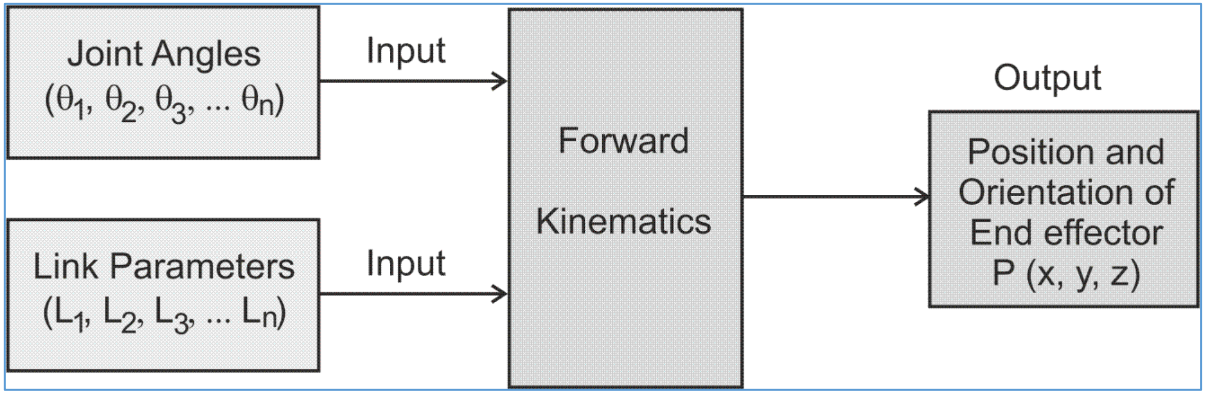
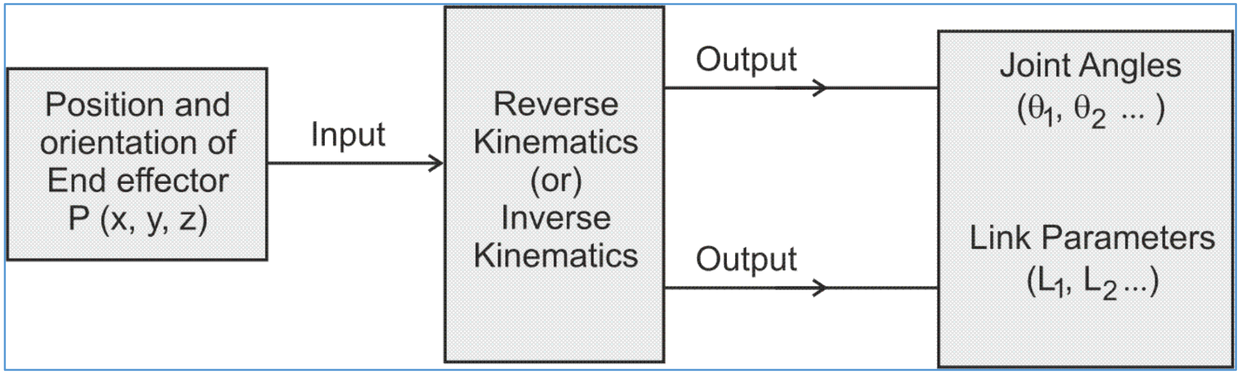
Forward and Inverse Kinematics

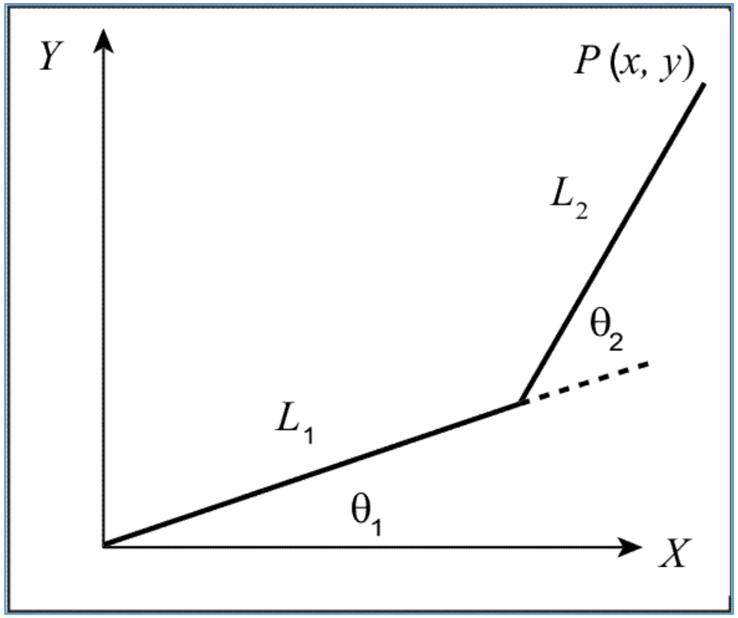
- **Definition:** Forward kinematics deals with solving the forward transformation equation to find the position and orientation of the end effector in terms of joint angles and link parameters.



- **Definition:** Inverse kinematics deals with calculation of all possible sets of joint angles (or offsets) which could be used to attain the given position and orientation of the end effector of the manipulator



A Two-Joint (2 DOF) Robot



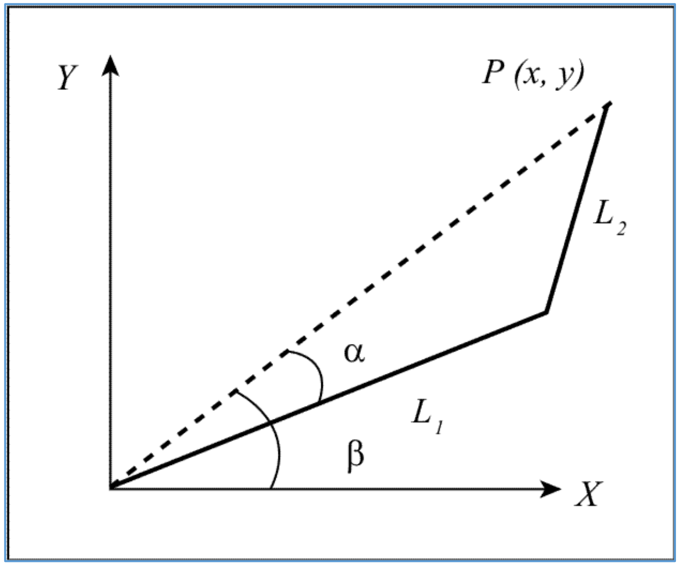
$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)
\end{aligned}
$$

Forward kinematic problem

- The position of end-effector *P(x,y)* can be computed using 2 dimensional trigonometric method as follows
- This can be solved using the following set of equations

