Last lecture I introduced the idea that any function defined on $x \in 0, \ldots, N-1$ could be written a sum of sines and cosines. There are two different reasons why this is useful. The first is a general one, that sines and cosines behave nicely under convolution and so we can sometimes understand better what filtering does if we understand its effects on sines and cosines. The second is more specific, that sines and cosines are a natural set of functions for describing sounds.

Today I will begin with the basic theory of Fourier analysis. This is a particular way of writing a signal as a sum of sines and cosines.

Discrete Fourier Transform

Consider 1D signals $I(x)$ which are defined on $x \in \{0, 1, ..., N-1\}$. Define the $N \times N$ *Fourier transform* matrix $\mathbf{F}$ whose k^{th} row and x^{th} column is:

$$\begin{aligned} \mathbf{F}_{k,x} &= \cos(\frac{2\pi}{N}kx) \; - \; i \; \sin(\frac{2\pi}{N}kx) \\ &\equiv e^{-i\frac{2\pi}{N}kx} \end{aligned}$$

Note that this matrix is symmetric since $e^{-i\frac{2\pi}{N}kx} = e^{-i\frac{2\pi}{N}xk}$. Also note that each row and column of the matrix $\mathbf{F}$ has a real part and an imaginary part. The real part is a sampled cosine function. The imaginary part is a sampled sine function. Note that the leftmost and rightmost column of the matrix ($x = 0$ and $x = N-1$) are not identical. You would need to go to $x = N$ to reach the same value as at $x = 0$, but $x = N$ is not represented. Similarly, the first and last row ($k = 0$ and $k = N-1$) are not identical.

Right multiplying the matrix $\mathbf{F}$ by the $N \times 1$ vector $I(x)$ gives a vector $\hat{I}(k)$

$$\hat{I}(k) \;\equiv\; \mathbf{F} \; I(x) = \sum_{x=0}^{N-1} I(x) e^{-i\frac{2\pi}{N}kx} \tag{1}$$

which is called the *discrete Fourier transform* of $I(x)$. In general, $\hat{I}(k)$ is a complex number for each k. We can write it using Euler's equation:

$$\hat{I}(k) \; = A(k) e^{i\phi(k)}$$

$|\hat{I}(k)| = A(k)$ is called the *amplitude spectrum* and $\phi(k)$ is called the *phase spectrum.*

Inverse Fourier transform

One can show (see Appendix A) that

$$\mathbf{F}^{-1} = \frac{1}{N}\overline{\mathbf{F}}$$

where $\bar{\mathbf{F}}$ is the matrix of complex conjugates of $\mathbf{F}$.

$$\overline{\mathbf{F}}_{k,x} \;\equiv\; e^{i\frac{2\pi}{N}kx}.$$

So, $\frac{1}{N}\mathbf{F}\overline{\mathbf{F}}$ is the identity matrix.

Periodicity properties of the Fourier transform

The Fourier transform definition assumed that the function was defined on $x \in 0, \ldots, N-1$, and for frequencies k in $0, \ldots, N-1$. However, sometimes we will want to be more flexible with our range of x and k.

For example, we may want to consider functions $h(x)$ that are defined on negative values of x such as the local difference function $D(x)$, the local average function $B(x)$, the Gaussian function which has mean 0, Gabor functions, etc. The point of the Fourier transform is to be able to write a function as a sum of sinuosoids. Since sine and cosine functions are defined over *all* integers, there is no reason why the Fourier transform needs to be defined only on functions that are defined on x in 0 to $N-1$.

We can define the Fourier transform of any function that is defined on a range of N consecutive values of x. For example, if we have a function defined on $-\frac{N}{2}, \cdots, -1, 0, 1, \frac{N}{2}-1$, then we can just write the Fourier transform as

$$\hat{I}(k) \;\equiv\; \mathbf{F}\; I(x) = \sum_{x=-\frac{N}{2}}^{\frac{N}{2}-1} h(x) e^{-i\frac{2\pi}{N}kx}$$

Essentially what we are doing here is treating this function $h(x)$ as periodic with period N, just like sine and cosine are, and compute the Fourier transform over a convenient sequence of N sample points. Later this lecture I will calculate the Fourier transform of $D(x)$ and $B(x)$, so look ahead to see how that is done.

