Game Theory II

COMP4418 Knowledge Representation and Reasoning

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These slides are based on lecture slides by Prof. Felix Brandt.

Maximin

Example: Battle of the Sexes

	boxing	ballet
boxing	2, 1	0, 0
ballet	0, 0	1, 2

The maximin strategy of the row player is to play "boxing" with probability $\frac{1}{3}$ and "ballet" with probability $\frac{2}{3}$.

If the column player knows that the row player plays the maximin strategy, he can play "ballet" with probability 1 and achieve an expected utility of $\frac{4}{3}$!

Best Responses and Nash Equilibria

Best Responses

- A strategy s_i is a best response to a strategy profile s_{-i} if $u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i})$ for all $t_i \in S_i$.
- The set of best responses for a strategy profile s_{-i} is $B(s_{-i})$.
- Theorem: In two-player games, s_i is never a best response if and only if it is dominated.
- Can we solve games by just letting players repeatedly change their strategy to a best response for the current strategy profile?

Best Responses

Example: Matching pennies

		$\downarrow$	$\downarrow$
		heads	tails
$\rightarrow$	heads	1, -1	-1, 1
$\rightarrow$	tails	-1, 1	1, -1

- We start in the strategy profile where both players play heads.
- The best response of the column player is to play tails.
- The best response of the row player is to play tails.
- The best response of the column player is to play heads.
- The best response of the row player is to play heads.

Nash Equilibria

- Are there strategy profiles where no player has an incentive to deviate?
- A strategy profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium if $u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i})$ for all $t_i \in S_i$ and all $i \in N$.
  - In a Nash equilibrium every player plays a best response to the strategies of the other players.
  - Nash equilibria are the predominant solution concept in game theory.
- A Nash equilibrium is pure if all players play an action with probability 1.
  - Pure Nash equilibria are not guaranteed to exist.
- Nash equilibria only randomize over actions that survive the iterated removal of dominated actions.

Nash Equilibria

Example: Matching pennies

	heads	tails
heads	1, -1	-1, 1
tails	-1, 1	1, -1

The strategy profile given by $s_1(\text{heads}) = s_1(\text{tails}) = 0.5$ and $s_2(\text{heads}) = s_2(\text{tails}) = 0.5$ is a Nash equilibrium.

Nash Equilibria

Example: Battle of the Sexes

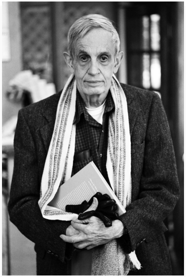
	boxing	ballet
boxing	2, 1	0, 0
ballet	0, 0	1, 2

The strategy profile given by $s_1(\text{boxing}) = \frac{2}{3}$, $s_1(\text{ballet}) = \frac{1}{3}$ and $s_2(\text{boxing}) = \frac{1}{3}$, $s_2(\text{ballet}) = \frac{2}{3}$ is a Nash equilibrium.

Indifference Principle

- Lemma: A strategy profile s is a Nash equilibrium if and only if $u_i(a_i, s_{-i}) = u_i(b_i, s_{-i}) \geq u_i(c_i, s_{-i})$ for all players $i \in N$ and actions $a_i, b_i, c_i \in A_i$ with $s_i(a_i) > 0$, $s_i(b_i) > 0$, and $s_i(c_i) = 0$.
  - In a Nash equilibrium, every player is indifferent between all actions in the support of his strategy, and weakly prefers these actions to actions outside of the support of his strategy.
- The payoff of player i is the same regardless how he randomizes over the actions in his support!
  - "Players randomize for other players!"
- Based on the indifference principle, one can efficiently verify whether a strategy profile is a Nash equilibrium.

Existence of Nash equilibria



Theorem (Nash, 1950)
Every normal-form game has a Nash equilibrium.

- Non-constructive proof via Brower's fix-point theorem.
- We finally have a nice solution concept that always exist!

Problems of Nash Equilibria

Computing Nash Equilibria

- Theorem (Daskalakis et al, Cheng and Deng, 2005): Finding a Nash equilibrium in a normal-form game is PPAD-complete.
  - Even holds if there are only two players.
  - It is believed that PPAD$\neq$ FP, which would imply that there is no efficient algorithm for computing Nash equilibria.
- The following problems are NP-hard, even if there are only 2 players (Gilboa and Zemel, 1989; Abbott et al., 2005): Determine whether there is a Nash equilibrium
  - where player i gets a utility of x.
  - whose expected outcome is Pareto-optimal.
  - where player i plays action a_i with probability 0.
  - ...

Multiplicity of Nash Equilibria

There may be multiple Nash equilibria in a game!

	boxing	ballet
boxing	2, 1	0, 0
ballet	0, 0	1, 2

- There are three Nash equilibria in this game:
  - $s_1(\text{boxing}) = \frac{2}{3}$, $s_1(\text{ballet}) = \frac{1}{3}$ and $s_2(\text{boxing}) = \frac{1}{3}$, $s_2(\text{ballet}) = \frac{2}{3}$
  - $s_1(\text{boxing}) = s_2(\text{boxing}) = 1$
  - $s_1(\text{ballet}) = s_2(\text{ballet}) = 1$
- Which Nash equilibrium should we choose?

