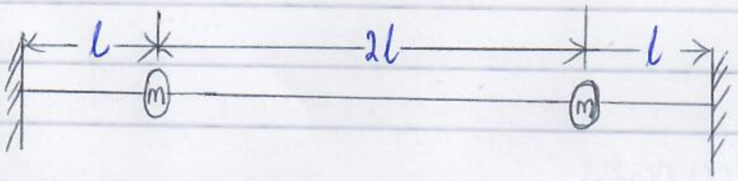
$x^2 + 6y + 9 = 0$

$x+3$
$x+3$
$x^2 + 3x$ $3x + 9$
$x^2 + 6x + 9$

2 dof system

Practice Question 1

1a Find the natural frequencies of the system shown below



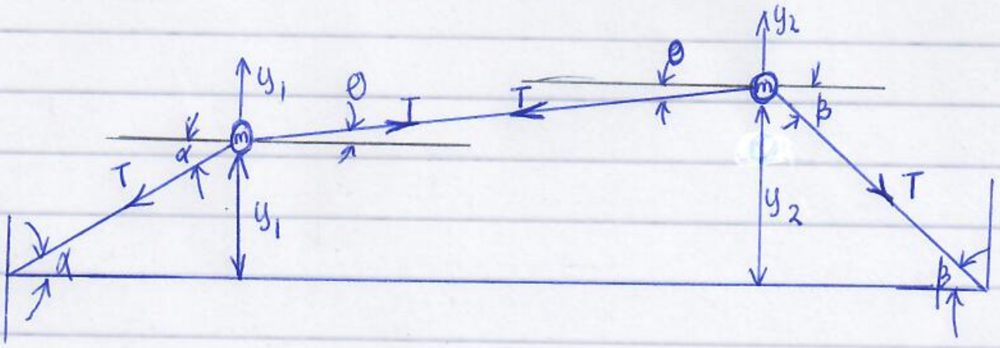
(b) Determine the ratio of amplitudes ~~and locate the nodes for each mode~~ of vibration

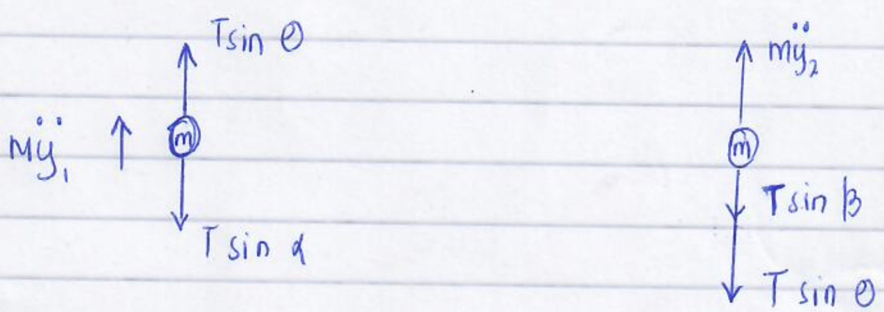
(c) Draw the mode shapes

(d) Repeat the analysis using the eigen value & eigen vector approach

Solution

Give displacement y_1 to mass m and y_2 to the other mass m assumming $y_2 > y_1$





Note that T is the tension in the string

Assume $y_1 = A \sin \omega t$ $\quad y_2 = B \sin \omega t$

Ans

(a) natural frequencies

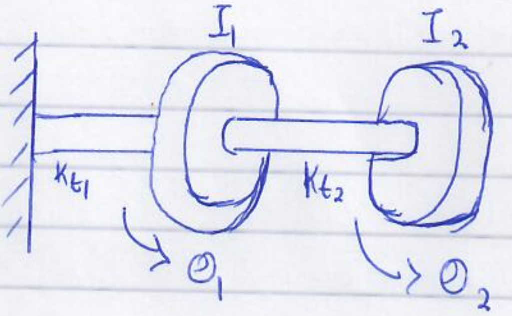
$\omega_{1n} = \sqrt{\frac{T}{ml}}$ rad/s $\qquad \omega_{2n} = 1.41\sqrt{\frac{T}{ml}}$ rad/s

