Exercise Session: Game Theory II

COMP4418 Knowledge Representation and Reasoning

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These slides are based on lecture slides by Prof. Felix Brandt.

Exercise I: Computing Nash Equilibria

a) Find a Nash equilibrium in the following game.

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

Indifference principle: The row player must randomizes such that the column player is indifferent between all actions in his support.

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

Indifference principle: The row player must randomizes such that the column player is indifferent between all actions in his support.

Since no action is dominated, it must hold that

$$1 \cdot s(a) + 3 \cdot s(b) = 4 \cdot s(a) + 2 \cdot s(b).$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$1 \cdot s(a) + 3 \cdot s(b) = 4 \cdot s(a) + 2 \cdot s(b)$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$1 \cdot s(a) + 3 \cdot s(b) = 4 \cdot s(a) + 2 \cdot s(b)$$

$$s(a) + 3(1 - s(a)) = 4s(a) + 2(1 - s(a))$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$1 \cdot s(a) + 3 \cdot s(b) = 4 \cdot s(a) + 2 \cdot s(b)$$

$$s(a) + 3(1 - s(a)) = 4s(a) + 2(1 - s(a))$$

$$3 - 2s(a) = 2 + 2s(a)$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$1 \cdot s(a) + 3 \cdot s(b) = 4 \cdot s(a) + 2 \cdot s(b)$$
$$s(a) + 3(1 - s(a)) = 4s(a) + 2(1 - s(a))$$
$$3 - 2s(a) = 2 + 2s(a)$$
$$1 = 4s(a)$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$1 \cdot s(a) + 3 \cdot s(b) = 4 \cdot s(a) + 2 \cdot s(b)$$

$$s(a) + 3(1 - s(a)) = 4s(a) + 2(1 - s(a))$$

$$3 - 2s(a) = 2 + 2s(a)$$

$$1 = 4s(a)$$

$$s(a) = \frac{1}{4}$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2
$\frac{1}{4}a + \frac{3}{4}b$	$\frac{9}{4}$, $\frac{10}{4}$	$\frac{3}{4}$, $\frac{10}{4}$

$$\begin{aligned}
1 \cdot s(a) + 3 \cdot s(b) &= 4 \cdot s(a) + 2 \cdot s(b) \\
s(a) + 3(1 - s(a)) &= 4s(a) + 2(1 - s(a)) \\
3 - 2s(a) &= 2 + 2s(a) \\
1 &= 4s(a) \\
s(a) &= \frac{1}{4}
\end{aligned}$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$3 \cdot s(x) + 0 \cdot s(y) = 2 \cdot s(x) + 1 \cdot s(y)$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$3 \cdot s(x) + 0 \cdot s(y) = 2 \cdot s(x) + 1 \cdot s(y)$$
$$3s(x) = 2s(x) + 1 - s(x)$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$3 \cdot s(x) + 0 \cdot s(y) = 2 \cdot s(x) + 1 \cdot s(y)$$
$$3s(x) = 2s(x) + 1 - s(x)$$
$$2s(x) = 1$$

Exercise I: Computing Nash Equilibria

	x	y
a	3, 1	0, 4
b	2, 3	1, 2

$$
\begin{aligned}
3 \cdot s(x) + 0 \cdot s(y) &= 2 \cdot s(x) + 1 \cdot s(y) \\
3s(x) &= 2s(x) + 1 - s(x) \\
2s(x) &= 1 \\
s(x) &= \frac{1}{2}
\end{aligned}
$$

Exercise I: Computing Nash Equilibria

	x	y	$\frac{1}{2}x + \frac{1}{2}y$
a	3, 1	0, 4	$\frac{3}{2}$, $\frac{5}{2}$
b	2, 3	1, 2	$\frac{3}{2}$, $\frac{5}{2}$

$$
\begin{aligned}
3 \cdot s(x) + 0 \cdot s(y) &= 2 \cdot s(x) + 1 \cdot s(y) \\
3s(x) &= 2s(x) + 1 - s(x) \\
2s(x) &= 1 \\
s(x) &= \frac{1}{2}
\end{aligned}
$$

Exercise I: Computing Nash Equilibria

	x	y	$\frac{1}{2}x + \frac{1}{2}y$
a	3, 1	0, 4	$\frac{3}{2}$, $\frac{5}{2}$
b	2, 3	1, 2	$\frac{3}{2}$, $\frac{5}{2}$
$\frac{1}{4}a + \frac{3}{4}b$	$\frac{9}{4}$, $\frac{10}{4}$	$\frac{3}{4}$, $\frac{10}{4}$	$\frac{3}{2}$, $\frac{5}{2}$

$$
\begin{aligned}
3 \cdot s(x) + 0 \cdot s(y) &= 2 \cdot s(x) + 1 \cdot s(y) \\
3s(x) &= 2s(x) + 1 - s(x) \\
2s(x) &= 1 \\
s(x) &= \frac{1}{2}
\end{aligned}
$$

Exercise I: Computing Nash Equilibria

b) Find a Nash equilibrium in the following game.

	x	y	z
a	5, 3	2, 4	1, 3
b	2, 5	2, 5	2, 6
c	0, 2	0, 1	8, 0
d	1, 4	3, 2	6, 1

Exercise I: Computing Nash Equilibria

	x	y	z
a	5, 3	2, 4	1, 3
b	2, 5	2, 5	2, 6
c	0, 2	0, 1	8, 0
d	1, 4	3, 2	6, 1

Exercise I: Computing Nash Equilibria

	x	y	z
a	5, 3	2, 4	1, 3
b	2, 5	2, 5	2, 6
c	0, 2	0, 1	8, 0
d	1, 4	3, 2	6, 1
$\frac{1}{2}a + \frac{1}{2}d$	$\frac{6}{2}$, $\frac{7}{2}$	$\frac{5}{2}$, $\frac{6}{2}$	$\frac{7}{2}$, $\frac{7}{2}$

Exercise I: Computing Nash Equilibria

	x	y	z
a	5, 3	2, 4	1, 3
b	2, 5	2, 5	2, 6
c	0, 2	0, 1	8, 0
d	1, 4	3, 2	6, 1
$\frac{1}{2}a + \frac{1}{2}d$	$\frac{6}{2}$, $\frac{7}{2}$	$\frac{5}{2}$, $\frac{6}{2}$	$\frac{7}{2}$, $\frac{7}{2}$

