Riemann–Stieltjes Integrability

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0.1 Motivation

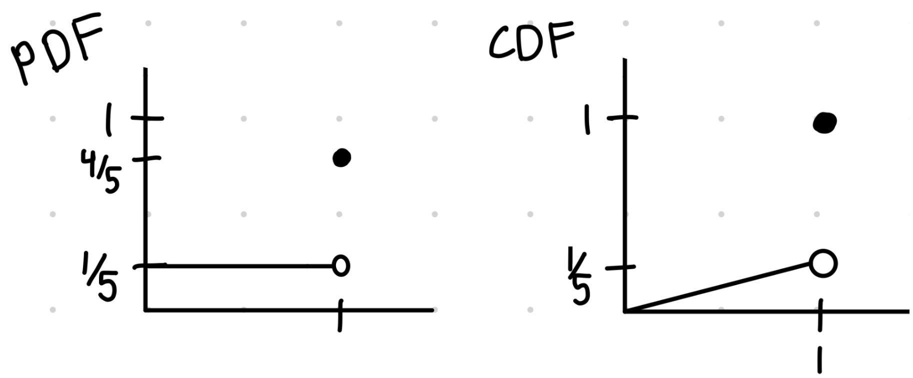
Previously, we have discussed and defined Riemann Sums and Integrability, and this can intuitively be thought of as a sum of a continuous function, f, with respect to a partition along x, where $P = \{t_i \in [a, b]\}$. We would like to extend this concept of summation but now with respect to another function, F(x). This has many different applications, and we will discuss the applications to probability.

0.1.1 Random Variable, PDF, CDF, Motivating Example

A random variable is a function that assigns events in the space of all outcomes to probabilities associated with those outcomes. For example, a coin flip could map X=1(heads) to 1/2 and X=0(tails) to 1/2. A random variable has an associated PDF (the probability that $X = x$, $P(X = x)$) and CDF (the probability that $X <= x$, $P(X <= x)$). It's also a fact that the sum of all possible pdf values across all different events must equal one (e.g. P(X=0) + P(X=1) = 1/2 + 1/2 = 1)

So, consider a random variable with a pdf (namely F'(x)), defined as follows:

$$F'(x) = \begin{cases} 1/5 \text{ if } x \in [0, 1) \\ 4/5 \text{ if } x = 1 \end{cases}$$



Now, to consider taking a sum of all possible values, we must consider integration since we are discussing a continous interval from [0,1). So, we must integrate the function and consider the appropriate weight.

If we are considering integration, we should first ask if this is Riemann-Integrable as discussed in class. It is Riemann-Integrable because the function is only discontinuous at a finite number of points (namely when x = 1). So, we can integrate the pdf and arrive at $\int_0^1 F'(x) = 1/5$ since the integral excludes finite discontinuities from the final result. However, this contradicts the stated fact that the sum of all pdf values must equal one. So, we must consider an improved method of integration that can account for the weight applied at x=1 (namely adding in that extra weight of 4/5 to achieve a final result where $\int_0^1 F'(x) = 1$)

0.2 Definitions and Statements

0.2.1 Darboux–Stieltjes Integrable Definition:

Let F be an increasing function on a closed interval [a,b] and $F(a) < F(b)$. F may have jumps and be discontinuous. Let f be a bounded function on [a,b].

Recall that the Upper Darboux Sum for a partition $P = \{t_0 < ... < t_i < ... < t_n\}$ is $U(f,P) = \sum_{i=1}^{n} sup\{f(x) : x \in [t_{i-1}, t_i]\}(t_i - t_{i-1})$. If we wanted to generalize this to a summation with respect to F(x), we can rewrite this as $\sum_{i=1}^{n} sup\{f(x) : x \in [t_{i-1}, t_i]\}(F(t_i) - F(t_{i-1}))$. However, this does not account for discontinuities/jumps we previously allowed. So, we revise this and establish the following defintion that handles discontinuous jumps in F (Note that $F(x^-)$ is the left-sided limit of F at x and $F(x^+)$ is the right-sided limit. This is necessary because if discontinuities exist, then the limits may be undefined because left-sided and right-sided limits might not agree). Also, we will use $F(a^-) = F(a)$ and $F(b^+) = F(b)$ as the limits of the endpoints.

$$J_F(f,P) = \sum_{i=0}^{n} f(t_i)(F(t_i^+) - F(t_i^-))$$

$$U_F(f,P) = J_F(f,P) + \sum_{i=1}^{n} sup\{f(x) : x \in (t_{i-1}, t_i)\}(F(t_i^-) - F(t_{i-1}^+))$$

