Lecture 15

Least squares estimation

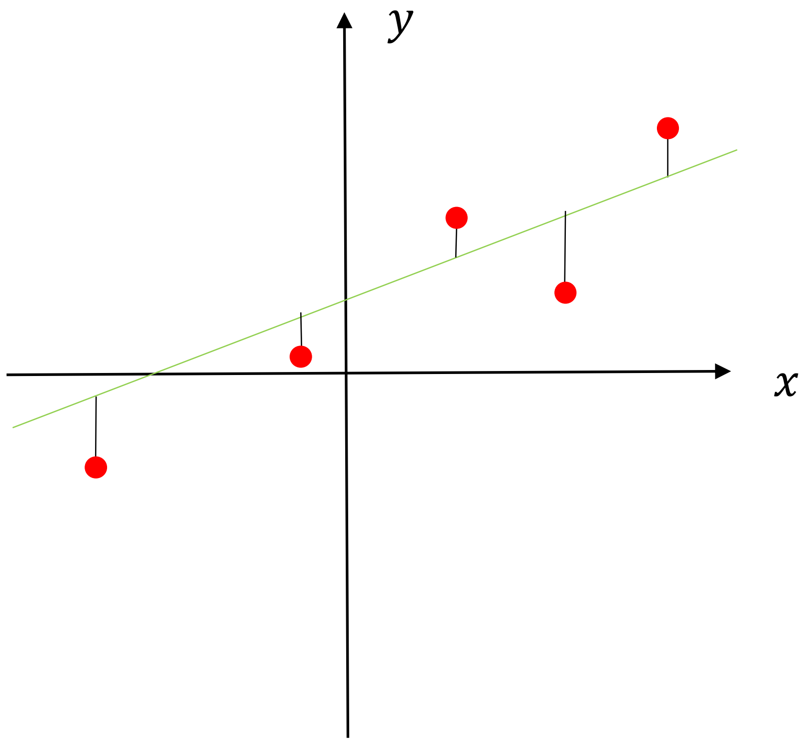
Singular value decomposition (SVD)

Wed. Oct. 28, 2020

Recall from lecture 5.

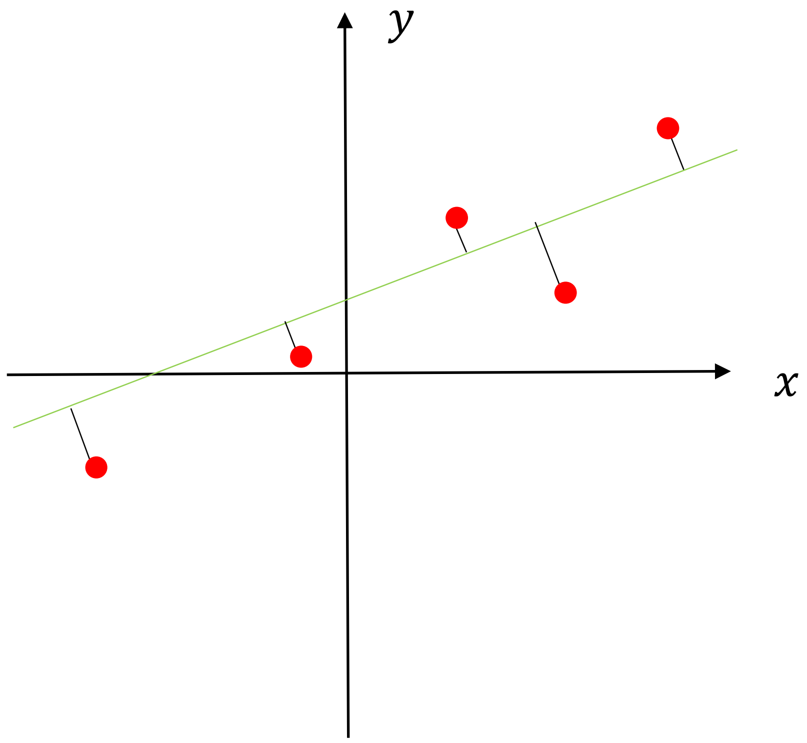
Version 1: linear regression

Error is distance to line in y direction only.



Version 2: "total least squares"

Error is distance perpendicular to line.



