Exercise Session: Social Choice Theory II

COMP4418 Knowledge Representation and Reasoning

Patrick Lederer[1]

[1]School of Computer Science and Engineering, UNSW Australia

Exercise I: Compute RSCFs

Compute the lottery chosen by the uniform random dictatorship, the randomized Borda rule (which randomizes proportional to the Borda scores), and a maximal lottery for the subsequent profiles.

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$

Uniform Random Dictatorship:

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$

Uniform Random Dictatorship:

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$

Uniform Random Dictatorship:

- a is top-ranked by 3 voters.
- b is top-ranked by 2 voters.
- c is top-ranked by 2 voters.
- d is top-ranked by 0 voters.

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$

Uniform Random Dictatorship:

- a is top-ranked by 3 voters.
- b is top-ranked by 2 voters.
- c is top-ranked by 2 voters.
- d is top-ranked by 0 voters.
- The uniform random dictatorship chooses the lottery $[\frac{3}{7} : a, \frac{2}{7} : b, \frac{2}{7} : c]$

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

Randomized Borda Rule:

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

$3 \succ 2 \succ 1 \succ 0$

Randomized Borda Rule:

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

$3 \succ 2 \succ 1 \succ 0$

Randomized Borda Rule:

- Borda score of a: $2 \cdot 0 + 2 \cdot 3 + 2 \cdot 1 + 1 \cdot 3 = 11$

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

$3 \quad 2 \quad 1 \quad 0$

Randomized Borda Rule:

- Borda score of a: $2 \cdot 0 + 2 \cdot 3 + 2 \cdot 1 + 1 \cdot 3 = 11$
- Borda score of b: $2 \cdot 3 + 2 \cdot 2 + 2 \cdot 0 + 1 \cdot 0 = 10$

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$
3 $\succ$ 2 $\succ$ 1 $\succ$ 0

Randomized Borda Rule:

- Borda score of a: $2 \cdot 0 + 2 \cdot 3 + 2 \cdot 1 + 1 \cdot 3 = 11$
- Borda score of b: $2 \cdot 3 + 2 \cdot 2 + 2 \cdot 0 + 1 \cdot 0 = 10$
- Borda score of c: $2 \cdot 2 + 2 \cdot 1 + 2 \cdot 3 + 1 \cdot 1 = 13$

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$
$3 \succ 2 \succ 1 \succ 0$

Randomized Borda Rule:

- Borda score of a: $2 \cdot 0 + 2 \cdot 3 + 2 \cdot 1 + 1 \cdot 3 = 11$
- Borda score of b: $2 \cdot 3 + 2 \cdot 2 + 2 \cdot 0 + 1 \cdot 0 = 10$
- Borda score of c: $2 \cdot 2 + 2 \cdot 1 + 2 \cdot 3 + 1 \cdot 1 = 13$
- Borda score of d: $2 \cdot 1 + 2 \cdot 0 + 2 \cdot 2 + 1 \cdot 2 = 8$

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

3 $\succ$ 2 $\succ$ 1 $\succ$ 0

Randomized Borda Rule:

- Borda score of a: $2 \cdot 0 + 2 \cdot 3 + 2 \cdot 1 + 1 \cdot 3 = 11$
- Borda score of b: $2 \cdot 3 + 2 \cdot 2 + 2 \cdot 0 + 1 \cdot 0 = 10$
- Borda score of c: $2 \cdot 2 + 2 \cdot 1 + 2 \cdot 3 + 1 \cdot 1 = 13$
- Borda score of d: $2 \cdot 1 + 2 \cdot 0 + 2 \cdot 2 + 1 \cdot 2 = 8$
- The randomized Borda rule chooses the lottery $[\frac{11}{42} : a, \frac{10}{42} : b, \frac{13}{42} : c, \frac{8}{42} : d]$

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

Maximal Lottery - Approach 1:

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

Maximal Lottery - Approach 1:

- Compute the matrix containing the values $n_{xy}(R) = |\{i \in N : x \succ_i y\}|$ for all $x, y \in A$.

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$

	a	b	c	d
a	0	5	3	3
b	2	0	4	4
c	4	3	0	6
d	4	3	1	0

Maximal Lottery - Approach 1:

- Compute the matrix containing the values $n_{xy}(R) = |\{i \in N : x \succ_i y\}|$ for all $x, y \in A$.

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$

	a	b	c	d
a	0	5	3	3
b	2	0	4	4
c	4	3	0	6
d	4	3	1	0

Maximal Lottery - Approach 1:

- Compute the matrix containing the values $n_{xy}(R) = |\{i \in N : x \succ_i y\}|$ for all $x, y \in A$.
- Solve the inequality system

$$0 \cdot p(a) + 2 \cdot p(b) + 4 \cdot p(c) + 4 \cdot p(d) \geq 0 \cdot p(a) + 5 \cdot p(b) + 3 \cdot p(c) + 3 \cdot p(d)$$
$$5 \cdot p(a) + 0 \cdot p(b) + 3 \cdot p(c) + 3 \cdot p(d) \geq 2 \cdot p(a) + 0 \cdot p(b) + 4 \cdot p(c) + 4 \cdot p(d)$$
$$3 \cdot p(a) + 4 \cdot p(b) + 0 \cdot p(c) + 1 \cdot p(d) \geq 4 \cdot p(a) + 3 \cdot p(b) + 0 \cdot p(c) + 6 \cdot p(d)$$
$$4 \cdot p(a) + 3 \cdot p(b) + 1 \cdot p(c) + 0 \cdot p(d) \geq 3 \cdot p(a) + 4 \cdot p(b) + 6 \cdot p(c) + 0 \cdot p(d)$$

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

Maximal Lottery - Approach 2:

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$
1: $a \succ d \succ c \succ b$

Maximal Lottery - Approach 2:

- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.

Exercise I: Compute RSCFs

a) R^1: 2: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

1: $a \succ d \succ c \succ b$

Maximal Lottery - Approach 2:

- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.
  $\rightarrow$ we can cancel out completely reversed preference relations.

Exercise I: Compute RSCFs

a) R^1: 1: $b \succ c \succ d \succ a$

2: $a \succ b \succ c \succ d$

2: $c \succ d \succ a \succ b$

Maximal Lottery - Approach 2:

- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.
  $\rightarrow$ we can cancel out completely reversed preference relations.

Exercise I: Compute RSCFs

a) R^1: 1: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$

Maximal Lottery - Approach 2:

- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.
  $\rightarrow$ we can cancel out completely reversed preference relations.
- Maximal lotteries assign probability 0 to Pareto-dominated alternatives and are invariant under removing alternatives with probability 0.

Exercise I: Compute RSCFs

a) R^1: 1: $b \succ c \succ d \succ a$
2: $a \succ b \succ c \succ d$
2: $c \succ d \succ a \succ b$

Maximal Lottery - Approach 2:

- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.
  $\rightarrow$ we can cancel out completely reversed preference relations.
- Maximal lotteries assign probability 0 to Pareto-dominated alternatives and are invariant under removing alternatives with probability 0.
  $\rightarrow$ We can remove Pareto-dominated alternatives.

Exercise I: Compute RSCFs

a) R^1: 1: $b \succ c \succ a$

2: $a \succ b \succ c$

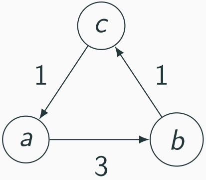
2: $c \succ a \succ b$

Maximal Lottery - Approach 2:

- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.
  $\rightarrow$ we can cancel out completely reversed preference relations.
- Maximal lotteries assign probability 0 to Pareto-dominated alternatives and are invariant under removing alternatives with probability 0.
  $\rightarrow$ We can remove Pareto-dominated alternatives.

Exercise I: Compute RSCFs

a) R^1: 1: $b \succ c \succ a$
2: $a \succ b \succ c$
2: $c \succ a \succ b$



Maximal Lottery - Approach 2:

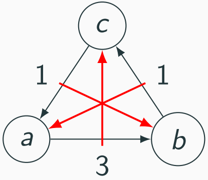
- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.
  $\rightarrow$ we can cancel out completely reversed preference relations.
- Maximal lotteries assign probability 0 to Pareto-dominated alternatives and are invariant under removing alternatives with probability 0.
  $\rightarrow$ We can remove Pareto-dominated alternatives.
- Triangle trick:

Exercise I: Compute RSCFs

a) R^1: 1: $b \succ c \succ a$

2: $a \succ b \succ c$

2: $c \succ a \succ b$

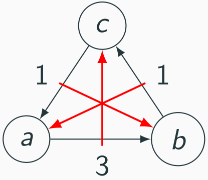


Maximal Lottery - Approach 2:

- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.
  $\rightarrow$ we can cancel out completely reversed preference relations.
- Maximal lotteries assign probability 0 to Pareto-dominated alternatives and are invariant under removing alternatives with probability 0.
  $\rightarrow$ We can remove Pareto-dominated alternatives.
- Triangle trick:

Exercise I: Compute RSCFs

a) R^1: 1: $b \succ c \succ a$
   2: $a \succ b \succ c$
   2: $c \succ a \succ b$



Maximal Lottery - Approach 2:

- Maximal lotteries can be computed based on the values $n_{xy}(R) - n_{yx}(R)$ for all $x, y \in A$.
  $\rightarrow$ we can cancel out completely reversed preference relations.
- Maximal lotteries assign probability 0 to Pareto-dominated alternatives and are invariant under removing alternatives with probability 0.
  $\rightarrow$ We can remove Pareto-dominated alternatives.
- Triangle trick:The maximal lottery is $[\frac{1}{5} : a, \frac{1}{5} : b, \frac{3}{5} : c]$.

