Social Choice Theory II

COMP4418 Knowledge Representation and Reasoning

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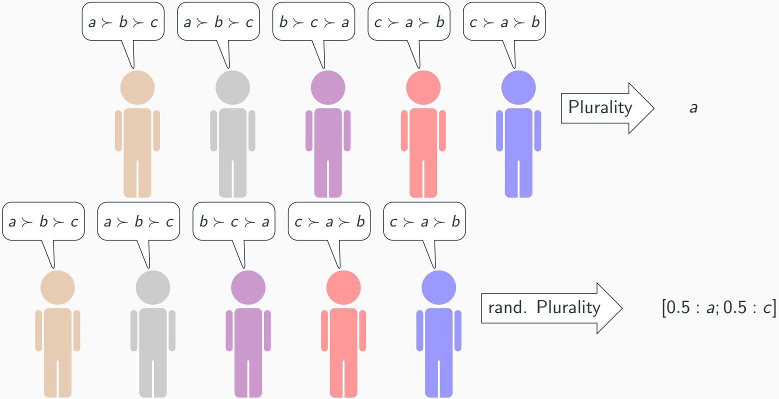
Randomized Social Choice

Randomized Social Choice

- Last week: Voters report preferences and we choose a single winner deterministically.
- Now: Voters report preferences and we may use chance to select a single winner.



Randomized Social Choice

Why should we randomize?

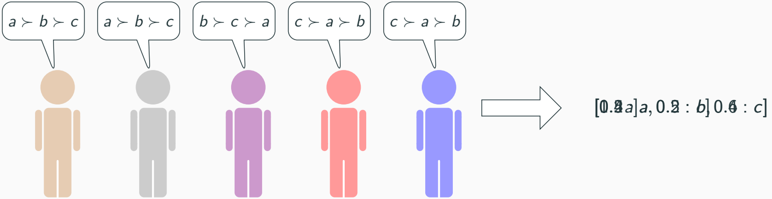
- Breaking ties between alternatives
  - The notion of a tie depends on the voting rule!
- Repeated decision-making
  - E.g.: Worker of the month, next song in a playlist
- Resource allocation: based on the voters' preferences, we assign a resource (e.g., money) to public projects.
- Better axiomatic properties.
  - We may be able to escape impossibility theorems by allowing for randomization.

Randomized Social Choice - The Formal Model

- Finite set of voters $N = \{1, \ldots, n\}$.
- Finite set of alternatives $A = \{x_1, \ldots, x_m\}$
- Every voter $i \in N$ reports a preference relation, which is a strict total order $\succ_i$ over A.
  - E.g.: $\succ_i =$ Harris $\succ_i$ Stein $\succ_i$ Trump
- A preference profile $R = (\succ_1, \ldots, \succ_n)$ contains the preference relations of all voters $i \in N$.
- A lottery is a probability distribution over the alternatives.
  - Recall from Week 1: $\Delta(A)$ is the set of all lotteries over A.
- A randomized social choice function (RSCF) maps every preference profile to a lottery over the alternatives.

Randomized Social Choice Functions

- Every deterministic social choice function is an RSCF that always assigns probability 1 to some alternative.
  - E.g.: Plurality rule with tie-breaking
- Every social choice correspondence can be turned into an RSCF by, e.g., randomizing uniformly over the chosen alternatives.
  - E.g.: Randomize over the set of plurality winners
- Randomize proportional to some scores.
  - E.g.: Uniform random dictatorship, randomized Copeland
- Construct new RSCFs tailored for the use of randomization
  - E.g.: Maximal lotteries



The Random Dictatorship Theorem

Strategyproof Social Choice

- Last week: When $|A| \geq 3$, the only strategyproof and onto social choice functions are dictatorships.
- Can we circumvent this impossibility theorem by allowing for randomization?
- Yes! The uniform random dictatorship (select a voter uniformly at random and return his favorite alternative) is strategyproof.
    - If a voter is not chosen, he cannot influence the outcome.
    - If a voter is chosen, his favorite alternative is chosen and he cannot benefit by lying.