The second aspect of periodicity in the Fourier transform is that $\hat{I}(k)$ is well-defined for *any* integer k (cycles per N pixels). The definition of the Fourier transform doesn't just allow k in 0 to $N-1$, but rather k can be any integer. In that case, $\hat{I}(k)$ may be considered periodic in k with period N,

$$\hat{I}(k) \;=\; \hat{I}(k+mN)$$

since, for any integer m,

$$e^{i2\pi m} = \cos(2\pi m) + i\sin(2\pi m) = 1$$

and so

$$e^{i\frac{2\pi}{N}kx} = e^{i\frac{2\pi}{N}k}\; e^{i\frac{2\pi}{N}mN} = e^{i\frac{2\pi}{N}(k+mN)x}$$

Thus, if we use frequency $k+mN$ instead of k in the definition of the Fourier transform, we get the same value.

Conjugacy property of the Fourier transform

It is a bit strange that our function $I(x)$ has N points and we will write it in terms of $2N$ functions, namely N cosines and N sines. I mentioned this point last lecture as well, and showed that indeed only N functions are needed, namely $\frac{N}{2}+1$ cosines and $\frac{N}{2}-1$ sines. This suggests that there is a redundancy in $\hat{I}(k)$ values. The redundancy is that $\cos(\frac{2\pi}{N}kx) = \cos(\frac{2\pi}{N}(N-k)x)$ and so taking the inner product with $I(x)$ will give the same value for frequency k as $N-k$. Similarly, $\sin(\frac{2\pi}{N}kx) = -\sin(\frac{2\pi}{N}(N-k)x)$ and so taking the inner product of $I(x)$ with these two functions will give the same value but with opposite sign.

Conjugacy property: If $I(x)$ is a real valued function, then

$$\overline{\hat{I}(k)} = \hat{I}(N-k).$$

The property does <u>*not*</u> *apply if $I(x)$ has imaginary components. We will see an example later, namely if we take the Fourier transform of $e^{i\frac{2\pi}{N}k_0 x}$, for some fixed frequency k_0*

For the proof of the Conjugacy Property, see Appendix B.

Linear Filtering

The visual and auditory systems analyze signals by *filtering* them into bands (ranges of different frequencies) of sines and cosines. The idea of a filter should be intuitive to you. You can imagine having a large bag of rocks and wanting to sort the rocks into ranges of different sizes. You could first pass the rocks through a fine mesh that has small holes only, so only the small rocks would pass through. Then take the bigger rocks that didn't pass through, and pass them through a mesh filter that has slightly larger holes so that now the medium size rocks pass through, but not the large rocks. This would give you three sets of rocks of a different range of sizes.

You are also intuitively familiar with filtering from color vision where the L, M, and S receptors selectively absorb the incoming light by wavelength[1]. There is some frequency overlap in the sensitivity functions, so we don't have a perfect separation of frequency bands by the three cones.

The figure below shows a more concrete example of the filtering that we will be considering. Here we have 1D signal in the upper left panel. We can write this signal as a *sum* of signals that have different ranges of frequencies. In this example, the original signal is exactly the sum of the other five signals. We will see shortly how this can be done.

Convolution Theorem

A very useful property of the Fourier transform is the *Convolution Theorem*: for any two functions $I(x)$ and $h(x)$ that are defined on 0 to $N-1$,

$$\mathbf{F}(I(x) * h(x))) = \mathbf{F}I(x)\ \mathbf{F}h(x) \ = \ \hat{I}(k)\ \hat{h}(k).$$

For the proof see Appendix C.