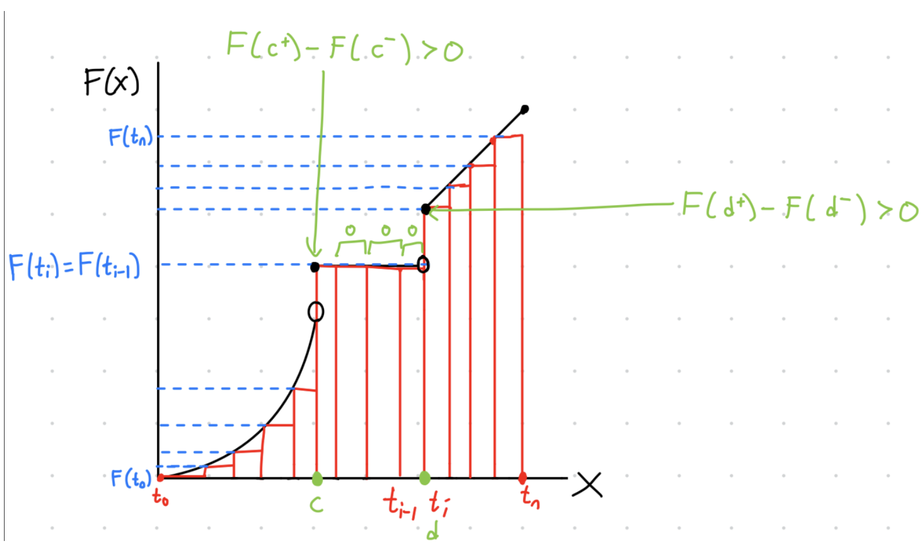
$$L_F(f,P) = J_F(f,P) + \sum_{i=1}^{n} inf\{f(x) : x \in (t_{i-1}, t_i)\}(F(t_i^-) - F(t_{i-1}^+))$$

These are called the Upper and Lower Darboux–Stieltjes Sums, we can define the Upper and Lower Darboux–Stieltjes Integrals as the infimum and supremem (respectively) of these sums of all possible partitions P. Similar to Darboux Integrability, we say f is Darboux–Stieltjes Integrable if $L_F(f) = U_F(f) = \int_a^b f dF$.

Note 1: $J_F(f,P)$ represents the total weight of the jumps, if F has jumps at the chosen partition points t_i. Then, naturally, if F is continuous, $J_F(f,P) = 0$ since $(F(t_i^+) - F(t_i^-))$ would always be 0, regardless of how the partition is chosen. It is also important to note that F can be discontinuous and a partition P can be chosen such that $J_F(f,P)$ is still 0 (i.e. none of the chosen points t_i are the points where jumps occur).

Note 2: We are stating the defintion of Darboux Integrability, which is synonymous with the Riemann Integrability definiton (where the function f(x) is being evaluated at some arbitrary x in the interval between t_{i-1} and t_i). It is also important to notice that since we are explicitly adding in the jumps/discontinuities, we must use open intervals when discussing $sup\{f(x) : x \in (t_{i-1}, t_i)\}$ and $inf\{f(x) : x \in (t_{i-1}, t_i)\}$. This was not the case with Riemann-Integrability from the previous section, where the discontinuity at a finite number of points did not affect the final result of the summation/integral. In other words, we are weighing the jumps as part of our summation unlike the previously defined version of Riemann integration, so we must not include those points when taking the summation of the $sups$ and $infs$ at those subinterval endpoints.

Note 3: Clearly, we are taking partitions on [a,b] and then applying $F(t_i)$ to those paritition points. Then it is natural to ask what if F is not one-to-one, which is possible if we are only restricting F to being increasing and not strictly increasing. In this case, we have distinct t_i and t_{i-1} where $F(t_{i-1}) = F(t_i)$. Intuitively, this is how the Riemann–Stieltjes accounts for the weights of jumps at distinct points. Consider the following graph depicting F(x) vs x. Then notice the instance from c to d where F is not one-to-one. Then, we can see from the definition above that $(F(t_i^-) - F(t_{i-1}^+)) = 0$ whenever $t_{i-1}, t_i \in (c, d)$. Then we consider $J_F(f, P)$ and determine that the only instances where $(F(t_i^+) - F(t_i^-)) \neq 0$ are $(F(c^+) - F(c^-))$ and $(F(d^+) - F(d^-))$, and so we see that the weight associated with these intervals become $f(c)(F(c^+) - F(c^-)) + f(d)(F(d^+) - F(d^-))$.



