Fair Allocations

COMP 4418 – Assignment 3

Sample Solution
Due 14 November 2024, 16:00
Total Marks: 100
Late Penalty: 10 marks per day

Worth: 15% of the course

Question 1 (20 marks) Consider a fair division instance $\langle N, M, v \rangle$ with n agents and m items. Prove or disprove the following:

1. **(5 marks)** Any Pareto Optimal allocation must also be Leximin Optimal.
2. **(5 marks)** Given any two allocations, one must pareto dominate the other.
3. **(5 marks)** For $n = 2$, any allocation that satisfies PROP is also EF.
4. **(5 marks)** Greedy round robin algorithm will return an EF1 allocation.

Solution.

1. False. For example, consider the case where the maximum Utilitarian Social Welfare allocation is not Leximin Optimal.
2. False. Consider the case with two agents $1, 2$ and one indivisible item g and $v_1(g) > 0, v_2(g) > 0$. There are two allocations, one is to assign the item to agent 1 and the other is to assign the item to agent 2. No allocation Pareto dominates the other allocation.
3. True. For any PROP allocation $A = (A_1, A_2)$, w.l.o.g, consider agent 1,

$$\begin{aligned} v_1(A_2) = v_1(M \setminus A_1) &= v_1(M) - v_1(A_1) && (\because \text{additivity}) \\ &\leq 2v_1(A_1) - v_1(A_1) = v_1(A_1). && (\because \text{PROP}) \end{aligned}$$

   Similarly, we can get $v_2(A_1) \leq v_2(A_2)$. Therefore, when $n = 2$, any PROP allocation satisfies EF.

4. False. Consider the following example. Recall the *Greedy Round Robin* algorithm,

	g_1	g_2	g_3	g_4
v_1	100	80	80	80
v_2	100	80	80	80

Initialize $I = \{1,2\}$ and $P = \{g_1, g_2, g_3, g_4\}$. Compute $\frac{\beta_1}{2} = \frac{\beta_2}{2} = 85$. W.l.o.g, for the while loop, assign item g_1 to agent 1, i.e., $A_1 = \{g_1\}$ and update $I = \{2\}, P = \{g_2, g_3, g_4\}$. For agent 2, $\frac{\beta_2}{2} = 120$. The algorithm completes the while loop and proceeds to execute the standard *Round Robin* algorithm, resulting in $A_2 = \{g_2, g_3, g_4\}$. This allocation is not EF1, as agent 1 always envies agent 2, regardless of which item is removed from A_2.

Question 2 (20 marks) Consider the following instance with $n = 4$ and $m = 8$.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
v_1	1	5	4	4	0	1	1	1
v_2	5	9	5	5	0	0	5	5
v_3	5	7	5	10	0	4	0	5
v_4	10	10	5	5	5	5	5	5

For this instance, consider running the standard round robin algorithm to find an EF1 allocation. We shall look at how different orderings over agents can lead to different allocations. For the given instance, identify:

1. **(10 marks)** The ordering over agents which leads to the following allocation: $A = (A_1, A_2, A_3, A_4)$, where $A_1 = \{g_1, g_5\}$, $A_2 = \{g_4, g_8\}$, $A_3 = \{g_3, g_7\}$ and $A_4 = \{g_2, g_6\}$.
2. **(5 marks)** An alternate EF1 allocation that can result from the same ordering which would Pareto dominate A.
3. **(5 marks)** An alternate ordering for the standard round robin algorithm that would result in the allocation identified in the previous part.

Solution.

1. Ordering $4,2,3,1$. Agent 4 first gets g_2, agent 2 gets g_4, agent 3 gets g_3, agent 1 gets g_1; Agent 4 gets g_6, agent 2 gets g_8, agent 3 gets g_7, agent 1 gets g_5.
2. One possible alternate allocation: $A_1 = \{g_3, g_6\}$, $A_2 = \{g_2, g_7\}$, $A_3 = \{g_4, g_8\}$ and $A_4 = \{g_1, g_5\}$. Any allocation that does pareto dominate the allocation in the question is fine.
3. Alternate ordering: $3,2,1,4$. Agent 3 first gets g_4, agent 2 gets g_2, agent 1 gets g_3, agent 4 gets g_1; Agent 3 gets g_8, agent 2 gets g_7, agent 1 gets g_6, agent 4 gets g_5.

