COMP 546

Lecture 16

Linear Systems 1:

convolution,

complex numbers, Fourier transform

Thurs. March 15, 2018

Many computations on images (and sounds) begin by performing local weighted sums. e.g.

Local Difference:

$$D\, I(x) \equiv \frac{1}{2} I(x+1) - \frac{1}{2} I(x-1)$$

Local Average:

$$B\, I(x) \equiv \frac{1}{4} I(x+1) + \frac{1}{2} I(x) + \frac{1}{4} I(x-1)$$

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_u f(u-x)\, I(u)$$

Convolution

$$f(x) * I(x) \equiv \sum_u f(x-u)\, I(u)$$

Any cross-correlation can be written as a convolution (and vice-versa) just by flipping the function $f(\,)$.

$$I(x) * f(x) = \sum_{u} I(x-u)\, f(u)$$

$$= \quad I(x)\, f(0) + \ ...$$

$$+\ I(x+1)\, f(-1) + \ I(x+2)\, f(-2) + \ \ ...$$

$$+\ I(x-1)\, f(1) \ + \ I(x-2)\, f(2) \ \ + \ ...$$

Some algebraic properties of convolution

For any $f_1(x), f_2(x), f_3(x)$:

$$f_1 * f_2 = f_2 * f_1$$

Cross correlation does not have this property

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$$

$$(f_1 + f_2) * f_3 = f_1 * f_3 + f_2 * f_3$$

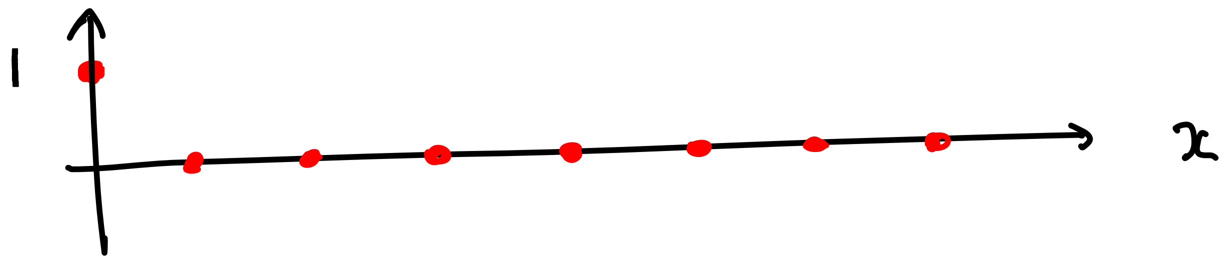
Boundary conditions

$$I(x) * f(x) = \sum_{u=0}^{N-1} I(x-u)\, f(u)$$

Assume $f(x) = 0$ and $I(x) = 0$ outside range 0 to N-1.

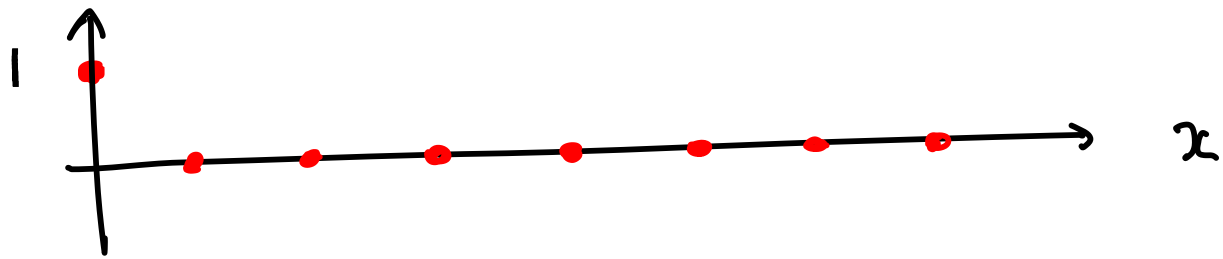
Impulse function

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

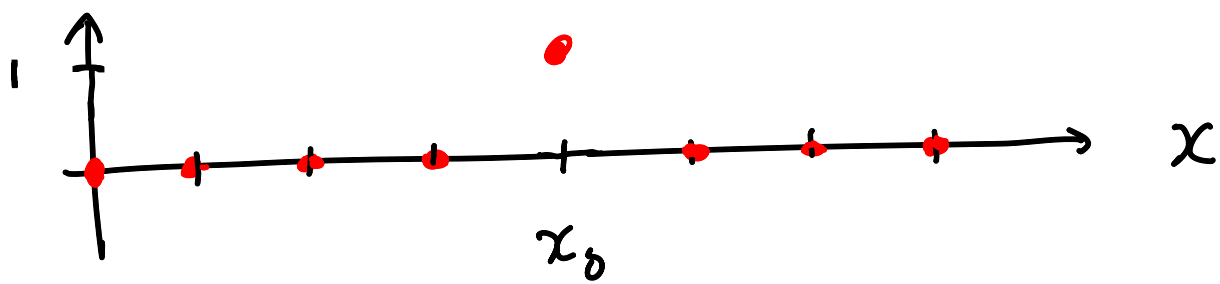


Impulse function

$$\delta(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$



$$\delta(x - x_0) = \begin{cases} 1, & x = x_0 \\ 0, & \text{otherwise} \end{cases}$$



$$\delta(x) * I(x) = \sum_{u} \delta(x-u)\, I(u)$$

$$= ?$$

$$\delta(x) * I(x) = \sum_{u} \delta(x-u)\, I(u)$$

$$= I(x)$$

An image can be thought of as a sum of delta functions.