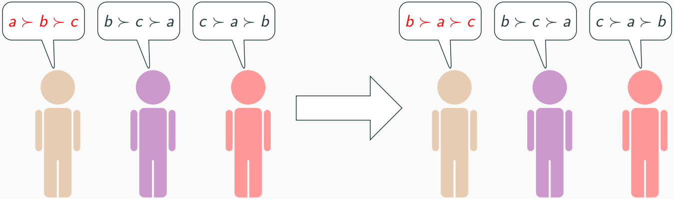
Strategyproofness for RSCFs

- How should we define strategyproofness in the presence of lotteries?
    - Voters only report ordinal preferences over the alternatives, but need to compare lotteries!
- We will assume that voters use vNM utility functions to compare lotteries by their expected utility!
- Recall from Week 1:
    - A vNM utility function u maps every alternative $x \in A$ to a numerical value $u(x) \in \mathbb{R}$.
    - The expected utility of a lottery p is $\mathbb{E}[u(p)] = \sum_{x \in A} p(x)u(x)$.
    - A vNM utility function u is consistent with a preference relation $\succ$ if $x \succ y \iff u(x) > u(y)$ for all $x, y \in A$.

Strategyproofness for RSCFs

- A voter i prefers lottery p to lottery q, denoted by $p \succsim_i q$, if $\mathbb{E}[u(p)] \geq \mathbb{E}[u(q)]$ for all utility functions u that are consistent with $\succ_i$.
  - Voters' preferences between lotteries are incomplete, i.e., there are lotteries that we cannot compare!
  - E.g.: if $a \succ_i b \succ_i c$, the lottery $p = [1 : b]$ and $q = [0.5 : a, 0.5 : c]$ are incomparable.
    - If $u(a) = 3$, $u(b) = 2$, $u(c) = 0$, then $\mathbb{E}[u(p)] = 2 > 1.5 = \mathbb{E}[u(q)]$.
    - If $u(a) = 3$, $u(b) = 1$, $u(c) = 0$, then $\mathbb{E}[u(p)] = 1 < 1.5 = \mathbb{E}[u(q)]$.
- An RSCF f is strategyproof if $f(R) \succsim_i f(R')$ for all preference profiles R, R' such that $\succ_j = \succ'_j$ for all $j \in N \setminus \{i\}$.
  - Every voter prefers the lottery obtained when voting honestly to every lottery he could obtain by lying about his preferences.
  - Voters cannot increase their expected utility for every utility function that is consistent with their true preferences.
  - An RSCF is manipulable if it is not strategyproof.

Strategyproofness for RSCFs - Example



$[\frac{1}{3} : a, \frac{1}{3} : b, \frac{1}{3} : c]$

$[\frac{2}{3} : b, \frac{1}{3} : c]$
$[1 : b]$

No manipulation! For every utility function u consistent with $a \succ b \succ c$:
$u(a) > u(b)$ implies that $\frac{1}{3}u(a) + \frac{1}{3}u(b) + \frac{1}{3}u(c) > \frac{2}{3}u(b) + \frac{1}{3}u(c)$

Manipulation! For the utility function u with $u(a) = 3$, $u(b) = 2$, $u(c) = 0$:
$\frac{1}{3}u(a) + \frac{1}{3}u(b) + \frac{1}{3}u(c) = \frac{5}{3} < 2 = 1u(b)$

The Random Dictatorship Theorem

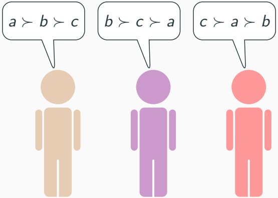
- An RSCF is unanimous if it selects an alternative with probability 1 whenever it is the favorite alternative of all voters.
- An RSCF f is a random dictatorship if there is a probability distribution $\alpha = (\alpha_1, \ldots, \alpha_n)$ over the voters such that, for each profile, f draws a voter from α and returns his favorite alternative.
  - Let $T_x(R)$ denote the set of voters who top-rank x in the profile R. The probability that a random dictatorship chooses x is $\sum_{i \in T_x(R)} \alpha_i$.
  - If $\alpha_i = 1$, then the corresponding random dictatorship picks the top ranked alternative of voter i.
  - If $\alpha_i = \frac{1}{n}$, we have the uniform random dictatorship.