Least squares: version 2

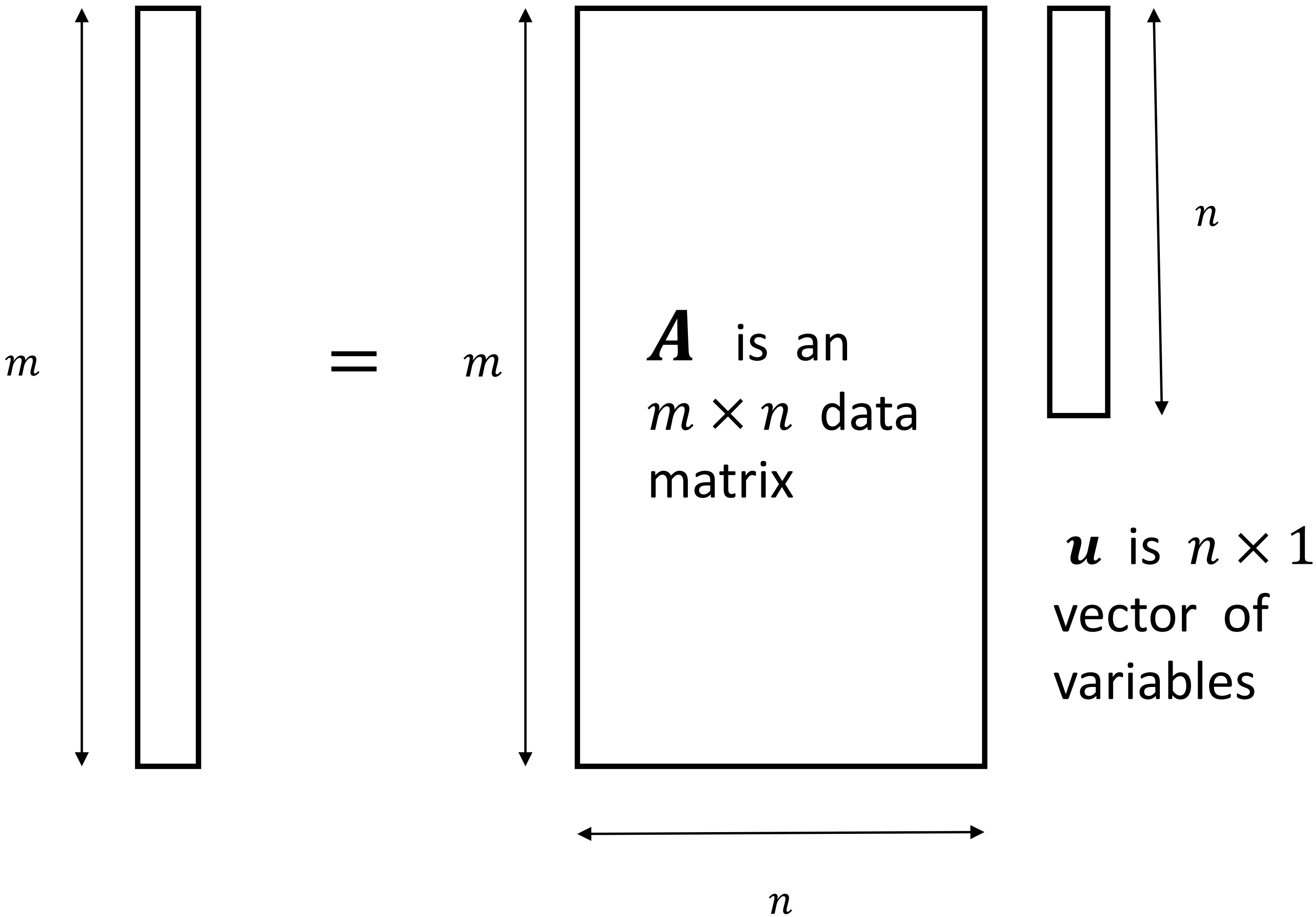
Find the $\boldsymbol{u}$ that minimizes L2 norm $\| \boldsymbol{A}\, \boldsymbol{u} \|^2$

subject to $\| \boldsymbol{u} \| = 1.$

where

- $\boldsymbol{A}$ is an $m \times n$ data matrix, $m \geq n$
- $\boldsymbol{u}$ is a $n \times 1$ vector of variables

Find unit vector $\boldsymbol{u}$ that minimizes the L2 norm of $\boldsymbol{A}\boldsymbol{u}$.



Examples of Version 2 Problems (next two weeks)

- Camera calibration
- Image Stitching for panoramas
- Binocular Stereo

Least squares: version 2

Find the $\boldsymbol{u}$ that minimizes L2 norm $\| \boldsymbol{A}\,\boldsymbol{u} \|^2$

subject to $\| \boldsymbol{u} \| = 1$, where

- $\boldsymbol{A}$ is an $m \times n$ data matrix, $m \geq n$
- $\boldsymbol{u}$ is a $n \times 1$ vector of variables

Solution (claimed back in lecture 5):

Compute the eigenvectors of $\boldsymbol{n} \times \boldsymbol{n}$ matrix $\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A}$.

Take the unit eigenvector that has the smallest eigenvalue.

Why does the eigenvector of $\boldsymbol{A}^T\boldsymbol{A}$ with the minimum eigenvalue solve this problem ?

Main idea: first suppose $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}^T\boldsymbol{A}$:

$$\| \boldsymbol{A}\,\boldsymbol{u} \|^2 = \boldsymbol{u}^T\boldsymbol{A}^T\boldsymbol{A}\,\boldsymbol{u}$$

$$= \lambda\, \boldsymbol{u}^T\boldsymbol{u}$$

if $\boldsymbol{u}$ is an eigenvector (and λ is its eigenvalue)

$$= \lambda$$

when $\boldsymbol{u}$ has unit length

$$\geq 0$$

because L2 norm is non-negative.

So we want the eigenvector with smallest eigenvalue.

Why does the eigenvector of $A^T A$ with the minimum eigenvalue solve the problem ?

Linear algebra tells us that the $n \times n$ matrix $\boldsymbol{A}^T\boldsymbol{A}$ has

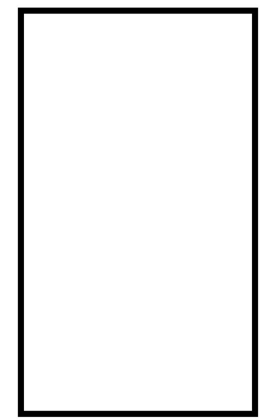
- n orthogonal eigenvectors
- non-negative eigenvalues