(b) Mode 1 $\quad \frac{A}{B} = 1$

Mode 2 $\quad \frac{A}{B} = -1$

Practice question 2

(a) Determine the natural frequencies of the torsional system shown



(b) ~~Draw the~~ Find the ratios of the amplitude of vibration and hence draw the mode shapes

(c) Repeat the analysis with the eigen value & eigen vector method

S/n : Recall

$$\Sigma M = I\ddot{\theta}$$

Note: Moment of inertia of $I_1 = I$

" " " " $I_2 = 2I$

The stiffness of the connecting rod $K_{t_1} = K_{t_2} = K_t$.

Give I_1 a rotation θ_1 and I_2 a rotation θ_2 assuming $\theta_2 > \theta_1$

Ans $\omega_{1n} = 0.56\sqrt{\frac{K_t}{I}}$ rad/s $\qquad \omega_{2n} = 1.76\sqrt{\frac{K_t}{I}}$ rad/s

Mode 1 $\frac{A}{B} = \frac{1}{2.69}$

Mode 2 $\frac{A}{B} = \frac{1}{-0.189}$

2 dof systems

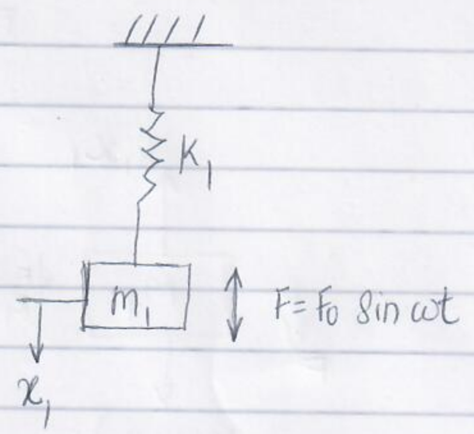
Vibration Absorbers.

If unwanted vibrations exist in a structure that is a single degree of freedom system, then it is possible to eliminate the vibrations by coupling a second vibrating system to it.

If the force exciting the structure has a frequency that is equal or close to that of the structure then resonance and hence severe damage can occur.

The idea of attaching the second mass is to allow it to vibrate freely while killing of the vibration of the main system / structure.

Consider a single D.O.F system subjected to a harmonic force $F = F_0 \sin \omega t$ (Fig 1)



Then, consider attaching a second spring mass system (Fig 2)

The sub-system $K_1 - m_1$ is the main system or structure whose vibrations we want to kill while $K_2 - m_2$ is the auxilliary system or the absorber.

