Questions

1. You might naively think that panning a camera would produce a constant image motion in the direction opposite to the rotation motion e.g. panning to the right would give uniform motion to the left. However, this is not quite so. To understand why, we need to think more precisely about what happens in image projection.

   (a) Consider a 3D point (X_0, Y_0, Z_0) and the line from the origin through that point. If the camera rotates around the Y axis (pan), then in camera coordinates this line will appear to sweep out a 3D cone that is centered on the Y axis. What is the equation of this 3D cone?

   (b) The cone will intersect the image projection plane $Z = f$. What is the image plane curve defined by this intersection? This defines the path of the 3D point as seen in the image projection.

2. Consider the plane
$$aX + bY + cZ + d = 0 \ .$$
   Suppose the 3D space undergoes a rotation, defined by a matrix $\mathbf{R}$. What is the new equation of the plane?

   Hint: The normal vector to the plane is (a, b, c).

3. Consider a rotation by θ degrees about an axis whose direction is some unit vector $\mathbf{n}$.

   What is the rotation matrix? Answer this question in two steps.

   (a) For any vector $\mathbf{X}$, write it as a sum of a component in the direction $\mathbf{n}$ and a component perpendicular to $\mathbf{n}$.

   (b) For the component of $\mathbf{X}$ that is perpendicular to $\mathbf{n}$, rotate it by θ around the axis defined by $\mathbf{n}$. To do so, think how you could get a rotation by $\theta = 90$ degrees, and then figure out the case of general θ.

   Hint: Use a cross product.

4. (a) Let two lines l_i be
$$a_i x + b_i y + c_i = 0$$
   where $i = 1, 2$. You can find the intersection of the two lines using Gaussian elimination, but there is a quicker way, namely take the cross product, $(a_1, b_1, c_1) \times (a_1, b_1, c_1)$ and treat the result as a homogenous vector; to get the intersection point, divide by the 3rd coordinate.

   Give a geometric argument that explains the result indeed gives the intersection of the lines.

(b) Suppose we have the two lines

$$3x + 4y + 2 = 0$$

$$2x - y = 0.$$

Compute their intersection using the method of (a).

If you want to brush up a bit on linear algebra, then see my Exercises from COMP 557 (Computer Graphics) for lectures 1 and 2.

Answers

1. (a) A cone that is centered on the Y axis has the form

$$Y^2 = m(X^2 + Z^2)$$

for some value of m. Since the cone passes through (X_0, Y_0, Z_0), we have

$$m = \pm \frac{Y_0}{\sqrt{X_0^2 + Z_0^2}}$$

(b) This cone will intersect the image projection plane $Z = f$ on two curves that satisfy:

$$y^2 = m(x^2 + f^2)$$

where (x, y) are the image plane coordinates. You can try to visualize these curves for yourself.

2. Since the normal to the plane is $\mathbf{n} = (a, b, c)$, the normal of the rotated plane is $\mathbf{n}' = \mathbf{R}\mathbf{n}$. It is not difficult to show from here that the rotated plane is $\mathbf{n}' \cdot (X, Y, Z) + d = 0$.

3. (a) We can write any 3D point $\mathbf{X}$ as

$$\begin{aligned} \mathbf{X} &= (\mathbf{X} \cdot \mathbf{n})\mathbf{n} + (\mathbf{X} - (\mathbf{X} \cdot \mathbf{n})\mathbf{n}) \\ &= \mathbf{n}\mathbf{n}^T\mathbf{X} + (\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{X} \end{aligned}$$

which decomposes it into a vector in direction $\mathbf{n}$ and a vector that is perpendicular to $\mathbf{n}$.

(b) Notice that if we were to rotate the second component $(\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{X}$ by 90 degrees about the axis $\mathbf{n}$, then we would get the vector $\mathbf{n} \times \mathbf{X}$. Therefore, if we were instead to rotate the vector $(\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{X}$ by θ degrees, then we would get:

$$\cos\theta(\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{X} + \sin\theta(\mathbf{n} \times \mathbf{X})$$

Thus,

$$\mathbf{R}\mathbf{X} = (\mathbf{X} \cdot \mathbf{n})\mathbf{n} + \cos\theta(\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{X} + \sin\theta(\mathbf{n} \times \mathbf{X}).$$

The formula is known as *Rodriguez'es formula for rotations.*

4. (a) Consider the 3D vectors (a_i, b_i, c_i) and $(x, y, 1)$, where the latter lies on a projection plane $Z = 1$. Because their dot product is 0, vectors (a_i, b_i, c_i) and $(x, y, 1)$ are orthogonal to each other. In particular, (a_i, b_i, c_i) is orthogonal to the plane π_i spanned by the origin and the line l_i, and so (a_i, b_i, c_i) is a normal vector to that plane. The two lines $l_i, i = 1, 2$ thus define two planes π_1 and π_2, both of which pass through the origin (0,0,0). The intersection of these two planes must therefore be a line that passes through the origin i.e. a "line of sight". This line meets $Z = 1$ at precisely the intersection of l_1 and l_2.

Since (a_i, b_i, c_i) is orthogonal to plane π_i for both $i = 1, 2$, it follows that their cross product vector

$$(a_1, b_1, c_1) \times (a_2, b_2, c_2)$$

must lie in *both planes* π, and hence it lies on the intersection of the two planes. Thus, to get the intersection of the two lines l_1 and l_2, we compute their cross product – call it (x_0, y_0, z_0) – and then divide by z so that it intersects $Z = 1$ i.e. so the intersection point is $(\frac{x_0}{z_0}, \frac{y_0}{z_0})$

(b) Then $(3, 4, 2) \times (2, -1, 0) = (2, 4, -11)$ and so the intersection point is $(-\frac{2}{11}, -\frac{4}{11})$.