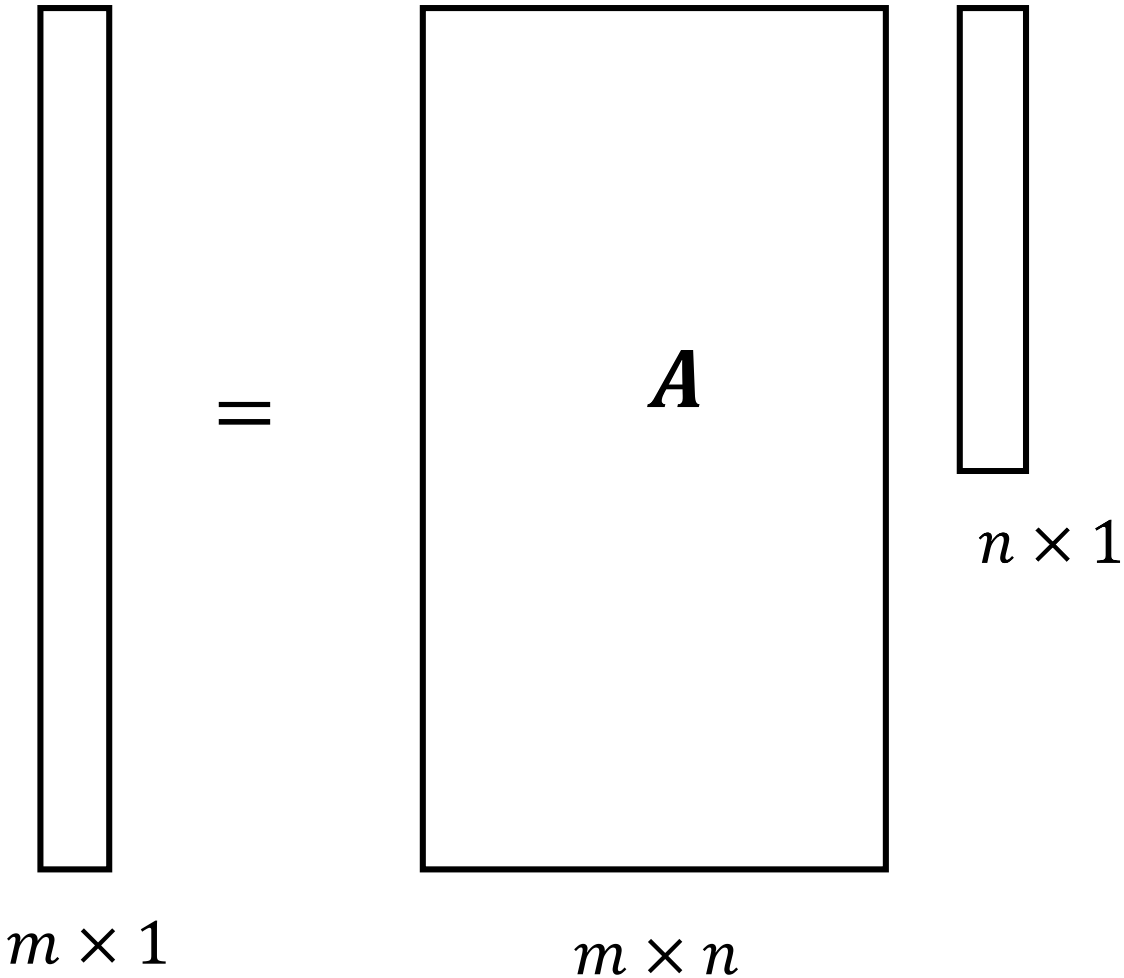
Therefore we can write any vector $\boldsymbol{u}$ as a sum of these eigenvectors. By inspection, $\boldsymbol{u}^T\boldsymbol{A}^T\boldsymbol{A}\,\boldsymbol{u}$ will be smallest when $\boldsymbol{u}$ is the eigenvector with smallest eigenvalue.

Singular Value Decomposition (SVD)

Let $\boldsymbol{A}$ be any $m \times n$ real data matrix.

In our examples, $m \geq n$.





$=$
$\boldsymbol{A}$
$n \times 1$
$m \times 1$
$m \times n$

Since the eigenvectors of $\boldsymbol{A^T A}$ are orthogonal and the eigenvalues are non-negative, we can write:

$$\boldsymbol{A^T A \; V \;=\; V \; D}$$

where

- the columns of $\boldsymbol{V}$ are ortho*normal* eigenvectors of $\boldsymbol{A^T A}$
- $\boldsymbol{D}$ is a diagonal matrix of (non-negative) eigenvalues

Since the eigenvectors of $A^T A$ are orthogonal and the eigenvalues are non-negative, we can write:

$$A^T A \; V \;=\; V \; \boldsymbol{\Sigma}^2$$

where

- the columns of V are ortho*normal* eigenvectors of $A^T A$

- $\boldsymbol{\Sigma}$ is a diagonal matrix, whose elements are called the *singular* values of A

$$\boldsymbol{A^T A \; V = V \, \Sigma^2}$$

Multiplying on the left by $\boldsymbol{V^T}$ gives:

$$\boldsymbol{V^T A^T A \; V = \Sigma^2}$$

$$A^T A \; V \; = \; V \, \Sigma^2$$

Multiplying on the left by V^T gives:

$$V^T A^T A \; V \; = \Sigma^2$$

By inspection, the columns of $A \; V$ are orthogonal.
Therefore we can uniquely define a matrix U such that:

$$A \; V \; = U \; \Sigma$$

where the columns of U are parallel to columns of $A \, V$ (and orthonormal).

Q: What are the magnitudes of the columns of $A \, V$?

Singular Value Decomposition (SVD)

From the previous slide:

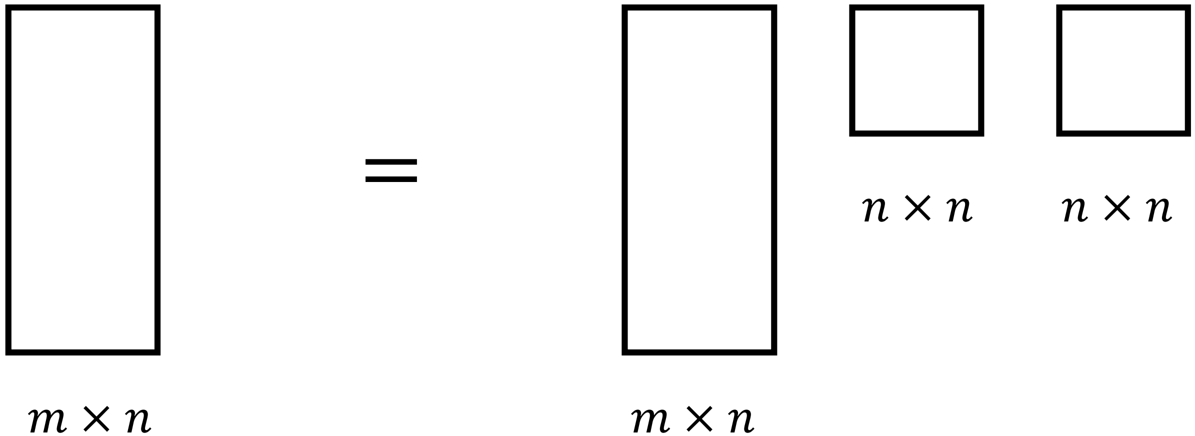
$$\boldsymbol{A}\ \boldsymbol{V} = \boldsymbol{U}\ \boldsymbol{\Sigma}$$

Since the columns of $\boldsymbol{V}$ are orthonormal, right multiplying by $\boldsymbol{V}^{\boldsymbol{T}}$ gives:

$$\boxed{\boldsymbol{A} = \boldsymbol{U}\ \boldsymbol{\Sigma}\ \boldsymbol{V}^{\boldsymbol{T}}}$$

This is known as the Singular Value Decomposition (SVD) of $\boldsymbol{A}$.