Theorem (Gibbard, 1977)
Assume $|A| \geq 3$. An RSCF is strategyproof and unanimous if and only if it is a random dictatorship.

Maximal Lotteries

Condorcet Paradox

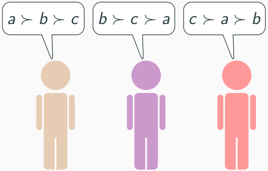
- Recall from last week: a Condorcet winner is an alternative that beats every other alternative in a pairwise majority comparison.
- Condorcet's paradox shows that Condorcet winners may not exist.
  - Condorcet's paradox is often considered the reason for all important impossibility theorems in social choice theory.



- Can we circumvent Condorcet's paradox by allowing for randomization?

Randomized Condorcet Winners

- Let $n_{xy}(R) = |\{i \in N \colon x \succ_i y\}|$ denote the number of voters who prefer x to y in R.
- An alternative x is a weak Condorcet winner in R if $n_{xy}(R) \geq n_{yx}(R)$ for all alternatives $y \in A$.
- Let $n_{pq}(R) = \sum_{x \in A} \sum_{y \in A} p(x)q(y)n_{xy}(R)$ denote the expected number of voters in R that prefer an alternative drawn from p to an alternative drawn from q.
- A lottery p is a randomized Condorcet winner in R if $n_{pq}(R) \geq n_{qp}(R)$ for all lotteries $q \in \Delta(A)$.



No weak Condorcet winner

The lottery p with $p(a) = p(b) = p(c) = \frac{1}{3}$ is the (unique) randomized Condorcet winner:

$$\begin{aligned} n_{pq}(R) - n_{qp}(R) &= p(a)q(b) + p(b)q(c) + p(c)q(a) \\ &\quad - p(a)q(c) - p(b)q(a) - p(c)q(b) \\ &= \tfrac{1}{3}(q(a) + q(b) + q(c)) - \tfrac{1}{3}(q(a) + q(b) + q(c) \\ &= 0 \end{aligned}$$

Randomized Condorcet Winners

- Theorem (Fishburn, 1984): A randomized Condorcet winner is guaranteed to exist!
  - Follows from the minimax theorem of von Neumann.
  - Randomized Condorcet winners are also called maximal lotteries.
- We define by $ML(R)$ the set of maximal lotteries/ randomized Condorcet winners in R.
  - $ML(R)$ is always non-empty but not always a singleton.
  - There is a unique maximal lottery if the number of voters is odd.
- To check whether a lottery p is maximal, it suffices to compare it to degenerate lotteries:.
  - p is maximal iff $\sum_{x \in A} p(x)(n_{xy}(R) - n_{yx}(R)) \geq 0$ for all $y \in A$.
  - Computing a maximal lottery can be done via linear programming.

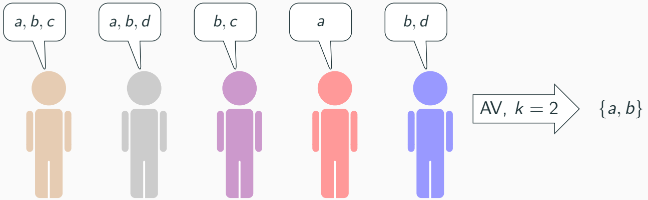
Maximal Lotteries

- An RSCF f is called a maximal lottery rule if $f(R) \in ML(R)$ for all profiles R.
- Maximal lottery rules satisfy many desirable properties:
  - Condorcet-consistency: if a deterministic and strict Condorcet winner exists, they always choose this alternative.
  - Clone-consistency: Cloning some alternatives does not change the probabilities assigned to other alternatives.
  - Mild forms of strategyproofness and participation: risk-averse voters cannot benefit by lying about their preferences or by abstaining from the election.
  - Very little randomization as Condorcet winners often exist in practice.

