Lecture 13

Rotations &

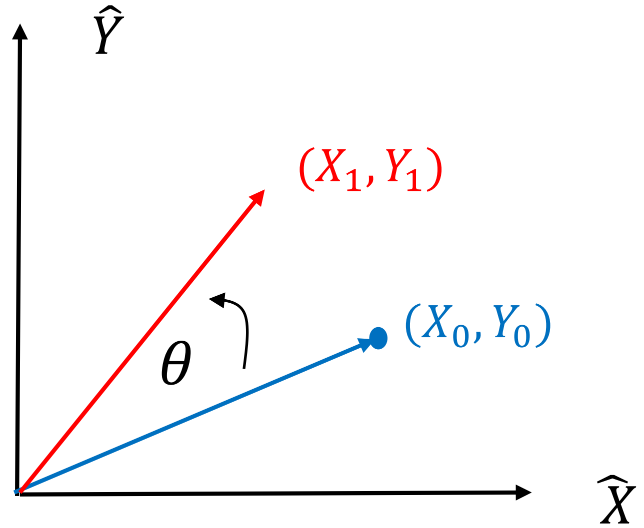
Homogeneous Coordinates

Wed. Oct. 21, 2020

Rotations

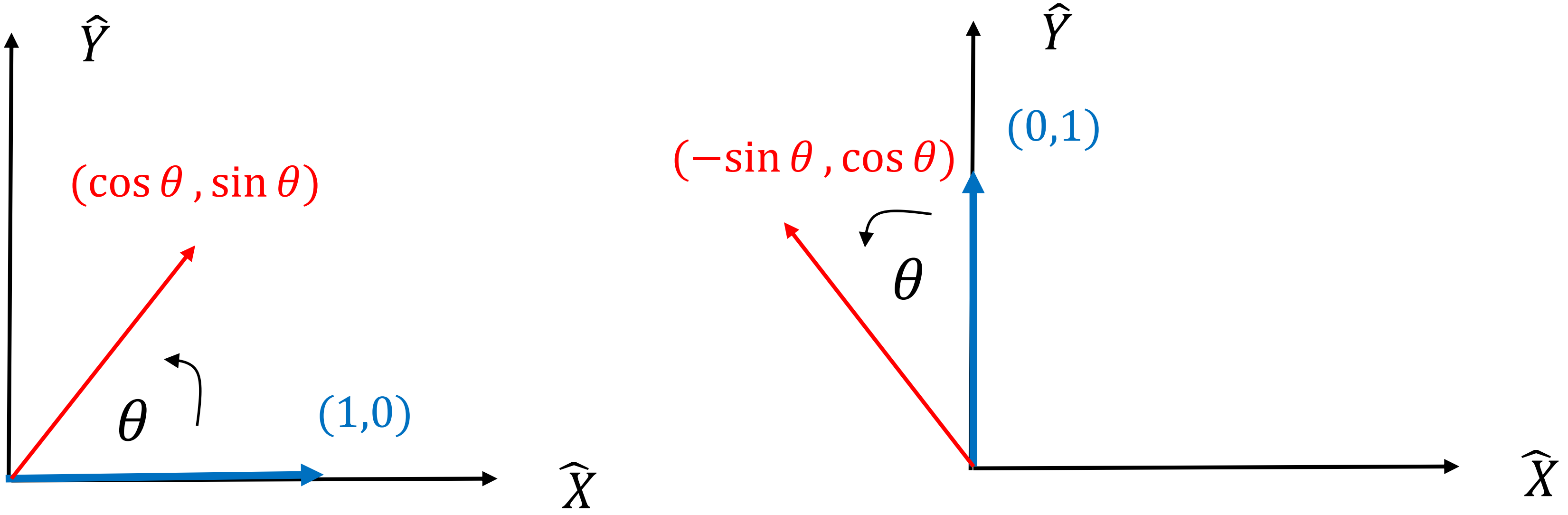
- 2D rotations
- 3D rotations + projection (continuous)
- 3D rotations (discrete)
- review of cross product (left vs. right hand coordinates)

2D Rotation (discrete)



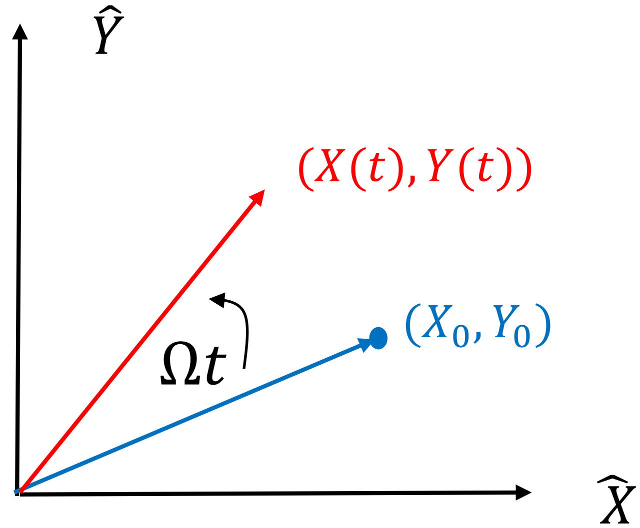
$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

2D Rotation (discrete)