Inverse kinematic problem

- For the given position of the end-effector *P(x,y)*, compute the joint coordinates (θ_1, θ_2)
- Consider the following figure



- From which θ_1 is computed as

$$x^2 + y^2 = L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2$$

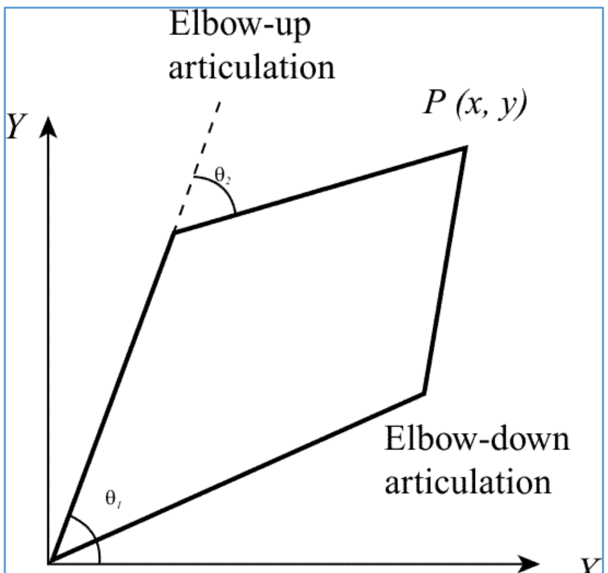
- Substitute θ_2 and find α from

$$\tan\alpha = (L_2 \sin\theta_2)/(L_2\cos\theta_2 + L_2)$$

- Find β from

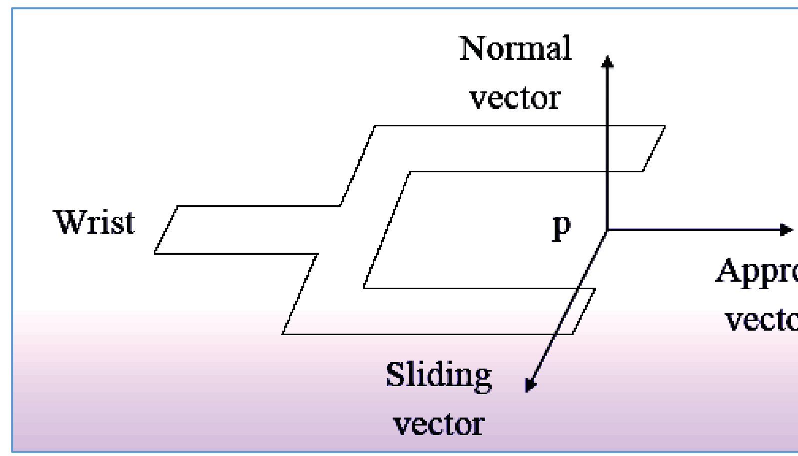
$$\tan\beta = \frac{y}{x}$$

- Then θ_1 is evaluated as $\theta_1 = (\beta - \alpha)$
- From this example, two solutions are possible for θ_2
- Hence point *P(x,y)* can be reached through two articulations as shown in the figure



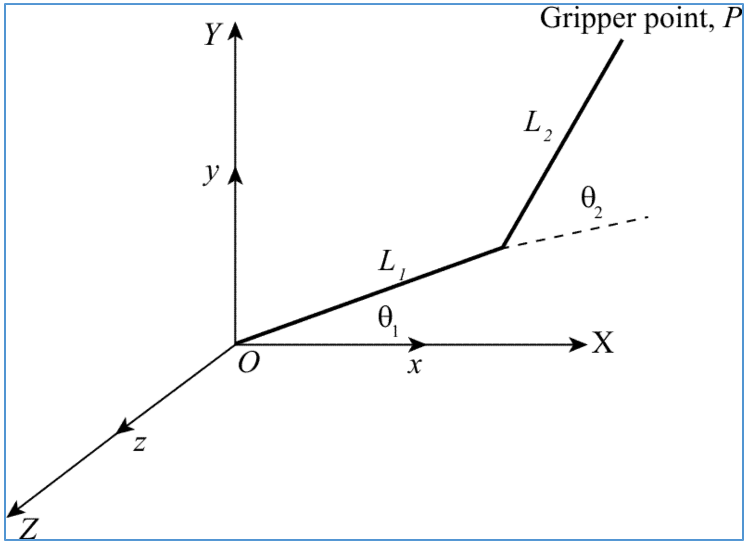
Normal, sliding, and approach vectors (end-effector)

- The approach vector is aligned with the tool roll axis and point way from wrist
- The sliding vector is orthogonal to the approach vector and aligned with the open-close axis of the tool
- The normal vector is orthogonal to the plane defined by the approach and sliding vectors and completes a right-handed orthonormal coordinate frame



Use of Homogeneous Transformation

- Consider the following figure



- For the **forward kinematic**, the following articulation are perform to reach the gripper point *P*
  - A rotation of θ_1° about the z-axis
  - A translation of L_1 units along the x-axis
  - A rotation of θ_2° about the z-axis
  - A translation of L_2 units along the x-axis

- The homogeneous transformation matrix from *O* to *p* is given by

$$H_0^P = \mathrm{Rot}(z,\theta_1)\mathrm{Trans}(L_1,0,0)\mathrm{Rot}(z,\theta_2)\mathrm{Trans}(L_2,0,0)$$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The simplified transformation of point P with respect to the O is given by

$$H_0^P = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & L_2\cos(\theta_1+\theta_2)+L_1\cos\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & L_2\sin(\theta_1+\theta_2+L_1\sin\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For the **inverse kinematics,** if the gripper position $P(x,y,z)$ is given in the homogeneous vector form $\begin{bmatrix} P_x & P_y & P_z & 1 \end{bmatrix}^T$, Then

$$P(x,y,x) = \begin{bmatrix} P_x & P_y & P_z & 1 \end{bmatrix}^T = \begin{bmatrix} L_2\cos(\theta_1+\theta_2)+L_1\cos\theta_1 \\ L_2\sin(\theta_1+\theta_2)+L_1\sin\theta_1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix}$$

