EMT 2542 Industrial Robotics

Course Purpose
The purpose of this course to enable the learner to understand the theoretical basis of the differential kinematics, statics, Lagrange, and Newton-Euler dynamics as well as carrying out robotic inverse kinematic.

Learning Outcomes
At the end of this course, the student should be able to:

1. Carry out kinematics and dynamic analysis of a robotic system;
2. Lay out simple systems and more complex systems of robotic movement by using suitable computer tools;
3. Select adequate notation, model and description for solving practical problems;
4. Apply the acquired knowledge to design a multi-degrees of freedom robot arm manipulator.

Course Description
Overview of key technologies in robotics and kinematic arrangements of robotic manipulators. Spatial Descriptions and Transformations: Descriptions of position and orientation, frames, rotations of a frame, homogeneous transformations, transform arithmetic. Manipulator Kinematics: Link description, link connection, Denavit-Hartenberg parameters, forward kinematics. Inverse Kinematics: Solvability, algebraic and geometric approaches, inverse instantaneous kinematics. Velocities and Static Forces: Linear and angular velocity of a rigid body, velocity propagation, Jacobian and singularities, equivalent joint torques, transformation of forces and moments, kinematics and statics duality, stiffness. Manipulator Dynamics: Important properties of dynamic model, Newton-Euler formulation for single and multi-body systems, Lagrangian formulation of robot dynamics. Case study: Manipulator-mechanism design.

Teaching Methodology: 3 credits (2-hours lecture and 3-hours practical session per week). The practical session comprises of laboratory work and/or tutorial.

Instruction Materials/Equipment: Textbooks, Journals, White Boards and Markers, Computers, LCD projector, Overhead projector, Control laboratory.

Mode of Assessment: The course shall be examined by both course work (accounting for 40%) and an end of semester written examination constituting 60%.

Course Textbooks

1. Craig, J.J. (2017). *Introduction to Robotics-Mechanics and Control* (4th ed.). Pearson, Prentice Hall. ISBN-13: 978-0133489798
2. Ramachandran, S., Lazarus B., Lakshmi, V. (2017). *Robotics.* Airwalk Publications. ISBN: B0194FVULY
3. Choset, H., Lynch, K.M., Hutchinson, S., Kantor, G., Burgard, W., Kavraki, L. E., Thrun, S. (2005). *Principles of Robot Motion: Theory, Algorithms, and Implementations.* MIT Press. ISBN-13: 978-0262033275

Course Journals

1. Journal of Robotics and Autonomous Systems, ISSN: 0921-8890
2. Journal of Mechanisms and Robotics, ISSN: 1942-4302

Course References

1. Sciavico, L. & Siciliano B. (2001). *Modeling and Control of Robot Manipulators.* The McGraw ? Hill Companies, Inc.: New York. ISBN-13: 978-1852332211
2. Siciliano, B., Khatib, O. (2017). *Handbook of Robotics* (2nd ed.). Springer-Verlag Berlin Heidelberg. ISBN-13: 978-3319325507
3. Sahin, F. & Kachroo, P. (2007). *Practical and Experimental Robotics.* CRC Press. ISBN-13: 978-1420059090

Contents

1 Introduction to Robotics

An industrial robot is a general-purpose, programmable machine possessing certain anthropomorphic characteristics. The most obvious anthropomorphic characteristic characteristics of an industrial robot is its mechanical arm,which is used to perform various industrial tasks. Other human-like characteristics are the robot's capabilities to respond to sensory inputs, communicate with other machines, and make decisions.
The development of robotic technology followed the development of numerical control and the two technologies are quite similar. They both involve coordinated control of multiple axes (the axes are called *joints* in robotics) and they both use dedicated digital computers as controllers.
Whereas NC machines are designed to perform specific processes (eg machining sheet metal hole punching and thermal cutting) robots are designed for a wide variety of tasks. Typical production applications of industrial robots include spot welding, material transfer, machine loading, spray painting and assembly.
Some of the qualities that make industrial robots commercially and technologically important includes

1. Robots can be substituted for humans in hazardous or uncomfortable work environments.

2. Robots can perform its work cycle with a consistency and repeatability that cannot be attained by humans

3. Robots can be reprogrammed. When the production run of current task is completed, a robot can be reprogrammed and equipped with the necessary tooling to perform an altogether different task.

4. Robots are controlled by computers and can therefore be connected to other computer systems to achieve computer integrated manufacturing.

1.1 Basic terms used in robotics

There is a set of basic terminology and concepts common to all robots. These terms follow with brief explanations of each.

1. **Links and Joints** - Links are the solid structural members of a robot, and joints are the movable couplings between them.

2. **Degree of Freedom (dof)** - Each joint on the robot introduces a degree of freedom. Each dof can be a slider, rotary, or other type of actuator. Robots typically have 5 or 6 degrees of freedom. 3 of the degrees of freedom allow positioning in 3D space, while the other 2or 3 are used for orientation of the end effector. 6 degrees of freedom are enough to allow the robot to reach all positions and orientations in 3D space. 5 dof requires a restriction to 2D space, or else it limits orientations. 5 dof robots are commonly used for handling tools such as arc welders.

3. **Orientation Axes** - Basically, if the tool is held at a fixed position, the orientation determines which direction it can be pointed in. Roll, pitch and yaw are the common orientation axes used. Looking at the figure below it will be obvious that the tool can be positioned at any orientation in space. (imagine sitting in a plane. If the plane rolls you will turn upside down. The pitch changes for takeoff and landing and when flying in a crosswind the plane will yaw.)

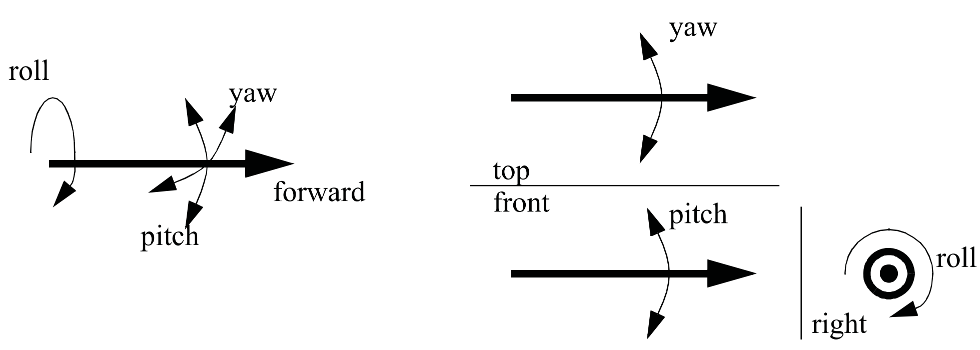


Figure 1: Orientation

4. **Position Axes** - The tool, regardless of orientation, can be moved to a number of positions in space. Various robot geometries are suited to different work geometries. (more later)

5. **Tool Centre Point (TCP)** - The tool centre point is located either on the robot, or the tool. Typically the TCP is used when referring to the robots position, as well as the focal point of the tool. (e.g. the TCP could be at the tip of a welding torch) The TCP can be specified in cartesian, cylindrical, spherical, etc. coordinates depending on the robot. As tools are changed we will often reprogram the robot for the TCP.