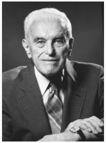
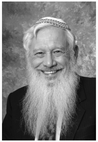
Coalitions of Players

Example: Prisoner's Dilemma

	cooperate	defect
cooperate	2, 2	0, 3
defect	3, 0	1, 1

- The only Nash equilibrium is (defect, defect).
- If both players coordinate, they are both better off by playing (cooperate, cooperate).

Refinements of Nash Equilibria

- One can consider related solution concepts to fix these flaws of Nash equilibria!
- A strategy profile s is a quasi-strict Nash equilibrium if it is a Nash equilibrium and $u_i(a_i, s_{-i}) > u_i(b_i, s_{-i})$ for all $i \in N$ and $a_i, b_i \in S_i$ with $s_i(a_i) > 0$, $s_i(b_i) = 0$ (Harsanyi, 1973).
    - Guaranteed to exist in 2-player games (Norde, 1999).
    - Finding Quasi-strict equilibria is computationally hard.
- A strategy profile s is a strong Nash equilbirium if, for all coalitions of players $C \subseteq N$, there is no t_C such that $u_i(t_C, s_{-C}) > u_i(s_C, s_{-C})$ for all $i \in C$ (Aumann, 1959).
    - Coalitions of players cannot jointly deviate to improve the utility of all players in the coalition.
    - Not guaranteed to exist.

Zero-sum Games

Zero-sum games

- A zero-sum game is a two-player normal-form game such that $u_1(a) + u_2(a) = 0$ for all action profiles $a \in A$.
  - The interests of the players are diametrically opposed: the benefit of player 1 is the loss of player 2.
  - Since $u_1(a) = -u_2(a)$ for all $a \in A$, we can represent zero-sum games by matrices showing only $u_1(a)$.

	heads	tails
heads	1	-1
tails	-1	1

Matching pennies

	rock	paper	scissors
rock	0	-1	1
paper	1	0	-1
scissors	-1	1	0

Rock-paper-scissors

The Minimax Theorem

- Let v_1 denote the security level of player 1 and v_2 the security of player 2.
  - $v_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$
  - $v_2 = \max_{s_2} \min_{s_1} u_2(s_1, s_2) = \max_{s_2} \min_{s_1} -u_1(s_1, s_2)$
- Theorem (von Neumann, 1928): It holds in every zero-sum game that $v_1 = -v_2$.
  - If player 1 can ensure to gain at least v_i, player 2 can ensure to lose at most $-v_1$.
- Zero-sum games are fully determined, i.e., there is a value v that is the unique rational outcome.
  - We define the value of a zero-sum game as v_1.

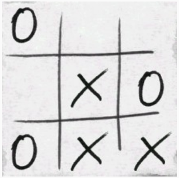
Consequences of the Minimax Theorem

- All combinations of maximin strategies are Nash equilibria in zero-sum games.
- The set of Nash equilibria is convex for zero-sum games.
- All Nash equilibria yield the same outcome in zero-sum games.
- Nash equilibria can be efficiently computed in zero-sum games.
- For zero-sum games, Nash equilibria satisfy all desiderata!

Extensive-form Games

Sequential Moves

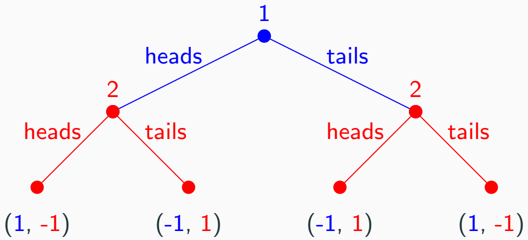
- So far we have assumed that all players choose their strategies simulatenously.
- In many real-world situations, players act sequential.

- These can, in principle, be modelled as normal-form game by defining actions as functions that map every game state to a move.
  - This results in an inefficient presentation and slow algorithms.
  - It is also counterintuitive.

Sequential Moves

Example: Sequential Matching Pennies

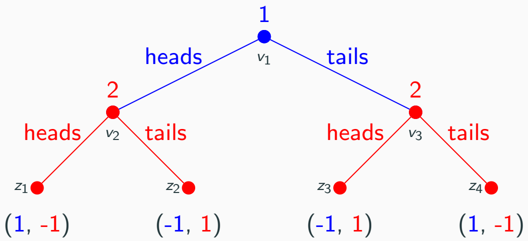


Extensive-Form Games

- An extensive-form game is a tuple $(N, A, H, Z, \chi, \rho, \sigma, (u_i)_{i \in N})$
  - $N = \{1, \ldots, n\}$ is the set of players.
  - $A = \{a_1, \ldots, a_m\}$ is the set of all possible actions.
  - H is a set of intermediate game states.
  - Z is a set of terminal game states.
  - $\chi : H \to 2^A \setminus \{\emptyset\}$ states for every intermediate game state the set of feasible actions.
  - $\rho : H \to N$ states for every intermediate game state the player whose turn it is.
  - $\sigma : H \times A \to H \cup Z$ states the new game state when action $a \in A$ is played at the game state $h \in H$
  - $u_i : Z \to \mathbb{R}$ states the utility of player i for every terminal game state.
- The set of strategies of player i is $S_i = \prod_{h \in H :\ \rho(h) = i} \chi(h)$.