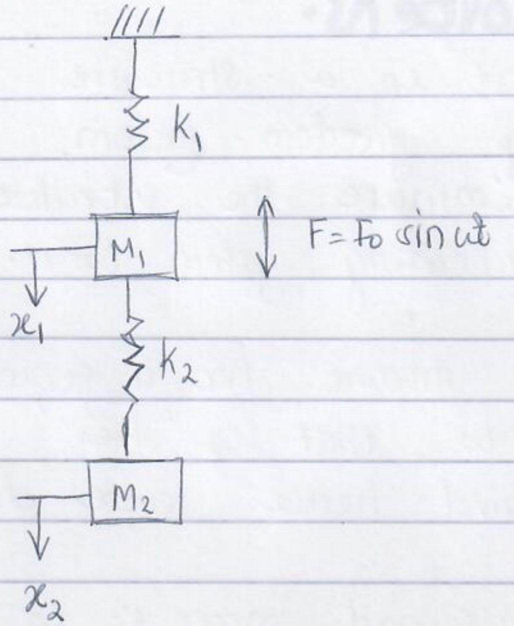


Fig 2

The free body diagrams for Fig 2 are shown in Fig 3

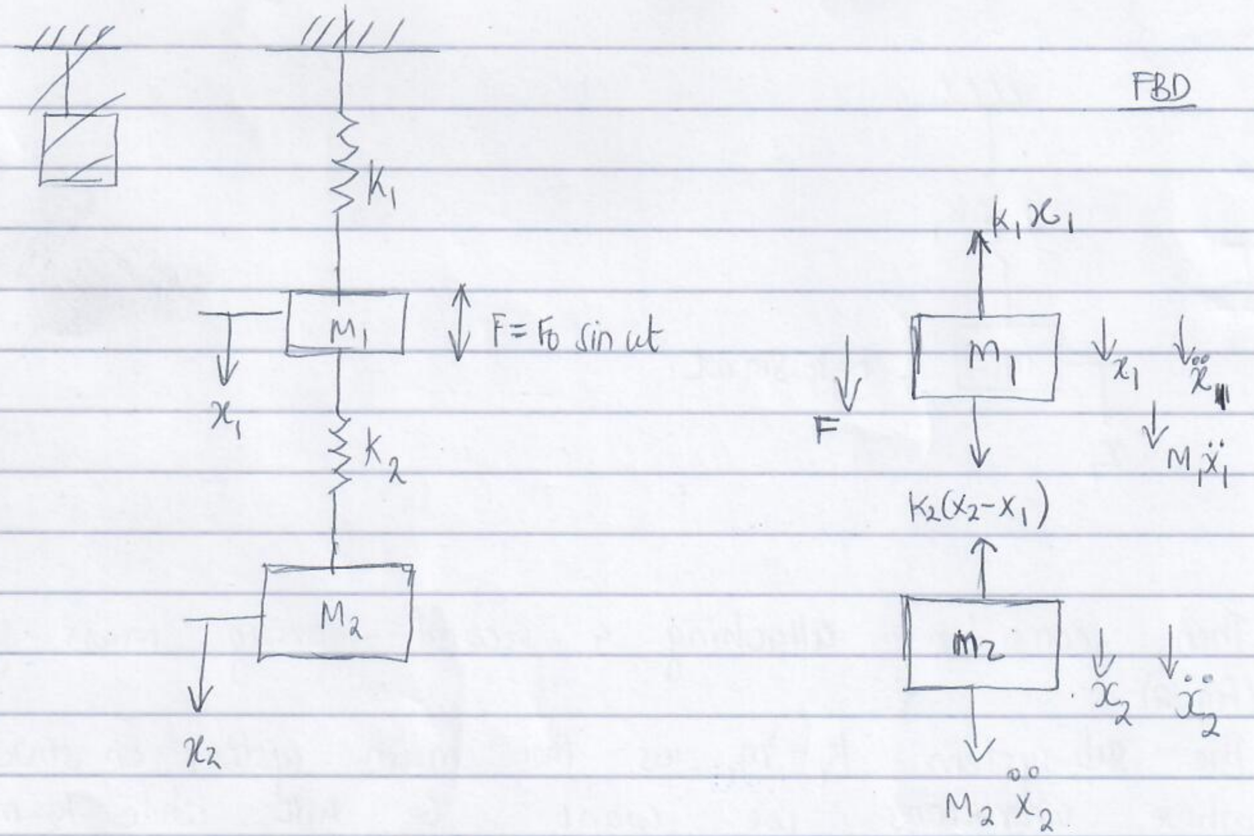


Fig 3

The equations of motion for mass m_1

$\Sigma F = m\ddot{x} \Rightarrow F + k_2(x_2 - x_1) - k_1 x_1 = m_1\ddot{x}_1$

$$m_1\ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = F$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F \quad (1)$$

For mass m_2

$\Sigma F = m\ddot{x} \Rightarrow -k_2(x_2 - x_1) = m_2\ddot{x}_2$

$$m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \quad (2)$$

Assuming due to the forcing function F, the masses will oscillate at the frequency of the external excitation.
In which case we assume

$x_1 = X_1 \sin\omega t$ $\qquad$ $x_2 = X_2 \sin\omega t$

$\dot{x}_1 = \omega X_1 \cos\omega t$ $\qquad$ $\dot{x}_2 = \omega X_2 \cos\omega t$ $\quad (3)$

$\ddot{x}_1 = -\omega^2 X_1 \sin\omega t$ $\qquad$ $\ddot{x}_2 = -\omega^2 X_2 \sin\omega t$

$\cancel{= -\omega^2 x_1}$

X_1, X_2 are amplitudes.