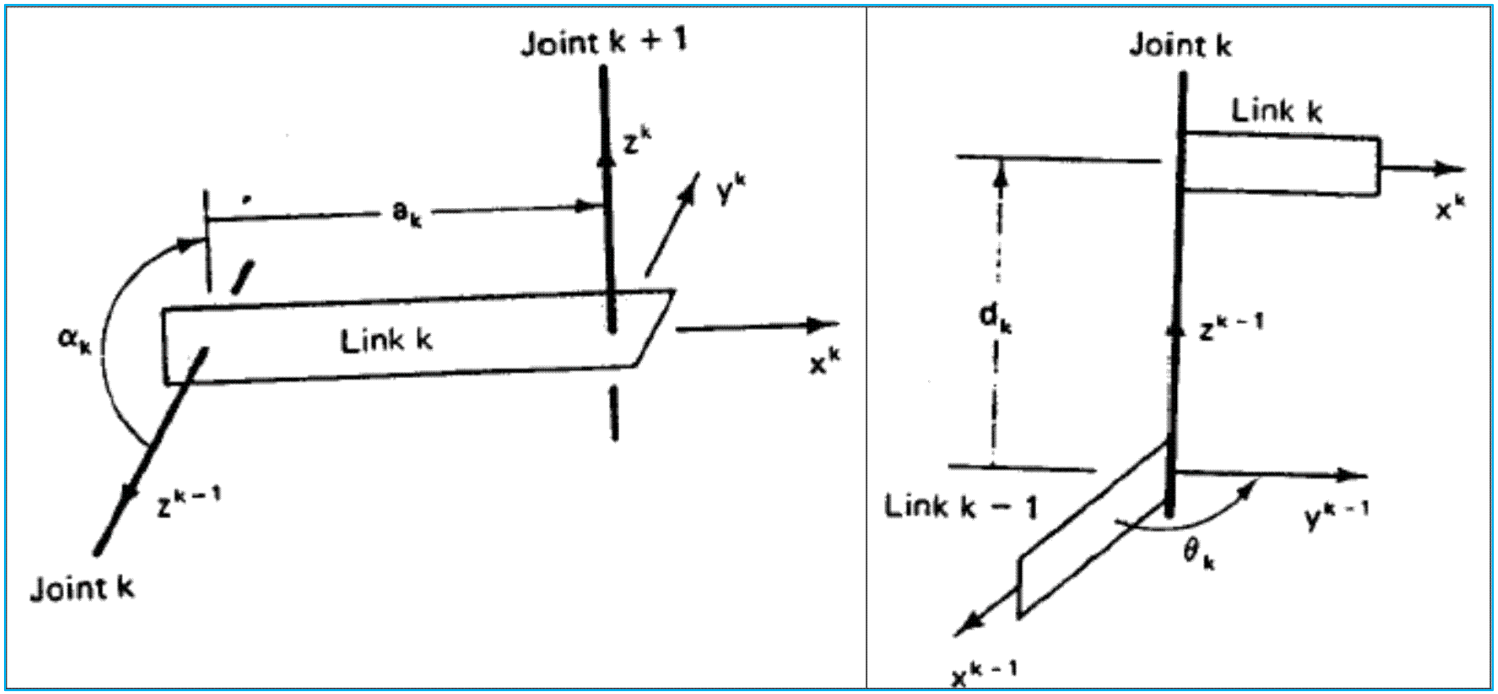
- To solve the inverse kinematic problem the following equations are used

$$\begin{aligned} P_x &= L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ P_y &= L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \end{aligned}$$

D-H (Denavit-Hartenberg) Representation

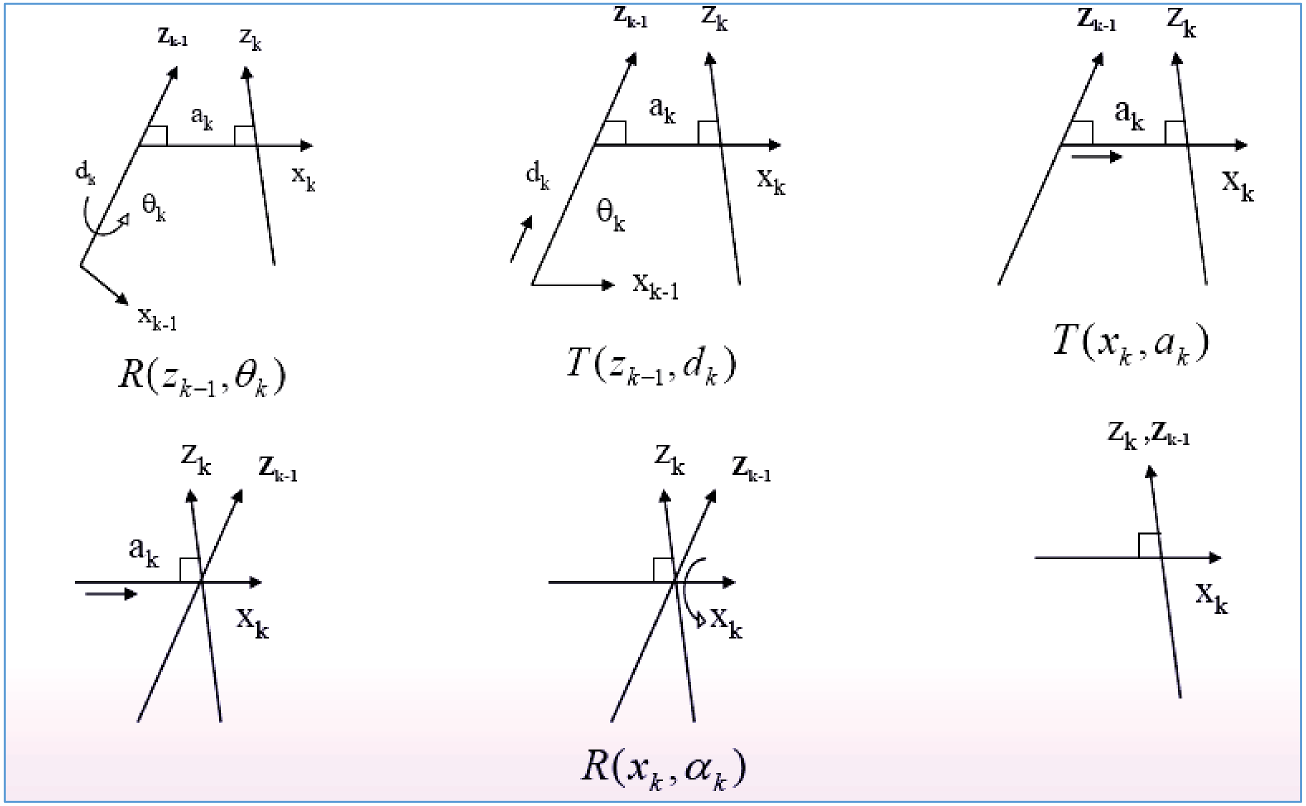
- Kinematic (Link) Parameters: The following four parameters are used to completely describe revolute or prismatic joints (or links)
  - Link length, a_k
  - Link twist, α_k
  - Link offset, d_k
  - Link angle, θ_k

a_k and α_k describe the structure of the link
d_k and θ_k describe the relative position of the neighboring links



D-H (Denavit-Hartenberg) Derivation (*k-1* frame to *k* frame)