Exercise I: Computing Nash Equilibria

	x	y	z
a	5, 3	2, 4	1, 3
c	0, 2	0, 1	8, 0
d	1, 4	3, 2	6, 1

Exercise I: Computing Nash Equilibria

	x	y	z
a	5, 3	2, 4	1, 3
c	0, 2	0, 1	8, 0
d	1, 4	3, 2	6, 1

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
c	0, 2	0, 1
d	1, 4	3, 2

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
c	0, 2	0, 1
d	1, 4	3, 2

Exercise I: Computing Nash Equilibria

	x	y
a	5 3	2 4
d	1 4	3 2

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
d	1, 4	3, 2

$$3 \cdot s(a) + 4 \cdot s(d) = 4 \cdot s(a) + 2 \cdot s(d)$$

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
d	1, 4	3, 2

$$3 \cdot s(a) + 4 \cdot s(d) = 4 \cdot s(a) + 2 \cdot s(d)$$

$$3s(a) + 4(1 - s(a)) = 4s(a) + 2(1 - s(a))$$

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
d	1, 4	3, 2

$$3 \cdot s(a) + 4 \cdot s(d) = 4 \cdot s(a) + 2 \cdot s(d)$$

$$3s(a) + 4(1 - s(a)) = 4s(a) + 2(1 - s(a))$$

$$4 - s(a) = 2 + 2s(a)$$

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
d	1, 4	3, 2
$\frac{2}{3}a + \frac{1}{3}d$	$\frac{11}{3}$, $\frac{10}{3}$	$\frac{9}{3}$, $\frac{10}{3}$

$$3 \cdot s(a) + 4 \cdot s(d) = 4 \cdot s(a) + 2 \cdot s(d)$$

$$3s(a) + 4(1 - s(a)) = 4s(a) + 2(1 - s(a))$$

$$4 - s(a) = 2 + 2s(a)$$

$$s(a) = \frac{2}{3}$$

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
d	1, 4	3, 2

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
d	1, 4	3, 2

$$5 \cdot s(x) + 2 \cdot s(y) = 1 \cdot s(x) + 3 \cdot s(y)$$

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
d	1, 4	3, 2

$$5 \cdot s(x) + 2 \cdot s(y) = 1 \cdot s(x) + 3 \cdot s(y)$$
$$5s(x) + 2(1 - s(x)) = 1s(x) + 3(1 - s(x))$$

Exercise I: Computing Nash Equilibria

	x	y
a	5, 3	2, 4
d	1, 4	3, 2

$$5 \cdot s(x) + 2 \cdot s(y) = 1 \cdot s(x) + 3 \cdot s(y)$$

$$5s(x) + 2(1 - s(x)) = 1s(x) + 3(1 - s(x))$$

$$2 + 3(x) = 3 - 2s(x)$$

Exercise I: Computing Nash Equilibria

	x	y	$\frac{1}{5}x + \frac{4}{5}y$
a	5, 3	2, 4	$\frac{13}{5}$, $\frac{19}{5}$
d	1, 4	3, 2	$\frac{13}{5}$, $\frac{12}{5}$

$$5 \cdot s(x) + 2 \cdot s(y) = 1 \cdot s(x) + 3 \cdot s(y)$$
$$5s(x) + 2(1 - s(x)) = 1s(x) + 3(1 - s(x))$$
$$2 + 3(x) = 3 - 2s(x)$$
$$s(x) = \frac{1}{5}$$

Exercise I: Computing Nash Equilibria

	x	y	$\frac{1}{5}x + \frac{4}{5}y$
a	5, 3	2, 4	$\frac{13}{5}$, $\frac{19}{5}$
d	1, 4	3, 2	$\frac{13}{5}$, $\frac{12}{5}$
$\frac{2}{3}a + \frac{1}{3}d$	$\frac{11}{3}$, $\frac{10}{3}$	$\frac{9}{3}$, $\frac{10}{3}$	$\frac{13}{5}$, $\frac{50}{15}$

$$5 \cdot s(x) + 2 \cdot s(y) = 1 \cdot s(x) + 3 \cdot s(y)$$

$$5s(x) + 2(1 - s(x)) = 1s(x) + 3(1 - s(x))$$

$$2 + 3(x) = 3 - 2s(x)$$

$$s(x) = \frac{1}{5}$$

Exercise I: Computing Nash Equilibria

c) Find a Nash equilibrium in the following zero-sum game.

	x	y	z
a	2	1	0
b	1	3	5
c	-2	4	2

Exercise I: Computing Nash Equilibria

	x	y	z
a	2	1	0
b	1	3	5
c	-2	4	2

Exercise I: Computing Nash Equilibria

	x	y	z
a	2	1	0
b	1	3	5
c	-2	4	2

$$\max u$$

$$\text{subject to } 2 \cdot s(a) + 1 \cdot s(b) - 2 \cdot s(c) \geq u \quad (1)$$

$$1 \cdot s(a) + 3 \cdot s(b) + 4 \cdot s(c) \geq u \quad (2)$$

$$0 \cdot s(a) + 5 \cdot s(b) + 2 \cdot s(c) \geq u \quad (3)$$

$$s(a) + s(b) + s(c) = 1$$

$$s(a) \geq 0, s(b) \geq 0, s(c) \geq 0$$

Exercise I: Computing Nash Equilibria

	x	y	z
a	2	1	0
b	1	3	5
c	-2	4	2

$$\begin{aligned} &\max u \\ \text{subject to } & 2 \cdot s(a) + 1 \cdot s(b) - 2 \cdot s(c) \geq u && (1)\\ & 1 \cdot s(a) + 3 \cdot s(b) + 4 \cdot s(c) \geq u && (2)\\ & 0 \cdot s(a) + 5 \cdot s(b) + 2 \cdot s(c) \geq u && (3)\\ & s(a) + s(b) + s(c) = 1 \\ & s(a) \geq 0, s(b) \geq 0, s(c) \geq 0 \end{aligned}$$

Exercise I: Computing Nash Equilibria

$$
\begin{aligned}
&\max u \\
\text{subject to } \; & 2 \cdot s(a) + 1 \cdot s(b) - 2 \cdot s(c) \geq u && (1) \\
& 1 \cdot s(a) + 3 \cdot s(b) + 4 \cdot s(c) \geq u && (2) \\
& 0 \cdot s(a) + 5 \cdot s(b) + 2 \cdot s(c) \geq u && (3) \\
& s(a) + s(b) + s(c) = 1 \\
& s(a) \geq 0, s(b) \geq 0, s(c) \geq 0
\end{aligned}
$$

Exercise I: Computing Nash Equilibria

$$\begin{aligned}
&\max u \\
&\text{subject to } 2 \cdot s(a) + 1 \cdot s(b) - 2 \cdot s(c) \geq u && (1)\\
&1 \cdot s(a) + 3 \cdot s(b) + 4 \cdot s(c) \geq u && (2)\\
&0 \cdot s(a) + 5 \cdot s(b) + 2 \cdot s(c) \geq u && (3)\\
&2 \cdot s(a) + 6 \cdot s(b) + 0 \cdot s(c) \geq 2u && (4)\\
&s(a) + s(b) + s(c) = 1 \\
&s(a) \geq 0, s(b) \geq 0, s(c) \geq 0
\end{aligned}$$

Exercise I: Computing Nash Equilibria

$$
\begin{aligned}
& \max u \\
\text{subject to } & 2 \cdot s(a) + 1 \cdot s(b) - 2 \cdot s(c) \geq u && (1) \\
& 1 \cdot s(a) + 3 \cdot s(b) + 4 \cdot s(c) \geq u && (2) \\
& 0 \cdot s(a) + 5 \cdot s(b) + 2 \cdot s(c) \geq u && (3) \\
& \color{red}{1 \cdot s(a) + 3 \cdot s(b) + 0 \cdot s(c) \geq u} && (4) \\
& s(a) + s(b) + s(c) = 1 \\
& s(a) \geq 0, s(b) \geq 0, s(c) \geq 0
\end{aligned}
$$

