Manipulator Dynamics

- **Defition:** Dynamics deals with mathematical formulation of the equations of robot arm motion.
- Robot manipulator is inherently multiple-input multiple-output dynamical system, with actuator torques and forces as the inputs and its continuous joint position and joint velocities as the output.
- The dynamic of *n*-link manipulator can be represented by *n*-coupled differential equations.
- The **Newton-Euler** dynamics and **Lagrangian** dynamics are normally used to derive the manipulator dynamics of motion.
- The **Newton-Euler** dynamics utilizes the second law of motion to come up with system differential equations by considering all forces and moments acting on the manipulator.ikjl
- The **Lagrangian** dynamics uses the Lagrangian function, which is the difference between the total kinetic energy and total potential energy stored in the system
- The Lagrangian function is obtained without considering work forces and constraint forces, while Newton-Euler considers the constraints in robot motion.

Jacobians

- The position and orientation of the manipulator's end-effector can be evaluated in relation to joint displacements and this can be obtained by performing forward-kinematics

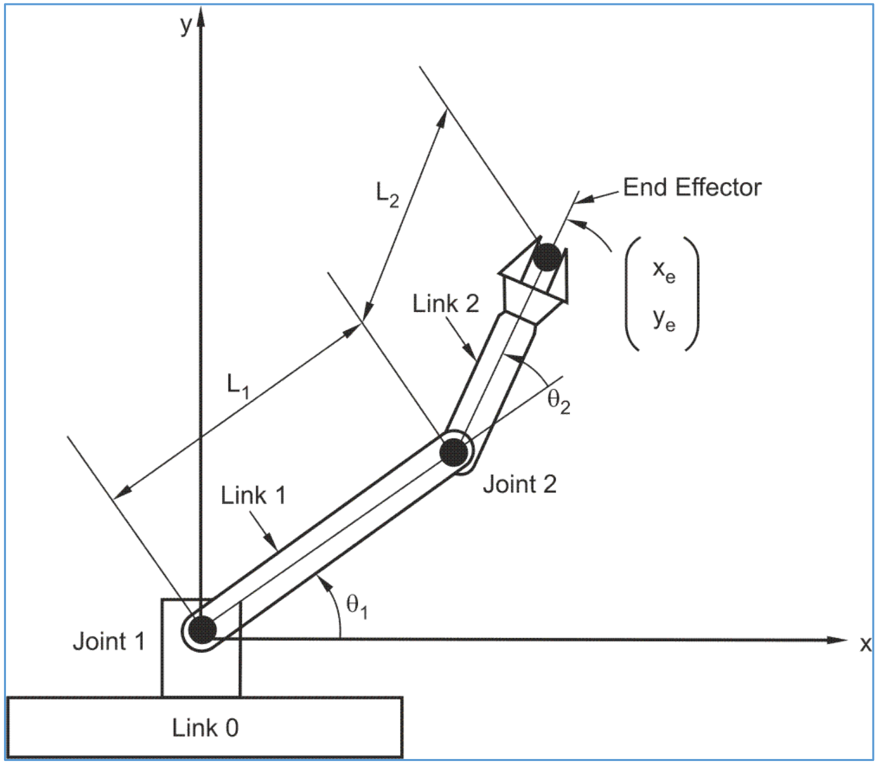
- It is important to know/evaluate the **velocity** at which the end-effector at which the end-effector moves to a given position in the work envelop.
- To move the end-effector in a specified position at a specified speed, it is necessary to coordinate the motion of individual joint to realize a coordinated motion in multiple joint robotic system.
- For coordinated joint motions, **differential relationship** between the joint displacements and the end-effector location should be derived to find out the individual joint motion.

