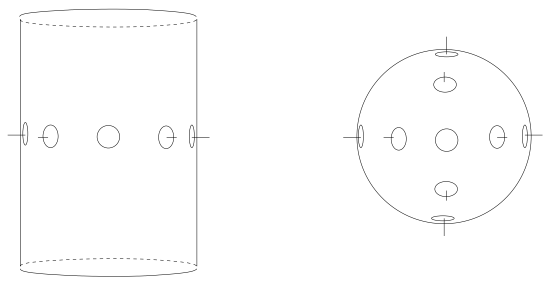
Questions

1. When a face is photographed from the front and from a small distance, the nose appears much larger than it should in comparison to the other parts of the face. Why?

2. Below are sketches of a cylinder and sphere. What are the slant and tilt for the various points marked on these surfaces?

   Note: To define slant and tilt, you need a plane. The plane being marked here is the "tangent plane", namely the plane that touches the surface at one point only. This plane defines the normal vector to the surface.



3.

   Below is a photograph some (upside-down) dessert plates on my dining room table. The plates are all the same size in 3D and they are spaced evenly on the table. You can see that both the image size and aspect ratio of the plates vary in the image. The image width Δx of the near plate is about twice as great as the far plate. The aspect ratio $\Delta y/\Delta x$ of the far plate is about 1/3 whereas the aspect ratio of the near plate is close to 1.

Motivated by the above image, let's try to derive an expression for the aspect ratio of the disks. The general equation of a plane in camera coordinates is $AX + BY + CZ = D$. Take the case such as in the above photograph in which the tilt is 90 degrees, and so the equation plane reduces to

$$BY + CZ = D.$$

(a) Write the equation of the plane in terms of slant σ.

(b) Suppose you drop a disk of diameter 1 onto the plane. (For example, if the XYZ coordinates were cm, then the disk would be 1 cm in diameter.) What is the δY and δZ for the closest and farthest points on the disk ? What is the corresponding image distance Δy between these two points?

(c) What is the image width Δx of the disk ?

(d) What is the aspect ratio $\frac{\Delta y}{\Delta x}$ as a function of $\frac{1}{Z}$?

4. True or false? Explain why.

   (a) "The bidirectional reflectance distribution function (BRDF) of a surface in a given scene depends on the lighting of that scene."

   (b) "If a curved surface is illuminated by a parallel source (e.g. sunlight), then all points on the surface have the same irradiance."

   (c) "Two points on the sensor plane have the same RGB irradiance $E_{RGB}(x, y)$ if and only if they have the same irradiance spectra $E(x, y, \lambda)$."

5. For the camera calibration matrix $\mathbf{K}$,

   (a) Describe what happens to the appearance of an image if $\alpha_x < \alpha_y$.

   (b) What are typical values of p_x and p_y ? (Your answer must be in terms of other camera parameters.)

6. Suppose a photographer chooses the following camera settings to get an acceptable exposure for a scene: focal length $f = 80$ mm, shutter speed $\frac{1}{t} = 32$, f-number $= 5.6$.

   Give an example of alternative settings that result in a greater range of depths that are in-focus. Do not change the exposure or the field of view angle.

7. Suppose

$$I(x) = \begin{cases} 80, & x > x_0 \\ 40, & x = x_0 \\ 20, & x < x_0 \end{cases}$$

and

$$B(x) = \begin{cases} \frac{1}{4}, & |x| = 1, \; i.e. \; x = \pm 1 \\ \\ \frac{1}{2}, & x = 0 \\ \\ 0, & otherwise \end{cases}$$

$$D(x) = \begin{cases} -1, & x = 1 \\ 1, & x = -1 \\ 0, & otherwise \end{cases}$$

(a) What is $I(x) * B(x)$?

(b) What is $I(x) * B(x) * D(x)$?

8. Suppose we have a plane

$$aX + bY + cZ = d$$

that is observed by a laterally moving observer ($T_z = 0$). Give an expression for the image velocity field as a function of x, y, T_x, T_y, f and any other scene variables.

9. Consider a wall at constant depth Z_0. What is the region of this wall that contributes to the image at position (x_0, y_0, Z_s) on the sensor plane, assuming the image is formed under a thin lens model ?

10. The projection matrix

$$\mathbf{P} = \mathbf{KR}[\ \mathbf{I} \mid -\mathbf{C}]$$

is a 3×4 matrix, i.e. $\mathbf{I}$ is 3×3 and $\mathbf{C}$ is 3×1.

(a) You might think that it has 12 degrees of freedom, but in fact it has only 11. Why?