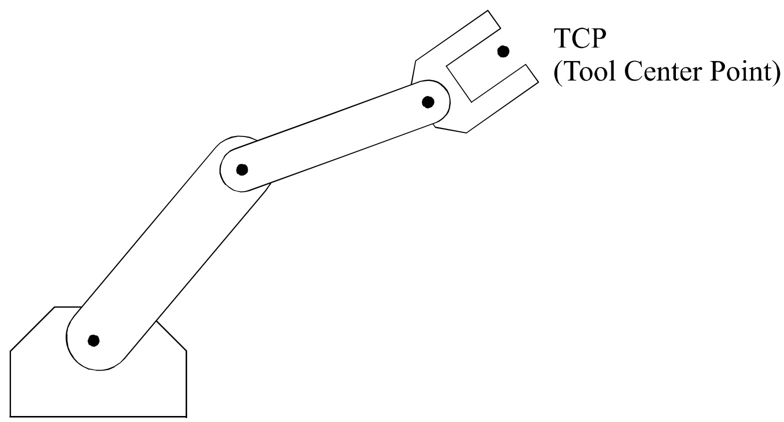


Figure 2: The Tool center point TCP

6. **Work envelope/Workspace** - The robot tends to have a fixed, and limited geometry. The work envelope is the boundary of positions in space that the robot can reach. For a cartesian robot (like an overhead crane) the workspace might be a square, for more sophisticated robots the workspace might be a shape that looks like a clump of intersecting bubbles.

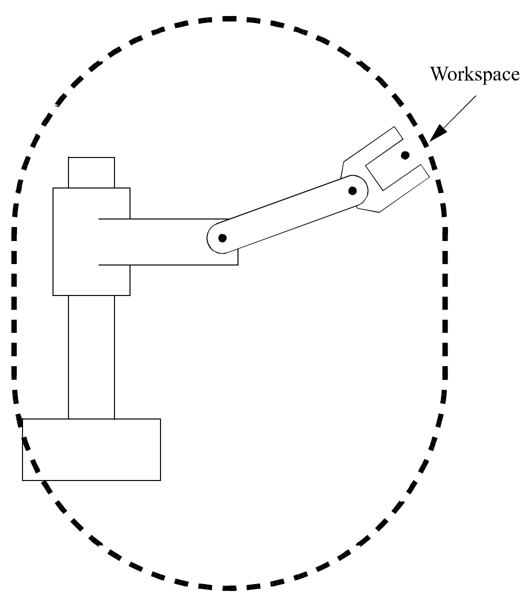


Figure 3: Work envelop

7. **Speed** - refers either to the maximum velocity that is achievable by the TCP, or by individual joints. This number is not accurate in most robots, and will vary over the workspace as the geometry of the robot changes (and hence the dynamic effects). The number will often reflect the maximum safest speed possible. Some robots allow the maximum rated speed (100%) to be passed, but it should be done with great care.

8. **Payload** - The payload indicates the maximum mass the robot can lift before either failure of the robots, or dramatic loss of accuracy. It is possible to exceed the maximum payload, and still have the robot operate, but this is not advised. When the robot is accelerating fast, the payload should be less than the maximum mass. This is affected by the ability to firmly grip the part, as well as the robot structure, and the actuators. The end of arm tooling should be considered part of the payload.

9. **Repeatability** - The robot mechanism will have some natural variance in it. This means that when the robot is repeatedly instructed to return to the same point, it will not always stop at the same position. Repeatability is considered to be +/-3 times the standard deviation of the position, or where 99.5% of all repeatability measurements fall. This figure will vary over the workspace, especially near the boundaries of the workspace, but manufacturers will give a single value in specifications.

10. **Accuracy**- This is determined by the resolution of the workspace. If the robot is commanded to travel to a point in space, it will often be off by some amount, the maximum distance should be considered the accuracy. This is an effect of a control system that is not necessarily continuous.

11. **Settling Time** - During a movement, the robot moves fast, but as the robot approaches the final position is slows down, and slowly approaches. The settling time is the time required for the robot to be within a given distance from the final position.

12. **Control Resolution** - This is the smallest change that can be measured by the feedback sensors, or caused by the actuators, whichever is larger. If a rotary joint has an encoder that measures every 0.01 degree of rotation, and a direct drive servo motor is used to drive the joint, with a resolution of 0.5 degrees, then the control resolution is about 0.5 degrees (the worst case can be 0.5+0.01).

13. **Coordinates** - The robot can move, therefore it is necessary to define positions. Note that coordinates are a combination of both the position of the origin and orientation of the axes.

1.2 Robot types

- Robots come in a wide variety of shapes, and configurations.

- The major classes of robots include,
  arms - fixed in place, but can reach and manipulate parts and tools
  mobile - these robots are free to move

