EMT 2431: Spatial Mechanisms

Compiled by:

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Students are reminded not to treat these lecture notes as comprehensive and solely sufficient for their studies. These notes are aimed at providing a quick reference and a brief guidance for the students, NOT a substitute for attending regular classes and reading recommended textbooks.

Expected Outcomes

At the end of this course, you should be able to;

- determine the linkage arrangement for 3D motion
- solve problems in spatial mechanisms using 3D transformation matrices
- carry out forward kinematic analysis and computer simulation of a given spatial mechanism

Course Outline

Introduction: Possible link connection types allowing for three dimensional motion, revolute, prismatic slides, helix pair cylindrical pair, spherical and plane joints, agitation mechanism, Hookes joint, universal joint. Degrees of freedom. Planar finite transformations using complex analysis. Identity transformation. Planar matrix operator for finite rotation. Homogeneous coordinates and finite planar translation. Concatenation of finite displacements. Rotation about an axis away from the origin. Rigid body transformation. Spatial transformations: Rotation, translation, rotation matrix with axis through the origin. Analysis of spatial mechanisms; 4x4 transformation matrix. Link and joint modeling with elementary matrices, spherical joint, Eulerian rotation transformation. Modeling of spatial mechanisms. Computer simulation.

Reference Textbooks

1. George N. S, Arthur G. E., (1988) *Advanced mechanism design: Analysis and synthesis*, Prentice Hall, volume 2
2. Sandier B.Z (1999) *Robotics Designing the Mechanisms for Automated Machinery Academic* Press, 2nd Ed.
3. Marghitu D.B. (2009) *Mechanisms and Robots Analysis with MATLAB*, Springer-Verlag London Limited
4. John J. Craig (2005), I*ntroduction to Robotics Mechanics and Control*, Pearson Education International, 3rd Ed

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1 Introduction

1.1 Kinematic Pairs used in Spatial Mechanisms

Although most mechanical linkages have planar motion, there are many cases where three-dimensional (or spatial) movement is required. The *revolute* and *prismatic (slider)* joints are quite familiar from planar linkages. They both allow single degree of freedom of motion between the links they connect. In spatial mechanisms the axes of these joints need not be parallel or perpendicular to the axes of other joints. Thus the general spatial motions may be obtained with these joints.

Another single-degree-of-freedom joint is *helix (screw)* joint. In this joint there is a linear relationship between the axial translation and the angle of rotation of the screw relative to the nut. The *cylindrical pair* has no coupling between the sliding and rotational positions so that the this pair permits two degrees of freedom of relative motion. The spherical and plane joints allow three degrees of freedom of relative motion- three rotations for spherical and two translations and one rotation for the planar joint. These joint pairs and three-dimensional links may be combined in countless combinations to yield spatial mechanisms.

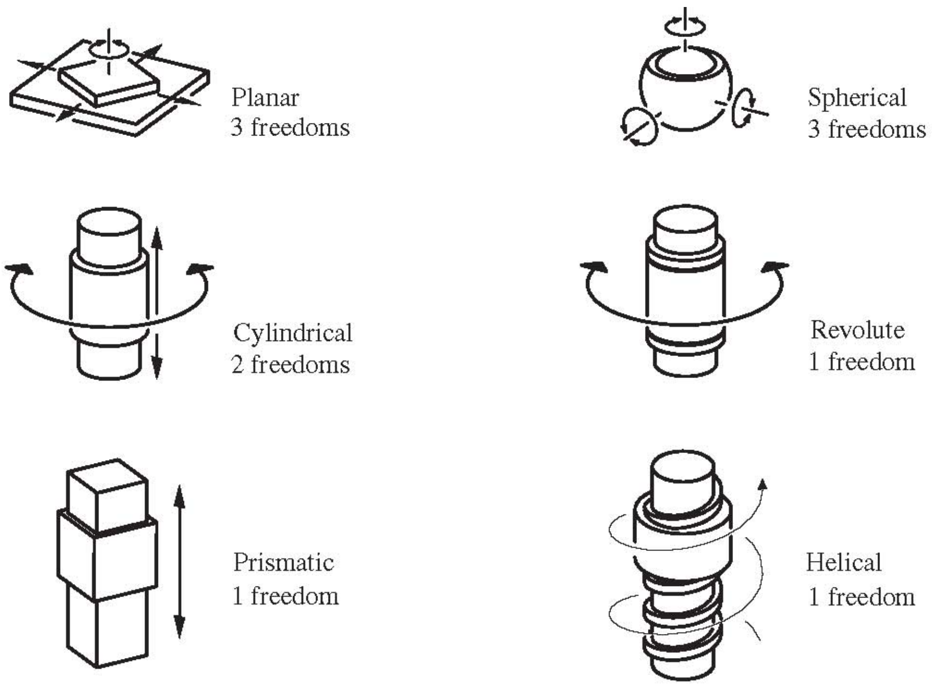


Figure 1: Spatial Kinematic Pairs

1.2 Degree of Freedom

Recall in planar mechanisms, Gruebler's equation was used to determine the degrees of freedom. For movement in three dimensions a different equation is required for determining degrees of freedom. Let

$F=$ degree of freedom of the mechanism

$n=$ number of links in the mechanism (including the fixed link; all links are considered as rigid bodies having at least two joints: if several machine parts are assembled as a rigid part, the assembly is considered as a single link)

$j=$ number of joints in the mechanism; each joint is assumed as binary (ie., connecting two links); joints connecting more than two links will be treated as was done with Gruebler's equation (i.e., a separate joint between each two links); thus a joint connecting 3 links would count as two joints

$f_i=$ degrees of freedom of the i^{th} joint, this is the number of degrees of freedom of the relative motion between the connected links

$\lambda=$ degrees of freedom the space within which the mechanism operates; for plane motion and for motion on a curved surface $\lambda = 3$ and for spatial motions $\lambda = 6$

$L_{IND}=$ number of independent circuits or closed loops in the mechanism

The following degree-of-freedom equations then apply to a large class of mechanisms:

$$F = \lambda(n - j - 1) + \sum_{i=1}^{j} f_i \tag{1}$$

$$L_{IND} = j - n + 1 \tag{2}$$

Combining (1) and (2), have

$$\sum f_i = F + \lambda L_{IND} \tag{3}$$

For example, for the plane slider-crank mechanism shown in Fig 2,

$$n = j = 4$$

$$\sum f_i = 1 + 1 + 1 + 1 = 4 \quad \lambda = 3$$

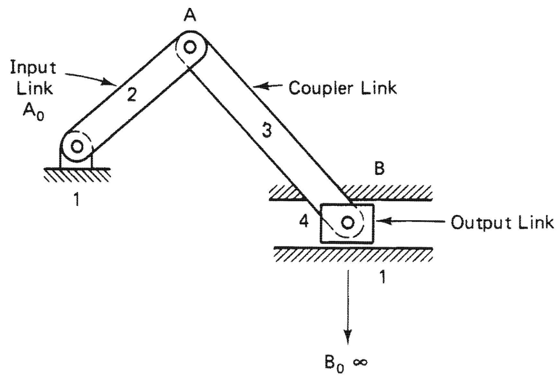


Figure 2: Plane Slider-Crank Mechanism

$$\text{Hence} \quad F = 3(4 - 4 - 1) + 4 = 1$$

In the application of these equations certain rules need to be kept in mind:

1. Joints connecting n links, where $n > 2$, are called multiple joints and are counted as $(n-1)$ binary joints.

2. Certain linkages commonly thought of having one degree of freedom may have an F value greater than 1. This can occur in spatial linkages, for example, having links with two joints of types spherical-spherical $(S-S)$, spherical-cylindrical $(S-C)$, and spherical-planar $(S-P_L)$. Such links can have a "redundant" or "passive" freedom of rotation about the axis connecting the joints, which is independent of the motion of the mechanism as a whole (e.g., see Ex. **??**.1),

3. Some highly significant mechanisms do not obey the general degree-of-freedom equations given above. These are mechanisms that depend on special dimensions or proportions for their mobility. Mechanisms with mixed plane/spatial portions (variable λ) are usually exceptional. In spatial linkages, special cases are often associated with parallel, intersecting, or perpendicular joint axes. There are no simple rules that will predict whether a mechanism obeys Eq. 1). Here experience is very helpful, and the presence of one or more of the above-listed special characteristics is a signal to be watchful.

Examples 1.1

Determine the degrees of freedom of the swash plate drive of Fig. 3.

Solution:

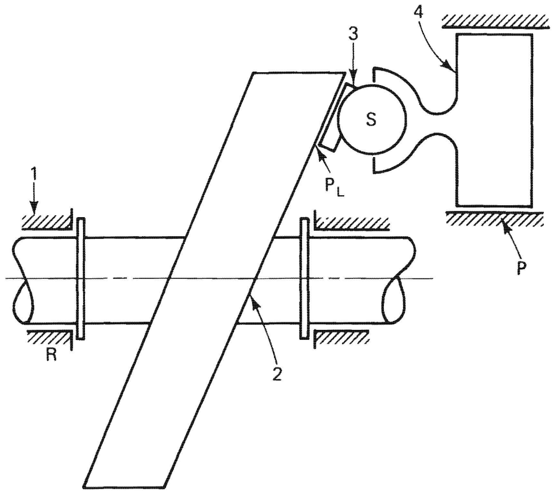


Figure 3: Example 1.1 Problem

$$n = j = 4$$

$$\sum f_i = 1 + 3 + 3 + 1$$

Since $\lambda = 6$, then from Eq. (3), $F = 2$. One would anticipate that this mechanism has a single degree of freedom (say the rotation of link 2 if the swash plate mechanism is used as a hydraulic motor). The second degree of freedom is the rotation of link 3 about an axis through the center of the sphere and normal to the face of the swash plate. This passive degree of freedom does not interfere with the desired input-output kinematic relationship of the drive (although it certainly plays a role in the lubrication and wear of the mating surfaces).

2 Planar Finite Transformation using Complex Analysis

Consider two finitely separated positions A_1 and A_2 of a rigid link shown that rotate about the origin O.

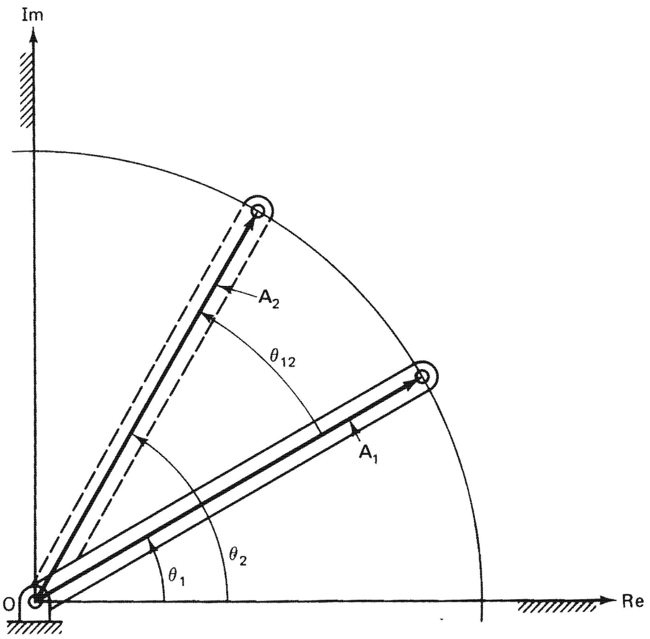


Figure 4: Two coplanar positions of a link pinned to ground

Define the position vectors of the points as $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively. Then we seek a transformation that rotate the link $\mathbf{A}_1$ by θ_{12} to a new position $\mathbf{A}_2$. We assume an unknown operator R_{12} whose product with $\mathbf{A}_1$ yield $\mathbf{A}_2$

$$\mathbf{A}_2 = \mathbf{R}_{12}\mathbf{A}_1 \tag{4}$$

This can be re-arranged by complex division

$$\mathbf{R}_{12} = \frac{\mathbf{A}_2}{\mathbf{A}_1} \tag{5}$$

The complex vector $\mathbf{A}_1$ and $\mathbf{A}_2$ can be written as

$$\begin{aligned} \mathbf{A}_1 &= |\mathbf{A}_1|(\cos\theta_1 + i\sin\theta_1) \\ \mathbf{A}_2 &= |\mathbf{A}_2|(\cos\theta_2 + i\sin\theta_2) \end{aligned}$$

Substituting in Eq. 40

$$\mathbf{R}_{12} = \frac{|\mathbf{A}_2|}{|\mathbf{A}_1|} = \frac{(\cos\theta_2 + i\sin\theta_2)}{(\cos\theta_1 + i\sin\theta_1)} \tag{6}$$

Since the initial position is known and we wish to rotate the link through a known angle θ_{1-2} then we strive to express Eq. 40 in terms of these known values. Now since the link is rigid and rotates about the origin then the length remains constant hence

$$|\mathbf{A}_1| = |\mathbf{A}_2|$$

Next we rationalize the denominator by multiplying through by the complex conjugate of the denominator to get

$$\mathbf{R}_{12} = \frac{\mathbf{A}_2}{\mathbf{A}_1}\frac{(\cos\theta_2 + i\sin\theta_2)}{(\cos\theta_1 + i\sin\theta_1)} \times \frac{(\cos\theta_1 - i\sin\theta_1)}{(\cos\theta_1 - i\sin\theta_1)} \tag{7}$$

Adopting the shorted notation $c\theta = \cos\theta$ and $s\theta = \sin\theta$. Expanding Eq. 46 and regrouping we have

$$\mathbf{R}_{12} = c\theta_2 c\theta_1 + s\theta_2 s\theta_1 + i(c\theta_1 s\theta_2 - c\theta_2 s\theta_1) \tag{8}$$

Now from Fig. 4 we have $\theta_2 = \theta_1 + \theta_{12}$ and using this in Eq. 47 yields

$$\mathbf{R}_{12} = c(\theta_1 + \theta_{12})c\theta_1 + s(\theta_1 + \theta_{12})s\theta_1 + i[s(\theta_1 + \theta_{12})c\theta_1 - c(\theta_1 + \theta_{12})s\theta_1] \tag{9}$$

Now from the trigonometric identities

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &\text{and} \\ \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha \end{aligned}$$

and letting $\alpha = (\theta_1 + \theta_{12})$ and $\beta = \theta_1$ hence Eq. 9 reduces to

$$\mathbf{R}12 = \cos\theta_{12} + i\sin\theta_{12} \tag{10}$$

Hence using Eulers equation we have

$$\mathbf{R}_{12} = e^{\theta_{12}} \tag{11}$$

This shows that the transformation describing the displacement is independent of the actual position of the link but dependent solely on the angle of rotation from initial position.

2.1 Identity Matrix

Consider a 2×2 identity matrix when used in transformation its effect is to leave the vector un-changed

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 1x + 0y \\ 0x + 1y \end{bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{12}$$

Note that the second entry in the first row of the transformation represents the contribution of y to the new value of x. Similarly, the first entry in the second row represents the contribution of x to the new value of y. Thus with 2×2 matrix operator, both final values are functions of each initial value.

2.2 Planar Matrix Operator for Finite Rotation

From the complex operator it was seen that the finite operator does not depend on either the initial or final position of the vector but only on the angle θ_{12}. But the coordinates of the final vector are each a function of both the real and imaginary coordinates of the initial vector. In matrix notation, it follows logically that there must be a similar arrangement.
Let $\mathbf{R}(\theta_{12})$ be a 2×2 matrix, whose product with $\mathbf{A}_1$, yields $\mathbf{A}_2$

$$\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \tag{13}$$

where x_1, y_1 and x_2, y_2 are the coordinate of $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively, both expressed in the same coordinate system. Expanding Eq. 13 by the way of matrix multiplication yields

$$\begin{aligned} x_2 &= r_{11}x_1 + r_{12}y_1 \\ y_2 &= r_{21}x_1 + r_{22}y_1 \end{aligned} \tag{14}$$

Expanding similar vectors in complex form and using the operator of Eq. 10 yields

$$\text{Real}_2 + i\text{Imaginary}_2 = \Re_2 + i\wp_{m_2} = (\cos\theta_{12} + i\sin\theta_{12})(\Re_1 + i\wp_{m_1})$$

when this is expanded

$$\begin{aligned} \Re_2 &= \cos\theta_{12}\Re_1 + i^2\sin\theta_{12}\wp_{m_1} \\ \wp_{m_2} &= \sin\theta_{12}\Re_1 + \cos\theta_{12}\wp_{m_1} \end{aligned} \tag{15}$$

comparing Eqs (14) and (15) yields

$$[R_(\theta_{12}) = \begin{bmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{bmatrix} \tag{16}$$

2.3 Homogeneous Coordinates and Finite Planar Translation

When a particle undergoes pure translation, the vector coordinates representing its location in a coordinate system will change. Consider the slider pivot of Fig. 5. It has undergone a pure translation (represented by ΔA) from its initial position at A_1 to a final position at A_2. We seek a transformation T whose product with A_1 yields A_2:

$$A_2 = [T]A_1$$

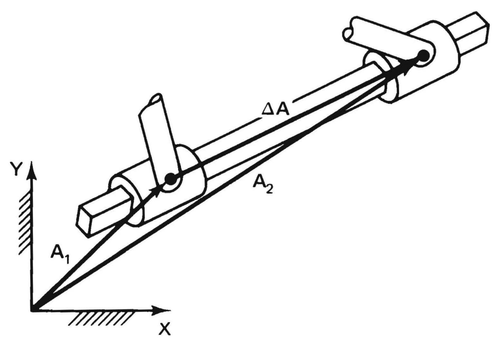


Figure 5: Translation

Using vector addition, it should be obvious that

$$A_2 = A_1 + \Delta A$$

or

$$x_2 = x_1 + \Delta x$$

and

$$y_2 = y_1 + \Delta x$$

One way that this translation can be represented is by using the 2×2 matrix with an additional column tacked on (representing ΔA) and converting A_1 from a 2×1 column vector to a 3×1 homogeneous column vector, which we will designate by $[A_1]$:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \quad \text{or} \quad A_2 = [T][A_1]$$

Multiplying this out yields

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 0y_1 + \Delta x \\ 0x_1 + y_1 + \Delta y \end{bmatrix}$$

which is what we wanted. But note that A_2 represented by its coordinates x_2 and y_2 only is a 2×1 column matrix rather than a 3×1 homogeneous column matrix like $[A_1]$. To remedy this, we may expand the 2 X 3 translation operator to a 3×3 matrix, keeping everything homogeneous coordinates:

$$[T] = \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix}$$

2.4 Concatenation of Finite Displacement

Suppose that a pivot or a particle goes through a "rotation" describing a circular path around the origin of a coordinate system and then a straight-line translation, both displacements being expressed in one and the same coordinate system. Can a single operator describe this motion? Consider the links of Fig. 6. The slider 2 on link 1, initially located by A_1 with respect to a fixed coordinate system attached to the center of rotation of link 1, rotates and translates to its final position at A_2.

Since positional kinematics is concerned only with the initial and final positions, this motion may be considered as a rotation to A_1', followed by a translation to A_2 by ΔA. Mathematically, we could represent this in two steps by

$$\begin{aligned} [A_1'] &= [R][A_1] \\ [A_2] &= [T][A_1'] \end{aligned}$$

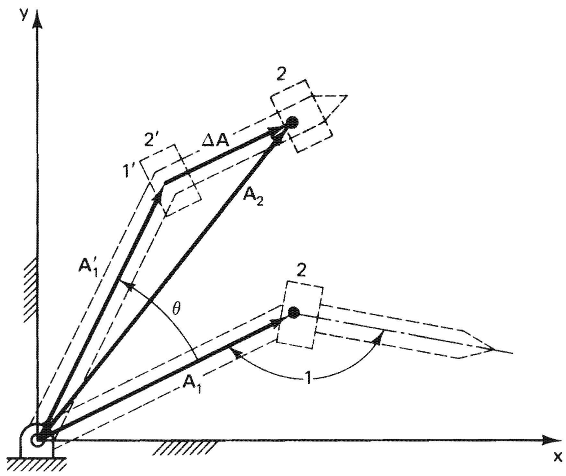


Figure 6: Concatenation

Or we could combine these two equations, yielding

$$[A_2] = [T][R][A_1]$$

If both the translation and the rotation operators are expressed as 3×3 matrices, an operation called concatenation may be performed. Concatenation is basically a matrix operation in which two or more transformations are combined by premultiplying the first operator by the second, the product of these two by the third, and so on, to reduce the number of operators to a single operator. For example the above operation can be represented by one single operator $[D]$ given by

$$\begin{aligned} [D] &= \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & \Delta x \\ \sin\theta & \cos\theta & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Thus, for the case of planar motion consisting of a rotation followed by a translation, both expressed in the same coordinate system, the final position of a vector representing a particle or a pivot may be found using a single 3×3 matrix operator when both the initial and final positions are expressed in homogeneous coordinates.