Replacing (3) in (1) and (2) we get

~~(5)~~

$$m_1(-\omega^2 X_1 \sin\omega t) + (k_1 + k_2)X_1 \sin\omega t - k_2 X_2 \sin\omega t = F_0 \sin\omega t$$

$$\Rightarrow (-m_1\omega^2 X_1) + (k_1 + k_2)X_1 - k_2 X_2 = F_0$$

$$[(k_1 + k_2) - m_1\omega^2]X_1 = F_0 + k_2 X_2 \quad (5)$$

for equation (8)

$$(-X_2 \omega^2 \sin \omega t) M_2 + K_2 (X_2 \sin \omega t) - K_2 (X_1 \sin \omega t) = 0$$

$$-X_2 \omega^2 M_2 + K_2 X_2 - K_2 X_1 = 0$$

$$\left[K_2 - m_2 \omega^2\right] X_2 = K_2 X_1$$

$$X_2 = \left[\frac{K_2}{K_2 - m_2 \omega^2}\right] X_1 \qquad (6)$$

Sub (6) in (5) to eliminate X_2

$$\left[(K_1 + K_2) - M_1 \omega^2\right] X_1 = F_0 + K_2 \left[\frac{K_2}{(K_2 - m_2 \omega^2)}\right] X_1$$

$$\left[(K_1 + K_2) - M_1 \omega^2\right] X_1 = F_0 + \left[\frac{K_2^2}{(K_2 - M_2 \omega^2)}\right] X_1$$

$$\left[K_1 + K_2 - M_1 \omega^2\right] X_1 - \left[\frac{K_2^2}{(K_2 - M_2 \omega^2)}\right] X_1 = F_0$$

$$\left[\frac{(K_1 + K_2 - M_1 \omega^2)(K_2 - M_2 \omega^2) - K_2^2}{K_2 - M_2 \omega^2}\right] X_1 = F_0$$

$$X_1 = \frac{F_0 (K_2 - M_2 \omega^2)}{(K_1 + K_2 - M_1 \omega^2)(K_2 - M_2 \omega^2) - K_2^2} \qquad (7)$$

Plugging (7) in (6) we get

$$X_2 = \frac{K_2 F_0}{(K_1 + K_2 - M_1\omega^2)(K_2 - M_2\omega^2) - K_2^2} \quad (8)$$

Now, to kill the vibration of mass M_1 ie $X_1 = 0$
then from the numerator equation (7), $K_2 - M_2\omega^2 = 0$

$$K_2 - M_2\omega^2 = 0$$

$$K_2 = M_2\omega^2$$

$$\omega^2 = \frac{K_2}{M_2} \Rightarrow$$

$$\omega = \sqrt{\frac{K_2}{M_2}} \quad \text{but} \quad \sqrt{\frac{K_2}{M_2}} = \omega_2$$

$\Rightarrow X_1 = 0$ when $\omega = \omega_2$ (9)

and ~~whe~~ because the absorber was required to avoid the situation in which the forcing frequency ω is close or equal to ω_1 thus causing resonance then

$X_1 = 0$ when $\omega = \omega_1 = \omega_2$. (10)

when $\omega = \omega_2$ then (8) becomes

$$X_2 = \frac{K_2 F_0}{\left(K_1 + K_2 - M_1 \omega_2^2\right)\left(K_2 - M_2 \omega_2^2\right) - K_2^2}$$

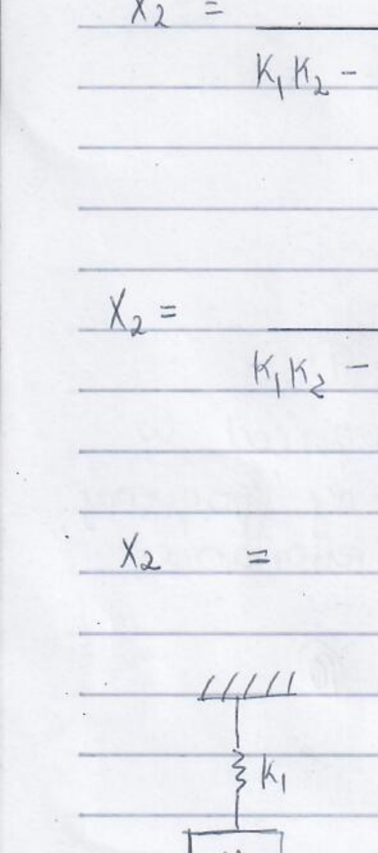
$$X_2 = \frac{K_2 F_0}{K_1 K_2 - M_2 K_1 \omega_2^2 + K_2^2 - M_2 K_2 \omega_2^2 - M_1 K_2 \omega_2^2 + M_1 M_2 \omega_2^4 - K_2^2}$$