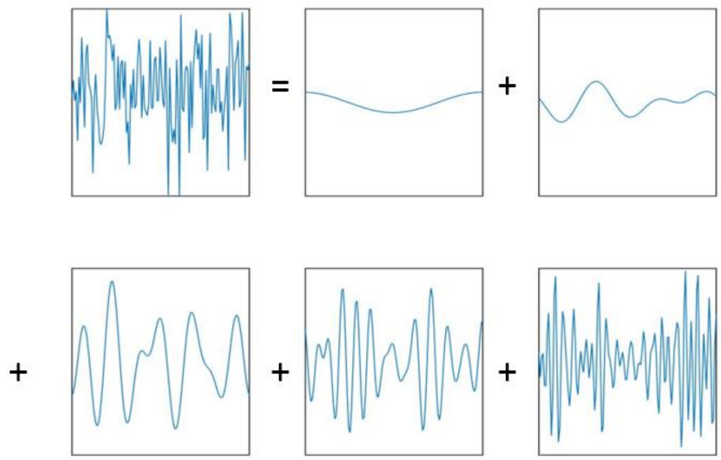
To prove this theorem, we need to deal with a similar issue that we mentioned before that the functions might be defined on values of x other than 0 to $N-1$. We do so by assuming the functions are periodic *i.e.* $I(x) = I(x+mN)$ and $h(x) = h(x+mN)$ for any integer m and we define the summation from 0 to $N-1$.

Filtering and bandwidth

Suppose we convolve an image $I(x)$ with a function $h(x)$. We have referred to $h(x)$ as an impulse response function. $h(x)$ is also called a *linear filter*. Recall that the Fourier transform of the filter $h(x)$ can be written

$$\hat{h}(k) \ = \ |\hat{h}(k)|\ e^{i\phi(k)}$$

[1] or frequency i.e. since light travels at a constant speed (called c), we can equivalently describe the sensitivity of L, M, and S cones to frequency (either spatial frequency λ or temporal frequency ω, where $c = \omega\lambda$.



where $|\hat{h}(k)|$ is called the *amplitude spectrum* and $\phi(k)$ is called the *phase spectrum*. By the convolution theorem,

$$\mathbf{F} I(x) = \mathbf{F}(I(x) * h(x)) = \hat{I}(k) \; |\hat{h}(k)| \; e^{i\phi(k)}$$

and $|\hat{h}(k)|$ amplifies or attenuates the frequency component amplitude $|\hat{I}(k)|$ and the phase $\phi(k)$ of the filter shifts each frequency component.

We can characterize filters by how they affect different frequencies. We will concern ourselves mainly with the amplitude spectrum for now. Let's first address the case of "ideal" filters. We say:

- $h(x)$ is an ideal *low pass filter* if there exists a frequency k_0 such that

$$\hat{h}(k) = \begin{cases} 1, & 0 \leq \; k \; \leq \; k_0 \\ 0, & k_0 < \; k \leq \frac{N}{2} \end{cases}$$

- $h(x)$ is an ideal *high pass filter* if there exists k_0 such that

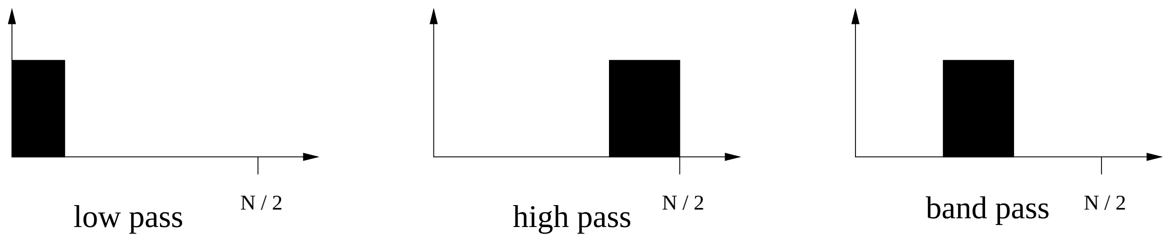
$$\hat{h}(k) = \begin{cases} 0, & 0 \leq \; k \; < \; k_0 \\ 1, & k_0 \leq \; k \leq \frac{N}{2} \end{cases}$$

- $h(x)$ is an ideal *bandpass filter* if there exists two frequencies k_0 and k_1 such that

$$\hat{h}(k) = \begin{cases} 0, & 0 \leq \; k \; < \; k_0 \\ 1, & k_0 \leq \; k \leq k_1 \\ 0, & k_1 < \; k \leq \frac{N}{2} \end{cases}$$

Note that these definitions above only concern $k \in \{0, \ldots, \frac{N}{2}\}$. Frequencies $k < 0$ and frequencies $k > \frac{N}{2}$ are ignored in the definition because the values of $\hat{h}(k)$ of these frequencies are determined by the conjugacy and periodicity properties.