$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

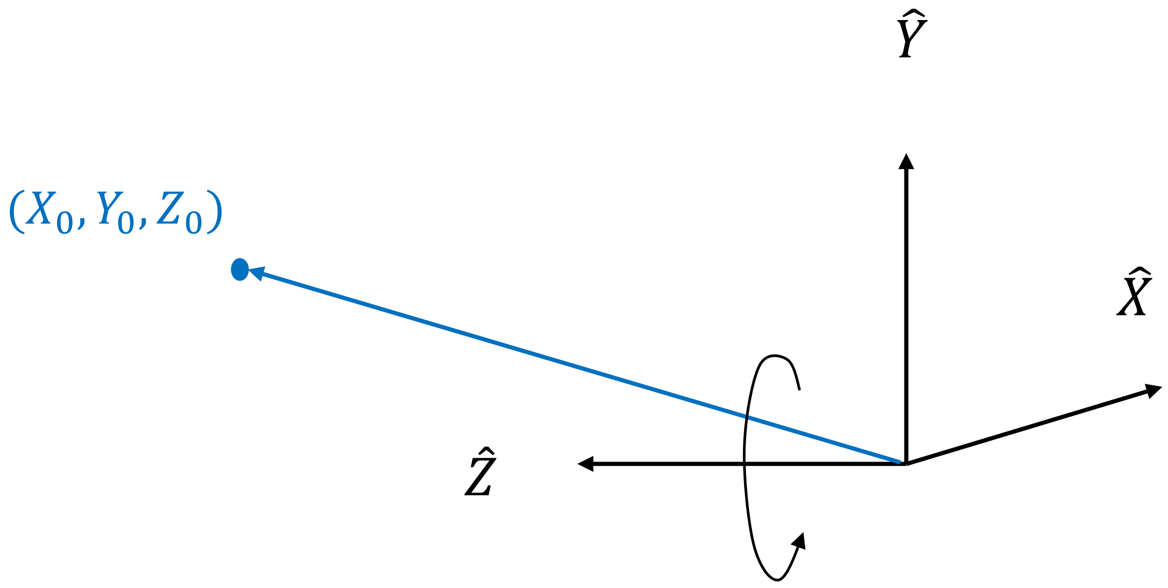
2D Rotation (continuous)



$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ \sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

where Ω is angular velocity (degrees or radians per unit time)

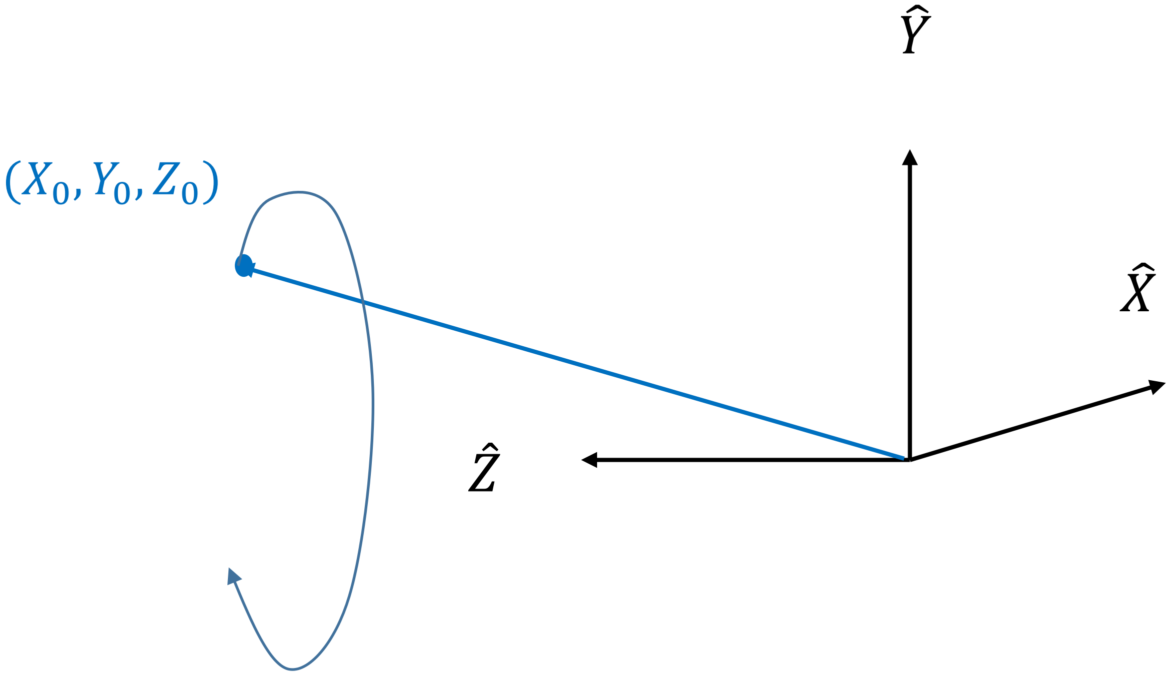
3D Camera Rotation (Z axis)



When the camera rotates about the Z axis, what motion does the camera see?

$$\begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} \cos(\Omega t) & -\sin(\Omega t) & 0 \\ \sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