Note:

- note that if $\theta = 0$ (no rotation), $\cos\theta = 1$, $\pm = \sin\theta = 0$, $[D]$ reduces to the translation operator.

- if $\Delta x = 0$ and $\Delta y = 0$ (no translation), $[D]$ reduces to the rotation operator.
- If there is no rotation $(0 = 0)$ and no translation $(\Delta x = \Delta y = 0)$, then [D] reduces to the identity matrix

Thus [D] is an operator representing four possible finite planar displacements of a point: (1) no motion, (2) pure rotation about the origin, (3) pure translation, and (4) rotation about the origin followed by translation, all defined in the same coordinate system.
For the fifth case, when translation occurs first and is followed by rotation, the concatenation is (using $c = \cos$ and $s = \sin$)

$$\begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta & c\theta\Delta x - s\theta\Delta y \\ s\theta & c\theta & s\theta\Delta x + c\theta\Delta x \\ 0 & 0 & 1 \end{bmatrix}$$

This is different from [D] for case 4

2.5 Rotation About an Axis not Through the Origin

Rotation of a link (vector) pivoted about the origin can be interpreted as a rotation of a free vector with respect to a fixedly oriented coordinate system with the origin attached to the tail of the free vector. An example of this is the case of synthesis of mechanism where the loop closure equation were applied to the vector representing the linkages. This involved the head-to-tail addition of the vector, each vector being a free-vector with its own origin and own coordinate system as shown in Figure 7. Note that the joints 1 and 4 are fixed. Here we note that each of these coordinate system have axes that are parallel to the global coordinate system i.e all coordinate systems have axes parallel to each other.
Now since the link 3 have a vector Z_3 whose tail is on a free coordinate system, we wish to find its rotation about a fixed co-ordinate e.g x_1, y_1 or x_2, y_2 etc. Consider a point A_1 located in the fixed x, y coordinate system in Fig. 8a by the vector $\mathbf{A_1}$. Now let the point A_1 describe the arc θ about point Q at a vector $\mathbf{Q}$. We seek a single operator that will describe the absolute motion of point A. We may construct a free vector $\mathbf{a}_1$ emanating from point Q to the particle initially at $\mathbf{A}_1$. If point Q (and the tail of $\mathbf{a}_1$ is translated back to the origin, $\mathbf{a}_1$ becomes a vector that is free to rotate about the origin. This step, shown in Fig. 8b, is represented mathematically by

$$\{a_1'\} = [T(-\mathbf{Q})]\{a_1\}$$

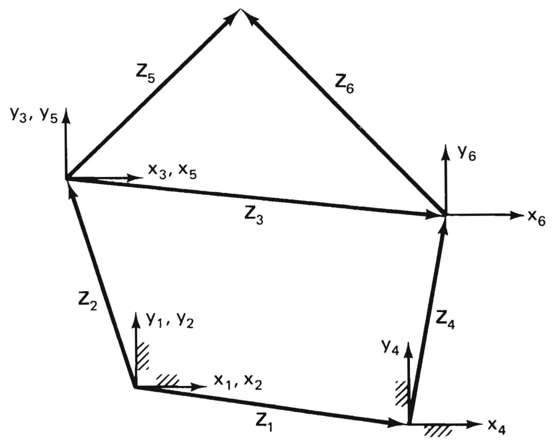


Figure 7: Coordinate systems for each vector representing a planar four bar linkage

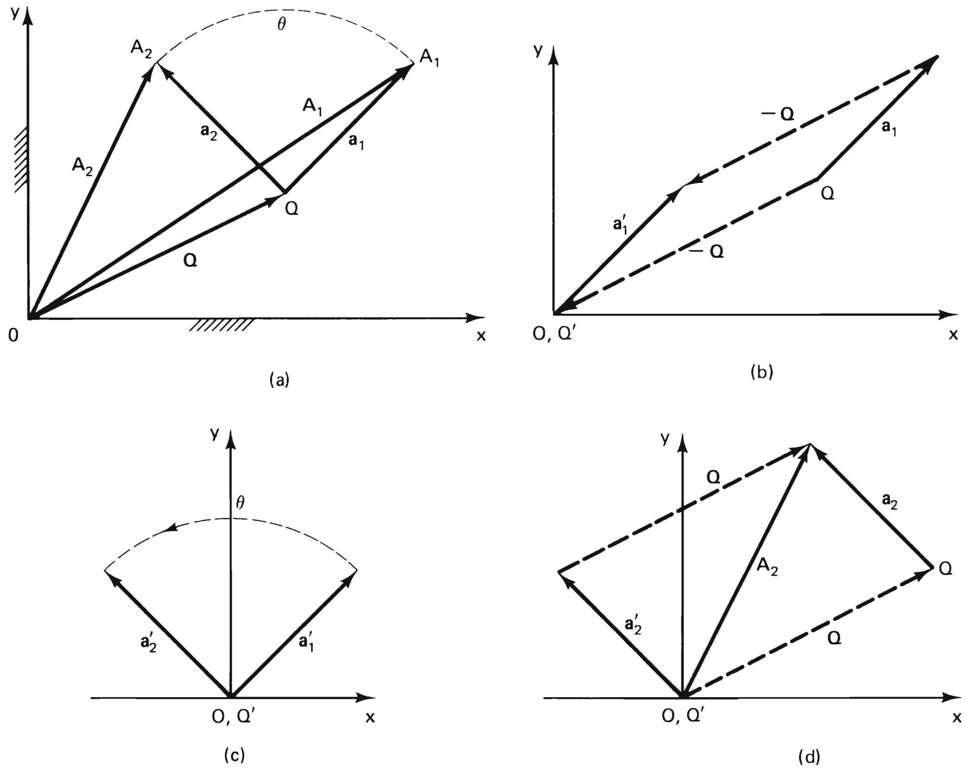


Figure 8: Illustration of transformation steps representing the movement of point A from position 1 to 2

Next the vector a_1' can be rotated about the origin by angle θ to give a_2' as shown in figure 8c

$$\{a_2'\} = [R(\theta)]\{a_1'\}$$

Finally, the point Q' (at origin) is translated with a_2' attached back by translation operator as shown

$$\{A_2\} = [T(+\mathbf{Q}]\{a_2'\}$$

All these steps may be concatenated to form the single 3×3 matrix operator as follows

$$\{A_2\} = [D(\theta, Q_x, Q_y)]\{A_1\}$$

where

$$\begin{aligned}
[D] &= [T(+\mathbf{Q})][R(\theta)][T(-\mathbf{Q})] \\
&= \begin{bmatrix} 1 & 0 & Q_x \\ 0 & 1 & Q_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -Q_x \\ 0 & 1 & -Q_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & -Q_x \\ 0 & 1 & -Q_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & -Q_y\cos\theta + Q_y\sin\theta \\ \sin\theta & \cos\theta & -Q_x\sin\theta - Q_y\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \\
[D] &= \begin{bmatrix} \cos\theta & -\sin\theta & Q_x - (Q_x\cos\theta - Q_y\sin\theta) \\ \sin\theta & \cos\theta & Q_y - (Q_x\sin\theta + Q_y\cos\theta) \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

3 Rigid-Body Transformations

In rigid body the distance of any point from the rest of the points in a rigid body remain constant. For example, the general planar motion of three points of a planar rigid body, from their initial positions (**A**, **B**, **C**) to their final positions (**A'**, **B'**, **C'**), can be represented by

$$\begin{bmatrix} A'_x & B'_x & C'_x \\ A'_y & B'_y & C'_y \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{bmatrix} \tag{17}$$

3.1 Introduction

Spatial transformations can basically be defined as rotations or translations. These may be defined as passing through the origin or not. This section will enable us know how to carry out various transformations in a 3-D space.

3.2 Rotation

All points on a rigid body undergoing pure rotation describes arcs in plane perpendicular to a fixed line. i.e the axis of rotation. In planar $x - y$ system the axis is always perpendicular to the $x - y$ plane, thus parallel to the z-axis. In the general spatial case, motion is not constrained to the xy plane and the axis of rotation may be oriented in any direction. Therefore, the location and direction of the axis must be incorporated into the spatial rotation operator.

An axis in space may be specified is by defining the location of a point on the axis and a unit vector in the positive direction along the axis. If, in deriving a spatial rotation operator, we specify that the axis must pass through the origin and if the three coordinates of the unit vector are used explicitly in the operator, the axis will be fully incorporated into the operator.

Using a unit vector to define a rotation allows the establishment of sign convection for the rotation. In a planar system, if the thumb of the right-hand points in the positive z direction (i.e., toward the observer), the fingers curl in a counterclockwise direction. Adopting this right hand rule, the right hand thumb points in the direction of the unit vector of the axis, a positive rotation about the axis is in the direction of the curled fingers. Considering a unit vector given by the symbol $\hat{u}$ where the 'hut' above indicates that it's a unit vector (i.e magnitude of 1). Referring to Fig. 9, the three coordinates are the direction cosine of the line given by

$$\begin{aligned} u_x &= \cos\alpha \\ u_y &= \cos\beta \\ u_z &= \cos\gamma \end{aligned} \tag{18}$$

where α, β and γ are angle of the vector measured from the positive branch of the x, y, z axes respectively.

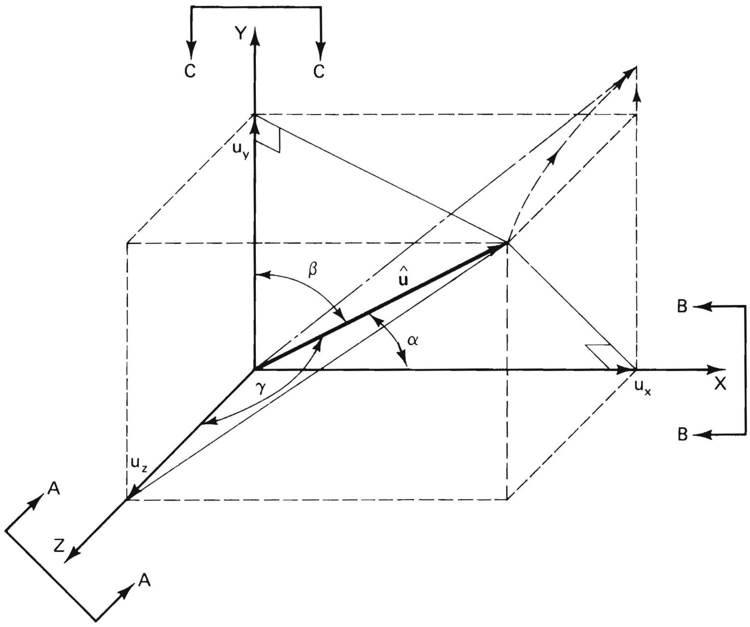


Figure 9: Unit vector and its direction cosine

A unit vector $\hat{u}$ pointing from a point **A** on an axis in space toward point **B** anywhere on that same axis may be found by normalizing the vector **AD**:

$$\begin{aligned} u &= \frac{B - A}{|\overrightarrow{AB}|} \\ u_x &= \frac{B_x - A_x}{|\overrightarrow{AB}|} \\ u_y &= \frac{B_y - A_y}{|\overrightarrow{AB}|} \\ u_z &= \frac{B_z - A_z}{|\overrightarrow{AB}|} \end{aligned}$$

where

$$|\overrightarrow{AB}| = [(B_x - A_x)^2 + (B_y - A_y)^2 + (B_z - A_z)^2]^{1/2}$$

hence the magnitude of a unit vector $\hat{u}$ is

$$|u| = [u_x^2 + u_y^2 + u_z^2]^{1/2}$$

3.2.1 Rotational Matrix with Axis Through the Origin

Consider the motion of a point P shown in figure 10. Pint P rotates about a point A on the axis through the origin. The axis is described by a unit vector $\hat{\mathbf{u}}$, which is equivalent to describing the slope of this line in space. The coordinate of $\hat{\mathbf{u}}$ are the direction cosine of the axis

$$\begin{aligned} u_x &= \cos\alpha \\ u_y &= \cos\beta \\ u_z &= \cos\gamma \end{aligned} \tag{19}$$

where α, β and γ are measured from positive x, y and z coordinate axes respectively to the vector $\hat{\mathbf{u}}$ erected at the origin.

Note that

$$u_x^2 + u_y^2 + u_z^2 = 1 \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

To drive the general expression for point P' in terms of P, $\hat{\mathbf{u}}$ and θ, we for instance take the coordinate as

$$P = [0,\ 1,\ 3]^T \quad A = [0,\ 2,\ 2]^T$$

So that axis in the yz plane is described by

$$\hat{\mathbf{u}} = [0,\ \cos 45,\ \cos 45]$$

Particle P rotates around the axis by an angle $\theta_{12} = 45^o$ (measured positive using the right hand rule with the thumb pointing in the same direction as $\hat{\mathbf{u}}$ at a radius:

$$P = [\mathbf{P} - \mathbf{A}] = \sqrt{2}$$

For a general derivation, we seek a spatial rotation operator R_{12} such that

$$\mathbf{r}_2 = R_{12}\mathbf{r}_1$$

where r_1 for this case is $\mathbf{P}$

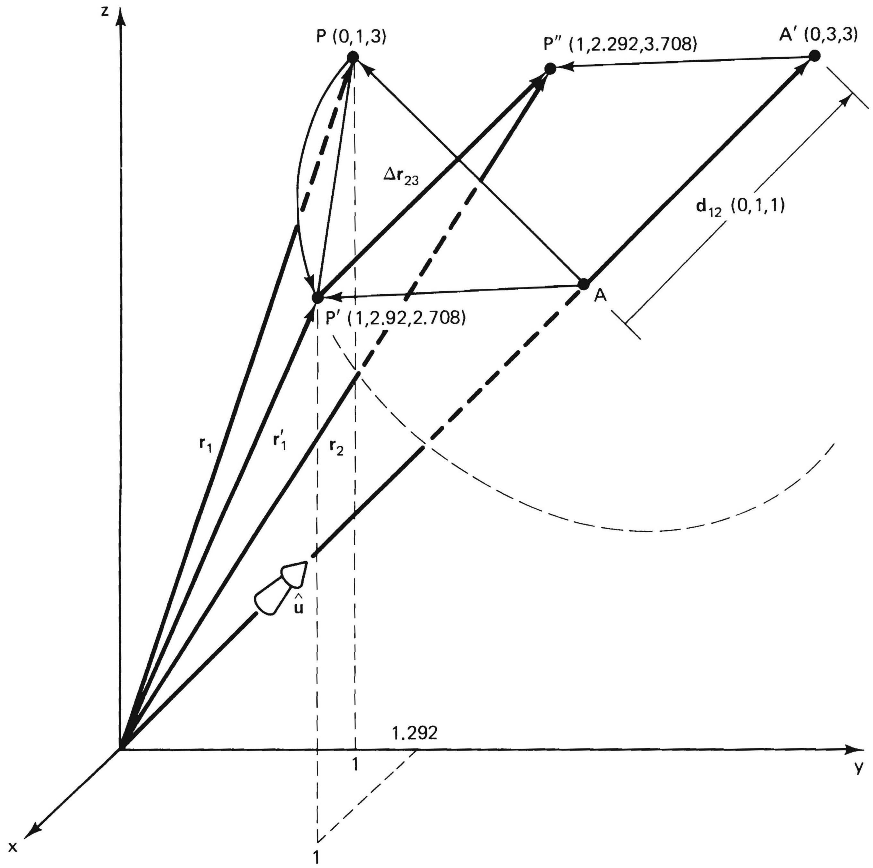


Figure 10: Screw displacement of point P about axis which goes through the origin

Define a vector through the origin to point A as $\mathbf{a}$, where $\mathbf{a}$ is the projection of r_1 onto the axis defined by a unit vector $\hat{\mathbf{u}}$. Since $\mathbf{a}$ and $\mathbf{r}_1$ form two sides of the right triangle POA then its magnitude is $|\mathbf{a}| = |\mathbf{r}_1 \cos \angle POA$ which is the dot product of $\mathbf{r}_1$ and $\hat{\mathbf{u}}$ Since $\mathbf{a}$ is in the same direction and sense as $\hat{u}$ and since $|\hat{\mathbf{u}}| = 1$, $\mathbf{a}$ is found to be

$$\mathbf{a} = (\mathbf{r}_1.\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{20}$$

Also needed are two unit vectors defining the direction of the components of $\Delta\mathbf{r}_{12}$ in the plane of rotation. These are the centripetal ($\Delta\mathbf{r}_c$) and the tangential ($\Delta\mathbf{r}_t$) component. To define this we also define an orthonormal coordinates systems which is established at A with mutually perpendicular unit vectors $\hat{\mathbf{m}}$, $\hat{\mathbf{s}}$ and $\hat{\mathbf{u}}$. The unit vector $\hat{\mathbf{s}}$ may be found by normalizing the vector $\overrightarrow{PA}$, which is $\mathbf{a} - \mathbf{r}_1$. To normalize this vector, divide it by its magnitude, the radius (P), therefore, $\hat{s}$ is defined as

$$\hat{s} = \frac{\mathbf{a} - \mathbf{r}_1}{\mathbf{P}} \tag{21}$$

Since $\hat{\mathbf{m}}$, $\hat{\mathbf{s}}$ and $\hat{\mathbf{u}}$ are orthonormal, $\hat{\mathbf{m}}$ may be defined as

$$\hat{\mathbf{m}} = \hat{\mathbf{s}} \times \hat{\mathbf{u}} = -\hat{\mathbf{u}} \times \hat{\mathbf{s}} \tag{22}$$

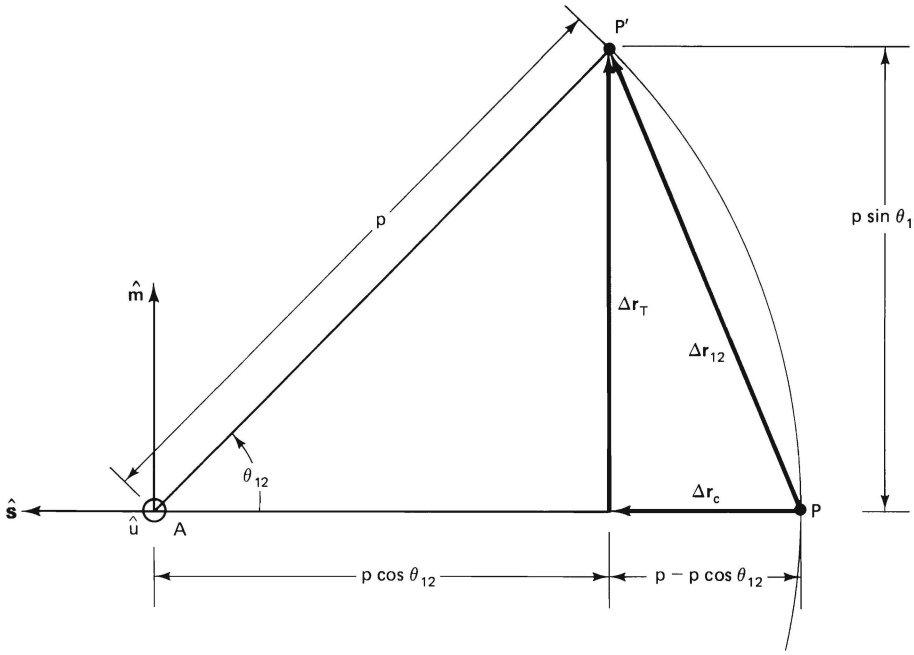


Figure 11: Screw displacement

The vector components now can be shown in Figure 11. Here the magnitude

$$\begin{aligned} |\Delta r_t| &= P \sin\theta_{12} \\ |\Delta r_c| &= P - P\cos\theta_{12} = P(1 - \cos\theta_{12}) \end{aligned}$$