 ©Michael Langer



Non-ideal filters and bandwidth

We typically work with filters that are not ideal i.e. filters that only approximately satisfy the above definitions. If we have an approximately bandpass filter, then we would like to describe the width of this filter i.e. the range of frequencies that it lets through. One often does this by considering the frequencies at which $|\hat{h}(k)|$ reaches *half* its maximum value. The *bandwidth at half-height* is defined to be $k_1 - k_0$, where $k_0 < k_1$ and

$$|\hat{h}(k_0)| = |\hat{h}(k_1)| = \frac{1}{2} \max_{k \in [0, \frac{N}{2}]} |\hat{h}(k)|$$

Bandwidth can also be defined in terms of the *ratio* of k_1 to k_0, specifically, the *octave bandwidth* at half height is:

$$\log_2(\frac{k_1}{k_0}) = \log_2(k_1) - \log_2(k_0)$$

For example, a filter with a bandwidth of one octave means that the k_1 frequency is twice the k_0 frequency.

Examples of filters and their Fourier transforms

Let's look at some examples, starting with an impulse function, and the local difference and local average. Some of our calculations of Fourier transforms below will use Euler's formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. In particular, you can verify for yourselves that:

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$i\sin(\theta) = \frac{1}{2}(e^{i\theta} - e^{-i\theta})$$

We will often take $\theta = \frac{2\pi}{N}kx$.

Example 1: Impulse function

Recall

$$\delta(x) \equiv \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

Its Fourier transform is

$$\begin{aligned}\hat{\delta}(k) &= \sum_{x=0}^{N-1} \delta(x) e^{\,-i\ (\frac{2\pi}{N} kx)} \\ &= 1 \cdot e^{i \frac{2\pi}{N} k \cdot 0} \\ &= 1\end{aligned}$$

This is rather surprising. It says that an impulse function can be written as sum of cosine functions over all frequencies $k \in 0, 1, \ldots, N-1$ and dividing by N, i.e.

$$\delta(x) = \frac{1}{N} \sum_{x=0}^{N-1} \hat{\delta}(k) e^{\ i\ (\frac{2\pi}{N} kx)}$$

Note that I write cosine functions, rather than cosine and sine functions, since $\hat{\delta}(k) = 1$ and so the phase is 0, i.e. $\phi(k) = 0$ for all k, i.e. purely real, and so there are no sine (imaginary) components. Basically, what happens is that all the cosine functions have the value 1 at $x = 0$, whereas at other values of x there are a range of values, some positive and some negative, and these other values cancel each other out when you take the sum.

To try to illustrate what is going on here, I have written a Matlab script

`http://www.cim.mcgill.ca/~langer/546/MATLAB/sumOfSinusoids.m`

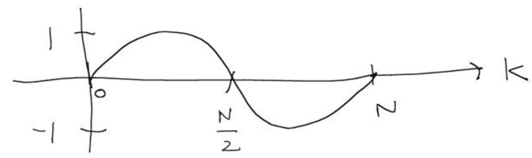
which shows what happens when you add up all the cosines (top) and sines (bottom) of frequency $k = 0, \ldots, N-1$ for some chosen N.

Example 2: local difference

Recall the local difference function $D(x)$ from last lecture. It has value -$\frac{1}{2}$ at $x = 1$ and value $\frac{1}{2}$ at $x = -1$. Let's compute its Fourier transform.

$$\begin{aligned}\hat{D}(k) &= \sum_{x} D(x) e^{-i\ (\frac{2\pi}{N} kx)} \\ &= \frac{1}{2}(-1 \cdot e^{-i\ \frac{2\pi}{N} k} + 1 \cdot e^{-i\ (\frac{2\pi}{N} k(-1))}) \\ &= \frac{1}{2}(-e^{-i\ \frac{2\pi}{N} k} + e^{i\ \frac{2\pi}{N} k}) \\ &= i \sin(\frac{2\pi}{N} k)\end{aligned}$$

Notice that $\hat{D}(k)$ is purely imaginary and the plot below shows the imaginary component only. The phase spectrum is constant $e^{i \frac{\pi}{2}} = \frac{\pi}{2}$.