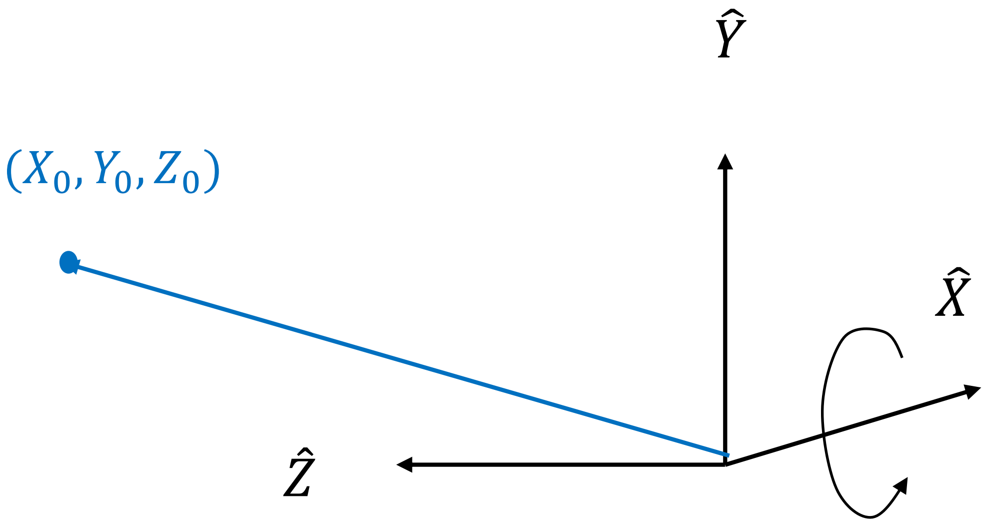
3D Camera Rotation (Z axis)



The camera sees the same motion as when it is static and the scene rotates about the Z axis with the opposite velocity.

$$\begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} \cos(\Omega t) & -\sin(\Omega t) & 0 \\ \sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

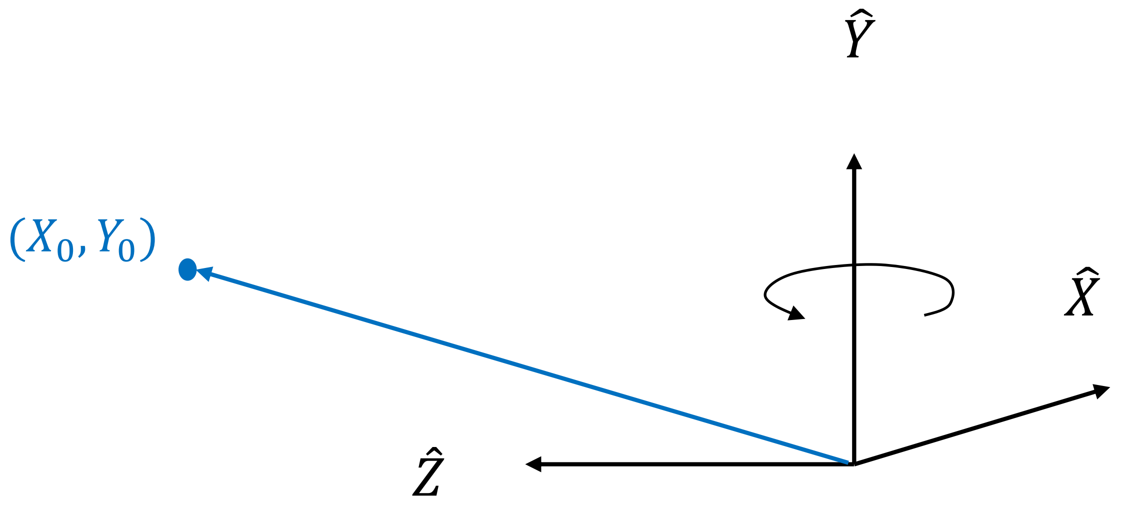
3D Camera Rotation (X axis)



When the camera rotates about the X axis (tilt), the motion observed is the same as when the scene rotates about the X axis with the opposite velocity.

$$\begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\Omega t) & \sin(\Omega t) \\ 0 & -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

3D Camera Rotation (Y axis)

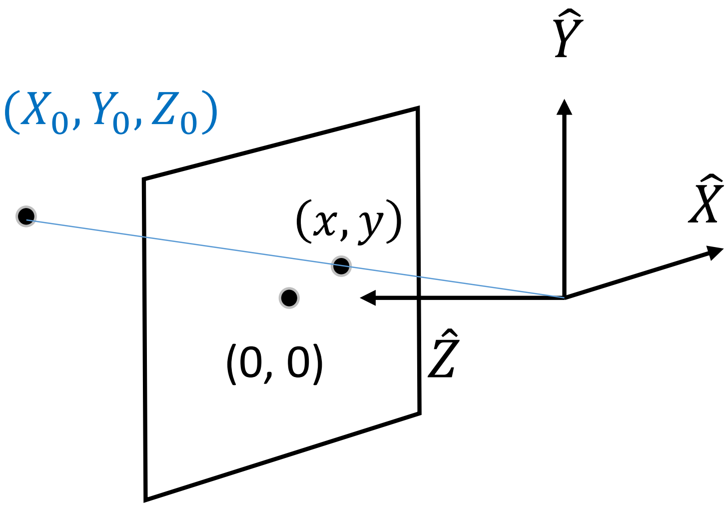


When the camera rotates about the X axis (tilt), the motion observed is the same as when the scene rotates about the X axis with the opposite velocity.

$$\begin{bmatrix} X(t) \\ Y(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} \cos(\Omega t) & 0 & \sin(\Omega t) \\ 0 & 1 & 0 \\ -\sin(\Omega t) & 0 & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

Image Projection (continued):

What is the image motion field seen by a rotating camera?