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

Uniform Random Dictatorship:

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

Uniform Random Dictatorship:

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

Uniform Random Dictatorship:

- a is top-ranked by 2 voters.
- b is top-ranked by 0 voters.
- c is top-ranked by 1 voters.
- d is top-ranked by 2 voters.

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

Uniform Random Dictatorship:

- a is top-ranked by 2 voters.
- b is top-ranked by 0 voters.
- c is top-ranked by 1 voters.
- d is top-ranked by 2 voters.
- The uniform random dictatorship chooses the lottery $[\frac{2}{5} : a, 0 : b, \frac{1}{5} : c, \frac{2}{5} : d]$

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

Randomized Borda Rule:

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

3 2 1 0

Randomized Borda Rule:

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

$3 \quad 2 \quad 1 \quad 0$

Randomized Borda Rule:

- Borda score of a: $2 \cdot 3 + 2 \cdot 0 + 1 \cdot 2 = 8$

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

$3 \quad 2 \quad 1 \quad 0$

Randomized Borda Rule:

- Borda score of a: $2 \cdot 3 + 2 \cdot 0 + 1 \cdot 2 = 8$
- Borda score of b: $2 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 9$

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

3 $\succ$ 2 $\succ$ 1 $\succ$ 0

Randomized Borda Rule:

- Borda score of a: $2 \cdot 3 + 2 \cdot 0 + 1 \cdot 2 = 8$
- Borda score of b: $2 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 9$
- Borda score of c: $2 \cdot 1 + 2 \cdot 1 + 1 \cdot 3 = 7$

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

$3 \succ 2 \succ 1 \succ 0$

Randomized Borda Rule:

- Borda score of a: $2 \cdot 3 + 2 \cdot 0 + 1 \cdot 2 = 8$
- Borda score of b: $2 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 9$
- Borda score of c: $2 \cdot 1 + 2 \cdot 1 + 1 \cdot 3 = 7$
- Borda score of d: $2 \cdot 3 + 2 \cdot 0 + 1 \cdot 0 = 6$

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

$3 \succ 2 \succ 1 \succ 0$

Randomized Borda Rule:

- Borda score of a: $2 \cdot 3 + 2 \cdot 0 + 1 \cdot 2 = 8$
- Borda score of b: $2 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 = 9$
- Borda score of c: $2 \cdot 1 + 2 \cdot 1 + 1 \cdot 3 = 7$
- Borda score of d: $2 \cdot 3 + 2 \cdot 0 + 1 \cdot 0 = 6$
- The randomized Borda rule chooses the lottery $[\frac{8}{30} : a, \frac{9}{30} : b, \frac{7}{30} : c, \frac{6}{30} : d]$

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

Maximal Lottery:

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$
2: $d \succ b \succ c \succ a$
1: $c \succ a \succ b \succ d$

Maximal Lottery:

- Maximal lotteries assign probability 0 to the alternative that loses all pairwise majority comparisons.

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c \succ d$

2: $d \succ b \succ c \succ a$

1: $c \succ a \succ b \succ d$

Maximal Lottery:

- Maximal lotteries assign probability 0 to the alternative that loses all pairwise majority comparisons.
  $\rightarrow$ we can remove this alternative from our profile.

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c$

2: $b \succ c \succ a$

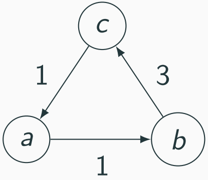
1: $c \succ a \succ b$

Maximal Lottery:

- Maximal lotteries assign probability 0 to the alternative that loses all pairwise majority comparisons.
  $\rightarrow$ we can remove this alternative from our profile.

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c$
2: $b \succ c \succ a$
1: $c \succ a \succ b$

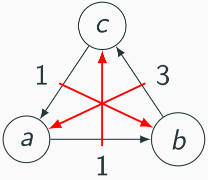


Maximal Lottery:

- Maximal lotteries assign probability 0 to the alternative that loses all pairwise majority comparisons.
  $\rightarrow$ we can remove this alternative from our profile.
- Triangle trick:

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c$
2: $b \succ c \succ a$
1: $c \succ a \succ b$

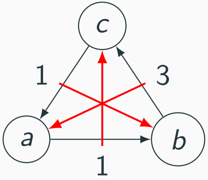


Maximal Lottery:

- Maximal lotteries assign probability 0 to the alternative that loses all pairwise majority comparisons.
  $\rightarrow$ we can remove this alternative from our profile.
- Triangle trick:

Exercise I: Compute RSCFs

b) R^2: 2: $a \succ b \succ c$
2: $b \succ c \succ a$
1: $c \succ a \succ b$



Maximal Lottery:

- Maximal lotteries assign probability 0 to the alternative that loses all pairwise majority comparisons.
  $\rightarrow$ we can remove this alternative from our profile.
- Triangle trick:The maximal lottery is $[\frac{3}{5} : a, \frac{1}{5} : b, \frac{1}{5} : c]$.

Exercise II: Strategyproofness for RSCFs

a) Show that no maximal lottery rule is strategyproof.

Exercise II: Strategyproofness for RSCFs

a) Show that no maximal lottery rule is strategyproof.

R^1: 1: $a \succ b \succ c$

1: $b \succ c \succ a$

1: $c \succ a \succ b$

Exercise II: Strategyproofness for RSCFs

a) Show that no maximal lottery rule is strategyproof.

R^1: 1: $a \succ b \succ c$

1: $b \succ c \succ a$

1: $c \succ a \succ b$

- The unique maximal lottery in R^1 is $p = [\frac{1}{3} : a, \frac{1}{3} : b, \frac{1}{3} : c]$

Exercise II: Strategyproofness for RSCFs

a) Show that no maximal lottery rule is strategyproof.

R^1: 1: $a \succ b \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

- The unique maximal lottery in R^1 is $p = [\frac{1}{3} : a, \frac{1}{3} : b, \frac{1}{3} : c]$
- Assume $u(a) = 3$, $u(b) = 2$, $u(c) = 0$. The expected utility of agent 1 is $\mathbb{E}[u(p)] = \frac{5}{3}$.

Exercise II: Strategyproofness for RSCFs

a) Show that no maximal lottery rule is strategyproof.

R^1: 1: $a \succ b \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

R^2: 1: $b \succ a \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

- The unique maximal lottery in R^1 is $p = [\frac{1}{3} : a, \frac{1}{3} : b, \frac{1}{3} : c]$
- Assume $u(a) = 3$, $u(b) = 2$, $u(c) = 0$. The expected utility of agent 1 is $\mathbb{E}[u(p)] = \frac{5}{3}$.

Exercise II: Strategyproofness for RSCFs

a) Show that no maximal lottery rule is strategyproof.

R^1: 1: $a \succ b \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

R^2: 1: $b \succ a \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

- The unique maximal lottery in R^1 is $p = [\frac{1}{3} : a, \frac{1}{3} : b, \frac{1}{3} : c]$
- Assume $u(a) = 3$, $u(b) = 2$, $u(c) = 0$. The expected utility of agent 1 is $\mathbb{E}[u(p)] = \frac{5}{3}$.
- The unique maximal lottery in R^2 is $q = [1 : b]$.

Exercise II: Strategyproofness for RSCFs

a) Show that no maximal lottery rule is strategyproof.

R^1: 1: $a \succ b \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

R^2: 1: $b \succ a \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

- The unique maximal lottery in R^1 is $p = [\frac{1}{3} : a, \frac{1}{3} : b, \frac{1}{3} : c]$
- Assume $u(a) = 3$, $u(b) = 2$, $u(c) = 0$. The expected utility of agent 1 is $\mathbb{E}[u(p)] = \frac{5}{3}$.
- The unique maximal lottery in R^2 is $q = [1 : b]$.
- Assume $u(a) = 3$, $u(b) = 2$, $u(c) = 0$. The expected utility of agent 1 for this utility is $\mathbb{E}[u(q)] = 2$.

Exercise II: Strategyproofness for RSCFs

a) Show that no maximal lottery rule is strategyproof.

R^1: 1: $a \succ b \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

R^2: 1: $b \succ a \succ c$
1: $b \succ c \succ a$
1: $c \succ a \succ b$

- The unique maximal lottery in R^1 is $p = [\frac{1}{3} : a, \frac{1}{3} : b, \frac{1}{3} : c]$
- Assume $u(a) = 3$, $u(b) = 2$, $u(c) = 0$. The expected utility of agent 1 is $\mathbb{E}[u(p)] = \frac{5}{3}$.
- The unique maximal lottery in R^2 is $q = [1 : b]$.
- Assume $u(a) = 3$, $u(b) = 2$, $u(c) = 0$. The expected utility of agent 1 for this utility is $\mathbb{E}[u(q)] = 2$.
- Voter 1 can manipulate!