Differential relationship

- The kinematic equation relating the end-effector coordinate (x_e,y_e) to the joint displacement ($\theta_1\theta_2$) are given by

$$\begin{aligned} x_e(\theta_1, \theta_2) &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ y_e(\theta_1, \theta_2) &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{aligned}$$

- Small movement of individual joints at current position and the resultant motion of the end-effector can be obtained by total derivatives of the above kinematic equations



Manipulator Dynamics

- Total derivative

$$\dot{x}_e = \frac{\partial x_e}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial x_e}{\partial \theta_2}\dot{\theta}_2$$
$$\dot{y}_e = \frac{\partial y_e}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial y_e}{\partial \theta_2}\dot{\theta}_2$$

- In vector form the above equation can be rewritten as

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \end{pmatrix} = \mathbf{J} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

- Here, **J** is a 2x2 **Jacobian matrix** given by

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x_e}{\partial \theta_1} & \frac{\partial x_e}{\partial \theta_2} \\ \frac{\partial y_e}{\partial \theta_1} & \frac{\partial y_e}{\partial \theta_2} \end{pmatrix}$$

- The **J** matrix comprises the partial derivatives of the function x_e and y_e wrt displacement θ_1 and $\theta_{2.}$
- Jacobian matrix is normally needed to describe the mapping of vectorial joint motion to the vectorial end-effector motion.
- For the 2DOF robot arm, the components of the Jacobian matrix are given as

$$\mathbf{J} = \begin{pmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

- Hence, the Jacobian collectively represents the sensitivity of individual end-effector coordinates to individual joint displacements.
- If the joint velocities is given by $\dot{q} = \begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{pmatrix}^T$ and the resultant end-effector velocity vector $\dot{v}_e = \begin{pmatrix} \dot{x}_e & \dot{y}_e \end{pmatrix}^T$, the Jacobian relationship is given by

$$\frac{dx_e}{dt} = \mathbf{J}\frac{dq}{dt}$$

- Thus, the Jacobian determines the velocity relationship between the joints and the end-effector

Singularities

- If there is a linear transformation relating joint velocity to the end-effector velocity, then the matrix is **nonsingular** if there exists an inversion to calculate joint velocities from a given Cartesian end-effector velocities, ie.,

$$\dot{q} = \mathbf{J}^{-1}\dot{v}_e$$

- If the end effector of the robot is to move with a certain velocity vector in Cartesian space, then using above equation, the necessary joint velocities at each instant along the path can be calculated
- Most manipulators have values of $\dot{q}$ where the Jacobian becomes singular. Such locations are called **singularities of the mechanism or simply singularities**.
- All manipulators have singularities at the boundary of their workspace, and most have loci of singularities inside their workspace.
  - **Workspace-boundary singularities** occur when the manipulator is fully stretched out or folded back on itself in such a way that the end-effector is at or very near the boundary of the workspace.
  - **Workspace-interior singularities** occur away from the workspace boundary; they generally are caused by a lining up of two or more joint axes
- When a manipulator is in a singular configuration, it has lost one or more degrees of freedom (as viewed from Cartesian space)
- This means that there is some direction (or subspace) in Cartesian space along which it is impossible to move the hand of the robot, no matter what joint velocities are selected.

Singularities (Cont')

- To find the singularity for 2 link arm the determinant of its Jacobian should be equal to zero and hence Jacobian has lost full rank and is singular.

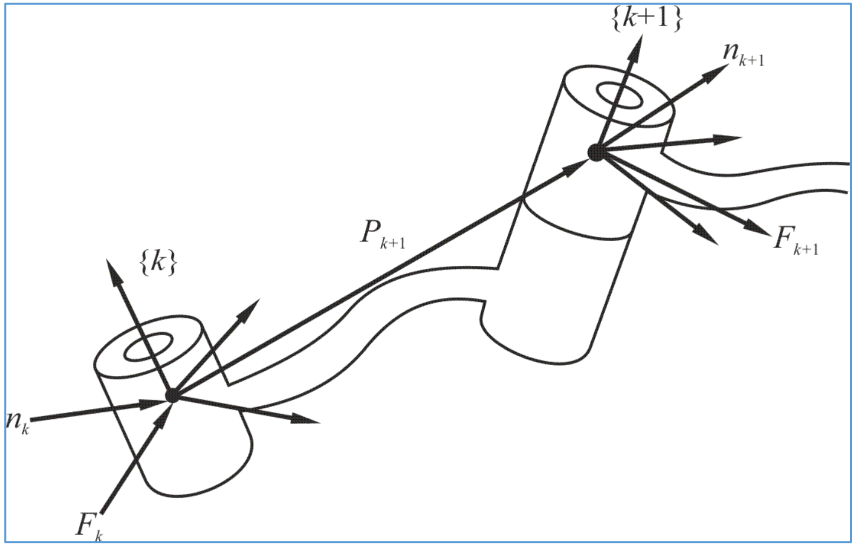
$$\det(\mathbf{J}) = \begin{bmatrix} L_1 \sin\theta_2 & 0 \\ L_1 \cos\theta_2 + L_2 & L_2 \end{bmatrix} = L_1 L_2 \sin\theta_2 = 0$$

- So, a singularity of the mechanism exists when θ_2 is 0 or 180 degrees.
- Physically, when $\theta_2=0$, the arm is stretched straight out, while $\theta_2=180$, the arm is folded completely back on itself.

