Cooperative Game Theory and Matchings

COMP 4418 – Assignment 2

Sample Solution
Total Marks: 100
Due 25 Oct. 2024, 23:59

Question 1 (20 marks) Consider a Student Proposing Deferred Acceptance (SPDA) algorithm on an one-one matching instance $\langle S, C \prec \rangle$ where $|S| = |C| = n$. Prove or disprove the following:

1. **(5 marks)** For any instance, at least one student always makes multiple proposals.
2. **(5 marks)** There is an instance where the number of proposals made is $\frac{n(n+1)}{2}$.
3. **(5 marks)** For any instance, there is always one college that receives exactly one proposal.
4. **(5 marks)** For an instance n students and n colleges, the maximum number of proposals that can be made is $n(n-1)+1$.

Solution:

1. False. For example,

$$
\begin{array}{ll}
s_1 : c_1 \succ c_2 \succ c_3 \succ c_4 & \quad c_1 : s_1 \succ s_2 \succ s_3 \succ s_4 \\
s_2 : c_2 \succ c_3 \succ c_4 \succ c_1 & \quad c_2 : s_2 \succ s_3 \succ s_4 \succ s_1 \\
s_3 : c_3 \succ c_4 \succ c_2 \succ c_1 & \quad c_3 : s_3 \succ s_4 \succ s_2 \succ s_1 \\
s_4 : c_4 \succ c_2 \succ c_1 \succ c_3 & \quad c_4 : s_4 \succ s_2 \succ s_1 \succ s_3
\end{array}
$$

2. True. Consider the following instance with $n = 4$,

$$
\begin{array}{ll}
s_1 : c_1 \succ c_2 \succ c_3 \succ c_4 & \quad c_1 : s_1 \succ s_2 \succ s_3 \succ s_4 \\
s_2 : c_1 \succ c_2 \succ c_3 \succ c_4 & \quad c_2 : s_1 \succ s_2 \succ s_3 \succ s_4 \\
s_3 : c_1 \succ c_2 \succ c_3 \succ c_4 & \quad c_3 : s_1 \succ s_2 \succ s_3 \succ s_4 \\
s_4 : c_1 \succ c_2 \succ c_3 \succ c_4 & \quad c_4 : s_1 \succ s_2 \succ s_3 \succ s_4
\end{array}
$$

3. True. Once a college receives a proposal it will be matched for the remaining SPDA. SPDA only stops when all colleges are matched. When all colleges receive at least one proposal, SPDA will end. Thus, there must be at least one college to receive exactly one proposal.

4. True. Consider the following instance with $n = 5$,

$$
\begin{array}{ll}
s_1 : c_2 \succ c_3 \succ c_4 \succ c_5 \succ c_1 & c_1 : s_4 \succ s_1 \succ s_2 \succ s_3 \succ s_5 \\
s_2 : c_2 \succ c_4 \succ c_5 \succ c_3 \succ c_1 & c_2 : s_3 \succ s_4 \succ s_5 \succ s_2 \succ s_1 \\
s_3 : c_3 \succ c_4 \succ c_5 \succ c_2 \succ c_1 & c_3 : s_2 \succ s_4 \succ s_5 \succ s_1 \succ s_3 \\
s_4 : c_4 \succ c_5 \succ c_2 \succ c_3 \succ c_1 & c_4 : s_5 \succ s_1 \succ s_2 \succ s_3 \succ s_4 \\
s_5 : c_5 \succ c_2 \succ c_3 \succ c_4 \succ c_1 & c_5 : s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5
\end{array}
$$

In the following instance, consider SPDA. For colleges c_2, c_3, c_4, c_5, each college receives 5 proposals while college c_1 receives exactly 1 proposal.

The statement is True in general. For SPDA, consider the instances in which for $(n-1)$ colleges, each receives n proposals, and one college receives 1 proposal.

Question 2 (10 marks) Consider the following one-one matching instance with $n = 5$.

$$
\begin{array}{ll}
s_1 : c_2 \succ c_1 \succ c_3 \succ c_4 \succ c_5 & c_1 : s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \\
s_2 : c_1 \succ c_2 \succ c_5 \succ c_3 \succ c_4 & c_2 : s_2 \succ s_1 \succ s_4 \succ s_5 \succ s_3 \\
s_3 : c_3 \succ c_4 \succ c_5 \succ c_2 \succ c_1 & c_3 : s_4 \succ s_5 \succ s_3 \succ s_2 \succ s_1 \\
s_4 : c_4 \succ c_5 \succ c_3 \succ c_1 \succ c_2 & c_4 : s_5 \succ s_3 \succ s_4 \succ s_1 \succ s_2 \\
s_5 : c_1 \succ c_2 \succ c_5 \succ c_3 \succ c_4 & c_5 : s_3 \succ s_4 \succ s_5 \succ s_1 \succ s_2
\end{array}
$$

Identify all colleges who can manipulate under the SPDA and explain why. Give an inconspicuous optimal manipulation for each, if any. Analogously identify all students who can manipulate under the CPDA and give an inconspicuous optimal manipulation for each, if any.