Since $\hat{\mathbf{m}}$ and $\hat{\mathbf{s}}$ are unit vectors, the two components of Δr_{12} are found to be

$$\Delta r_t = P\sin\theta_{12}\hat{\mathbf{m}} \quad \Delta r_c = P\text{vers}\theta_{12}\hat{\mathbf{s}} \tag{23}$$

where $\text{vers}\theta_{12} \triangleq 1 - \cos\theta_{12}$ and where vers is the short form of versine. substituting for the unit vectors s and m in Equation 23

$$\Delta \mathbf{r}_t = P\sin\theta_{12}(\hat{s} \times \hat{u}) = P\sin\theta_{12}\frac{(a - r_1) \times u}{P}$$

and

$$\Delta \mathbf{r}_c = P\text{vers}\theta_{12}\frac{\mathbf{a}-\mathbf{r}_1}{p}$$

which simplifies to

$$\Delta \mathbf{r}_t = \sin\theta_{12}\left[(\mathbf{a}\times\hat{\mathbf{u}}) - (\mathbf{r}_1\times\hat{\mathbf{u}})\right]$$

$$\Delta \mathbf{r}_c = \text{vers}\theta_{12}(\mathbf{a}-\mathbf{r}_1) \tag{24}$$

Since $\mathbf{a}$ and $\hat{\mathbf{u}}$ are in the same direction, $\mathbf{a}\times\hat{\mathbf{u}} = 0$ and because $-\mathbf{r}_1\times\hat{\mathbf{u}} = \hat{\mathbf{u}}\times\mathbf{r}_1$, $\Delta\mathbf{r}_1$ can be further simplified to

$$\Delta \mathbf{r}_t = \sin\theta_{12}(\hat{\mathbf{u}}\times\mathbf{r}_1) \tag{25}$$

substituting for $\mathbf{a}$ in equation 24 yields

$$\Delta \mathbf{r}_c = \text{vers}\theta_{12}[(\mathbf{r}_1.\hat{\mathbf{u}})\hat{\mathbf{u}} - \mathbf{r}_1] \tag{26}$$

Hence the rotation of r_1 and r_2 can be described by displacement vector Δr_{12}, where $\Delta\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ or

$$\begin{aligned}
\mathbf{r}_2 &= \mathbf{r}_1 + \Delta\mathbf{r}_{12} \\
&= r_1 + (\Delta\mathbf{r}_t + \Delta\mathbf{r}_c) \\
\mathbf{r}_2 &= \mathbf{r}_1 + \sin\theta_{12}\,(\hat{u}\times\mathbf{r}_1) + \text{vers}\theta_{12}\left[(\mathbf{r}_1.\hat{u})\,\hat{u} - \mathbf{r}_1\right] \qquad (27)
\end{aligned}$$

Hence, we now require a rotational matrix $R_{12}(\hat{u},\theta_{12}$ which yield the product, $r_2 = R_{12}(\hat{u},\theta_{12})r_1$

In this equation we require to factor out the vector $\mathbf{r}_1$ and hence find matrices which when added up will yield the 3×3 spatial rotational operator $R_{12}(\mathbf{u},\theta_{12})$

$$[\mathbf{r}_2] = [M_1]\,[\mathbf{r}_1] + [M_2]\,[\mathbf{r}_1] + [M_3]\,[\mathbf{r}_1]$$

The first matrix will be

$$[M_1][r_1] = [r_1], \quad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, the second matrix

$$(\hat{u} \times \mathbf{r}_1) = \det = \begin{vmatrix} i & j & k \\ u_x & u_y & u_z \\ \mathbf{r}_{1x} & \mathbf{r}_{1y} & \mathbf{r}_{1z} \end{vmatrix}$$

This determinant is equal to

$$\begin{aligned} &= \underline{\mathrm{i}}(u_y\mathbf{r}_{1z} - u_z\mathbf{r}_{1y}) - \underline{\mathrm{j}}(u_x\mathbf{r}_{1z} - u_z\mathbf{r}_{1x}) + \underline{\mathrm{k}}(u_x\mathbf{r}_{1y} - u_y\mathbf{r}_{1x}) \qquad (28) \\ &= \{(u_y\mathbf{r}_{1z} - u_z\mathbf{r}_{1y}), (u_z\mathbf{r}_{1x} - u_x\mathbf{r}_{1z}), (u_x\mathbf{r}_{1y} - u_y\mathbf{r}_{1x})\} \qquad (29) \end{aligned}$$

This cross product can also be found by multiplying 3×3 matrix with r_1

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} \mathbf{r}_{1x} \\ \mathbf{r}_{1y} \\ \mathbf{r}_{1z} \end{bmatrix} = \begin{bmatrix} m_{11}\mathbf{r}_{1x} + m_{12}\mathbf{r}_{1y} + m_{13}\mathbf{r}_{1z} \\ m_{21}\mathbf{r}_{1x} + m_{22}\mathbf{r}_{1y} + m_{23}\mathbf{r}_{1z} \\ m_{31}\mathbf{r}_{1x} + m_{32}\mathbf{r}_{1y} + m_{33}\mathbf{r}_{1z} \end{bmatrix} \qquad (30)$$

Comparing equations 45 and 30 gives,

$$\begin{array}{lll} m_{11} = 0 & m_{12} = -u_z & m_{13} = u_y \\ m_{21} = u_z & m_{22} = 0 & m_{23} = -u_z \\ m_{31} = -u_y & m_{32} = u_x & m_{33} = 0 \end{array}$$

Hence the cross product is given by

$$\hat{u} \times \mathbf{r}_1 = \begin{bmatrix} 0 & -u_z & u_y \\ u_x & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{1x} \\ \mathbf{r}_{1y} \\ \mathbf{r}_{1z} \end{bmatrix} \qquad (31)$$

Finally the dot product

$$[(\mathbf{r}_1.\hat{u})\hat{u} - \mathbf{r}_1] = \begin{bmatrix} u_x(\mathbf{r}_{1x}u_x + \mathbf{r}_{1y}u_y + \mathbf{r}_{1z}u_z) - \mathbf{r}_{1x} \\ u_y(\mathbf{r}_{1x}u_x + \mathbf{r}_{1y}u_y + \mathbf{r}_{1z}u_z) - \mathbf{r}_{1y} \\ u_z(\mathbf{r}_{1x}u_x + \mathbf{r}_{1y}u_y + \mathbf{r}_{1z}u_z) - \mathbf{r}_{1z} \end{bmatrix}$$

Now factoring out $\mathbf{r}_1$, we have,

$$= \begin{bmatrix} \mathbf{r}_{1x}u_x^2 + \mathbf{r}_{1y}u_xu_y + \mathbf{r}_{1z}u_xu_z \\ \mathbf{r}_{1x}u_xu_y + \mathbf{r}_{1y}u_y^2 + \mathbf{r}_{1z}u_yu_z - r_{1y} \\ \mathbf{r}_{1x}u_xu_z + \mathbf{r}_{1y}u_yu_z + \mathbf{r}_{1z}u_z^2 - r_{1z} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{r}_{1x}(u_x^2 - 1) + \mathbf{r}_{1y}u_xu_y + \mathbf{r}_{1z}u_xu_z \\ \mathbf{r}_{1x}u_xu_y + \mathbf{r}_{1y}(u_y^2 - 1) + \mathbf{r}_{1z}u_yu_z \\ \mathbf{r}_{1x}u_xu_z + \mathbf{r}_{1y}u_yu_z + \mathbf{r}_{1z}(u_z^2 - 1) \end{bmatrix}$$

This dot product can be found by matrix multiplication

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} \mathbf{r}_{1x} \\ \mathbf{r}_{1y} \\ \mathbf{r}_{1z} \end{bmatrix} = \begin{bmatrix} u_{11}\mathbf{r}_{1x} + u_{12}\mathbf{r}_{1y} + u_{13}\mathbf{r}_{1z} \\ u_{21}\mathbf{r}_{1x} + u_{22}\mathbf{r}_{1y} + u_{23}\mathbf{r}_{1z} \\ u_{31}\mathbf{r}_{1x} + u_{32}\mathbf{r}_{1y} + u_{33}\mathbf{r}_{1z} \end{bmatrix}$$

Similarly comparing this yields

$$\begin{bmatrix} u_{11} = (u_x^2 - 1) & u_{12} = u_xu_y & u_{13} \\ u_{21} = u_xu_y & u_{22} = (u_y^2 - 1) & u_{23} \\ u_{31} = u_xu_z & u_{32} = u_yu_z & u_{33} = (u_z^2 - 1) \end{bmatrix}$$

Hence dot product

$$[(\mathbf{r}_1.\hat{u})\hat{u} - r_1] = \begin{bmatrix} u_x^2 - 1 & u_xu_y & u_xu_z \\ u_xu_y & u_y^2 - 1 & u_yu_z \\ u_xu_z & u_yu_z & u_z^2 - 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{1x} \\ \mathbf{r}_{1y} \\ \mathbf{r}_{1z} \end{bmatrix}$$

Hence adding this together yields

$$\begin{Bmatrix} \mathbf{r}_{2x} \\ \mathbf{r}_{2y} \\ \mathbf{r}_{2z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{1x} \\ \mathbf{r}_{1y} \\ \mathbf{r}_{1z} \end{bmatrix} + \sin\theta_{12} \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{r}_{1x} \\ \mathbf{r}_{1y} \\ \mathbf{r}_{1z} \end{Bmatrix} + \text{vers}\theta_{12}[-] \begin{Bmatrix} \mathbf{r}_{1x} \\ \mathbf{r}_{1y} \\ \mathbf{r}_{1z} \end{Bmatrix}$$

where

$$[-] = \begin{bmatrix} u_x^2 - 1 & u_xu_y & u_xu_z \\ u_xu_y & u_y^2 - 1 & u_yu_z \\ u_xu_z & u_yu_z & u_z^2 - 1 \end{bmatrix}$$

That is,

$$\begin{Bmatrix} \mathbf{r}_{2x} \\ \mathbf{r}_{2y} \\ \mathbf{r}_{2z} \end{Bmatrix} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta_{12} \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} + \mathrm{vers}\theta_{12}[-] \right\} \begin{Bmatrix} \mathbf{r}_{1x} \\ \mathbf{r}_{1y} \\ \mathbf{r}_{1z} \end{Bmatrix}$$

Hence, the three matrices can be added together to yield the operator

$$[R_{12}(\hat{u},\theta_{12})] = \begin{bmatrix} u_x^2\mathrm{v}\theta_{12} + \mathrm{c}\theta_{12} & u_x u_y \mathrm{v}\theta_{12} - u_z\mathrm{s}\theta_{12} & u_x u_z\mathrm{v}\theta_{12} + u_y\mathrm{s}\theta_{12} \\ u_x u_y\mathrm{v}\theta_{12} + u_z\mathrm{s}\theta_{12} & u_y^2\mathrm{v}\theta_{12} + \mathrm{c}\theta_{12} & u_y u_z\mathrm{v}\theta_{12} - u_x\mathrm{s}\theta_{12} \\ u_x u_z\mathrm{v}\theta_{12} - u_y\mathrm{s}\theta_{12} & u_y u_z\mathrm{v}\theta_{12} + u_x\mathrm{s}\theta_{12} & u_z^2\mathrm{v}\theta_{12} + \mathrm{c}\theta_{12} \end{bmatrix} \tag{32}$$

Example 3.1

Using the spatial 3×3 matrix rotational operator Equation 32, find the operators representing the rotation about the x, y, and z-axes

Solution:

Rotation about x-axis. We have,

$$u_x = 1, \quad u_y = u_z = 0$$

The general rotational spatial matrix reduces to

$$R_{12}(\hat{u},\theta_{12}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - \mathrm{v}\theta_{12} & \mathrm{s}\theta_{12} \\ 0 & \mathrm{s}\theta_{12} & 1 - \mathrm{v}\theta_{12} \end{bmatrix}$$

But $\mathrm{v}\theta_{12} = 1 - \cos\theta_{12}$, therefore, $1 - v\theta_{12} = 1 - (1 - \cos\theta_{12}) = \cos\theta_{12}$

i.e

$$[R_{12}(\hat{u},\theta_{12})] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\theta_{12} & -\mathrm{s}\theta_{12} \\ 0 & \mathrm{s}\theta_{12} & \mathrm{c}\theta_{12} \end{bmatrix}$$

Note that this rotation about the x-axis all points move within a plane perpendicular to the x-axis, hence the x-coordinate remain unchanged such that the matrix operator is of the form

$$\{a'\} = [R(\theta_{12}, u_x)]\{a\}$$

$$\left\{\begin{array}{c} a'_x \\ a'_y \\ a'_z \end{array}\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\theta & -\mathrm{s}\theta \\ 0 & \mathrm{s}\theta & \mathrm{c}\theta \end{bmatrix} \left\{\begin{array}{c} a_x \\ a_y \\ a_z \end{array}\right\} = \begin{bmatrix} a_x \\ a_y \cos\theta - a_z \sin\theta \\ a_y \sin\theta + a_z \cos\theta \end{bmatrix}$$

Similarly for rotation about y-axis

$$u_x = u_z = 0 \quad u_y = 1$$

$$[R(\theta_{12}, u_y)] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Finally for rotation about z-axis (x-y plane rotation) we have

$$u_x = u_y = 0 \quad u_z = 1$$

$$[R(\theta_{12}, u_z)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3.2

Determine the combined rotational matrix for a rotation of a free vector in the fixed $Oxyz$ system by the following rotations in this order α about the z-axis, β about the y-axis, and γ about the x-axis

Solution:

$$\begin{aligned} R(\alpha, u_x) &= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ R(\beta, u_y) &= \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \\ R(\gamma, u_z) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \end{aligned}$$

concatenating these matrices in the given order results in

$$
\begin{aligned}
R(\theta,\hat{u}) &= [R(\gamma,u_z)][R(\beta,u_y)][R(\alpha,u_x)] \\
&= \begin{bmatrix} \mathrm{c}\beta\mathrm{c}\alpha & -\mathrm{c}\beta\mathrm{s}\alpha & \mathrm{s}\beta \\ \mathrm{c}\alpha\mathrm{s}\beta\mathrm{s}\gamma+\mathrm{c}\gamma\mathrm{s}\alpha & -\mathrm{s}\alpha\mathrm{s}\beta\mathrm{s}\gamma+\mathrm{c}\gamma\mathrm{c}\alpha & -\mathrm{s}\gamma\mathrm{c}\beta \\ -\mathrm{c}\alpha\mathrm{s}\beta\mathrm{c}\gamma+\mathrm{s}\gamma\mathrm{s}\alpha & \mathrm{s}\alpha\mathrm{s}\beta\mathrm{c}\gamma+\mathrm{s}\gamma\mathrm{c}\alpha & \mathrm{c}\gamma\mathrm{c}\beta \end{bmatrix}
\end{aligned}
$$

3.2.2 Assignment 1

1. Using the results of example 3.2, determine $\hat{\mathbf{u}}$ and θ for the specific case where $\alpha=\beta=\gamma=30^o$
2. Given three points on a planar rigid body ($\mathbf{A}=0.866+5i,\ \mathbf{B}=1+i,\ \mathbf{C}=1.5+i$) and its motion given by $\theta=90^o$, $\Delta X=1$, and $\Delta Y=0.5$. Find the final positions (A', B', C'). Make a step-by-step scale drawing of the transformation.
3. Points A, B and C are on rigid body that is rotated by 30^0 about an axis that passes through the origin pointing toward point D. If the coordinate of these points are: $A=[2,\ 2,\ 1]^T$, $B=[2,\ 2,\ 0]^T$ and $D=[2,\ 2,\ 2\sqrt{2}]^T$. Find the new coordinates of points A', B' and C'

3.2.3 Coordinate transformation

If $P_{xyz}=RP_{uvw}$ and $P_{uvw}=QP_{xyz}$, then $Q=R^{-1}=R^T$
Therefore: $QR=R^TR=R^{-1}.R=1$, but $R_AR_B\neq R_BR_A$

3.2.4 Composite rotation algorithm

Using composite rotations (multiple rotations), we can establish an arbitrary single orientation.

1. Initialize the rotation matrix to $R=1$, which corresponds to the orthonormal coordinate frame F(fixed coordinate frame) and M(mobile coordinate frame) being coincident.
2. If the mobile coordinate frame M is to be rotated by an amount φ about the k-th unit vector of the fixed coordinate frame F, then **premultiply** R by $R_k(\varphi)$

3. If the mobile coordinate frame M is to be rotated by an amount φ about its own k-th unit vector, then **postmultiply** R by $R_k(\varphi)$

4. If there are more fundamental rotations to be performed go to step 2, else stop. The resulting composite rotation matrix R maps mobile M coordinates into fixed F coordinates

3.3 Translation

Recall in planar systems, the motion included two translational degree of freedom. It is desired that a matrix operator operating on a vector would add the Δx and Δy translations to the coordinates of that vector.
This is accomplished by combining Δx and Δy with a 3×3 identity matrix and using homogeneous rotation to represent the vector. A spatial translation operator may be formed, using the same logic, by including the three translational degrees of freedom defined by $[D] = \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix}$ in a 4×4 identity matrix and using the 4×1 homogeneous rotation to represent the vector D. Thus the translation operator is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The homogeneous representation of a simple vector locating point A in 3-D space with respect to a x, y, z coordinate system is

$$r_A = \{r_A\} = \begin{Bmatrix} x_A \\ y_A \\ z_A \\ 1 \end{Bmatrix}$$

Example 3.3

A point P is to be translated by $\Delta x = -1$, $\Delta y = -2$ and $\Delta z = 2$. If the initial position is $P_1 = [1, 4, 2]^T$. Find P_2.

Solution:

$$\begin{bmatrix} P_{2x} \\ P_{2y} \\ P_{2z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} o \\ 2 \\ 4 \\ 1 \end{Bmatrix}$$

Hence, $P_2 = [0, 2, 4]$

3.4 Homogeneous coordinates

During coordinate transformations, instead of using both rotation matrix (3X3) and a translation vector (3X1), the homogeneous coordinates system seeks to use a single 4X4 matrix. This way the computation is simplified. For a variety of reasons, it is desirable to keep transformation matrices in square form, either 3X3 or 4X4. For instance, it is much easier to calculate the inverse of square matrices than rectangular matrices and in order to multiply two matrices, their dimensions must match, such that the number of columns of the first matrix must be the same as the number of rows of the second matrix. In four dimensional space of homogeneous coordinates, representing both position and rotation of the frame, the transformation is expressed as:

$$^{A}P = {}^{B}_{A}H^{B}P$$

Matrix H is called homogeneous transformation operator (matrix). A homogeneous transformation matrix is therefore a 4X4 matrix that maps a position vector in homogeneous coordinates from one coordinate system to another. The physical meaning of homogeneous transformations is depicted below.

Example 3.4

For the following Assembly workspace shown in figure 12, describe point P with respect to the base frame. For given (x_{4p}, y_{4p}, z_{4p}) with respect to frame $\{4\}$. *Note drawn to scale*

Solution:
To be solved in class.

