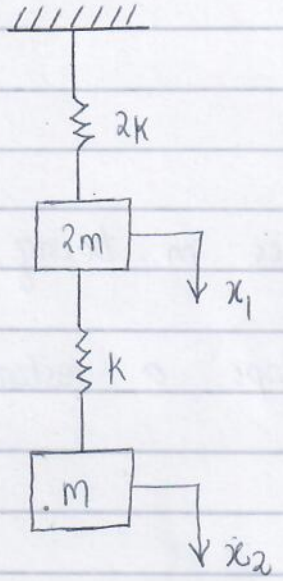
Vibration of

CHAPTER 2 = 2 degrees of freedom systems



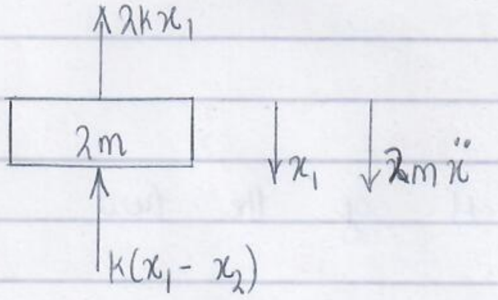
a) Find the natural frequencies of this system

(b) Determine the ratio of amplitudes

(c) Draw the mode shapes

solution

Give a displacement x_1 to mass 2m and a displacement x_2 to mass m



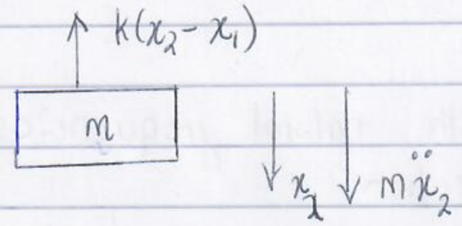
spring 2K is stretched by x_1 and develops a restoring force $2Kx_1$ (opposing the accn) spring K is compressed by $(x_1 - x_2)$ and develops a restoring force ~~$k(x_1 - x_1)$~~ $K(x_1 - x_2)$

$$2m\ddot{x}_1 = -2kx_1 - k(\cancel{x_2} - \cancel{x_1})$$

$$\cancel{2m\ddot{x}_1 + 2kx_1 + k(x_1 - x_2) = 0} \quad 2m\ddot{x}_1 + 2kx_1$$

$$2m\ddot{x}_1 + \cancel{3k\,x_1 - k\,x_2 = 0} \qquad 1a$$

$$2m\ddot{x}_1 = -2kx_1 - k(x_1 - x_2)$$
$$2m\ddot{x}_1 + 2kx_1 + k(x_1 - x_2) = 0$$
$$2m\ddot{x}_1 + 3kx_1 - kx_2 = 0$$



spring k is stretched $(x_2 - x_1)$ by mass m being displaced by x_2. It thus develops a restoring force $k(x_2 - x_1)$ that opposes $m\ddot{x}_2$

$$m\ddot{x}_2 = -k(x_2 - x_1)$$

$$m\ddot{x}_2 + k(x_2 - x_1) = 0$$

$$m\ddot{x}_2 + kx_2 - kx_1 = 0 \qquad 2a$$

Assume that the S.H.M of the two masses is of the form

$$x_1 = A\sin(\omega t + \phi) \qquad x_2 = B\sin(\omega t + \phi)$$

SHM1

$$\dot{x}_1 = A\omega\cos(\omega t + \phi) \qquad \dot{x}_2 = B\omega\sin(\omega t + \phi)$$

$$\ddot{x}_1 = -A\omega^2\sin(\omega t + \phi) \qquad \ddot{x}_2 = -B\omega^2\cos(\omega t + \phi)$$