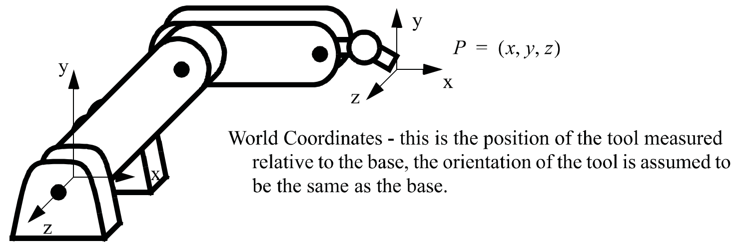


Figure 4: World Coordinates - To Locate the TCP

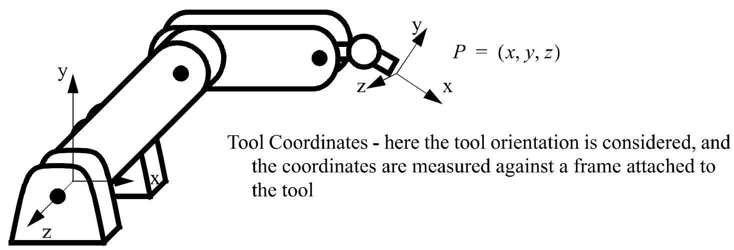


Figure 5: Tool Coordinates - Describing Positions Relative to the Tool

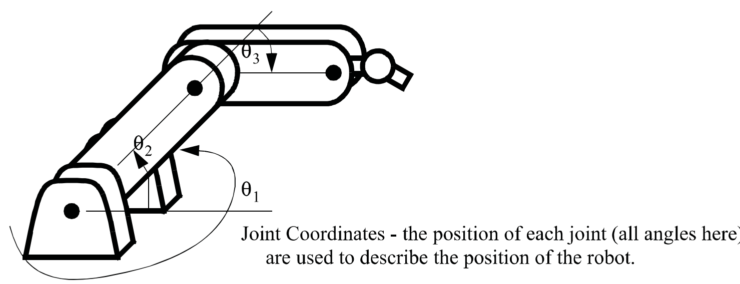


Figure 6: Joint Coordinates - the Positions of the Actuators

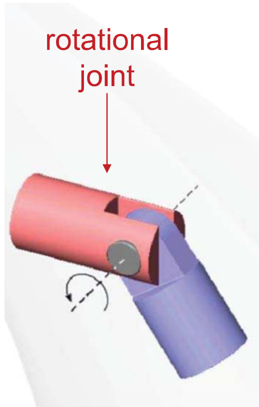
1.2.1 Robotic Arms

- Typical joint types are,
  Revolute - rotary joints often driven by electric motors and chain/belt/gear transmissions, or by hydraulic cylinders and levers.
  Prismatic - slider joints in which the link is supported on a linear slider bearing, and linearly actuated by ball screws and motors or cylinders.

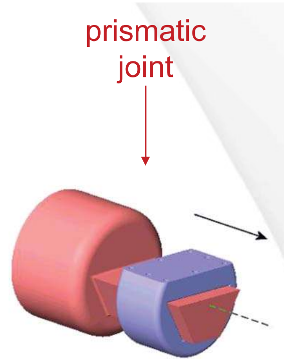
- Basic configurations are,
  Cartesian/Rectilinear/Gantry - Positioning is done in the workspace with prismatic joints. This configuration is well used when a large workspace must be covered, or when consistent accuracy is expected from the robot.

  Cylindrical - The robot has a revolute motion about a base, a prismatic joint for height, and a prismatic joint for radius. This robot is well suited to round workspaces.

  Spherical - Two revolute joints and one prismatic joint allow the robot to point



(a) Revolute



(b) Prismatic

Figure 7: Typical joint types in robotics

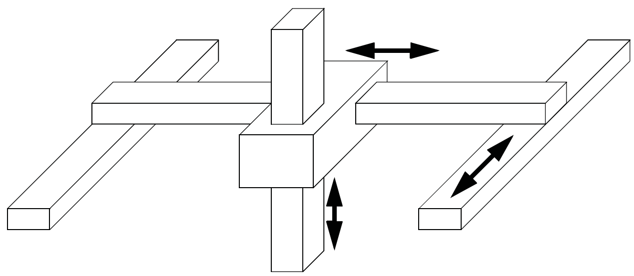


Figure 8: Cartesian Robot

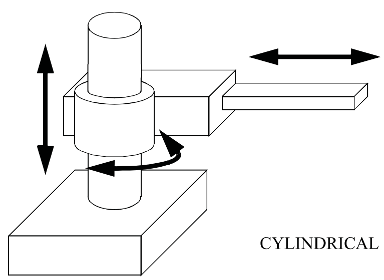


Figure 9: Cylindrical Robot

in many directions, and then reach out some radial distance.

Articulated/Jointed Spherical/Revolute-The robot uses 3 revolute joints to position the robot. Generally the work volume is spherical. This robot most resembles the human arm, with a waist, shoulder, elbow, wrist.

Scara (Selective Compliance Arm for Robotic Assembly) - This robot conforms to cylindrical coordinates, but the radius and rotation is obtained by a two planar links with revolute joints.

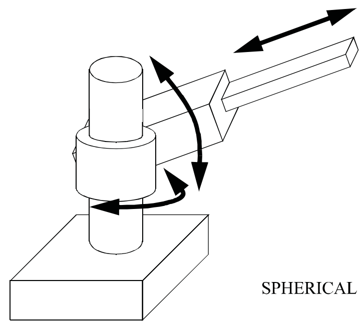


Figure 10: Spherical Robot

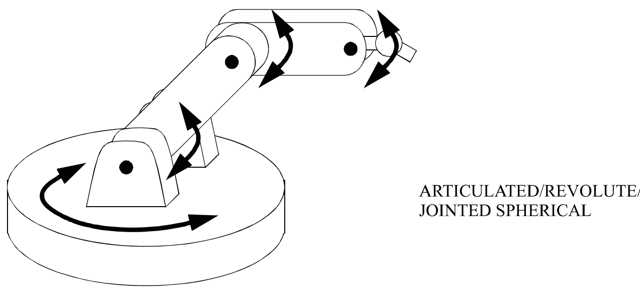


Figure 11: Articulated/Revolute/ Jointed Spherical Robot

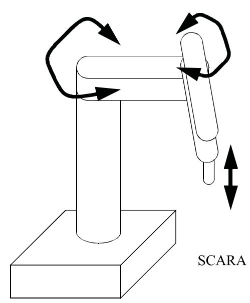


Figure 12: Scara robot

1.3 Joint drive systems

Robot joints are actuated using any of three types of drive systems

- Electric
- Hydraulic
- Pneumatic

Electric drive systems use electric motors as joints actuators (eg servo motors or stepper motors). Hydraulic and pneumatic systems use devices such as linear pistons and rotary vane actuators to accomplish the motion of the joints.

Pneumatic drive is typically limited to smaller robots used in simple material transfer applications. Electric drive and hydraulic drives are used on more sophisticated industrial robots. Electric drive has become the preferred drive system in commercially available robots as electric motor technology has advanced in recent years. It is more readily adaptable to computer control which is the dominant technology used today on robot controllers. Electric drive robots are relatively accurate compared with hydraulically powered robots. By contrast, the advantage of hydraulic drive include greater speed and strength.