Impulse Response function

If we think of $I(x) * f(x)$ as a mapping from a input function $I(x)$ to an output function, then we often call $f(x)$ an "impulse response" function since:

$$\delta(x) * f(x) = f(x)$$

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_{u} f(u-x)\, I(u)$$

Sliding a template across an image, and taking inner product.

Convolution

$$f(x) * I(x) \equiv \sum_{u} f(x-u)\, I(u)$$

Summing the impulse responses from all the pixels.

Cross correlation

$$f(x) \otimes I(x) \equiv \sum_{u} f(u-x)\, I(u)$$

How well does the filter match the image, when filter is placed at position x ?

Convolution

$$f(x) * I(x) \equiv \sum_{u} f(x-u)\, I(u)$$

How much does image intensity at position x contribute to the filtered output?

Towards Fourier Analysis

$$cos\left(\frac{2\pi k}{N}x\right) * \;\; h(x) = \;\; ?$$

$$sin\left(\frac{2\pi k}{N}x\right) * \;\; h(x) = \;\; ?$$

Claim 1: (See lecture notes for proof.)

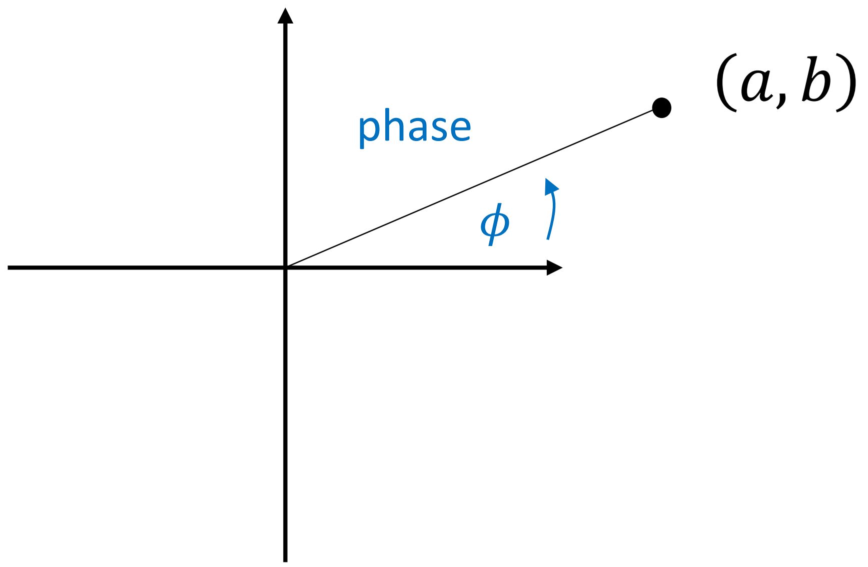
$$cos\left(\frac{2\pi k}{N}x\right) * \ h(x) = \ a\ cos\left(\frac{2\pi k}{N}x\right) + b\ sin\left(\frac{2\pi k}{N}x\right)$$

where $a,\ b$ depend on $h(x)$ and frequency k.

Claim 2:

$$a\ cos\left(\frac{2\pi k}{N}x\right) + b\ sin\left(\frac{2\pi k}{N}x\right) = \sqrt{a^2+b^2}\ \cos\left(\frac{2\pi k}{N}x - \phi\right)$$

To prove Claim 2, write (a, b) in polar coordinates



amplitude

$$(a, b) = \sqrt{a^2 + b^2} \quad \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right)$$

$$= \sqrt{a^2 + b^2} \quad (\cos \phi \ , \sin \phi)$$

$$\cos\left(\frac{2\pi k}{N}x\right) * \; h(x)$$

Claim 1

$$= \; a \; \cos\left(\frac{2\pi k}{N}x\right) + b \sin\left(\frac{2\pi k}{N}x\right)$$

$$= \sqrt{a^2+b^2} \; \left(\cos\phi \; \cos\left(\frac{2\pi k}{N}x\right) + \; \sin\phi \; \sin\left(\frac{2\pi k}{N}x\right)\right)$$

Claim 2

$$= \sqrt{a^2+b^2} \; \cos\left(\frac{2\pi k}{N}x - \phi\right)$$

Using identity from Calculus 1 that $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$.

Exercise: Show

$$sin\left(\frac{2\pi k}{N}x\right) * \ h(x) \quad = \quad \sqrt{a^2+b^2} \ \ sin\left(\frac{2\pi k}{N}x \ - \phi\right)$$

same amplitude and phase

Using identity from Calculus 1 that $\sin(A-B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

Claim 3: (proof omitted)

