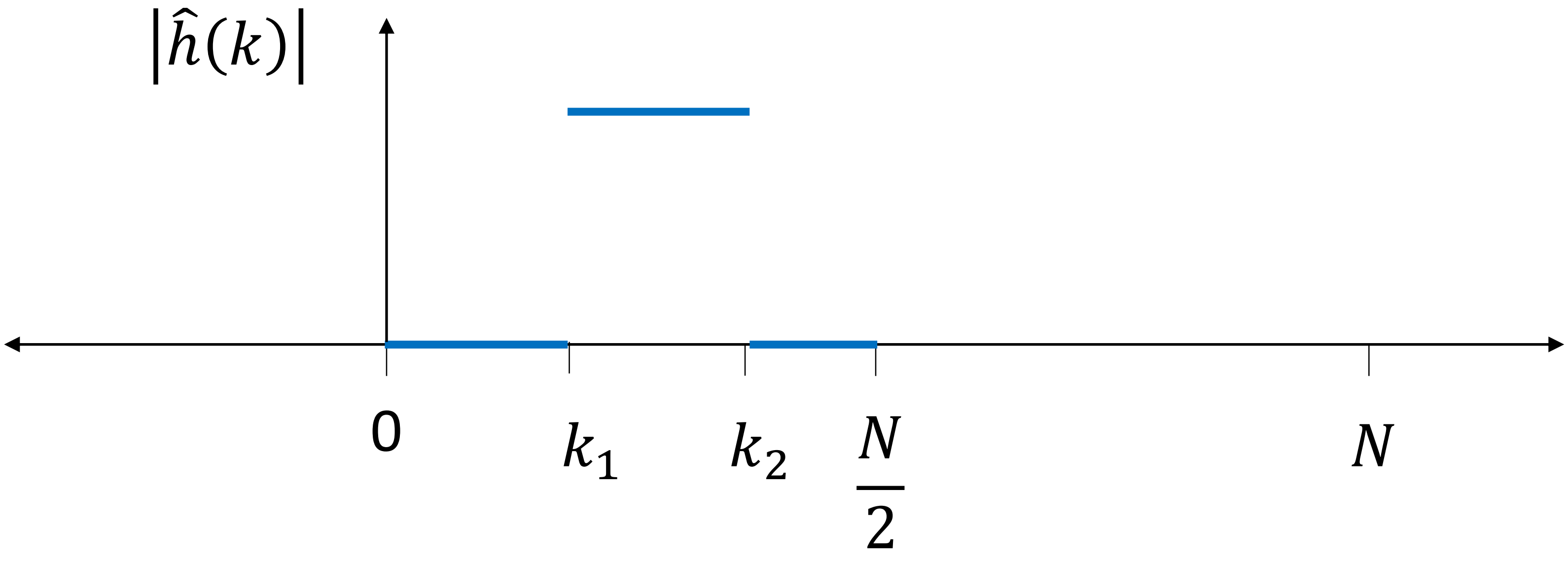


Only consider up to N/2 because of the conjugacy property (coming soon).