Static forces in manipulator

- Deals with finding/evaluating the joint torques to keep the system in static equilibrium if the end-effector is supporting a load
- To find static forces in a manipulator, all the joints should be locked so that the manipulator becomes a structure.
- Then consider each link in this structure and write a force-moment balance relationship in terms of the link frames.
- Static torque acting about the joint axis can be computed in order to keep manipulator in static equilibrium.
- The static forces and torques acting at the joints are considered when the manipulator has its end-effector with the load

- Consider the figure below



- Here, F_k is the force exerted on the link k by neighbor link k-1 and n_k is the torque exerted on link k by neighbor link k-1
- Summing the forces and setting them to zero
  $$F_k - F_{k+1} = 0$$
- Summing the torque about the origin of frame $\{k\}$
  $$n_k - n_{k+1} - P_{k+1} \times F_{k+1} = 0$$
  $$n_k = n_{k+1} + P_{k+1} \times F_{k+1}$$
- To write these equations in terms of only forces and moments defined within their own link frames, the rotation matrix describing frame $\{k+1\}$ relative to frame $\{k\}$ is transformed
  $$F_k = RF_{k+1}$$
  $$n_k = Rn_{k+1} + P_{k+1} \times F_{k+1}$$
- To find the joint torque required to maintain the static equilibrium, the dot product of the joint-axis vector with the moment vector acting on the link is computed

Jacobian in the force domain

- Torques will exactly balance forces at the end effector in the static situation
- When forces act on a mechanism, work is done if the mechanism moves through a displacement
- Work is the dot product of a vector force or torque and a vector displacement.

  $F.dx = \tau.d\theta$
- Here, F is a force vector acting at the end-effector, dx is the displacement of the end-effector τ is a vector of torque at the joints, and $d\theta$ is a vector of infinitesimal joint displacement

  $$F^T.dx = \tau^T.d\theta$$
- The definition of the Jacobian is

  $dx = \mathbf{J}d\theta$
- Therefore

  $F^T\mathbf{J}d\theta = \tau^T d\theta$
- From which

  $\tau = \mathbf{J}^T F$
- The Jacobian transpose maps Cartesian forces acting at the end effector into equivalent joint torques.

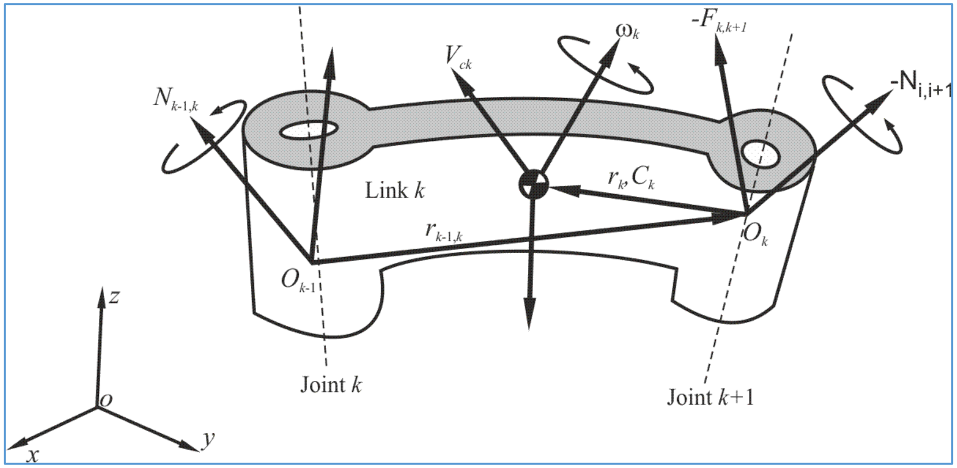
Manipulator dynamics

- The manipulator dynamics behavior is described in terms of the time rate of change of the robot configuration in relation to the joint torques