Replace the relevant terms from SHM into 1a and 2a

$$2m\ddot{x}_1 + 3kx_1 - kx_2 = 0 \qquad 1a$$

$$-2m\omega^2 A \sin\omega t + 3kA\sin\omega t - kB\sin\omega t = 0$$
$$(3kA - 2m\omega^2 A)\sin\omega t = kB\sin\omega t$$
$$(3k - 2m\omega^2)A = kB$$
$$\frac{A}{B} = \frac{k}{3k - 2m\omega^2} \qquad 1b$$

$$m\ddot{x}_2 + kx_2 - kx_1 = 0 \qquad 2a$$
$$-m\omega^2 B\sin\omega t + kB\sin\omega t - kA\sin\omega t = 0$$
$$(kB - m\omega^2 B)\sin\omega t = kA\sin\omega t$$
$$(k - m\omega^2)B = kA$$
$$\frac{k - m\omega^2}{k} = \frac{A}{B} \qquad 2b$$

$$1a = 2b \Rightarrow \frac{k}{3k - 2m\omega^2} = \frac{k - m\omega^2}{k}$$

$$(3k - 2m\omega^2)(k - m\omega^2) = k^2$$
$$3k^2 - 3km\omega^2 - 2km\omega^2 + 2m^2\omega^4 - k^2 = 0$$
$$2m^2\omega^4 - 5km\omega^2 + 2k^2 = 0$$
$$\omega^4 - \frac{5km\omega^2}{2m^2} + \frac{2k^2}{2m^2} = 0$$
$$\omega^4 - \frac{5k}{2m}\omega^2 + \frac{k^2}{m^2} = 0$$

let $p = \omega^2 \Rightarrow p^2 = \omega^4$

$$\omega^4 - \frac{5k\omega^2}{2m} + \frac{k^2}{m^2} = 0 \Rightarrow p^2 - \frac{5k}{2m}p + \frac{k^2}{m^2} = 0$$

4a

To solve for p using the quadratic formula

$a = 1 \qquad b = -\frac{5K}{2m} \qquad c = \frac{K^2}{m^2}$

$$P_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5K}{4m} \pm \frac{1}{2}\sqrt{\left(\frac{5K}{2m}\right)^2 - \frac{4K^2}{m^2}}$$

$$= \frac{5K}{4m} \pm \frac{1}{2}\sqrt{\frac{25K^2}{4m^2} - \frac{4K^2}{m^2}}$$

$$= \frac{5K}{4m} \pm \frac{1}{2}\sqrt{\frac{25K^2 - 16K^2}{4m^2}} = \frac{5K}{4m} \pm \frac{1}{2}\sqrt{\frac{9K^2}{4m^2}}$$

$$= \frac{5K}{4m} \pm \frac{1}{2}\left(\frac{3K}{2m}\right)$$

$$= \frac{5K}{4m} \pm \frac{3K}{4m}$$

$P_1 = \frac{5K}{4m} + \frac{3K}{4m} = \frac{8K}{4m} = \frac{2K}{m} \qquad P_2 = \frac{5K}{4m} - \frac{3K}{4M} = \frac{2K}{4M} = \frac{1}{2}\frac{K}{m}$