1.4 Robot control systems

The actuation of the individual joints must be controlled in a coordinated fashion for the manipulator to perform a desired motion cycle. Microprocessor-based controllers are commonly used today in robotics as the control system hardware. The controller is organized in a hierarchical structure as shown in Fig. **??** so that each joint has its own feedback control system, and a supervisory controller to coordinate the combined actuation of the joints according to the sequence of robot program. Different types of controls are required for different applications. Robots controllers can be classified into four categories

1. Limit sequence control
2. Playback with point-to-point control
3. Playback with continuous path control
4. Intelligent control

Limit sequence control:

This is the most elementary control type. It can be utilized only for simple motion cycles, such as pick and place operations (i.e picking an object up at one location and placing it at another location). It is usually implemented by setting limits or mechanical stops for each joint and sequencing the actuation of the joints to accomplish the cycle. Feedback loops are sometimes used to indicate that the particular joint actuation has been accomplished so that the next step in the sequence can be initiated. However, there is no servo-control to accomplish precise positioning of the joint. Many pneumatically driven robots are limited sequence robots.

Playback with point-to-point control:

Playback robots represent more sophisticated form of control than limited sequence robots. Playback control means that the controller has a memory to record the sequence of motion in a given work cycle as well as the locations and other parameters (such as speed0 associated with each motion and the to subsequently play back the work cycle during execution of the program. In point-to-point (PTP) control, individual position of the robot arm are recorded into memory. These positions are not limited to mechanical stops for each joint as in limited sequence robots. Instead, each position in the robot program consists of a set of values representing locations in the range of each joint of the manipulator. Thus, each "point" consists of five or six values corresponding to the positions of each of the five or six joints of the manipulator. Feedback control is used during the motion cycle to confirm that the individual joints achieve the specified location in the program.

Playback with continuous path control

Continuous path robots have the same playback capability as the previous type. The difference between the continuous path and point-to-point is the same in robotics as it is in NC. A playback robot with continuous path control is capable of one or both of the following:

1. *Greater storage capacity* The controller has a far greater storage capacity than its point-to-point counterpart, so the number of locations that can be recorded into memory is far greater than for point-to-point Thus, the points constituting the motion cycle can be spaced very closely together to permit the robot to accomplish a smooth continuous motion. In PTP, only the final location of the individual motion elements are controlled, so the path taken by the arm to reach the final location is not controlled. In a continuous path motion, the movement of the arm and wrist is controlled during the motion.

2. *Interpolation calculations* The controller computes the path between the starting point and the ending point for each move using interpolation routines similar to those used in NC. These routines generally include linear and circular interpolation.

The difference between PTP and continuous path control can be distinguished in the following mathematical way. Consider a three axes cartesian coordinate manipulator in which the end-of-arm is moved in $x-y-z$ space. In the point-to-point systems, the x, y and z axes are controlled to achieve a specified point location within the robot's work volume. In continuous path systems, not only are the x, y and z axes controlled, but the velocitiesdx/dt, dy/dt and dz/dt are controlled simultaneously to achieve the specified linear or curvilinear path. Servo-control is used to continuously regulate the position and speed of the manipulator. It should be mentioned that a playback robot with continuous path control has the capacity for PTP control.

Intelligent control

An intelligent robot is one that exhibit behavior that makes it seem intelligent. Some of the characteristics that make a robot appear intelligent include the capacities to interact with its environment, make decisions when things go wrong during the work cycle, communicate with humans, make computations during the motion cycle, and respond to advanced sensor inputs such as machine vision.

In addition, robot with intelligent control possess playback capability for both PTP or continuous path control. These features require (1) a relatively high level of computer control and (2) an advanced programming language to input the decision-making logic and other “intelligence” into memory.

1.5 End Effectors

The end effector enables the robot to accomplish a specific task. There are two categories of end effector namely grippers and tools

Grippers

Grippers are end effectors used to grasp and manipulate objects during the work cycle. Owing to the variety of part shapes, sizes and weights, most grippers must be custom designed. Types of grippers used in industrial robot applications include the following

- *Mechanical grippers*, consisting of two or more fingers that can be actuated by the robot controller to open and close to grasp the workpart
- *Vacuum gripper*, in which suction cups are used to hold flat objects
- *Magnetized devices*, for holding ferrous parts
- *Adhesive devices*, which use an adhesive substance to hold a flexible material such as a fabric
- *Simple mechanical devices* such as hooks and scoops.

Mechanical grippers are the most common gripper type. Some of the innovations and advantages in mechanical gripper technology include:

- *Interchangeable fingers* that can be used on one gripper mechanism. To accommodate different parts, different fingers are attached to the gripper.
- *Sensory feedback* in the fingers that provide the gripper with capabilities such as (1) sensing the presence of the workpart or (2) applying a specified limited force to the workpart during gripping (for fragile workparts)
- *Multiple fingered grippers* that possess the general anatomy of human hand.
- *Standard gripper products* that are commercially available, thus reducing the need to custom-design a gripper for each separate robot application.

Tools

The robot uses tools to perform processing operations on the workpart. The robot manipulates the tool relative to stationary or slowly moving object (e.g., workpart or subassembly). Examples of the tools used as end effectors by robots to perform processing applications include spot welding gun, arc welding tool; spray painting gun; rotating spindle for drilling, routing, grinding, and similar operations; assembly tool (eg automatic screwdriver); heating torch; ladle (for metal casting); and water jet cutting tool. In each case, the robot must not only control the relative position of the tool with respect to the work as a function of time, it must also control the operation of the tool

1.6 Sensors in robotics

Sensors used in industrial robotics can be classified into two categories (1) internal and (2) external. Internal sensors are components of the robot and are used to control to control the positions and velocities of the various joints of the robot. These sensors form a feedback control loop with the robot controller. Typical sensors used to control the position of the robot arm include potentiometers and optical encoders. Tachometers of various types are used to control the speed of the robot arm.
External sensors are external to the robot and are used to coordinate the operation of the robot with the the other equipments in the cell. In many cases, these external sensors are relatively simple devices, such as limit switches that determines whether a part has been positioned properly in a fixture or that a part is ready to be picked up at a conveyor. Other situations require more advanced sensor technologies, including the following:

- *Tactile sensors.* These are used to determine whether contact is made between the sensor and another object. Tactile sensors can be divided into two types in robot applications: (1) touch sensors and (2) force sensors. Touch sensors indicate simply that contact has been made with the object. *Force sensors* indicate the magnitude of the force with the object. This might be useful in gripper to measure and control the force being applied to grasp a delicate object.

- *Proximity sensors.* These indicate when an object is close to the sensor. When this type of sensor is used to indicate the actual distance of the object, it is called a *range sensor*

- *Optical sensors.* Photocells and other photometric devices can be utilize to detect the presence or absence of objects and are often used for proximity detection.

- *Machine vision.* Machine vision is used in robotics for inspection, parts identification, guidance, and other uses.