0.2.2 Darboux–Stieltjes Integrability Theorem:

Following from the definition of $U_F(f, P)$ and $L_F(f, P)$ and the assumptions about f and F, we say that f is Darboux-Stieltjes Integrable (or F-Integrable) iff $\forall \varepsilon > 0$, $\exists P$ such that $U_F(f, P) - L_F(f, P) < \varepsilon$.

Note 1: This is very similar to the Darboux Integrability Theorem discussed in the course but this time using Upper and Lower Darboux–Stieltjes Sums and Integrals. Many other theorems that apply for Darboux Integrals can also be extended for Darboux–Stieltjes Integrals (such as continuity implying integrability, monotonic implying integrability, linearity of integration, etc.).

0.2.3 Theorem (If F is differentiable)

If F is differentiable on [a,b] and F' is Riemann-Integrable on [a,b], then f (a bounded function on [a,b]) is F-Integrable iff fF' is Riemann-Integrable. Then we have $\int_a^b f(x)dF = \int_a^b f(x)F'(x)dx$.

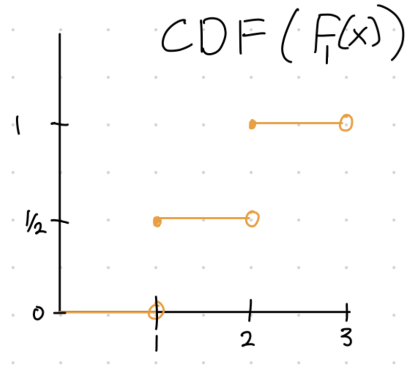
Note 1: If F is not differentiable (such as when we are dealing with a step function that is not continuous) then this theorem does not apply, but the integral $\int_a^b f(x)dF$ is still well-defined as long as F is an increasing function as previously stated.

0.2.4 Riemann–Stieltjes Integrable as a Generalization of Riemann Integrable

Riemann–Stieltjes Integrability is a generalization of Riemann-integrability, and we can retrieve Riemann-Integrability when we set F(x) = x. Then, because x is differentiable everywhere, we can say that its derivative F'(x) = 1, which (after applying the previous theorem) leaves us with $\int_a^b f(x)dx$.

0.3 Exercises

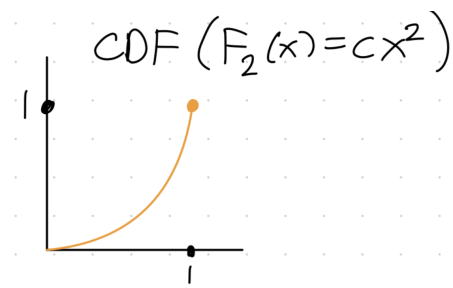
Given the following graphs of the CDFs of random variables, answer the following questions.



a. What is P(X = 1) = P(X = 2)?
Since the only points when the CDF increases are x=1 and x=2, and since each has a jump of 1/2, then P(X = 1) = P(X = 2) = 1/2.

b. Can we apply theorem 0.2.3 with f(x) = x?
F_1 is a step function with discontinuous jumps, so it is not differentiable at those points where jumps occur. Since it is not differentiable, we cannot apply the theorem.

c. The expectation value of a random variable with CDF function F(x) is given by $\int_a^b x dF$. In this case, a = 0, b = 1, and $F_1(x)$ is the given graph. What is the expectation value for the random variable? We notice that because the function values at F are the same except at a finite number of points (namely x = 1 and x = 2), we only apply the weights to those values. In other words, the difference of $(F(t_i^-) - F(t_{i-1}^+))$ will be 0 everywhere else, so only consider the weights of x = 1 and x = 2. Then we see that $\int_0^1 x dF = 1(1/2) + 2(1/2) = 3/2$.



a. It is a known that the PDF of a random variable is the derivative of its CDF (if the derivative exists). Given $F_2(x) = cx^2$ is differentiable and $\int_0^1 F_2'(x)dx = 1$, what must c equal?

Since $F_2(x) = cx^2$, we can say that $F_2'(x) = 2cx$. Since $\int_0^1 2cx = 1$, it follows that c = 1.

b. Can we apply theorem 0.2.3 with f(x) = x?

$F_2 = x^2$ is a quadratic function, and since x is differentiable and continous, it follows that x^2 must be differentiable and continuous as well. Its derivative $F_2'(x) = 2x$ is also continous and differentiable. Since f(x) = x is continous, then $fF_2' = 2x^2$ is continous and therefore Riemann-Integrable. So the theorem does apply.

c. What is the expectation value for the random variable? In other words, evaluate $\int_0^1 xdF$.

We first apply theorem 0.2.3, and we see that $\int_0^1 xdF = \int_0^1 xF_2'(x)dx = \int_0^1 2x^2dx$ $= 2/3$.