$$\omega_2^2 = \left(\frac{K_2}{M_2}\right)$$

$$X_2 = \frac{K_2 F_0}{K_1 K_2 - \cancel{M_2} K_1 \left(\frac{K_2}{\cancel{M_2}}\right) + K_2^2 - \cancel{M_2} K_2 \left(\frac{K_2}{\cancel{M_2}}\right) - M_1 K_2 \left(\frac{K_2}{M_2}\right) + M_1 \cancel{M_2} \left(\frac{K_2}{\cancel{M_2}}\right)^2 - K_2^2}$$

$$X_2 = \frac{K_2 F_0}{K_1 K_2 - K_1 K_2 + K_2^2 - K_2^2 - \frac{M_1}{M_2} K_2^2 + \frac{M_1}{M_2} K_2^2 - K_2^2}$$

$$X_2 = \frac{\cancel{K_2} F_0}{-K_2^2} = \frac{-F_0}{K_2} \qquad (10b)$$



the net force on m_1 is then $F_0 + (-F_0) = 0$ because spring K_2 is attached to mass m_1 and exerts a force $-F_0$ on m_1 which at the same time has a force F_0 impressed on it.

We can divide equation (7) in its expanded form by $K_1 K_2$ to get

$$X_1 = \frac{(K_2 - m_2\omega^2)\dfrac{F_0}{K_1K_2}}{\dfrac{1}{K_1K_2}\left[M_1M_2\omega^4 - [M_1K_2 + M_2(K_1+K_2)]\omega^2 + K_1K_2\right]}$$

$$X_1 = \frac{\left[\dfrac{K_2}{K_2} - \dfrac{m_2}{K_2}\omega^2\right]\dfrac{F_0}{K_1}}{\dfrac{M_1M_2}{K_1K_2}\omega^4 - \left[\dfrac{M_1K_2}{K_1K_2} + \dfrac{M_2(K_1+K_2)}{K_1K_2}\right]\omega^2 + \dfrac{K_1K_2}{K_1K_2}}$$

$$* \; X_1 = \frac{\left[1 - \dfrac{\omega^2}{\omega_2^2}\right]\dfrac{F_0}{K_1}}{\dfrac{\omega^4}{\omega_1^2\omega_2^2} - \left[\dfrac{1}{\omega_1^2} + \dfrac{1}{\omega_2^2} + \dfrac{M_2}{K_1}\right]\omega^2 + 1}$$

let's define μ to be the ratio of the masses
ie $\mu = \dfrac{m_2}{m_1}$ such that $\dfrac{M_2}{K_1} = \dfrac{M_2}{M_1} \times \dfrac{M_1}{K_1} = \mu \times \dfrac{1}{\omega_1^2}$

$$X_1 = \frac{\left[1 - \dfrac{\omega^2}{\omega_2^2}\right]\dfrac{F_0}{K_1}}{\dfrac{\omega^4}{\omega_1^2\omega_2^2} - \left[\dfrac{1}{\omega_1^2} + \dfrac{1}{\omega_2^2} + \dfrac{\mu}{\omega_1^2}\right]\omega^2 + 1}$$

$\cdot\ 0.4096 - (2+u)\ 0.64 + 1 = 0$

$1.4096 - (2+u)\ 0.64 = 0$

$2+u = \frac{1.4096}{0.64}$ $\quad 2.2025$

$u = 0.2$

$$X_1 = \frac{\left[1 - \frac{\omega^2}{\omega_2^2}\right] \frac{F_0}{K_1}}{\frac{\omega^4}{\omega_1^2 \omega_2^2} - \left[(1+\mu)\frac{\omega^2}{\omega_1^2} + \frac{\omega^2}{\omega_2^2}\right] + 1} \qquad (11)$$