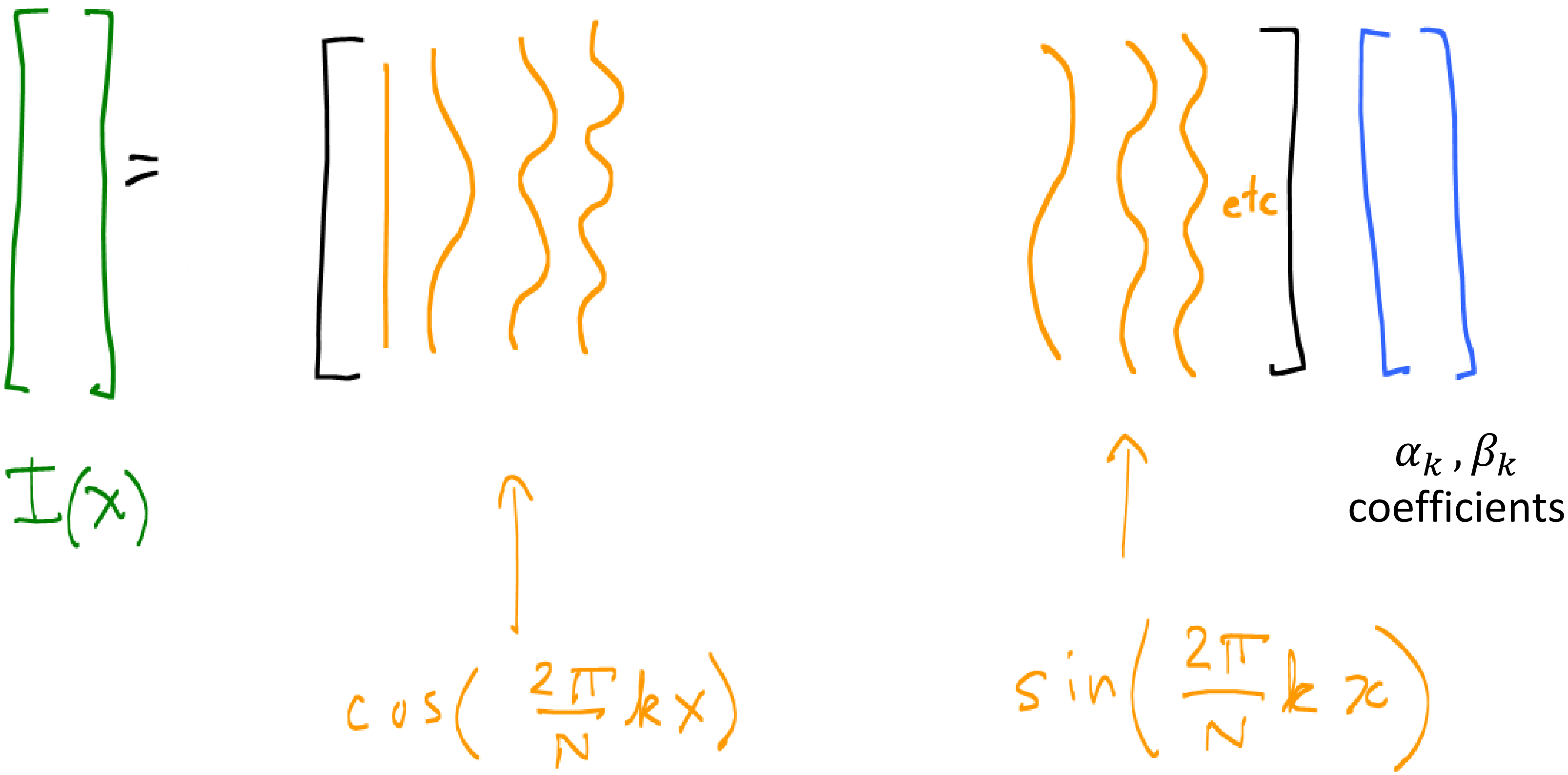
We can write any function $I(x)$ defined on $x = 0$ to $\frac{N}{2} - 1$ as:

$$I(x) = \sum_{k=0}^{\frac{N}{2}} \alpha_k \cos\left(\frac{2\pi}{N} kx\right) + \sum_{k=1}^{\frac{N}{2}-1} \beta_k \sin\left(\frac{2\pi}{N} kx\right)$$

Thus, sines and cosines define an basis for functions $I(x)$.
One can show that this is an orthogonal basis.

Note that the frequency range k differs for sine and cosine.

$$I(x) = \sum_{k=0}^{\frac{N}{2}} \alpha_k \cos\left(\frac{2\pi}{N} kx\right) + \sum_{k=1}^{\frac{N}{2}-1} \beta_k \sin\left(\frac{2\pi}{N} kx\right)$$



Brief Summary

$$cos\left(\frac{2\pi k}{N}x\right) \ast \ h(x) \quad = \quad \sqrt{a^2+b^2} \ \ cos\left(\frac{2\pi k}{N}x \ - \phi\right)$$

$$sin\left(\frac{2\pi k}{N}x\right) \ast \ h(x) \quad = \quad \sqrt{a^2+b^2} \ \ sin\left(\frac{2\pi k}{N}x \ - \phi\right)$$

$$I(x) \ = \ \sum_{k=0}^{\frac{N}{2}} \alpha_k \cos\left(\frac{2\pi}{N}kx\right) + \sum_{k=1}^{\frac{N}{2}-1} \beta_k \sin\left(\frac{2\pi}{N}kx\right)$$