Approval-based Committee Elections

Approval-based Committee Elections

- Last week and before: we aim to choose a single winner.
- Now: we aim to choose a committee (i.e., a fixed-size set of alternatives).
  - Committee elections can be modeled as single-winner elections when voters report preferences over all committees.
  - Since this is impractical, we assume that voters only report approval ballots that indicate the alternatives they like.



Approval-based Committee Elections

Why should we study committee elections?

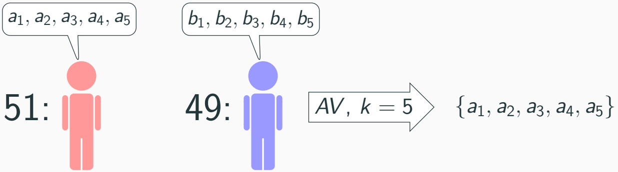
- Elections of parliaments, city councils, committees for various tasks in companies
- Short-listing of job applicants, competitors in some competition.
- Proof-of-Stake in blockchains
- Recommender systems (e.g., given your data, suggest the three next movies to watch).

Approval-based Committee Elections - The Model

- Finite set of voters $N = \{1, \ldots, n\}$
- Finite set of alternatives $A = \{x_1, \ldots, x_m\}$
- Each voter i reports an approval ballot A_i, which are formally non-empty subsets over A.
- An approval profile $\boldsymbol{\mathcal{A}} = (A_1, \ldots, A_n)$ contains the approval ballots of all voters.
- A size-k committee W is a subset of A with $|W| = k$.
- An approval-based committee (ABC) voting rule maps every profile $\mathcal{A}$ and target committee size k to a (set of) size-k committee(s).

Multiwinner Approval Voting

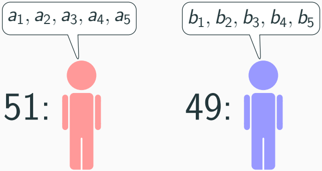
- Multiwinner approval voting (AV) chooses the k-candidates that are approved by the most voters.
- Formally, $AV(\mathcal{A}, k) = \max_{W \subseteq A \colon |W|=k} \sum_{i \in N} |A_i \cap W|$.
- AV is unfair in the sense that a large group of voters can fully specify the outcome.



Fairness Axioms

- Idea: Large groups of voters with consistent preferences deserve some representation.
- A size-k committee W satisfies
  - justified representation (JR) for an approval profile $\boldsymbol{\mathcal{A}}$ if there is no group of voters $G \subseteq N$ such that $|\bigcap_{i \in G} A_i| \geq 1$, $|G| \geq \frac{|N|}{k}$, and $|W \cap \bigcup_{i \in G} A_i| = 0$.
  - proportional justified representation (PJR) for an approval profile $\boldsymbol{\mathcal{A}}$ if there is no group of voters $G \subseteq N$ and integer $\ell \in \mathbb{N}$ such that $|\bigcap_{i \in G} A_i| \geq \ell$, $|G| \geq \frac{\ell|N|}{k}$, and $|W \cap \bigcup_{i \in G} A_i| < \ell$.
  - extended justified representation (EJR) for an approval profile $\boldsymbol{\mathcal{A}}$ if there is no group of voters $G \subseteq N$ and integer $\ell \in \mathbb{N}$ such that $|\bigcap_{i \in G} A_i| \geq \ell$, $|G| \geq \frac{\ell|N|}{k}$, and $|W \cap A_i| < \ell$ for all $i \in G$.
- An ABC voting rule f satisfies JR/ PJR/ EJR if $f(A, k)$ satisfies JR/ PJR/ EJR for all approval profiles A and all committee sizes k.