$P_1 = W_{1n}^2 = \frac{2K}{m} \qquad P_2 = W_{2n}^2 = \frac{K}{2m}$

$W_{1n} = \sqrt{\frac{2K}{m}} \qquad W_{2n} = \sqrt{\frac{K}{2m}}$

W_{1n} and W_{2n} are the natural frequencies of the system

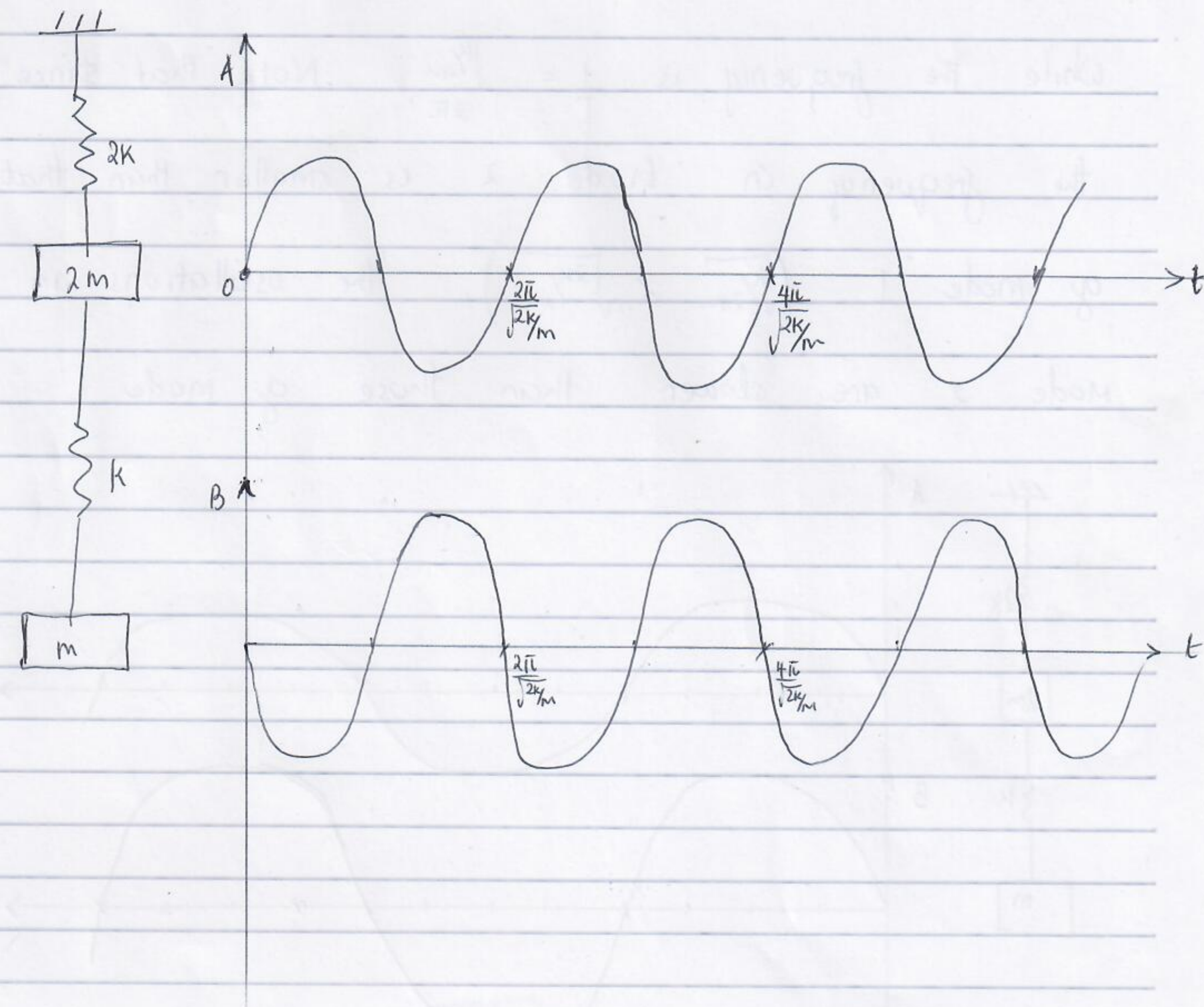
Mode shape 1

$$\frac{A}{B} = \frac{k}{3k - 2m\omega^2} \qquad \text{when } \omega^2 = \omega_{1n}^2 = \frac{2k}{m}$$

$$\Rightarrow \frac{A}{B} = \frac{k}{3k - 2m(2k/m)} = \frac{k}{3k-4k} = \frac{k}{-k} = \frac{1}{-1}$$

The period of oscillation is $\frac{2\pi}{\omega_{1n}} = \frac{2\pi}{\sqrt{2k/m}}$