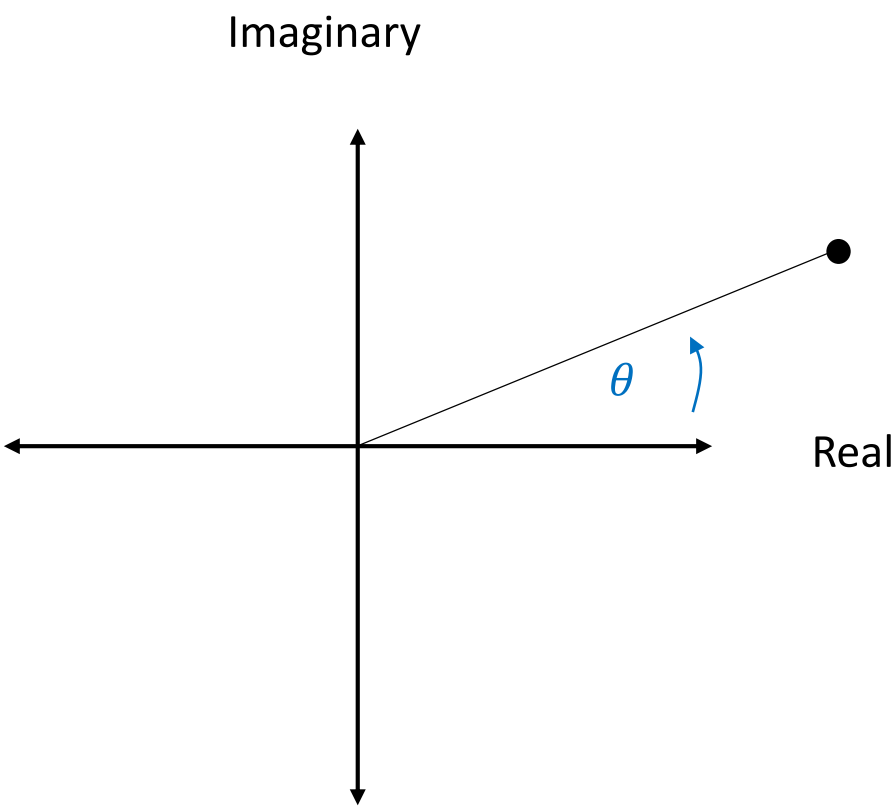
Putting these together gives us the idea of Fourier transforms and filtering.

Towards Fourier analysis

- an alternative way of writing the above equations

- based on complex numbers (which I review next).

What is a complex number ?