3.5 The 4×4 Translation and Rotation Matrix for Axis Through the Origin

When a combination of 3×3 rotation matrix through the origin with a 4×4 translation matrix is required, we expand the 3×3 rotation matrix $R(u, \theta)$ to a 4×4 matrix by

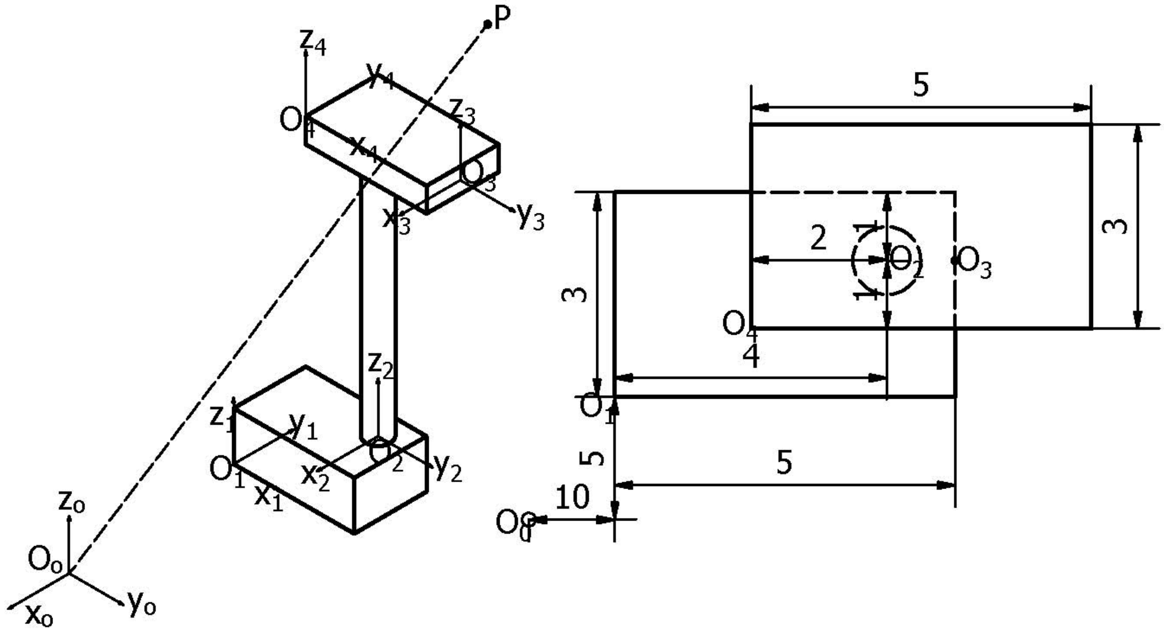


Figure 12:

adding a 4th row and a 4th identity column.
Finally this new 4×4 rotation matrix is premultiplied by 4×4 translation matrix to get a screw operator for an axis through the origin

$$[S_{12}] = [T][R(\theta, u)] = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \Delta x \\ r_{21} & r_{22} & r_{23} & \Delta y \\ r_{31} & r_{32} & r_{33} & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

The resulting screw matrix in Equation 33 can be expressed in symbolic form by partitioning the matrix

$$s_{12} = \left[\begin{array}{c|c} R(\theta, u) & D \\ \hline 0 & I \end{array}\right]$$

where $R(\theta, u)$, $[D]$ and $[I]$ carry the usual meaning.
Note $[D] = D\hat{u}$ or $\Delta x = |D|u_x$, $\Delta y = |D|u_y$ and $\Delta z = |D|u_z$.

Example 3.5

A particle P describes a 45^0 arc in space from an initial position at $(0, 1, 3)$ about an axis through the origin with unit vector $\hat{u} = [0,\ \cos 45,\ \cos 45]^T$. The particle is also translated parallel to this axis by $[D] = \left\{ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right\} = \sqrt{2}\hat{u}$. Find the rotation matrix $R(\theta, u)$ and the final position of the particle.

Solution:

$u_x = 0 \quad u_y = 0.707 \quad u_z = 0.707$ and $\theta_{12} = 45^o$. Then

$$[R(\hat{u}, \theta) = \begin{bmatrix} 0.707 & -0.500 & 0.500 \\ 0.500 & 0.854 & 0.146 \\ -0.500 & 0.146 & 0.854 \end{bmatrix}$$

The screw matrix hence

$$s_{12} = \begin{bmatrix} 0.707 & -0.500 & 0.500 & 0 \\ 0.500 & 0.854 & 0.146 & 1 \\ -0.500 & 0.146 & 0.854 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \{P_2\} &= [s_{12}]\{P_1\} \\ &= \begin{bmatrix} 0.707 & -0.500 & 0.500 & 0 \\ 0.500 & 0.854 & 0.146 & 1 \\ -0.500 & 0.146 & 0.854 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} 0 \\ 1 \\ 2 \\ 1 \end{array} \right\} \\ &= \begin{bmatrix} 0.5 \\ 2.146 \\ 2.854 \\ 1 \end{bmatrix} \end{aligned}$$

Hence, $[P] = [0.5, 2.146, 2.854]^T$

3.5.1 Screw displacement of a point through axis that does not pass through the origin

Let a point $P(x, y, z)$ be a point on the screw axis which does not pass through the origin. Let point Q_1 undergo a screw displacement to Q_2

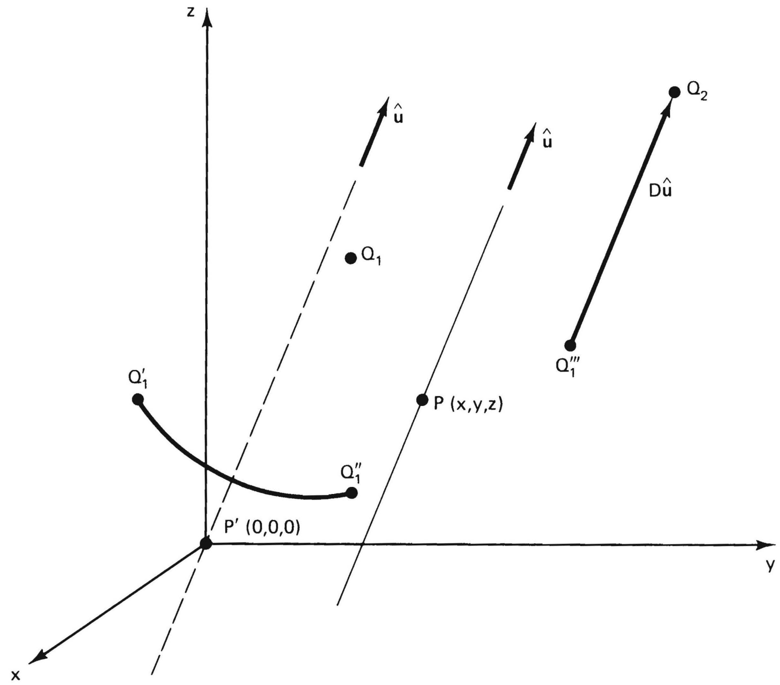


Figure 13: Screw displacement not through axis

To perform this transformation, we first translate P to the origin and obtain

$$[Q_1'] = \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix} [Q]$$

we then obtain Q_1'' from Q_1' by applying the rotation operator

$$[Q_1''] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [Q_1']$$

Next, we translate P back to its original position and obtain Q_1'''

$$[Q_1'''] = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} [Q_1'']$$

Finally, we translate parallel to the screw axis to obtain Q_2

$$[Q_2] = \begin{bmatrix} 1 & 0 & 0 & \Delta u_x \\ 0 & 1 & 0 & \Delta u_y \\ 0 & 0 & 1 & \Delta u_z \\ 0 & 0 & 0 & 1 \end{bmatrix} [Q_1''']$$

Concatenating these four matrices in their proper order results in the general screw displacement matrix with its axis passing through the point (x, y, z)

$$[s_{12}] = \begin{bmatrix} 1 & 0 & 0 & \Delta u_x \\ 0 & 1 & 0 & \Delta u_y \\ 0 & 0 & 1 & \Delta u_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[s_{12}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \Delta u_x + x - (r_{11}x + r_{12}y + r_{13}z) \\ r_{21} & r_{22} & r_{23} & \Delta u_y + y - (r_{21}x + r_{22}y + r_{23}z) \\ r_{31} & r_{32} & r_{33} & \Delta u_z + z - (r_{31}x + r_{32}y + r_{33}z) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3.6

A certain point Q_1 is translated to a point Q_2 by the screw matrix s_{12} given below

$$s_{12} = \begin{bmatrix} -0.748 & 0.263 & 0.612 & -3.17 \\ 0.608 & -0.093 & 0.788 & 5.99 \\ 0.263 & 0.963 & -0.092 & 4.94 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If this screw matrix is for a motion not through the origin find the $\hat{u}$, θ, the Δ and also the point P at which the screw matrix intersects the yz-plane

Solution:

As for the case of rotational matrix $[R(\theta, u)]$ the diagonal of the screw vector gives

$$\begin{aligned} \theta &= \arccos \frac{1}{2}(r_{11} + r_{22} + r_{33} - 1) \\ &= \arccos \frac{1}{2}(-0.784 - 0.093 - 0.092 - 1) \\ \theta &= 165^0 \end{aligned}$$

Also we have,

$$\begin{aligned}
u_x &= \frac{r_{32} - r_{23}}{2\sin\theta} = 0.333 \\
u_y &= \frac{r_{13} - r_{31}}{2\sin\theta} = 0.667 \\
u_z &= \frac{r_{21} - r_{12}}{2\sin\theta} = 0.667
\end{aligned}$$

To determine the point at which the screw axis intersects the yz plane, we set x equal to zero, and solve the three equations simultaneously for D, y and z

$$\begin{aligned}
\Delta(0.333) - (0.263)y - (0.612)z &= -3.17 \\
\Delta(0.667) + (1.093)y - (0.788)z &= 5.99 \\
\Delta(0.667) - (0.963)y + (1.092)z &= 4.96
\end{aligned}$$

The screw axis intersect the yz plane at $(0, 6, 6)$

$$\Delta = 6.25$$

★★★To fully understand the concept of rotating an object about an arbitrary axis, the following outline gives step by step procedure used to accomplish this transformation.
(Note the procedure is similar to the one explained above)

1. Translate the given axis so that it will pass through the origin.
2. Rotate the axis about x-axis (or y-axis) so that it will lie in the xz-plane (angle α).
3. Rotate the axis about the y-axis so that it will coincide with the z-axis (angle φ).
4. Rotate the geometric object about the z-axis (angle θ).
5. Reverse of step 3.
6. Reverse of step 2.
7. Reverse of step 1.

We will illustrate this procedure by the following example.

Example 3.6

Rotate the rectangle shown in figure 14, 30^0 ccw about the line EF and find the new coordinates of the rectangle

Solution:

We will make use of the seven-step procedure outlined above and write the applicable

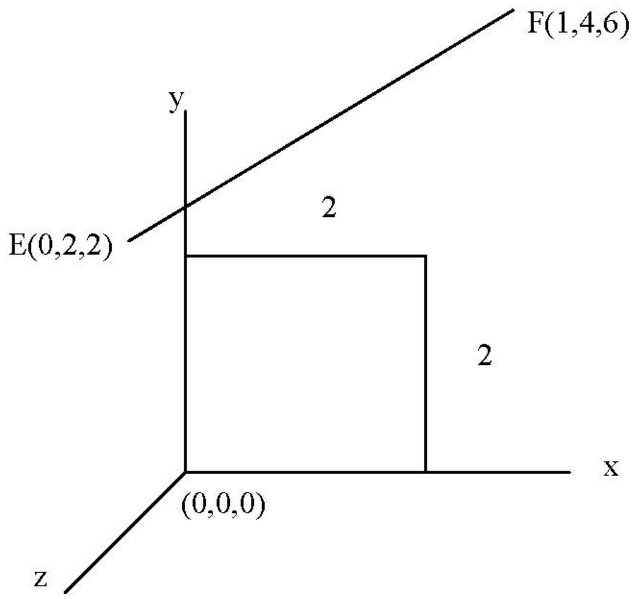


Figure 14: Example 3.6 Problem

transformation matrix in each step. After we have generated all the transformation matrices, we will solve for the new coordinates of the rectangle at the end of the 7$^{\text{th}}$ step.

- **Translate the given axis so that it will pass through the origin**
  Translation of the line EF to origin is given as,

$$[P^*]_1 = [P][T_t], \quad \text{where}[P] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [T_t] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix}$$

- **Rotate the axis so that it will lie in the xz-plane**
  The line EF is now rotated an angle α, about the x-axis so that it will lie in the xz-plane. The angle α is calculated with trigonometric relations, shown in the figure 15.

  (To view this, use the projection such that the vector will appear on the left wall)

$$\cos\alpha = \frac{c}{d} = \frac{c}{\sqrt{(b2 + c2)}} = 4/(4.4721) = 0.8944$$

$$\sin\alpha = \frac{b}{d} = 2/(4.4721) = 0.4472$$

$$\text{Now,} \quad [P^*]_2 = [P][T_t][T_r]_\alpha, \quad \text{where,}$$

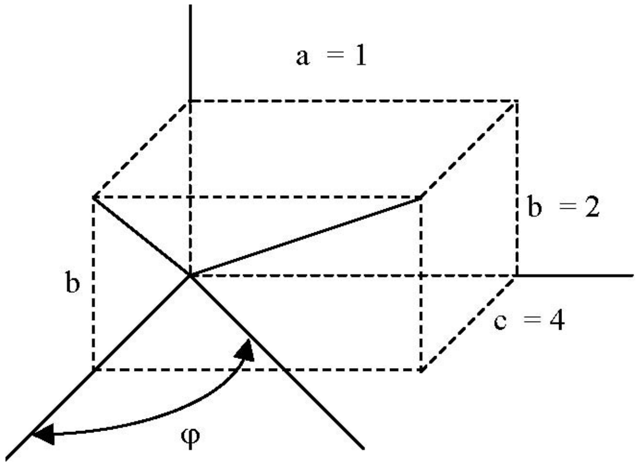


Figure 15: Example 3.6 Solution

$$[T_r]_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.8944 & 0.4472 & 0 \\ 0 & -0.4472 & 0.8944 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Rotate the line so that it will coincide with the z-axis**

  We will now rotate the line an angle φ about the y-axis so that it will coincide with the z-axis. The value of the angle φ is calculated from the trigonometry of the figure shown.

$$\sin\varphi = \frac{a}{L} = \frac{1}{\sqrt{(a^2 + b^2 + c^2)}} = \frac{1}{(4.5825)} = 0.2182$$

$$\cos\varphi = \frac{d}{L} = \frac{4.4721}{4.5825} = 0.9759$$

  Now the point matrix at this step is $[P^*]_3 = [P^*]_2[T_r]_\varphi$, and

$$[T_r]_\varphi = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.9759 & 0 & -0.2182 & 0 \\ 0 & 1 & 0 & 0 \\ 0.2182 & 0 & 09759 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Rotate the Geometric Object about the z-axis**

  Up to this point we have translated and rotated the rectangle so that its original position is changed and the line is coincident with the z-axis. To understand the effect of these steps, imagine that the rectangle and the line are frozen in space in a box. Now, move (translate) the line to the origin so that the new coordinates of the point E are $(0,0,0)$, rotate the box about the x-axis, so that the line EF lies in the xz-plane, finally, rotate the box about the y-axis so that it coincides with the z-axis. The noteworthy point in this analogy is that the transformation carried out in steps 1 through 3, affect both the coordinates of the line as well as that of the rectangle. Now we are ready to carry out the rotation of the rectangle about line EF. Since the axis of rotation is now coincident with the z-axis, we can apply the equation of rotation about the z-axis, defined earlier. Therefore,

$$[T_r]_\theta = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 & 0 & 0 \\ -0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Reverse of Step 3**
  In this step we will rotate the frozen box an angle $-\varphi$, about the y-axis. Since $\cos(-\varphi)$ =$\cos\varphi$, and $\sin(-\varphi) = -\sin(\varphi)$, the transformation matrix is,

$$[T_r]_{-\varphi} = \begin{bmatrix} \cos(-\varphi) & 0 & -\sin(-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\varphi) & 0 & \cos(-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.9759 & 0 & 0.2182 & 0 \\ 0 & 1 & 0 & 0 \\ -0.2182 & 0 & 0.9759 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- **Reverse of Step 2**
  Rotate the box an angle $-\alpha$ about the x-axis. The transformation matrix is,

$$[T_r]_{-\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\alpha) & \sin(-\alpha) & 0 \\ 0 & -\sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.8944 & -0.4472 & 0 \\ 0 & 0.4472 & 0.8944 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Reverse of Step 1**
  In this final step, we will translate the box so that the corner E will move back to its original coordinates (0, 2, 2). The transformation matrix is,

$$[T_{-t}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

This completes all the seven steps that are necessary to rotate the rectangle about the line EF. The new coordinates of the rectangle are given by the equation,

$$[P^*] = [P][T_t][T_r]_\alpha[T_r]_\varphi[T_r]_\theta[T_r]_{-\varphi}[T_r]_{-\alpha}[T_{-t}]$$

The concatenated transformation matrix is,

$$[T]_c = [T_t][T_r]_\alpha[T_r]_\varphi[T_r]_\theta[T_r]_{-\varphi}[T_r]_{-\alpha}[T_{-t}] = \begin{bmatrix} 0.9312 & 0.1634 & -0.3256 & 0 \\ -0.1743 & 0.9846 & -0.0044 & 0 \\ 0.3199 & 0.0609 & 0.9454 & 0 \\ -0.2913 & -0.0909 & 0.1179 & 1 \end{bmatrix}$$

and

$$\begin{aligned} [P^*] = [P][T]_c &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9312 & 0.1634 & -0.3256 & 0 \\ -0.1743 & 0.9846 & -0.0044 & 0 \\ 0.3199 & 0.0609 & 0.9454 & 0 \\ -0.2913 & -0.0909 & 0.1179 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.2913 & -0.0909 & 0.1179 & 1 \\ 1.5712 & 0.2359 & -0.5334 & 1 \\ 1.2226 & 2.2051 & -0.5421 & 1 \\ -0.6399 & 1.8783 & 0.1092 & 1 \end{bmatrix} \end{aligned}$$

3.6 Practical-Exercise 1 (Simulation)

The following dimensions are given for the inverted slider-crank mechanism (shown in Fig. 16): AC=0.15 m and BC=0.2 m. The length AD is selected as 0.35 m ($AD = AC + BC$). The driver link 1 rotates with a constant speed of $n = n_1 = 30$ rpm. Write MATLAB program to find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 60^o$.

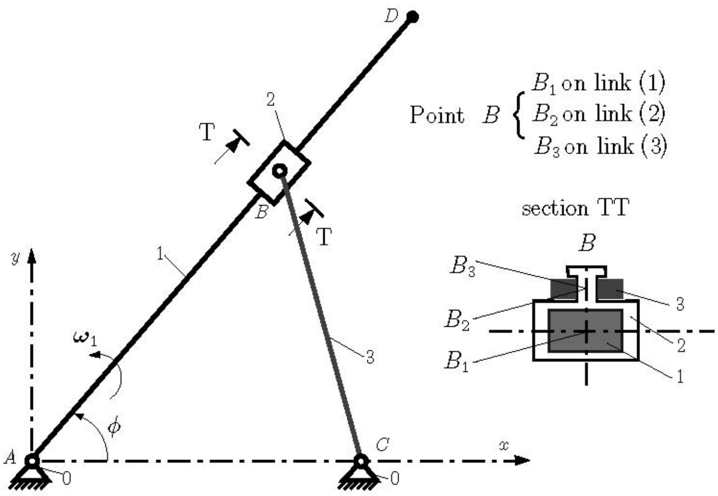


Figure 16: Practical Exercise Problem

4 Analysis of Spatial Mechanisms

The 4×4 transformation matrices introduced in previous sections are adopted to model spatial mechanisms for the purposes of analysis. The analysis begin by establishing a local cartesian coordinate system for each link. The location and orientation of the local coordinate system on each link is arbitrary; however, it is usually chosen at a joint center. The modeling task then becomes one of describing how the coordinate systems of connected links move relative to one another. Consequently, transformation matrices must be defined describe to relative motions.