Ideal High Pass Filter $h(x)$



Ideal bandpass filter



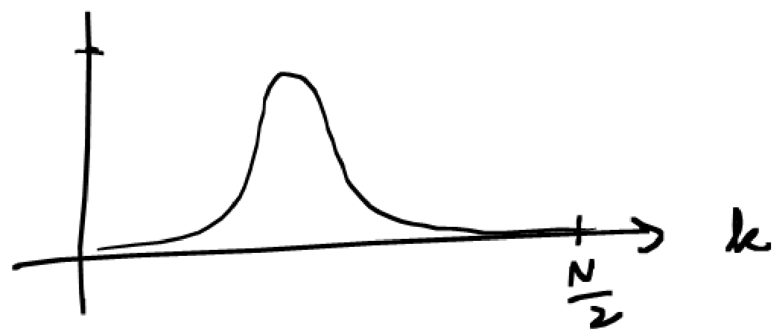
$$\text{Bandwidth} \equiv k_2 - k_1$$

$$\text{Bandwidth (octaves)} \equiv log_2(k_2) - log_2(k_1)$$

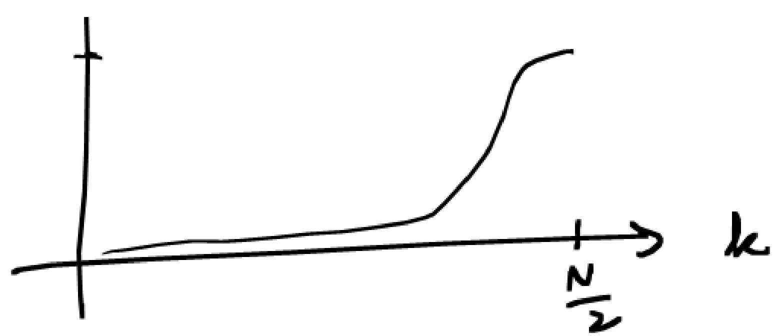
Non-Ideal Filters

low pass

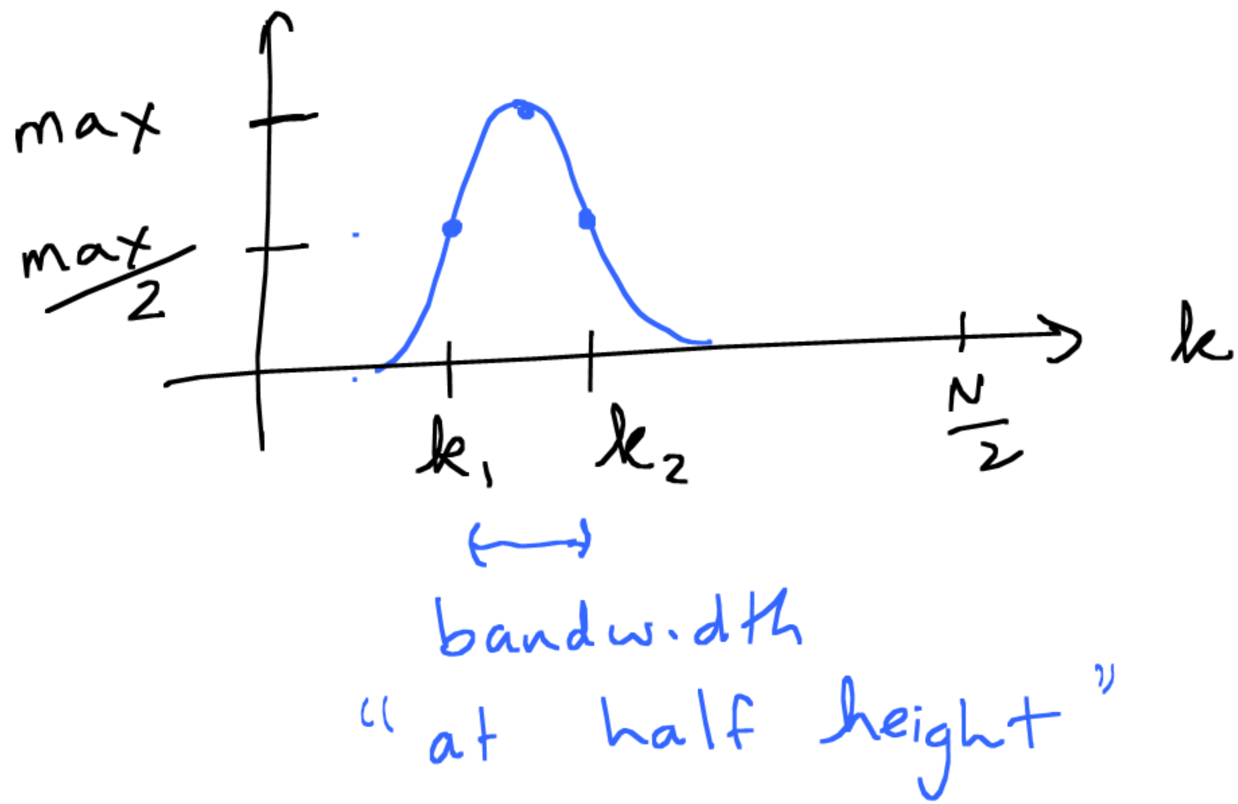
band pass



high pass



Bandwidth of Non-Ideal Bandpass Filter