New-Euler Formulation Equations of Motion

- The dynamic equations of a rigid body can also be represented by two equations
  - One describes the translational motion of the centroid – Newton's equation of motion
  - The other describes the rotational motion about the centroid – Euler's equation of motion

- Consider the figure below



- Let V_{ck} be the linear velocity of the centroid of link k with reference to the base coordinate frame O-xyz, which is an inertia reference frame
- The inertia force is given by $-m_k\dot{V}_{ck}$, where m_k is the mass of the link
- Based on D'Alembert's principle, the equation of motion is then obtained by adding the inertia force to the static balance of forces so that

$$F_{k-1,k} - F_{k,k+1} + m_k g - m_k \dot{V}_{ck} = 0, \quad k = 1, \ldots, n$$

- Here, $F_{k-1,k}$ and $-F_{k,k+1}$ are the coupling forces applied to link k by links k-1 and k+1, respectively, and g is the gravitation acceleration
- Rotational motions are described by **Euler's equations** by adding "inertia torques" to the static balance of moments
- Newton-Euler equations for **link 1** are given by

$$F_{0,1} - F_{1,2} + m_1 g - m_1 \dot{V}_{c1} = 0$$

$$N_{0,1} - N_{1,2} + r_{1,c1} \times F_{1,2} - r_{0,c1} \times F_{0,1} - I_1\dot{\omega}_1 = 0$$

$$F_{1,2} + m_2 g - m_2 \dot{V}_{c2} = 0$$

$$N_{1,2} - r_{1,c2} \times F_{1,2} - I_2\dot{\omega}_2 = 0$$

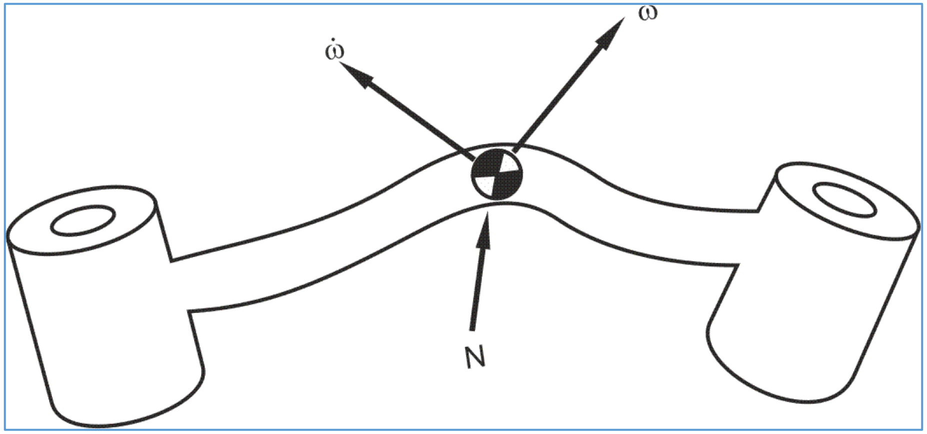
Newton's Equation in Simple format

- Consider a right body whose center of mass is accelerating with acceleration $\dot{V}$
- The force, F, acting at the center of mass and causing this acceleration is given by Newton's equation

- That is,
  $F = m\dot{V}$
- Where m is the total mass of the body and $\dot{V}$ is the acceleration

Euler's equation in simple format

- Consider the figure below, where a rigid body rotating with angular velocity ω and with angular acceleration $\dot{\omega}$



- In such a situation, the moment N, which must be acting on the body to cause this motion, is given by Euler's equation
  $N = I\dot{\omega} + \omega \times I\omega$
- Where I is the inertia tensor of the body