For instance, let point P be attached to link 2 (see Fig. 17) and its location be defined in the local coordinate system x_2, y_2, z_2 embedded in the moving link 2, by the vector $\mathbf{P}^2$. To find the location of P with respect to the global coordinate system x_1, y_1, z_1 embedded in the fixed link 1, namely $\mathbf{P}^1$, after link 2 has been rotated by the angle θ_z, we must account for the fact that system x_2, y_2, z_2 has been obtained from x_1, y_1, z_1 by combined rotation and translation. In this section necessary transformation matrices for various spatial link shapes and joint motion are derived to accomplish the transformation as follows

$$\mathbf{P}^1 = T_{12}\mathbf{P}^2$$

The matrix operator T_{12} is constructed to obtain $\mathbf{P}^1$ from $\mathbf{P}^2$. $\mathbf{P}^2$ is defined in the x_2, y_2 coordinate system. Also note x_2, y_2 coordinate system is obtained from x_1, y_1 by rotation through the angle θ_z about z_1. Therefore, the argument of $\mathbf{P}^2$ expressed in the x'_1, y'_1 system is increased by θ_2 from its argument in the x_2, y_2 system. Hence $(\mathbf{P}^2)' = [R(\theta_2)]\mathbf{P}^2$

Note that x'_1, y'_1 came from x_1, y_1 by a translation L_{1x}. Thus, the x coordinate of any vector in the x'_1, y'_1 system is increased by L_{1x} when the vector is expressed in the x_1, y_1

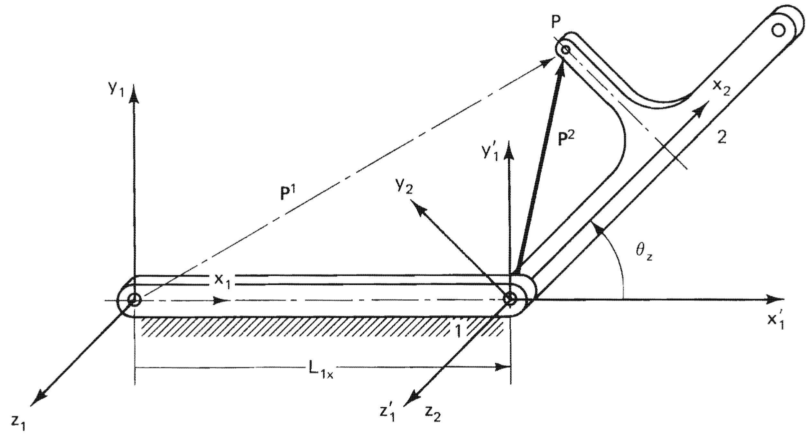


Figure 17:

system. As a result,

$$\mathbf{P}' = [T(L_{1x}](\mathbf{P}^2) = [T(L_{1x}][R(\theta_z)]\mathbf{P}^2$$

So we see that the operator T_{12} is obtained by the concatenation of a rotation and a translation matrix in that order.

$$T_{12} = [T(L_{1x})][R(\theta_z)]$$

Elementary 4×4 Transformation Matrices

There are seven elementary forms of 4×4 transformation matrices which when used in suitable combination, describes most joints found in mechanisms (spatial or planar). They can also be used to describe complex link shapes between the joints
The seven matrices are shown schematically in Figure 18. These matrices are intended for transforming a position vector $\mathbf{P}^j$ of point P embedded in the O_j, x_j, y_j, z_j coordinate system to the vector $\mathbf{P}^i$ locating the same point P in the O_i, x_i, y_i, z_i coordinate system.

Case 1: System x_j, y_j, z_j is obtained from x_i, y_i, z_i by translation$D = [a, b, c,]^T$ followed by rotation about its translated origin at (a, b, c) by the angle θ_z around $\hat{u} = [u_x, 0, 0]^T$. Thus for this case, the transformation to obtain $\mathbf{P}^i$ from $\mathbf{P}^j$ is: $\mathbf{P}^i = T_{tj}\mathbf{P}^j$ where,

$$T_{ij}(a,\, b,\, c,\, \theta_x) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & \cos\theta_x & -\sin\theta_x & b \\ 0 & \sin\theta_x & \cos\theta_x & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

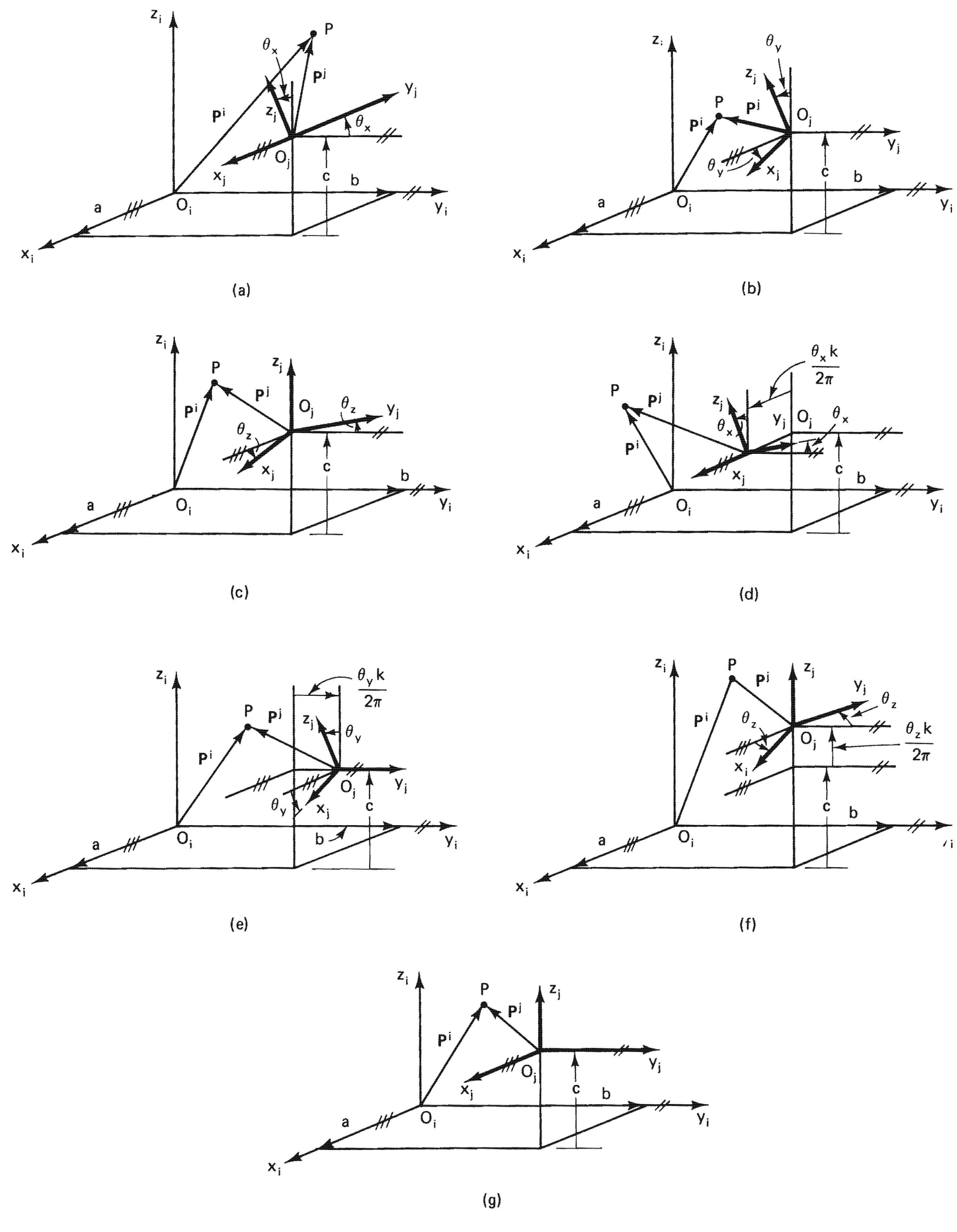


Figure 18:

Case 2: Similar to case 1, except the rotation is θ_y. Here $\mathbf{P}^i = \mathbf{T}_{ij}^2\mathbf{P}^j$, where

$$\mathbf{T}_{ij}^2(a,b,c,\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & a \\ 0 & 1 & 0 & b \\ -\sin\theta_y & 0 & \cos\theta_y & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Case 3: is also similar to case 1 except for rotation about the z axis. Again, $\mathbf{P}^i = \mathbf{T}_{ij}^3\mathbf{P}^j$,

where

$$\mathbf{T}^3_{ij}(a,b,c,\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & a \\ \sin\theta_z & \cos\theta_z & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Case 4: The system x_j, y_j, z_j is obtained from x_i, y_i, z_i by translation $D = [a,b,c]^T$ followed by a screw motion in the x direction. Now $\mathbf{P}^i = \mathbf{T}^4_{ij}\mathbf{P}^j$, where

$$\mathbf{T}^4_{ij}(a,b,c,\theta_x,k) = \begin{bmatrix} 1 & 0 & 0 & a + \frac{\theta_z k}{2\pi} \\ 0 & \cos\theta_x & -\sin\theta_x & b \\ 0 & \sin\theta_x & \cos\theta_x & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and where k is the lead of the screw.

Case 5: is similar to case 4, except that the screw motion of the coordinate system takes place in the y direction. Here $\mathbf{P}^i = \mathbf{T}^5_{ij}\mathbf{P}^j$, where

$$\mathbf{T}^5_{ij}(a,b,c,\theta_y,k) = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & a \\ 0 & 1 & 0 & b + \frac{\theta_y k}{2\pi} \\ -\sin\theta_y & 0 & \cos\theta_y & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Case 6: applies when the screw motion in the z direction of the coordinate system follows its a, b, c translation. Thus $\mathbf{P}^i = \mathbf{T}^6 ij \mathbf{P}^j$

$$\mathbf{T}^6_{ij}(a,b,c,\theta_z,k) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & a \\ \sin\theta_z & \cos\theta_z & 0 & b \\ 0 & 0 & 1 & c + \frac{\theta_z k}{2\pi} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Case 7: applies when the system x_j, y_j, z_j is obtained from x_i, y_i, z_i by translation $D = [a,b,c]^T$ only. For this case $\mathbf{P}^i = \mathbf{T}^7_{ij}\mathbf{P}^j$ where

$$T^7_{ij}(a,b,c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is important to note that the translations and rotations in these matrices describe relative motion of a coordinate system moving from coincidence with the i^{th} system to coincidence with the j^{th} system.

5 Link and Joint Modeling with Elementary Matrices

Some illustrative examples demonstrating the use of the elementary matrices for modeling links and joints of mechanisms are shown in this section.

1. **Revolute joint**
   Perhaps the most common joint is the revolute. Figure 19 shows an example of two connected links numbered i and j. Before the joint can be modeled, local coordinate systems on the two links must be defined. In the figure these coordinate systems are denoted as (x_ℓ, y_ℓ, z_ℓ) and (x_m, y_m, z_m).

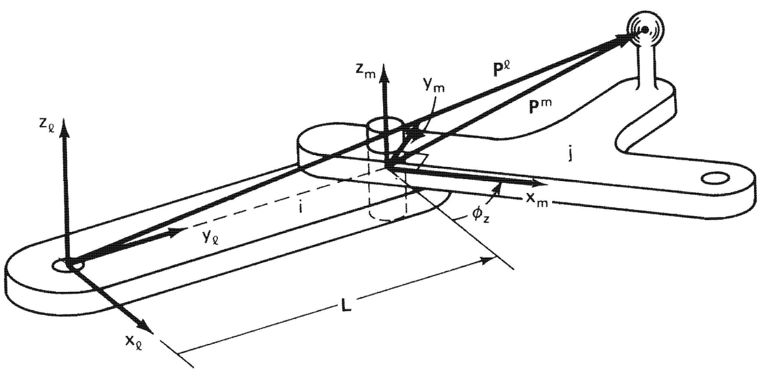


Figure 19: Revolute Joint

Link i and the revolute joint ij in figure 19 is described using a type 3 (Case 3) elementary matrix, since the rotation is about a z axis. Prior to the rotation of this joint, there is a translation in the y_ℓ direction of length L. Therefore, the description of the ith link with the ij joint is

$$s_{\ell,m}(\phi_z) = T^3_{\ell,m}(0, L, 0, \phi_z) = \begin{bmatrix} \cos\phi_z & -\sin\phi_z & 0 & 0 \\ \sin\phi_z & \cos\phi_z & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The symbol $s_{\ell,m}(\phi_z)$ is used to represent a general 4×4 transformation matrix, the quantities in parenthesis following the symbol s denote the relative motion variables in the matrix. Suppose we know the location of point P of link j with respect to coordinate system x_m, y_m, z_m embedded in link j. Let this position vector be denoted as $\mathbf{P}^m$. Now if we want to know the position of P with respect to the x_ℓ, y_ℓ, z_ℓ system of link i, namely P^ℓ, we obtain it by

$$P^\ell = s_{\ell,m}(\phi_z)P^m \tag{34}$$

This notion indicates that, while P^m is a constant (because it is embedded in the rigid link j). P^ℓ is a function of ϕ_z and therefore changes as link j rotates relative to link i.

2. **Cylindrical joint**
   Figure 20 shows a cylindrical joints between links m and n. To describe link m and this joint, two elementary matrices are required. One matrix describes the translation along link m and the translation along the axis of the joint. A second matrix covers the rotation of the joint. Thus the link m and the cylindrical joint between links m and 11 shown in Fig. 20 are defined as follows:

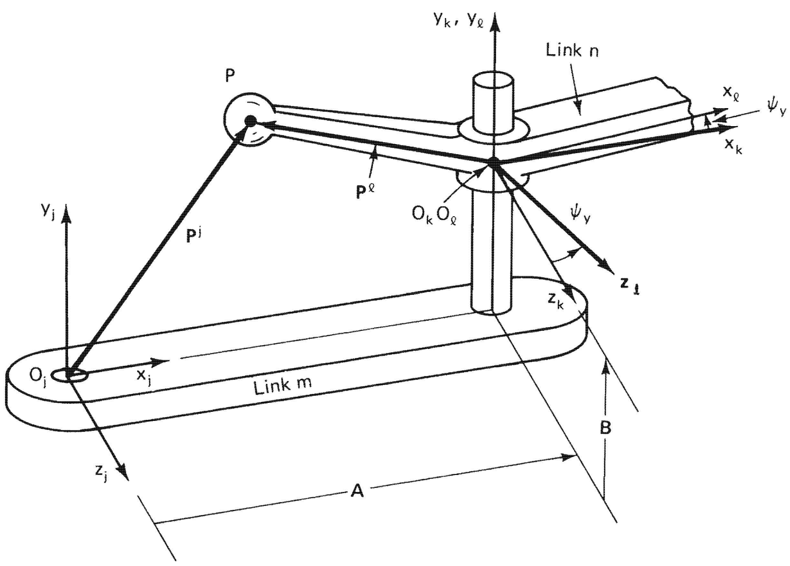


Figure 20: Cylindrical Joint

$$\begin{aligned}
s_{j,\ell}(B,\psi_y) &= T^7_{jk}(A,B,0)T^2_{k,\ell}(0,0,0,\psi_y) \\
&= \begin{bmatrix} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi_y & 0 & \sin\psi_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\psi_y & 0 & \cos\psi_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$s_{j,\ell}(B,\psi_y) = \begin{bmatrix} \cos\psi_y & 0 & \sin\psi_y & A \\ 0 & 1 & 0 & B \\ -\sin\psi_y & 0 & \cos\psi_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

With this,

$$P^j = s_{j,\ell}(B, \psi_y)P^\ell \tag{36}$$

Note that for this case B and ψ_y represent the degrees of relative freedom in this joint.

3. **Screw joint** The screw axis is in the z_j direction so case 6 matrix, describes this joint. This matrix is written as

$$S_{ij}(\theta_z) = T_{ij}^6(A, B, \theta_z, k) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & A \\ \sin\theta_z & \cos\theta_z & 0 & B \\ 0 & 0 & 1 & C + \frac{k\theta_z}{2\pi} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and in figure 21,

$$P^i = S_{ij}(\theta_z)P^j$$

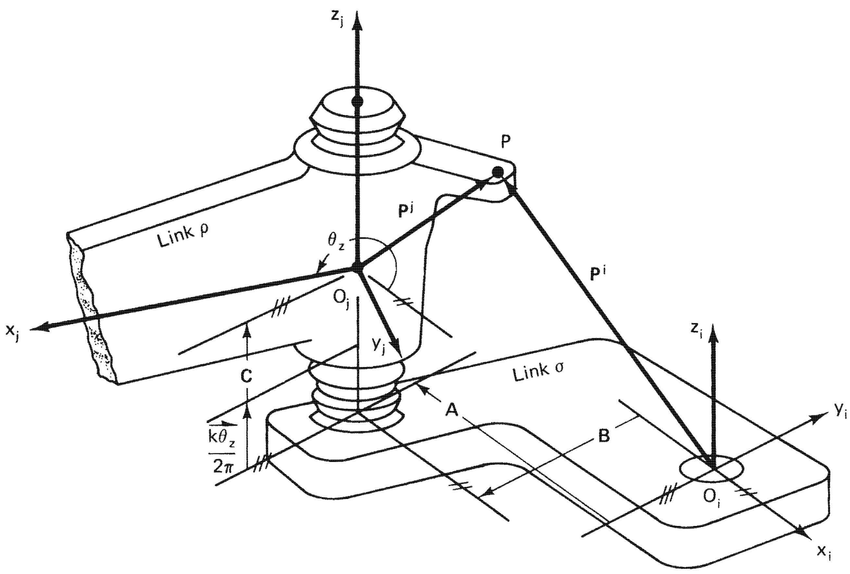


Figure 21: Screw Joint

Note that for this joint two of the constant translations are in the negative directions of x_i and y_i; consequently, A and B would have negative values. Also, the screw joint has rotated by a little less than 270[?]. This leaves the x_j axis pointing in nearly the negative direction of the y_i axis and the rotation has caused a translation in the direction of the screw of $k\theta_z/2\pi$. Thus, if this joint were reverse-rotated from its current position by the amount θ_z, the x_j, y_j, z_j coordinate system would be parallel to x_i, y_i, z_i and its origin would be a distant C above the x_i, y_i plane. The single degree of relative freedom in this joint is θ, because k, the lead of the screw, is constant. For a right-hand screw, k is positive; for a left-hand screw, k is negative.