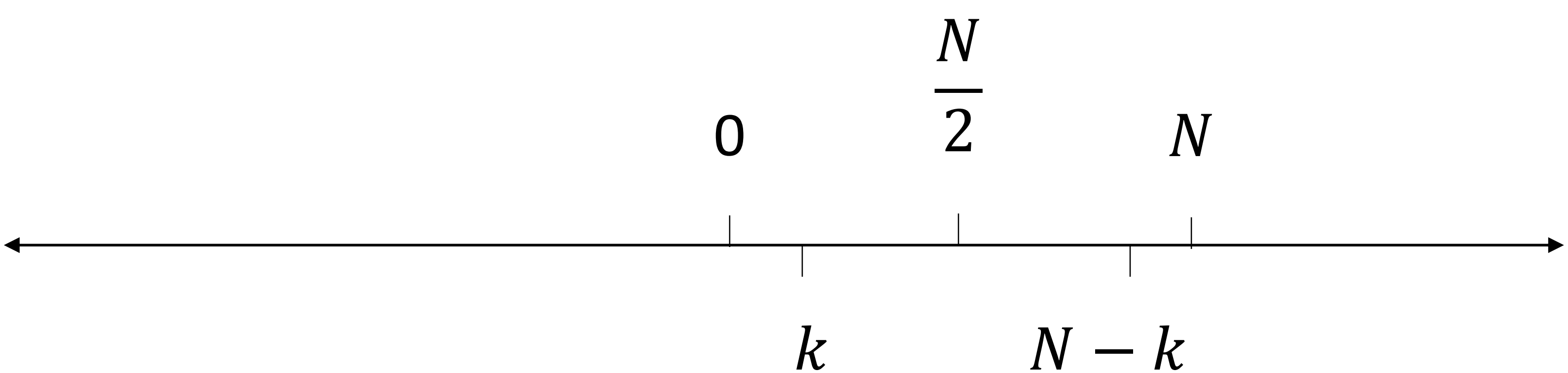
Why are we defining the low/band/high pass filters according to their properties on frequencies k in 0, .. N/2 only ?

Conjugacy Property of Fourier transform

Let $h(x)$ be a real valued function.

Then, for any integer k, $\qquad \hat{h}(k) = \overline{\hat{h}(N-k)}.$

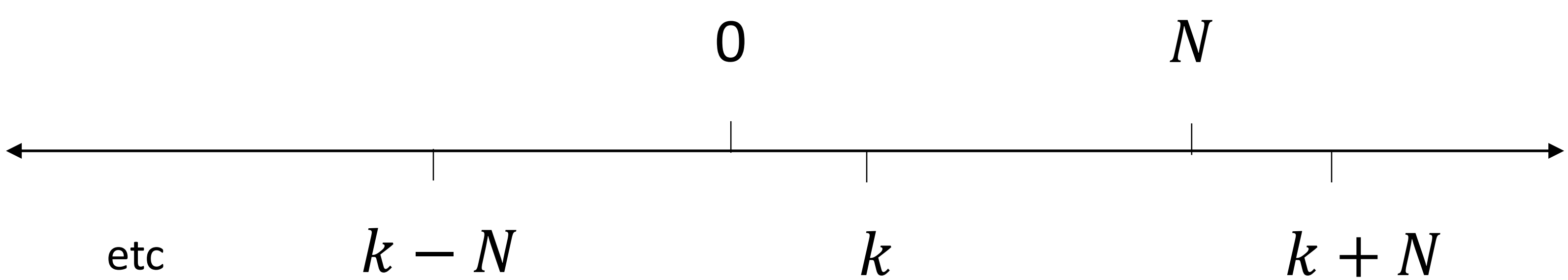
Proof: see the lecture notes.