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let R^1 and R^2 denote two profiles that only differ in the preference relation of voter i.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let R^1 and R^2 denote two profiles that only differ in the preference relation of voter i.
- Let p and q denote the lotteries chosen by the randomized Borda rule for R^1 and R^2.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let R^1 and R^2 denote two profiles that only differ in the preference relation of voter i.
- Let p and q denote the lotteries chosen by the randomized Borda rule for R^1 and R^2.
- Let u denote a utility function that is consistent with the preference relation $\succ_i^1$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let R^1 and R^2 denote two profiles that only differ in the preference relation of voter i.
- Let p and q denote the lotteries chosen by the randomized Borda rule for R^1 and R^2.
- Let u denote a utility function that is consistent with the preference relation $\succ_i^1$.
- To show: $\mathbb{E}[u(p)] \geq \mathbb{E}[u(q)]$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let $b(R^1, x)$ denote the Borda score of alternative x in R^1, $B^1 = \sum_{x \in A} b(R^1, x)$ the total Borda score in R^1, and $B^2 = \sum_{x \in A} b(R^2, x)$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let $b(R^1, x)$ denote the Borda score of alternative x in R^1, $B^1 = \sum_{x \in A} b(R^1, x)$ the total Borda score in R^1, and $B^2 = \sum_{x \in A} b(R^2, x)$.
- Since R^1 and R^2 have the same number of voters, $B^1 = B^2$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let $b(R^1, x)$ denote the Borda score of alternative x in R^1, $B^1 = \sum_{x \in A} b(R^1, x)$ the total Borda score in R^1, and $B^2 = \sum_{x \in A} b(R^2, x)$.
- Since R^1 and R^2 have the same number of voters, $B^1 = B^2$.
- It suffices to show that $\sum_{x \in A} u(x) b(R^1, x) \geq \sum_{x \in A} u(x) b(R^2, x)$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let $b(R^1, x)$ denote the Borda score of alternative x in R^1, $B^1 = \sum_{x \in A} b(R^1, x)$ the total Borda score in R^1, and $B^2 = \sum_{x \in A} b(R^2, x)$.
- Since R^1 and R^2 have the same number of voters, $B^1 = B^2$.
- It suffices to show that $\sum_{x \in A} u(x) b(R^1, x) \geq \sum_{x \in A} u(x) b(R^2, x)$.
- Since $\succ_j^1 = \succ_j^2$ for all voters $j \neq i$, it holds that $b(R^1, x) - b(R^2, x) = b(\succ_i^1, x) - b(\succ_i^2, x)$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- Let $b(R^1, x)$ denote the Borda score of alternative x in R^1, $B^1 = \sum_{x\in A} b(R^1, x)$ the total Borda score in R^1, and $B^2 = \sum_{x\in A} b(R^2, x)$.
- Since R^1 and R^2 have the same number of voters, $B^1 = B^2$.
- It suffices to show that $\sum_{x\in A} u(x)b(R^1, x) \geq \sum_{x\in A} u(x)b(R^2, x)$.
- Since $\succ_j^1 = \succ_j^2$ for all voters $j \neq i$, it holds that $b(R^1, x) - b(R^2, x) = b(\succ_i^1, x) - b(\succ_i^2, x)$.
- It suffices to show that $\sum_{x\in A} u(x)b(\succ_i^1, x) \geq \sum_{x\in A} u(x)b(\succ_i^2, x)$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- We claim that $\succ_i^1$ maximizes $\sum_{x \in A} u(x) b(\succ, x)$

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- We claim that $\succ_i^1$ maximizes $\sum_{x \in A} u(x) b(\succ, x)$
- Consider a preference relation $\succ \neq \succ_i^1$. Wlog, let y denote the alternative that maximizes $b(\succ_i^1, y)$ subject to $b(\succ, y) \neq b(\succ_i^1, y)$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- We claim that $\succ_i^1$ maximizes $\sum_{x\in A} u(x)b(\succ, x)$
- Consider a preference relation $\succ \neq \succ_i^1$. Wlog, let y denote the alternative that maximizes $b(\succ_i^1, y)$ subject to $b(\succ, y) \neq b(\succ_i^1, y)$.
- Since $b(\succ, x) = b(\succ_i^1, x)$ for all x with $x \succ_i^1 y$, it holds that $b(\succ, y) < b(\succ_i^1, y)$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- We claim that $\succ_i^1$ maximizes $\sum_{x \in A} u(x) b(\succ, x)$
- Consider a preference relation $\succ \neq \succ_i^1$. Wlog, let y denote the alternative that maximizes $b(\succ_i^1, y)$ subject to $b(\succ, y) \neq b(\succ_i^1, y)$.
- Since $b(\succ, x) = b(\succ_i^1, x)$ for all x with $x \succ_i^1 y$, it holds that $b(\succ, y) < b(\succ_i^1, y)$.
- Let $\succ'$ denote the preference relation derived from $\succ$ by moving y at position $b(\succ_i^1, y)$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- We claim that $\succ_i^1$ maximizes $\sum_{x \in A} u(x) b(\succ, x)$
- Consider a preference relation $\succ \neq \succ_i^1$. Wlog, let y denote the alternative that maximizes $b(\succ_i^1, y)$ subject to $b(\succ, y) \neq b(\succ_i^1, y)$.
- Since $b(\succ, x) = b(\succ_i^1, x)$ for all x with $x \succ_i^1 y$, it holds that $b(\succ, y) < b(\succ_i^1, y)$.
- Let $\succ'$ denote the preference relation derived from $\succ$ by moving y at position $b(\succ_i^1, y)$.
- Let $\ell = b(\succ', y) - b(\succ, y)$

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- By going from $\succ$ to $\succ'$:

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- By going from $\succ$ to $\succ'$:
  - We increase the utility gained by y by $u(y)\ell$.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- By going from $\succ$ to $\succ'$:
  - We increase the utility gained by y by $u(y)\ell$.
  - We decrease the Borda score of ℓ alternatives $z_1, \ldots, z_\ell$ by 1.

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- By going from $\succ$ to $\succ'$:
  - We increase the utility gained by y by $u(y)\ell$.
  - We decrease the Borda score of ℓ alternatives $z_1, \ldots, z_\ell$ by 1.
  - This decreases the utility by $\sum_{j=1}^{\ell} u(z_i)$.
- It holds that $y \succ_i^1 z_j$ for all $j \in \{1, \ldots, \ell\}$ (as y is only weakened when going from $\succ_i^1$ to $\succ$)

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- By going from $\succ$ to $\succ'$:
  - We increase the utility gained by y by $u(y)\ell$.
  - We decrease the Borda score of ℓ alternatives $z_1, \ldots, z_\ell$ by 1.
  - This decreases the utility by $\sum_{j=1}^{\ell} u(z_i)$.
- It holds that $y \succ_i^1 z_j$ for all $j \in \{1, \ldots, \ell\}$ (as y is only weakened when going from $\succ_i^1$ to $\succ$)
- This implies that $u(y) > u(z_j)$ for all $j \in \{1, \ldots, \ell\}$

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- By going from $\succ$ to $\succ'$:
  - We increase the utility gained by y by $u(y)\ell$.
  - We decrease the Borda score of ℓ alternatives $z_1, \dots, z_\ell$ by 1.
  - This decreases the utility by $\sum_{j=1}^{\ell} u(z_i)$.
- It holds that $y \succ_i^1 z_j$ for all $j \in \{1, \dots, \ell\}$ (as y is only weakened when going from $\succ_i^1$ to $\succ$)
- This implies that $u(y) > u(z_j)$ for all $j \in \{1, \dots, \ell\}$
- Thus $u(y)\ell > \sum_{j=1}^{\ell} u(z_i)$

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- By going from $\succ$ to $\succ'$:
  - We increase the utility gained by y by $u(y)\ell$.
  - We decrease the Borda score of ℓ alternatives $z_1, \dots, z_\ell$ by 1.
  - This decreases the utility by $\sum_{j=1}^{\ell} u(z_i)$.
- It holds that $y \succ_i^1 z_j$ for all $j \in \{1, \dots, \ell\}$ (as y is only weakened when going from $\succ_i^1$ to $\succ$)
- This implies that $u(y) > u(z_j)$ for all $j \in \{1, \dots, \ell\}$
- Thus $u(y)\ell > \sum_{j=1}^{\ell} u(z_i)$
- This proves that $\succ_i^1$ maximizes $\sum_{x \in A} u(x) b(\succ, x)$

Exercise II: Strategyproofness for RSCFs

b) Show that the randomized Borda rule is strategyproof.

- By going from $\succ$ to $\succ'$:
  - We increase the utility gained by y by $u(y)\ell$.
  - We decrease the Borda score of ℓ alternatives $z_1, \dots, z_\ell$ by 1.
  - This decreases the utility by $\sum_{j=1}^{\ell} u(z_i)$.
- It holds that $y \succ_i^1 z_j$ for all $j \in \{1, \dots, \ell\}$ (as y is only weakened when going from $\succ_i^1$ to $\succ$)
- This implies that $u(y) > u(z_j)$ for all $j \in \{1, \dots, \ell\}$
- Thus $u(y)\ell > \sum_{j=1}^{\ell} u(z_i)$
- This proves that $\succ_i^1$ maximizes $\sum_{x \in A} u(x)b(\succ, x)$
- Thus, $\sum_{x \in A} u(x)b(\succ_i^1, x) \geq \sum_{x \in A} u(x)b(\succ_i^2, x)$ and the randomized Borda rule is strategyproof.

Exercise II: Strategyproofness for RSCFs

c) Given a preference relation $\succ$ and an alternative x, let $U(\succ, x) = \{x\} \cup \{y \in A\colon y \succ x\}$. Show that, for all preference relations $\succ$ and all lotteries $p, q \in \Delta(A)$, it holds that $\mathbb{E}[p(u)] \geq \mathbb{E}[q(u)]$ for all u that are consistent with $\succ$ if and only if $\sum_{y \in U(\succ, x)} p(y) \geq \sum_{y \in U(\succ, x)} q(y)$ for all $x \in A$.

Exercise II: Strategyproofness for RSCFs

c) Given a preference relation $\succ$ and an alternative x, let $U(\succ, x) = \{x\} \cup \{y \in A \colon y \succ x\}$. Show that, for all preference relations $\succ$ and all lotteries $p, q \in \Delta(A)$, it holds that $\mathbb{E}[p(u)] \geq \mathbb{E}[q(u)]$ for all u that are consistent with $\succ$ if and only if $\sum_{y \in U(\succ,x)} p(y) \geq \sum_{y \in U(\succ,x)} q(y)$ for all $x \in A$.

- Assume that $\succ$ is given by $x_1 \succ x_2 \succ \cdots \succ x_m$ and fix two lotteries p and q.