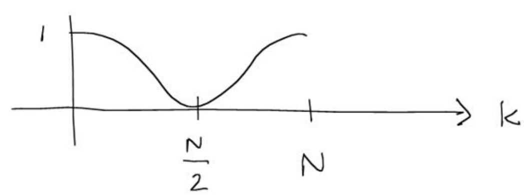
Example 3: local average

$$B(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{4}, & x = -1 \\ \frac{1}{4}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Taking its Fourier transform,

$$\begin{aligned} \mathbf{F}\ B(x) &= \frac{1}{2} + \frac{1}{4}(e^{-i\frac{2\pi}{N}k} + e^{-i\frac{2\pi}{N}k(-1)}) \\ &= \frac{1}{2} + \frac{1}{4}(e^{-i\frac{2\pi}{N}k} + e^{i\frac{2\pi}{N}k}), \\ &= \frac{1}{2}(1 + \cos(\frac{2\pi}{N}k)) \end{aligned}$$

Notice that $\hat{B}(k)$ is real, i.e. it has no imaginary component. Moreover it is non-negative. Thus, the phase spectrum $\phi(k)$ is 0.



Example 4: the "complex exponential"

Let $h(x) = e^{\,i\frac{2\pi}{N}k_0 x}$ for some integer frequency k_0. Then,

$$\mathbf{F}\ e^{\,i\frac{2\pi}{N}k_0 x} \ = \ N\delta(k - k_0).$$

See the Appendix A for a proof.

Is this result surprising. In hindsight, no. Taking the Fourier transform of a function amounts to finding out what are the coefficients on the complex exponentials $e^{\,i\frac{2\pi}{N}kx}$ for various k such that you can add these complex exponentials up and get the function. But if the function itself *is* a single complex exponential, then there is just one non-zero complex exponential needed!

We will use this result below when we compute the Fourier transforms of a cosine and sine function.

Example 5: constant function $h(x) = 1$

This is just a special case of the last example, namely if we take $k_0 = 0$. In this case,

$$\hat{h}(k) = N\ \delta(k).$$

Thus, the Fourier transform of the constant function $h(x) = 1$ is a delta function *in the frequency domain*, namely it has value N at $k = 0$ and has value 0 for all values of k in $1, \ldots, N-1$.

Examples 6 and 7: cosine and sine

We use Euler's formula to rewrite cosine and sine as a sum of complex exponentials.

$$\begin{aligned}
\mathbf{F}\cos(\frac{2\pi}{N}k_0 x) &= \sum_{x=0}^{N-1} \cos(\frac{2\pi}{N}k_0 x) e^{-i\ (\frac{2\pi}{N}kx)} \\
&= \sum_{x=0}^{N-1} \frac{1}{2}(e^{i\frac{2\pi}{N}k_0 x} + e^{-i\frac{2\pi}{N}k_0 x}) e^{-i\ \frac{2\pi}{N}kx} \\
&= \frac{N}{2}(\delta(k_0 - k) + \delta(k_0 + k))
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}\sin(\frac{2\pi}{N}k_0 x) &= \sum_{x=0}^{N-1} \sin(\frac{2\pi}{N}k_0 x) e^{-i\ (\frac{2\pi}{N}kx)} \\
&= \sum_{x=0}^{N-1} \frac{1}{2i}(e^{i\frac{2\pi}{N}k_0 x} - e^{-i\frac{2\pi}{N}k_0 x}) e^{-i\ \frac{2\pi}{N}kx} \\
&= -\frac{Ni}{2}(\delta(k_0 - k) - \delta(k_0 + k))
\end{aligned}$$

Example 7: Gaussian

If we sample a Gaussian function

$$G(x, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

on integer values of x, and take the Fourier transform, we get the following approximation:

$$\hat{G}(k, \sigma) \approx e^{-\frac{1}{2}(\frac{2\pi}{N})^2 \sigma^2 k^2}$$

This approximation becomes exact in the limit as $N, \sigma \to \infty$, with $\frac{\sigma}{N}$ held constant. (This amounts to taking the continuous instead of discrete Fourier transform. The proof of these claims are beyond the scope of this course.)