(b) What is the null space of $\mathbf{P}$? ($\mathbf{P}$ is not invertible since it has more columns than rows.)

(c) What are the four columns of $\mathbf{P}$? Geometrically, what do they represent?

(d) What are the three rows of $\mathbf{P}$? i.e. Geometrically, what do they represent?

11. For many artificial light sources such as candles and light bulbs, it is common to approximate the source as an infinitely bright single point (or a very bright small sphere) and to describe the illumination from that point in terms of the power of light radiated over the sphere of directions centered at that point. Define the *radiant intensity* of such a source to be power per unit solid angle. In general, radiant intensity is a function defined on the unit sphere.

Suppose you have a spherical light source, centered at position $\mathbf{X}_{src} = (0, 4, 3)$ in a scene, where distances are measured in meters. The sphere is of radius 0.1 m and suppose the radiant intensity in direction $\mathbf{l}$ is $(1 + \mathbf{l} \cdot \mathbf{g})^2$ where $\mathbf{g} = (0, -1, 0)$ is the unit vector in the direction of gravity.

Suppose the light source is illuminating a plane $Y = 0$.

(a) What is the irradiance of a surface at the origin.

(b) What is the solid angle subtended by the sphere, as seen from the origin?

(c) What is the radiance of the sphere as seen from the origin $(X, Y, Z) = (0, 0, 0)$?

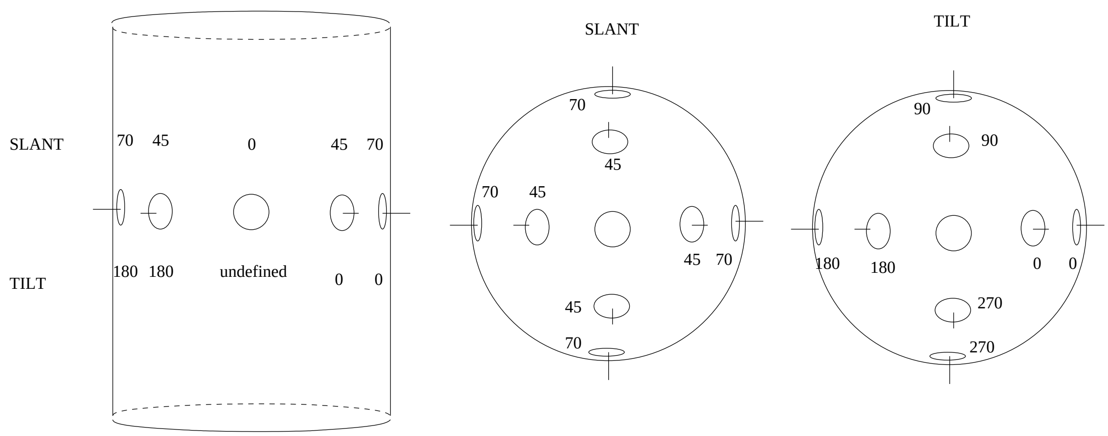
Answers

1. Since the nose is the closest part of the face, the depth Z of the nose is less than for other parts of the face. For fixed ΔX, note that

$$\Delta x = f \frac{\Delta X}{Z}.$$

Thus for two neighboring points on the nose that are separated by ΔX and are at roughly at the same depth Z_{nose}, the Δx will be greater than the Δx value for two neighboring non-nose points with the same ΔX but greater Z.

2.



3. (a) Multiplying the equation of the plane by f/Z and dividing by C gives

$$\frac{B}{C} y + f = \frac{Df}{CZ}$$

so

$$y \tan \sigma + f = \frac{Df}{CZ}.$$

(b)

Taking the derivative of y with respect to Z,

$$\begin{aligned} \Delta y \tan \sigma &= \frac{Df}{C} \Delta(\frac{1}{Z}) \\ &\approx -\frac{Df}{CZ^2} \Delta Z. \end{aligned}$$

The farthest and nearest points on the disk have a Z difference $\Delta Z = \sin \sigma$. Thus

$$|\Delta y| \approx |\frac{fD}{CZ^2} \cos \sigma|.$$

(c) The width of the disk in the X direction is $\Delta X = 1$, and $x = f\frac{X}{Z}$, so we get

$$\Delta x = \frac{f\Delta X}{Z} = \frac{f}{Z}.$$

(d) The image aspect ratio of the projected disk is

$$\frac{\Delta y}{\Delta x} \approx |\frac{D\cos\sigma}{CZ}| = |\frac{D}{Z\sqrt{B^2+C^2}}|.$$

As we approach the horizon ($Z \to \infty$), the aspect ratio goes to 0.
For example, for the case $C = 0$, we have $Y = D/B$, and so

$$\frac{\Delta y}{\Delta x} \approx |\frac{D}{ZB}|.$$

But for this case, $y = \frac{Df}{BZ}$ so $\frac{\Delta y}{\Delta x} \approx \frac{y}{f}$.