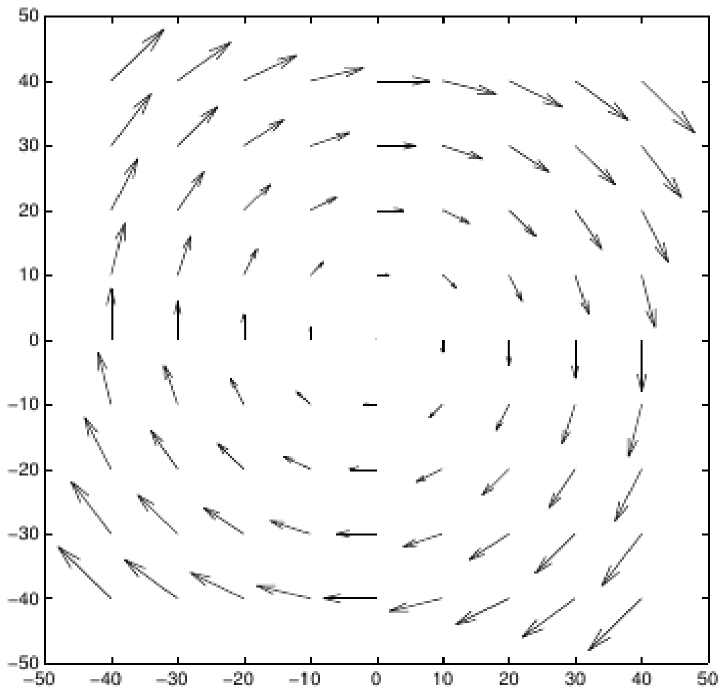


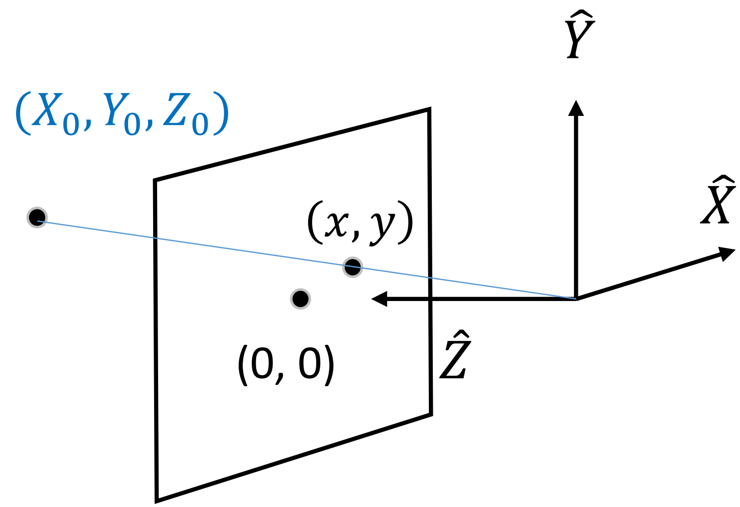
$$(x(t), y(t)) = (\frac{X(t)}{Z(t)}, \frac{Y(t)}{Z(t)})f$$

$$(v_x, v_y) = \frac{d}{dt}(x(t), y(t))|_{t=0}$$

[See Lecture Notes (Appendix TODO) for derivations.
On the following slides, I will give results only.]

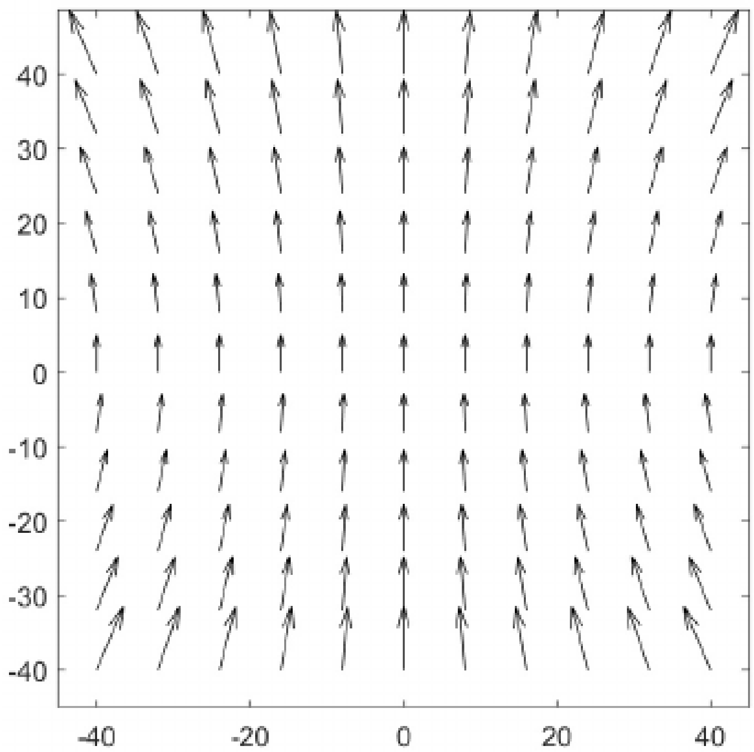
3D Camera Rotation about Z axis: "roll"