What does the matrix $\boldsymbol{A}$ do when you multiply it by an n vector: $\boldsymbol{Ax}$?



$$\boldsymbol{A} = \boldsymbol{U} \; {\color{red}\boldsymbol{\Sigma}} \; \boldsymbol{V^T}$$

Use the coefficients as weights on $\boldsymbol{n}$ columns of $\boldsymbol{U}$ in $\boldsymbol{R^m}$ and sum up.

Scale coefficients by multiplying by singular values.

Rotate to coordinates defined by orthonormal columns of $\boldsymbol{V}$ in $\boldsymbol{R^n}$ (orthonormal rows of $\boldsymbol{V^T}$)

Matlab

```
[U, S, V]  =  svd(A)
```

n singular values
returned as a vector
in decreasing order

Examples of Version 2 Problems (coming soon)

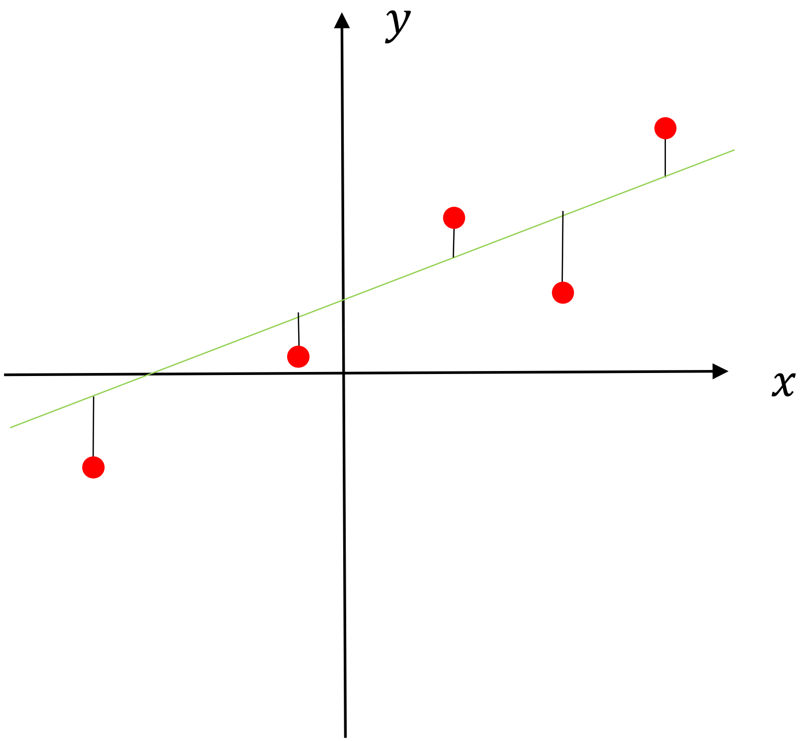
- Camera calibration
- Image Stitching for panoramas
- Binocular Stereo

For each of these problems, we will set up a data matrix `A` and solve the problem by taking the SVD. The solution will be the column of matrix `V` that corresponds to the smallest singular value.

Recall from lecture 5.

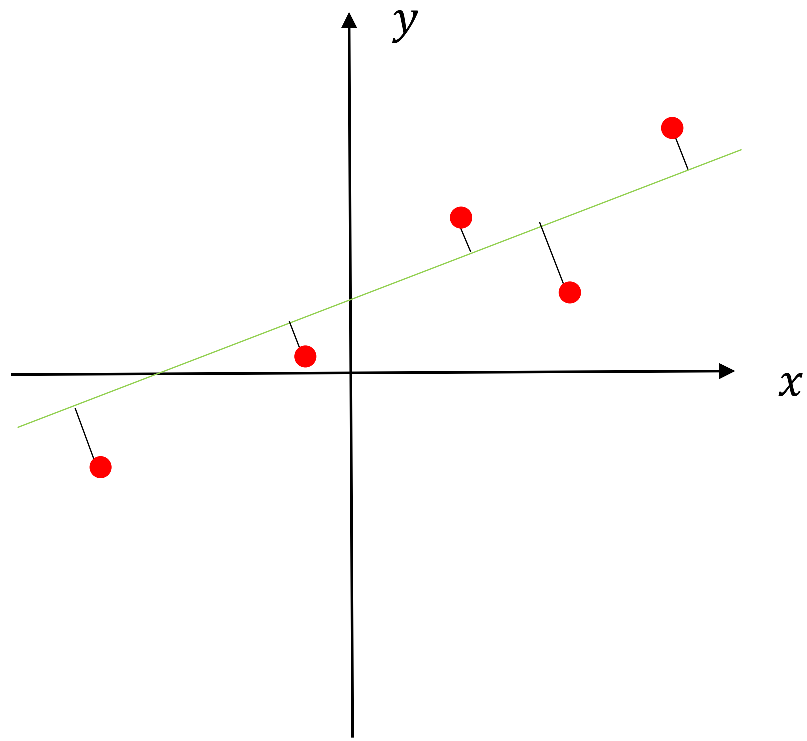
Version 1: linear regression

Error is distance to line in y direction only.



Version 2: "total least squares"

Error is distance perpendicular to line.