~~At~~ when the denominator of (11) goes to zero X_1 becomes infinite. In other words resonance occurs.
In the special case where $\omega_1 = \omega_2$ we can find the resonant frequencies.

$$\frac{\omega^4}{\omega_2^2 \omega_2^2} - \left[\frac{\omega^2}{\omega_2^2} + \mu \frac{\omega^2}{\omega_2^2} + \frac{\omega^2}{\omega_2^2}\right] + 1 = 0$$

$$\frac{\omega^4}{\omega_2^4} - (2+u)\frac{\omega^2}{\omega_2^2} + 1 = 0$$

$$\left(\frac{\omega}{\omega_2}\right)^4 - (2+u)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0 \qquad (12)$$

let $P = \left(\frac{\omega}{\omega_2}\right)^2$

Then (12) becomes

$$P^2 - (2+u)P + 1 = 0$$

Using the quadratic formula where $a = 1$ $b = -(2+u)$ and $c = 1$ we have

$$P_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2+u \pm \sqrt{(2+u)^2 - 4}}{2}$$

$$= \frac{2+u}{2} \pm \frac{\sqrt{4 + u^2 + 4u - 4}}{2}$$

$$= 1 + \frac{u}{2} \pm \frac{\sqrt{u^2 + 4u}}{2}$$

$$= 1 + \frac{u}{2} \pm \sqrt{\frac{4u}{4} + \frac{u^2}{4}}$$

$$= \left[1 + \frac{u}{2}\right] \pm \sqrt{u + \frac{u^2}{4}}$$

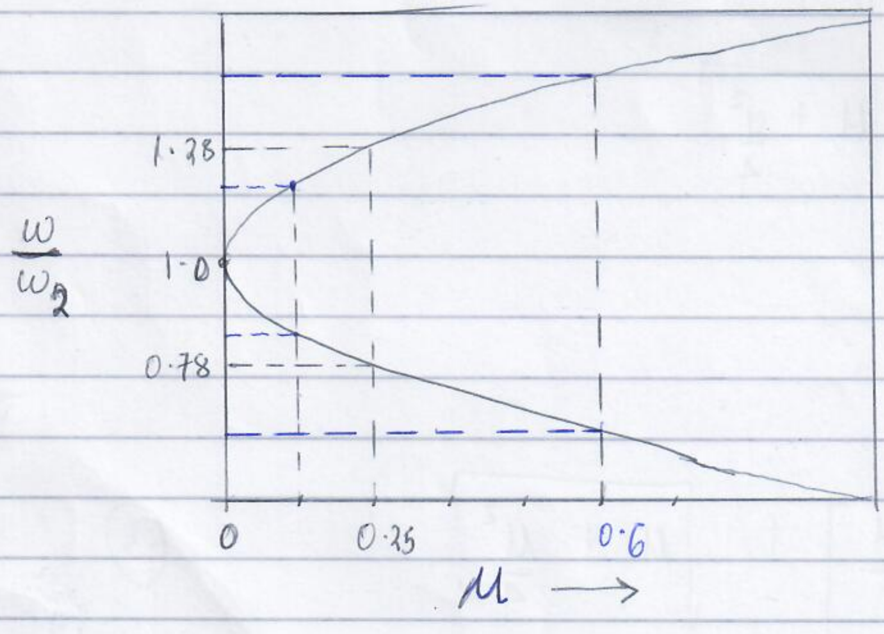
but $P = \left(\frac{\omega}{\omega_2}\right)^2$

$$\left(\frac{\omega}{\omega_2}\right)^2 = \left[1 + \frac{u}{2}\right] \pm \sqrt{u + \frac{u^2}{4}} \qquad (13)$$

when $\mu = 0$ ~~ie there is no second mass~~
then $\frac{\omega}{\omega_2} \approx 1$ ie the resonant frequency is
equal to the natural frequency of the main system