- *Other sensors.* A miscellaneous category includes other types of sensors that might be used in robotics, such as devices for measuring temperature, fluid pressure, fluid flow, electrical voltage, current, and various other physical properties

Read about Industrial applications

2 Rigid Motion

2.1 Relative position and orientation of two coordinate systems

Figure 13 shows the pair of coordinate systems A and B. The position and orientation of system B relative to A are defined by the vector $\mathbf{V}_{A_o \to B_o}$ which gives the position of the origin of the B coordinate system relative to the origin of the A system, and the three unit vectors $\mathbf{x}_B$, $\mathbf{y}_B$, and $\mathbf{z}_B$, which point along the coordinate axes of the B coordinate system. Knowledge of these four vectors as measured in the A coordinate system (written as ${}^A\mathbf{V}_{A_O \to B_O}$, ${}^A\mathbf{x}_B$, ${}^A\mathbf{y}_B$, ${}^A\mathbf{z}_B$) completely defines the position and orientation of the B coordinate system measured with respect to the A coordinate system.
The three unit vectors ${}^A\mathbf{x}_B$, $\mathbf{y}_B$, ${}^A\mathbf{z}_B$ each of which has three scalar components, represent a total of nine scalar quantities. However, these are not independent because the vectors are unit vectors and they are also mutually perpendicular. Thus, the following constraint equations may be written:

$$|{}^A\mathbf{x}_B| = 1 \tag{1}$$

$$|{}^A\mathbf{y}_B| = 1 \tag{2}$$

$$|{}^A\mathbf{z}_B| = 1 \tag{3}$$

$${}^A\mathbf{x}_B{}^A\mathbf{y}_B = 0 \tag{4}$$

$${}^A\mathbf{x}_B{}^A\mathbf{z}_B = 0 \tag{5}$$

$${}^A\mathbf{y}_B{}^A\mathbf{z}_B = 0 \tag{6}$$

The unit vectors ${}^A\mathbf{x}_B$, ${}^A\mathbf{y}_B$, ${}^A\mathbf{z}_B$ thus represent 9-6=3 independent scalar quantities that specify the orientation of the coordinate system B relative to A.
Consider now that the coordinate system B is attached to a rigid body. The vectors ${}^A\mathbf{V}_{A_O \to B_O}$ ${}^A\mathbf{x}_B$, ${}^A\mathbf{y}_B$ and ${}^A\mathbf{z}_B$, which define the position and orientation of the B coordinate system with respect to the A system and which consist of six independent parameters, can be used to locate the rigid body in space with respect to the A reference frame. Because six independent parameters must be specified to define position and orientation, it is said that a rigid body in space possesses six degrees of freedom.

2.2 Point transformations

The notation ${}^{\mathbf{I}}\mathbf{P}_{\mathbf{j}}$ is used to indicate the coordinates of a point $\mathbf{j}$ as measured in a coordinate system $\mathbf{I}$. As such, ${}^{\mathbf{I}}\mathbf{P}_{\mathbf{j}}$ is a vector that begins at the origin of the $\mathbf{I}$ coordinate system and ends at point $\mathbf{j}$ and is thus equivalent to ${}^{\mathbf{I}}\mathbf{V}_{IO} \to j$.
In many kinematic problems, the position of a point is known in terms of one coordinate system, and it is necessary to determine the position of the same point measured in

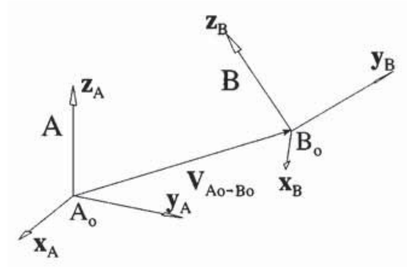


Figure 13: Two coordinate systems

another coordinate system. The problem statement is presented as follows (see Figure 14):

Given: ${}^{\mathbf{B}}\mathbf{P}_1$, the coordinates of point 1 measured in the B coordinate system (i.e. ${}^{B}\mathbf{V}_{B_o\to 1}$)
${}^{A}\mathbf{P}_{B_o}$, the location of the origin of the B coordinate system measured with respect to the A coordinate system (i.e., ${}^{A}\mathbf{V}_{AO\to BO}$)
${}^{A}\mathbf{x}_B$, ${}^{A}\mathbf{y}_B$, ${}^{A}\mathbf{z}_B$, the orientation of the B coordinate system measured with respect to the A coordinate system,

find: ${}^{A}\mathbf{P}_1$, the coordinates of point 1 measured in the A coordinate system (i.e., ${}^{A}V_{A_o\to 1}$)

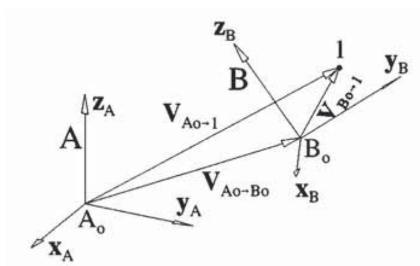


Figure 14: Depiction of point transformation problem

From triangle $A_0 - B_0 - 1$ in Figure 14, it may be written that

$$\mathbf{V}_{AO\to 1} = \mathbf{V}_{AO\to BO} + \mathbf{V}_{BO\to 1} \tag{7}$$

Evaluating all the vectors in terms of the A coordinate system gives

$${}^{A}\mathbf{V}_{AO\to 1} = {}^{A}\mathbf{V}_{AO\to BO} + {}^{A}\mathbf{V}_{BO\to 1} \tag{8}$$

It is thus necessary to solve Eq. 8 for ${}^{A}V_{AO\to 1}$ $(= {}^{A}P_1)$. The first term on the right side of Eq. 8 is a given quantity, that is, the coordinates of the origin of the B coordinate system as measured with respect to the A system. The second term on the right side of Eq. 8 is yet to be obtained.

