Questions

1. A vanishing point in an image defines a pattern of line and edge fragments that are all oriented along radial lines through that point. This radial pattern reminds us of the motion field generated by a translating observer, which consists of vectors (2D velocities) pointing radially away from the direction of heading. What is the relationship, if any, between these radial patterns? Do they have a common cause?

2. Recall the case of a ground plane (lecture 1) $Y = h < 0$:
$$\frac{y}{f} = \frac{h}{Z}.$$
Suppose the ground plane has a checkerboard pattern on it, which is aligned with the XZ axes. Assume each square has size 1×1 unit.

   (a) Show (mathematically) that for squares that are further in the distance, their image width varies with $\frac{1}{Z}$.
   Hint: Consider taking a step $\Delta X = 1$ on the 3D surface. What is the image step that results?

   (b) A more subtle effect is *foreshortening*. This is a change in the ratio of height to width of each square as it appears in the image. (sometimes called the *aspect ratio*).
   How does the foreshortening of squares vary with Z ?
   Hint: Consider taking a step $\Delta Z = 1$. What is the image step that results?

3. (a) Suppose the ground plane surface is slanted at some slope $m > 0$ relative to the observer's Z axis, so
$$Z = Z_0 + mY$$
   where $m = \tan\sigma$ and σ is called the *slant.*
   Suppose you would like to rescale the scene by changing the units. For example, suppose you would like to change from meters to millimeters (shrink) or to kilometers (expand). You would like to do so while *not changing the resulting image.* Could you use the same equation but just have different units? Or would you have to change the slope m to account for the unit change?

   (b) Give an expression for the Δx and Δy that results from steps on the surface that correspond to the two edges of a square on the checkerboard, namely $(\Delta X, \Delta Y, \Delta Z) = (1, 0, 0)$ and $(0, \cos\sigma, \sin\sigma)$.

   (c) Write the ratio $\frac{\Delta y}{\Delta x}$ in terms of slant σ and any other quantities that it depends on.

   (d) In Question 2, the user's Z axis was parallel to the ground, and the horizon was at $y = 0$. Where is the horizon for the plane in this question?
   Hint: It depends on m. Test your intuition before working out the math. What if m is 0? What if m is very large?

4. Give an example of a object whose surface has convex regions and saddle regions, but no concave regions.

Solutions

1. The patterns arise for the same reason: parallel lines in space project lines in an image that intersect at a common point. For an observer translating with 3D velocity (T_X, T_Y, T_Z), points move in 3D along lines parallel to that velocity vector.

2. (a) The step $\Delta X = 1$ does not change the Z value, so

$$\frac{\Delta x}{f} = \frac{\Delta X}{Z} = \frac{1}{Z}$$

Thus, the angular width $\frac{\Delta x}{f}$ of each square decreases as Z increases.

(b) What happens when we take a small step in depth on the surface, $\Delta Z = 1$? This changes the y value in the image.

$$\frac{\Delta y}{f} = \Delta(\frac{h}{Z})$$

Approximating the derivative of $\frac{d}{dZ}(\frac{1}{Z}) \approx -\frac{\Delta Z}{Z^2}$ gives:

$$\frac{\Delta y}{f} \approx -(\frac{h \Delta Z}{Z^2})$$

so when $\Delta Z = 1$ we have

$$\frac{\Delta y}{f} \approx -\frac{h}{Z^2}.$$

Note that a positive step in Z corresponds to a positive step in y since $h < 0$ for ground plane.

Taking the ratio of x and y steps gives:

$$\frac{\Delta y}{\Delta x} = \frac{h}{Z}$$

So as depth Z increases, each projected square *in the image* becomes smaller, and it also become more compressed in the y direction. However, as $Z \to 0$, the opposite occurs and the image of the squares become relatively elongated in the y direction. This is somewhat counterintuitive; the challenge here is to visualize what happens when you project a square on a ground plane in front of your toes onto a perpendicular (frontoparalle) plane $Z = f$ that passes through the tip of your nose.

3. (a) You can use the same units. The slope m is $\frac{\Delta Y}{\Delta Z}$ and so it has no units.

(b) Dividing $Z = Z_0 + mY$ by Z gives

$$1 = m\frac{Y}{Z} + \frac{Z_0}{Z}.$$

Then, since $\frac{y}{f} = \frac{Y}{Z}$, we have

$$1 = m\frac{y}{f} + \frac{Z_0}{Z}.$$

Taking a small step size Δy in the image and approximating $\frac{dy}{dZ}$ as in Question 2 gives

$$m\frac{\Delta y}{f} \approx Z_0 \frac{\Delta Z}{Z^2}$$

For a step size 1 in the direction of increasing Z *on the surface*, we move $\Delta Z = \sin\sigma$. Since $m = \tan\sigma$, we get

$$\frac{\Delta y}{f} \approx \frac{Z_0 \sin\sigma}{\tan\sigma \; Z^2} = \frac{Z_0 \cos\sigma}{Z^2}$$

What about steps in the x direction? A step $\Delta X = 1$ along the surface does not lead to a change in Z for this surface. So, since $\frac{x}{f} = \frac{X}{Z}$, we have $\frac{\Delta x}{f} = \frac{1}{Z}$. This is the same is for the ground plane discussed in class.

(c) The compression ratio for the obliquely slanted plane is thus:

$$\frac{\Delta y}{\Delta x} = \frac{Z_0}{Z}\cos\sigma.$$

Notice that the compression ratio has no units. This means that if we rescale the scene by expressing X, Y, Z in mm or km has no effect on the compression ratio. *This means that we cannot use this cue to judge the scale of the scene.*

Also note that as $Z \to \infty$, the compression ratio is small, so points near the horizon are highly compressed. This is similar to Question 2. Also, as $Z \to 0$, the compression ratio becomes very large, again similar to Question 2.

(d) From

$$1 = m\frac{y}{f} + \frac{Z_0}{Z}$$

we let $Z \to \infty$. This gives $y = \frac{1}{m}$. So as the slope decreases toward frontoparallel (wall), the horizon height in the image increases.

4. The points in the "neck" region in the middle are saddle points since they are concave in a constant Z plane of the image and they are convex in a constant X plane, i.e. a YZ slice through the object.