For each value of μ, there are two resonant frequencies: one below and another above the natural frequency of the main system ~~as shown by~~ corresponding to the plus and minus signs of (13)

The general ~~plot~~ behavior indicated by equation (13) is shown in Fig 4



eg for $\mu = 0.25$
we have resonant frequencies at
$\frac{\omega}{\omega_2} = 0.78$

$\Rightarrow \omega_r = 0.78\omega_2$

and

$\frac{\omega}{\omega_2} = 1.28$

$\rightarrow \omega_r = 1.28\,\omega_2$

Fig 4

The larger μ is the wider the range between the resonant frequencies and thus the wider the operating range. Even if ω/ω_2 is not exactly equal to 1, the wider range means the system does not operate anywhere close to resonance.

Example ~~1~~

A shaper runs at 5000 rpm
Its forcing frequency is very near the machine's natural frequency. If it is desired that the nearest frequency of the machine is to be at least 20% from the forcing frequency, design a suitable absorber. Mass of the machine is 30 kg

Solution

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 5000}{60} = 523.3 \text{ rad/s}$$

~~for~~ ~~the~~ Requirements $\frac{\omega}{\omega_2} = 0.8$

or $\frac{\omega}{\omega_2} = 1.2$

Using eqn (12)

for $\frac{\omega}{\omega_2} = 0.8 \Rightarrow \left(\frac{\omega}{\omega_2}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$

$$0.8^4 - (2+\mu)(0.8)^2 + 1 = 0$$

$$\mu = 0.2$$

for $\frac{\omega}{\omega_2} = 1.2 \Rightarrow (1.2)^4 - (2+\mu)(1.2)^2 + 1 = 0$

$$\mu = 0.13$$

From fig 4, the bigger the ratio of the masses μ, the bigger the separation between the resonant frequencies and hence the wider the operating range. Thus we pick the larger value of μ for design

$$\mu = 0.2 = \frac{m_2}{m_1} = \frac{m_2}{30kg}$$

$$\Rightarrow m_2 = 6.0 \ kg$$

We know that for a vibration absorber

$\omega_1 = \omega_2 = \omega_n$

$$\omega_1 = \sqrt{\frac{K_1}{m_1}} \qquad \omega_1^2 = \frac{K_1}{m_1} \quad \Rightarrow K_1 = +$$

$$\omega_1^2 = \frac{K_1}{m_1} \qquad K_1 = \omega_1^2 \times m_1$$

$$= (523.33)^2 \times 30$$

$$= 8216.22 \ KN/M$$

$$\omega_2^2 = \frac{K_2}{m_2} \qquad K_2 = (523.33)^2 \times 6$$

$$= 1643.24 \ KN/M.$$

Example 2

A diesel engine weighing 3000N is placed on a pedestal mount which transmit vibration to the surrounding area.
The machine operates at a speed of 6000 rpm. If the magnitude of the exciting force is 250N and the amplitude of the absorber mass is to be limited to 2mm determine the parameters of the absorber

Solution

$$\omega_f = \frac{2\pi f}{60} = \frac{2\pi \times 6000}{60} = 628.32 \text{ rad/s}$$

$$F_0 = |K_2 X_2| \quad \text{from eqn (10 b)}$$

but $K_2 = m_2 \omega^2$

$$F_0 = m_2 \omega^2 X_2$$

$$m_2 = \frac{F_0}{\omega^2 X_2} = \frac{250}{(628.32)^2 (0.002)} = 0.3166 \text{ Kg}$$