The force and torque acting on a link

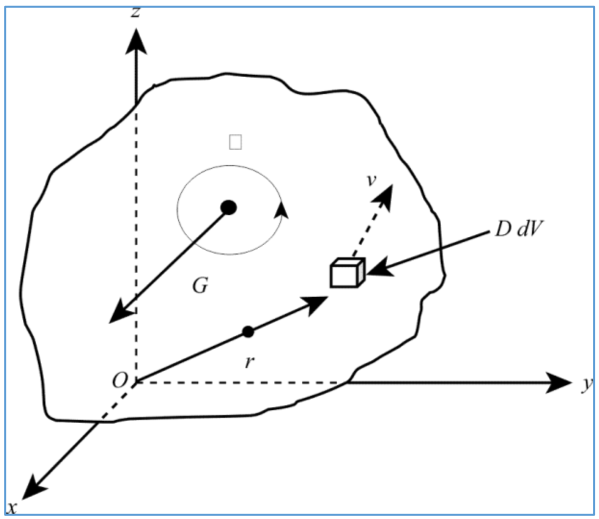
- If the linear and angular accelerations of the mass center of each link is computed, Newton-Euler equations can be applied to compute the inertia force and torque acting at the center of mass of each link.
- That is,

$$\boxed{F = m\dot{V}}$$

$$\boxed{N = I\dot{\omega} + \omega \times I\omega}$$

N-Link robot manipulator

- Using the **Lagrange** equations, a set of differential equations for each of an n-link robot manipulator is obtained
- An **inertia tensor** of a rigid body is a 3x3 matrix used to describe the mass distribution of a rigid body wrt selected coordinate frame



- Let D and v be the mass density of the rigid body and volume occupied by the body coordinate frame, respectively

- The inertia tensor of the rigid body wrt body coordinate frame is expressed as

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

- Here elements are called moment of inertia about x,y,z axes, respectively
- Then, the **mass moments of inertia** are given by

$$I_{xy} = I_{yx} = -\int\int\int xyDdv$$

$$I_{zx} = I_{xz} = -\int\int\int zxDdv$$

$$I_{yz} = I_{zy} = -\int\int\int yzDdv$$

- Then, the **mass products of inertia** are given by

$$I_{xx} = \int\int\int (y^2+z^2)Ddv$$

$$I_{yy} = \int\int\int (x^2+z^2)Ddv$$

$$I_{zz} = \int\int\int (x^2+y^2)Ddv$$

- The elemental part of angular momentum is expressed as
  $dH = (r \times v)Ddv$
- Where Ddv is the elemental mass, $v = \omega x r$ is the linear velocity of the mass (Ddv), and ω is the angular velocity vector
- The angular momentum of the whole rigid body is given by
  $H = I\omega$
- Where I is the inertia tensor and ω is the angular velocity
- The angular (rotational) kinetic energy of the whole rigid body

  $$K_{rot} = \frac{1}{2} I \omega^2$$
- Here, the inertia tensor I is obtained wrt origin of the body coordinate frame (x,y, z)
- If the inertia tensor is obtained with respect to G, the center of gravity of rigid body, the rotational kinetic energy, K_{rot} , of rigid body can then be expressed as

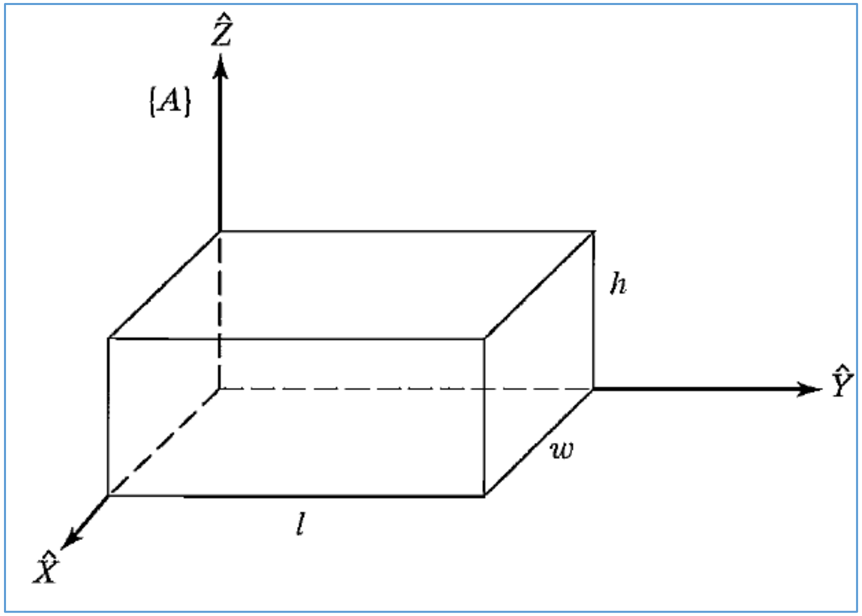
  $$K_{rot} = \frac{1}{2} m v_g^2 + \frac{1}{2} I \omega^2$$
- Where m is the total mass of rigid body, v_g is the velocity of center of mass, and I_g is the inertia tensor wrt the center of mass
- f the rigid body undertakes not only the rotation but also the translation, then the translational kinetic energy, K_{tran} , of the rigid body is given by