Exercise I: Computing Nash Equilibria

$$
\begin{aligned}
&\max u \\
\text{subject to } & 2 \cdot s(a) + 1 \cdot s(b) - 2 \cdot s(c) \geq u && (1) \\
& 0 \cdot s(a) + 5 \cdot s(b) + 2 \cdot s(c) \geq u && (3) \\
& 1 \cdot s(a) + 3 \cdot s(b) + 0 \cdot s(c) \geq u && (4) \\
& s(a) + s(b) + s(c) = 1 \\
& s(a) \geq 0, s(b) \geq 0, s(c) \geq 0
\end{aligned}
$$

Exercise I: Computing Nash Equilibria

$$\begin{aligned}
&\max u \\
&\text{subject to } 2 \cdot s(a) + 1 \cdot s(b) - 2 \cdot s(c) \geq u && (1) \\
&0 \cdot s(a) + 5 \cdot s(b) + 2 \cdot s(c) \geq u && (3) \\
&s(a) + s(b) + s(c) = 1 \\
&s(a) \geq 0, s(b) \geq 0, s(c) \geq 0
\end{aligned}$$

Exercise I: Computing Nash Equilibria

$$
\begin{aligned}
& \max u \\
\text{subject to } & 2 \cdot s(a) + 1 \cdot s(b) \geq u && (1) \\
& 0 \cdot s(a) + 5 \cdot s(b) \geq u && (3) \\
& s(a) + s(b) = 1 \\
& s(a) \geq 0, s(b) \geq 0, s(c) = 0
\end{aligned}
$$

Exercise I: Computing Nash Equilibria

$$\max u$$

$$\text{subject to } 2 \cdot s(a) + 1 - 1s(a) \geq u \tag{1}$$

$$0 \cdot s(a) + 5 - 5s(a) \geq u \tag{3}$$

$$0 \leq s(a) \leq 1$$

Exercise I: Computing Nash Equilibria

$$
\begin{aligned}
&\max u \\
&\text{subject to } 1 + 1s(a) \geq u && (1)\\
&5 - 5s(a) \geq u && (3)\\
&0 \leq s(a) \leq 1
\end{aligned}
$$

Exercise I: Computing Nash Equilibria

If $s(a) = \frac{2}{3}$, then $u = \frac{5}{3}$ since

$$1 + \frac{2}{3} = \frac{5}{3} \tag{1}$$

$$5 - 5 \cdot \frac{2}{3} = \frac{5}{3} \tag{3}$$

Exercise I: Computing Nash Equilibria

If $s(a) = \frac{2}{3}$, then $u = \frac{5}{3}$ since

$$1 + \frac{2}{3} = \frac{5}{3} \tag{1}$$

$$5 - 5 \cdot \frac{2}{3} = \frac{5}{3} \tag{3}$$

The maximin strategy of the row player is $s(a) = \frac{2}{3}$, $s(b) = \frac{1}{3}$, and $s(c) = 0$.

Exercise I: Computing Nash Equilibria

	x	y	z
a	2	1	0
b	1	3	5
c	-2	4	2
$\frac{2}{3}a + \frac{1}{3}b$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

Exercise I: Computing Nash Equilibria

	x	y	z
a	2	1	0
b	1	3	5
c	-2	4	2

Exercise I: Computing Nash Equilibria

	x	y	z
a	2	1	0
b	1	3	5
c	-2	4	2

$$-2 \cdot s(x) - 1 \cdot s(y) + 0 \cdot s(z) \geq -\frac{5}{3} \tag{1}$$

$$-1 \cdot s(x) - 3 \cdot s(y) - 5 \cdot s(z) \geq -\frac{5}{3} \tag{2}$$

$$2 \cdot s(x) - 4 \cdot s(y) - 2 \cdot s(z) \geq -\frac{5}{3} \tag{3}$$

$$s(x) + s(y) + s(z) = 1$$

$$s(x) \geq 0, s(y) \geq 0, s(z) \geq 0$$

Exercise I: Computing Nash Equilibria

	x	y	z
a	2	1	0
b	1	3	5
c	-2	4	2

$$2 \cdot s(x) + 1 \cdot s(y) - 0 \cdot s(z) \leq \frac{5}{3} \tag{1}$$

$$1 \cdot s(x) + 3 \cdot s(y) + 5 \cdot s(z) \leq \frac{5}{3} \tag{2}$$

$$-2 \cdot s(x) + 4 \cdot s(y) + 2 \cdot s(z) \leq \frac{5}{3} \tag{3}$$

$$s(x) + s(y) + s(z) = 1$$

$$s(x) \geq 0, s(y) \geq 0, s(z) \geq 0$$

Exercise I: Computing Nash Equilibria

$$2 \cdot s(x) + 1 \cdot s(y) - 0 \cdot s(z) \leq \frac{5}{3} \tag{1}$$

$$1 \cdot s(x) + 3 \cdot s(y) + 5 \cdot s(z) \leq \frac{5}{3} \tag{2}$$

$$-2 \cdot s(x) + 4 \cdot s(y) + 2 \cdot s(z) \leq \frac{5}{3} \tag{3}$$

$$s(x) + s(y) + s(z) = 1$$

$$s(x) \geq 0, s(y) \geq 0, s(z) \geq 0$$

Exercise I: Computing Nash Equilibria

$$2 \cdot s(x) + 1 \cdot s(y) - 0 \cdot s(z) \leq \frac{5}{3} \tag{1}$$

$$1 \cdot s(x) + 3 \cdot s(y) + 5 \cdot s(z) \leq \frac{5}{3} \tag{2}$$

$$-2 \cdot s(x) + 4 \cdot s(y) + 2 \cdot s(z) \leq \frac{5}{3} \tag{3}$$

$$s(x) + s(y) + s(z) = 1$$

$$s(x) \geq 0, s(y) \geq 0, s(z) \geq 0$$

It can be checked that $s(x) = \frac{2}{3}$ and $s(y) = \frac{1}{3}$ satisfies the conditions.

Exercise I: Computing Nash Equilibria

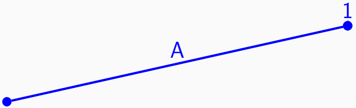
	x	y	z	$\frac{2}{3}x + \frac{1}{3}y$
a	2	1	0	$\frac{5}{3}$
b	1	3	5	$\frac{5}{3}$
c	-2	4	2	0
$\frac{2}{3}a + \frac{1}{3}b$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

Exercise II: Extensive-Form Games

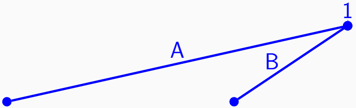
a) Alice (1), Bob (2), and Charlie (3) want to go on a trip together. Their possible destinations are Adelaide (A), Brisbane (B), Canberra (C), and Darwin (D). To decide on a final destination, they agree on an elimination process: Alice, Bob, and Charlie (in this order) each get to veto one of the cities and the last city will be their destination.