Given $\mathbf{A}$ and $\mathbf{b}$ defined below, find the $\mathbf{u}$ that minimizes:

$$\| \mathbf{A}\, \mathbf{u} - \mathbf{b} \|^2$$

where

$\mathbf{A}$ is an $m \times n$ matrix, where $m \geq n$ $\boxed{\boldsymbol{A}}$

$\mathbf{u}$ is a $n \times 1$ vector of variables

$\mathbf{b}$ is a $m \times 1$ data vector (lecture 5) To find the $\mathbf{u}$ that minimizes:

$$\| \mathbf{A}\,\mathbf{u} - \mathbf{b} \|^2$$

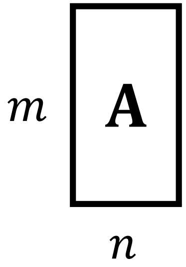
we solve for:

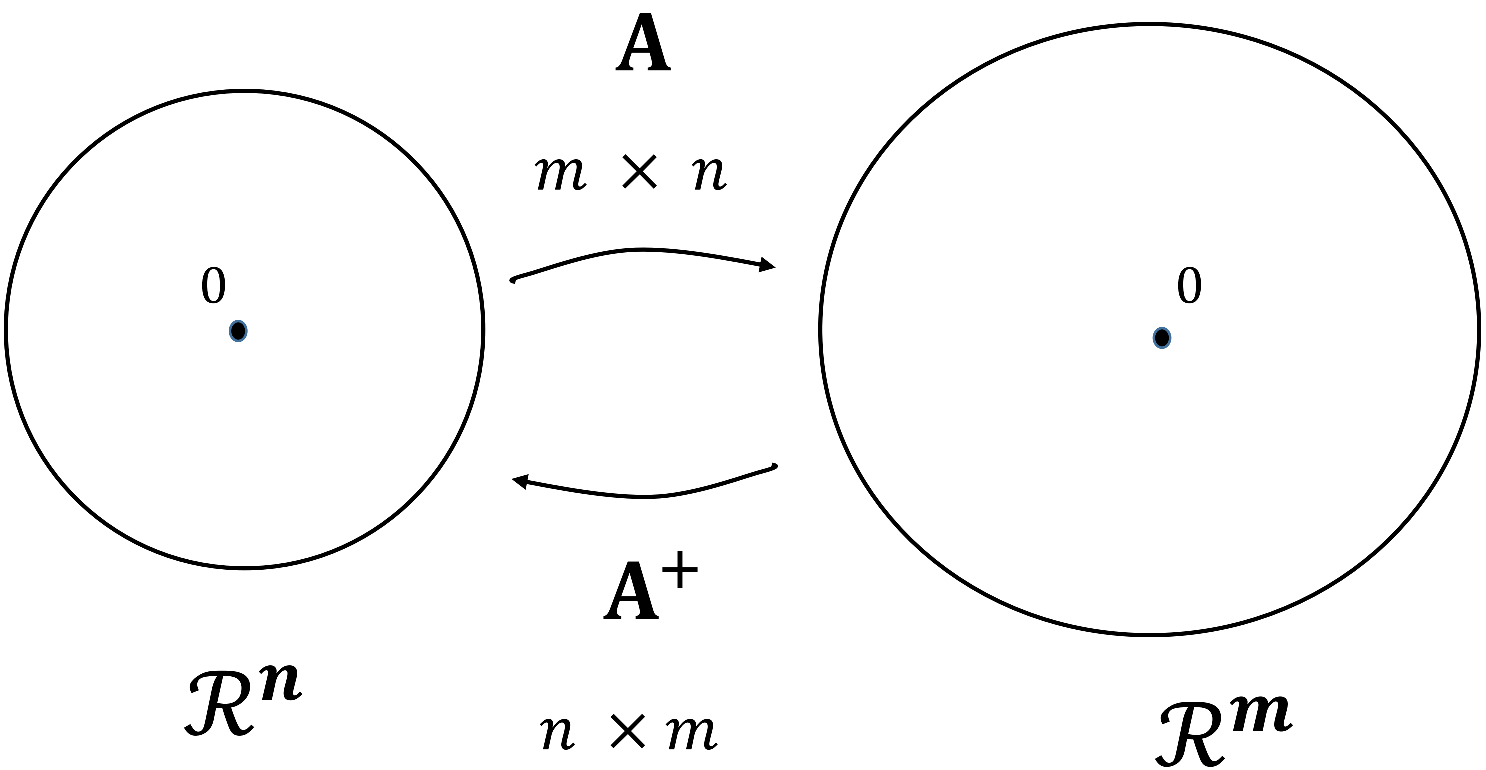
$$\mathbf{u} = \underbrace{(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T}\mathbf{b}$$

This matrix is called the *pseudoinverse* of **A**.
It is typically written $\boldsymbol{A}^+$.

Let's give a geometric interpretation of this matrix and then relate it to the SVD.

pseudoinverse $\mathbf{A}^{+} \equiv \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}$



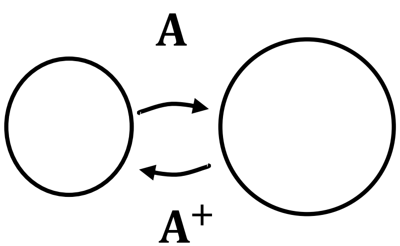


What does the pseudoinverse do?

$$\mathbf{A}^{+} \equiv \left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-\mathbf{1}}\mathbf{A}^{\mathbf{T}}$$