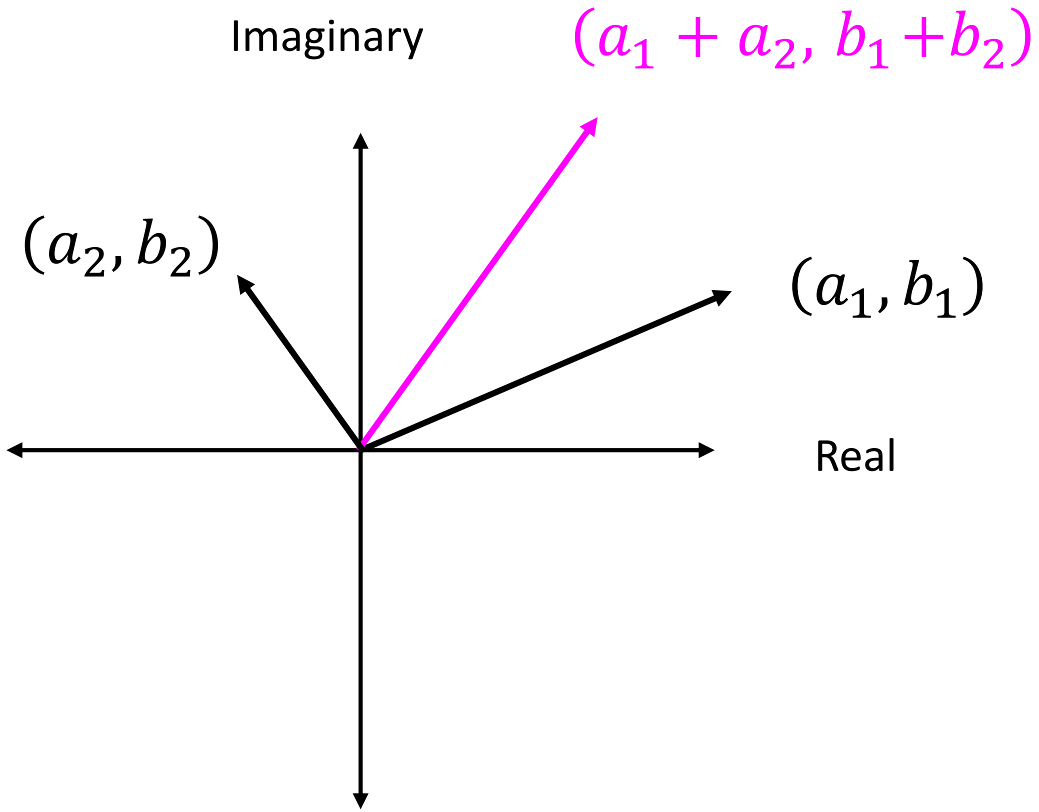


$$c = a + b\,i$$

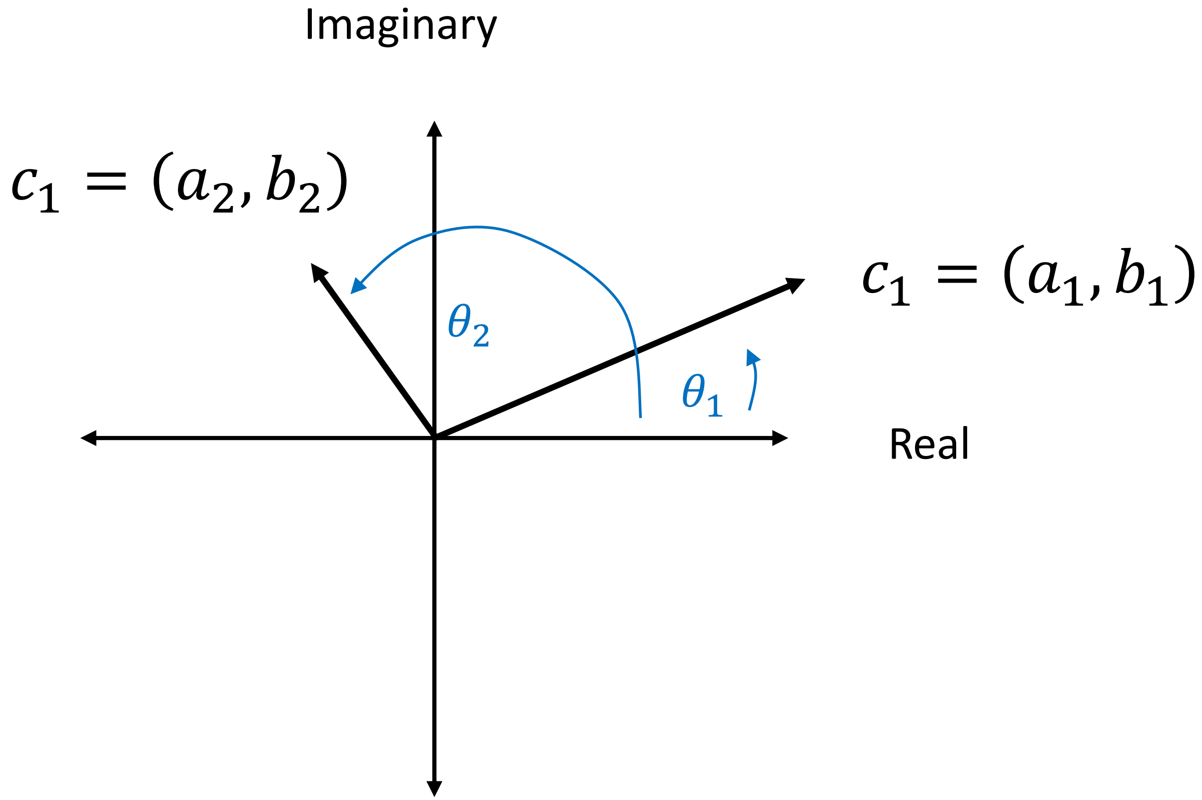
$$= \; r\cos\theta + i\,r\sin\theta$$

$$|\,c\,| = \; r \; = \sqrt{a^2 + b^2}$$

Addition of complex numbers

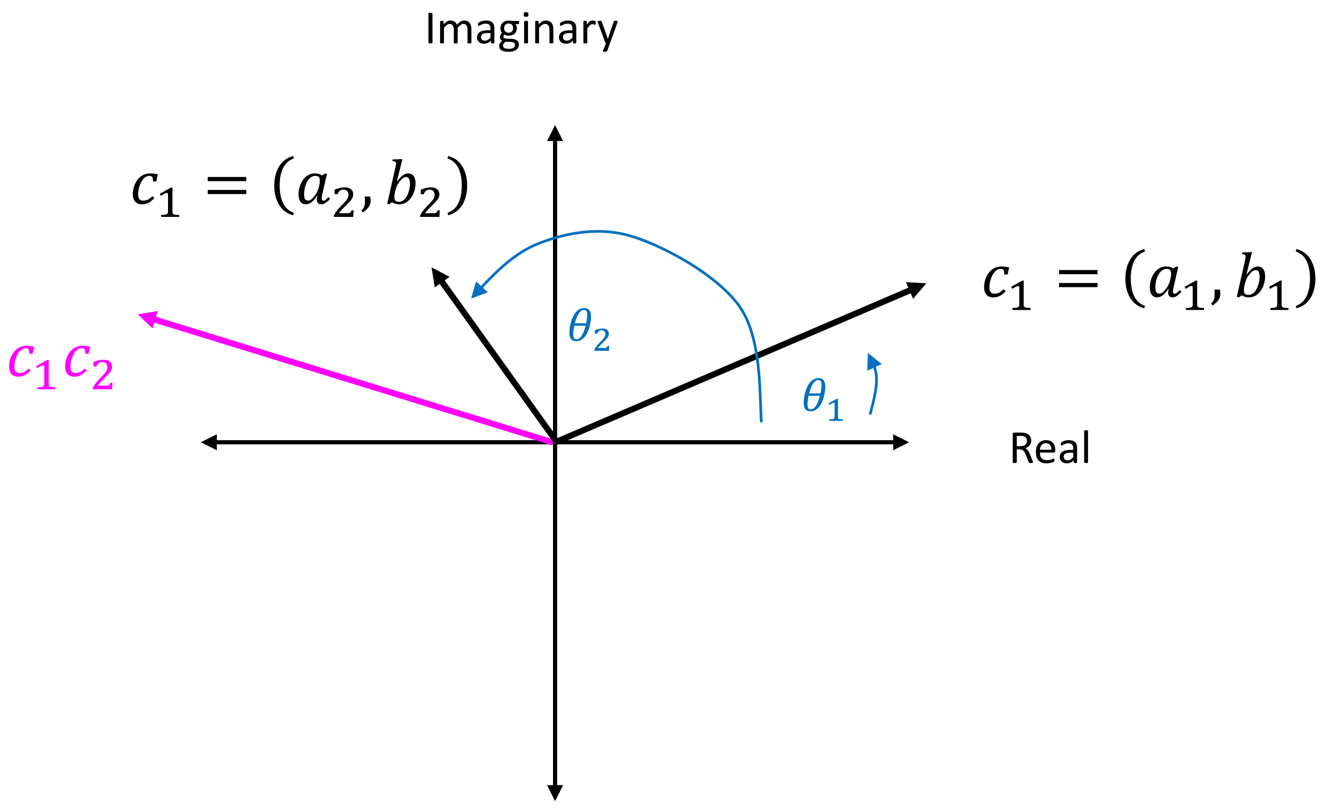


Multiplication of complex numbers



$$c_1 c_2 = \quad ?$$