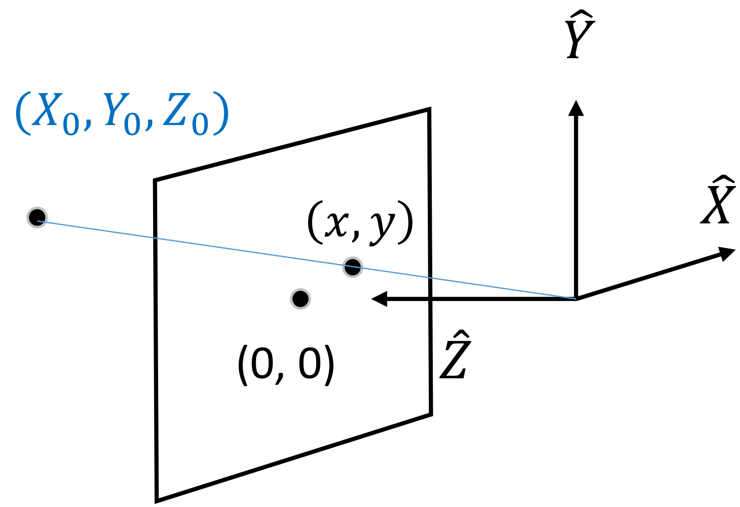




$$(v_x, v_y) = \Omega_Z(-y, x)$$ where Ω_Z is rotational velocity about Z axis

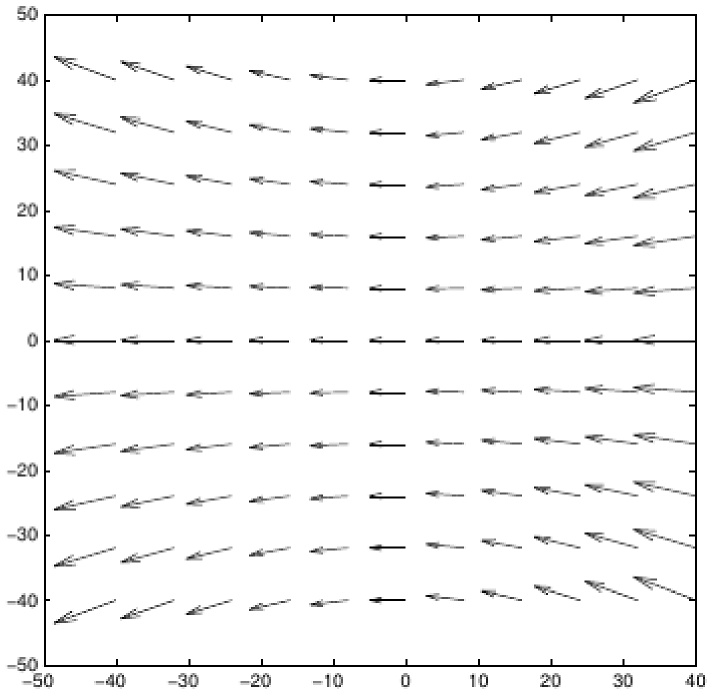
3D Camera Rotation about X axis: "pitch" or "tilt"

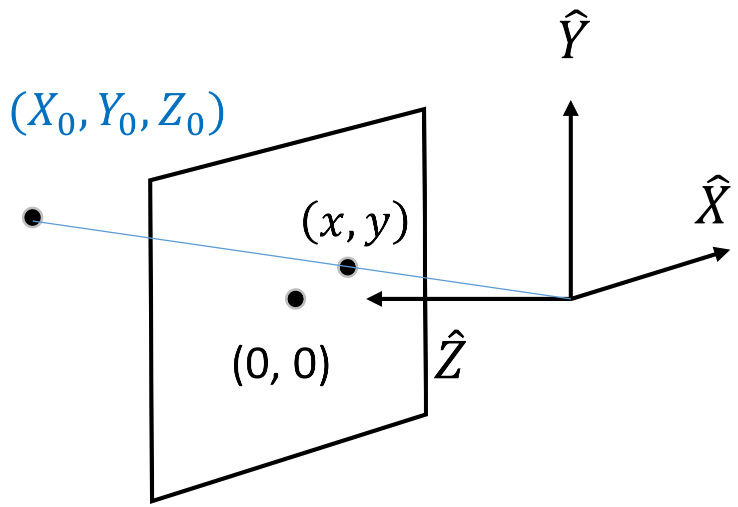




$$(v_x, v_y) = \Omega_X\left(\frac{xy}{f}, f\left(1 + \left(\frac{y}{f}\right)^2\right)\right)$$ where Ω_X is rotational velocity about X axis

3D Camera Rotation about Y axis: "pan"





$$(v_x, v_y) \;=\; \Omega_Y(f(1 + (\frac{x}{f})^2),\; \frac{xy}{f})$$ where Ω_Y is rotational velocity about Y axis

Note:

- One can define motion fields from rotation about arbitrary axis $(\Omega_X, \Omega_Y, \Omega_Z)$. Details omitted.