   Draw the game tree for this scenario.

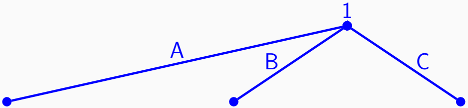
Exercise II: Extensive-Form Games



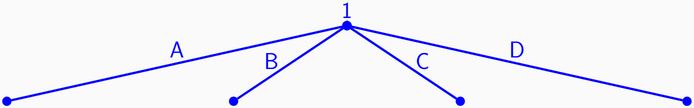
Exercise II: Extensive-Form Games



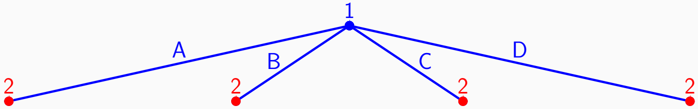
Exercise II: Extensive-Form Games



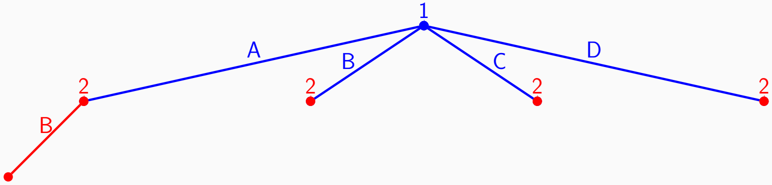
Exercise II: Extensive-Form Games



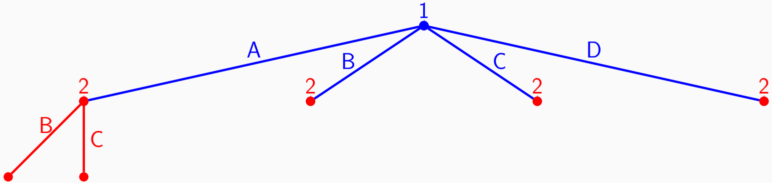
Exercise II: Extensive-Form Games



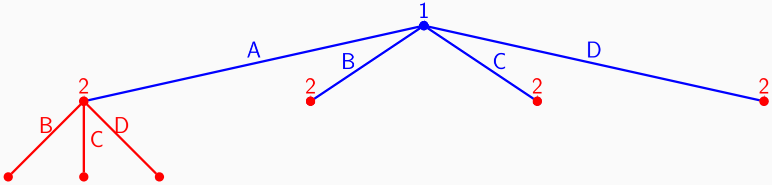
Exercise II: Extensive-Form Games



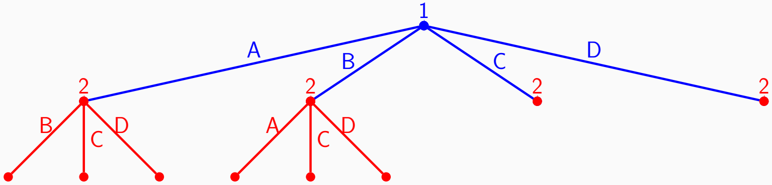
Exercise II: Extensive-Form Games



Exercise II: Extensive-Form Games



Exercise II: Extensive-Form Games



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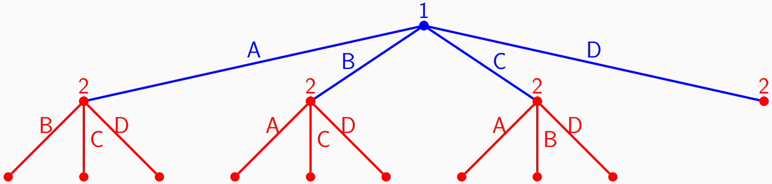

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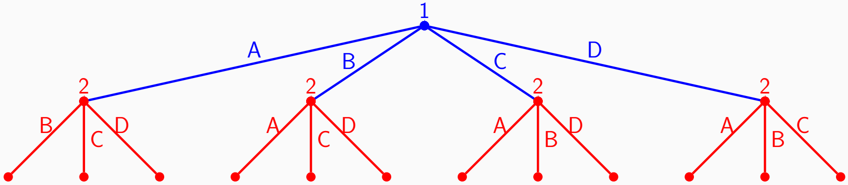

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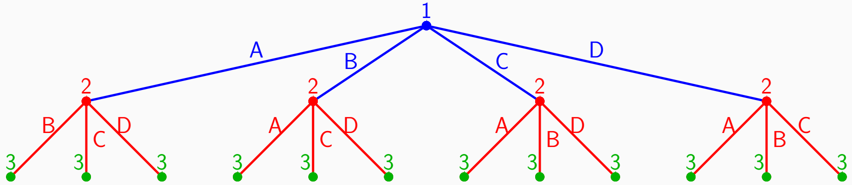

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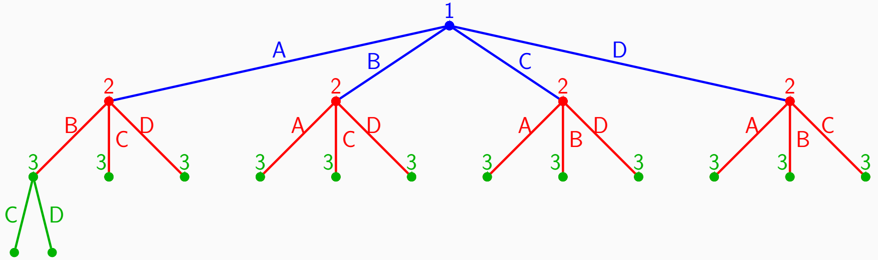

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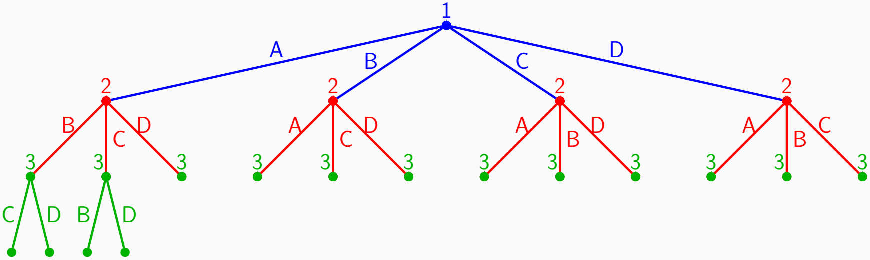
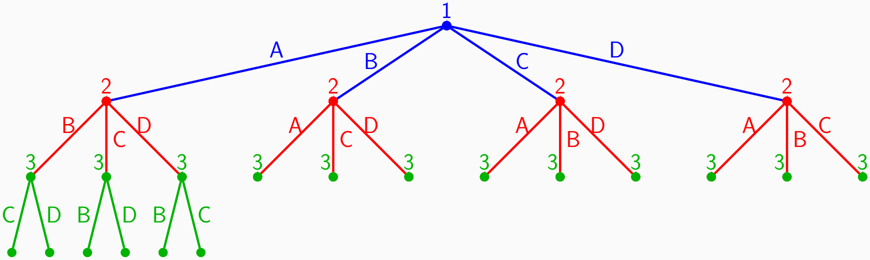
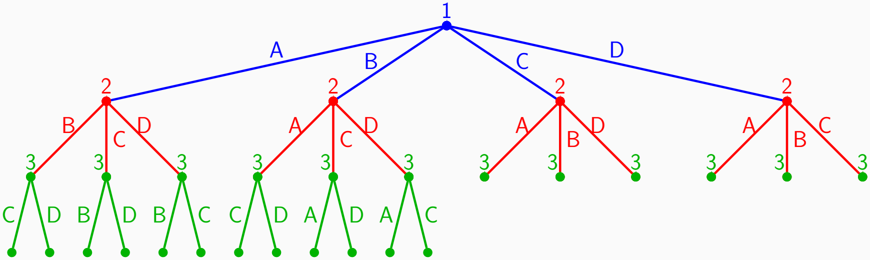