Exercise II: Strategyproofness for RSCFs

c) Given a preference relation $\succ$ and an alternative x, let $U(\succ, x) = \{x\} \cup \{y \in A\colon y \succ x\}$. Show that, for all preference relations $\succ$ and all lotteries $p, q \in \Delta(A)$, it holds that $\mathbb{E}[p(u)] \geq \mathbb{E}[q(u)]$ for all u that are consistent with $\succ$ if and only if $\sum_{y \in U(\succ,x)} p(y) \geq \sum_{y \in U(\succ,x)} q(y)$ for all $x \in A$.

- Assume that $\succ$ is given by $x_1 \succ x_2 \succ \dots \succ x_m$ and fix two lotteries p and q.
- Assume that $\sum_{y \in U(\succ,x)} p(y) \geq \sum_{y \in U(\succ,x)} q(y)$ for all $x \in A$.

Exercise II: Strategyproofness for RSCFs

c) Given a preference relation $\succ$ and an alternative x, let $U(\succ, x) = \{x\} \cup \{y \in A \colon y \succ x\}$. Show that, for all preference relations $\succ$ and all lotteries $p, q \in \Delta(A)$, it holds that $\mathbb{E}[p(u)] \geq \mathbb{E}[q(u)]$ for all u that are consistent with $\succ$ if and only if $\sum_{y \in U(\succ, x)} p(y) \geq \sum_{y \in U(\succ, x)} q(y)$ for all $x \in A$.

- Assume that $\succ$ is given by $x_1 \succ x_2 \succ \dots \succ x_m$ and fix two lotteries p and q.
- Assume that $\sum_{y \in U(\succ, x)} p(y) \geq \sum_{y \in U(\succ, x)} q(y)$ for all $x \in A$.
- Let u denote a utility function consistent with $\succ$. Define $\Delta_m = u(x_m)$ and $\Delta_i = u(x_i) - u(x_{i+1})$ for $i \in \{1, \dots, m-1\}$.

Exercise II: Strategyproofness for RSCFs

c) Given a preference relation $\succ$ and an alternative x, let $U(\succ, x) = \{x\} \cup \{y \in A \colon y \succ x\}$. Show that, for all preference relations $\succ$ and all lotteries $p, q \in \Delta(A)$, it holds that $\mathbb{E}[p(u)] \geq \mathbb{E}[q(u)]$ for all u that are consistent with $\succ$ if and only if $\sum_{y \in U(\succ,x)} p(y) \geq \sum_{y \in U(\succ,x)} q(y)$ for all $x \in A$.

- Assume that $\succ$ is given by $x_1 \succ x_2 \succ \cdots \succ x_m$ and fix two lotteries p and q.
- Assume that $\sum_{y \in U(\succ,x)} p(y) \geq \sum_{y \in U(\succ,x)} q(y)$ for all $x \in A$.
- Let u denote a utility function consistent with $\succ$. Define $\Delta_m = u(x_m)$ and $\Delta_i = u(x_i) - u(x_{i+1})$ for $i \in \{1, \ldots, m-1\}$.
- It holds that $u(x_i) = \sum_{j=i}^{m} \Delta_j$ and that $x_i \in U(\succ, x_j)$ for all $j \in \{1, \ldots, m\}$

Exercise II: Strategyproofness for RSCFs

- Hence, we have that $\sum_{i=1}^m p(x_i)u(x_i) = \sum_{i=1}^m p(x_i) \sum_{j=i}^m \Delta_j = \sum_{j=1}^m \Delta_j \sum_{x_i \in U(\succ, x_j)} p(x_i)$

Exercise II: Strategyproofness for RSCFs

- Hence, we have that $\sum_{i=1}^{m} p(x_i)u(x_i) = \sum_{i=1}^{m} p(x_i) \sum_{j=i}^{m} \Delta_j = \sum_{j=1}^{m} \Delta_j \sum_{x_i \in U(\succ, x_j)} p(x_i)$
- By a symmetric argument for q, we conclude that

$$\begin{aligned}
\sum_{i=1}^{m} p(x_i)u(x_i) &= \sum_{j=1}^{m} \Delta_j \sum_{x_i \in U(\succ, x_j)} p(x_i) \\
&\geq \sum_{j=1}^{m} \Delta_j \sum_{x_i \in U(\succ, x_j)} q(x_i) \\
&= \sum_{i=1}^{m} q(x_i)u(x_i)
\end{aligned}$$

Exercise II: Strategyproofness for RSCFs

- Hence, we have that $\sum_{i=1}^{m} p(x_i)u(x_i) = \sum_{i=1}^{m} p(x_i) \sum_{j=i}^{m} \Delta_j = \sum_{j=1}^{m} \Delta_j \sum_{x_i \in U(\succ, x_j)} p(x_i)$
- By a symmetric argument for q, we conclude that

$$\begin{aligned}
\sum_{i=1}^{m} p(x_i)u(x_i) &= \sum_{j=1}^{m} \Delta_j \sum_{x_i \in U(\succ, x_j)} p(x_i) \\
&\geq \sum_{j=1}^{m} \Delta_j \sum_{x_i \in U(\succ, x_j)} q(x_i) \\
&= \sum_{i=1}^{m} q(x_i)u(x_i)
\end{aligned}$$

- We hence conclude that $\mathbb{E}[p(u)] \geq \mathbb{E}[q(u)]$ for all u.

Exercise II: Strategyproofness for RSCFs

- Next, assume that there is an alternative x_i such that $\sum_{x_j \in U(\succ, x_i)} p(x_j) < \sum_{x_j \in U(\succ, x_i)} q(x_j)$.

Exercise II: Strategyproofness for RSCFs

- Next, assume that there is an alternative x_i such that $\sum_{x_j \in U(\succ, x_i)} p(x_j) < \sum_{x_j \in U(\succ, x_i)} q(x_j)$.
- Let $\delta = \sum_{x_j \in U(\succ, x_i)} q(x_j) - \sum_{x_j \in U(\succ, x_i)} p(x_j)$

Exercise II: Strategyproofness for RSCFs

- Next, assume that there is an alternative x_i such that $\sum_{x_j \in U(\succ, x_i)} p(x_j) < \sum_{x_j \in U(\succ, x_i)} q(x_j)$.
- Let $\delta = \sum_{x_j \in U(\succ, x_i)} q(x_j) - \sum_{x_j \in U(\succ, x_i)} p(x_j)$
- Choose $\epsilon > 0$ such that $m^2 \epsilon < \delta$.

Exercise II: Strategyproofness for RSCFs

- Next, assume that there is an alternative x_i such that $\sum_{x_j \in U(\succ, x_i)} p(x_j) < \sum_{x_j \in U(\succ, x_i)} q(x_j)$.
- Let $\delta = \sum_{x_j \in U(\succ, x_i)} q(x_j) - \sum_{x_j \in U(\succ, x_i)} p(x_j)$
- Choose $\epsilon > 0$ such that $m^2\epsilon < \delta$.
- Let u denote the utility function given by $u(x_j) = 1 + (m-j)\epsilon$ if $x_j \in U(\succ, x_i)$ and $u(x_j) = (m-j)\epsilon$ if $x_j \notin U(\succ, x_i)$. Note that u is consistent with $\succ$.

Exercise II: Strategyproofness for RSCFs

- Next, assume that there is an alternative x_i such that $\sum_{x_j \in U(\succ, x_i)} p(x_j) < \sum_{x_j \in U(\succ, x_i)} q(x_j)$.
- Let $\delta = \sum_{x_j \in U(\succ, x_i)} q(x_j) - \sum_{x_j \in U(\succ, x_i)} p(x_j)$
- Choose $\epsilon > 0$ such that $m^2\epsilon < \delta$.
- Let u denote the utility function given by $u(x_j) = 1 + (m-j)\epsilon$ if $x_j \in U(\succ, x_i)$ and $u(x_j) = (m-j)\epsilon$ if $x_j \notin U(\succ, x_i)$. Note that u is consistent with $\succ$.
- It holds that

$$\mathbb{E}[u(p)] = \sum_{x_j \in A} u(x_j)p(x_j) < m^2\epsilon + \sum_{x_j \in U(x_i)} u(x_j)p(x_j)$$
$$< \sum_{x_j \in U(x_i)} u(x_j)q(x_j) < \mathbb{E}[q(u)].$$

Exercise III: Computing ABC Voting Rules

Compute *AV*, *PAV*, *CCAV*, *Phragmen*, and *MES* for the subsequent profile and the target committee size $k = 3$.

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

AV:

- a is approved by 8 voters.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

AV:

- a is approved by 8 voters.
- b is approved by 5 voters.

Exercise III: Computing ABC Voting Rules

$3\colon \{a, b\} \quad 3\colon \{a, c\} \quad 2 : \{a, b, d\} \quad 2\colon \{e\} \quad 1\colon \{f\}$

AV:

- a is approved by 8 voters.
- b is approved by 5 voters.
- c is approved by 3 voters.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

AV:

- a is approved by 8 voters.
- b is approved by 5 voters.
- c is approved by 3 voters.
- d is approved by 2 voters.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

AV:

- a is approved by 8 voters.
- b is approved by 5 voters.
- c is approved by 3 voters.
- d is approved by 2 voters.
- e is approved by 2 voters.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

AV:

- a is approved by 8 voters.
- b is approved by 5 voters.
- c is approved by 3 voters.
- d is approved by 2 voters.
- e is approved by 2 voters.
- f is approved by 1 voters.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

AV:

- a is approved by 8 voters.
- b is approved by 5 voters.
- c is approved by 3 voters.
- d is approved by 2 voters.
- e is approved by 2 voters.
- f is approved by 1 voters.
- $AV(\mathcal{A}, 3) = \{a, b, c\}$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:
  - $\{a, b, c\}$ has a *CCAV* score of 8.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:
  - $\{a, b, c\}$ has a *CCAV* score of 8.
  - $\{a, b, d\}$ has a *CCAV* score of 8.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:
  - $\{a, b, c\}$ has a *CCAV* score of 8.
  - $\{a, b, d\}$ has a *CCAV* score of 8.
  - ...