Fairness Axioms



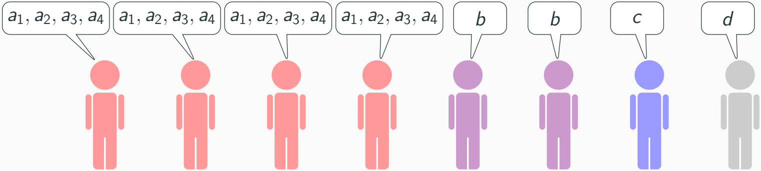
- The committee $\{a_1, a_2, a_3, a_4, a_5\}$ fails JR (and thus PJR and EJR).
- The committee $\{a_1, a_2, a_3, a_4, b_5\}$ satisfies JR but fails PJR (and EJR).
- The committee $\{a_1, a_2, a_3, b_4, b_5\}$ satisfies EJR (and thus PJR and JR).
- The committee $\{a_1, a_2, b_3, b_4, b_5\}$ satisfies EJR (and thus PJR and JR).

Thiele Rules

- Thiele rules are defined by a scoring function $s : \mathbb{N}_0 \to \mathbb{R}$ and return the size-k committee W that maximizes $\sum_{i \in N} s(|A_i \cap W|)$, i.e., $f(\mathcal{A}, k) = \max_{W \subseteq A\colon |W|=k} \sum_{i \in N} s(|A_i \cap W|)$.
  - Thiele rules are the equivalent of positional scoring rules for ABC elections.

- Multiwinner approval voting (AV): $s_{AV}(x) = x$.
  - Maximize the number of total approvals
  - "excellence-oriented"
- Chamberlin-Courant approval voting ($CCAV$): $s_{CCAV}(x) = 1$ for all $x > 0$ and $s_{CCAV}(0) = 0$.
  - Maximize the number of voters that approve an elected alternative
  - "diversity-oriented'
- Proportional approval voting (PAV): $s_{PAV}(x) = \sum_{y=1}^{x} \frac{1}{y}$ for all $x > 0$ and $s_{PAV}(0) = 0$.
  - Idea of diminishing returns
  - "proportionality-oriented"

Thiele Rules - Example



- $AV(\mathcal{A}, 4) = W_1 = \{a_1, a_2, a_3, a_4\}$
  - $\sum_{i \in N} s_{AV}(|A_i \cap W_1|) = 4 \cdot 4 + 4 \cdot 0 = 16$
  - W_1 fails JR as the purple voters reporting deserve representation.
- $CCAV(\mathcal{A}, 4) = W_2 = \{a_1, b, c, d\}$
  - $\sum_{i \in N} s_{CCAV}(|A_i \cap W_2|) = 8 \cdot 1 = 8.$
  - W_2 satisfies JR but fails PJR since the red voters deserve to be represented by two committee members.
- $PAV(\mathcal{A}, 4) = W_3 = \{a_1, a_2, a_3, b\}$
  - $\sum_{i \in N} s_{PAV}(|A_i \cap W_2|) = 4 \cdot \sum_{y=1}^{3} \frac{1}{y} + 2 \cdot 1 + 2 \cdot 0 = \frac{28}{3}.$
  - W_3 satisfies EJR.

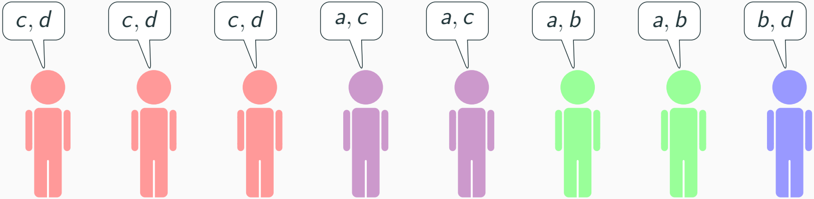
Thiele Rules - Properties

- Theorem: *PAV* is the only Thiele rule satisfying EJR.
- *CCAV* satisfies JR, *AV* satisfies no fairness notion.
- All Thiele rules but *AV*
  - are NP-hard to compute
  - fail committee-monotonicity: increasing the target committee size by 1 can completely change the outcome.
  - fail strategyproofness: voters can increase their number of approved committee members by lying about their approval ballot.
- A straightforward fix to the first two problems are sequential Thiele rules, which iteratively add the alternative to the committee that increases the score the most.
  - Let $W^0 = \emptyset$ and $W^{i+1} = \max_{x \in A \setminus W^i} \sum_{i \in N} s(|A_i \cap (W^i \cup \{x\})|)$ for $i \in \mathbb{N}$. The sequential Thiele rule f induced by s is defined by $f(A, k) = W^k$.
  - No sequential Thiele rule satisfies JR.