- Rotate L_{k-1} about L_{k-1} by $\theta_k \Rightarrow R(z_{k-1}, \theta_k)$
- Translate L_{k-1} along z_{k-1} by $d_k \Rightarrow T(z_{k-1}, d_k)$
- Translate L_{k-1} along x_{k-1} by $a_k \Rightarrow T(x_k, a_k)$
- Rotate L_{k-1} about x_{k-1} by $\alpha_k \Rightarrow R(x_k, \alpha_k)$



$$ {}^{k}_{k-1}H = R(z_{k-1}, \theta_k) T(z_{k-1}, d_k) T(x_k, a_k) R(x_k, \alpha_k) $$

D-H (Denavit-Hartenberg) Derivation (*k-1* frame to *k* frame)

- The generalized homogeneous coordinate frame becomes

$$
{}^{k}_{k-1}H = R(z_{k-1}, \theta_k) T(z_{k-1}, d_k) T(x_k, a_k) R(x_k, \alpha_k)
$$

$$
= \begin{bmatrix} C\theta_k & -S\theta_k & 0 & 0 \\ S\theta_k & C\theta_k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_k & -S\alpha_k & 0 \\ 0 & S\alpha_k & C\alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} C\theta_k & -C\alpha_k S\theta_k & S\alpha_k S\theta_k & a_k C\theta_k \\ S\theta_k & C\alpha_k C\theta_k & -S\alpha_k C\theta_k & a_k S\theta_k \\ 0 & S\alpha_k & C\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- For prismatic joint $\alpha_k = 0$, hence

$$
{}^{k}_{k-1}H = R(z_{k-1}, \theta_k) T(z_{k-1}, d_k) R(x_k, \alpha_k)
$$

$$
= \begin{bmatrix} C\theta_k & -C\alpha_k S\theta_k & S\alpha_k S\theta_k & 0 \\ S\theta_k & C\alpha_k C\theta_k & -S\alpha_k C\theta_k & 0 \\ 0 & S\alpha_k & C\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

D-H (Denavit-Hartenberg) Derivation (*k-1* frame to *k* frame)

- The generalized inverse homogeneous coordinate frame becomes

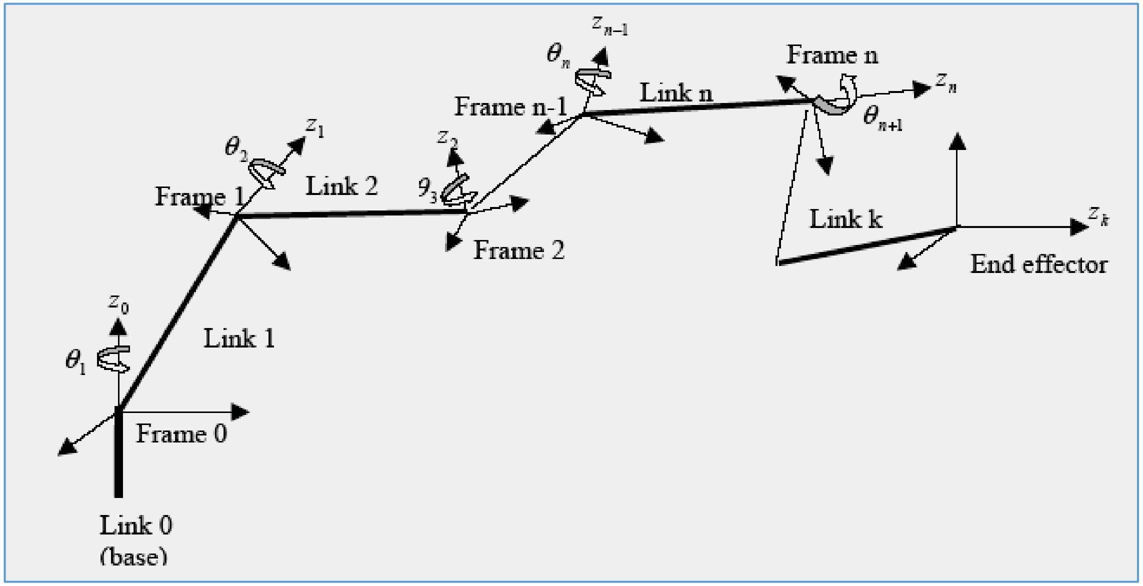
$$
{}^{k-1}_{k}H = {}^{k}_{k-1}H^{-1} = \begin{bmatrix} C\theta_k & -C\alpha_k S\theta_k & S\alpha_k S\theta_k & a_k C\theta_k \\ S\theta_k & C\alpha_k C\theta_k & -S\alpha_k C\theta_k & a_k S\theta_k \\ 0 & S\alpha_k & C\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}
$$

$$
= \begin{bmatrix} C\theta_k & S\theta_k & 0 & -a_k \\ -C\alpha_k S\theta_k & C\alpha_k C\theta_k & S\alpha_k & -d_k S\alpha_k \\ S\alpha_k S\theta_k & -S\alpha_k C\theta_k & C\alpha_k & -d_k C\alpha_k \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

D-H Algorithm for assigning coordinate frames

- Number the joints from 1 to n starting with the base and ending with the tool yaw, pitch, and roll, in that order
- Assign a right-handed orthonormal coordinate frame 0 to the robot base making sure that z_0 aligns with the axis of joint 1. Set $k = 1$
- Align z^k with the joint axis of joint $k + 1$
- Select x^k to be orthogonal to both z^k and z^{k-1}. If z^k and z^{k-1} are parallel, point x^k away from z^{k-1}
- Set $k = k + 1$. If $k < n$, go to step 3; else continue
- Set the origin of frame k at the tool tip. Align z^k with the approach vector, y^k with the sliding vector, and x^k with the normal vector of the tool.