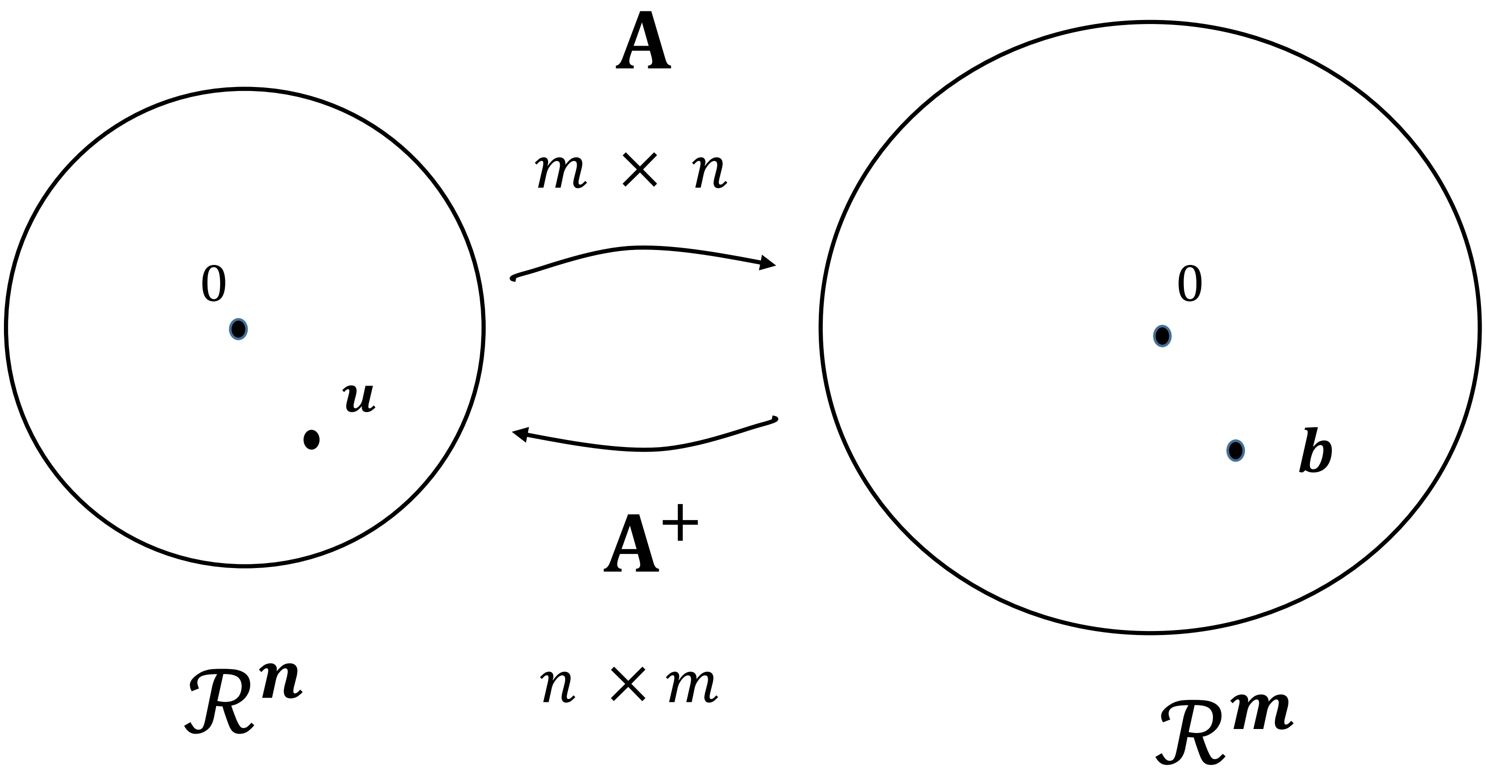
Thus,

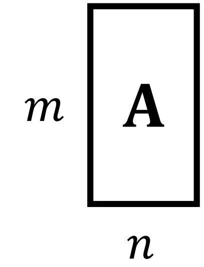
$$\mathbf{A}^{+}\mathbf{A} = \left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-\mathbf{1}}\mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{I}$$



$$\mathbf{A}\,\mathbf{A}^{+} = \mathbf{A}\left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-\mathbf{1}}\mathbf{A}^{\mathbf{T}} \neq \mathbf{I}$$

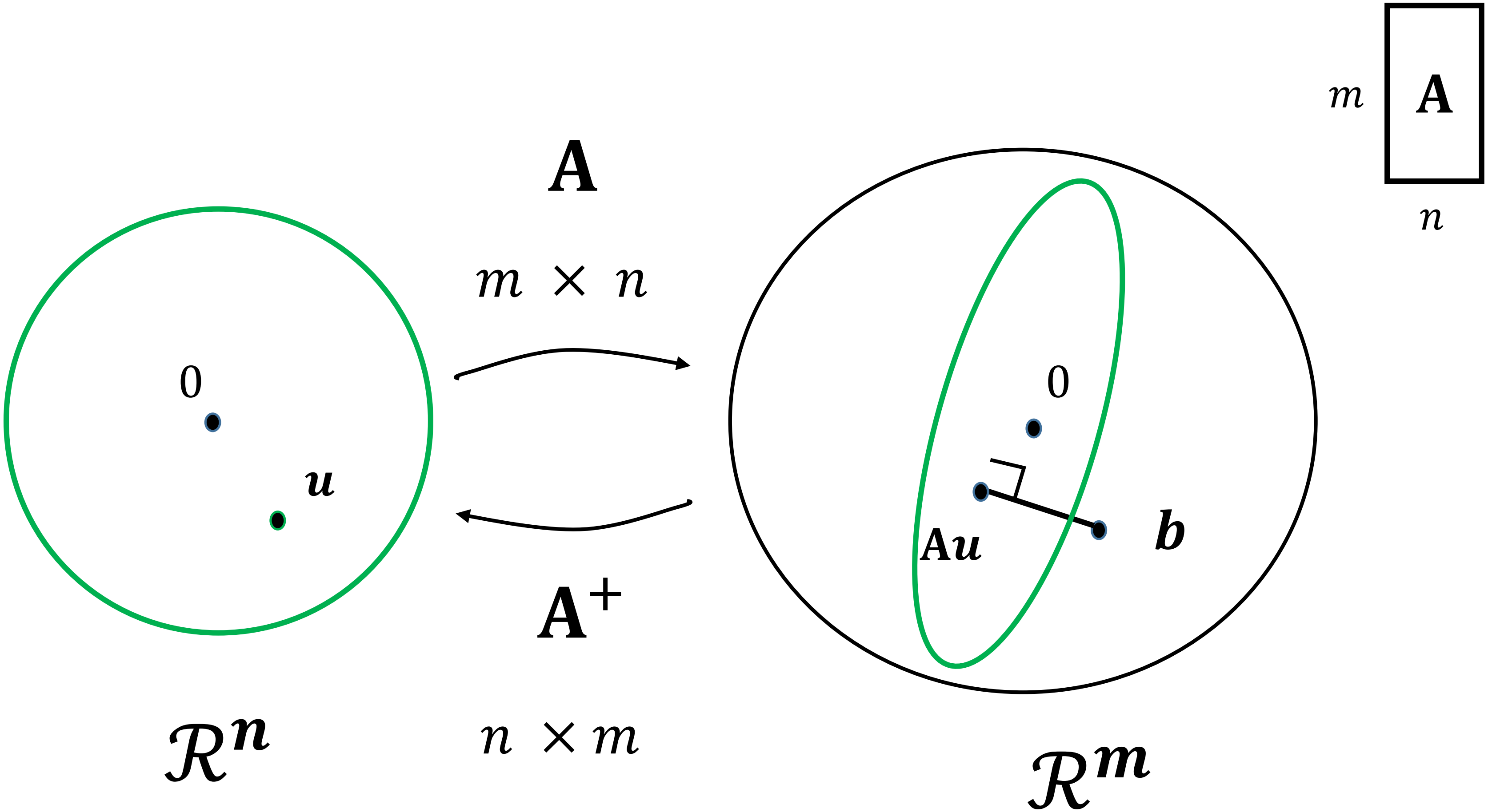
equality if $\boldsymbol{A}$ is invertible.