Periodicity Property of Fourier transform

For any positive or negative integer m,

$$\hat{h}(k) = \hat{h}(k + mN) .$$



Periodicity Property of Fourier transform

For any positive or negative integer m,

$$\hat{h}(k) = \hat{h}(k + mN) .$$

Proof: Use this:

$$e^{-i\frac{2\pi}{N}(k+mN)\,x} = e^{-i\frac{2\pi}{N}kx} \; \underbrace{e^{-i\frac{2\pi}{N}mNx}}_{1}$$

The Fourier transform is well defined for *any* k (not just in 0, ..., $N-1$.)

$$\hat{I}(k) = \sum_{x=0}^{N-1} e^{-i\frac{2\pi}{N}k\,x} \quad I(x)$$

The Fourier transform is well defined for *any* range of N consecutive values of x.

e.g.

$$\hat{f}(k) = \sum_{x=-\frac{N}{2}}^{\frac{N}{2}-1} e^{-i\frac{2\pi}{N}kx} \quad f(x)$$

Essentially we are treating $f(x)$ as periodic.

$$e^{-i\frac{2\pi}{N}(x+mN)k} = e^{-i\frac{2\pi}{N}kx} \; \underbrace{e^{-i\frac{2\pi}{N}kmN}}_{1}$$

Example 1

$$\delta(x) \equiv \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\hat{\delta}(k) \quad = \quad ?$$

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$$\delta(x) \equiv \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \hat{\delta}(k) &= \sum_{x=0}^{N-1} \delta(x) e^{-i \left(\frac{2\pi}{N} kx\right)} \\ &= 1 \cdot \bar{e}^{i \frac{2\pi}{N} k \cdot 0} \\ &= 1 \end{aligned}$$

Examples 2:

Local Difference:

$$I(x) * D(x) \equiv \frac{1}{2} I(x+1) - \frac{1}{2} I(x-1)$$

$$D(x) \equiv \begin{cases} -\frac{1}{2}, & x = 1 \\ \frac{1}{2}, & x = -1 \\ 0, & \text{otherwise} \end{cases}$$

$$D(x) \equiv \begin{cases} -\dfrac{1}{2}, & x = 1 \\[2ex] \dfrac{1}{2}, & x = -1 \\[2ex] 0, & \text{otherwise} \end{cases}$$

$$\widehat{D}(k) = \sum_{x=0}^{N-1} e^{-i\frac{2\pi}{N}kx} \; D(x)$$

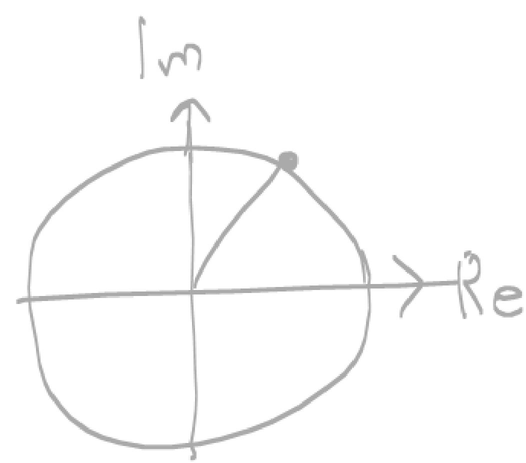
= ? **(Done on blackboard. See lecture notes)**

Useful trick

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$i\sin\theta = \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$



Example 3:

Local Average:

$$I(x) * B(x) \equiv \frac{1}{4} I(x+1) + \frac{1}{2} I(x) + \frac{1}{4} I(x-1)$$

$$B(x) \equiv \begin{cases} \frac{1}{4}, & x = -1, 1 \\ \frac{1}{2}, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$B(x) \equiv \begin{cases} \dfrac{1}{4}, & x = -1, 1 \\ \dfrac{1}{2}, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\hat{B}(k) = \sum_{x=0}^{N-1} e^{-i\frac{2\pi}{N}kx} B(x)$$

$$= \quad ?$$ (*Sketched* on blackboard. See lecture notes)