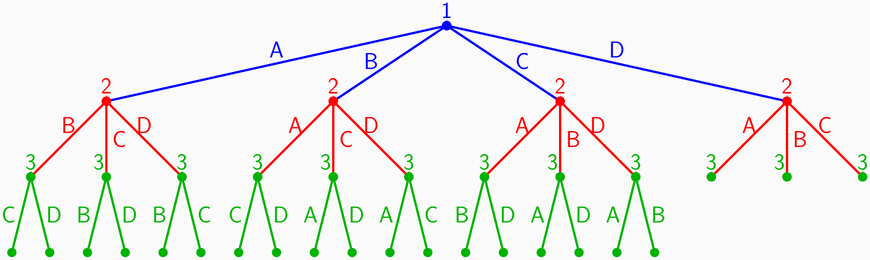
Exercise II: Extensive-Form Games



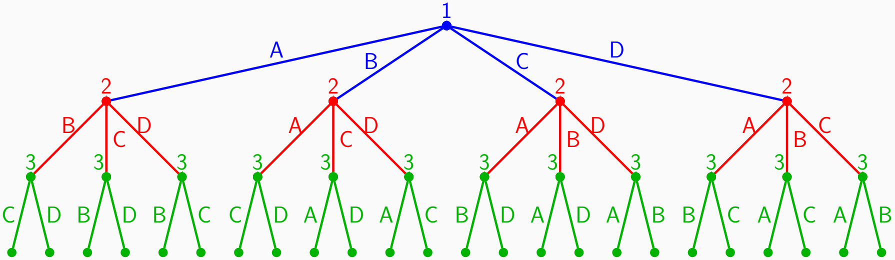
Exercise II: Extensive-Form Games



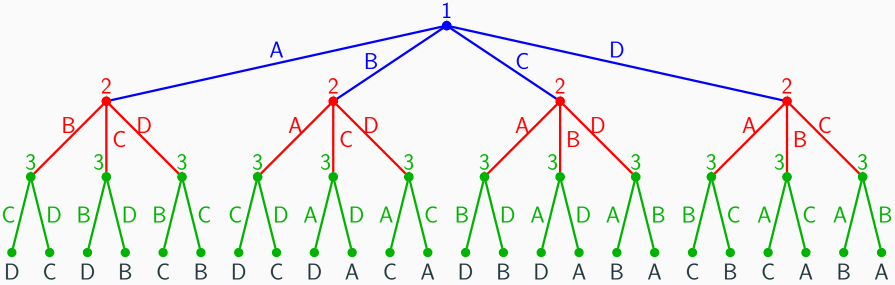
Exercise II: Extensive-Form Games



Exercise II: Extensive-Form Games



Exercise II: Extensive-Form Games



Exercise II: Extensive-Form Games

b) Assume that

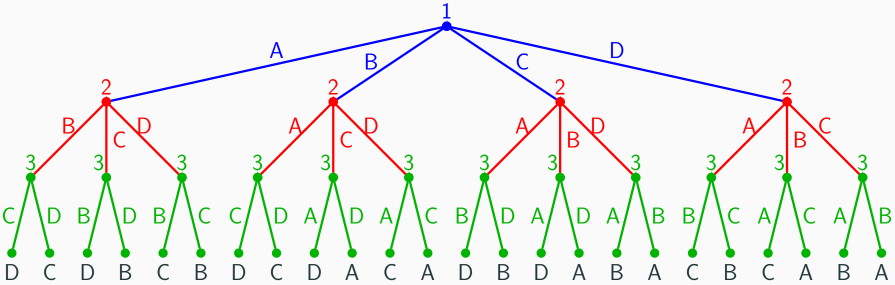
- Alice prefers A to C to B to D,
- Bob prefers B to D to A to C,
- Charlie prefers A to D to B to C.

Compute a subgame-perfect Nash equilibrium.

Exercise II: Extensive-Form Games

- Alice prefers A to C to B to D,
- Bob prefers B to D to A to C,
- Charlie prefers A to D to B to C.

	A	B	C	D
u_1	3	1	2	0
u_2	1	3	0	2
u_3	3	1	0	2



Exercise II: Extensive-Form Games

- Alice prefers A to C to B to D,
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	A	B	C	D
u_1	3	1	2	0
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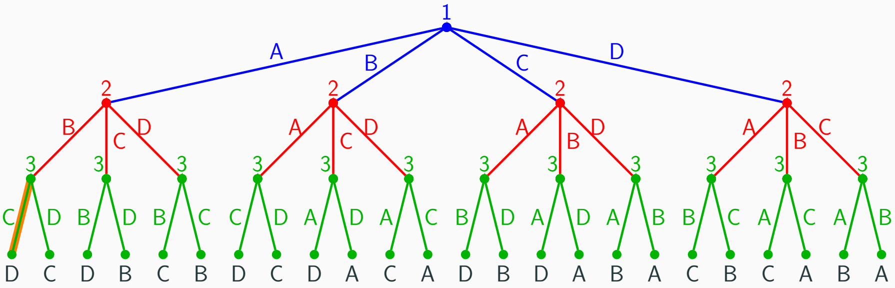
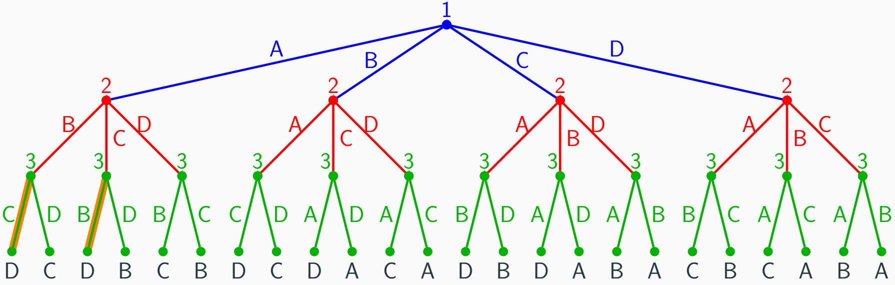


Exercise II: Extensive-Form Games

- Alice prefers A to C to B to D,
- Bob prefers B to D to A to C,
- Charlie prefers A to D to B to C.