Solution: Only c_1 and c_2 can beneficially manipulate. For c_3, c_4, c_5, each of the college only receives exactally one proposal during SPDA.

Inconspicuous Optimal Manipulation for c_1: $s_1 \succ s_5 \succ s_2 \succ s_3 \succ s_4$.

Inconspicuous Optimal Manipulation for c_2: $s_2 \succ s_5 \succ s_1 \succ s_4 \succ s_3$.

For CPDA, each student only receives exactaly one proposal, so no student can manipulate.

Question 3 (10 marks) Give a polynomial time algorithm to check if a given one-one matching instance has a unique stable matching. Prove its correctness and running time.

Solution: Consider the following algorithm:

Input: An instance $I = (S, C, \succ)$.

Output: True or False.

Given I, run SPDA on I and CPDA on I compare the matchings returned.

If SPDA and CPDA return the same stable matching, return True, else return False.

Correctness: SPDA returns student optimal/college pessimal matching. CPDA returns student pessimal college optimal. Thus, if more than one matching is stable, at least one student/college would have a different optimal and pessimal partner.

Running time: $O(N^2)$ (N is the number of agents in each side.)

Question 4 (10 marks) Given a one-one matchings instance $\langle S, C \succ \rangle$, let μ and μ' be two distinct stable matchings. Suppose for each student $s \in S$ if $\mu(s) \neq \mu'(s)$ then $\mu(s) \succ_s \mu'(s)$. Prove that for each college if $\mu(c) \neq \mu'(c)$ then $\mu(c) \prec \mu'(c)$.

Solution: If there is some c s.t. $\mu(c) \succ_c \mu'(c)$ then $(\mu(c), c)$ would block μ'.

Complete proof needs to specify the following: Consider $c \in C$ s.t. $\mu(c) \neq \mu'(c)$.

Assume for contradiction that $\mu(c) \succ_c \mu'(c)$.

Let $s = \mu(c)$. As $\mu(c) \neq \mu'(c)$, thus, $\mu(s) \neq \mu'(s)$.

We know that/recall/ thus it must be that $c \succ_s \mu'(s)$.

Thus, we have that (s, c) form a **blocking pair** under μ'.

This contradicts the fact that μ' is a stable matching.

Hence, for all c s.t. $\mu(c) \neq \mu'(c)$, we have that $\mu(c) \prec_c \mu'(c)$.

Question 5 (10 marks) Consider a Shapley-Scarf housing market with a set of agents $N = \{0,1,2,3,4\}$, a set of items $O = \{o_0, o_1, o_2, o_3, o_4\}$, and an endowment function $\omega : N \to 2^O$ such that $\omega(i) = \{o_i\}$. The preferences of the agents are as follows from left to right in decreasing order of preference.

$$
\begin{aligned}
0 &: o_0, o_4, o_2, o_1, o_3 \\
1 &: o_0, o_2, o_4, o_1, o_3 \\
2 &: o_3, o_0, o_2, o_4, o_1 \\
3 &: o_0, o_2, o_3, o_1, o_4 \\
4 &: o_3, o_2, o_1, o_4, o_0
\end{aligned}
$$

Find the outcome of the TTC (top trading cycles) algorithm. Can agent 4 misreport her preference to get a more preferred allocation? Prove or disprove that the outcome is individually rational.

Answer

Detailed procedures of TTC:

- Step 1: One self cycle of $0 \to o_0 \to 0$.
- Step 2: One cycle between $2 \to o_3 \to 3 \to o_2 \to 2$.
- Step 3: One cycle between $1 \to o_4 \to 4 \to o_4 \to 1$.

The final outcome is $\{(0, o_0), (1, o_4), (2, o_3), (3, o_2), (4, o_1)\}$.

Agent 4 cannot misreport her preference to get a more preferred allocation as TTC algorithm satisfies strategyproofness.