The frequency of oscillation is $\frac{1}{T} = \frac{\sqrt{2k/m}}{2\pi}$



4c

Mode shape 2

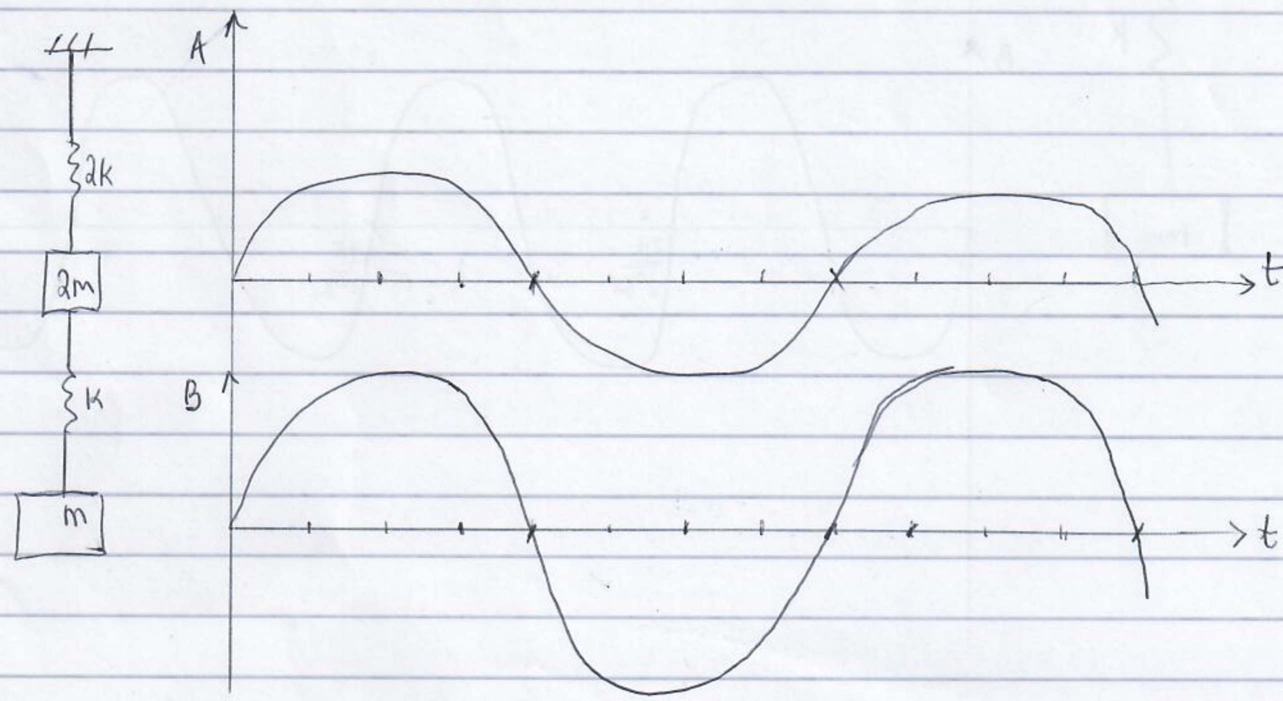
$$\frac{A}{B} = \frac{K - m\omega^2}{K} \quad \text{when } \omega^2 = \omega_{2n}^2 = \frac{K}{2m}$$

$$\frac{A}{B} = \frac{K - m(K/2m)}{K} = \frac{K - K/2}{K} = \frac{2K - K}{2K} = \frac{1}{2}$$

A and B are in phase but the amplitude of B is twice that of A.