	A	B	C	D
u_1	3	1	2	0
u_2	1	3	0	2
u_3	3	1	0	2



Exercise II: Extensive-Form Games

- Alice prefers A to C to B to D,
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	A	B	C	D
u_1	3	1	2	0
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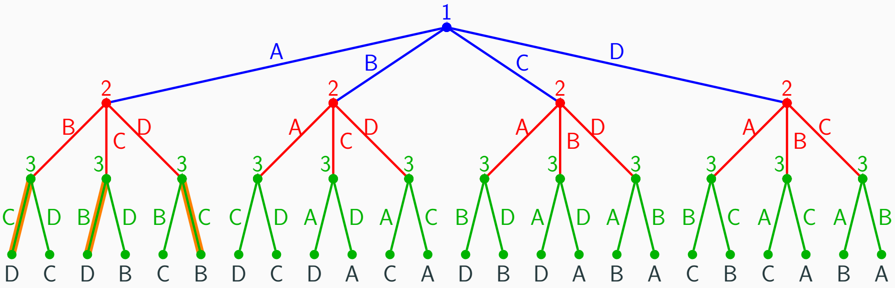

Exercise II: Extensive-Form Games

- Alice prefers A to C to B to D,
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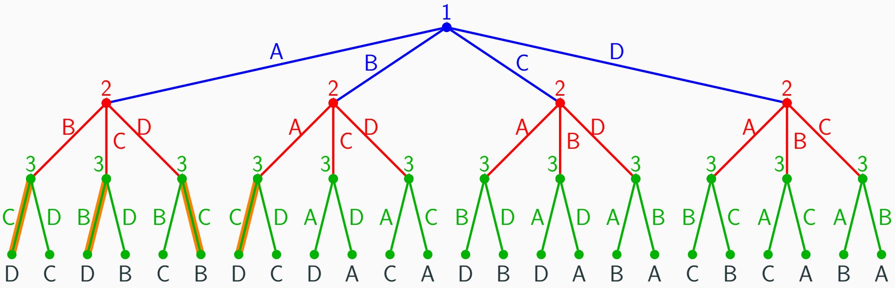
	A	B	C	D
u_1	3	1	2	0
u_2	1	3	0	2
u_3	3	1	0	2



Exercise II: Extensive-Form Games

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	A	B	C	D
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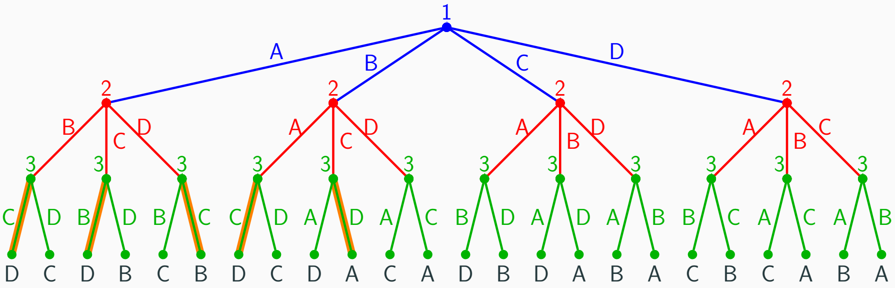

Exercise II: Extensive-Form Games

- Alice prefers A to C to B to D,
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	A	B	C	D
u_1	3	1	2	0
u_2	1	3	0	2
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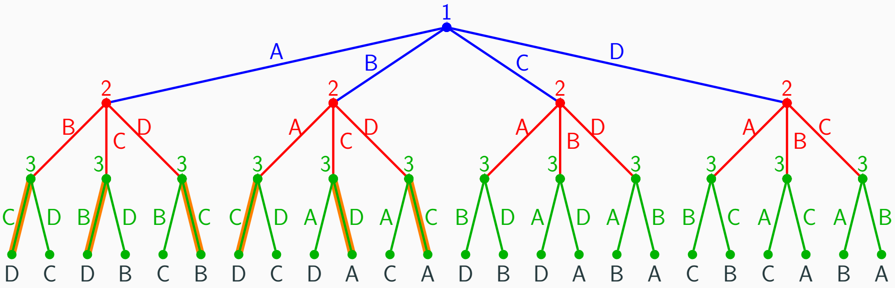

Exercise II: Extensive-Form Games

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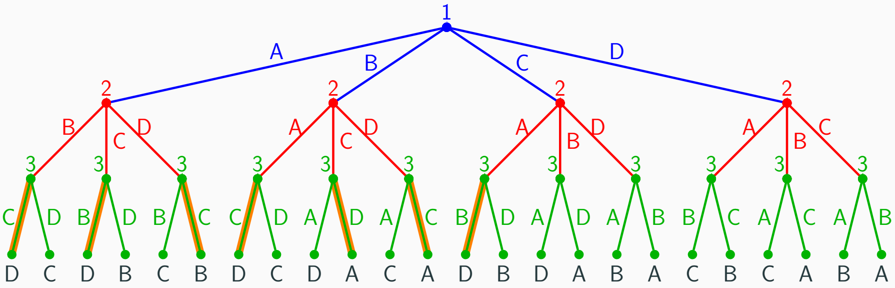

Exercise II: Extensive-Form Games

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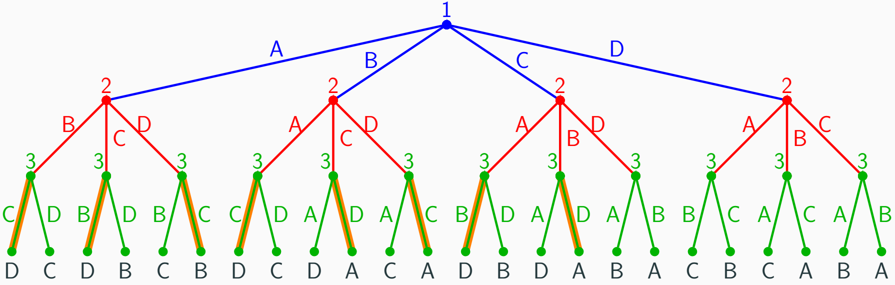

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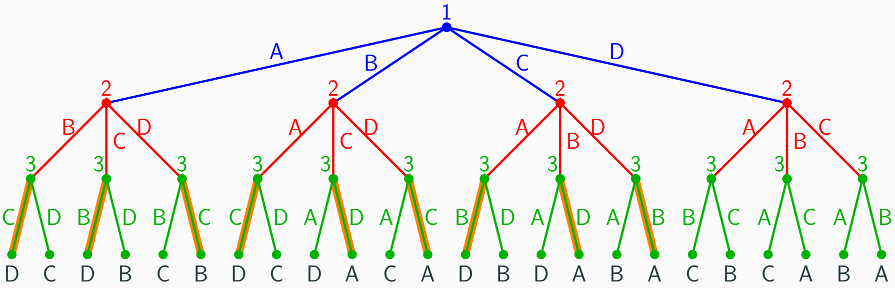