For the outcome, TTC algorithm satisfies individual rationality. For agent 0, $o_0 \succeq o_0$; For agent 1, $o_4 \succeq o_1$; For agent 2, $o_3 \succeq o_2$; For agent 3, $o_2 \succeq o_3$; For agent 4, $o_1 \succeq o_4$.

Question 6 (20 marks) Consider Shapley-Scarf housing markets in which we are only allowed to obtain allocations in which at most two agents are a part of a trading cycle and each agent can be a part of at most of one trading cycle. (a) Design a polynomial-time algorithm for the problem and that is individually rational and Pareto optimal among feasible outcomes. Prove its properties. (b) Design a polynomial-time algorithm for the problem and that is strategyproof and Pareto optimal among feasible outcomes. Prove its properties.

Solution Hints

a) Construct a graph of agents as nodes. There is an edge is there an individually rational exchange. The weight of an edge $w(i,j) = v_i(j) + v_j(i)$ where the values are consistent with ordinal preferences. We then compute a maximum weight matching of the graph.

b) Consider the Random Serial Dictatorship, initialize the set of houses with set H and randomly order the agents with premutation π, for each agent $\pi(i)$ from $\pi(1), \pi(2), \dots, \pi(n)$, assign her favorite house $h \in H$ to $\pi(i)$, delete h from H.

Question 7 (20 points) Consider the coalitional game (N, v) such that $N = \{1,2,3\}$ and v is defined as follows:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	4	3	2	4	3	2	12

For this game, compute the nucleolus and explain why it is the nucleolus.

Also compute the Shapley value and explain how you computed it.

We first compute the nucleolus. The nucleolus: (5,4,3). The payoff of agent 1 is 5, payoff of agent 2 is 4, payoff of agent 3 is 3. The justification of why the nucleolus is (5,4,3) is as follows. Consider the excess vector for (5,4,3).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	4	3	2	4	3	2	12
$x(S)$	5	4	3	9	8	7	12
$e(x,S)$	1	1	1	5	5	5	12

If any player gets more payoff, then another player gets less payoff and the minimum excess is less than 1.

Next, we compute the Shapley value. The solution is (5,4,3). The solution is computed as follows. We compute the marginal value of each agent in each permutation:

Agent 1

- 123: $v(\{1\}) - v(\emptyset) = 4 - 0 = 4$
- 132: $v(\{1\}) - v(\emptyset) = 4 - 0 = 4$
- 213: $v(\{1,2\}) - v(\{2\}) = 4 - 3 = 1$
- 231: $v(\{1,2,3\}) - v(\{2,3\}) = 12 - 2 = 10$
- 312: $v(\{1,3\}) - v(\{3\}) = 3 - 2 = 1$
- 321: $v(\{1,2,3\}) - v(\{2,3\}) = 12 - 2 = 10$

Shapley value of agent 1 is $(4+4+1+10+1+10)3! = 30/6 = 5$

Agent 2

- 123: $v(\{1,2\}) - v(\{1\}) = 4 - 4 = 0$
- 132: $v(\{1,2,3\}) - v(\{1,3\}) = 12 - 3 = 9$

- 213: $v(\{2\}) - v(\emptyset) = 3 - 0 = 3$
- 231: $v(\{2\}) - v(\emptyset) = 3 - 0 = 3$
- 312: $v(\{1,2,3\}) - v(\{1,3\}) = 12 - 3 = 9 =$
- 321: $v(\{2,3\}) - v(\{3\}) = 2 - 2 = 0$

Shapley value of agent 2 is $(0+9+3+3+9+0)3! = 24/6 = 4$
Agent 2

- 123: $v(\{1,2,3\}) - v(\{1,2\}) = 12 - 4 = 8$
- 132: $v(\{1,3\}) - v(\{1\}) = -1$
- 213: $v(\{1,2,3\}) - v(\{1,2\}) = 12 - 4 = 8$
- 231: $v(\{2,3\}) - v(\{2\}) = 2 - 3 = -1$
- 312: $v(\{3\}) - v(\emptyset) = 2 - 0 = 2$
- 321: $v(\{3\}) - v(\emptyset) = 2 - 0 = 2$

Shapley value of agent 3 is $(8-1+8-1+2+2)3! = 18/6 = 3$

SUBMISSION

- Submit your solution directly via Moodle in the assessment hub at the end of the Moodle page. Please make sure that your manuscript contains your name and zID.
- Your answers are to be submitted in a single PDF file. We will not accept any other file formats. Please make sure that your solutions are clearly readable.
- The deadline for this submission is 25 October 2024, 23:59.

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