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:
  - $\{a, b, c\}$ has a *CCAV* score of 8.
  - $\{a, b, d\}$ has a *CCAV* score of 8.
  - ...
- Better approach: Greedy optimize and think how to improve

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:
  - $\{a, b, c\}$ has a *CCAV* score of 8.
  - $\{a, b, d\}$ has a *CCAV* score of 8.
  - ...
- Better approach: Greedy optimize and think how to improve
  - If we choose a, the 8 voters on the left are satisfied.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:
  - $\{a, b, c\}$ has a *CCAV* score of 8.
  - $\{a, b, d\}$ has a *CCAV* score of 8.
  - ...
- Better approach: Greedy optimize and think how to improve
  - If we choose a, the 8 voters on the left are satisfied.
  - If we choose e and f, the 3 voters on the right are satisfied.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:
  - $\{a, b, c\}$ has a *CCAV* score of 8.
  - $\{a, b, d\}$ has a *CCAV* score of 8.
  - ...
- Better approach: Greedy optimize and think how to improve
  - If we choose a, the 8 voters on the left are satisfied.
  - If we choose e and f, the 3 voters on the right are satisfied.
  - The *CCAV* score of $\{a, e, f\}$ is 11 (which is maximal).

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

CCAV:

- In principle: check the *CCAV* score of every committee:
  - $\{a, b, c\}$ has a *CCAV* score of 8.
  - $\{a, b, d\}$ has a *CCAV* score of 8.
  - ...
- Better approach: Greedy optimize and think how to improve
  - If we choose a, the 8 voters on the left are satisfied.
  - If we choose e and f, the 3 voters on the right are satisfied.
  - The *CCAV* score of $\{a, e, f\}$ is 11 (which is maximal).
- $CCAV(\mathcal{A}, 3) = \{a, e, f\}$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:
  - $\{a, b, c\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot (1 + \frac{1}{2}) + 2 \cdot (1 + \frac{1}{2}) = 11$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:
  - $\{a, b, c\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot (1 + \frac{1}{2}) + 2 \cdot (1 + \frac{1}{2}) = 11.$
  - $\{a, b, d\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot 1 + 2 \cdot (1 + \frac{1}{2} + \frac{1}{3}) = 11 + \frac{1}{6}.$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:
  - $\{a, b, c\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot (1 + \frac{1}{2}) + 2 \cdot (1 + \frac{1}{2}) = 11$.
  - $\{a, b, d\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot 1 + 2 \cdot (1 + \frac{1}{2} + \frac{1}{3}) = 11 + \frac{1}{6}$.
  - ...

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:
  - $\{a, b, c\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot (1 + \frac{1}{2}) + 2 \cdot (1 + \frac{1}{2}) = 11$.
  - $\{a, b, d\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot 1 + 2 \cdot (1 + \frac{1}{2} + \frac{1}{3}) = 11 + \frac{1}{6}$.
  - $\ldots$
- Better approach: Greedy optimize and think how to improve

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:
  - $\{a, b, c\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot (1 + \frac{1}{2}) + 2 \cdot (1 + \frac{1}{2}) = 11$.
  - $\{a, b, d\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot 1 + 2 \cdot (1 + \frac{1}{2} + \frac{1}{3}) = 11 + \frac{1}{6}$.
  - ...
- Better approach: Greedy optimize and think how to improve
  - If we choose a, the 8 voters on the left return 1 point.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:
  - $\{a, b, c\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot (1 + \frac{1}{2}) + 2 \cdot (1 + \frac{1}{2}) = 11$.
  - $\{a, b, d\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot 1 + 2 \cdot (1 + \frac{1}{2} + \frac{1}{3}) = 11 + \frac{1}{6}$.
  - ...
- Better approach: Greedy optimize and think how to improve
  - If we choose a, the 8 voters on the left return 1 point.
  - If we choose b, we increase the score of 5 voters by $\frac{1}{2}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:
  - $\{a, b, c\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot (1 + \frac{1}{2}) + 2 \cdot (1 + \frac{1}{2}) = 11$.
  - $\{a, b, d\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot 1 + 2 \cdot (1 + \frac{1}{2} + \frac{1}{3}) = 11 + \frac{1}{6}$.
  - ...
- Better approach: Greedy optimize and think how to improve
  - If we choose a, the 8 voters on the left return 1 point.
  - If we choose b, we increase the score of 5 voters by $\frac{1}{2}$.
  - If we choose e, we increase the score of 2 voters by 1.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

PAV:

- In principle: check the *PAV* score of every committee:
  - $\{a, b, c\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot (1 + \frac{1}{2}) + 2 \cdot (1 + \frac{1}{2}) = 11$.
  - $\{a, b, d\}$: $3 \cdot (1 + \frac{1}{2}) + 3 \cdot 1 + 2 \cdot (1 + \frac{1}{2} + \frac{1}{3}) = 11 + \frac{1}{6}$.
  - ...
- Better approach: Greedy optimize and think how to improve
  - If we choose a, the 8 voters on the left return 1 point.
  - If we choose b, we increase the score of 5 voters by $\frac{1}{2}$.
  - If we choose e, we increase the score of 2 voters by 1.
  - $\{a, b, e\}$ has a *PAV* score of 12.5
- $PAV(\mathcal{A}, 3) = \{a, b, e\}$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
- Compute time when next alternative can be afforded.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
- Compute time when next alternative can be afforded.
  - a is approved by 8 voters $\rightarrow$ affordable at $t = \frac{1}{8}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
- Compute time when next alternative can be afforded.
  - a is approved by 8 voters $\rightarrow$ affordable at $t = \frac{1}{8}$.
  - b is approved by 5 voters $\rightarrow$ affordable at $t = \frac{1}{5}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
- Compute time when next alternative can be afforded.
    - a is approved by 8 voters $\rightarrow$ affordable at $t = \frac{1}{8}$.
    - b is approved by 5 voters $\rightarrow$ affordable at $t = \frac{1}{5}$.
    - c is approved by 3 voters $\rightarrow$ affordable at $t = \frac{1}{3}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
- Compute time when next alternative can be afforded.
  - a is approved by 8 voters $\rightarrow$ affordable at $t = \frac{1}{8}$.
  - b is approved by 5 voters $\rightarrow$ affordable at $t = \frac{1}{5}$.
  - c is approved by 3 voters $\rightarrow$ affordable at $t = \frac{1}{3}$.
  - d is approved by 2 voters $\rightarrow$ affordable at $t = \frac{1}{2}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
- Compute time when next alternative can be afforded.
  - a is approved by 8 voters $\rightarrow$ affordable at $t = \frac{1}{8}$.
  - b is approved by 5 voters $\rightarrow$ affordable at $t = \frac{1}{5}$.
  - c is approved by 3 voters $\rightarrow$ affordable at $t = \frac{1}{3}$.
  - d is approved by 2 voters $\rightarrow$ affordable at $t = \frac{1}{2}$.
  - e is approved by 2 voters $\rightarrow$ affordable at $t = \frac{1}{2}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
- Compute time when next alternative can be afforded.
  - a is approved by 8 voters $\rightarrow$ affordable at $t = \frac{1}{8}$.
  - b is approved by 5 voters $\rightarrow$ affordable at $t = \frac{1}{5}$.
  - c is approved by 3 voters $\rightarrow$ affordable at $t = \frac{1}{3}$.
  - d is approved by 2 voters $\rightarrow$ affordable at $t = \frac{1}{2}$.
  - e is approved by 2 voters $\rightarrow$ affordable at $t = \frac{1}{2}$.
  - f is approved by 1 voters $\rightarrow$ affordable at $t = 1$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- Initial budget vector: $b(0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.
- Compute time when next alternative can be afforded.
  - a is approved by 8 voters $\rightarrow$ affordable at $t = \frac{1}{8}$.
  - b is approved by 5 voters $\rightarrow$ affordable at $t = \frac{1}{5}$.
  - c is approved by 3 voters $\rightarrow$ affordable at $t = \frac{1}{3}$.
  - d is approved by 2 voters $\rightarrow$ affordable at $t = \frac{1}{2}$.
  - e is approved by 2 voters $\rightarrow$ affordable at $t = \frac{1}{2}$.
  - f is approved by 1 voters $\rightarrow$ affordable at $t = 1$.
- We will first add a at $t = \frac{1}{8}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying a, the budget vector is : $b(\frac{1}{8}) = (0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying a, the budget vector is :
  $b(\frac{1}{8}) = (0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.
- Compute when the next alternative can be afforded.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying a, the budget vector is : $b(\frac{1}{8}) = (0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.
- Compute when the next alternative can be afforded.
  - b is approved by 5 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{5}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying a, the budget vector is :
  $b(\frac{1}{8}) = (0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.
- Compute when the next alternative can be afforded.
  - b is approved by 5 voters with no budget
    $\rightarrow$ affordable at $t + \frac{1}{5}$.
  - c is approved by 3 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{3}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying a, the budget vector is : $b(\frac{1}{8}) = (0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.
- Compute when the next alternative can be afforded.
  - b is approved by 5 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{5}$.
  - c is approved by 3 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{3}$.
  - d is approved by 2 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{2}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying a, the budget vector is : $b(\frac{1}{8}) = (0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.
- Compute when the next alternative can be afforded.
  - b is approved by 5 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{5}$.
  - c is approved by 3 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{3}$.
  - d is approved by 2 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{2}$.
  - e is approved by 2 voters with total budget of $\frac{2}{8}$ $\rightarrow$ affordable at $t + \frac{3}{8}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying a, the budget vector is : $b(\frac{1}{8}) = (0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.
- Compute when the next alternative can be afforded.
    - b is approved by 5 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{5}$.
    - c is approved by 3 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{3}$.
    - d is approved by 2 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{2}$.
    - e is approved by 2 voters with total budget of $\frac{2}{8}$ $\rightarrow$ affordable at $t + \frac{3}{8}$.
    - f is approved by 1 voters with total budget of $\frac{1}{8}$ $\rightarrow$ affordable at $t + \frac{7}{8}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying a, the budget vector is :
  $b(\frac{1}{8}) = (0, 0, 0, 0, 0, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.
- Compute when the next alternative can be afforded.
    - b is approved by 5 voters with no budget
      $\rightarrow$ affordable at $t + \frac{1}{5}$.
    - c is approved by 3 voters with no budget $\rightarrow$ affordable at $t + \frac{1}{3}$.
    - d is approved by 2 voters with no budget
      $\rightarrow$ affordable at $t + \frac{1}{2}$.
    - e is approved by 2 voters with total budget of $\frac{2}{8}$
      $\rightarrow$ affordable at $t + \frac{3}{8}$.
    - f is approved by 1 voters with total budget of $\frac{1}{8}$
      $\rightarrow$ affordable at $t + \frac{7}{8}$.
- We add b at $t = \frac{1}{8} + \frac{1}{5} = \frac{13}{40}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying b, the budget vector is : $b(\frac{13}{40}) = (0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{13}{40}, \frac{13}{40}, \frac{13}{40})$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying b, the budget vector is :
  $b(\frac{13}{40}) = (0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{13}{40}, \frac{13}{40}, \frac{13}{40})$.
- Compute when the next alternative can be afforded.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying b, the budget vector is :
  $b(\frac{13}{40}) = (0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{13}{40}, \frac{13}{40}, \frac{13}{40})$.
- Compute when the next alternative can be afforded.
  - c is approved by 3 voters with total budget of $\frac{3}{5}$
    $\rightarrow$ will be affordable at $t + \frac{2}{5} \cdot \frac{1}{3} = t + \frac{2}{15}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying b, the budget vector is : $b(\frac{13}{40}) = (0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{13}{40}, \frac{13}{40}, \frac{13}{40})$.
- Compute when the next alternative can be afforded.
  - c is approved by 3 voters with total budget of $\frac{3}{5}$ $\rightarrow$ will be affordable at $t + \frac{2}{5} \cdot \frac{1}{3} = t + \frac{2}{15}$.
  - d is approved by 2 voters with no budget $\rightarrow$ will be affordable at $t + \frac{1}{2}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying b, the budget vector is : $b(\frac{13}{40}) = (0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{13}{40}, \frac{13}{40}, \frac{13}{40})$.
- Compute when the next alternative can be afforded.
  - c is approved by 3 voters with total budget of $\frac{3}{5}$ $\rightarrow$ will be affordable at $t + \frac{2}{5} \cdot \frac{1}{3} = t + \frac{2}{15}$.
  - d is approved by 2 voters with no budget $\rightarrow$ will be affordable at $t + \frac{1}{2}$.
  - e is approved by 2 voters with total budget of $\frac{26}{40}$ $\rightarrow$ will be affordable at $t + \frac{14}{40} \cdot \frac{1}{2} = t + \frac{7}{40}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying b, the budget vector is :
  $b(\frac{13}{40}) = (0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{13}{40}, \frac{13}{40}, \frac{13}{40})$.
- Compute when the next alternative can be afforded.
  - c is approved by 3 voters with total budget of $\frac{3}{5}$
    $\rightarrow$ will be affordable at $t + \frac{2}{5} \cdot \frac{1}{3} = t + \frac{2}{15}$.
  - d is approved by 2 voters with no budget
    $\rightarrow$ will be affordable at $t + \frac{1}{2}$.
  - e is approved by 2 voters with total budget of $\frac{26}{40}$
    $\rightarrow$ will be affordable at $t + \frac{14}{40} \cdot \frac{1}{2} = t + \frac{7}{40}$.
  - f is approved by 1 voters with total budget of $\frac{13}{40}$
    $\rightarrow$ will be affordable at $t + \frac{27}{40}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen:

