Questions

1. The motion constraint equation requires measuring partial derivatives $\frac{\partial I}{\partial x}$, $\frac{\partial I}{\partial y}$, $\frac{\partial I}{\partial t}$ of the image $I(x, y, t)$. Given a digital video, how could you estimate these gradients?

   (This question came up in class, so I am adding it here rather than in the lecture notes.)

2. For a smooth noise-free image $I(x, y, t)$, the 2D *normal velocity* vector $\vec{v}_n$ at (x, y, t) is defined to be the image velocity component in the direction parallel to the intensity gradient. Show that the normal velocity vector $\vec{v}_n$ is

$$\vec{v}_n \equiv \frac{-\frac{\partial I}{\partial t}}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2} \, (\frac{\partial I}{\partial x} , \frac{\partial I}{\partial y}) \ .$$

   Hint: The definition of v_n gives its direction. You just need to find its length, i.e. find a scalar α such that

$$\vec{v}_n = \alpha \, (\frac{\partial I}{\partial x} , \frac{\partial I}{\partial y}) \ .$$

3. Consider a checkerboard pattern moving with image velocity $(v_x, 0)$.

   What are the two motion constraint lines available to the vision system from this moving pattern? Assume the checkerboard pattern is oriented in the usual way namely with horizontal and vertical edges.

4. (a) Consider a video that is a sine function

$$I(x, y, t) = I_0 + \sin(\frac{2\pi}{N} k_0 x + \frac{2\pi}{T} \omega_0 t).$$

   For fixed t, this is a sine function over x namely k_0 cycles over N samples. For fixed x, it is a sine function of time, namely it oscillates over time t with frequency ω cycles per T time steps.

   To make this sinusoidal image *translate* with 2D velocity $(v_x, 0)$, we must find suitable values for k_0 and ω_0. Use the motion constraint equation to show that these values satisfy:

$$v_x = -\frac{\omega_0}{k_0} \frac{N}{T}.$$

   (b) What are the units for v_x in the above formula ?

   (c) Same question, but now let the frequency be (k_0, k_1, ω_0) so

$$I(x, y, t) = I_0 + \sin(\frac{2\pi}{N}(k_0 x + k_1 y) + \frac{2\pi}{T} \omega_0 t).$$

   What is the relationship between (v_x, v_y) and the frequency (k_0, k_1, ω_0) ?

(d) For the case in (c), what is the normal velocity vector v_n ?
Hint: first ask yourself what is the direction of v_n ?

5. Suppose a set of vertically oriented lines is travelling to the right at speed v_x and a set of horizontally oriented lines is travelling upwards at speed v_y.

   What is the velocity of the points of intersection of these lines? (This should be intuitively obvious. But make sure you understand what this has to do with motion constraint lines.)

6. Consider a bright disk on a dark background. Suppose the disk is moving with $(v_x, v_y) = (1, 0)$. What are the normal velocities of points on the boundary of the disk? Answer this question for specific $(\cos\theta, \sin\theta)$ on the disk boundary, namely $\theta = 0, 45, 90, 135, 180$ degrees.

Solutions

1. There are many ways to estimate the partial derivatives that are needed to define a gradient. The basic idea is to take a discrete approximation. For a time varying image $I(x, y, t)$, we could approximate the partial derivative in x, y, t as follows:

$$\begin{aligned}
\frac{\partial I(x,y,t)}{\partial x} &\approx \frac{I(x+1,y,t) - I(x-1,y,t)}{2} \\
\frac{\partial I(x,y,t)}{\partial y} &\approx \frac{I(x,y+1,t) - I(x,y-1,t)}{2} \\
\frac{\partial I(x,y,t)}{\partial t} &\approx \frac{I(x,y,t+1) - I(x,y,t-1)}{2}.
\end{aligned}$$

2. The normal velocity satisfies the motion constraint equation, so we substitute v_n into

$$\frac{\partial I}{\partial x}\ v_x + \frac{\partial I}{\partial y}\ v_y + \frac{\partial I}{\partial t}\ = 0$$

which gives

$$\alpha\ ((\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y}\ )^2) + \frac{\partial I}{\partial t}\ = 0\ .$$

So

$$\alpha = -\frac{\frac{\partial I}{\partial t}}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}.$$

3. For the horizontal lines of the checkerboard, $\frac{\partial I}{\partial x}$. Since there is no motion component in the y direction, $\frac{\partial I}{\partial t}\ = 0$ across any horizontal edge. Substituting into the motion constraint equation gives the motion constraint line

$$v_y = 0$$

which corresponds to the v_x axis.