The period of oscillation is $\frac{2\pi}{\omega_{2n}} = \frac{2\pi}{\sqrt{K/2m}}$

while the frequency is $\frac{1}{T} = \frac{\sqrt{K/2m}}{2\pi}$. Note that since the frequency in mode 2 is smaller than that of mode 1 $\left(\sqrt{K/2m} < \sqrt{2K/m}\right)$, the oscillations in mode 2 are slower than those of mode 1



Alternative solution approach - Use of Eigen value & Eigen vector

Recall equations 1a and 2a

$$2m\ddot{x}_1 + 3kx_1 - kx_2 = 0 \quad 1a$$
$$m\ddot{x}_2 + kx_2 - kx_1 = 0 \quad 2a$$

The motion of undamped systems such as this one is sinusoidal of ~~them~~ the form

$$x_1 = A\sin(\omega t + \phi) \qquad x_2 = B\sin(\omega t + \phi)$$
$$\dot{x}_1 = A\omega\cos(\omega t + \phi) \qquad \dot{x}_2 = B\omega\cos(\omega t + \phi)$$
$$\ddot{x}_1 = -A\omega^2\sin(\omega t + \phi) \qquad \ddot{x}_2 = -B\omega^2\sin(\omega t + \phi)$$
$$= -\omega^2 x_1 \qquad = -\omega^2 x_2$$

substituting for $\ddot{x}_1$ and $\ddot{x}_2$ in 1a and 2a

$$-2m\omega^2 x_1 + 3kx_1 - kx_2 = 0$$
$$-m\omega^2 x_2 + kx_2 - kx_1 = 0$$

$$-\omega^2 x_1 + \frac{3k}{2m}x_1 - \frac{k}{2m}x_2 = 0$$

$$-\omega^2 x_2 + \frac{k}{m}x_2 - \frac{k}{m}x_1 = 0$$

$$\left(\frac{3k}{2m} - \omega^2\right)x_1 - \left(\frac{k}{2m}\right)x_2 = 0$$

$$-\left(\frac{k}{m}\right)x_1 + \left(\frac{k}{m} - \omega^2\right)x_2 = 0$$

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$$\begin{bmatrix} \frac{3k}{2m} - \omega^2 & \frac{-k}{2m} \\ \frac{-k}{m} & \frac{k}{m} - \omega^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad Q$$

A matrix is multiplied by a vector to yield the null / zero vector which is a form of the eigen value / eigen vector equation with $\lambda = \omega^2$

$Q = (A - I\lambda)(x) = 0 \quad \Rightarrow \lambda =$ eigen value

The only way Q is possible is if the determinant of $(A - I\lambda)$ is zero if we disregard the trivial solution $[x] = 0$

$\det(A - I\lambda) = 0$

$$\begin{vmatrix} \frac{3k}{2m} - \omega^2 & \frac{-k}{2m} \\ \frac{-k}{M} & \frac{k}{M} - \omega^2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} \frac{3k}{2m} - \lambda & \frac{-k}{2m} \\ \frac{-k}{M} & \frac{k}{M} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{3k}{2m} - \lambda\right)\left(\frac{k}{M} - \lambda\right) - \frac{k^2}{2M^2} = 0$$

$$\frac{3K^2}{2m^2} - \frac{3K}{2m}\lambda - \frac{K}{m}\lambda + \lambda^2 - \frac{K^2}{M^2} = 0$$

$$\lambda^2 - \left[\frac{3K}{2m} + \frac{K}{m}\right]\lambda + \frac{3K^2}{2m^2} - \frac{K^2}{m^2} = 0$$

$$\lambda^2 - \left[\frac{3K + 2K}{2m}\right]\lambda + \frac{K^2}{m^2} = 0$$

$$\lambda^2 - \frac{5K}{2m}\lambda + \frac{K^2}{m^2} = 0 \quad (Q_1)$$

We solved (Q_1) earlier where it was a polynomial in p.

$$\Rightarrow \lambda_{1,2} = \frac{5K}{4M} \pm \frac{3K}{4M} \Rightarrow \lambda_1 = \frac{2K}{M} \qquad \lambda_2 = \frac{K}{2m}$$

$$\lambda_1 = W_{1n}^2 \Rightarrow W_{1n} = \sqrt{\frac{2K}{M}} \qquad \lambda_2 = W_{2n}^2 \Rightarrow W_{2n} = \sqrt{\frac{K}{2m}}$$