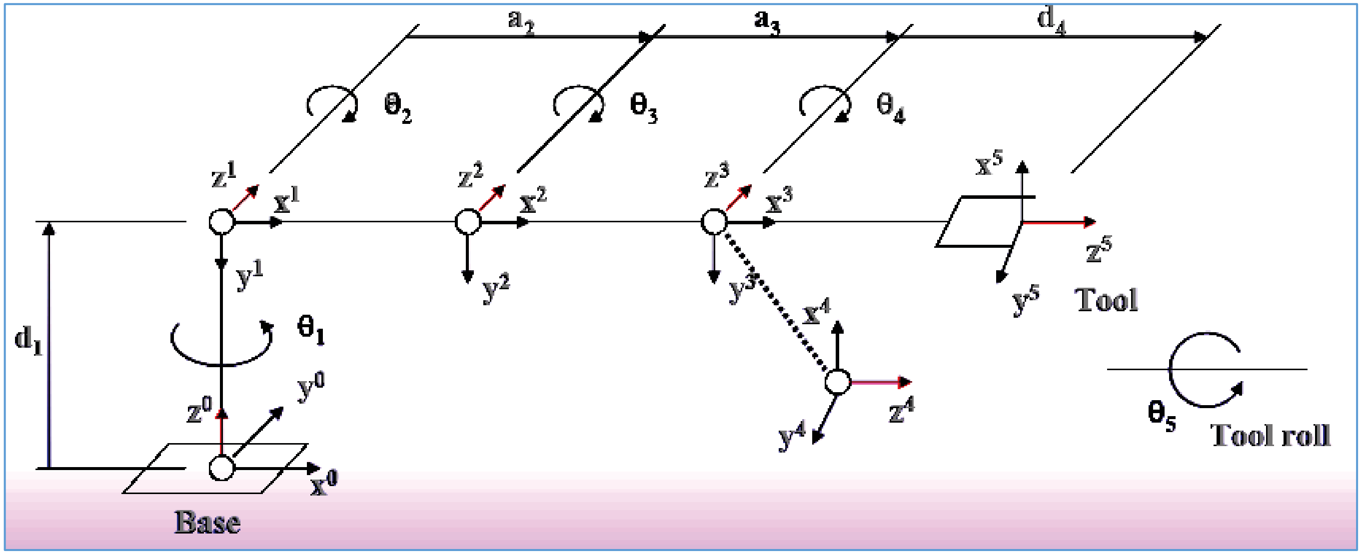
Figure illustrating D-H Algorithm for assigning coordinate frames



D-H Algorithm for Assigning Link Parameters

- Set $k=1$
- Locate point b^k at the intersection of the x^k and z^{k-1} axes. If they don't intersect, use the intersection of x^k with common normal between x^k and z^{k-1}
- Compute θ_k as the angle of rotation from x^{k-1} to x^k measure about z^{k-1}
- Compute d_k as the distance from the origin of frame k-1 to point b^k measured along z^{k-1}
- Compute a_k as the distance from point bk to the origin of the origin of frame k measured along x^k
- Compute α_k as the angle of rotation from z^{k-1} to z^k measured about x^k

Figure illustrating D-H Algorithm for assigning Link parameters



Procedure to solve a forward kinematic Problem

- For a given manipulator, assign the coordinate frames for each link
- Find the link parameters. Fill in the parameters table

Axis	θ_k	d_k	a_k	α_k
1	…	…	…	…
⋮	⋮	⋮	⋮	⋮
n	…	…	…	…

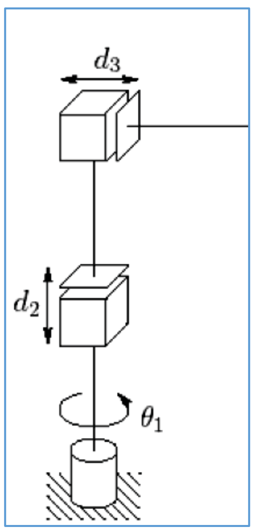
- Find ${}^{k}_{k-1}H$ using the following formula

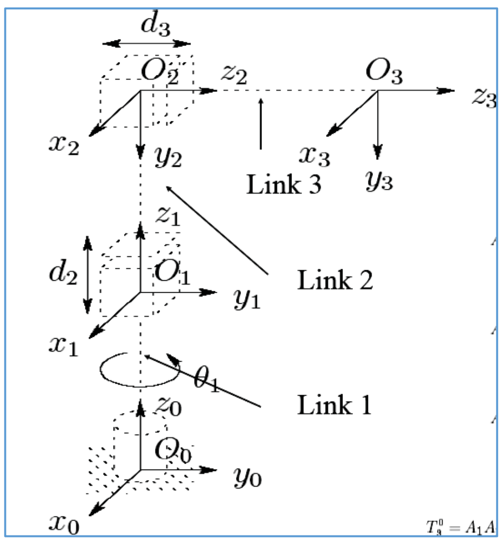
$$
{}^{k}_{k-1}H = \underbrace{\begin{bmatrix} C\theta_k & -C\alpha_k S\theta_k & S\alpha_k S\theta_k & a_k C\theta_k \\ S\theta_k & C\alpha_k C\theta_k & -S\alpha_k C\theta_k & a_k S\theta_k \\ 0 & S\alpha_k & C\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{For rotary joint}}, \; {}^{k}_{k-1}H = \underbrace{\begin{bmatrix} C\theta_k & -C\alpha_k S\theta_k & S\alpha_k S\theta_k & 0 \\ S\theta_k & C\alpha_k C\theta_k & -S\alpha_k C\theta_k & 0 \\ 0 & S\alpha_k & C\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{For prismatic joint}}
$$

- Find ${}^{tool}_{base}H$ using chain rule based on the relation ${}^{k}_{k-1}H$

$$
{}^{k}_{0}H = {}^{1}_{0}H\,{}^{2}_{1}H \ldots {}^{k}_{k-1}H = \prod_{j=1}^{k} {}^{k}_{j-1}H \quad \text{for} j = 1, 2, 3 \ldots k
$$

Example (RPP Cylindrical Robot)





D-H parameter table

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

Example (RPP Cylindrical Robot, Cont')

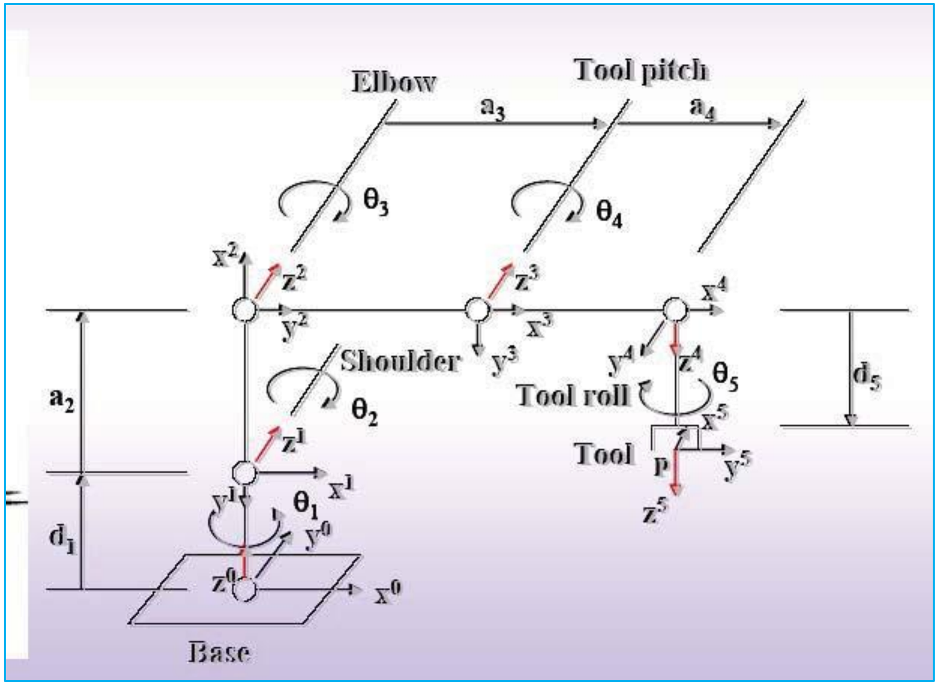
- The corresponding matrices

$$A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Forward kinematics

$$\boxed{{}^0_3T = A_1 A_2 A_3 = \begin{bmatrix} C_1 & 0 & -S_1 & -S_1 d_3 \\ S_1 & 0 & C_1 & C_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Example (A Five Axis Articulated Education Robot, Rhino XR-3)