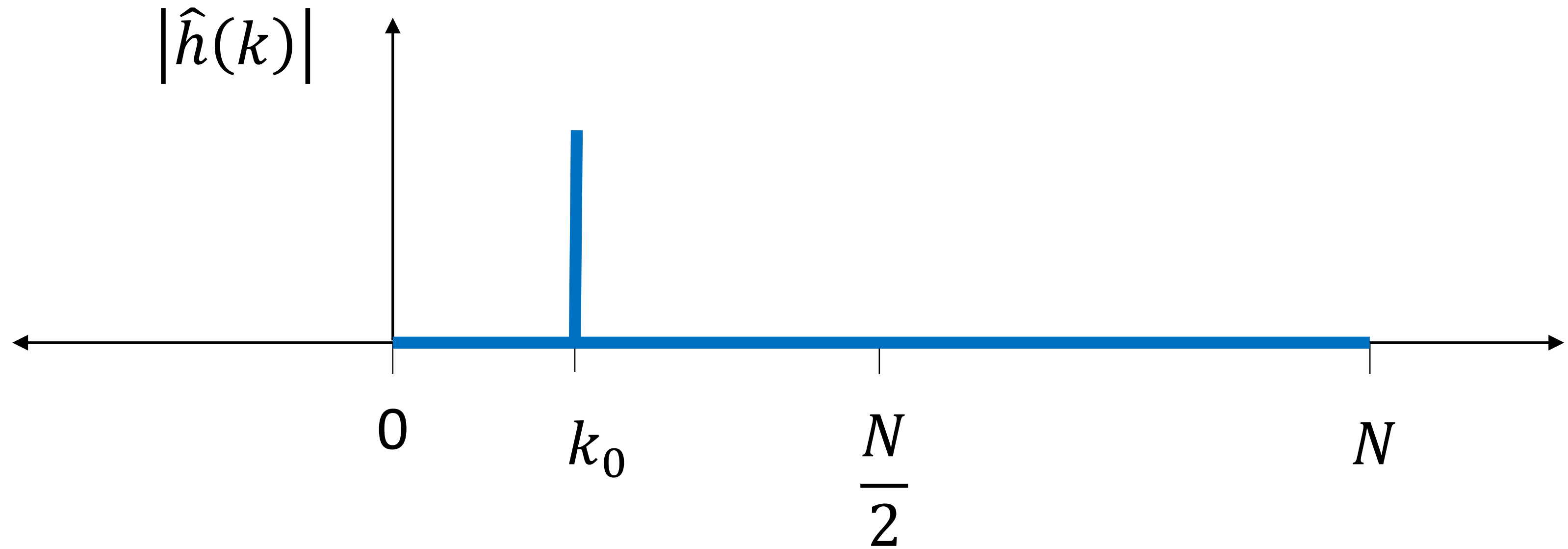
Stopped here

(will finish next class)

Example 4:

$$\mathbf{F}\ e^{i\frac{2\pi}{N}k_0 x} = N\,\delta(k - k_0)$$

(Done on blackboard. See lecture notes.)



Example 5 & 6: cosine and sine

$$\mathbf{F}\ \cos\left(\frac{2\pi}{N}k_0 x\right) = \ ?$$

$$\mathbf{F}\ \sin\left(\frac{2\pi}{N}\ k_0 x\right) = \ ?$$

Example 5 & 6: cosine and sine

$$\mathbf{F}\cos\left(\frac{2\pi}{N}k_0x\right) = \frac{N}{2}(\delta(k - k_0) + \delta(k + k_0))$$

$$\mathbf{F}\sin\left(\frac{2\pi}{N}k_0x\right) = \frac{N}{2i}(\delta(k - k_0) - \delta(k + k_0))$$

Use Euler’s formula and Example 4.
Also, recall the conjugacy property.

Example 7: Gaussian

$$G(x, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\; e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

What is its Fourier transform?