  $$K_{tran} = \frac{1}{2} m v_m^2$$
- Where v_m is the velocity of translation of whole rigid body
- The total kinetic energy is expressed as

  $$K = K_{tran} + K_{rot} = \frac{1}{2} m v_m^2 + \frac{1}{2} m v_g^2 + I_g \omega^2$$

Example

- Find the inertia tensor for the rectangular body of uniform mass density D with respect to the coordinate system shown in the figure below



- Ixx is computed using volume element $dv = dxdydz$

$$
\begin{aligned}
I_{xx} &= \int_0^h \int_0^l \int_0^w (y^2 + z^2) D dx dy dz \\
&= \int_0^h \int_0^l (y^2 + z^2) w D dy dz \\
&= \int_0^h \left(\frac{l^3}{3} + z^2 l \right) w D dz \\
&= \left(\frac{hl^3 w}{3} + \frac{h^3 l w}{3} \right) D \\
&= \frac{m}{3}(l^2 + h^2)
\end{aligned}
$$

- Where m is the total mass of the body
- Permuting the terms, Iyy and Izz can be gotten by inspection

$$
\begin{aligned}
I_{yy} &= \frac{m}{3}(w^2 + h^2) \\
I_{zz} &= \frac{m}{3}(l^2 + w^2)
\end{aligned}
$$

Example (cont')

- Next, I_{xy} are computed as follow

$$\begin{aligned} I_{xy} &= \int_0^h \int_0^l \int_0^w xyD dx dy dz \\ &= \int_0^h \int_0^l \frac{w^2}{2} yD dy dz \\ &= \int_0^h \frac{w^2 l^2}{4} D dz \\ &= \frac{m}{4} wl \end{aligned}$$

- Permuting the terms, I_{xz} and I_{yz} are given as

$$\begin{aligned} I_{yz} &= \frac{m}{4} hw \\ I_{yz} &= \frac{m}{4} hl \end{aligned}$$

- Hence, the inertia tensor for this object is given by

$$I = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & -\frac{m}{4}wl & -\frac{m}{4}hw \\ -\frac{m}{4}wl & \frac{m}{3}(w^2+h^2) & -\frac{m}{4}hl \\ -\frac{m}{4}hw & -\frac{m}{4}hl & \frac{m}{3}(l^2+w^2) \end{bmatrix}$$

- ***NB**: The inertia tensor is a function of the location and orientation of the reference frame*
- *A well-known result, the **parallel-axis theorem**, is one way of computing how the inertia tensor changes under **translations** of the reference coordinate system*

Lagrange's Equation of Motion

- Lagrange's equation of motion for a conservative system are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

- Here, q is an n-vector of generalized coordinates qi, τ is an n-vector of generalized forces τ_i, and the ***Lagrangian*** is the difference between the kinetic and potential energies

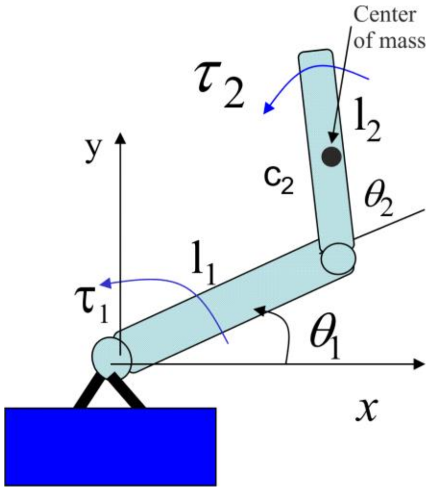
$L = K - P$

- The q will be the joint-variable vector, consisting of joint angles θ_i; (in deg or rads) and joint offsets d_i (in m).
- Then τ is a vector that has components n_i of torque (Nm) corresponding to the joint angles, and f_i of force (N) corresponding to the joint offsets

- The Jacobian transpose maps Cartesian forces acting at the end effector into equivalent joint torques.