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u_1	3	1	2	0
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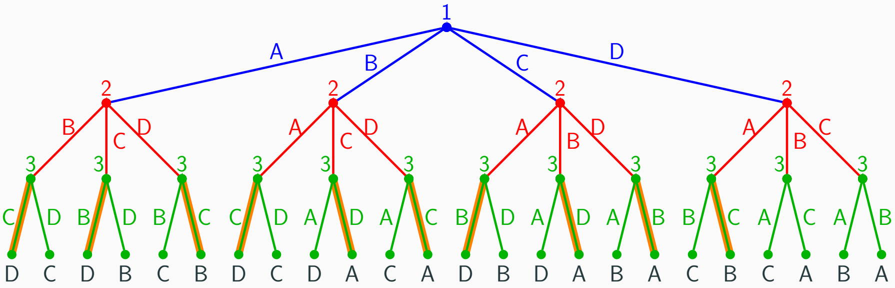

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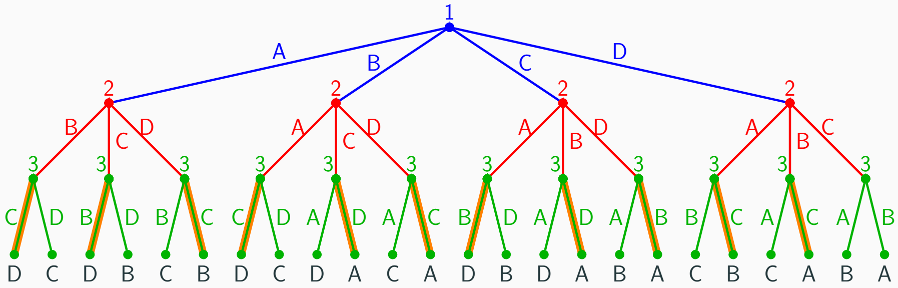

Exercise II: Extensive-Form Games

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u_1	3	1	2	0
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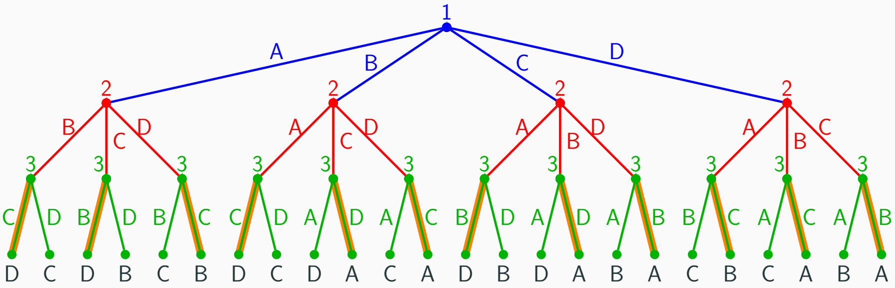

Exercise II: Extensive-Form Games

- Alice prefers A to C to B to D,
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- Charlie prefers A to D to B to C.

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u_1	3	1	2	0
u_2	1	3	0	2
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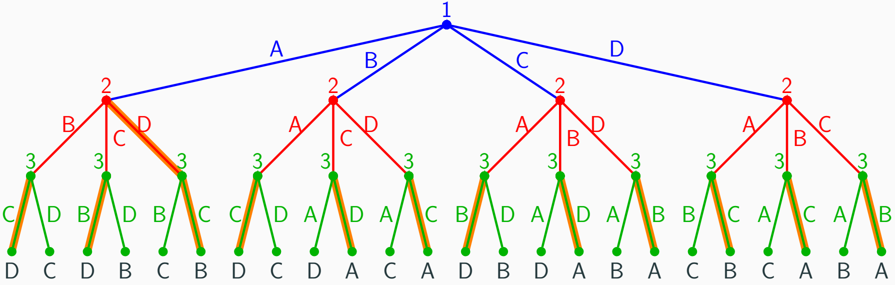

Exercise II: Extensive-Form Games

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u_1	3	1	2	0
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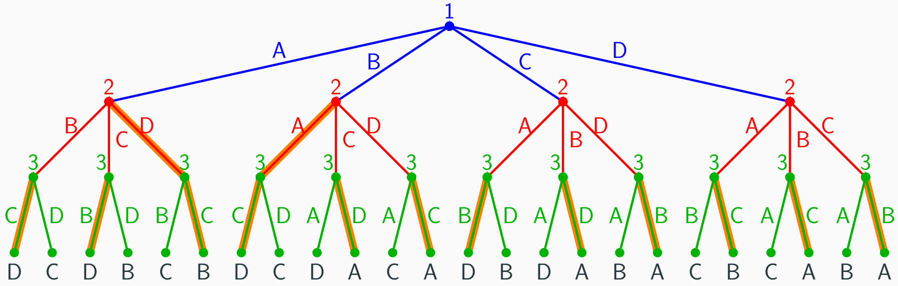

Exercise II: Extensive-Form Games

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u_1	3	1	2	0
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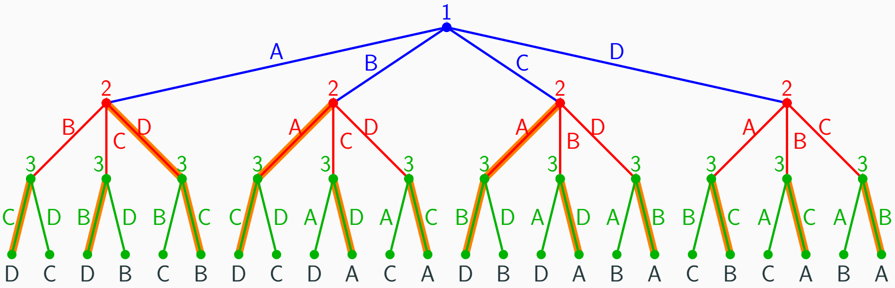

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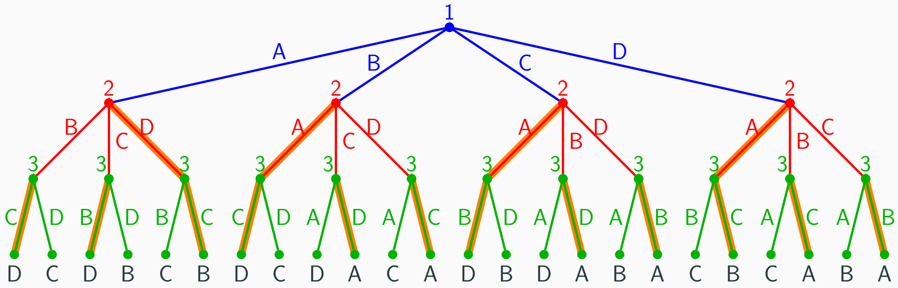