Phragmén's Rule

- Idea: Voters continuously earn money and buy alternatives once they can afford them.
  - The cost of each alternative is 1.
  - Each voter i has a budget $b_i(t)$ which is initially 0 and increases at unit rate (i.e., $b_i(t) = t$ unless agents spend their budget).
  - An alternative x is bought (and added to the winning committee) once $\sum_{i \in N:\, x \in A_i} b_i(t) = 1$. When we buy x, we set the budget of each voter approving x to 0.
  - We run this process until k alternatives have been bought.

Phragmén's Rule



- Assume $k = 3$. The initial budget vector is $b(0) = (0, 0, 0, 0, 0, 0, 0, 0)$.
- At $t = \frac{1}{5}$, $b(t) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$. We select alternative c and the new budget vector is $b(t) = (0, 0, 0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.
- At $t = \frac{1}{3} = \frac{1}{5} + \frac{2}{15}$, $b(t) = (\frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We select alternative b and the new budget vector is $b(t) = (\frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, 0, 0, 0)$.
- At $t = \frac{29}{60} = \frac{1}{3} + \frac{9}{60}$, $b(t) = (\frac{17}{60}, \frac{17}{60}, \frac{17}{60}, \frac{17}{60}, \frac{17}{60}, \frac{9}{60}, \frac{9}{60}, \frac{9}{60})$. We select alternative d and the new budget vector is $b(t) = (0, 0, 0, \frac{17}{60}, \frac{17}{60}, \frac{9}{60}, \frac{9}{60}, 0)$.
- The winning committee is $\{b, c, d\}$.

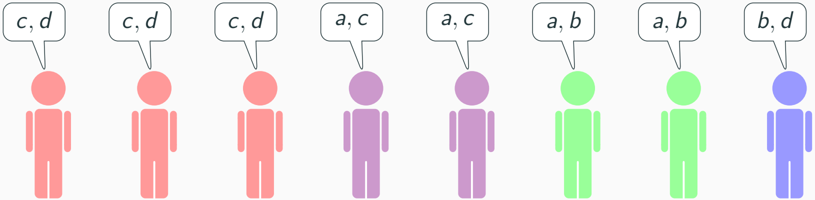
Phragmén's Rule

- Phragmén's rule satisfies PJR but not EJR.
- It is committee monotone: all candidates that are selected for the target committee size k are also selected for the target committee size $k + 1$.
- It can be computed in polynomial time.
- It is not strategyproof.

Method of Equal Shares

- Recently suggested by Peters and Skowron (2020).
- Idea: similar as for Phragmén's rule, but now voters start with a budget.
    - The cost of each alternative is 1; the budget of each voter i initially is $b_i = \frac{k}{n}$.
    - Identify the alternative x that is not yet selected and that minimizes the value ρ such that $\sum_{i \in N\colon x \in A_i} \min(\rho, b_i) = 1$.
    - Add x to the winning committee, reduce the budget b_i of each voter i who approves x by $\min(\rho, b_i)$.
    - Repeat until k alternatives are selected.
- The Method of Equal Shares (*MES*) may not be able to select k candidates. In this case, we complete the committee by running Phragmén's rule, where the remaining budget of *MES* are used as the starting budget of *Phragmen*.