Multiplication of complex numbers

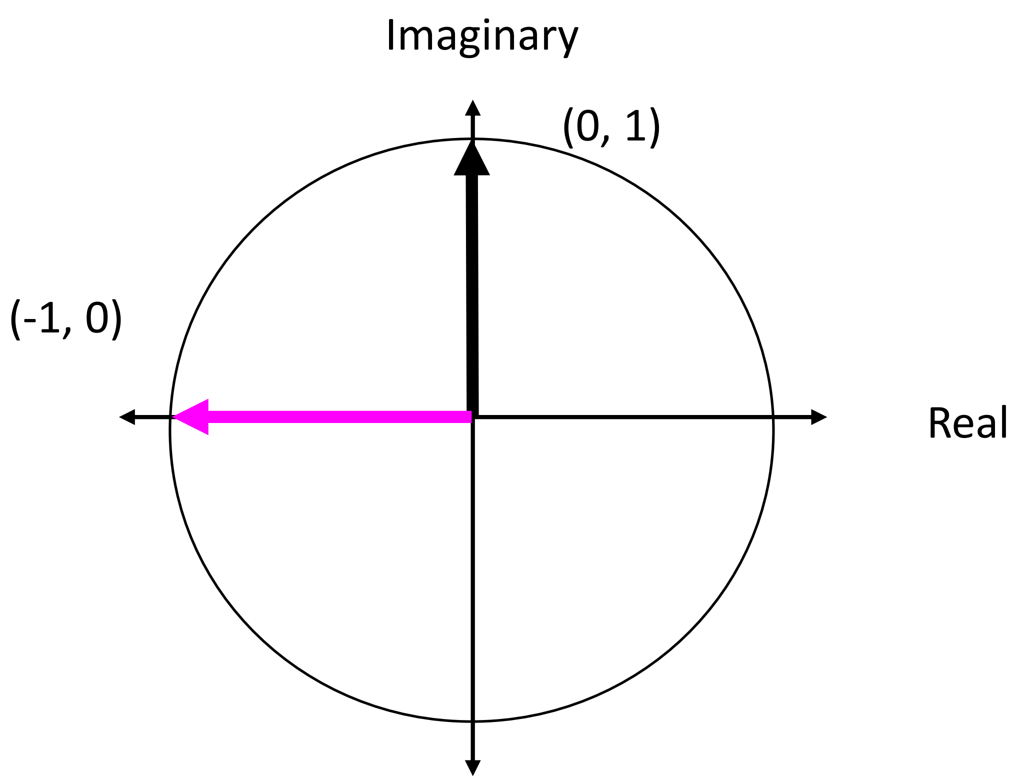


$$c_1 c_2 = |c_1||c_2| \left(cos(\theta_1 + \theta_2),\ sin(\theta_1 + \theta_2) \right)$$

Multiply lengths

Add angles

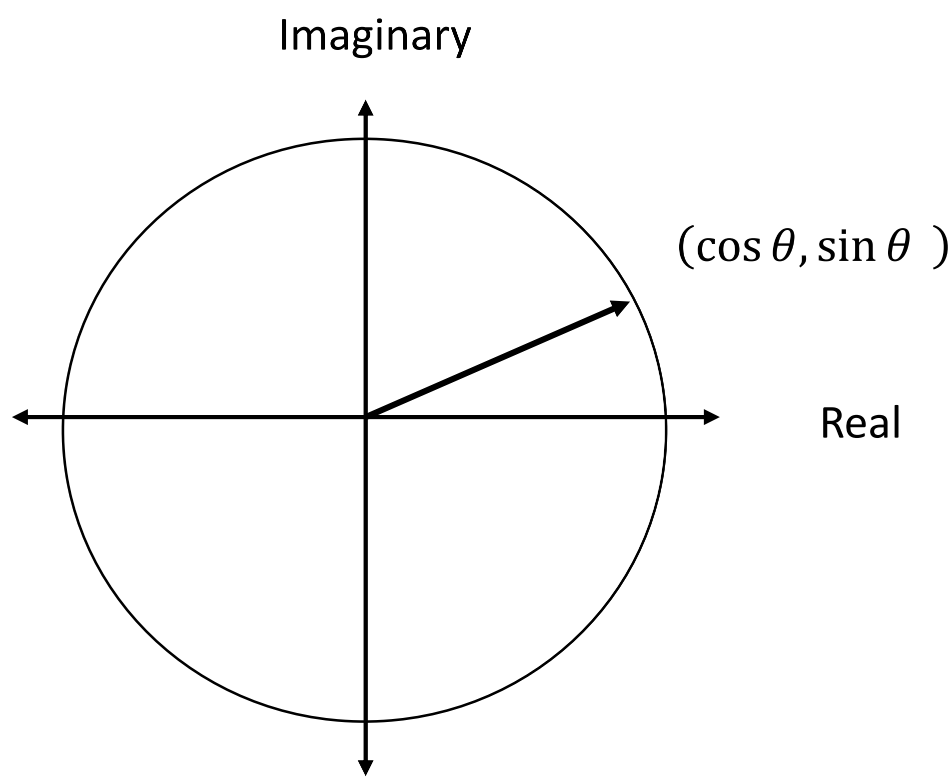
Example $i * i = -1$



Euler's equation

(Why? See Appendix to lecture notes)