- D-H parameter table

Axis	θ	d	a	α
1	θ_1	$d_1(260.4\text{mm})$	0	$\frac{-\pi}{2}$
2	θ_2	0	$a_2(228.6\text{mm})$	0
3	θ_3	0	$a_3(228.6\text{mm})$	0
4	θ_4	0	$a_4(9.5\text{mm})$	$\frac{-\pi}{2}$
5	θ_5	$d_5(171.5\text{mm})$	0	0

- Corresponding matrices

$$
{}^1_0H = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^2_1H = \begin{bmatrix} C_2 & -S_1 & 0 & a_2C_2 \\ S_2 & C_1 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3_2H = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{\text{Wrist}}_{\text{Base}}H = {}^1_0H\,{}^2_1H\,{}^3_2H
$$

Example (A Five Axis Articulated Education Robot, Rhino XR-3, Cont')

- Forward kinematic (Part 1)

$$
{}^{\text{Wrist}}_{\text{Base}}H = {}^{1}_{0}H\,{}^{2}_{1}H\,{}^{3}_{2}H
$$

$$
= \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_2 & -S_1 & 0 & a_2C_2 \\ S_2 & C_1 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} C_1C_{23} & -C_1C_{23} & -S_1 & C_1(a_2C_2 + a_3C_3) \\ S_1C_{23} & -S_1S_{23} & C_1 & S_1(a_2C_2 + a_3C_3) \\ -S_{23} & -C_{23} & 0 & d_1 - a_2S_2 - a_3S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Forward kinematic (part 2)

$$
{}^{\text{Tool}}_{\text{Wrist}}H = {}^{4}_{3}H\,{}^{5}_{4}H
$$

$$
= \begin{bmatrix} C_4 & 0 & -S_4 & a_4C_4 \\ S_4 & 0 & C_4 & a_4S_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_5 & -S_5 & 0 & 0 \\ S_5 & C_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
= \begin{bmatrix} C_4C_5 & -C_4C_5 & -S_4 & a_4C_4 - d_5S_4 \\ S_4C_5 & -S_4S_5 & C_4 & a_4S_4 + d_5C_4 \\ -S_5 & -C_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Example (A Five Axis Articulated Education Robot, Rhino XR-3, Cont')

- Forward kinematic (Overall)

$$ {}^{\text{Tool}}_{\text{Base}}H = {}^{\text{Wrist}}_{\text{Base}}H \, {}^{\text{Tool}}_{\text{Wrist}}H $$

$$ = \begin{bmatrix} C_1C_{234}C_5 + S_1S_5 & -C_1C_{234}S_5 + S_1C_5 & -C_1S_{234} & C_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ S_1C_{234}C_5 - C_1S_5 & -S_1C_{234}S_5 - C_1C_5 & -S_1S_{234} & S_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ S_{234}C_5 & S_{234}S_5 & -C_{234} & d_1 - a_2S_2 - a_3S_{23} - a_4S_{234} - d_5C_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Inverse Kinematics in Trajectory Planning

- The information provided by the overhead camera is specified in terms of position and orientation rather than joint variables
- For proper control and trajectory planning, an accurate inverse kinematic model is required
- Challenges faced in modelling inverse kinematics
  - Modeling many degrees of freedom. The full inverse kinematics algorithm is usually supplied (for the specific robot) by the manufacturer
  - No closed-form solution guaranteed for the inverse kinematics. Numerical approach may need more calculation time
  - Navigating with obstacles

Existence of Inverse Kinematics Solution

- For an inverse kinematic solution to exist
  - The specified goal point must lie within the workspace
  - A manipulator will be considered solvable if the joint variables can be determined by an algorithm which allows one to determine all the sets of joint variables associated with a given position and orientation, i.e.,

$$
{}^6_0H = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_0H\,{}^2_1H\,{}^3_2H\,{}^4_3H\,{}^5_4H\,{}^6_5H
$$

Approached for Inverse Kinematics

- Analytical approach (Closed form solutions): The specific approach depends on the type of robots
  - Algebraic Approach
  - Geometric Approach
- Numerical approach (Numerical solutions)

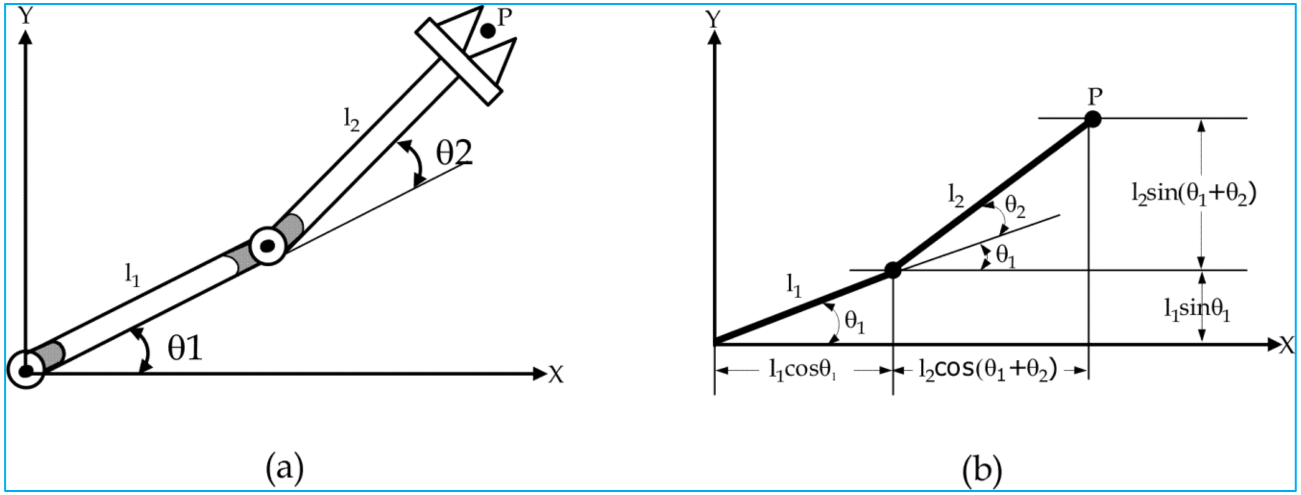
NB: *The scope is restricted to closed form solution methods!!*