If you wish to see this approximation for yourself, run the Matlab script

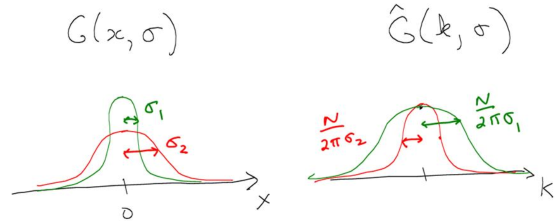
`http://www.cim.mcgill.ca/~langer/546/MATLAB/plotFourierTransformGaussian.m`

which generates the figure

`http://www.cim.mcgill.ca/~langer/546/MATLAB/plotFourierTransformGaussian.jpg`

A few key properties to notice are:

- If the standard deviation of the Gaussian in the space (x) domain is σ then the standard deviation of the Gaussian in the frequency (k) domain is proportional to $\frac{1}{\sigma}$
- $\hat{G}(k, \sigma)$ has a Gaussian shape, but it does not integrate to 1, namely there is no scaling factor present. The max value occurs at $k = 0$ and the max value is always 1.
- The Fourier transform is periodic, with period N. This is always true.



Example 8: Gabor

To compute the Fourier transform of Gabor, we use a property which is similar to the convolution theorem:

$$\mathbf{F}(I(x)h(x))) = \frac{1}{N}\mathbf{F}I(x) * \mathbf{F}h(x) \ .$$

See Appendix B for a proof, if you are interested (not on exam).

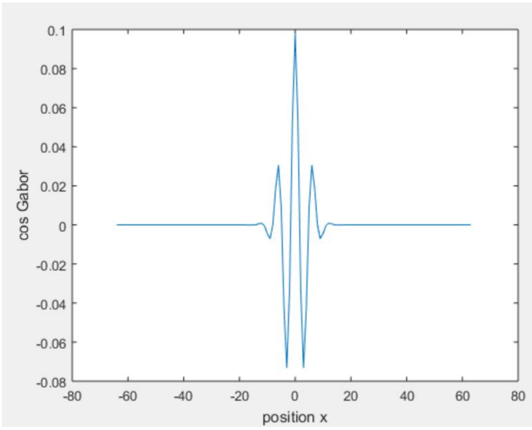
Thus the Fourier transform of a cosine Gabor is the convolution in the frequency domain of the Fourier transforms of a Gaussian and the Fourier transform of a cosine:

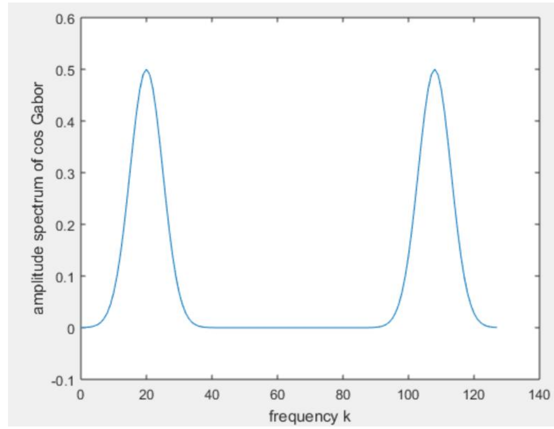
$$\begin{aligned}
\mathbf{F}\ cosGabor(x, k_0, \sigma) &= \mathbf{F}\ \{\ G(x,\sigma)\cos(\frac{2\pi}{N}k_0 x)\ \} \\
&= \frac{1}{N}\ e^{-\frac{1}{2}(\frac{2\pi\sigma\ k}{N}\ )^2} * \frac{N}{2}(\delta(k_0 - k) + \delta(k_0 + k)) \\
&= \frac{1}{2}\{e^{-\frac{1}{2}(\frac{2\pi\sigma\ (k-k_0)}{N}\ )^2} + e^{-\frac{1}{2}(\frac{2\pi\sigma\ (k+k_0)}{N}\ )^2}\ \}
\end{aligned}$$

which is the sum of two Gaussians, centered at $k = \pm k_0$.

[**ADDED: April 12, 2018**]

An example is shown below which was computed using Matlab. The cosine Gabor is defined on a vector of size $N = 128$ and has a central frequency of 20 cycles and a Gaussian with a standard deviation of 5. The amplitude spectrum has a peak at $k_0 = \pm 20$. In the amplitude spectrum plot on the right below, I plot the frequency range from 0 to $N-1$ instead of $-\frac{N}{2}$ to $\frac{N}{2}+1$. The Fourier transform of a sine Gabor can be calculated similarly. (See Exercises.)