$$e^{i\theta} = \cos\theta + i\,\sin\theta$$



Examples of Euler's equation

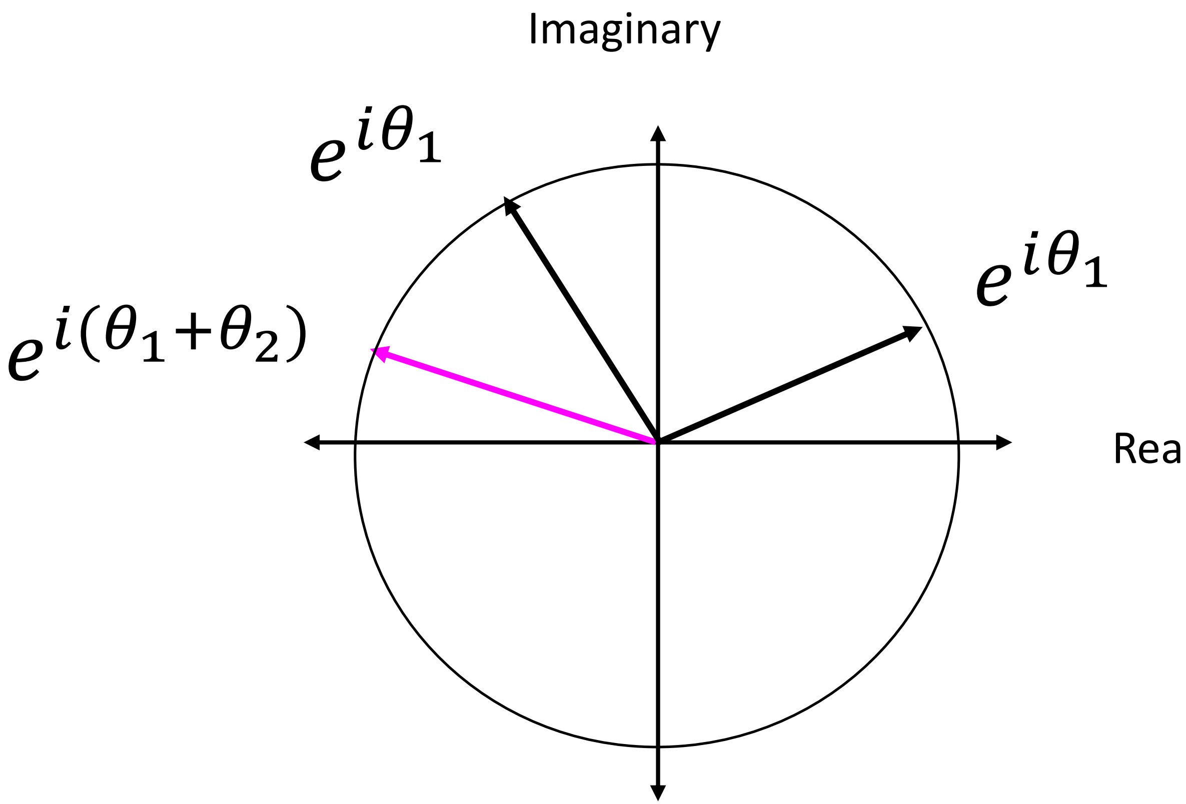
$$e^{i\cdot 0} = \cos(0) + i\sin(0) = 1$$

$$e^{i\cdot\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

$$e^{i\cdot\pi} = \cos(\pi) + i\sin(\pi) = -1$$

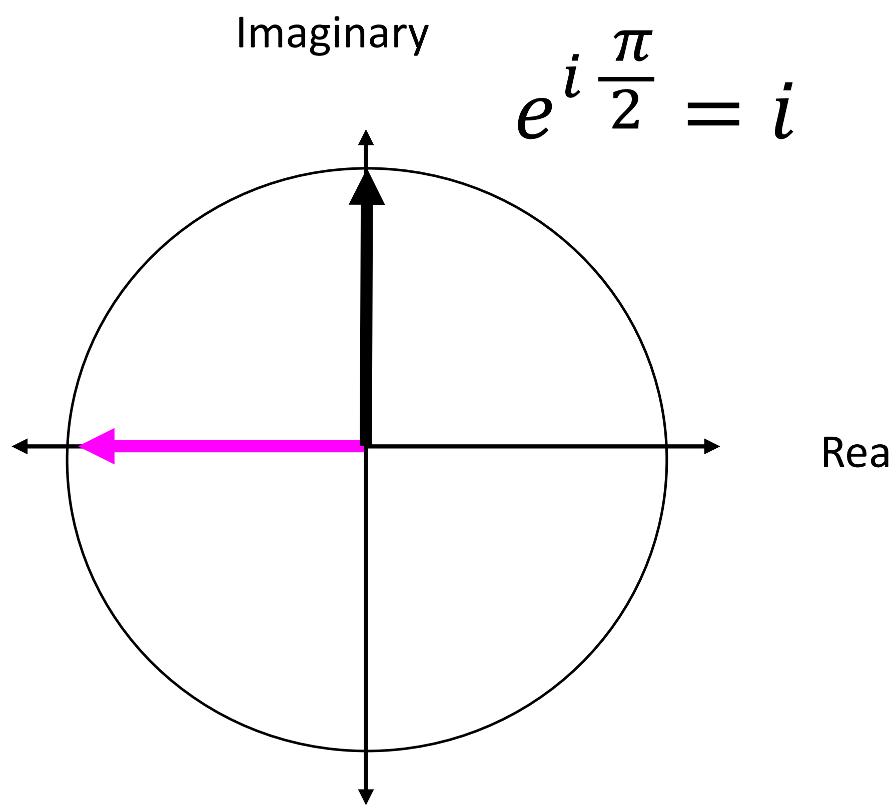
$$e^{i\cdot 2\pi} = \cos(2\pi) + i\sin(2\pi) = -i$$