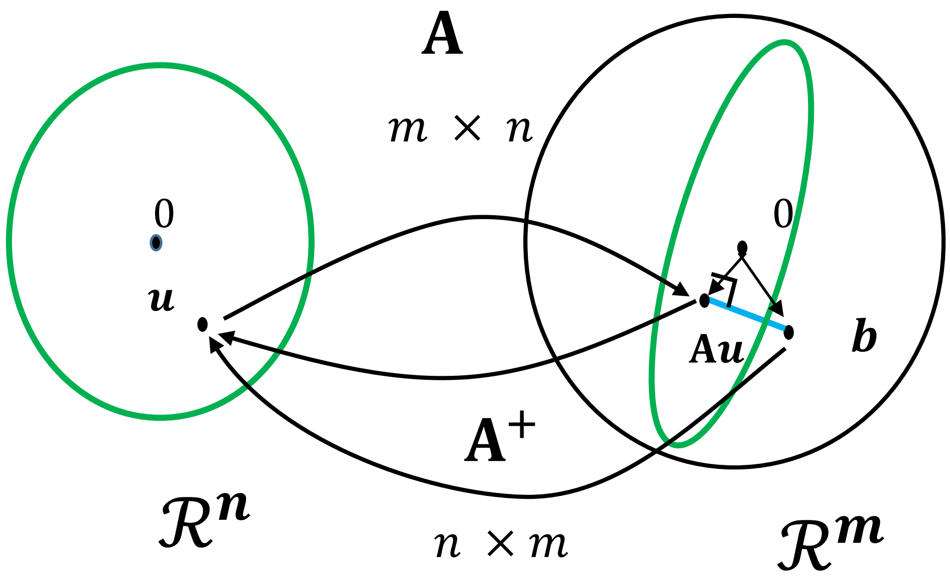
Given $\mathbf{A}$ and $\mathbf{b}$ in $\mathcal{R}^{\boldsymbol{m}}$, find the $\mathbf{u}$ in $\mathcal{R}^{\boldsymbol{n}}$ that minimizes $\| \mathbf{A}\,\mathbf{u} - \mathbf{b} \|^2$.

The solution is $\mathbf{u} = \mathbf{A}^{+}\mathbf{b}$.



Given $\mathbf{A}$ and $\mathbf{b}$ in $\mathcal{R}^m$, find the $\mathbf{u}$ in $\mathcal{R}^n$ that minimizes $\| \mathbf{A}\,\mathbf{u} - \mathbf{b} \|^2$.

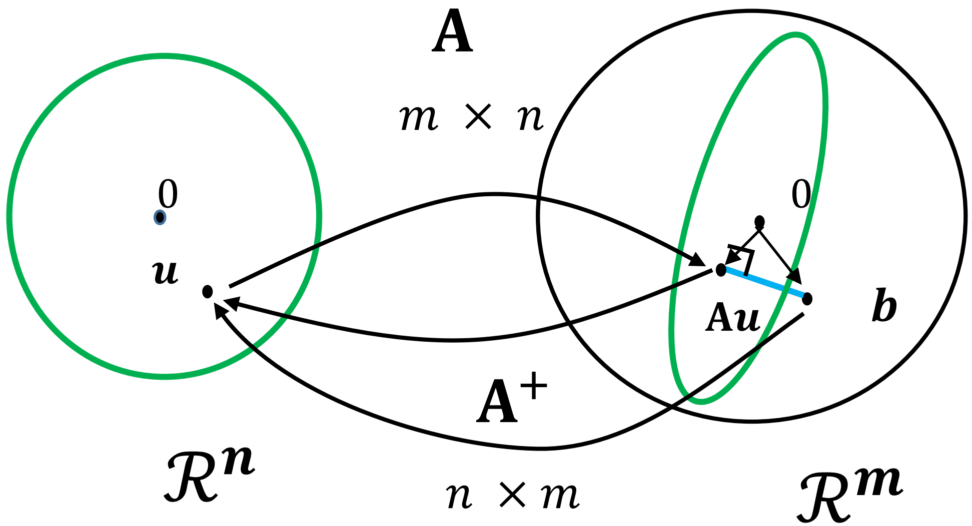
The solution is $\mathbf{u} = \mathbf{A}^+\mathbf{b}$. Intuitively (and as we'll argue next), $\mathbf{A}u$ is the orthogonal projection of $\boldsymbol{b}$ onto the column space of $\mathbf{A}$.