The convolution theorem tells us that convolving a function $I(x)$ with a cosine (or sine) Gabor will give you a function that has only a band of frequencies remaining, namely the frequencies near the center frequency k_0 of the Gabor. The width of the band depends on the σ of the Gaussian of the Gabor. We will return to this idea of filtering a signal into bands of different frequencies (different Gabor filters can be used, or other bandpass filteres) when we discuss audition.

Appendix A

We will use the following claim to show what the inverse Fourier transform is (bottom of page).

Claim (Example 4): For any frequency k_0,

$$\mathbf{F}\ e^{\ i\frac{2\pi}{N}k_0 x} \ = \ N\delta(k-k_0).$$

That is,

$$\sum_{x=0}^{N-1} e^{\ i\ \frac{2\pi}{N}k_0 x} \ \ e^{-i\frac{2\pi}{N}kx} = \begin{cases} N, & k=k_0 \\ 0, & k \neq k_0 \end{cases}$$

Note that this claim essentially is essentially equivalent to saying that two cosine (or sine) functions of different frequencies are orthogonal; their inner product is 0.

Proof: Rewrite the left side of the above summation as

$$\sum_{x=0}^{N-1} e^{\ i\ \frac{2\pi}{N}(k_0-k)x} \quad . \tag{2}$$

If $k = k_0$, then the exponent is 0 and so we are just summing $e^0 = 1$ and the result is N.

That doesn't yet give us the result of the claim, because we still need to show that the summation is 0 when $k \neq k_0$. So, for the case $k \neq k_0$, observe that the summation is a finite geometric series and thus we can use the following identity which you know from Calculus:[2]: let γ be any number (real or complex) then

$$\sum_{x=0}^{N-1} \gamma^x = \frac{1-\gamma^N}{1-\gamma}.$$

Applying this identity for our case, namely $\gamma = e^{\ i\ \frac{2\pi}{N}(k-k_0)}$, lets us write (2) as

$$\sum_{x=0}^{N-1} e^{\ i\ \frac{2\pi}{N}(k-k_0)x} = \frac{1-e^{\ i\ 2\pi(k-k_0)}}{1-e^{\ i\ \frac{2\pi}{N}(k-k_0)}} \quad . \tag{3}$$

The numerator on the right hand side vanishes because $k - k_0$ is an integer and so

$$e^{\ i2\pi(k-k_0)} \ = 1 \ .$$

What about the denominator? Since k and k_0 are both in $0, \ldots, N-1$ and since we are considering the case that $k \neq k_0$, we know that $|k - k_0| < N$ and so $e^{\ i\ \frac{2\pi}{N}(k-k_0)} \neq 1$. Hence the denominator does not vanish. Since the numerator is 0 but the denominator is not 0, we can conclude that the right side of Eq. (3) is 0. Thus, the summation of (2) is 0, and so $\mathbf{F}\ e^{\ i\frac{2\pi}{N}k_0 x} = 0$ when $k \neq k_0$. This completes the derivation for the case $k \neq k_0$.

Claim (inverse Fourier transform): $\mathbf{F}^{-1} = \frac{1}{N}\overline{\mathbf{F}}$

Proof:

The matrix $\frac{1}{N}\mathbf{F}\bar{\mathbf{F}}$ is $N \times N$. The above example says that row k_0 and column k of this matrix is 1 when $k_0 = k$ and 0 when $k_0 \neq k$ and hence this matrix is the unit diagonal.

[2]If you are unsure where this comes from, see equations (1)-(6) of `http://mathworld.wolfram.com/GeometricSeries.html`.

Appendix B: Conjugacy property of the Fourier transform

Claim: If $I(x)$ is a real valued function, then

$$\overline{\hat{I}(k)} = \hat{I}(N-k).$$

Proof: (not on final exam)