- The rotation field does *not* depend on depth.

  Recall the translation field does depend on depth as we saw last lecture.

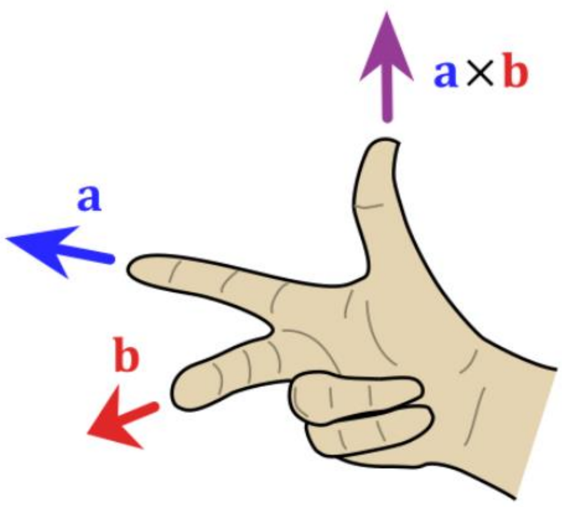
Classic computer vision problem:

- Given two image frames $I(x, y)$ and $J(x, y)$ taken by a two nearby cameras, estimate the "image motion" $\left(v_x, v_y\right)$ between frames.

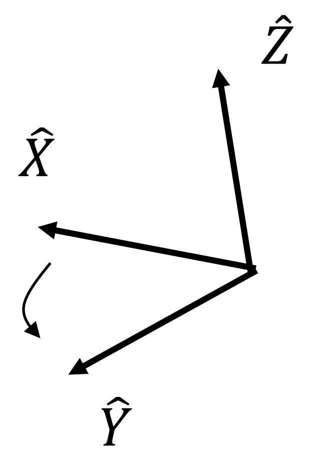
- Estimate the relative camera translation (T_X, T_Y, T_Z) and rotation $(\Omega_X, \Omega_Y, \Omega_Z)$ and the depth map $Z(x, y)$ that best explains the image motion.

We will cover fundamental elements of this problem in the coming weeks…

Cross Product



"Right hand"



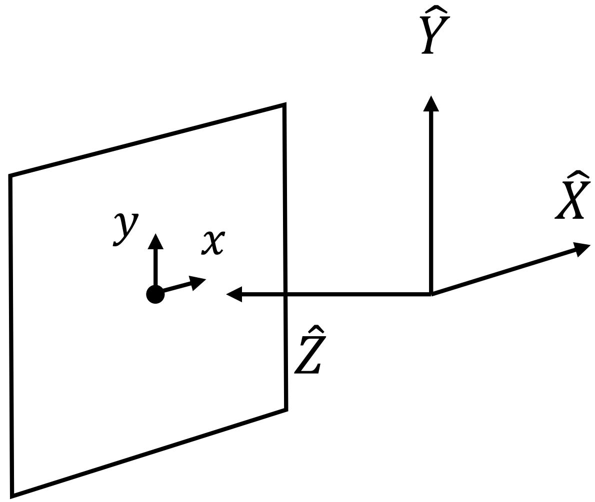
$$\hat{X} \times \hat{Y} = \hat{Z}$$

$$\hat{Y} \times \hat{Z} = \hat{X}$$

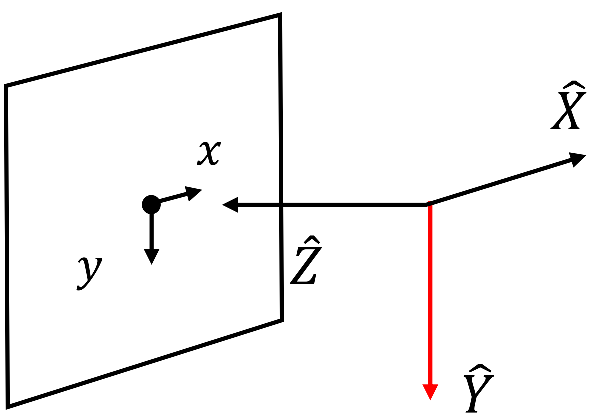
$$\hat{Z} \times \hat{X} = \hat{Y}$$

Left versus right hand coordinate systems

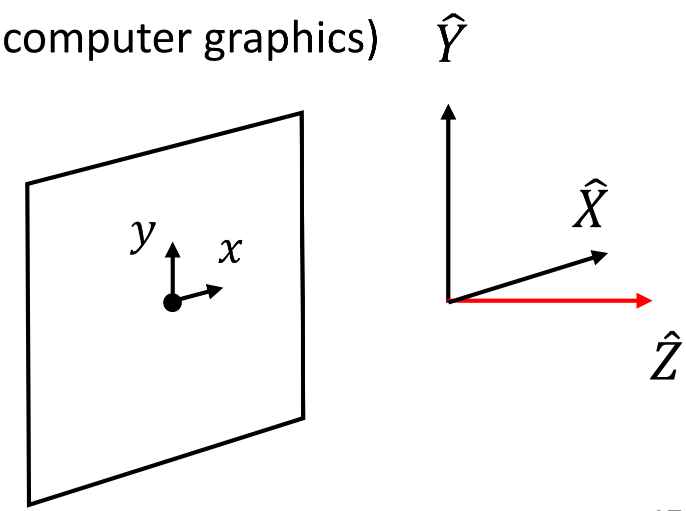
Left hand coordinates
(what I used last lecture)