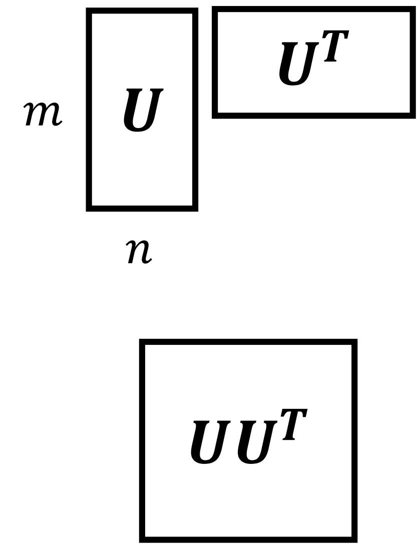


Substitute $\mathbf{A} = \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^{\mathbf{T}}$ into $\mathbf{A}^{+} \equiv \left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}$

Exercise : Show this gives $\mathbf{A}^{+} = \mathbf{V}\,\boldsymbol{\Sigma}^{-1}\mathbf{U}^{\mathbf{T}}$.

Exercise : Show $\mathbf{A}\,\mathbf{A}^{+} = \mathbf{U}\mathbf{U}^{\mathbf{T}}$.





$$\mathbf{V}\,\boldsymbol{\Sigma}^{-1}\mathbf{U}^{\mathbf{T}}\,.$$

The crossed out statement is not true.

$$\boldsymbol{A}\,\boldsymbol{A}^{+} = \boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}\,\boldsymbol{b}$$

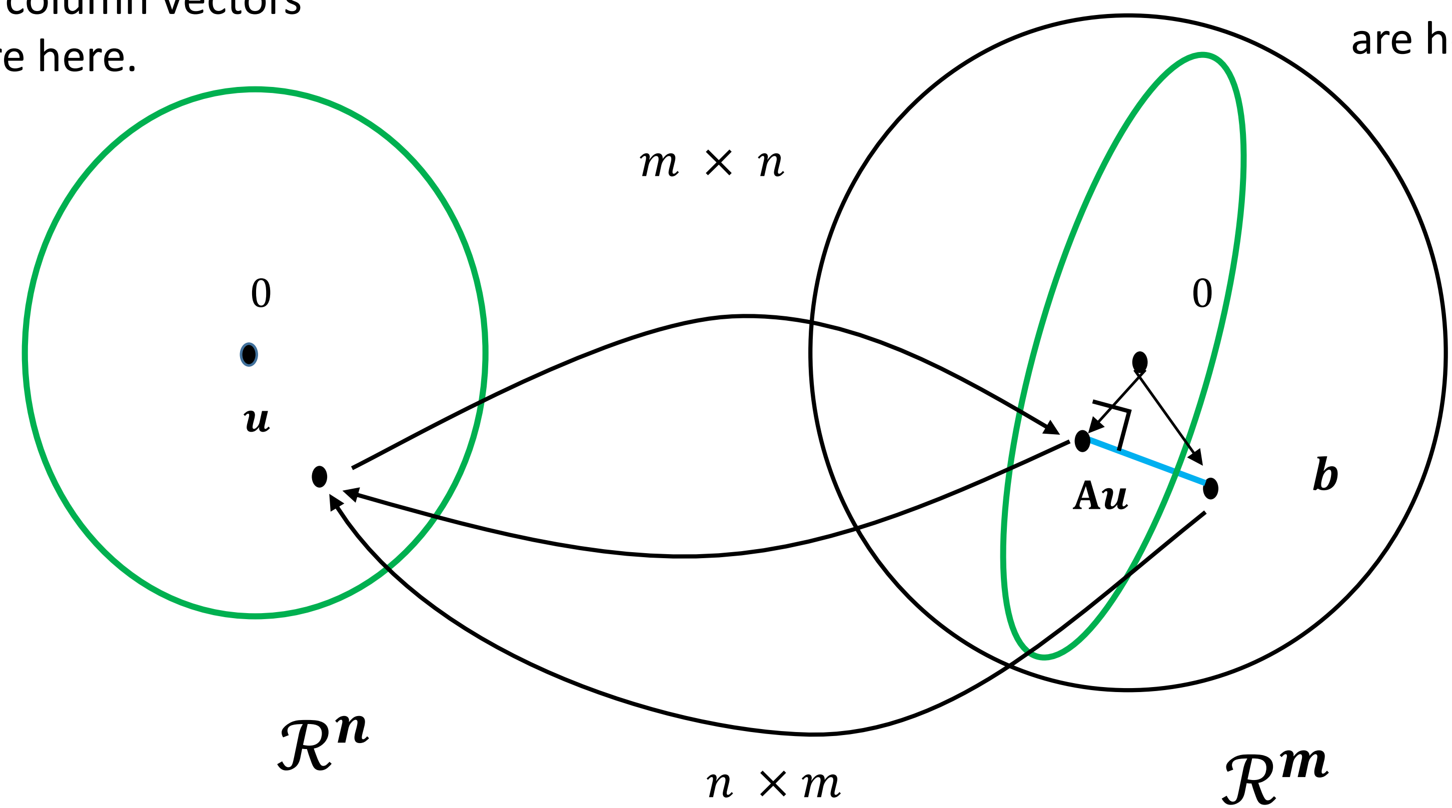
~~$\boldsymbol{UU^T}$ is an $m \times m$ diagonal matrix, with 1's on the first n diagonals and 0's on the last $m - n$ diagonals. Why ?~~

$\boldsymbol{UU^T}$ is an $m \times m$ matrix which projects $\boldsymbol{b}$ to the column space of $\boldsymbol{A}$.

$$\mathbf{A} = \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^{\mathbf{T}}$$

V column vectors are here.

U column vectors are here.



$$\mathbf{A}^{+} = \boldsymbol{V}\,\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{U}^{\boldsymbol{T}}$$

Next Two Weeks…

- Camera calibration
- Image Stitching for panoramas
- Binocular Stereo