Figure 15 shows the vector $\mathbf{V}_{BO\to 1}$ projected onto the coordinate axes of the B coordinate system. The directions of the coordinate axes of the A coordinate system are also shown

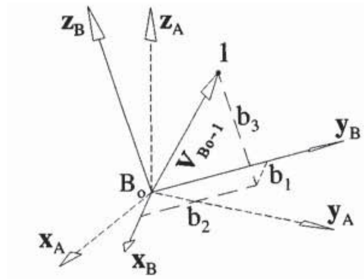


Figure 15: Point 1 projected onto the B coordinate system

as intermittent lines. The vector $V_{BO\to 1}$ may be written in terms of the B coordinate system as

$$^B\mathbf{V}_{BO\to 1} = b_1{}^B\mathbf{x}_B + b_2{}^B\mathbf{y}_B + b_3{}^B\mathbf{z}_B \tag{9}$$

where the components of $^B\mathbf{x}_B$, $^B\mathbf{y}_B$, and $^B\mathbf{z}_B$ are respectively $[1,0,0]^T$, $[0,1,0]^T$, and $[0,0,1]^T$. Because the vector $^B\mathbf{V}_{BO\to 1}$ is a given quantity, the values of the scalars b_1, b_2, and b_3 are known. Further, the vector $V_{B0\to 1}$ can now be expressed in terms of the A coordinate system, and thus

$$^A\mathbf{V}_{BO\to 1} = b_1{}^A\mathbf{x}_B + b_2{}^A\mathbf{y}_B + b_3{}^A\mathbf{z}_B \tag{10}$$

Finally, substituting Eq. (10) into Eq. (8) gives

$$^A\mathbf{V}_{A0\to 1} = {}^A\mathbf{V}_{AO\to BO} + b_1{}^A\mathbf{x}_B + b_2{}^A\mathbf{y}_B + b_3{}^A\mathbf{z}_B \tag{11}$$

which can be further arranged in matrix form as

$$^AV_{A0\to 1} = {}^AV_{AO\to BO} + \left[{}^A\mathbf{x}_B \;\; {}^A\mathbf{y}_B \;\; {}^A\mathbf{z}_B\right] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{12}$$

where $[{}^A\mathbf{x}_B \; {}^A\mathbf{y}_B \; {}^A\mathbf{z}_B]$ represents a 3 x 3 matrix that will be designated as

$$^A_B\mathbf{R} = \left[{}^A\mathbf{x}_B{}^A\mathbf{y}_B{}^A\mathbf{z}_B\right] \tag{13}$$

Substituting $^B\mathbf{P}_1 = [b_1 b_2, b_3]^T$, $^A\mathbf{P}_1 = {}^A\mathbf{V}_{A0\to 1}$, and $^A\mathbf{P}_{B0} = {}^A\mathbf{V}_{AO\to BO}$ yields

$$^A\mathbf{P}_1 = {}^A\mathbf{P}_{BO} + {}^A_B\mathbf{R}{}^B\mathbf{P}_1 \tag{14}$$

All terms on the right-hand side of Eqn. 14 are given directly where the vectors $^A\mathbf{x}_B$, $^A\mathbf{y}_B$, and $^A\mathbf{z}_B$ are the columns of the matrix $^A_B\mathbf{R}$.

2.3 4 x 4 transformation matrices

Equation 14 expresses the transformation of any point in one coordinate system to a reference coordinate system when the relative position and orientation of the pair of coordinate systems are known. The notation will be slightly modified, however, by introducing homogeneous coordinates. In homogeneous coordinates, a three-dimensional point given by **X**, **Y**, and **Z** is represented by four scalar values, that is, **x**, **y**, **z**, and **w**. The three-dimensional and homogeneous coordinates are related by

$$\mathbf{X} = \frac{\mathbf{x}}{w} \tag{15}$$

$$\mathbf{Y} = \frac{\mathbf{y}}{w} \tag{16}$$

$$\mathbf{Z} = \frac{\mathbf{z}}{w} \tag{17}$$

Thus, when $w = 1$, the first three components of the homogeneous coordinates of a point are the same as the three-dimensional coordinates of the point. By using homogeneous coordinates, Eq. 14 may be written as

$$\begin{bmatrix} {}^{A}\mathbf{P}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^{A}_{B}\mathbf{R} & & {}^{A}\mathbf{P}_{BO} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}\mathbf{P}_1 \\ 1 \end{bmatrix} \tag{18}$$

where the matrix ${}^{A}_{B}\mathbf{R}$ and the vector ${}^{A}\mathbf{P}_{BO}$ form the first three rows of a 4 x 4 matrix. The equivalency of Eqs. 14 and 18 is most easily seen by representing the components of ${}^{A}_{B}\mathbf{R}$, ${}^{A}\mathbf{P}_{BO}$ and ${}^{B}\mathbf{P}_1$ symbolically and then performing the indicated multiplications and additions. The results will be the same for both equations. The notation ${}^{A}_{B}T$ will be used to represent the 4 x 4 matrix as

$${}^{A}_{B}\mathbf{T} \begin{bmatrix} & {}^{A}_{B}\mathbf{R} & & {}^{A}\mathbf{P}_{BO} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

The point transformation problem can now be written as

$${}^{A}\mathbf{P}_1 = {}^{A}_{B}\mathbf{T}\,{}^{B}\mathbf{P}_1 \tag{20}$$

It should be noted that in Eq. 14 all the vectors such as ${}^{A}\mathbf{P}_1$ are three dimensional, whereas in Eq. 20 each vector is expressed in homogeneous coordinates with $w = 1$.

2.4 Inverse of a transformation

Quite often during robot analysis, it will be necessary to obtain the inverse of a 4 x 4 transformation. In other words, given ${}^{A}_{B}\mathbf{T}$ it will be necessary to obtain ${}^{B}_{A}\mathbf{T}$. The definition of ${}^{A}_{B}\mathbf{T}$ was presented in Eq. 19. The matrix ${}^{A}_{B}\mathbf{R}$ is a 3 x 3 matrix whose columns are ${}^{A}\mathbf{x}_B$, ${}^{A}\mathbf{y}_B$, ${}^{A}\mathbf{z}_B$, that is, the coordinates of the unit axis vectors of the B coordinate system measured in the A coordinate system. The vector ${}^{A}\mathbf{P}_{BO}$ represents the coordinates of the origin of the B coordinate system measured with respect to the A

coordinate system. It should be clear that the inverse of ${}^A_B\mathbf{T}$ can be obtained from Eq. 19 by interchanging the letters A and B and that

$$ {}^B_A\mathbf{P} = \begin{bmatrix} & {}^B_A\mathbf{R} & & {}^B\mathbf{P}_{AO} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{21} $$