Method of Equal Shares



- Assume $k = 3$. The initial budget vector is $b = (\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8})$.
- Alternative c gets selected for $\rho = \frac{1}{5}$. The new budget vector is $b = (\frac{7}{40}, \frac{7}{40}, \frac{7}{40}, \frac{7}{40}, \frac{7}{40}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8})$.
- Alternative a gets selected at $\rho = \frac{13}{40}$. The purple voters pay $\frac{7}{40}$ and the green voters pay $\frac{13}{40}$. The new budget vector is $b = (\frac{7}{40}, \frac{7}{40}, \frac{7}{40}, 0, 0, \frac{2}{40}, \frac{2}{40}, \frac{3}{8})$.
- No alternative is affordable. We start increasing the budgets of all voters uniformly and add the next affordable alternative. At $t = \frac{1}{40}$, d becomes affordable as $b = (\frac{8}{40}, \frac{8}{40}, \frac{8}{40}, \frac{1}{40}, \frac{1}{40}, \frac{3}{40}, \frac{3}{40}, \frac{16}{40})$.
- The winning committee is $\{a, c, d\}$.

Method of Equal Shares

- MES is the only known rule that satisfies EJR and that can be computed in polynomial time!
- It also satisfies further fairness notions such as priceability.
- MES does not satisfy committee-monotonicity.
- It is not strategyproof. No rule satisfying JR is strategyproof.
- The current literature often views MES as the best ABC voting rule regarding fairness.

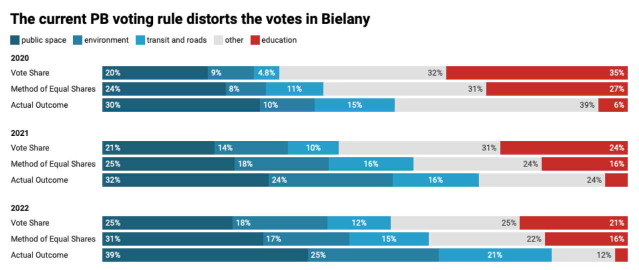
Outlook: Participatory Budgeting

Participatory Budgeting

- Participatory budgeting is a modern democratic decision-making process to decide how to spend a communal budget on various projects.
  - There is a maximum budget B and each project x comes with a cost $c(x)$.
  - The voters cast approval ballots over the projects to indicate the projects the support.
  - The goal is to choose a set of projects W such that $\sum_{x \in W} c(x) \leq B$ that fairly represents the voters' preferences.
  - Participatory budgeting generalizes ABC elections by introducing cost to alternatives!

Participatory Budgeting

- In practice, the standard rule for participatory budgeting is a greedy rule: sort the projects by the number of approvals, go through that list and add a project to the winning set if it is still affordable.
  - Direct adaption of (sequential) multi-winner approval voting.
  - Outcomes tend to be highly unfair.
- The Method of Equal Shares can be extended to participatory budgeting while preserving its desirable properties.
  - This method has recently been used in some elections and results in much fairer results.
  - See `https://equalshares.net` for more information.



Further Reading

Reading

The following books and articles:

- F. Brandt. Rolling the Dice: Recent Results in Probabilistic Social Choice. In: Trends in Computational Social Choice. AI Access. 2017.
  `https://pub.dss.in.tum.de/brandt-research/psc_chapter.pdf`
- M. Lackner and P. Skowron. Multi-Winner Voting with Approval Preferences. Springer. 2023.
  `https://library.oapen.org/bitstream/handle/20.500.12657/60149/978-3-031-09016-5.pdf?sequence=1&isAllowed=y`
- S. Rey and J. Maly. The (Computational) Social Choice Take on Indivisible Participatory Budgeting. 2024.
  `https://arxiv.org/abs/2303.00621`

Image References

- Slide 3: https://www.shutterstock.com/image-vector/lottery-machine-balls-inside-lotto-260nw-458501299.jpg
- Slide 10: https://www3.nd.edu/~pweithma/Justice%20Seminar%20Images/gibbard.jpg
- Slide 13: https://cdn.lifestorynet.com/obituaries/01c/136161/136161-00-2x.jpg
- Slide 20: https://upload.wikimedia.org/wikipedia/commons/thumb/7/75/Thiele1.jpg/400px-Thiele1.jpg
- Slide 23: https://media.springernature.com/lw685/springer-static/image/art%3A10.1007%2Fs10107-023-01926-8/MediaObjects/10107_2023_1926_Fig1_HTML.jpg
- Slide 30: https://equalshares.net/benefits/categories