Example

- Denote the mass of link k by m_k. If the mass is uniformly distributed, derive the dynamics of the 2 DOF arm



Solution

- Generalized coordinates

$$q = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

- Non-conservative forces/torques that do work $\tau_1, \ \tau_2$
- The kinetic energy
  Link 1 rotating about 0,

$$K_1 = \frac{1}{2} I_0 \dot{\theta}_1^2$$

Solution (cont')

Link 2 in general plane motion

$$K_2 = \frac{1}{2}m_2V_{c2}^2 + \frac{1}{2}I_{c2}\omega_2^2$$

- Position of the center of mass

$$\begin{aligned} x_{c2} &= l_1\cos\theta_1 + \frac{l_2}{2}\cos(\theta_1+\theta_2) \\ y_{c2} &= l_1\sin\theta_1 + \frac{l_2}{2}\sin(\theta_1+\theta_2) \end{aligned}$$

- Implying

$$\begin{aligned} V_{c2}^2 &= \dot{x}_{c2}^2 + \dot{y}_{c2}^2 \\ &= l_1^2\dot{\theta}_1^2 + \frac{l_2^2}{4}(\dot{\theta}_1+\dot{\theta}_2)^2 + \dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)l_1l_2\cos\theta_2 \end{aligned}$$

- The total kinetic energy

$$\begin{aligned} K &= K_1 + K_2 = \frac{1}{2}(I_0 + m_2I_1^2)\dot{\theta}_1^2 \\ &\quad + \frac{1}{2}m_2l_1l_2C_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2) \\ &\quad + \left(\frac{1}{8}m_2l_2^2 + \frac{1}{2}I_{c2}\right)(\dot{\theta}_1+\dot{\theta})^2 \end{aligned}$$

- The potential energy

$$\begin{aligned} P &= m_1gy_{c1} + m_2gy_{c2} \\ &= m_1g\frac{l_1}{2}S_1 + m_2g(l_1S_1 + \frac{l_2}{2}S_{12}) \end{aligned}$$

- Calculation of the partial derivatives

$$\begin{aligned} \frac{\partial L}{\partial\dot{\theta}_1} &= (I_0 + m_2l_1^2)\dot{\theta}_1 + m_2l_1l_2C_2\left(\dot{\theta}_1 + \frac{1}{2}\dot{\theta}_2\right) \\ &\quad + \left(\frac{1}{4}m_2l_2^2 + I_{c2}\right)(\dot{\theta}_1+\dot{\theta}_2) \end{aligned}$$

$$\frac{\partial L}{\partial\dot{\theta}_2} = \frac{1}{2}m_2l_1l_2C_2\dot{\theta}_1 + \left(\frac{1}{4}m_2l_2^2 + I_{c2}\right)(\dot{\theta}_1+\dot{\theta}_2)$$

$$\frac{\partial L}{\partial\theta_1} = -m_1g\frac{l_1}{2}C_1 - m_2g(l_1C_1 + \frac{l_2}{2}C_{12})$$

$$\frac{\partial L}{\partial\theta_2} = -m_2g\frac{l_2}{2}C_{12} - \frac{1}{2}m_2gl_1l_2S_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)$$

Example (Cont')

- Applying the Lagrange equation, $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = \tau$, the final dynamic equation is expressed as

$$\begin{pmatrix} I_{01} + m_2 l_1^2 + 0.25 m_2 l_2^2 + I_{c2} + m_2 l_1 l_2 C_2 & 0.25 m_2 l_2^2 + I_{c2} + 0.5 m_2 l_1 l_2 C_2 \\ 0.25 m_2 l_2^2 + I_{c2} + 0.5 m_2 l_1 l_2 C_2 & 0.25 m_2 l_2^2 + I_{c2} \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}$$
$$+ \begin{pmatrix} -m_2 l_1 l_2 S_2 (\dot{\theta}_1 + 0.5\dot{\theta}_2)\dot{\theta}_2 + (0.5 m_1 + m_2) l_1 g C_1 + 0.5 m_2 l_2 g C_{12} \\ 0.5 m_2 l_1 l_2 S_2 \dot{\theta}_1^2 + 0.5 m_2 l_2 g C_{12} \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$