Example 7: Gaussian

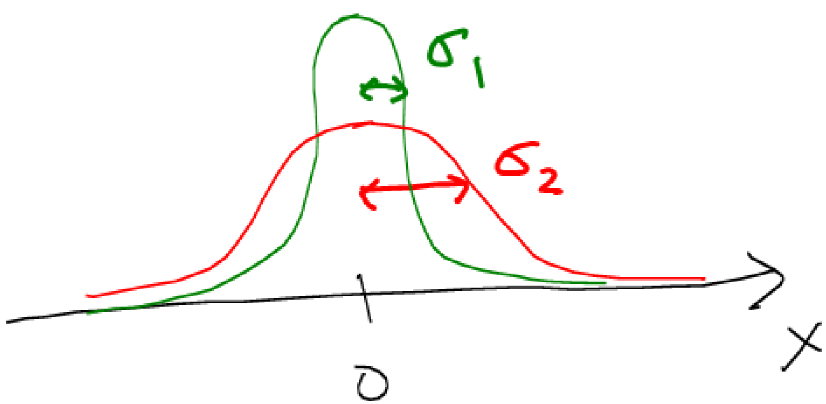
$$G(x,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\; e^{-\frac{1}{2}\left(\frac{x}{\color{red}{\sigma}}\right)^2}$$

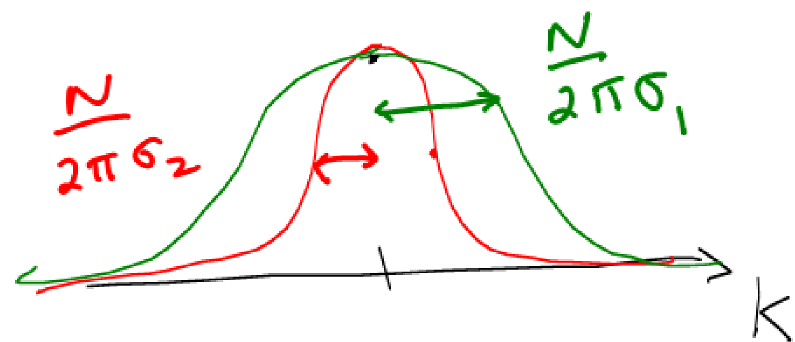
Note the inverse relationship

$$\boldsymbol{F}\, G(x,\sigma) \approx \qquad e^{-\frac{1}{2}\left(\frac{2\,\pi\,\color{red}{\sigma}\,k}{N}\right)^2}$$

with equality in the limit as distance between samples goes to 0 and N goes to infinity, i.e. continuous Fourier transform.

$G(x, \sigma)$ $\hat{G}(k, \sigma)$





Example 8: cosine Gabor

$$\mathbf{F} \;\; \cos\left(\frac{2\pi}{N} k_0 x\right) = \; \frac{N}{2}\left(\delta(k \, - \, k_0) + \, \delta(k + \, k_0)\right)$$

$$\mathbf{F} \;\; G(x, \sigma) \approx \;\; e^{-\frac{1}{2}\left(\frac{2\,\pi\,\sigma\,k}{N}\right)^2}$$

$$\mathbf{F} \;\; cosGabor(x, \sigma) = \quad \mathbf{F} \; \{ \cos\left(\frac{2\pi}{N} k_0 x\right) \; G(x, \sigma) \; \}$$

$$= \;\; ?$$

Convolution Theorem (version 2)

$$\boldsymbol{F} \{ \; I(x) \;\; h(x) \; \} \; = \; \frac{1}{N} \; \boldsymbol{F}\, I(x) * \boldsymbol{F}\, h(x)$$

Proof: see Appendix in lecture notes

Example 8: cosine Gabor

$$\mathbf{F} \quad \cos\left(\frac{2\pi}{N} k_0 x\right) = \frac{N}{2}(\delta(k - k_0) + \delta(k + k_0))$$

$$\mathbf{F} \quad G(x, \sigma) \approx e^{-\frac{1}{2}\left(\frac{2\,\pi\,\sigma\,k}{N}\right)^2}$$

$$\mathbf{F} \quad cosGabor(x, \sigma) \approx \frac{N}{2}\left(e^{-\frac{1}{2}\left(\frac{2\,\pi\,\sigma\,(k - k_0)}{N}\right)^2} + e^{-\frac{1}{2}\left(\frac{2\,\pi\,\sigma\,(k + k_0)}{N}\right)^2} \right)$$

See lecture notes for proof, and formula for sine Gabor.

Example: cosine Gabor
[ADDED: April 12]

N = 128, k0 = 20, sigma = 5