- After buying b, the budget vector is : $b(\frac{13}{40}) = (0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{13}{40}, \frac{13}{40}, \frac{13}{40})$.
- Compute when the next alternative can be afforded.
  - c is approved by 3 voters with total budget of $\frac{3}{5}$ $\rightarrow$ will be affordable at $t + \frac{2}{5} \cdot \frac{1}{3} = t + \frac{2}{15}$.
  - d is approved by 2 voters with no budget $\rightarrow$ will be affordable at $t + \frac{1}{2}$.
  - e is approved by 2 voters with total budget of $\frac{26}{40}$ $\rightarrow$ will be affordable at $t + \frac{14}{40} \cdot \frac{1}{2} = t + \frac{7}{40}$.
  - f is approved by 1 voters with total budget of $\frac{13}{40}$ $\rightarrow$ will be affordable at $t + \frac{27}{40}$.
- We will add c at $t = \frac{13}{40} + \frac{2}{15} = \frac{55}{120}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

Phragmen: $Phragmen(\mathcal{A}, 3) = \{a, b, c\}$

- After buying b, the budget vector is :
  $b(\frac{13}{40}) = (0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0, \frac{13}{40}, \frac{13}{40}, \frac{13}{40})$.
- Compute when the next alternative can be afforded.
  - c is approved by 3 voters with total budget of $\frac{3}{5}$
    $\rightarrow$ will be affordable at $t + \frac{2}{5} \cdot \frac{1}{3} = t + \frac{2}{15}$.
  - d is approved by 2 voters with no budget
    $\rightarrow$ will be affordable at $t + \frac{1}{2}$.
  - e is approved by 2 voters with total budget of $\frac{26}{40}$
    $\rightarrow$ will be affordable at $t + \frac{14}{40} \cdot \frac{1}{2} = t + \frac{7}{40}$.
  - f is approved by 1 voters with total budget of $\frac{13}{40}$
    $\rightarrow$ will be affordable at $t + \frac{27}{40}$.
- We will add c at $t = \frac{13}{40} + \frac{2}{15} = \frac{55}{120}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- Initially the budget vector is
  $b = (\frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- Initially the budget vector is $b = (\frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- Compute ρ for all affordable candidates.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- Initially the budget vector is $b = (\frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- Compute ρ for all affordable candidates.
    - c, d, e, and f are not affordable as their supporters do not have enough money.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- Initially the budget vector is $b = (\frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- Compute ρ for all affordable candidates.
    - c, d, e, and f are not affordable as their supporters do not have enough money.
    - a is affordable at $\rho = \frac{1}{8}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- Initially the budget vector is
  $b = (\frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- Compute ρ for all affordable candidates.
  - c, d, e, and f are not affordable as their supporters do not have enough money.
  - a is affordable at $\rho = \frac{1}{8}$.
  - b is affordable at $\rho = \frac{1}{5}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- Initially the budget vector is $b = (\frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- Compute ρ for all affordable candidates.
    - c, d, e, and f are not affordable as their supporters do not have enough money.
    - a is affordable at $\rho = \frac{1}{8}$.
    - b is affordable at $\rho = \frac{1}{5}$.
- We buy a for $\rho = \frac{1}{8}$.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- After we buy a, the budget vector is $b = (\frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- After we buy a, the budget vector is $b = (\frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- No candidate is affordable anymore!

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- After we buy a, the budget vector is $b = (\frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- No candidate is affordable anymore!
- We start running *Phragmen* with the remaining budgets.

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- After we buy a, the budget vector is $b = (\frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- No candidate is affordable anymore!
- We start running *Phragmen* with the remaining budgets.
    - b will be bought at $t = (1 - \frac{65}{88}) \cdot \frac{1}{5} = \frac{23}{440}$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ 2 : $\{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- After we buy a, the budget vector is $b = (\frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- No candidate is affordable anymore!
- We start running *Phragmen* with the remaining budgets.
    - b will be bought at $t = (1 - \frac{65}{88}) \cdot \frac{1}{5} = \frac{23}{440}$
    - c will be bought at $t = (1 - \frac{39}{88}) \cdot \frac{1}{3} = \frac{49}{272}$

Exercise III: Computing ABC Voting Rules

3: $\{a, b\}$ 3: $\{a, c\}$ $2 : \{a, b, d\}$ 2: $\{e\}$ 1: $\{f\}$

MES:

- After we buy a, the budget vector is $b = (\frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{13}{88}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11})$
- No candidate is affordable anymore!
- We start running *Phragmen* with the remaining budgets.
  - b will be bought at $t = (1 - \frac{65}{88}) \cdot \frac{1}{5} = \frac{23}{440}$
  - c will be bought at $t = (1 - \frac{39}{88}) \cdot \frac{1}{3} = \frac{49}{272}$
- $MES(\mathcal{A}, 3) = \{a, b, c\}$.

Exercise IV: Extended Justified Representation

a) Show that *PAV* satisfies EJR.

Exercise IV: Extended Justified Representation

a) Show that *PAV* satisfies EJR.

- Assume for contradiction that there is a profile $\boldsymbol{\mathcal{A}}$ and a target committee size k such that the committee W chosen by *PAV* fails EJR.

Exercise IV: Extended Justified Representation

a) Show that *PAV* satisfies EJR.

- Assume for contradiction that there is a profile $\mathcal{A}$ and a target committee size k such that the committee W chosen by *PAV* fails EJR.
- There is a set of voters S and an integer ℓ such that $|S| \geq \frac{\ell|N|}{k}$, $|\bigcap_{i \in S} A_i| \geq \ell$, and $|W \cap A_i| < \ell$ for all $i \in S$.

