

The issue of scale has come up in several lectures. For example, in the Canny edge analysis, we considered what happens when we stretch the size of a filter. Then, we discussed image registration and Harris corners, we considered distances (length scales) over which we can or cannot assume a first order model of image intensity.

## Gaussian scale space

We begin with a 1D image  $I(x)$ . Define family of images

$$I(x, \sigma) = I(x) * G(x, \sigma)$$

which is called a *Gaussian scale space*. This is a sequence of blurred 1D images, indexed by the amount of blur  $\sigma$  which is the standard deviation of the Gaussian.

A related scale space uses the function  $g(x) = e^{-\frac{x^2}{2}}$  and its scaled version  $g_\sigma(x) = e^{-\frac{x^2}{2\sigma^2}}$ . For the given image  $I(x)$ , we define the scale space:

$$I(x) * g_\sigma(x).$$

Notice that since  $G(x, \sigma) = \frac{1}{\sqrt{2\pi}\sigma}g_\sigma(x)$ , the above two scale spaces are related by

$$I(x) * G(x, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} I(x) * g_\sigma(x)$$

(See the slides, which show examples of these and other scale spaces.)

## Edge detection

Consider a noise-free image edge  $I(x) = au(x - x_0)$ . To detect edges previously, we convolved the image with a derivative of  $g_\sigma(x)$  and then looked for a peak in the response. Take the second scale space above,  $I(x) * \frac{dg_\sigma}{dx}(x)$ , and note

$$I(x) * \frac{dg_\sigma}{dx}(x) = ag_\sigma(x) * \delta(x - x_0) = ag_\sigma(x - x_0).$$

At the location of the edge  $x = x_0$ , we have

$$(I * \frac{dg_\sigma}{dx})(x_0) = ag_\sigma(0) = a$$

which depends only on the amplitude of the edge. The filter  $\frac{dg_\sigma}{dx}(x)$  is sometimes called the *normalized Gaussian derivative* and the resulting scale space as the *normalized Gaussian derivative scale space*.

If we filter a (noise-free) edge,  $I(x) = au(x - x_0)$ , with  $\frac{dg_\sigma(x)}{dx}$  then we get a peak response  $a$  at the location of the edge. It follows that if we were to filter with the second derivative  $\frac{d^2g_\sigma(x)}{dx^2}$ , we would get a value 0 at the location of the edge. More generally, at any point  $x$ , filtering with the

second derivative would yield

$$\begin{aligned}
 au(x - x_0) * \frac{d^2 g_\sigma(x)}{dx^2} &= a \frac{d}{dx} u(x - x_0) * \frac{dg_\sigma(x)}{dx} \\
 &= a \frac{dg_\sigma}{dx}(x - x_0) \\
 &= a \frac{-(x - x_0)}{\sigma^2} e^{-\frac{(x - x_0)^2}{2\sigma^2}}. \quad (*)
 \end{aligned}$$

Where do the peaks of  $au(x - x_0) * \frac{d^2 g_\sigma(x)}{dx^2}$  occur? To answer this question, we take the derivative again:

$$\frac{d}{dx} \left( au(x - x_0) * \frac{d^2 g_\sigma(x)}{dx^2} \right) = -\frac{1}{\sigma^2} \left( 1 - \frac{(x - x_0)^2}{\sigma^2} \right) e^{-\frac{(x - x_0)^2}{2\sigma^2}}$$

and set it to 0. The peak thus occurs when

$$1 - \frac{(x - x_0)^2}{\sigma^2} = 0$$

that is,  $x = x_0 \pm \sigma$ .

Substituting into (\*), we see that the value of the peak of the filtered edge  $au(x - x_0) * \frac{d^2 g_\sigma(x)}{dx^2}$  is  $-\frac{a}{\sigma} e^{-\frac{1}{2}}$ . If we want the value of the peak to be independent of  $\sigma$  then we need to use the filter  $\sigma \frac{d^2 g_\sigma(x)}{dx^2}$  instead. Notice that  $\sigma g_\sigma(x) = \sqrt{2\pi} \sigma^2 G(x, \sigma)$ .

## Box detection

Suppose we have a 1D intensity function that has value  $a$  for  $x \in [-\sigma_0, \sigma_0]$  and zero otherwise. We can define such an image as the difference of step edges, namely

$$I(x) = au(x + \sigma_0) - au(x - \sigma_0).$$

If we filter this  $I(x)$  with  $\frac{dg_\sigma^2(x)}{dx^2}$ , then the scale space has a very nice property. When  $\sigma = \sigma_0$ , there will be a positive peak in  $u(x - \sigma_0) * \sigma \frac{dg_\sigma^2(x)}{dx^2}$  at  $x = 0$  and there will be negative peak in  $u(x + \sigma_0) * \sigma \frac{dg_\sigma^2(x)}{dx^2}$  at  $x = 0$ . The two peaks coincide, leading to a minimum response at the center of the box ( $x = 0$ ).

What is the value of the peak?

$$I(x) * \frac{d^2 g_\sigma(x)}{dx^2} = a \frac{-(x + \sigma_0)}{\sigma^2} e^{-\frac{(x + \sigma_0)^2}{2\sigma^2}} - a \frac{-(x - \sigma_0)}{\sigma^2} e^{-\frac{(x - \sigma_0)^2}{2\sigma^2}}.$$

At  $x = 0$  and  $\sigma_0 = \sigma$ ,

$$\left( I * \frac{d^2 g_\sigma}{dx^2} \right)(0, \sigma_0) = -a \frac{a}{\sigma} e^{-\frac{1}{2}} - \frac{a}{\sigma} e^{-\frac{1}{2}} = -\frac{2a}{\sigma} e^{-\frac{1}{2}}$$

If we were to use the filter  $\sigma \frac{d^2 g_\sigma(x)}{dx^2}$ , *instead of* the filter  $\frac{d^2 g_\sigma(x)}{dx^2}$ , then the same argument about peak alignment would still hold, but now the peak response would be

$$\left( I * \sigma \frac{d^2 g_\sigma}{dx^2} \right)(0, \sigma_0) = 2ae^{-\frac{1}{2}}$$

which depends only on the amplitude  $a$  of the box but does not depend on  $\sigma_0$ .

One can now easily see that the minimum of  $I(x) * \sigma \frac{d^2 g_\sigma}{dx^2}(x, \sigma)$  at  $(x, \sigma) = (0, \sigma_0)$  is a global minimum. The two peaks discussed above can only align when  $\sigma = \sigma_0$  and  $x = 0$ . At any other  $\sigma$  or  $x$  value, we will be summing two values whose magnitudes are less than or equal to  $a e^{-\frac{1}{2}}$  and one of whose magnitudes is strictly less than  $a e^{-\frac{1}{2}}$ .

If we used  $\frac{dg_\sigma^2(x)}{dx^2}$  instead of  $\sigma \frac{dg_\sigma^2(x)}{dx^2}$ , we would not have this unique global minimum property.

Next class, we will look at some 2D scale spaces and discuss how to use them in problems we have seen, namely corner detection, 2D box detection, and registration (Lucas-Kanade).