Multiplication of complex numbers



$$e^{i\theta_1} \; e^{i\theta_1} \; = \; e^{i(\theta_1+\theta_2)}$$

Example $\quad i * i = -1$



$$e^{i\,\frac{\pi}{2}} = i$$

$$e^{i\,\pi} = -1$$

$$e^{i\,\frac{\pi}{2}}\ e^{i\,\frac{\pi}{2}} = e^{i\,\pi}$$

Trigonometric identities follow from Euler's Rule

$$e^{i\theta_1} e^{i\theta_2}$$

$$= e^{i(\theta_1+\theta_2)}$$

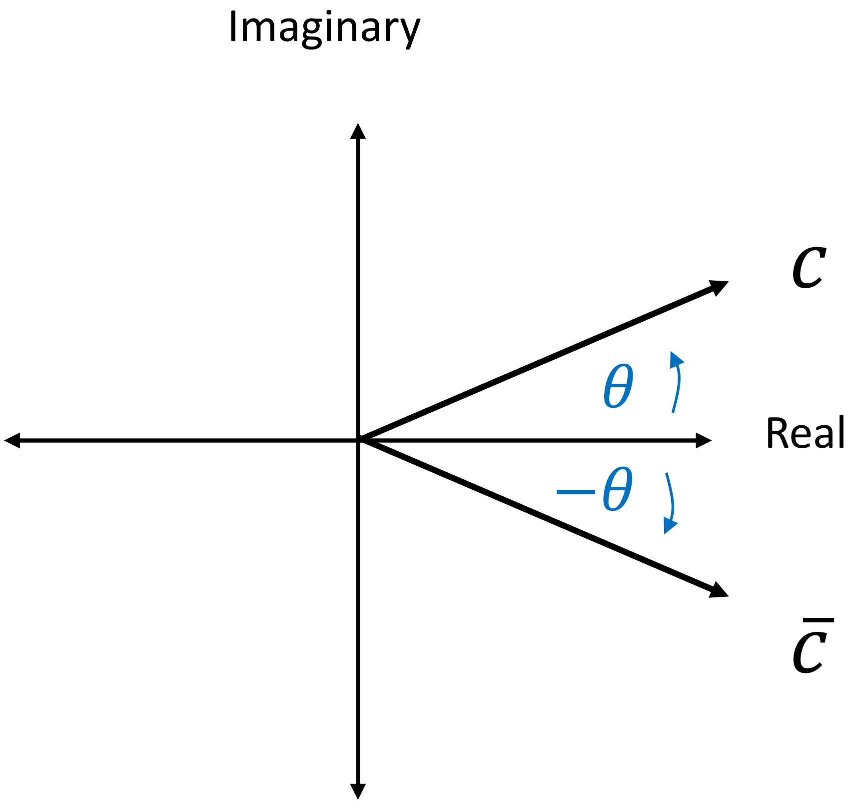
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$= (\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2)$$

$$= (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)$$

$$+ i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)$$

Complex conjugate

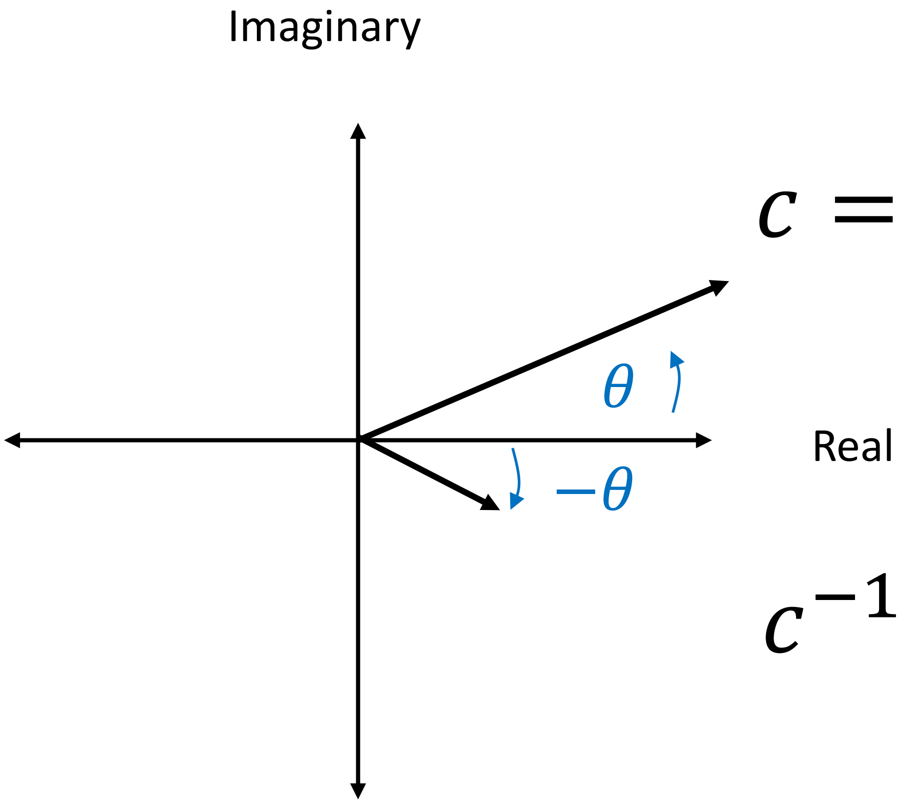


$$c = a + b\,i \;=\; r\,e^{i\theta}$$

$$\bar{c} = a - b\,i \;=\; r\,e^{-i\theta}$$

$$\bar{c}\,c \;= r^{2}$$

Multiplicative Inverse



$$c = r\, e^{i\theta}$$

$$c^{-1} \equiv \frac{1}{r} e^{-i\theta}$$

$$c\ c^{-1} = 1$$