Exercise IV: Extended Justified Representation

a) Show that *PAV* satisfies EJR.

- Assume for contradiction that there is a profile $\boldsymbol{\mathcal{A}}$ and a target committee size k such that the committee W chosen by *PAV* fails EJR.
- There is a set of voters S and an integer ℓ such that $|S| \geq \frac{\ell|N|}{k}$, $|\bigcap_{i \in S} A_i| \geq \ell$, and $|W \cap A_i| < \ell$ for all $i \in S$.
- There is at least one alternative $c \in \bigcap_{i \in S} A_i$ with $c \notin W$.

Exercise IV: Extended Justified Representation

a) Show that *PAV* satisfies EJR.

- Assume for contradiction that there is a profile $\boldsymbol{\mathcal{A}}$ and a target committee size k such that the committee W chosen by *PAV* fails EJR.
- There is a set of voters S and an integer ℓ such that $|S| \geq \frac{\ell|N|}{k}$, $|\bigcap_{i \in S} A_i| \geq \ell$, and $|W \cap A_i| < \ell$ for all $i \in S$.
- There is at least one alternative $c \in \bigcap_{i \in S} A_i$ with $c \notin W$.
- Let $s(X)$ denote the *PAV* score of a committee X

Exercise IV: Extended Justified Representation

a) Show that *PAV* satisfies EJR.

- Assume for contradiction that there is a profile $\boldsymbol{\mathcal{A}}$ and a target committee size k such that the committee W chosen by *PAV* fails EJR.
- There is a set of voters S and an integer ℓ such that $|S| \geq \frac{\ell|N|}{k}$, $|\bigcap_{i \in S} A_i| \geq \ell$, and $|W \cap A_i| < \ell$ for all $i \in S$.
- There is at least one alternative $c \in \bigcap_{i \in S} A_i$ with $c \notin W$.
- Let $s(X)$ denote the *PAV* score of a committee X
- We will show that there is an alternative $d \in W$ such that $s((W \setminus \{d\}) \cup \{c\}) > s(W)$.

Exercise IV: Extended Justified Representation

a) Show that *PAV* satisfies EJR.

- Assume for contradiction that there is a profile $\boldsymbol{\mathcal{A}}$ and a target committee size k such that the committee W chosen by *PAV* fails EJR.
- There is a set of voters S and an integer ℓ such that $|S| \geq \frac{\ell|N|}{k}$, $|\bigcap_{i \in S} A_i| \geq \ell$, and $|W \cap A_i| < \ell$ for all $i \in S$.
- There is at least one alternative $c \in \bigcap_{i \in S} A_i$ with $c \notin W$.
- Let $s(X)$ denote the *PAV* score of a committee X
- We will show that there is an alternative $d \in W$ such that $s((W \setminus \{d\}) \cup \{c\}) > s(W)$.
- For this, we define $\Delta(X, Y) = s(X) - s(Y)$.

Exercise IV: Extended Justified Representation

- It holds for c and every $d \in W$ that

$$\Delta((W \setminus \{d\}) \cup \{c\}, W \setminus \{d\})$$
$$= \sum_{i \in N} \sum_{y=1}^{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \frac{1}{y} - \sum_{i \in N} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y}$$

Exercise IV: Extended Justified Representation

- It holds for c and every $d \in W$ that

$$\Delta((W \setminus \{d\}) \cup \{c\}, W \setminus \{d\})$$

$$= \sum_{i \in N} \sum_{y=1}^{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \frac{1}{y} - \sum_{i \in N} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y}$$

$$= \sum_{i \in N : c \in A_i} \frac{1}{|A_i \cap (W \setminus \{d\} \cup \{c\})|}$$

Exercise IV: Extended Justified Representation

- It holds for c and every $d \in W$ that

$$
\begin{aligned}
&\Delta((W \setminus \{d\}) \cup \{c\}, W \setminus \{d\}) \\
&= \sum_{i \in N} \sum_{y=1}^{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \frac{1}{y} - \sum_{i \in N} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y} \\
&= \sum_{i \in N : c \in A_i} \frac{1}{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \\
&\geq \sum_{i \in S} \frac{1}{|A_i \cap (W \setminus \{d\} \cup \{c\})|}
\end{aligned}
$$

Exercise IV: Extended Justified Representation

- It holds for c and every $d \in W$ that

$$
\begin{aligned}
&\Delta((W \setminus \{d\}) \cup \{c\}, W \setminus \{d\}) \\
&= \sum_{i \in N} \sum_{y=1}^{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \frac{1}{y} - \sum_{i \in N} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y} \\
&= \sum_{i \in N : c \in A_i} \frac{1}{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \\
&\geq \sum_{i \in S} \frac{1}{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \\
&\geq \sum_{i \in S} \frac{1}{\ell}
\end{aligned}
$$

Exercise IV: Extended Justified Representation

- It holds for c and every $d \in W$ that

$$
\begin{aligned}
&\Delta((W \setminus \{d\}) \cup \{c\}, W \setminus \{d\}) \\
&= \sum_{i \in N} \sum_{y=1}^{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \frac{1}{y} - \sum_{i \in N} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y} \\
&= \sum_{i \in N : c \in A_i} \frac{1}{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \\
&\geq \sum_{i \in S} \frac{1}{|A_i \cap (W \setminus \{d\} \cup \{c\})|} \\
&\geq \sum_{i \in S} \frac{1}{\ell} \\
&\geq \frac{|N|}{k}
\end{aligned}
$$

Exercise IV: Extended Justified Representation

- It holds for every $d \in W$ that

$$\Delta(W, W \setminus \{d\}) = \sum_{i \in N} \sum_{y=1}^{|A_i \cap W|} \frac{1}{y} - \sum_{i \in N} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y}$$

Exercise IV: Extended Justified Representation

- It holds for every $d \in W$ that

$$\Delta(W, W \setminus \{d\}) = \sum_{i \in N} \sum_{y=1}^{|A_i \cap W|} \frac{1}{y} - \sum_{i \in N} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y}$$

$$= \sum_{i \in N : d \in A_i} \sum_{i=1}^{|A_i \cap W|} \frac{1}{y} - \sum_{i \in N : d \in A_i} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y}$$

Exercise IV: Extended Justified Representation

- It holds for every $d \in W$ that

$$\begin{aligned}
\Delta(W, W \setminus \{d\}) &= \sum_{i \in N} \sum_{y=1}^{|A_i \cap W|} \frac{1}{y} - \sum_{i \in N} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y} \\
&= \sum_{i \in N: d \in A_i} \sum_{i=1}^{|A_i \cap W|} \frac{1}{y} - \sum_{i \in N: d \in A_i} \sum_{i=1}^{|A_i \cap (W \setminus \{d\})|} \frac{1}{y} \\
&= \sum_{i \in N: d \in A_i} \frac{1}{|A_i \cap W|}
\end{aligned}$$

Exercise IV: Extended Justified Representation

- It holds that

$$\sum_{d \in W} \Delta(W, W \setminus \{d\}) = \sum_{d \in W} \sum_{i \in N : d \in A_i} \frac{1}{|A_i \cap W|}$$

Exercise IV: Extended Justified Representation

- It holds that

$$\sum_{d\in W} \Delta(W, W\setminus\{d\}) = \sum_{d\in W} \sum_{i\in N: d\in A_i} \frac{1}{|A_i\cap W|}$$
$$= \sum_{i\in N:\ A_i\cap W\neq\emptyset} |A_i\cap W|\cdot\frac{1}{|A_i\cap W|}$$

- There is an alternative $d\in W$ such that

$$\Delta(W, W\setminus\{d\}) \leq \frac{|\{i\in N\colon A_i\cap W\neq\emptyset\}|}{k} \leq \frac{|N|}{k}.$$

Exercise IV: Extended Justified Representation

- It holds that

$$\begin{aligned}\sum_{d\in W} \Delta(W, W\setminus\{d\}) &= \sum_{d\in W}\sum_{i\in N: d\in A_i} \frac{1}{|A_i\cap W|}\\ &= \sum_{i\in N:\ A_i\cap W\neq\emptyset} |A_i\cap W|\cdot\frac{1}{|A_i\cap W|}\\ &= |\{i\in N\colon A_i\cap W\neq\emptyset\}|.\end{aligned}$$

- There is an alternative $d \in W$ such that

$$\Delta(W, W\setminus\{d\}) \leq \frac{|\{i\in N\colon A_i\cap W\neq\emptyset\}|}{k} \leq \frac{|N|}{k}.$$

Exercise IV: Extended Justified Representation

- If there is $d \in W$ with $\Delta(W, W \setminus \{d\}) < \frac{|N|}{k}$, then

$$
\begin{aligned}
&\Delta(W \setminus \{d\} \cup \{c\}, W) \\
&= \Delta(W \setminus \{d\} \cup \{c\}, W \setminus \{d\}) - \Delta(W, W \setminus \{d\}) \\
&> \frac{|N|}{k} - \frac{|N|}{k} \\
&= 0.
\end{aligned}
$$

Exercise IV: Extended Justified Representation

- If there is $d \in W$ with $\Delta(W, W \setminus \{d\}) < \frac{|N|}{k}$, then

$$\begin{aligned}
&\Delta(W \setminus \{d\} \cup \{c\}, W) \\
&= \Delta(W \setminus \{d\} \cup \{c\}, W \setminus \{d\}) - \Delta(W, W \setminus \{d\}) \\
&> \frac{|N|}{k} - \frac{|N|}{k} \\
&= 0.
\end{aligned}$$

- This proves that $W \setminus \{d\} \cup \{c\}$ has a higher *PAV* score than W.