Question 3 (20 marks) Consider an indivisible item setting with $m > n$ where agents are indifferent between the items. That is, for any $i \in N$ and $g \neq g' \in M$, we have that $v_i(g) = v_i(g') > 0$. However, agent valuations are not (guaranteed to be) identical. That is, there may be $i \neq j$ and $g \in M$, s.t. $v_i(g) \neq v_j(g)$. For this setting:

1. **(5 marks)** Show that an MMS allocation always exists.
2. **(5 marks)** Show that an EF1 allocation will always be MMS.
3. **(10 marks)** Give examples of instances in this setting such that:

a. Any allocation with maximum ESW is not MMS.
b. There is at least one allocation with maximum USW which is $\frac{1}{2}-$MMS in this instance.

Solution.

1. Divide the items as equally as possible among all agents, i.e., each agent gets $\lfloor \frac{m}{n} \rfloor$ or $\lceil \frac{m}{n} \rceil$ items. This allocation will be MMS for any agent because in this setting, the MMS valuation for each agent i will be $\lfloor \frac{m}{n} \rfloor \cdot v_i(g)$.

2. First, we show that for any EF1 allocation, the difference in the sizes of agents' bundles must be at most 1. We prove this by contradiction. Assume there exists an EF1 allocation such that there are agents i and j where $|A_j| - |A_i| \geq 2$. For agent i, consider any item $g \in A_j$. Removing g from A_j, the value of the remaining bundle for agent i is given by
$$v_i(A_j \setminus \{g\}) = (|A_j| - 1) \cdot v_i(g).$$
Since $|A_j| - 1 > |A_i|$, it follows that:
$$v_i(A_j \setminus \{g\}) > |A_i| \cdot v_i(g) = v_i(A_i),$$
which contradicts the condition for EF1 allocation. Therefore, the difference in the sizes of agents' bundles in any EF1 allocation must be at most 1. Now, recall that in this setting, the MMS valuation for each agent i is given by $\lfloor \frac{m}{n} \rfloor \cdot v_i(g)$. Since any EF1 allocation ensures that the difference in bundle sizes is at most 1, it follows that every EF1 allocation will be an MMS allocation.

3. a) An example of an instance where one agent has very low value compared to anyone else, and there are at least $2n$ items. For instance,

	g_1	g_2	g_3	g_4
v_1	100	100	100	100
v_2	1	1	1	1

In this example, the MMS values for agent 1 and 2 will be $\text{MMS}_1 = 200, \text{MMS}_2 = 2$. For any allocation maximizing ESW, it will be case that agent 1 gets 1 item while agent 2 gets 3 items. That is, for any maximum ESW allocation $A = (A_1, A_2)$, we have $v_1(A_1) = 100 < \text{MMS}_1$ and $v_2(A_2) = 3 > \text{MMS}_2$. Therefore, any maximum ESW allocation in this example will not be an MMS allocation.

b) Consider the example where all agents are identical and indifferent across items.

Question 4 (20 marks) Consider the random assignment problem with 3 agents with the following preferences over 3 items.

$$\succ_1: \quad g_1 \succ_1 g_2 \succ_1 g_3$$
$$\succ_2: \quad g_1 \succ_2 g_2 \succ_2 g_3$$
$$\succ_3: \quad g_2 \succ_3 g_1 \succ_3 g_3$$

Find the random assignment as a result of the following rules.