Obtaining the eigen vectors corresponding to $\lambda_1 = \frac{2K}{M}$

from Q

$$\left(\frac{3K}{2m} - \frac{2K}{M}\right)x_1 - \frac{K}{2m}x_2 = 0 \Rightarrow -\frac{K}{2m}x_1 - \frac{K}{2m}x_2 = 0 \quad i)$$

$$-\frac{K}{m}x_1 + \left(\frac{K}{m} - \frac{2K}{M}\right)x_2 = 0 \Rightarrow -\frac{K}{m}x_1 - \frac{K}{M}x_2 = 0 \quad ii)$$

You see that i) and ii) are multiples of each other and therefore we can use either to get x_1 and x_2

Using ii)

$$-\frac{k}{m}x_1 - \frac{k}{m}x_2 = 0 \quad \Rightarrow -\frac{k}{m}x_1 = k/m\, x_2$$

$$-x_1 = x_2$$

Choose $x_1 = 1 \quad \Rightarrow x_2 = -1$

The first eigen vector is thus $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or if we had chosen $x_1 = -1$ then $x_2 = 1 \Rightarrow$ Eigen Vector $= \pm\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Obtaining the eigen vector corresponding to $\lambda_2 = \frac{k}{2m}$ from Q

$$\begin{bmatrix} \frac{3k}{2m} - \frac{k}{2m} & -\frac{k}{2m} \\ -\frac{k}{m} & \frac{k}{m} - \frac{k}{2m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{2K}{2m}x_1 - \frac{K}{2m}x_2 = 0 \qquad iii)$$