4. **Spherical joint** A common joint appearing in many spatial mechanisms is the spherical joint. This joint allows freedom of rotation around all three axes of one link relative to another. Typically, these rotations are described using *Euler angles* For the purpose of discussion, the spherical joint is represented by its kinematic equivalent of 3 rotational joints whose axes of rotation intersects at the center of spherical joint. The transformation matrices defining this joints are;

$$S_{i\ell} = T^3_{ij}(0, L_i, 0, \theta_{zj})T^1_{jk}(0, 0, 0, \psi_{xj})T^3_{k\ell}(0, 0, 0, \phi_{zk})$$

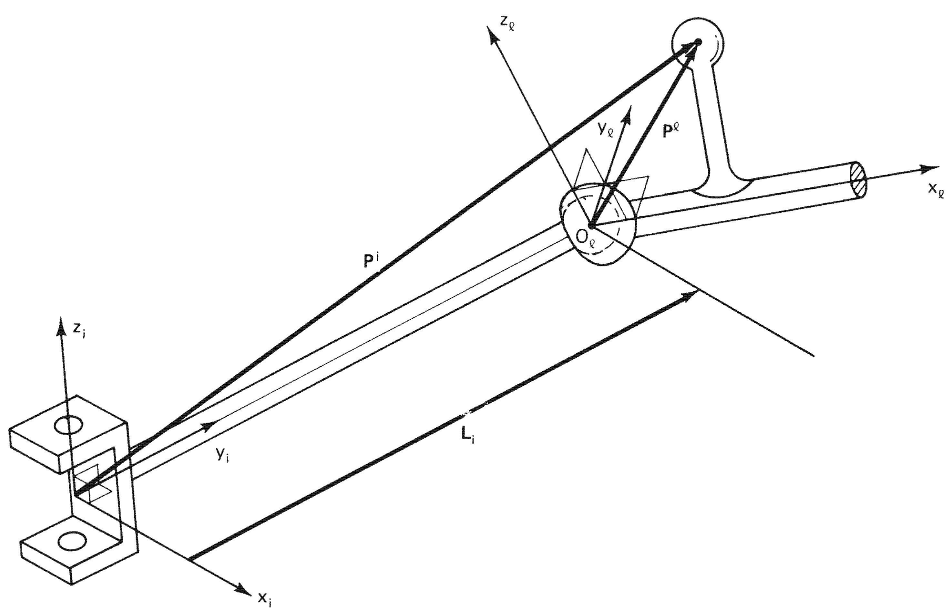


Figure 22: Spherical Joint

The first matrix defines the translation of the coordinate system along the y_i axis by distance L_i followed by a rotation θ_{zj} about an axis parallel to the z_i axis. This defines the $x_jy_jz_j$ coordinate system. The second matrix defines a rotation of the jth system about the x_j axis by the angle ψ_{xj},which locates the $x_ky_kz_k$ coordinate system. The final matrix represents a rotation of the kth system about z_k by the angle ϕ_{zk}, which defines the final $x_\ell y_\ell z_\ell$ coordinate system. Upon concatenation, the following operator is found.

$$S_{i\ell}(0, L_i, 0, \theta_{zj}, \psi_{xj}, \phi_{zk}) =$$
$$\begin{bmatrix} c\theta_{zj}c\phi_{zk} - s\theta_{zj}c\psi_{xj}s\phi_{zk} & -c\theta_{zj}s\phi_{zk} - s\theta_{zj}c\psi_{xj}c\phi_{zk} & s\theta_{zj}s\psi_{xj} & 0 \\ s\theta_{zj}c\phi_{zk} + c\theta_{zj}c\psi_{xj}s\phi_{zk} & -s\theta_{zj}s\phi_{zk} + c\theta_{zj}c\psi_{xj}c\phi_{zk} & -c\theta_{zj}s\psi_{xj} & L_i \\ s\psi_{xj}s\phi_{zk} & s\psi_{xj}c\phi_{zk} & c\psi_{xj} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With this

$$P^i = S_{i\ell}(\theta_{zj}, \psi_{xj}, \phi_{zk})P^\ell$$

5.0.1 Tutorial Questions

1. If in figure 23, $\mathbf{A}_1 = 0.707 + 0.707\mathbf{i}$

   (a) Find $\mathbf{A}_2$, if $\theta_{12} = 165^o$. Use the matrix rotation operator method.

   (b) What is θ_1?

   (c) What is θ_2?

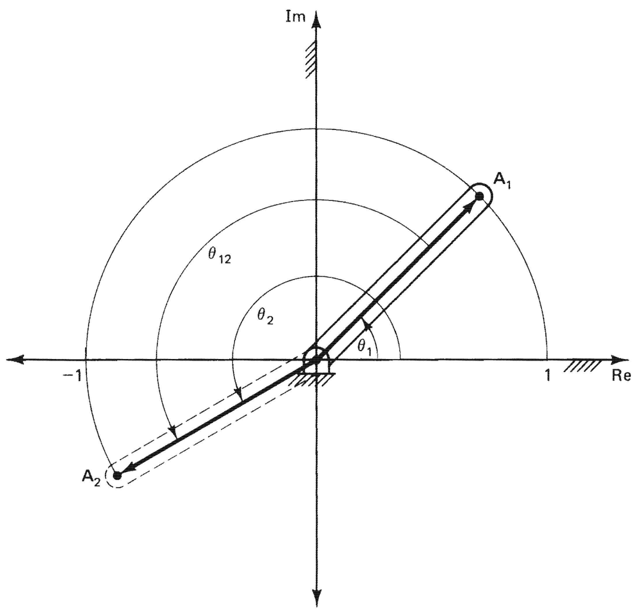


Figure 23:

2. (a) Determine the transformation operator for the screw joint mechanism in Fig. 24, which will transform a vector $\mathbf{P}^j$ expressed in the jth moving system to the vector $\mathbf{P}^i$ expressed in the ith "fixed" system.

   (b) Find $\mathbf{P}^i$ for the screw joint mechanism of Fig. 24 if $\mathbf{P}^j = (0, 2, 0)^T$, $\theta_{zj} = 180^o$, $A = 2$, $B = -3$, $C = 2$, and the lead $k = 0.125$.

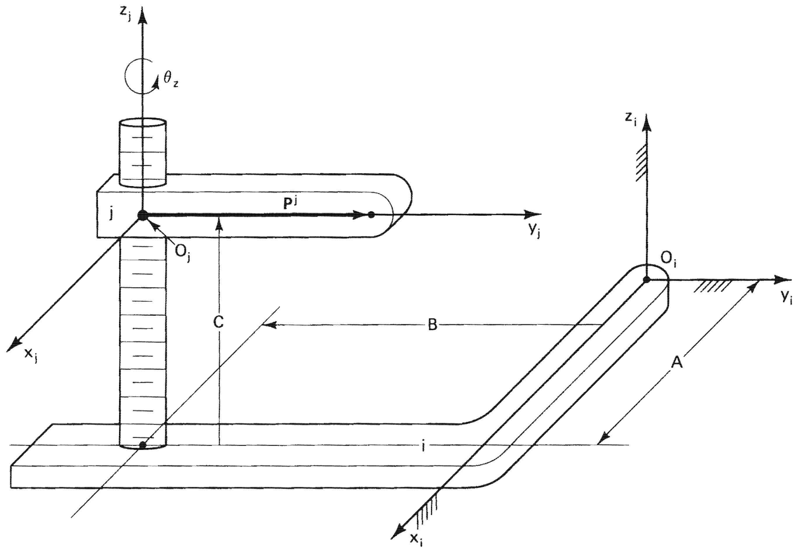


Figure 24:

3. A frame {B} is described as initially coincident with frame {A}. Then frame {B} is rotated about vector ${}^{A}k$ = [0.707, 0707, 0.0], passing through the point ${}^{A}P = [1.0, 2.0, 3.0]$ by an amount $\theta = 30^o$. Give the frame description of {B}. *Superscript* ^{A} *refers to frame* {A}

6 Vector Algebra

Vector: Vector quantity has both magnitude and direction eg. Velocity, acceleration, force
Scalar: Scalar quantity has magnitude only, eg Mass, volume, work.

6.1 Vector Sum

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C$$

$$A - B = A + (-B)$$

6.2 Dot Product (Scalar)

$A.B + |A||B|\cos\theta$ where $|A|$ represents the magnitude of a vector A

$$A.(B + C) = A.B + A.C$$

$$A.B = B.A$$

If $A.B = 0$ either one of the vectors A, B is zero or A and B are perpendicular to each other.

6.3 Cross Product (vector)

$A \times B$, where the magnitude $=|A||B|\sin\theta$ and direction given by the right hand rule.

$$A \times (B + C) = A \times B + A \times C \tag{37}$$

$$A \times B = -B \times A \tag{38}$$

If $A \times B = 0$, either one of the vectors A and B is zero or A and B are parallel.

6.4 Unit Vectors: i, j, k

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

If $A = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $B = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, then

- $A + B = (a_x + b_x)\mathbf{i} + (a_x + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$
- $A \times B = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
- $A.B = a_x b_x + a_y b_y + a_z b_z$

Note

- $A.(A \times B) = B.(A \times B) = 0$
- $A.(B \times C) = B.(C \times A) = C.(A \times B) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$
- $A \times (B \times C) = (A.C)B - (A.B)C$
- $(A \times B) \times C = (A.C)B - (B.C)A$

6.5 Vector Differentiation

For vector function of time $A(t)$, $B(t)$, $C(t)$, and scalar function of time $m(t)$

- $\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t}$
- $\frac{d}{dt}(A + B) = \frac{dA}{dt} + \frac{dB}{dt}$
- $\frac{d}{dt}(A \times B) = \frac{dA}{dt} \times B + A \times \frac{dB}{dt}$
- $\frac{d}{dt}(A.B) = \frac{dA}{dt}.B + A.\frac{dB}{dt}$
- $\frac{d}{dt}(mA) = \frac{dm}{dt}A + m\frac{dA}{dt}$
- $\frac{d}{dt}[A \times (B \times C)] = \frac{dA}{dt} \times (B \times C) + A \times \left(\frac{dB}{dt} \times C\right) + A \times \left(B \times \frac{dC}{dt}\right)$

7 Vector Analysis of Spatial Mechanisms

7.1 Motion in the Stationary Coordinate System i.e $\frac{d}{dt}(\mathbf{i}, \mathbf{j}, \mathbf{k}) = 0$

Consider any point P in space relative to a fixed point O as shown in figure 25. To reach P we must use a straight line $\mathbf{r}$ directed from O to P. This is the position vector. A vector may be expressed conveniently in terms of its components and the three unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, whose directions are those of the axes of the cartesian frame of reference Ox, Oy, and Oz.

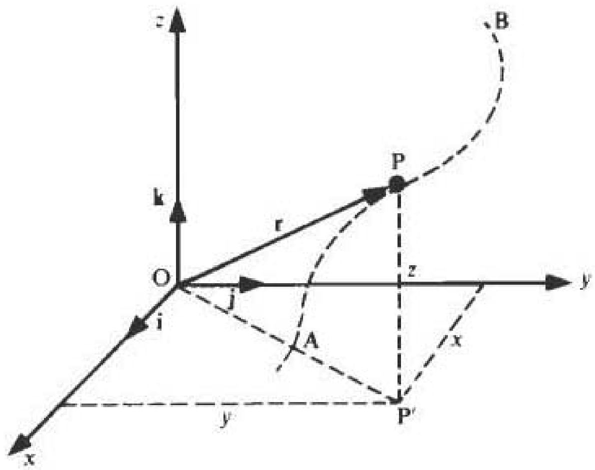


Figure 25: Position vector of a point in Cartesian coordinate

If x, y and z are the directed lengths of the projections of the vector $\mathbf{r}$ on these axes, we then have the following vector equation:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{39}$$

Furthermore, if the particle is moving along the space curve AB then x, y, and z must be functions of time, i.e

$$x = x(t) \quad y = y(t) \quad z = z(t)$$

Hence, the vector equation for the position of the particle at any instant may be written as:

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{40}$$

The magnitude of the position vector $\mathbf{r}$, denoted by $|\mathbf{r}|$ or simply r, is

$$r = \sqrt{x^2 + y^2 + z^2}$$

The velocity of the particle is obtained by differentiating Eq. 40 with respect to time:

$$\dot{\mathbf{r}} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}$$

$\dot{\mathbf{i}} = \dot{\mathbf{j}} = \dot{\mathbf{k}} = 0$, since the axes are Cartesian axes, fixed in space and of constant magnitude. Similarly, the acceleration of the particle is

$$\ddot{\mathbf{r}} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k}$$

7.2 Motion of a Rigid Body about a Fixed Axis (without translation)

A point, which is fixed in the body rotating with ω about an axis that is fixed in a stationary coordinate system is described by a vector R.
The velocity of P is

$$\begin{aligned}\dot{R} = v_P &= \omega \times R = (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times (R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}) \\ &= (\omega_y R_z - \omega_z R_y)\mathbf{i} + (\omega_z R_x - \omega_x R_z)\mathbf{j} + (\omega_x R_y - \omega_y R_x)\mathbf{k}\end{aligned}$$

Direction: $(thumb \times Index) = third \quad finger$
The acceleration of P is,

$$a = v_P = \dot{\omega} \times R + \omega \times \dot{R} = \alpha \times R + \omega \times (\omega \times R) = a^t + a^n$$

7.3 Moving Coordinate Systems

The velocity of point P

$$v_P = \dot{R} = \dot{R}_o + \dot{r} = \dot{R}_{ox}\mathbf{I} + \dot{R}_{oy}\mathbf{J} + \dot{R}_{oz}\mathbf{K} + \dot{r}_x\mathbf{i} + \dot{r}_y\mathbf{j} + \dot{r}_z\mathbf{k} + \left(\frac{di}{dt}\right) r_x + \left(\frac{dj}{dt}\right) r_y + \left(\frac{dk}{dt}\right) r_z$$

By using $R = \omega \times R \Rightarrow (di/dt) = \omega \times i$

$$v_P = \dot{R} = \dot{R}_o + \dot{r}_r + \omega \times r$$

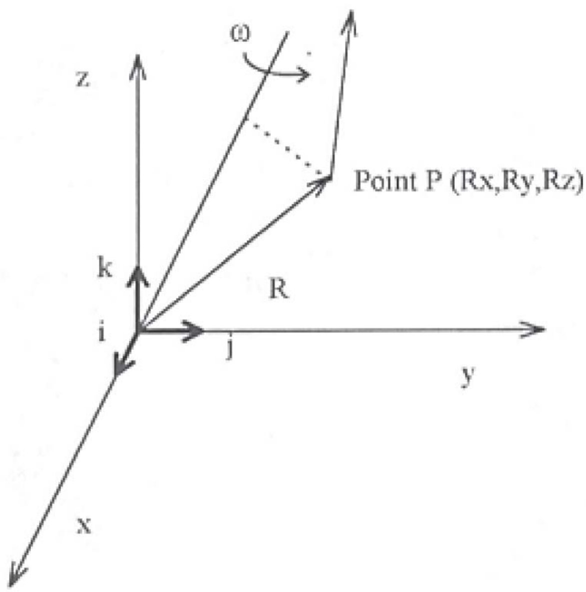
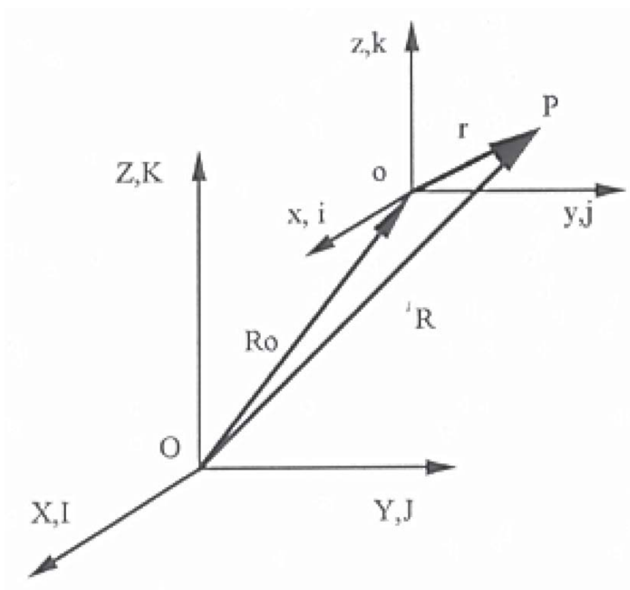


Figure 26:

Where:

- ω is the angular velocity of the xyz coordinate system,
- r is the position vector of P in xyz'
- $\dot{R}$ is the absolute velocity of P in XYZ,
- $\dot{R}_o$ is the velocity of the origin o of xyz, and
- $\dot{r}_r$ is the velocity of point P relative to xyz.



The acceleration of point P

$$a_P = \ddot{R} = \ddot{R}_o + \alpha \times r + \omega \times (\omega \times r) + \ddot{r}_r + 2\omega \times \dot{r}_r$$

$$= \ddot{R}_o + \ddot{r}_r + a^t + a^n + a^c$$

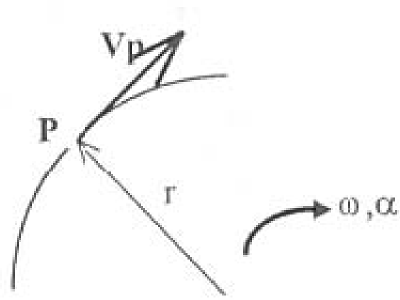
Where:

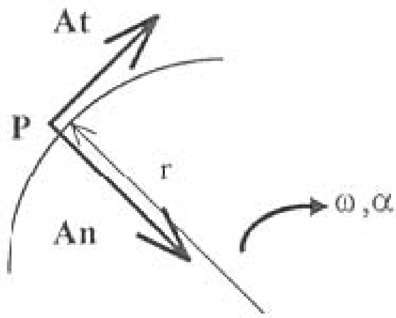
- ω and α are the angular velocity and angular acceleration of the moving system ($oxyz$) in the $OXYZ$ system
- r is the position vector of P in the $oxyz$ system,
- $\dot{r}_r$ is the velocity of point P relative to the $oxyz$ system

Note
If the $oxyz$ system has no translation relative to the $OXYZ$ system ($\dot{R}_o = 0$), the point P is fixed in the body whose coordinate system is $oxyz$ ($\dot{r}_r = 0$), and the body rotates with ω and α in the $OXYZ$ system, the velocity and acceleration of point P become,

$$v_P = \dot{R} = \omega \times r \quad \text{and} \quad a_P = \ddot{R} = \alpha \times r + \omega \times (\omega \times r)$$

For planar motion $\omega = \omega\mathbf{k}$, $\alpha = \alpha\mathbf{k}$, $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j}$ Normal acceleration: $a^n = v^2/R =$





$R\omega^2 = v\omega$ (Direction- Toward the center of curvature)
Tangential acceleration: $a^t = R\alpha$ (Direction- Tangential to the path)
Coriolis acceleration: $a^c = 2v_r\omega$

Example 7.1

Two sliders A and B are constrained to move in slots at right angles to each other, as shown in Fig. 27, and are connected by the rigid link AB of length $450mm$. At the instant when $\theta = 30^o$ slider A is moving with a velocity of $0.6m/s$ and acceleration of $1.2m/s^2$ in the direction shown. Calculate the velocity and acceleration of the slider B at that instant and the angular velocity and acceleration of the link AB

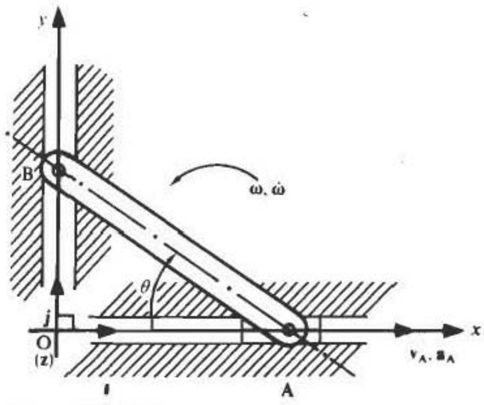


Figure 27: Example 6.1

Solution 7.1

$$r_{B/A} = -0.45\cos 30i + 0.45\sin 30j = -0.39i + 0.225j$$
$$v_B = -v_B j \quad v_A = 0.6i \quad \omega = \omega k$$

$$v_B = v_A + \omega \times r_{B/A}$$
$$-v_B j = 0.6i + \omega k \times (-0.39i + 0.225j)$$
$$= 0.6i - 0.39\omega j - 0.225\omega i$$

Comparing like terms

$$0.6 - 0.225\omega = 0 \tag{41}$$

$$0.39\omega - v_B = 0 \tag{42}$$

From Eqn. 41

$$\omega = \frac{0.6}{0.225} = 2.67$$
$$\therefore \omega = (2.67k)rad/s$$

And from Eqn. 42

$$v_B = 0.39 \times 2.67 = 1.04$$
$$v_B = (-1.04j)m/s$$

To obtain acceleration

$$a_B = a_A + \alpha \times r_{B/A} + \Omega \times (\Omega \times r_{B/A})$$
$$a_B = -a_B j \quad a_A = 1.2i \quad \alpha = \alpha k$$

$$\begin{aligned}
-a_B j &= 1.2i + \alpha k \times (-0.39i + 0.225j) + 2.67k \times [2.67k \times (-0.39i + 0.225j)] \\
&= 1.2i + -0.39\alpha j - 0.225\alpha i + 2.67k \times [-1.04j - 0.6i] \\
&= 1.2i + -0.39\alpha j - 0.225\alpha i + 2.78i - 1.6j \\
&= (3.98 - 0.225\alpha)i - (1.6 + 0.39\alpha)j + 2.67k
\end{aligned}$$

Comparing the like terms, i.e. i, j and k

$$3.98 - 0.225\alpha = 0 \tag{43}$$

$$1.6 - 0.39\alpha = -a_B \tag{44}$$

From Eqn. 43

$$\alpha = 17.69 rad/s^2 \quad (\alpha = 17.69k) rad/s^2$$

And Eqn. 44

$$-a_B = 1.6 + 0.39(17.69) = 8.5 \quad a_B = (-8.5j)m/s^2$$

7.4 Kinematics of a Typical Four-Bar Spatial Linkage

Figure 28 shows a four-bar $RSSR$ spatial linkage in which the input crank A_oA rotates about the y-axis so that A moves in a circular path in the $x-z$ plane. The output link B_oB rotates about an axis parallel to the x-axis in the $x-y$ plane and oscillates through an angle $\Delta\psi$ in a plane parallel to the $y-z$ plane giving the coupler AB motion in three dimensions.