Geometric Solution Approach

- Based on the decomposing the spatial geometry of the manipulator into several plane geometry problems
- Is applied to simple robot structure such as 2D planar robots

Example (2D Planar Robot)

- Solve the inverse kinematics based on trigonometry



- The position of end-effector can be expressed in terms of joint variables as

$$
\begin{aligned}
P_x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\
P_y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)
\end{aligned}
$$

Example (2D Planar Robot cont')

- The solution for θ_1 and θ_2 can be computed as

$$\begin{aligned} P_x^2 + P_y^2 &= l_1(C^2\theta_1 + S^2\theta_1) + l_2^2(C^2\theta_{12} + S^2\theta_{12}) + 2l_1l_2(C\theta_1C\theta_{12} + S\theta_1S\theta_{12}) \\ &= l_1^2 + l_2^2 + 2l_1l_2C\theta_2 \end{aligned}$$

- So,

$$\cos\theta_2 = \frac{P_x^2 + P_y^2 - l_1^2 - l_2^2}{2l_1l_2} \quad \text{or} \quad \sin\theta_2 = \pm\sqrt{1 - \left(\frac{P_x^2 + P_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)^2}$$

- Therefore the two possible solutions of θ_2 is given by

$$\boxed{\theta_2 = \arctan 2\left\{\pm\sqrt{1 - \left(\frac{P_x^2 + P_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)^2}, \frac{P_x^2 + P_y^2 - l_1^2 - l_2^2}{2l_1l_2}\right\}}$$

- To solve for θ_1, multiply the equations P_x and P_y with $C\theta_1$ and $S\theta_1$, respectively

$$\begin{aligned} C\theta_1P_x &= l_1C^2\theta_1 + l_2C^2\theta_1C\theta_2 - l_2C\theta_1S\theta_1S\theta_2 \\ S\theta_1P_y &= l_1S^2\theta_1 + l_2S^2\theta_1C\theta_2 + l_2S\theta_1C\theta_1S\theta_2 \end{aligned}$$

- Adding up the above two equations

$$\begin{aligned} C\theta_1P_x + S\theta_1P_y &= l_1(C^2\theta_1 + S^2\theta_1) + l_2C\theta_2(C^2\theta_1 + S^2\theta_1) \\ &= l_1 + l_2C\theta_2 \end{aligned}$$

Example (2D Planar Robot cont`)

- Again multiply the equations P_x and P_y with -$S\theta_1$ and $C\theta_1$, respectively

$$\begin{aligned} -S\theta_1 P_x &= -l_1 S\theta_1 C\theta_1 - l_2 S\theta_1 C\theta_1 C\theta_2 + l_2 S^2\theta_1 S\theta_2 \\ C\theta_1 P_y &= l_1 S\theta_1 C\theta_1 + l_2 C\theta_1 S\theta_1 C\theta_2 + l_2 C^2\theta_1 S\theta_2 \end{aligned}$$

- Which simplifies to

$$-S\theta_1 P_x + C\theta_1 P_y = l_2 S\theta_2$$

- Further manipulation gives

$$\begin{aligned} C\theta_1 P_x^2 + S\theta_1 P_x P_y &= P_x(l_1 + l_2 C\theta_2) \\ -S\theta_1 P_x P_y + C\theta_1 P_y^2 &= P_y l_2 S\theta_2 \\ C\theta_1(P_x^2 + P_y^2) &= P_x(l_1 + l_2 C\theta_2) + P_y l_2 S\theta_2 \end{aligned}$$

- From which

$$\cos\theta_1 = \frac{P_x(l_1 + l_2 C\theta_2) + P_y l_2 S\theta_2}{P_x^2 + P_y^2}$$ or $$\sin\theta_1 = \pm\sqrt{1 - \left(\frac{P_x(l_1 + l_2 C\theta_2) + P_y l_2 S\theta_2}{P_x^2 + P_y^2}\right)^2}$$

- Two possible solution for θ_1 can be expressed as

$$\boxed{\theta_1 = \arctan 2\left\{\pm\sqrt{1 - \left(\frac{P_x(l_1 + l_2 C\theta_2) + P_y l_2 S\theta_2}{P_x^2 + P_y^2}\right)^2}, \frac{P_x(l_1 + l_2 C\theta_2) + P_y l_2 S\theta_2}{P_x^2 + P_y^2}\right\}}$$

Algebraic Solution Approach

- For 3D manipulator, geometric solution approach becomes tedious
- Example of the inverse kinematic solution for a six-axis manipulator

$$
{}^{0}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{0}_{1}T(q_1)\,{}^{1}_{2}T(q_2)\,{}^{2}_{3}T(q_3)\,{}^{3}_{4}T(q_4)\,{}^{4}_{5}T(q_5)\,{}^{5}_{6}T(q_6)
$$

- To find the inverse kinematics solution for the first joint (q_1) as a function of the known elements of ${}^{\text{base}}_{\text{end-effector}}T$, the link transformation inverses are pre-multiplied as follows.

$$
\begin{aligned}
\left[{}^{0}_{1}T(q_1)\right]^{-1}\,{}^{0}_{6}T &= \left[{}^{0}_{1}T(q_1)\right]^{-1}\,{}^{0}_{1}T(q_1)\,{}^{1}_{2}T(q_2)\,{}^{2}_{3}T(q_3)\,{}^{3}_{4}T(q_4)\,{}^{4}_{5}T(q_5)\,{}^{5}_{6}T(q_6) \\
&= {}^{1}_{2}T(q_2)\,{}^{2}_{3}T(q_3)\,{}^{3}_{4}T(q_4)\,{}^{4}_{5}T(q_5)\,{}^{5}_{6}T(q_6)
\end{aligned}
$$

- To find the other variables, the following equations are obtained in a similar manner

$$
\begin{aligned}
\left[{}^{0}_{1}T(q_1)\,{}^{1}_{2}T(q_2)\right]^{-1}\,{}^{0}_{6}T &= {}^{2}_{3}T(q_3)\,{}^{3}_{4}T(q_4)\,{}^{4}_{5}T(q_5)\,{}^{5}_{6}T(q_6) \\
\left[{}^{0}_{1}T(q_1)\,{}^{1}_{2}T(q_2)\,{}^{2}_{3}T(q_3)\right]^{-1}\,{}^{0}_{6}T &= {}^{3}_{4}T(q_4)\,{}^{4}_{5}T(q_5)\,{}^{5}_{6}T(q_6) \\
\left[{}^{0}_{1}T(q_1)\,{}^{1}_{2}T(q_2)\,{}^{2}_{3}T(q_3)\,{}^{3}_{4}T(q_4)\right]^{-1}\,{}^{0}_{6}T &= {}^{4}_{5}T(q_5)\,{}^{5}_{6}T(q_6) \\
\left[{}^{0}_{1}T(q_1)\,{}^{1}_{2}T(q_2)\,{}^{2}_{3}T(q_3)\,{}^{3}_{4}T(q_4)\,{}^{4}_{5}T(q_5)\right]^{-1}\,{}^{0}_{6}T &= {}^{5}_{6}T(q_6)
\end{aligned}
$$