$$\begin{aligned}
\hat{I}(N-k) \quad &= \quad \sum_{x=0}^{N-1} I(x) e^{-i\frac{2\pi}{N}\ (N-k)x} \\
&= \quad \sum_{x=0}^{N-1} I(x)\ e^{-i\ 2\pi x}\ e^{i\ \frac{2\pi}{N}\ kx} \\
&= \quad \sum_{x=0}^{N-1} I(x)\ e^{i\ \frac{2\pi}{N}\ kx}, \quad \text{since } e^{i2\pi x}\ = 1 \quad \text{for any integer } x \\
&= \quad \sum_{x=0}^{N-1}\ I(x)\ \overline{e^{-i\frac{2\pi}{N}\ kx}} \\
&= \quad \sum_{x=0}^{N-1}\ \overline{I(x)}\ \overline{e^{-i\frac{2\pi}{N}\ kx}}\ \ \text{, if } I(x) \text{ is real} \\
&= \quad \sum_{x=0}^{N-1}\ \overline{I(x)\ e^{-i\frac{2\pi}{N}\ kx}} \\
&= \quad \overline{\hat{I}(k)}
\end{aligned}$$

Appendix C: Convolution Theorem

Claim: For any two functions $I(x)$ and $h(x)$ that are defined on N consecutive samples e.g. 0 to $N-1$,

$$\mathbf{F}(I(x) * h(x))) = \mathbf{F}I(x) \ \mathbf{F}h(x) \ = \ \hat{I}(k) \ \hat{h}(k).$$

Proof: (not on final exam)

$$\begin{aligned}
\mathbf{F} \ \ I * h(x) &= \sum_{x=0}^{N-1} e^{-i\frac{2\pi}{N}kx} \ \sum_{x'=0}^{N-1} I(x-x')h(x'), \quad \text{by definition} \\
&= \sum_{x'=0}^{N-1} \ h(x') \ \sum_{x=0}^{N-1} e^{-i\frac{2\pi}{N}kx} \ \ I(x-x'), \quad \text{by switching order of sums} \\
&= \sum_{x'=0}^{N-1} \ h(x') \ \sum_{u=0}^{N-1} e^{-i\frac{2\pi}{N}k(u+x')} I(u) \ , \quad \text{where } u = x - x' \\
&= \sum_{x'=0}^{N-1} \ h(x') \ e^{-i\frac{2\pi}{N} \ kx'} \ \sum_{u=0}^{N-1} e^{-i\frac{2\pi}{N}ku} \ I(u) \\
&= \hat{h}(k) \ \hat{I}(k)
\end{aligned}$$

 ©Michael Langer

Appendix D (another Convolution Theorem)

We will often work with filters such as Gabor functions that are the product of two functions. Suppose we have two 1D functions $I(x)$ and $h(x)$ and we take their product. What can we say about the Fourier transform? The answer is similar to the convolution theorem, and indeed is just another version of that theorem:

$$\mathbf{F}\ (I(x)h(x)) \ = \ \frac{1}{N}\ \hat{I}(k) * \hat{h}(k)$$

or, in words, the Fourier transform of the product of two functions is the convolution of the Fourier transforms of the two functions. Note that the convolution on the right hand side is between two complex valued functions, rather than real valued functions. But the same definition of convolution applies.

To prove the above property, we take the inverse Fourier transform of the right side and show that it gives $I(x)h(x)$. Note that the summations and functions below are defined on frequencies k, k', k'' mod N, since the Fourier transform of a function has period N.

$$\begin{aligned}
\mathbf{F}^{-1}\ \ \hat{I}(k) * \hat{h}(k) \ &= \ \frac{1}{N}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}kx} \ \sum_{k'=0}^{N-1} \hat{h}(k')\ \hat{I}(k-k') \qquad \text{...and rearrange...} \\
&= \ \frac{1}{N}\sum_{k'=0}^{N-1} \hat{h}(k') \ \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}kx} \ \ \hat{I}(k-k') \qquad \text{...and multiply by...} e^{i\frac{2\pi}{N}k'x} e^{-i\frac{2\pi}{N}k'x} \\
&= \ \frac{1}{N}\sum_{k'=0}^{N-1} \hat{h}(k') e^{i\frac{2\pi}{N}k'x} \ \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}(k-k')x} \ \ \hat{I}(k-k') \\
&= \ h(x) \sum_{k''=-k'}^{N-1-k'} e^{i\frac{2\pi}{N}(k'')x} \ \ \hat{I}(k''), \ \ \text{where } k'' = k - k' \\
&= \ N h(x)\ I(x)
\end{aligned}$$

Dividing both sides by N and we're done.

©Michael Langer