For the vertical lines of the checkerboard, $\frac{\partial I}{\partial y}\ = 0$, and so the motion constraint line is $\frac{\partial I}{\partial x}\ v_x + \frac{\partial I}{\partial t}\ = 0$ or

$$v_x = -\frac{\partial I}{\partial t}\ /\ \frac{\partial I}{\partial x}$$

which is parallel to the v_y axis. By definition, these two lines intersect at the true velocity $(-\frac{\partial I}{\partial t}\ /\ \frac{\partial I}{\partial x}, 0)$.

4. (a)

$$\begin{aligned}
\frac{\partial I(x(t),t)}{\partial x} &= \frac{2\pi}{N}k_0 \cos(\frac{2\pi}{N}k_0 x + \frac{2\pi}{T}\omega t) \\
\frac{\partial I(x(t),t)}{\partial y} &= 0 \\
\frac{\partial I(x(t),t)}{\partial t} &= \frac{2\pi}{T}\omega \cos(\frac{2\pi}{N}k_0 x + \frac{2\pi}{T}\omega t)
\end{aligned}$$

Substituting into the motion constraint equation gives

$$(\frac{2\pi}{N}k_0 v_x + \frac{2\pi}{T}\omega)\cos(\frac{2\pi}{N}k_0 x + \frac{2\pi}{T}\omega t) = 0$$

which implies the term on the left which is independent of x and t is always equal to 0. Isolating v_x gives the answer.

$$v_x = -\frac{\omega_0}{k_0}\,\frac{N}{T}.$$

(b) ω_0 is cycles per T frames and so ω_0/T is cycles per frame. k_0 is cycles per N pixels, so k_0/N is cycles per pixel. So, $v_x = \frac{\omega_0}{k_0}\,\frac{N}{T}$ is in units of pixels (of motion i.e. displacement) per frame (of video).

(c) The x and t derivatives of $I(x, y, t)$ are the same as in the previous question, but now the y derivative is

$$\frac{\partial I(x(t), t)}{\partial y} = \frac{2\pi}{N}k_1\cos(\frac{2\pi}{N}(k_0 x + k_1 y) + \frac{2\pi}{T}\omega t).$$

Substituting into the motion constraint equation gives

$$\frac{2\pi}{N}(k_0 v_x + k_1 v_y) + \frac{2\pi}{T}\omega = 0.$$

This is similar to the motion constraint equation, but note that the coefficients of this equation are proportional but *not* identical to the partial derivatives of the image intensity, since the latter include a cosine factor.

(d) To get the normal velocity vector, substitute as in Q1 into

$$\vec{v}_n \;\equiv\; \frac{-\frac{\partial I}{\partial t}}{(\frac{\partial I}{\partial x})^2 + (\frac{\partial I}{\partial y})^2}\;(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y})\;.$$

The cosine terms will all cancel, and we are left with:

$$v_n = -\frac{N}{T}\,\frac{\omega_0}{k_0^2 + k_1^2}\,(k_0, k_1).$$

which is a more general version of the one in the previous question.

5. The velocity is (v_x, v_y). There is nothing to compute here.

6. The normal velocity v_n is the true velocity (v_x, v_y) minus the component of the true velocity that is perpendicular to the spatial gradient $(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y})$. In this question, the true velocity is $(1, 0)$.

   For $\theta = 0$ and 180 deg, the spatial gradient is $(\pm 1, 0)$ which are both perpendicular to $(0, 1)$. The true velocity has no component in direction $(0, 1)$ so there is nothing to subtract, and so the normal velocity equals the true velocity.

For $\theta = 90$, the spatial gradient is in the y direction and hence perpendicular to the true velocity $(1, 0)$. So the normal velocity is the true velocity minuse the true velocity, so $v_n = (0, 0)$. Same for $\theta = -90$.

For $\theta = 45$, the spatial gradient is either $(1, 1)$ or $(-1, -1)$. Both are perpendicular to $(1, -1)$. We subtract from the true velocity the component in direction $(1, -1)$.

$$v_n = (1, 0) - \{(1, 0) \cdot (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = (\frac{1}{2}, \frac{1}{2}).$$

Note the normal velocity vector has length $\frac{1}{\sqrt{2}}$.

It is easy to make a calculation error for the last one, so let's think about it differently. The motion constraint line must contain the true velocity $(1, 0)$ and it also must have a slope in direction $(1, -1)$, namely perpendicular to the spatial gradient. Thus by inspection the motion constraint line must also pass through $(0, 1)$. The normal velocity is the closest point on this line to the origin which, by inspection, is $(\frac{1}{2}, \frac{1}{2})$.

Similar reasoning holds for the disk point at 135 deg. The motion constraint line again must pass through $(1, 0)$ but now it must have slope in direction $(1, 1)$. Thus the motion constraint line must pass through $(0, -1)$. The closest point on this line to the origin is $(\frac{1}{2}, -\frac{1}{2})$. This is the normal velocity.