$$=$$

Thus $k_2 = m_2 \omega^2$

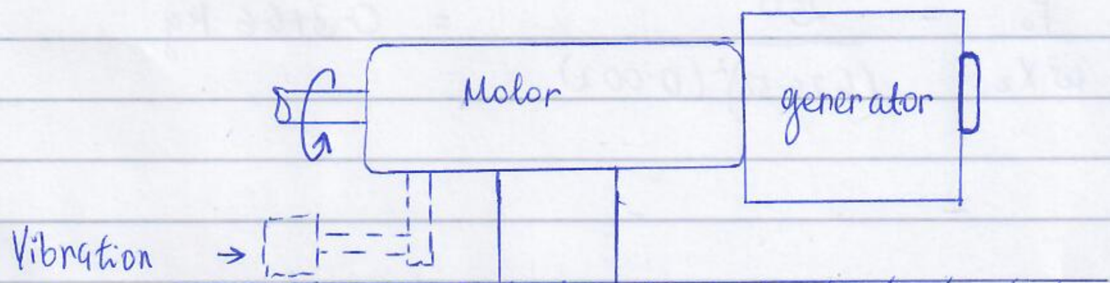
$= (0.3166)(628.32)^2$

$= 124.989$ KN/m

Example 3

A generator coupled to a motor is designed to operate in the speed range of 2000 rpm to 4000 rpm.
However, the combined generator - motor set vibrates violently at a speed of 3000 rpm due to an unbalance in the rotor
The proposal is to attach a cantilever type lumped mass absorber system to eliminate the issue.
when a cantilever carrying a trial mass of 2kg tuned to 3000 rpm is attached to the set, the resonant natural frequencies of the system are found to be 2500 rpm and 3500 rpm

Determine the mass and stiffness of the absorber such that the natural frequencies of the total system fall outside the operating - speed range of the generator - motor set.



Recall that

$$\left(\frac{\omega_r}{\omega_2}\right)^2 = \left(1 + \frac{\mu}{2}\right) \pm \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \qquad (A)$$

The absorber is tuned so that $\omega_1 = \omega_2 = \frac{3000 \times 2\pi}{60}$

$= 314.16 \text{ rad/s}$

$\omega_{r_1} = 2500 \text{ rpm} = \frac{2500 \times 2\pi}{60} = 261.80 \text{ rad/s}$

$\omega_{r_2} = 3500 \text{ rpm} = \frac{3500 \times 2\pi}{60} = 366.52 \text{ rad/s}$

$$\frac{\omega_{r_1}}{\omega_2} = \frac{261.80}{314.16} = 0.833$$

$$\frac{\omega_{r_2}}{\omega_2} = \frac{366.52}{314.16} = 1.1667$$

Now in equation (A) the lower frequency ratio is obtained when the first term subtracts the second

$$\Rightarrow \quad \frac{\omega_{r_1}}{\omega_2} = 1 + \frac{\mu}{2} \pm \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} = 0.8333$$

$$\Rightarrow \mu = 0.1345 = \frac{m_2}{m_1} = \frac{2\text{kg}}{m_1} \Rightarrow m_1 = \frac{2\text{kg}}{0.1345} = 14.869 \text{ kg}$$

However, the lowest operating speed of the set is 2000 rpm = 209.44 rad/s

$\frac{\omega}{\omega_2} = \frac{209.44}{314.16} = 0.6667$

for $\frac{\omega}{\omega_2} = 0.6667$ equation (A) gives $\mu = 0.6942$

$\mu = \frac{m_2}{m_1}$ $\quad M_2 = m_1 \cdot \mu$
$= 14.869 \times 0.6942$
$= 10.3227$ kg $\quad$ (why ?)

=> The second resonant frequency is obtained by using $\mu = 0.6942$

$$\left(\frac{\omega_{r2}}{\omega_2}\right)^2 = \left(1 + \frac{0.6942}{2}\right) + \sqrt{\left(1 + \frac{0.6942}{2}\right)^2 - 1}$$

$= 2.2497$

$\omega_{r2} = \sqrt{2.2497} \times \omega_2$
$= 471.209$ rad/s

$= \frac{471.209 \times 60}{2\pi} = 4499.71$ rpm

Therefore the second/higher resonance frequency is ~~outside~~ higher than the upper limit of the machine's operating speed ie 4000 rpm

$\Rightarrow \quad K_2 = m_2 \omega_2^2$

$= 10.3227 \times (314.16)^2$

$= 1.02 \ MN/m^2$

$\Rightarrow \quad k_2 = m_2 \omega_2^2$

$= 10.3227 \times (314.16)^2$

$= 1.02 \text{ MN/m}^2$