To analyse such a linkage we replace the links by the vectors a, b, c and d as shown in fig. 29 Writing the vector or loop equation for the linkage we have

$$\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d} \tag{45}$$

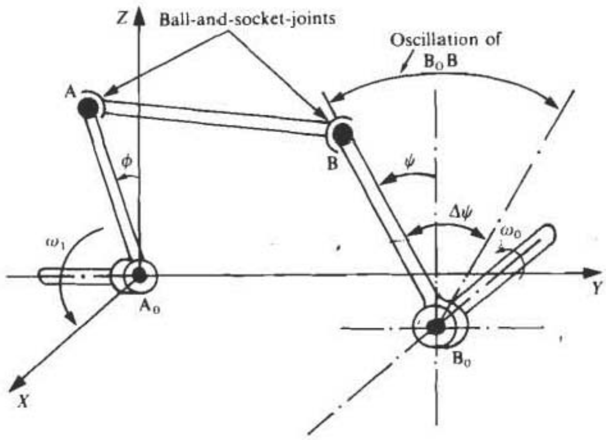


Figure 28: Typical four-bar spatial linkage

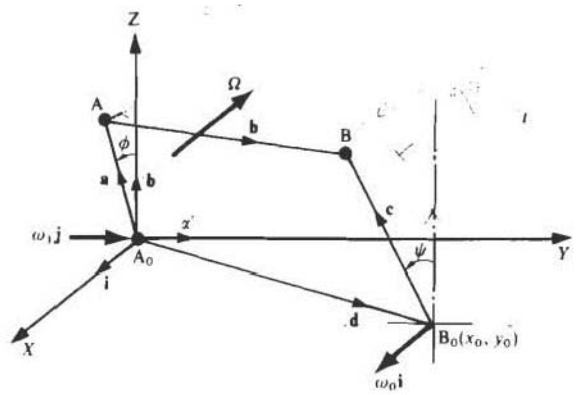


Figure 29: Vector representation of the linkage in Fig 28

Which upon differentiating becomes

$$\dot{\mathbf{a}} + \dot{\mathbf{b}} = \dot{\mathbf{c}}$$

since d is a vector of constant magnitude and direction. If ω_1 is the input angular velocity of the link A_oA, then in vector form we have

$$\boldsymbol{\omega}_1 = \omega_1 \mathbf{j}$$

Similarly, the output angular velocity of the link B_oB is

$$\boldsymbol{\omega}_o = \omega_o \mathbf{i}$$

Let v_A be the velocity of point A as a point on A_oA, then,

$$v_A = \boldsymbol{\omega}_1 \times \mathbf{a} = \omega_1 \mathbf{j} \times \mathbf{a}$$

Also, if $\mathbf{v}_B$ is the velocity of point B as a point on B_oB, then

$$v_B = \boldsymbol{\omega_o} \times \mathbf{c} = \omega_o \mathbf{i} \times \mathbf{c}$$

But the velocity of v_B of point B on the coupler AB is given by

$$v_B = v_A + v_{BA} = v_A + \Omega \times \mathbf{b}$$

where Ω is the angular velocity of the coupler AB which can be expressed in terms of its components ω_x, ω_y, and ω_z, i.e

$$\Omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$$

Hence,

$$\Omega \times \mathbf{b} = v_B - v_A = \omega_o \mathbf{i} \times \mathbf{c} - \omega_1 \mathbf{j} \times \mathbf{a} \tag{46}$$

Since v_{BA}, i.e., the velocity of B on AB as seen by an observer at A, is perpendicular to the coupler AB then by taking the 'dot' (scalar) product of v_{BA} with $\mathbf{b}$ we must have

$$v_{BA}.\mathbf{b} = 0$$

Substituting for v_{BA} yields

$$(\omega_o \mathbf{i} \times \mathbf{c} - \omega_1 \mathbf{a}).\mathbf{b} = 0 \tag{47}$$

To solve for ω_o we need to express $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the input angle ϕ and the output angle ψ. Referring to figure 29

$$\begin{aligned} \mathbf{a} &= a \sin\phi \mathbf{i} + a \cos\phi \mathbf{k} \\ \mathbf{c} &= -c \sin\psi \mathbf{j} + c \cos\psi \mathbf{k} \\ \mathbf{d} &= x_o \mathbf{i} + y_o \mathbf{j} \end{aligned} \tag{48}$$

x_o and y_o are the coordinates of B_o.

Solving for $\mathbf{b}$ in equation 45 we get,

$$\mathbf{b} = \mathbf{c} + \mathbf{d} - \mathbf{a}$$

and substituting for $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{d}$ from eq. 48 yields

$$b = (x_o - a\sin\phi)\mathbf{i} + (y_o - c\sin\psi)\mathbf{j} + (c\cos\psi - a\cos\phi)\mathbf{k} \tag{49}$$

Substituting Eq. 49 in Eq. 46 and performing the dot product will yield an expression for the output angular velocity ω_o.
An alternative approach is to consider the length $b = AB$ of the coupler and make use of the fact that since the link is rigid $\dot{b} = 0$, as follows. From Eq. 49,

$$b^2 = (x_o - a\sin\phi)^2 + (y_o - c\sin\psi)^2 + (c\cos\psi - a\cos\phi)^2$$

Differentiating yields

$$0 = (x_o - a\sin\phi)(-a\cos\phi\dot{\phi}) + (y_o - c\sin\psi)(-c\cos\psi)\dot{\psi}$$
$$+(c\cos\psi - a\cos\phi)(-c\sin\psi\dot{\psi} + a\sin\phi\dot{\phi})$$

Expanding and collecting terms the angular velocity ω_o of the output is given by

$$\omega_o = \frac{a(x_o\cos\phi - c\sin\phi\cos\psi)}{c(a\cos\phi\sin\psi - y_o\cos\psi)} \tag{50}$$

Example 7.2

Figure 30 shows a crank A_oA rotating in the $x-z$ plane at a constant angular velocity $\omega_1 = 10$ rad/s and driving the slider B on the rod PQ by means of the link AB. The rod is in the $y-z$ plane and parallel to A_oy. Calculate the velocity of the slider and the angular velocity of the link AB when $\phi = 90^o$.

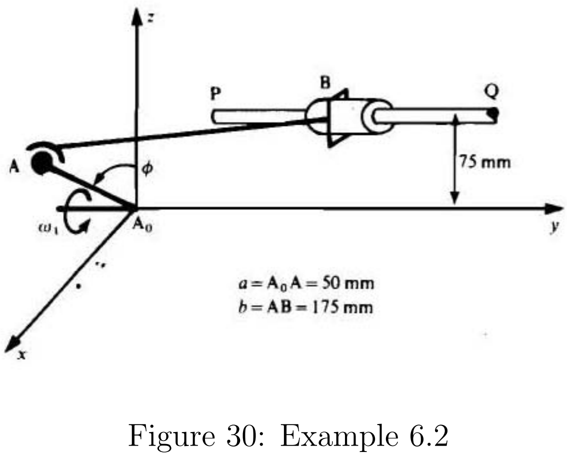


Figure 30: Example 6.2

Solution 7.2

When $\phi = 90^o$ the mechanism is in the position shown in Fig. 31

$$\begin{aligned}
\mathbf{v_B} &= v_{Bj} \\
\mathbf{v_A} &= \omega_1 \mathbf{j} \times \mathbf{a} \\
&= 10\mathbf{j} \times 50\mathbf{i} = -500\mathbf{k}
\end{aligned}$$

- Velocity of the slider: Let Ω be the angular velocity of the link AB, then we have

$$v_B = v_A + v_{BA} = v_A + \Omega \times \mathbf{b}$$

where $\mathbf{b} = \mathbf{R_B} - \mathbf{a} = 150j + 75k - 50i$
Hence,

$$v_B j = -500k + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ -50 & 150 & 75 \end{vmatrix}$$

$$v_B\mathbf{j} = (75\omega_x - 150\omega_z)\mathbf{i} + (-75\omega_x - 50\omega_z)\mathbf{j} + (150\omega_x + 50\omega_y)\mathbf{k} - 500\mathbf{k}$$

Equating the coefficients of the unit vectors, we get

$$\begin{aligned}
75\omega_y - 150\omega_z &= 0 && (51)\\
-75\omega_x - 50\omega_z &= v_B && (52)\\
150\omega_x + 50\omega_y &= 500 && (53)
\end{aligned}$$

Solving the above equations simultaneously, we get,

$$v_B = -250 \text{ mm/s}$$

That is, the collar is moving towards P at that particular instant.

- Angular velocity of link AB: From observation of mechanism, the slider cannot rotate about $A'B = \mathbf{R}$, so that the projection Ω on $\mathbf{R}$ must be zero, i.e., $\Omega.\mathbf{R} = 0$. From Fig. 31 we see that

$$\mathbf{R} + \mathbf{R_B} = \mathbf{d}$$

Hence

$$R = 50\mathbf{i} + 150\mathbf{j} - 150\mathbf{j} - 75\mathbf{k} = 50\mathbf{i} - 75\mathbf{k}$$

and

$$(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}).(50\mathbf{i} - 75\mathbf{k}) = 0$$

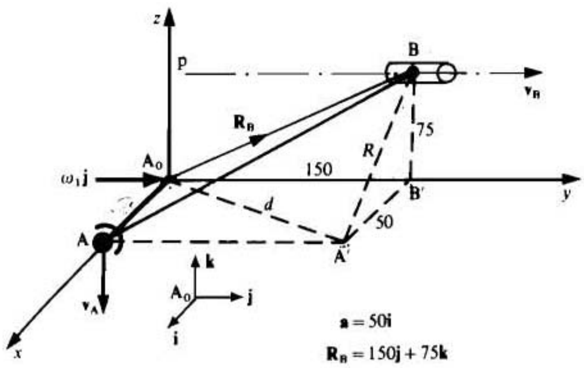


Figure 31: Example 6.2 solution

i.e.,

$$50\omega_x - 75\omega_z = 0 \tag{54}$$

Solving Eqs. 51 to 54 yields

$$v_B = -250 \text{ mm/s} \quad \omega_x = 57.7 \text{ rad/s} \quad \omega_y = 76.9 \text{ rad/s} \quad \omega_z = 38.5 \text{ rad/s}$$

Example 7.3 Figure 32 shows a four-bar spatial linkage where the input and output shafts are at 90^o to each other. Double revolutes are used at A and B. The input shaft at A_o has a constant angular velocity ω_1 in a clockwise direction looking along the shaft towards A_o. Links dimensions are; $A_oA = a = 102$ mm; $AB = b = 381$ mm; $B_oB = c = 254$ mm; $A_oB_o = d = 315$ mm.

(a) Obtain expression for angular velocity ratio and for the acceleration.

(b) When the input speed is 240 rev/min, calculate, the angular velocity of the coupler AB for $\phi = 60^o$.

(c) Calculate the angular acceleration of the coupler AB when the input is rotating at 1200 rev/min and accelerating at 3000 rev/min/s in the sense of ω_1 at the instant when $\phi = 0$.

Solution 7.3

- Output velocity and acceleration:
  Figure 33 shows the linkage with the links expressed as vectors given by

$$\begin{aligned}
\mathbf{a} &= -a\cos\phi\mathbf{i} + \sin\phi\mathbf{k} \\
\mathbf{b} &= -c\cos\psi\mathbf{i} + c\sin\psi\mathbf{j} \\
\mathbf{c} &= -d\mathbf{i}
\end{aligned}$$

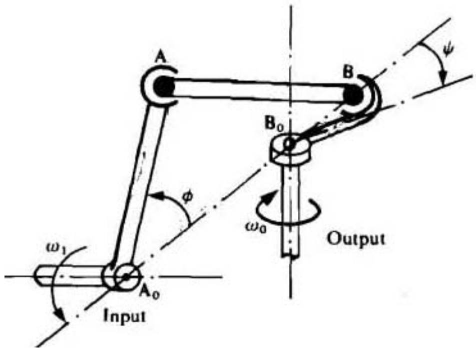


Figure 32: Example 6.3

Since $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$, then

$$\mathbf{b} = \mathbf{c} + \mathbf{d} - \mathbf{a}$$

Substituting for $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{a}$ we get

$$\mathbf{b} = (-c\cos\psi - d + a\cos\phi)\mathbf{i} + (c\sin\psi)\mathbf{j} - (a\sin\phi)\mathbf{k} \tag{55}$$

For the velocity of B we have $v_B = v_A + v_{BA}$

$$\begin{aligned} \therefore v_{BA} &= v_B - v_A = -\omega_o \times \mathbf{c} - \omega_1 \times \mathbf{a} \\ &= -(\omega_o\mathbf{k} \times \mathbf{c} + \omega_1\mathbf{j} \times \mathbf{a}) \end{aligned}$$

where ω_o is the angular velocity of the output link BB_o.
Since v_{BA} is perpendicular to AB then

$$v_{BA}.\mathbf{b} = 0$$

Hence

$$(\omega_o\mathbf{k} \times \mathbf{c} + \omega_1\mathbf{j} \times \mathbf{a}).\mathbf{b} = 0$$

but $v_B = -\omega_o\mathbf{k} \times \mathbf{c} = -\omega_o\mathbf{k} \times (c\cos\psi\mathbf{i} + c\sin\psi\mathbf{j}) = \omega_o c\cos\psi\mathbf{j} + \omega_o c\sin\psi\mathbf{i}$
and $v_A = \omega_1\mathbf{j} \times \mathbf{a} = \omega_1\mathbf{j} \times (-a\cos\phi\mathbf{i} + a\sin\phi\mathbf{k}) = \omega a\cos\phi\mathbf{k} + \omega_1 a\sin\phi\mathbf{i}$
It follows that

$$v_{BA} = (\omega_o c\sin\psi - \omega_1 a\sin\phi)\mathbf{i} + \omega_o c\cos\psi\mathbf{j} - \omega_1 a\cos\phi\mathbf{k}$$

Upon performing the dot product with $\mathbf{b}$ we get

$$-\omega_o c^2\sin\psi\cos\psi + (\omega_1 a\sin\phi - \omega_o c\sin\psi)(a\cos\phi - c\cos\psi - d) - \omega_1 a^2\sin\phi\cos\phi = 0$$

which upon expanding and solving for velocity ratio yields

$$\frac{\omega_o}{\omega_1} = \frac{a\sin\phi(d + c\cos\psi)}{c\sin\psi(d - a\cos\phi)} \tag{56}$$

In order to solve for this ratio we need the value of the output angle ψ. From Eq. 55 we have

$$b^2 = (-c\cos\psi + a\cos\phi - d)^2 + c^2\sin^2\psi + a^2\sin^2\phi \tag{57}$$

Expanding and solving for $\cos\psi$ yields

$$\cos\psi = \frac{b^2 - a^2 - c^2 - d^2 + 2ad\cos\phi}{2c(d - a\cos\phi)} \tag{58}$$

To obtain an expression for the acceleration α_o of the output BB_o it is best to differentiate Eq. 56

Let

$$\begin{aligned} u &= a\sin\phi(d + c\cos\psi) \\ v &= c\sin\psi(d - a\cos\phi) \end{aligned}$$

then

$$\omega_0 = \frac{u}{v}\omega_1 \tag{59}$$

Differentiating (59) with respect to time, we get

$$\alpha_o = \frac{vu' - uv'}{v^2}\omega_1^2 + \frac{u}{v}\alpha_1 \tag{60}$$

We also note that

$$\omega_o = \frac{d\psi}{dt} = \frac{d\psi}{d\phi}\frac{d\phi}{dt} = \frac{u}{v}\omega_1$$

Hence

$$\begin{aligned} u' &= a\cos\phi(d + c\cos\psi) - ac\sin\phi\sin\psi\left(\frac{u}{v}\right) \\ v' &= c\cos\psi(d - a\cos\phi)\left(\frac{u}{v}\right) + ac\sin\phi\sin\psi \end{aligned}$$

- Angular velocity of the coupler

  There are two situation to consider

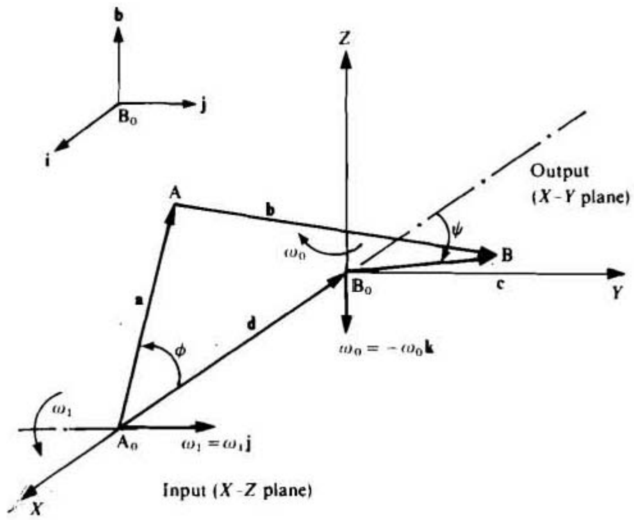


Figure 33: Example 6.3 Solution

(a) The magnitude only of the angular velocity of the coupler

$$\begin{aligned} v_{BA} &= v_B - v_A \\ &= -\omega_o \times \mathbf{c} - \omega \times \mathbf{a} \\ &= (\omega_o c \sin\psi - \omega_1 a \sin\phi)\mathbf{i} + \omega_o c \cos\psi \mathbf{j} - \omega_1 a \cos\phi \mathbf{k} \end{aligned}$$

The magnitude v_{BA} is given by

$$v_{BA} = \sqrt{(\omega_o c \sin\psi - \omega_1 a \sin\phi)^2 + (\omega_o c \cos\psi)^2 + (\omega_1 a \cos\phi)^2}$$

If Ω is the magnitude of the angular velocity of the coupler then

$$\Omega = \frac{v_{BA}}{b}$$

Since we already have expressions for ψ and ω_o we can readily calculate Ω. When $\phi = 60^o$, $\psi = 88.42^o$, $\omega_o = 10.7$ rad/s, $b = 0.381$ m.
Substitution in the above equation yields

$$\Omega = \frac{1}{0.381}\sqrt{\begin{array}{l}(10.7 \times 0.254 \sin 88.42 - 25.1 \times 0.102 \sin 60^o)^2 + \\ (10.7 \times 0.254 \cos 88.42)^2 + (25.1 \times 0.102 \cos 60)^2\end{array}}$$

Hence $\Omega = 3.61$ rad/s

(b) The components of the angular velocity of the coupler parallel to the axes We have

$$\Omega \times \mathbf{b} = v_B - v_A \tag{61}$$

Substituting for $\mathbf{b}$, v_B and v_A we get

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ (a\cos\phi - c\cos\psi - d) & (c\sin\psi) & (-a\sin\phi) \end{vmatrix} = \begin{array}{c} (\omega_o c\sin\psi - \omega_1 a\sin\phi)\mathbf{i} \\ +\omega_o c\cos\psi\mathbf{j} - \omega_1 a\cos\phi\mathbf{k} \end{array}$$