Extensive-Form Games

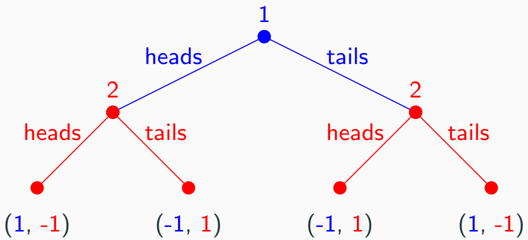
Example: Sequential Matching Pennies



- $N = \{1, 2\}$
- $A = \{\text{heads}, \text{tails}\}$
- $H = \{v_1, v_2, v_3\}$
- $Z = \{z_1, z_2, z_3, z_4\}$
- $\chi(v_1) = \chi(v_2) = \chi(v_3) = \{\text{heads}, \text{tails}\}$
- $\rho(v_1) = 1,\ \rho(v_2) = \rho(v_3) = 2$
- $\sigma(v_1, \text{heads}) = v_2,\ \sigma(v_2, \text{heads}) = z_1,\ \ldots$
- $u_1(z_1) = 1,\ u_1(z_2) = -1,\ \ldots$
- $S_2 = \{(\text{heads, heads}), (\text{heads, tails}), (\text{tails, heads}), (\text{tails, tails})\}$

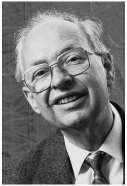
From Extensive-Form Games to Normal-Form Games

Example: Sequential Matching Pennies



	(h,h)	(h,t)	(t,h)	(t,t)
h	1, -1	1, -1	-1, 1	-1, 1
t	-1, 1	1, -1	-1, 1	1, -1

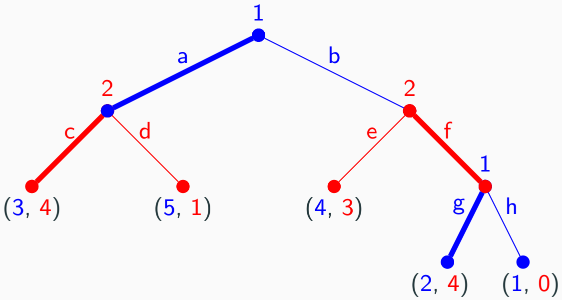
Sub-game Perfect Nash Equilibrium



- Extensive-form games admit pure Nash equilibria!
- However, some Nash equilibria may contain non-credible threats, i.e., actions that no rational player would play.
- A strategy profile is a subgame-perfect Nash equilibrium if it is a Nash equilibrium for every subtree of G.
- Theorem (Selten, 1965): Every extensive-form game contains a subgame-perfect Nash equilibrium in pure strategies.

Backwards induction

- For finding a subgame-perfect Nash equilibrium, we can use backwards induction.



Outlook: Solving Real Games

- For real games, the utilities are much more structured
  - E.g.: For chess, a game ends either with a win, lose, or draw. We can formalize this by only using 1, -1, and 0 as utilities!
- Theorem (Zermelo, 1913): Every zero-sum extensive-form game is uniquely determined, i.e., there is a unique value obtainable by pure strategies.
- For, e.g., chess, one of the following claims holds:
  - There is a strategy for player 1 that guarantees him a win,
  - There is a strategy for player 2 that guarantees him a win,
  - Both players have a strategy that guarantees them a draw.

Computers vs. Humans

- Today, computer programs are capable of beating grand masters in chess, Go, checkers etc.
    - 1992: Marion Tinsley (considered the greatest checkers player ever) wins 4 : 2 against Chinook (which marks 2 of his 7 official losses).
    - 2006: Wladimir Kramnik (the reigning chess world champion) loses against Deep Fritz (2:4)
    - 2016: Lee Sedol (considered one of the best Go players) loses against AlphaGo (4:1)
- Before AlphaGo, all of these computer programs were merely based on using clever heuristics for exploring the game tree.

AlphaZero

- AlphaZero Go combines self-play with reinforcment learning:
  - Start with some initial set of (randomized) policies.
  - Use self-play (letting computer play against each other) to produce new data.
  - Feed the data to a neuronal network to get new policies.
  - Repeat to infer good policies.

Further Reading

Further Reading

- Y. Shoham and K. Leyton-Brown. Multiagent Systems: Algorithmic Game-Theoretic, and Logical Foundations. 2009. Section 3. http://www.masfoundations.org/mas.pdf
- N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani. Algorithmic Game Theory. Cambridge University Press, 2007. Section 1. https://www.cs.cmu.edu/~sandholm/cs15-892F13/algorithmic-game-theory.pdf
- M. Maschler, E. Solan, and S. Zamir. Game Theory. Cambridge University Press. 2015. Sections 2 to 5.

Image References

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