$$-\frac{K}{m}x_1 + \frac{K}{2m}x_2 = 0 \qquad iv)$$

iii and iv are multiples of each other and we can use either. Using iv

$$\frac{K}{m}x_1 = \frac{K}{2m}x_2 \Rightarrow x_1 = \frac{1}{2}x_2$$

$$2x_1 = x_2$$

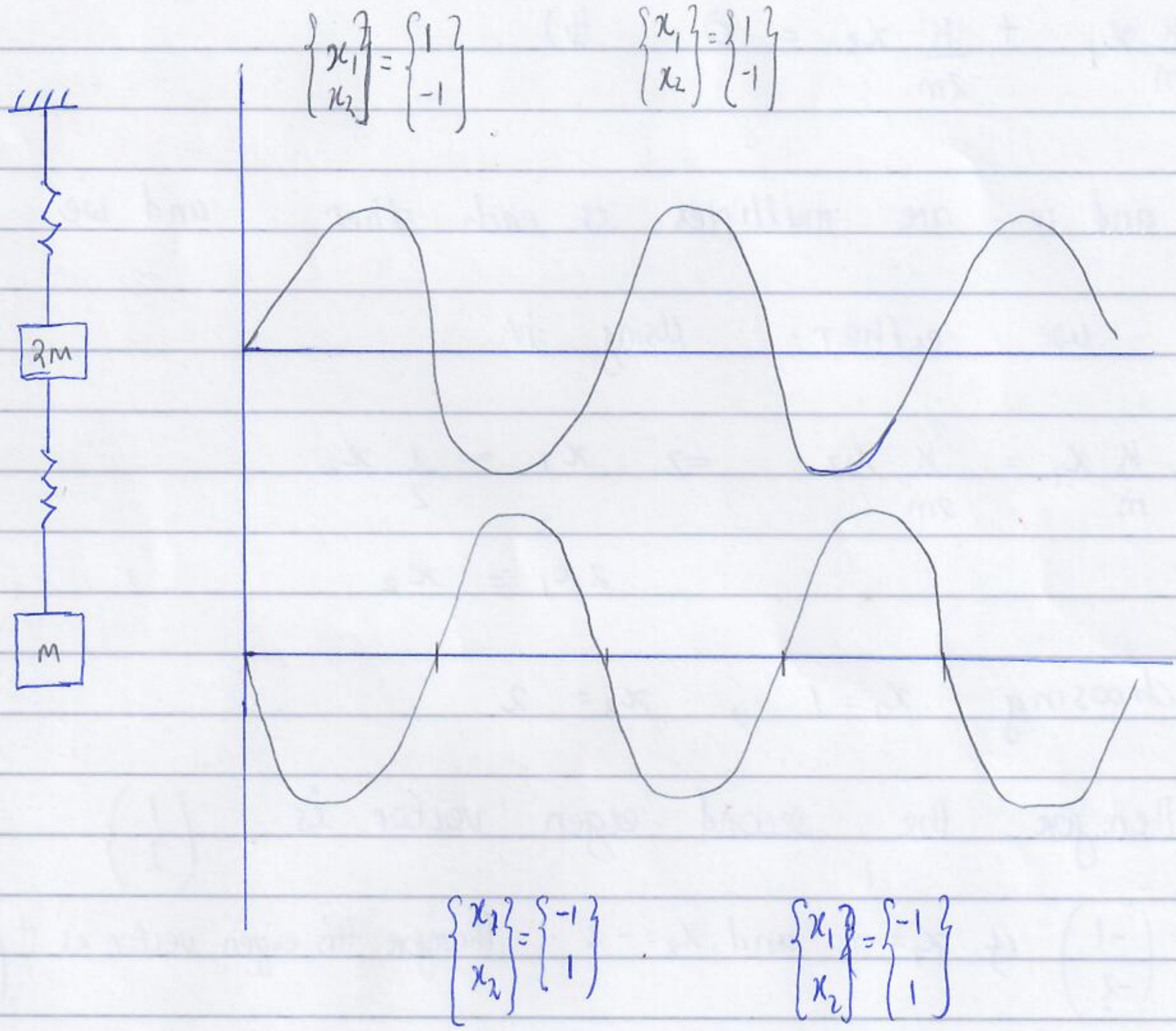
choosing $x_1 = 1$, $x_2 = 2$

There for the second eigen vector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ if $x_1 = -1$ and $x_2 = -2$. Therefore this eigen vector is $\pm\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