The inverse will be defined once the matrix ${}^B_A\mathbf{R}$ and the coordinates of the point ${}^B\mathbf{P}_{AO}$ are determined.
The matrix ${}^A_B\mathbf{R}$ can be written in the form

$$ {}^A_B = \begin{bmatrix} {}^A\mathbf{x}_B.{}^A\mathbf{x}_A & {}^A\mathbf{y}_B.{}^A\mathbf{x}_A & {}^A\mathbf{z}_B.{}^A\mathbf{x}_A \\ {}^A\mathbf{x}_B.{}^A\mathbf{y}_A & {}^A\mathbf{y}_B.{}^A\mathbf{y}_A & {}^A\mathbf{z}_B.{}^A\mathbf{y}_A \\ {}^A\mathbf{x}_B.{}^A\mathbf{z}_A & {}^A\mathbf{y}_B.{}^A\mathbf{z}_A & {}^A\mathbf{z}_B.{}^A\mathbf{z}_A \end{bmatrix} \tag{22} $$

where the components of ${}^Ax_A$, ${}^Ay_A$, and ${}^Az_A$ are respectively $[1,0,0]^T$, $[0,1,0]^T$, and $[0,0,1]^T$. Each of the nine scalar terms of the 3 x 3 matrix ${}^A_B\mathbf{R}$ has been expressed in terms of a scalar product. A scalar product is an invariant operator that can be physically interpreted as being the cosine of the angle between the two unit vectors. The value of the scalar product will remain constant no matter what coordinate system the two vectors are expressed in. Thus,

$$ \begin{aligned} {}^A\mathbf{x}_B.{}^A\mathbf{x}_A &= {}^B\mathbf{x}_B.{}^B\mathbf{x}_A \\ {}^A\mathbf{y}_B.{}^A\mathbf{y}_A &= {}^B\mathbf{y}_B.{}^B\mathbf{y}_A \\ {}^A\mathbf{z}_B.{}^A\mathbf{z}_A &= {}^B\mathbf{z}_B.{}^B\mathbf{z}_A \end{aligned} \tag{23} $$

Applying this to all the terms of ${}^A_B\mathbf{R}$ yields

$$ {}^A_B\mathbf{R} = \begin{bmatrix} {}^B\mathbf{x}_B.{}^B\mathbf{x}_A & {}^B\mathbf{y}_B.{}^B\mathbf{x}_A & {}^B\mathbf{z}_B.{}^B\mathbf{x}_A \\ {}^B\mathbf{x}_B.{}^B\mathbf{y}_A & {}^B\mathbf{y}_B.{}^B\mathbf{y}_A & {}^B\mathbf{z}_B.{}^B\mathbf{y}_A \\ {}^B\mathbf{x}_B.{}^B\mathbf{z}_A & {}^B\mathbf{y}_B.{}^B\mathbf{z}_A & {}^B\mathbf{z}_B.{}^B\mathbf{z}_A \end{bmatrix} \tag{24} $$

It can be seen that the rows of the 3 x 3 matrix ${}^A_B\mathbf{R}$ are ${}^B\mathbf{x}_A$, ${}^B\mathbf{y}_A$, ${}^B\mathbf{z}_A$ by recognizing that ${}^B\mathbf{x}_B = [1,0,0]^T$, ${}^B\mathbf{y}_B = [0,1,0]^T$, and ${}^B\mathbf{z}_B = [0,0,1]^T$. Thus, ${}^A\mathbf{R}$ can be written as

$$ {}^A_B\mathbf{R} = \begin{bmatrix} {}^A\mathbf{x}_B & {}^A\mathbf{y}_B & {}^A\mathbf{z}_B \end{bmatrix} \begin{bmatrix} {}^B\mathbf{x}_A^T \\ {}^B\mathbf{y}_A^T \\ {}^B\mathbf{z}_A^T \end{bmatrix} \tag{25} $$

The transpose of Eq. 25 is

$$ {}^A_B\mathbf{R}^T = \begin{bmatrix} {}^B\mathbf{x}_A & {}^B\mathbf{y}_A & {}^B\mathbf{z}_A \end{bmatrix} \tag{26} $$

The columns of the 3 x 3 matrix in Eq. 26 are the unit vectors of the A coordinate system measured in terms of the B coordinate system. This is precisely the definition of ${}^B_A\mathbf{R}$. Thus, it can be concluded that

$$ {}^{B}_{A}\mathbf{R} = {}^{A}_{B}\mathbf{R}^{T} \tag{27} $$

The remaining term to be determined is ${}^{B}P_{AO}$- This term can readily be calculated from ${}^{A}\mathbf{P}_{BO}$ now that ${}^{B}_{A}\mathbf{R}$ is known. First, the vector ${}^{A}P_{BO}$ will be transformed to the B coordinate system by utilizing Eq. 14 as

$$ {}^{B}\mathbf{P}_{BO} = {}^{B}_{A}\mathbf{R}{}^{A}\mathbf{P}_{BO} + {}^{B}\mathbf{P}_{AO} \tag{28} $$

Now ${}^{B}P_{BO} = [0, 0, 0]^{T}$, which are the coordinates of the origin of the B coordinate system measured in the B system. Substituting this result into Eq. 28 and rearranging yields

$$ {}^{B}\mathbf{P}_{AO} = -{}^{B}_{A}\mathbf{R}{}^{A}\mathbf{P}_{BO} = -{}^{A}_{B}\mathbf{R}^{T\,A}\mathbf{P}_{BO} \tag{29} $$

Substituting Eqs. 27 and 29 into Eq. 21 yields the final result

$$ {}^{B}_{A}\mathbf{T} = \left[\begin{array}{cccc} & {}^{A}_{B}\mathbf{R}^{T} & & -{}^{A}_{B}\mathbf{R}^{T\,A}\mathbf{P}_{BO} \\ 0 & 0 & 0 & 1 \end{array} \right] \tag{30} $$

2.5 Compound transformations

In Figure 16, the coordinates of point 1 are known in terms of the C coordinate system. The position and orientation of the C coordinate system is known relative to the B coordinate system. The position and orientation of the B coordinate system is known in terms of the A coordinate system. The objective is to determine the coordinates of point 1 in terms of the A coordinate system. From this problem description, it should be apparent that the transformations ${}^{B}_{C}\mathbf{T}$ and ${}^{A}_{B}\mathbf{T}$ are known. Thus, the problem can be solved in two steps. First, the coordinates of point 1 in the B coordinate system can be found from

$$ {}^{B}\mathbf{P}_{1} = {}^{B}_{C}\mathbf{T}{}^{C}\mathbf{P}_{1} \tag{31} $$

Then the final answer can be obtained from

$$ {}^{A}\mathbf{P}_{1} = {}^{A}_{B}\mathbf{T}{}^{B}\mathbf{P}_{1} \tag{32} $$

Combining these two equations yields

$$ {}^{A}\mathbf{P}_{1} = {}^{A}_{B}\mathbf{T}{}^{B}_{C}\mathbf{T}{}^{C}\mathbf{P}_{1} \tag{33} $$

The term ${}_{B}\mathbf{T}{}^{B}_{C}\mathbf{T}$ transfers a point directly from the C coordinate system to the A coordinate system. Thus, it can be inferred that

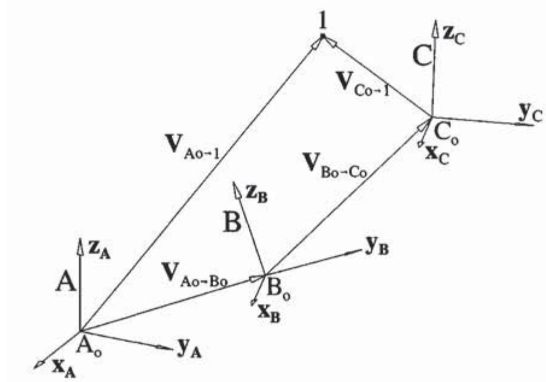


Figure 16: Compound Transformation

$$ {}^{A}_{C}\mathbf{T} = {}^{A}_{B}\mathbf{T}\,{}^{B}_{C}\mathbf{T} \tag{34} $$

The ability to perform matrix multiplication to yield compound transformations is the primary reason why the 4 x 4 transformation notation is used.