4. (a) False. $BRDF = f(\mathbf{x}, \mathbf{l}_{in}, \mathbf{l}_{out})$ and so the BRDF depends only on the position and on the *directions* of incoming and outgoing light. It does not depend on the incoming radiance of a particular scene.

   (b) False. On a curved surface, the normal $\mathbf{n}$ varies. So, the irradiance $E(\mathbf{x})$ will not be constant.

   (c) False. The question says "if and only if". Two points on the sensor plane have the same RGB irradiance $E_{RGB}(x, y)$ if they have the same irradiance spectra $E(x, y, \lambda)$," However, the converse is not true. Two points can have the same $E_{RGB}(x, y)$ but can have different irradiance spectra.

5. (a) α_x and α_y are the number of pixels per mm in the x, y directions. If $\alpha_x < \alpha_y$ then the density of pixels is less in the x direction. When the image is displayed on a square grid of pixels on a monior it will appear squished in the x direction.

   (b) (p_x, p_y) is the pixel coordinate of the optical axis (or principal point). This is supposed to be in the center of the image, so it should be about $(N_x/2, N_y/2)$ where (N_x, N_y) are the number of pixels in the x, y directions. [You needed to know this for Assignment 1.]

6. The question says to change neither the exposure nor the field of view angle. If you cannot change the field of view angle, then you cannot change the focal length. Thus, you change only the shutter speed and/or the aperture. You need to decrease the blur width ΔX at each point. So you need to *decrease* the aperture (*increase* the f-number, since focal length is fixed). To keep the exposure constant, you need to *decrease* the shutter speed.

   The exposure depends on the inverse of the shutter speed and on the square of the aperture. Consecutive f-numbers vary by a factor $\sqrt{2}$ whereas consecutive shutter speeds vary by a factor 2. So, a shutter speed of $1/t = 16$ (changed from 32) and f-number of $8 = \sqrt{64}$ (changed from $5.6 = \sqrt{32}$) would do the trick.

7. (a)
$$(I * B)(x_0 - 2, x_0 - 1, x_0, x_0 + 1, x_0 + 2) = (20, 25, 45, 70, 80)$$
and values are less than $x_0 - 2$ are 20, and all values greater than $x_0 + 2$ are 80.

   (b) Take the answer from (a) and convolve with $D(x)$. This is easy (and tedious) to do by hand and gives
$$(I * f * D)(x_0 - 2, x_0 - 1, x_0, x_0 + 1, x_0 + 2) = (\ldots, 0, 0, 5, 25, 45, 35, 10, 0, 0, \ldots).$$

8. The velocity field for lateral motion is
$$(v_x, v_y) = \frac{f}{Z}\,(-T_x, \ \ -T_y).$$
Multiplying the equation of a plane by $\frac{df}{Z}$ gives:
$$\frac{ax + by + cf}{d} = \frac{f}{Z}.$$
Substituting the left side into the velocity field expression gives:
$$(v_x, v_y) = \frac{ax + by + cf}{d}\,(-T_x, \ \ -T_y).$$
The field has a constant direction but the magnitude varies linearly with (x, y).

9. The center (X_0, Y_0) of the region on the wall is determined by:
$$(\frac{X_0}{Z_0}, \frac{Y_0}{Z_0}) = (\frac{x_0}{Z_s}, \frac{y_0}{Z_s})$$
To find the width ΔX of the region on the wall, use reverse projection:
$$\frac{Z_0 - Z_s^*}{\Delta X} = \frac{Z_s^*}{A}$$
where A is the diameter of the lens (the aperture), and
$$\frac{1}{f} = \frac{1}{Z_s} + \frac{1}{Z_s^*}.$$
A similar expression holds for ΔY.

10. (a) We can multiply the matrix $\mathbf{P}$ by any constant $a > 0$ but this will not change the projection, since $\mathbf{P}$ maps vectors that are represented in homogeneous coordinates, so $\mathbf{P}(a\mathbf{X})$ represents the same point as $\mathbf{PX}$. But, $\mathbf{PX}$ represents the same point as $a\mathbf{P})\mathbf{X})$.