Right hand coordinates
(what I will use from now on)



Right hand coordinates
(used in computer graphics)

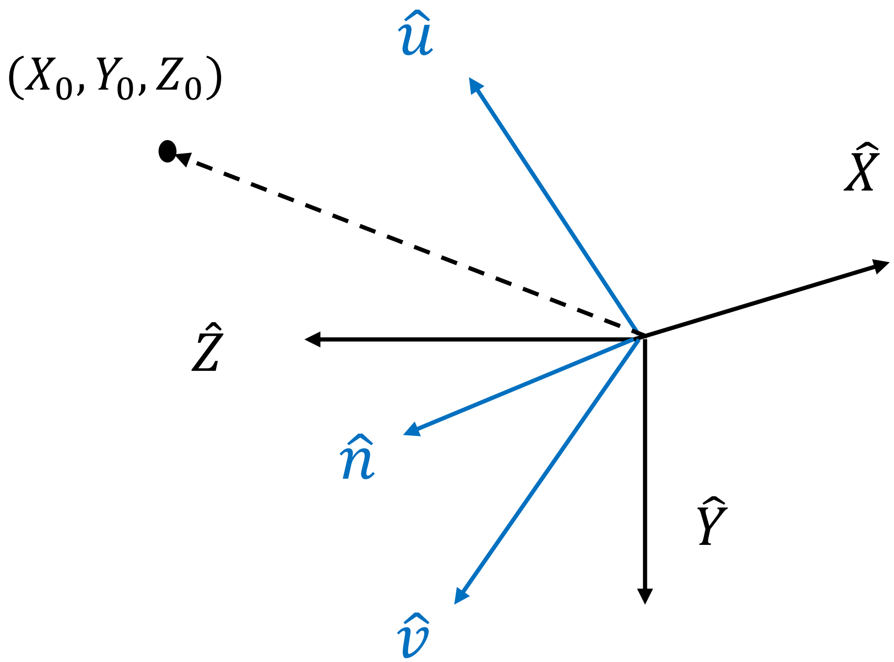


3D Camera Rotation (discrete)

A 3D rotation matrix is a 3x3 matrix that has orthonormal rows and columns and its determinant is 1.

$$\mathbf{R}^T = \mathbf{R}^{-1}$$

$$\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$$



The matrix $\boldsymbol{R}$ rotates a 3D point into different coordinate system, whose axes are the rows of $\boldsymbol{R}$.

The rotation takes the inner (dot) product with the rows of $\boldsymbol{R}$.

$$\mathbf{R} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{bmatrix}$$

3D Camera Rotation (discrete)

A 3D rotation matrix preserves the length of a vector.
It also preserves the angles between vectors.

Why?

$$(\mathbf{R}\mathbf{p}_1) \cdot (\mathbf{R}\mathbf{p}_2) = \mathbf{p}_1^T \mathbf{R}^T \mathbf{R} \mathbf{p}_2 = \mathbf{p}_1^T \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_2$$

3D Reflection

A 3D rotation matrix is a 3x3 matrix that has orthonormal rows and columns and its determinant is 1.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The above matrices are reflections.
Their determinant is -1.

Axis of Rotation

For any rotation matrix $\boldsymbol{R}$, one can show there is a unique vector $\boldsymbol{v}$ such that

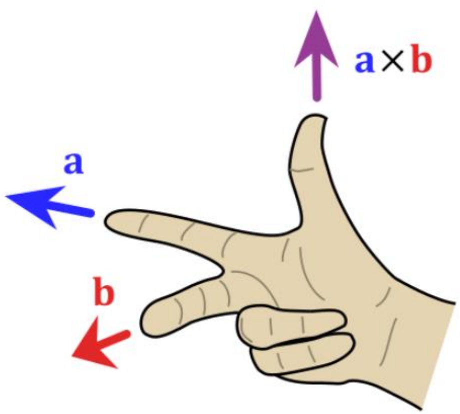
$$\boldsymbol{R}\boldsymbol{v} = \boldsymbol{v} \quad .$$

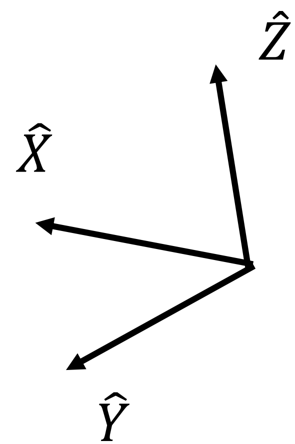
This (eigen)vector defines the *axis of rotation*.

$$\mathbf{R} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{bmatrix}$$

Cross Product

“Right hand”





$$\hat{X} \times \hat{Y} = \hat{Z}$$

$$\hat{Y} \times \hat{Z} = \hat{X}$$

$$\hat{Z} \times \hat{X} = \hat{Y}$$

$$(a_X \hat{X} + a_Y \hat{Y} + a_Z \hat{Z}) \times (b_X \hat{X} + b_Y \hat{Y} + b_Z \hat{Z}) = \ ?$$