Expanding the determinant and equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ yields the following equation

$$\begin{bmatrix} 0 & -a\sin\phi & -c\sin\psi \\ a\sin\phi & 0 & (a\cos\phi - c\cos\psi - d) \\ c\sin\psi & -(a\cos\phi - c\cos\psi - d) & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \omega_o c\sin\psi - \omega_1 a\sin\phi \\ \omega_o c\cos\psi \\ -\omega_1 a\cos\phi \end{bmatrix} \tag{62}$$

i.e.,$A\Omega = B$

An examination of these equations reveals that the matrix A is singular, i.e the determinant of the matrix is zero, hence this equation has either an infinite number of solutions or none at all. To obtain the angular velocity Ω we can proceed as follows. Let us consider Eq. 61. ie

$$\Omega \times \mathbf{b} = v_{BA}$$

If we pre-multiply by $\mathbf{b}$ we get

$$\mathbf{b} \times (\Omega \times \mathbf{b}) = b \times v_{BA}$$

$$\mathbf{b} \times (\Omega \times \mathbf{b}) = (\mathbf{b}.\mathbf{b})\Omega - (\mathbf{b}.\Omega)\mathbf{b} \tag{63}$$

The first term of Eq. (63) equals $b^2\Omega$ and the second term is zero because the components of Ω taken along body axes i.e axes fixed to the coupler AB at A (Fig. 34) are perpendicular to $\mathbf{b}$, except for ω'_y which is along AB and may be equated to zero since any spin of AB about its own axis does not affect the output motion of the mechanisms. Thus we have

$$b^2\Omega = \mathbf{b} \times v_{BA} = \mathbf{b} \times (v_B - v_A)$$

Substituting for Ω, $\mathbf{b}$, v_B and v_B and solving for Ω yields

$$(\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k}) = \frac{1}{b^2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (a\cos\phi - c\cos\psi - d) & c\sin\psi & -a\sin\phi \\ (\omega_o c\sin\psi - \omega_1 a\sin\phi) & \omega_o c\cos\psi & -\omega_1 a\cos\phi \end{vmatrix}$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ and using the numerical values of case (a) we find
$\omega_x = -2.239$ rad/s, $\omega_y = -2.685$ rad/s, and $\omega_z = -1.013$ rad/s
Hence $\Omega = \sqrt{\omega_x^2+\omega_y^2+\omega_z^2} = \sqrt{2.239^2+2.685^2+1.013^2} = 3.64$ rad/s
Which is close to that obtained for case (a)

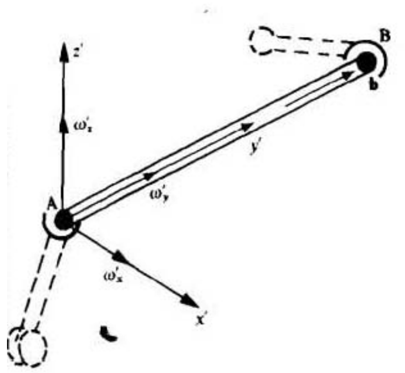


Figure 34: Example 6.3 Solution b

- For acceleration of a point P on a link AP, we have

$$\mathbf{a}_P = \mathbf{a}_A + \alpha\times\mathbf{R} + \omega\times(\omega\times\mathbf{R})$$

where
a_A=acceleration of link A
α=angular acceleration of the link
ω=angular velocity of the link
$\mathbf{R}$=length of the link

  – *Point A on link A_oA*

$$\begin{aligned}\mathbf{a} &= -a\cos\phi\mathbf{i} + a\sin\phi\mathbf{k} \\ &= -0.102\mathbf{i} \quad \text{since} \quad \phi = 0\end{aligned}$$

$$\begin{aligned} v_A &= \omega_1 \mathbf{j} \times \mathbf{a} \\ &= 125.7\mathbf{j} \times (-0.102\mathbf{i}) \\ &= 12.82\mathbf{k} \end{aligned}$$

$$\mathbf{a}_A = \underbrace{\alpha_1 \mathbf{j} \times \mathbf{a}}_{\text{Tangential}} + \underbrace{\omega_1 \mathbf{j} \times (\omega_1 \mathbf{j} \times \mathbf{a})}_{\text{Centripetal}}$$

But $\omega_1 \mathbf{j} \times \mathbf{a} = v_A$
We have $\omega_1 \mathbf{j} \times v_A = 125.7\mathbf{j} \times 12.82\mathbf{k} = 1611\mathbf{i}$
Hence $a_A = 1611\mathbf{i} + 32.05\mathbf{k}$

– *Position B on link B_oB*

$$\begin{aligned} \mathbf{c} &= -c\cos\psi\mathbf{i} + c\sin\psi\mathbf{j} \\ &= -0.254\cos 70.64\mathbf{i} + 0.254\sin 70.64\mathbf{j} \\ &= -0.0842\mathbf{i} + 0.240\mathbf{j} \end{aligned}$$

Since 70.64^o when $\phi = 0$

$$\begin{aligned} v_B &= -\omega_o \mathbf{k} \times \mathbf{c} \\ &= 0 \quad \textit{since} \quad \omega_o/\omega_1 = 0 \quad \textit{when} \quad \phi = 0 \end{aligned}$$

$$\begin{aligned} a_B &= -\alpha_o \mathbf{k} \times \mathbf{c} - \omega_o \mathbf{k} \times v_B = -\alpha_o \mathbf{k} \times \mathbf{c} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\alpha_o \\ -0.0842 & 0.240 & 0 \end{vmatrix} = 0.240\alpha_o\mathbf{i} + 0.0842\alpha_o\mathbf{j} \end{aligned} \tag{64}$$

– *Point B on link AB*
$\mathbf{b} = \mathbf{c} + \mathbf{d} - \mathbf{a} = (-c\cos\psi - d + a\cos\phi)\mathbf{i} + c\sin\psi\mathbf{j} - a\sin\phi\mathbf{k}$
Substituting values we get

$$\begin{aligned} \mathbf{b} &= (-0.254\cos 70.64 - 0.314 + 0.102)\mathbf{i} + 0.254\sin 70.64\mathbf{j} \\ &= -0.296\mathbf{i} + 0.240\mathbf{j} \\ \mathbf{a}_{BA} &= \alpha_c \times \mathbf{b} + \omega_c \times (\omega_c \times \mathbf{b}) = \alpha_c \times \mathbf{b} + \omega_c \times v_{BA} \end{aligned}$$

where

$$\begin{aligned}\omega_c &= \text{angular velocity of the coupler} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k} \\ \alpha_c &= \text{angular acceleration of the coupler} = \alpha_x\mathbf{i} + \alpha_y\mathbf{j} + \alpha_z\mathbf{k}\end{aligned}$$

$$\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.296 & 0.24 & 0 \\ 0 & 0 & -12.82 \end{vmatrix}$$

Hence $\omega_x = -21.2$, $\omega_y = -26.13$, and $\omega_z = 0$

$$v_{BA} = -\omega_1 a \cos\phi \mathbf{k} = -125.7 \times 0.102\mathbf{k} = -12.82\mathbf{k}$$

$$\omega_c \times v_{BA} = \begin{vmatrix} i & j & k \\ -12.2 & -26.13 & 0 \\ 0 & 0 & -12.82 \end{vmatrix} = 335\mathbf{i} - 272\mathbf{j} \tag{65}$$

$$\alpha_c \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha_x & \alpha_y & \alpha_z \\ -0.296 & 0.240 & 0 \end{vmatrix} = -0.240\alpha_z\mathbf{i} - 0.296\alpha_z\mathbf{j} + (0.240\alpha_x + 0.296\alpha_y)\mathbf{k} \tag{66}$$

Writing the vector equation for the acceleration of the point B on AB we have

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA}$$

i.e.,

$$\mathbf{a}^n_B + \mathbf{a}^t_B = \mathbf{a}^n_A + \mathbf{a}^t_A + \mathbf{a}^n_{BA} + \mathbf{a}^t_{BA} \tag{67}$$

Before we can solve for α_c we need to introduce the condition that $\alpha_c.\mathbf{b} = 0$ to eliminate any possible spin of the coupler about its axis. This leads to the following equation:

$$\begin{aligned}(\alpha_x\mathbf{i} + \alpha_y\mathbf{j} + \alpha_z\mathbf{k}).(-0.296\mathbf{i} + 0.240\mathbf{j}) &= 0 \\ -0.296\alpha_x + 0.240\alpha_y &= 0 \end{aligned} \tag{68}$$

Substituting Eqs.(64) and (66) into Eq. (67), we get

$$0.240\alpha_o\mathbf{i} + 0.0842\alpha_o\mathbf{j} =$$
$$1611\mathbf{i} + 32.05\mathbf{k} - 0.240\alpha_z\mathbf{i} - 0.296\alpha_z\mathbf{j} + 0.2400\alpha_x\mathbf{k} + 0.296\alpha_y\mathbf{k} + 335\mathbf{i} - 272\mathbf{j}$$

Equating the coefficients of **i**, **j**, and **k**, yields

$$0.240\alpha_o + 0.240\alpha_z = 1611 + 335 = 1946 \quad (69)$$

$$0.0842\alpha_o + 0.296\alpha_z = -292 \quad (70)$$

$$0.240\alpha_x + 0.296\alpha_y = -32.05 \quad (71)$$

$$-0.296\alpha_x + 0.240\alpha_y = 0 \quad (72)$$

Solving Eqs. (69) to (72), we have;

$$\alpha_o = 12.607 \text{ rad/s}, \quad \alpha_z = -4497 \text{ rad/s}, \quad \alpha_x = -53 \text{ rad/s}^2$$

Hence at the instant when $\phi = 0$, the angular acceleration of the output $\alpha_o = -12,607$ rad/s^2, and that of the coupler $\alpha_c = -53\mathbf{i} - 65.3\mathbf{j} - 4497\mathbf{k}$ rad/s^2

The output acceleration is given by

Example 7.4

At a given instant (Figure 35), the satellite dish has an angular motion $\omega_1 = 6rad/s$ and $\dot{\omega}_1 = 3rad/s^2$ about the z axis.At this same instant $\theta = 25°$, the angular motion about the x axis is $\omega_2 = 2rad/s$, and $\dot{\omega}_2 = 1.5rad/s^2$. Determine the velocity and acceleration of the signal horn A at this instant.

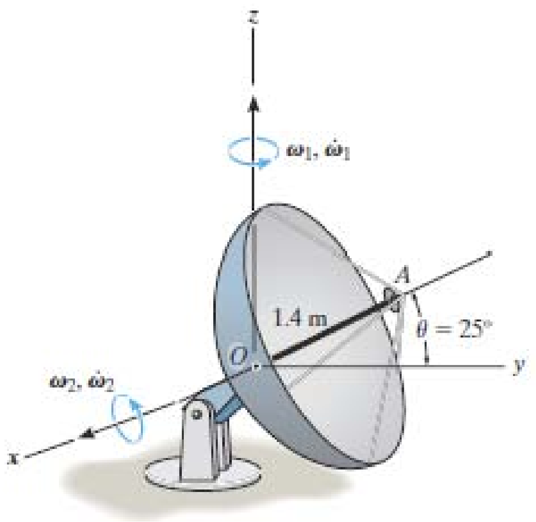


Figure 35:

Solution 7.4

Angular Velocity- Coordinates of the fixed frame and the rotating frame are coincident at this instant, thus expressing the angular velocity in terms of unit vectors i,j and k.

$$\omega = \omega_1 + \omega_2 = (2i + 6k)rad/s$$

Angular Acceleration- ω_2 is observed to have a constant direction from the rotating xyz frame but has a change in direction wrt the fixed XYZ, rotating at $\Omega = \omega_1 = 6k$ rad/s

Using

$$\begin{aligned}\dot{A} &= (\dot{A})_{xyz} + \Omega \times A \\ \dot{\omega}_2 &= (\dot{\omega}_2)_{xyz} + \omega_1 \times \omega_2 \\ &= 1.5i + 6k \times 2i = (1.5i + 12j)rad/s^2\end{aligned}$$

Since ω_1 is always directed along the Z-axis ($\Omega = 0$) then

$$\begin{aligned}\dot{\omega}_1 &= (\dot{\omega}_1)_{xyz} + 0 \times \omega_1 \\ &= (3k)rad/s^2\end{aligned}$$

The angular acceleration of the satellite is $\therefore$

$$\begin{aligned}\alpha &= \dot{\omega}_1 + \dot{\omega}_2 \\ &= (1.5i + 12j + 3k)rad/s^2\end{aligned}$$

Signal Horn A-Vector r_A is given as

$$\begin{aligned}r_A &= OA\cos\theta j + OA\sin\theta k \\ &= 1.4\cos 25j + 1.4\sin 25k \\ &= (1.27j + 0.59k)m\end{aligned}$$

The velocity of A is given as

$$\begin{aligned}v_A &= v_0 + \omega \times r_A \\ &= 0 + (2i + 6k) \times (1.27j + 0.59k) \\ &= \mathbf{(-7.6i - 1.18j + 2.54k)m/s}\end{aligned}$$

And Acceleration of horn A

$$
\begin{aligned}
a_A &= a_0 + \alpha \times r_A + \omega \times (\omega \times r_A) \\
&= (1.5i + 12j + 3k) \times (1.27j + 0.59k) + (2i + 6k) \times [(2i + 6k) \times (1.27j + 0.59k)] \\
&= \mathbf{(10.4i - 51.6j - 0.46k) m/s^2}
\end{aligned}
$$

7.5 Assignment 2

(a) At the instant shown (Figure 36), the motor rotates about the z axis with an angular velocity of $\omega_1 = 3rad/s$ and angular acceleration of $\dot{\omega}_1 = 1.5rad/s^2$. Simultaneously, shaft OA rotates with an angular velocity of $\omega_2 = 6rad/s$ and angular acceleration of $\dot{\omega}_2 = 3rad/s^2$ and collar C slides along rod AB with a velocity and acceleration of $6m/s$ and $3m/s^2$. Determine the velocity and acceleration of collar C at this instant.

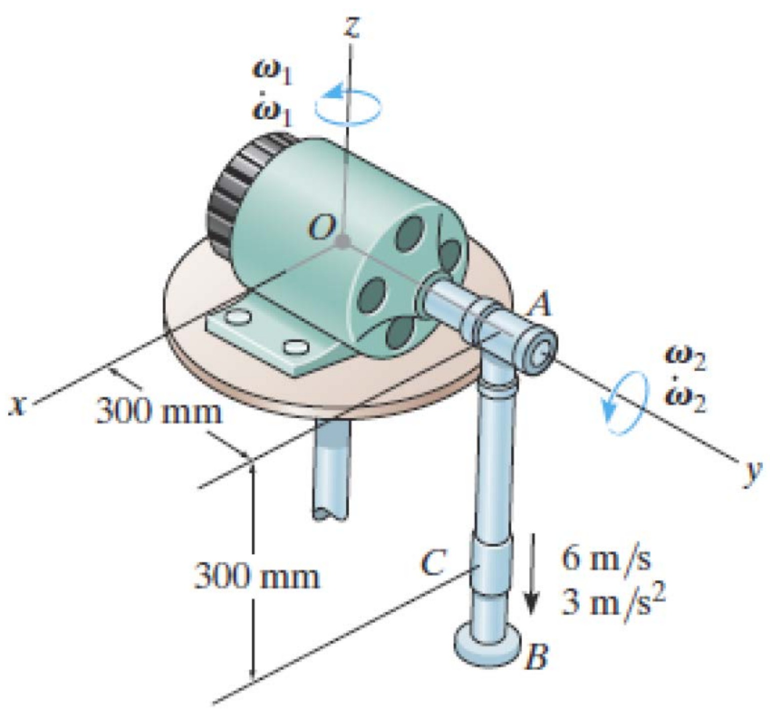


Figure 36: Assignment 2(a)

(b) At the instant shown in Figure 37, rod BD is rotating about the vertical axis with an angular velocity $\omega_{BD} = 7rad/s$ and an angular acceleration $\alpha_{BD} = 4rad/s^2$. Link AC is rotating downward. Determine the velocity and acceleration of point A on the link at this instant given $\theta = 60°$, $\dot{\theta} = 2rad/s$, $\ddot{\theta} = 3rad/s^2$ and $l = 0.8m$.

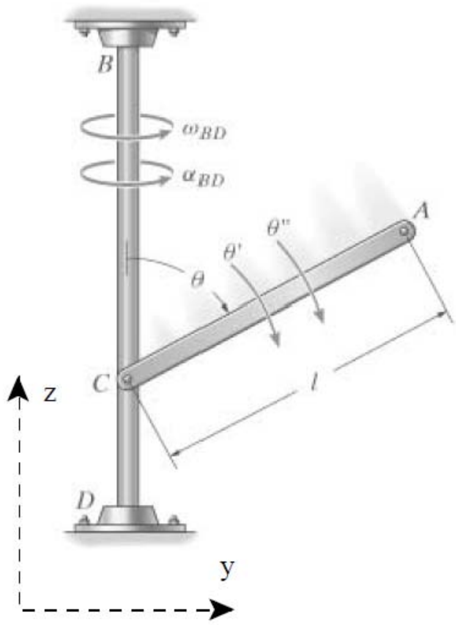


Figure 37: Assignment 2(b)

7.6 Practical-Exercise 2 (Simulation)

The crane shown in Figure 38 rotates with a constant angular velocity ω_1 of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity ω_2 of 0.50 rad/s relative to the cab. Knowing that the length of the boom OP is $l = 12m$, determine:

(i) the angular velocity ω of the boom,

(ii) the angular acceleration α of the boom,

(iii) the velocity v of the tip of the boom,

(iv) the acceleration a of the tip of the boom.

1. Solve the above question manually
2. Write a MATLAB program that will accept all the variable inputs and solve for the velocity and acceleration of the tip of the boom.

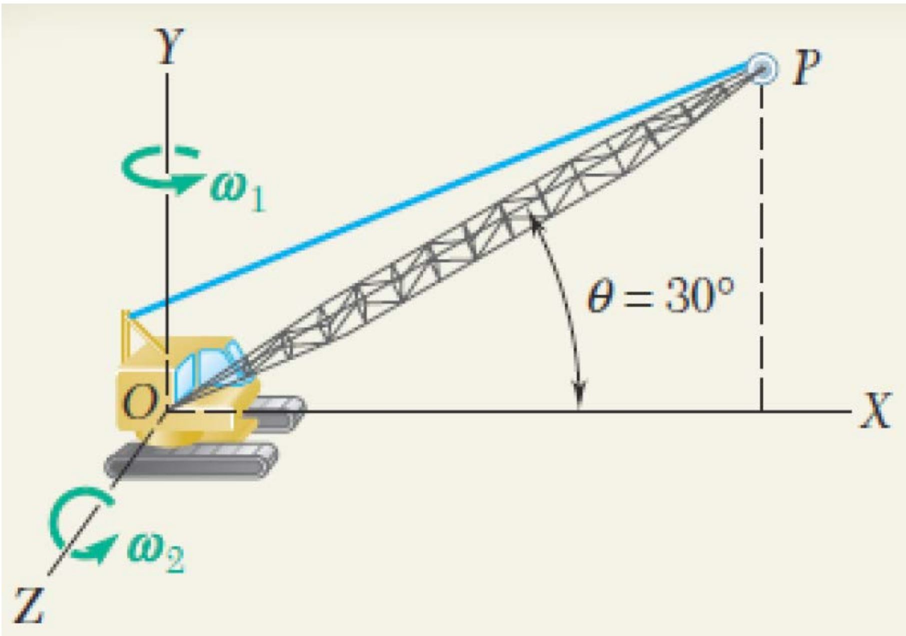


Figure 38: Practical Exercise 2