- The nonlinear matrix elements of the right hand of the following equation are either zero, constants, or functions of q_2 through q_6

$$\begin{aligned}\left[{}^0_1T(q_1)\right]^{-1}\,{}^0_6T &= \left[{}^0_1T(q_1)\right]^{-1}\,{}^0_1T(q_1)\,{}^1_2T(q_2)\,{}^2_3T(q_3)\,{}^3_4T(q_4)\,{}^4_5T(q_5)\,{}^5_6T(q_6)\\ &= {}^1_2T(q_2)\,{}^2_3T(q_3)\,{}^3_4T(q_4)\,{}^4_5T(q_5)\,{}^5_6T(q_6)\end{aligned}$$

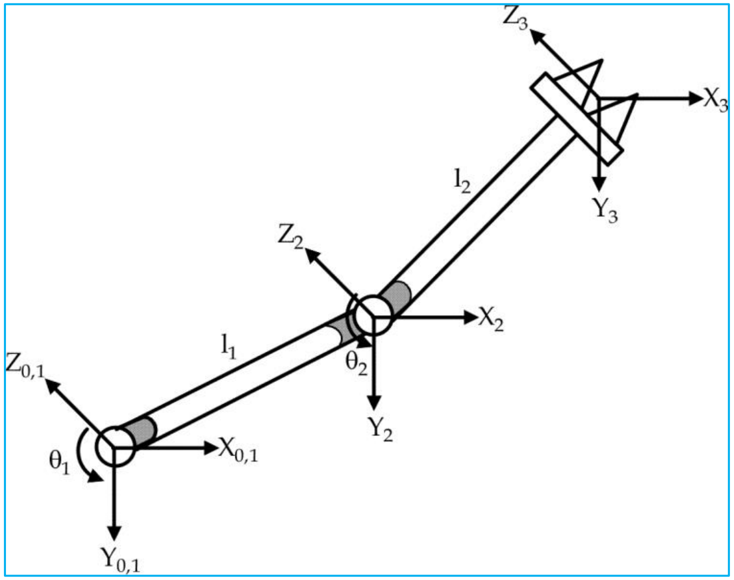
- If the elementd on the left hand side which are the funvtion of q_1 are equated with the elements of the right hand side, the joint varibles q_1 can be solved as a function of $r_{11}, r_{12},.....r_{33}, P_x, P_y, P_z$ and the fixed link parameters
- Once the q_1 is found, the other joint variables are solve the same way

Example (Planar Manipulator)

- Solve the inverse kinematics (Algebraic Approach)

D-H Parameter table

Link	θ_k	α_{k-1}	a_{k-1}	d_k
1	θ1	0	0	0
2	θ2	0	l1	0
3	0	0	l2	0



- The link transformation matrices are given by

$$ {}^{0}_{1}T = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}T = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & l_1 \\ S\theta_2 & C\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- To solve the inverse kinematic

$$ {}^{0}_{3}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T $$

- Multiplying above equation by ${}^{0}_{1}T^{-1}$

$$ {}^{0}_{1}T^{-1}\,{}^{0}_{3}T = {}^{0}_{1}T^{-1}\,{}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T = {}^{1}_{2}T\,{}^{2}_{3}T $$

- Where

$$ {}^{0}_{1}T^{-1} = \begin{bmatrix} {}^{0}_{1}R^{T} & -{}^{0}_{1}R^{T}\,{}^{0}P_1 \\ 000 & 1 \end{bmatrix} $$

here, ${}^{0}_{1}R^{T}$ and ${}^{0}P_1$ denote the transpose of rotation and positive vector of ${}^{0}_{1}T$, respectively

- Substituting transformation matrices gives

$$\begin{bmatrix} C\theta_1 & S\theta_1 & 0 & 0 \\ -S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta_1 & S\theta_1 & 0 & l_1 \\ -S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} . & . & . & C\theta_1 P_x + S\theta_1 P_y \\ . & . & . & -S\theta_1 P_x + C\theta_1 P_y \\ . & . & . & P_x \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} . & . & . & l_2 C\theta_2 + l_1 \\ . & . & . & l_2 S\theta_2 \\ . & . & . & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Squaring the (1,4) and (2,4) matrix elements of each side

$$\begin{aligned} C^2\theta_1 P_x^2 + S^2\theta_1 P_y^2 + 2P_x P_y C\theta_1 S\theta_1 &= l_2^2 C^2\theta_2 + 2l_1 l_2 C\theta_2 + l_1^2 \\ S^2\theta_1 P_x^2 + C^2\theta_1 P_y^2 - 2P_x P_y C\theta_1 S\theta_1 &= l_2^2 S^2\theta_2 \end{aligned}$$

- Adding the resulting equations above gives

$$P_x^2 + P_y^2 = l_2^2 + 2l_1 l_2 C\theta_2 + l_1^2$$

- From which

$$C\theta_2 = \frac{P_x^2 + P_y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

- The two possible solutions for θ_2 are given by

$$\theta_2 = \arctan 2\left\{\pm\sqrt{1-\left(\frac{P_x^2+P_y^2-l_1^2-l_2^2}{2l_1l_2}\right)^2}, \frac{P_x^2+P_y^2-l_1^2-l_2^2}{2l_1l_2}\right\}$$

- First joint variable θ_1 is determined by equating (1,4) elements, i.e,

$$C\theta_1 P_x + S\theta_1 P_y = l_2 C\theta_2 + l_1$$

- From the above equation two solutions for θ_1 can be computed using the trigonometric

$$\theta_1 = \arctan 2(P_x, P_y) \pm \arctan 2\left(\sqrt{P_y^2+P_x^2-(l_2C\theta_2+l_1)^2}, l_2C\theta_2+l_1\right)$$

Degenerancy and Dexterity

- **Degeneracy:** The robot looses a degree of freedom and thus cannot perform as desired
  - When the robot's joints reach their physical limits, and as a result, cannot move any further
  - In the middle point of its workspace if the *z*-axes of two similar joints becomes collinear.
- **Dexterity:** The volume of points where one can position the robot as desired, but no orientate it