Exercise IV: Extended Justified Representation

- Otherwise, we have that $|\{i \in N \colon A_i \cap W \neq \emptyset\}| = |N|$ and $\Delta(W, W \setminus \{d\}) = \frac{|N|}{k}$ for all $d \in W$.

Exercise IV: Extended Justified Representation

- Otherwise, we have that $|\{i \in N\colon A_i \cap W \neq \emptyset\}| = |N|$ and $\Delta(W, W \setminus \{d\}) = \frac{|N|}{k}$ for all $d \in W$.
- Choose d such that $d \in A_i$ for some $i \in S$. It holds that $|W \setminus \{d\} \cup \{c\}| \leq \ell - 1$.

Exercise IV: Extended Justified Representation

- Otherwise, we have that $|\{i \in N \colon A_i \cap W \neq \emptyset\}| = |N|$ and $\Delta(W, W \setminus \{d\}) = \frac{|N|}{k}$ for all $d \in W$.
- Choose d such that $d \in A_i$ for some $i \in S$. It holds that $|W \setminus \{d\} \cup \{c\}| \leq \ell - 1$.
- Using this in our previous analysis shows that $\Delta(W \setminus \{d\} \cup \{c\}, W \setminus \{d\}) > \frac{|N|}{k}$.

Exercise IV: Extended Justified Representation

- Otherwise, we have that $|\{i \in N\colon A_i \cap W \neq \emptyset\}| = |N|$ and $\Delta(W, W \setminus \{d\}) = \frac{|N|}{k}$ for all $d \in W$.
- Choose d such that $d \in A_i$ for some $i \in S$. It holds that $|W \setminus \{d\} \cup \{c\}| \leq \ell - 1$.
- Using this in our previous analysis shows that $\Delta(W \setminus \{d\} \cup \{c\}, W \setminus \{d\}) > \frac{|N|}{k}$.
- We can now derive again that $\Delta(W \setminus \{d\} \cup \{c\}, W) > 0$.

Exercise IV: Extended Justified Representation

- Otherwise, we have that $|\{i \in N \colon A_i \cap W \neq \emptyset\}| = |N|$ and $\Delta(W, W \setminus \{d\}) = \frac{|N|}{k}$ for all $d \in W$.
- Choose d such that $d \in A_i$ for some $i \in S$. It holds that $|W \setminus \{d\} \cup \{c\}| \leq \ell - 1$.
- Using this in our previous analysis shows that $\Delta(W \setminus \{d\} \cup \{c\}, W \setminus \{d\}) > \frac{|N|}{k}$.
- We can now derive again that $\Delta(W \setminus \{d\} \cup \{c\}, W) > 0$.
- Hence our initial assumption is wrong.

Exercise IV: Extended Justified Representation

b) Show that *MES* satisfies EJR.

- We will only focus on the first phase of *MES* (i.e., not take the completion into account).

Exercise IV: Extended Justified Representation

b) Show that *MES* satisfies EJR.

- We will only focus on the first phase of *MES* (i.e., not take the completion into account).
- Assume for contradiction that there is a profile $\boldsymbol{\mathcal{A}}$ and a target committee size k such that the committee W chosen by (the first phase of) *MES* fails EJR.

Exercise IV: Extended Justified Representation

b) Show that *MES* satisfies EJR.

- We will only focus on the first phase of *MES* (i.e., not take the completion into account).
- Assume for contradiction that there is a profile $\mathcal{A}$ and a target committee size k such that the committee W chosen by (the first phase of) *MES* fails EJR.
- There is a set of voters S and an integer ℓ such that $|S| \geq \frac{\ell|N|}{k}$, $|\bigcap_{i \in S} A_i| \geq \ell$, and $|W \cap A_i| < \ell$ for all $\ell \in S$.

Exercise IV: Extended Justified Representation

b) Show that *MES* satisfies EJR.

- We will only focus on the first phase of *MES* (i.e., not take the completion into account).
- Assume for contradiction that there is a profile $\boldsymbol{\mathcal{A}}$ and a target committee size k such that the committee W chosen by (the first phase of) *MES* fails EJR.
- There is a set of voters S and an integer ℓ such that $|S| \geq \frac{\ell|N|}{k}$, $|\bigcap_{i \in S} A_i| \geq \ell$, and $|W \cap A_i| < \ell$ for all $\ell \in S$.
- When *MES* stops, there must be at least one voter $i \in S$ with a budget $b_i < \frac{1}{|S|}$. Otherwise, $|W| < k$ and we can add a candidate from $\bigcap_{i \in S} A_i$ to the committee.

Exercise IV: Extended Justified Representation

- This implies that

$$\frac{\frac{k}{|N|} - b_i}{\ell - 1} > \frac{\frac{k}{|N|} - \frac{1}{S}}{\ell - 1} \geq \frac{\frac{k}{|N|} - \frac{k}{\ell|N|}}{\ell - 1} = \frac{k}{|N|} \cdot \frac{1}{\ell}.$$

Exercise IV: Extended Justified Representation

- This implies that

$$\frac{\frac{k}{|N|} - b_i}{\ell - 1} > \frac{\frac{k}{|N|} - \frac{1}{S}}{\ell - 1} \geq \frac{\frac{k}{|N|} - \frac{k}{\ell|N|}}{\ell - 1} = \frac{k}{|N|} \cdot \frac{1}{\ell}.$$

- Hence, agent i payed more than $\frac{k}{|N|} \cdot \frac{1}{\ell}$ for some candidate.

Exercise IV: Extended Justified Representation

- This implies that

$$\frac{\frac{k}{|N|} - b_i}{\ell - 1} > \frac{\frac{k}{|N|} - \frac{1}{S}}{\ell - 1} \geq \frac{\frac{k}{|N|} - \frac{k}{\ell|N|}}{\ell - 1} = \frac{k}{|N|} \cdot \frac{1}{\ell}.$$

- Hence, agent i payed more than $\frac{k}{|N|} \cdot \frac{1}{\ell}$ for some candidate.
- Let c denote the first candidate such that a voter $i \in S$ payed more than $\frac{1}{\ell}$ for this candidate.

Exercise IV: Extended Justified Representation

- This implies that

$$\frac{\frac{k}{|N|} - b_i}{\ell - 1} > \frac{\frac{k}{|N|} - \frac{1}{S}}{\ell - 1} \geq \frac{\frac{k}{|N|} - \frac{k}{\ell |N|}}{\ell - 1} = \frac{k}{|N|} \cdot \frac{1}{\ell}.$$

- Hence, agent i payed more than $\frac{k}{|N|} \cdot \frac{1}{\ell}$ for some candidate.
- Let c denote the first candidate such that a voter $i \in S$ payed more than $\frac{1}{\ell}$ for this candidate.
- Let b_i' denote the budgets of the voters immediately before c is chosen.

Exercise IV: Extended Justified Representation

- As each voter in b'_i payed for at most $\ell - 1$ alternatives with a price of at most $\frac{k}{|N|} \cdot \frac{1}{\ell}$, each voter $i \in S$ has a budget of $b'_i \geq \frac{k}{|N|} \cdot \frac{1}{\ell}$.

Exercise IV: Extended Justified Representation

- As each voter in b_i' payed for at most $\ell - 1$ alternatives with a price of at most $\frac{k}{|N|} \cdot \frac{1}{\ell}$, each voter $i \in S$ has a budget of $b_i' \geq \frac{k}{|N|} \cdot \frac{1}{\ell}$.
- Since $|W \cap A_i| < \ell - 1$ for all $i \in S$ and $|\bigcap_{i \in S} A_i| \geq \ell$, there is an alternative c' that is approved by all voters in S that is not yet selected.

Exercise IV: Extended Justified Representation

- As each voter in b'_i payed for at most $\ell - 1$ alternatives with a price of at most $\frac{k}{|N|} \cdot \frac{1}{\ell}$, each voter $i \in S$ has a budget of $b'_i \geq \frac{k}{|N|} \cdot \frac{1}{\ell}$.
- Since $|W \cap A_i| < \ell - 1$ for all $i \in S$ and $|\bigcap_{i \in S} A_i| \geq \ell$, there is an alternative c' that is approved by all voters in S that is not yet selected.
- This alternative can be bought at a price of at most $\frac{1}{|S|} \leq \frac{k}{|N|} \cdot \frac{1}{\ell}$.

Exercise IV: Extended Justified Representation

- As each voter in b'_i payed for at most $\ell - 1$ alternatives with a price of at most $\frac{k}{|N|} \cdot \frac{1}{\ell}$, each voter $i \in S$ has a budget of $b'_i \geq \frac{k}{|N|} \cdot \frac{1}{\ell}$.
- Since $|W \cap A_i| < \ell - 1$ for all $i \in S$ and $|\bigcap_{i \in S} A_i| \geq \ell$, there is an alternative c' that is approved by all voters in S that is not yet selected.
- This alternative can be bought at a price of at most $\frac{1}{|S|} \leq \frac{k}{|N|} \cdot \frac{1}{\ell}$.
- This contradicts that *MES* chooses c as next candidate.

Exercise IV: Extended Justified Representation

- As each voter in b'_i payed for at most $\ell - 1$ alternatives with a price of at most $\frac{k}{|N|} \cdot \frac{1}{\ell}$, each voter $i \in S$ has a budget of $b'_i \geq \frac{k}{|N|} \cdot \frac{1}{\ell}$.
- Since $|W \cap A_i| < \ell - 1$ for all $i \in S$ and $|\bigcap_{i \in S} A_i| \geq \ell$, there is an alternative c' that is approved by all voters in S that is not yet selected.
- This alternative can be bought at a price of at most $\frac{1}{|S|} \leq \frac{k}{|N|} \cdot \frac{1}{\ell}$.
- This contradicts that *MES* chooses c as next candidate.
- Hence, we now conclude that *MES* satisfies EJR.