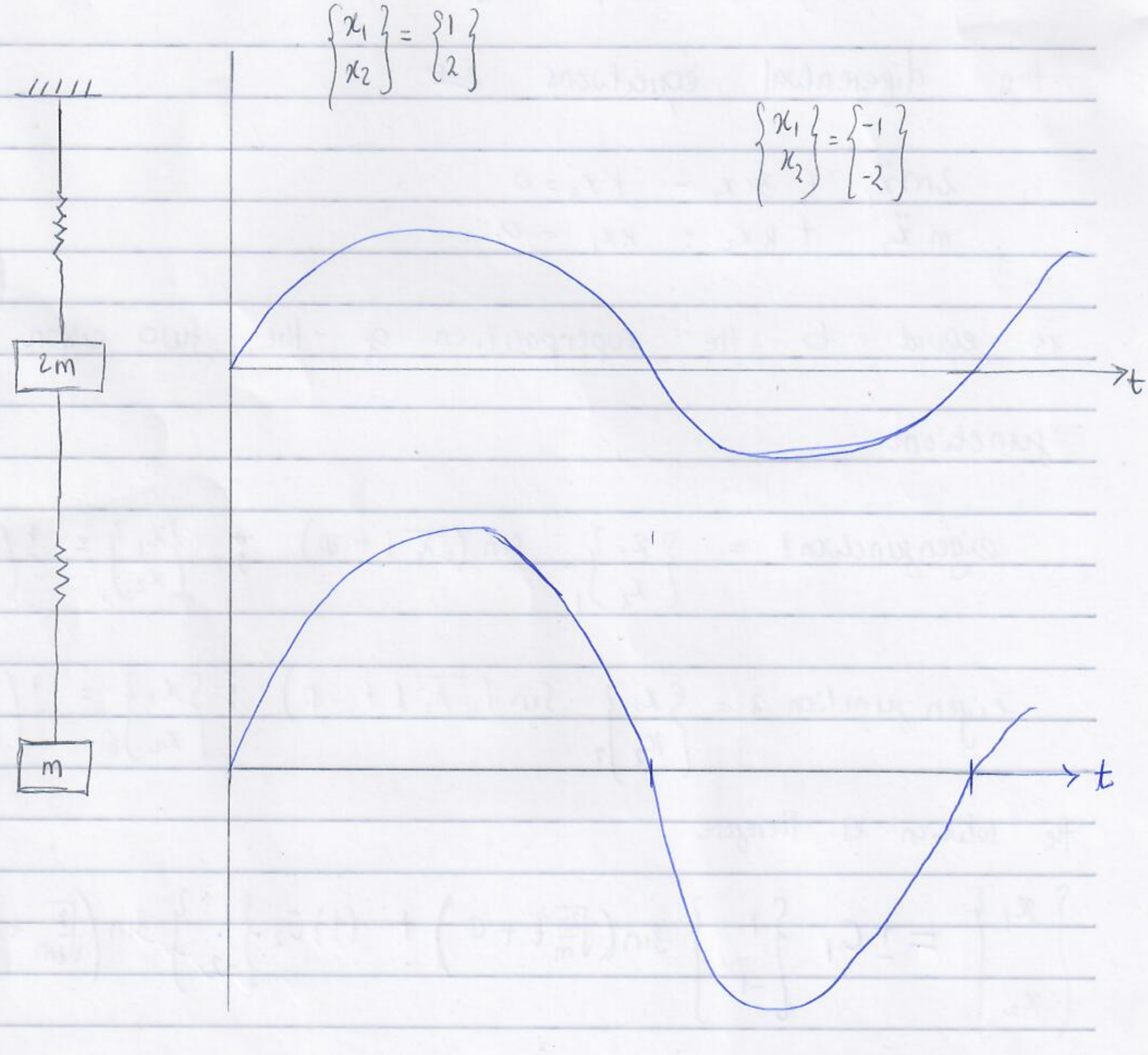
Note that these eigen vectors are equal to the amplitude ratios $\frac{A}{B}$ in the previous solution to the problem. Therefore eigen vectors represent the mode shapes of the system.

Also

The mode shape corresponding to $\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{2k}{m}}$ is indicated by eigen vector $V_1 = \pm \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



The mode shape corresponding to $\omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{K}{M}}$ is indicated by eigen vector $V_2 = \pm \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



Note that since $\omega_2 = \sqrt{\frac{K}{m}} < \omega_1 \sqrt{\frac{2K}{m}}$, mode 1 has a higher frequency ie more oscillations per second than mode 2.

An eigen function is a set of independent functions which are solutions to differential equations.

For this ~~system~~, coupled system, the solution to the differential equations ~~is~~

$$2m\ddot{x}_1 + 3kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + kx_2 - kx_1 = 0$$

is equal to the superposition of the two eigen functions

$$\text{eigenfunction } 1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 \sin(\sqrt{\lambda_1}\, t + \phi) \;;\; \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \pm\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{eigen function } 2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 \sin(\sqrt{\lambda_2}\, t + \phi) \;:\; \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \pm\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

the solution is therefore

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \underbrace{\pm C_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \sin\left(\sqrt{\tfrac{2k}{m}}\, t + \phi\right)}_{\text{mode } 1} + \underbrace{(\pm) C_2 \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} \sin\left(\sqrt{\tfrac{k}{2m}}\, t + \phi\right)}_{\text{mode } 2.}$$