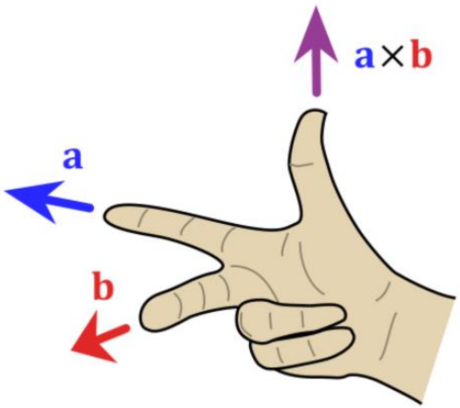
Cross Product

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} 0 & -a_Z & a_Y \\ a_Z & 0 & -a_X \\ -a_Y & a_X & 0 \end{bmatrix} \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix}$$

We will often write $\boldsymbol{a} \times \boldsymbol{b}$ as $[\boldsymbol{a}]_\times \boldsymbol{b}$. This treats this cross product as a linear transformation defined by $\boldsymbol{a}$ and applied to vector $\boldsymbol{b}$.

Cross Product

"Right hand"



Verify for yourself that

$\boldsymbol{a} \times \boldsymbol{a}$ is the 0 vector

$\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$

Lecture 13

Rotations &

Homogeneous Coordinates

Wed. Oct. 21, 2020

Homogenous Coordinates

To represent a 3D point, (X, Y, Z) we write the point in 4D as $(X, Y, Z, 1)$.

This allows us to represent various transformations in a similar way, namely using 4D matrix multiplication.

Translation

$$\begin{bmatrix} X + T_x \\ Y + T_y \\ Z + T_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} \boxed{} \\ 1 \end{bmatrix} = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

result of 3D rotation

3 x 3 rotation matrix

Scaling

$$\begin{bmatrix} \sigma_X X \\ \sigma_Y Y \\ \sigma_Z Z \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_X & 0 & 0 & 0 \\ 0 & \sigma_Y & 0 & 0 \\ 0 & 0 & \sigma_Z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}.$$

What if we have a value different than 1 in the 4^{th} coordinate?

$$\{\ (wX, wY, wZ, w)\ \ :\ \ w \neq 0\ \}$$

No problem. These 4D points all represent the same 3D point, namely (X, Y, Z).

Consider a 3D point (X, Y, Z) and scale the coordinates of this point by $s > 0$:

$$(sX, sY, sZ, 1) \equiv (X, Y, Z, \frac{1}{s})$$

For different values s, we get 3D points that all lie along a line from the origin through (X, Y, Z).

$$(sX, sY, sZ, 1) \equiv (X, Y, Z, \frac{1}{s})$$

As $s \rightarrow \infty$, we get a "point at infinity" in direction (X, Y, Z).

$$lim_{s \rightarrow \infty} (sX, sY, sZ, 1) = (X, Y, Z, 0)$$

What happens if we apply a rotation or translation or scaling transformation to a point at infinity?

Translating a point at infinity

$$? \quad = \begin{bmatrix} 1 & 0 & 0 & T_X \\ 0 & 1 & 0 & T_Y \\ 0 & 0 & 1 & T_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix}$$

Translating a point at infinity

$$\begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_X \\ 0 & 1 & 0 & T_Y \\ 0 & 0 & 1 & T_Z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix}$$

Rotating a point at infinity

$$\begin{bmatrix} \boxed{} \\ 0 \end{bmatrix} = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix}$$

result of 3D rotation

3 x 3 rotation matrix

So, it behaves similarly to the rotation of a finite point.

Scaling

$$\begin{bmatrix} \sigma_X X \\ \sigma_Y Y \\ \sigma_Z Z \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_X & 0 & 0 & 0 \\ 0 & \sigma_Y & 0 & 0 \\ 0 & 0 & \sigma_Z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix}.$$

Note the direction of the point at infinity will be changed when axes are scaled by different amounts.

Exercise:

How are (3D) points at infinity related to vanishing points?

Homogeneous Coordinates in 2D

Translation:

$$\begin{bmatrix} x + T_x \\ y + T_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation:

$$\begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates in 2D

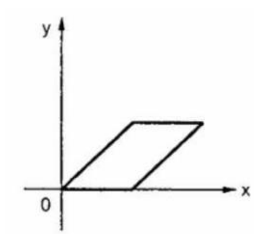
scaling

$$\begin{bmatrix} \sigma_x x \\ \sigma_y y \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_X & 0 & 0 \\ 0 & \sigma_Y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

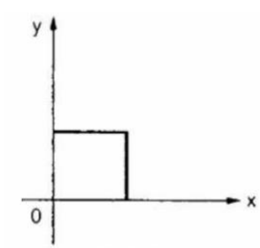
shear

$$\begin{bmatrix} x + sy \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Recall motion field of ground plane from last lecture



(b) Object after x shear



(a) Original object

Points at infinity in 2D homogenous coordinates

$$lim_{s\to\infty}(sx, sy, 1) = lim_{s\to\infty}\ (x, y, \frac{1}{s}) = (x, y, 0)$$

You can think of this as a *direction vector*.
(Its magnitude is undefined.)

We will use 2D points at infinity in coming weeks.