   (b) The null space of $\mathbf{P}$ is the set of vectors $\mathbf{X}$
$$\mathbf{PX} = \mathbf{0}.$$

By inspection (see lecture 4, page 3), the camera position $\mathbf{X} = \mathbf{C}$ is a null vector since

$$[\mathbf{I} \mid -\mathbf{C}]\mathbf{C} = \mathbf{O}$$

that is,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \end{bmatrix} \begin{bmatrix} c_x \\ c_y \\ c_z \\ 1 \end{bmatrix}$$

This is the only null vector since both $\mathbf{R}$ and $\mathbf{K}$ are invertible.

(c) The first three columns of $\mathbf{P}$ are, by definition, the image projection of the of the world coordinate axes $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, which are direction vectors. The last column of $\mathbf{P}$ is the image projection of the world origin. Note that the world origin is not the same thing generally as the camera origin (and if it is, then fourth column will be $(0,0,0)^T$).

(d) We begin with the third row of $\mathbf{P}$. If we treat this row as the set of coefficients of the 3D plane,

$$P_{3,1}X + P_{3,2}Y + P_{3,3}Z + P_{3,4} = 0$$

then any point (X,Y,Z) on this plane will map either to a point at infinity in the image, i.e. to some point $(*,*,0)$. Actually there is one exception to this. Since the camera center $\mathbf{C}$ lies in the null space of $\mathbf{P}$, the camera center must lie on this plane as well.

What scene points project to image points at infinity? These are the scene points that lie in a plane containing the camera's center of projection and whose normal is the optical axis. This is called the *principal plane* of the camera. Note that the 3-vector $(P_{3,1}, P_{3,2}, P_{3,3})$ must be in the direction of the optical axis of the camera, since it is normal to the principal plane.

The first row of $\mathbf{P}$ can be interpreted as the set of coefficients of a different plane,

$$P_{1,1}X + P_{1,2}Y + P_{1,3}Z + P_{1,4} = 0$$

whose points are projected to the line $x = 0$ in the image plane. Note that since the camera center $\mathbf{C}$ lies in the null space of $\mathbf{P}$, the camera center must lie on this plane as well.

Does the optical axis lie in this plane? Not necessarily, since the image coordinate coordinate system does not necessarily have the optical axis at $(0,0)$.

By similar reasoning, the second row of $\mathbf{P}$ corresponds to a plane containing the camera center and the x-axis of the image plane.

11. (a) Consider a small patch of area a containing the origin. If this patch were oriented so that it faced the source directly, then it would subtend a solid angle $\frac{a}{5^2}$ steradians, as viewed from the center of the spherical source. It is oriented obliquely, however, and so its solid angle is less: the area is foreshortened and so the solid angle is multiplied by $(0, \frac{4}{5}, \frac{3}{5}) \cdot (0,1,0) = \frac{4}{5}$. Thus, the solid angle of the patch as seen from the center of the sphere is $\frac{a}{5^2}\frac{4}{5}$ steradians.

To calculate the irradiance, we first calculate how many Watts this patch of area a would receive. The radiant intensity of the source is given in the question, and the unit vector $\mathbf{l}$ in question is $(0, -\frac{4}{5}, -\frac{3}{5})$. Plugging in, we get that the radiant intensity in the direction of our surface patch is $(1 + \mathbf{l} \cdot \mathbf{g})^2 = (1 + \frac{4}{5})^2 = (\frac{9}{5})^2$ Watts per steradian.

Thus, the patch receives $(\frac{9}{5})^2$ Watts per steradian * $\frac{a}{5^2}\frac{4}{5}$ steradians $= (\frac{9}{5})^2 \cdot \frac{a}{5^2} \cdot \frac{4}{5}$ Watts. Finally, to get the irradiance (Watts per m^2), we must divide by the area a.

(b) The sphere is at a distance $5 = \sqrt{3^2 + 4^2}$ from the origin. If the sphere were projected onto a unit sphere centered at the origin, it would cover an area of about $\frac{\pi(0.1)^2}{5^2}$, so this is the solid angle (in steradians).

(c) Since the source subtends a small angle only and has a smooth irradiant intensity (as a function of direction $\mathbf{l}$), we can assume that the radiance over incident directions is approximately constant:

$$E = L_{in} \mathbf{n} \cdot \mathbf{l}_{in} \Omega$$

where Ω was the answer to (b), E is the answer to (a), and $\mathbf{l} \cdot \mathbf{n} = (0, \frac{4}{5}, \frac{3}{5}) \cdot (0, 1, 0) = \frac{4}{5}$. i.e. same foreshortening factor as above. Substituting gives the solution for L_{in}.