1. **(10 marks)** probabilistic serial (PS)

2. **(10 marks)** random serial dictator (RSD)

Solution: For probabilistic serial (PS):

$$PS(N,M,\succ) = \begin{pmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/6 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}.$$

For random serival dictator (RSD):

$$RSD(N,M,\succ) = \begin{pmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/6 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}.$$

Question 5 (20 marks) Consider the following instance with $n = 4$ and $m = 8$.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
v_1	1	5	4	4	0	1	1	1
v_2	5	9	5	5	0	0	5	5
v_3	5	7	5	10	0	4	0	5
v_4	10	10	5	5	5	4	1	1

Consider the allocation A in which $A_1 = \{g_1, g_2\}$, $A_2 = \{g_3, g_4\}$, $A_3 = \{g_5, g_6\}$, and $A_4 = \{g_7, g_8\}$.

1. **(5 marks)** Prove or disprove that the allocation is envy-free.

2. **(5 marks)** Prove or disprove that the allocation is envy-freeable.

3. **(5 marks)** Compute the corresponding envy-graph with the amount of envy on the edge weights.

4. **(5 marks)** Find the subsidy needed to be given to each agent in order to make the allocation envy-free or show that no such subsidy exists.

Solution: For question (1), this allocation is not enyv-free. For example, agent 1 envies agent 2. For question (2) - (4), we first construct the envy graph and set up the weights. For the allocation $A : A_1 = \{g_1, g_2\}$, $A_2 = \{g_3, g_4\}$, $A_3 = \{g_5, g_6\}$, and $A_4 = \{g_7, g_8\}$. For any pair of agents $i, j \in N$, the weight of arc (i, j) is the envy agent i has for agent j under the allocation A: $w(i, j) = v_i(A_j) - v_i(A_i)$.

$$
\begin{aligned}
w(1,2) &= v_1(A_2) - v_1(A_1) = 8 - 6 = 2. \\
w(2,1) &= v_2(A_1) - v_2(A_2) = 14 - 10 = 4. \\
w(1,3) &= v_1(A_3) - v_1(A_1) = 1 - 6 = -5. \\
w(3,1) &= v_3(A_1) - v_3(A_3) = 12 - 4 = 8. \\
w(1,4) &= v_1(A_4) - v_1(A_1) = 2 - 6 = -4. \\
w(4,1) &= v_4(A_1) - v_4(A_4) = 20 - 2 = 18. \\
w(2,3) &= v_2(A_3) - v_2(A_2) = 0 - 10 = -10. \\
w(3,2) &= v_3(A_2) - v_3(A_3) = 15 - 4 = 11. \\
w(2,4) &= v_2(A_4) - v_2(A_2) = 10 - 10 = 0. \\
w(4,2) &= v_4(A_2) - v_4(A_4) = 10 - 2 = 8. \\
w(3,4) &= v_3(A_4) - v_3(A_3) = 5 - 4 = 1. \\
w(4,3) &= v_4(A_3) - v_4(A_4) = 9 - 2 = 7.
\end{aligned}
$$

The envy graph can be represented as follows:

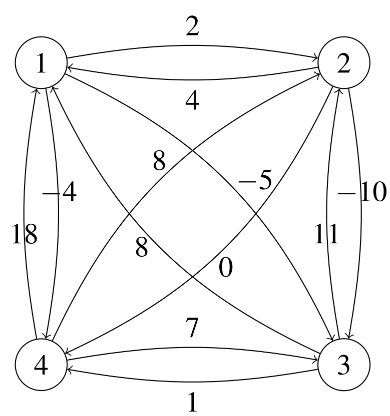

Figure 1: Envy graph of question 5

Recall the characterization of envy-freeability: Under positive additive utilities, the following conditions are equivalent for a given allocation:

- the allocation is envy-freeable.
- the allocation is reassignment-stable.
- for the allocation, there is no positive weight cycle in the corresponding envy-graph.

Obivously, there exist positive weight cycles in the envy graph in Figure 1. Hence the allocation A is not envy-freeable and no subsidy exists for the allocation A to be envy-free.

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- Your answers are to be submitted in a single PDF file. We will not accept any other file formats. Please make sure that your solutions are clearly readable.
- The deadline for this submission is 14th November 2024, 16:00.

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