2.6 Standard transformations

In many problems, the relationship between coordinate systems will be defined in terms of rotations about the **X**, **Y**, or **Z** axes. A typical problem statement would be as follows:

given:	(1) Coordinate system B is initially aligned with coordinate system A.
	(2) Coordinate system B is then rotated α degrees about the **X** axis,
find	${}^{A}_{B}\mathbf{R}$ (often written as $\mathbf{R}_{x,\alpha}$).

Figure 17 shows the A and B coordinate systems. By projection, it can be seen that

$$ {}^{A}\mathbf{x}_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{35} $$

$$ {}^{A}\mathbf{y}_B = \begin{bmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{bmatrix} \tag{36} $$

$$ {}^{A}\mathbf{z}_B = \begin{bmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{bmatrix} \tag{37} $$

Thus;

$$ {}^{A}_{B}\mathbf{R} = \mathbf{R}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{38} $$

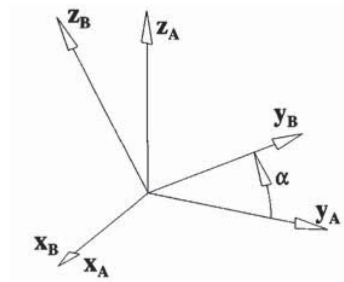


Figure 17: Rotation about the X axis

The problem will now be repeated for rotations about the Y and Z axes.

given:	(1) Coordinate system B is initially aligned with coordinate system A. (2) Coordinate system B is then rotated β degrees about the $\mathbf{Y}$ axis,
find:	${}^A_B\mathbf{R}$ (often written as $\mathbf{R}_{y,\beta}$).

Again by projection it can be shown that

$$
{}^A_B\mathbf{R} = \mathbf{R}_{y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \tag{39}
$$

Lastly,

given:	(1) Coordinate system B is initially aligned with coordinate system A. (2) Coordinate system B is then rotated γ degrees about the Z axis,
find	${}^A_B\mathbf{R}$ (often written as $\mathbf{R}_{z,\gamma}$).

By projection it can be shown that

$$
{}^A_B\mathbf{R} = \mathbf{R}_{z,\gamma} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{40}
$$

Example

Coordinate system B is initially aligned with coordinate system A. It is translated to the point $[5, 4, 1]^T$ and then rotated 30 degrees about its $\mathbf{X}$ axis. Lastly, the coordinate system is rotated 60 degrees about an axis that passes through the point $[2, 0, 2]^T$, measured in the current coordinate system, which is parallel to the $\mathbf{Y}$ axis. Find ${}^A_B\mathbf{T}$.

Solution

Figure 18 shows the A coordinate system and the B coordinate system after all the operations have been performed. Figure 19 also shows four other intermediate coordinate systems labeled C, D, E, and F.

Coordinate system C was initially aligned with coordinate system A and was then translated to the point $[5, 4, 1]^T$. Thus,

$$ {}^{A}_{C}\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{41} $$

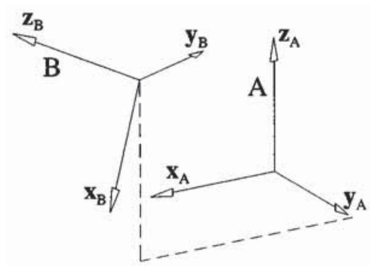


Figure 18: Initial and final coordinate systems

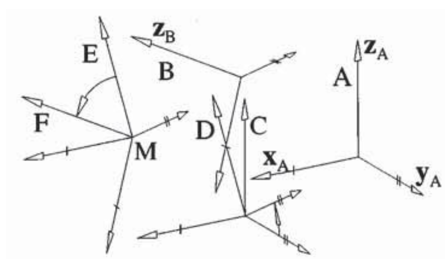


Figure 19: Intermediate coordinate systems

Coordinate system D was initially aligned with coordinate system C and was then rotated 30 degrees about the $\mathbf{X}$ axis. Thus,

$$ {}^{C}_{D}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 & 0 \\ 0 & \sin 30 & \cos 30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{42} $$

The last modification to the coordinate system was a rotation of 60 degrees about an axis parallel to the $\mathbf{Y}$ axis, which passes through the point $[2, 0, 2]^T$, measured in terms of the D coordinate system. The point that the axis of rotation passes through will be called point M. From observation (see Figure 19), it is apparent that ${}^{D}P_M = {}^{B}P_M = [2, 0, 2]^T$. Because of this fact, the transformation that relates coordinate system D and coordinate system B will be formed by translating to the point $[2, 0, 2]^T$ (see coordinate system E), rotating 60 degrees about the current $\mathbf{Y}$ axis (see coordinate system F), and then translating $[2, 0, 2]^T$ to obtain coordinate system B. The transformations that accomplish this are

$$ {}^{D}_{E} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{43} $$

$$
{}^{E}_{F} = \begin{bmatrix} \cos 60 & 0 & \sin 60 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 60 & 0 & \cos 60 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{44}
$$

$$
{}^{F}_{B} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{45}
$$

The overall transformation ${}^{A}_{B}\mathbf{T}$ can be calculated as

$$
{}^{A}_{B}\mathbf{T} = {}^{A}_{C}\mathbf{T}{}^{C}_{D}\mathbf{T}{}^{D}_{E}\mathbf{T}{}^{E}_{F}\mathbf{T}{}^{F}_{B}\mathbf{T} \tag{46}
$$

The numerical value of ${}^{A}_{B}T$ is

$$
{}^{A}_{B}\mathbf{T} = \begin{bmatrix} 0.5 & 0 & 0.866 & 4.268 \\ 0.433 & 0.866 & -0.25 & 2.634 \\ -0.75 & 0.5 & 0.433 & 3.366 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{47}
$$