Exercise II: Extensive-Form Games

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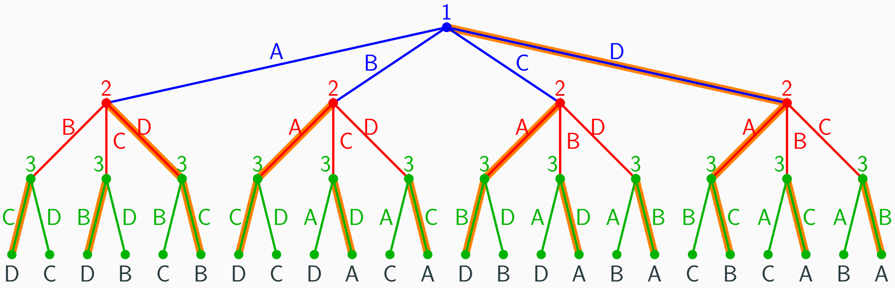
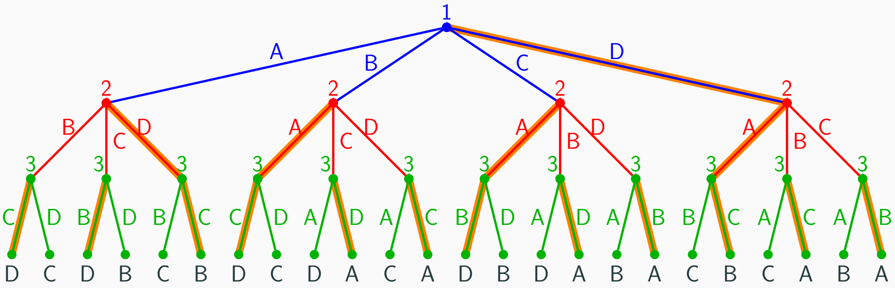


c) Assume the same preferences as before. Find a Nash equilibrium where the final destination is Adelaide.

Exercise II: Extensive-Form Games

- Alice prefers A to C to B to D,
- Bob prefers B to D to A to C,
- Charlie prefers A to D to B to C.



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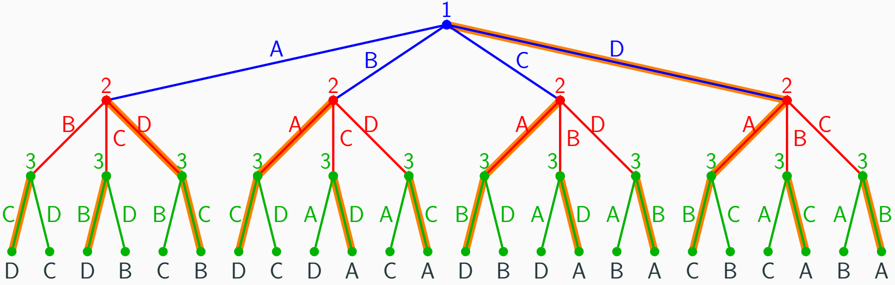

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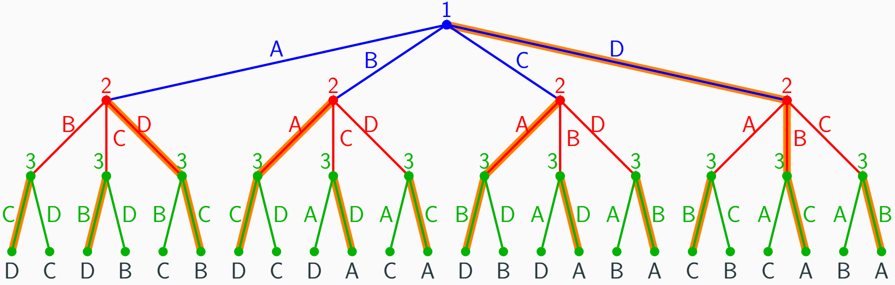

Exercise III: Congestion Games

a) 10000 agents wish to travel from city X to city Y as fast as possible. There are three options different routes: a highway (a), a connector street (b), and a rural street (c). The travel time t_x of each street depends on the street and the number of people using it:

- For the highway, $t_a(z) = 50 + \frac{z}{1000}$.
- For the connector street, $t_b(z) = 40 + \frac{z}{500}$.
- For the rural street, $t_c(z) = \frac{z}{100}$.

A action profile is defined by a partition of the agents (N_a, N_b, N_c) indicating which route each agent takes. The utility of every agent is the negative of his travel time, i.e., for each agent $i \in N_x$, we have that $u_i(N_a, N_b, N_c) = -t_x(|N_x|)$. Find a pure Nash equilibrium for his game.

Exercise III: Congestion Games

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An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

We will find a action profile (N_a, N_b, N_c) such that

$$
\begin{aligned}
t_a(n_a) &= t_c(n_c) \\
t_b(n_b) &= t_c(n_c) \\
n_a + n_b + n_c &= 10000
\end{aligned}
$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

$$50 + \frac{n_a}{1000} = \frac{n_c}{100}$$
$$40 + \frac{n_b}{500} = \frac{n_c}{100}$$
$$n_a + n_b + n_c = 10000$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

$$\begin{aligned} n_a &= 10n_c - 50000 \\ n_b &= 5n_c - 20000 \\ n_a + n_b + n_c &= 10000 \end{aligned}$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

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$$10n_c - 50000 + 5n_c - 20000 + n_c = 10000$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

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$$16n_c = 80000$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

$$n_c = 5000$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

$$
\begin{aligned}
50 + \frac{n_a}{1000} &= \frac{n_c}{100} \\
40 + \frac{n_b}{500} &= \frac{n_c}{100}
\end{aligned}
$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

$$50 + \frac{n_a}{1000} = \frac{5000}{100}$$
$$40 + \frac{n_b}{500} = \frac{5000}{100}$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

$$50 + \frac{n_a}{1000} = 50$$
$$40 + \frac{n_b}{500} = 50$$

Exercise III: Congestion Games

An action profile (N_a, N_b, N_c) is a pure Nash equilibrium if no agent can deviate to another (pure) strategy with a higher utility.

Hence, it must hold that $t_x(|N_x| + 1) \geq t_y(|N_y|)$ for all distinct $x, y \in \{a, b, c\}$.

$$n_a = 0$$
$$n_b = 5000$$
$$n_c = 5000$$

The travel time of each agent is 50 in this Nash equilibrium.

Exercise III: Congestion Games

b) Congestion games generalize the idea of our previous game: there are r different resources $R = \{x_1, \ldots, x_r\}$, and each player $i \in N$ can choose which of these resources to use (i.e., $A_i \subseteq 2^R \setminus \{\emptyset\}$). Moreover, let $c : R \times \mathbb{N} \to \mathbb{R}$ denote the cost of resource x depending on the number of agents that use x. The corresponding congestion game is the normal-form game $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ where

$$u_i(a) = -\sum_{x \in a_i} c(x, |\{j \in N \colon r \in a_j\}|)$$

for all agents $i \in N$ and action profiles $a \in A$.

Show that every congestion game has a pure Nash equilibrium.

Exercise III: Congestion Games

Exercise III: Congestion Games

- Fix some set of resources $R = \{x_1, \ldots, x_r\}$ and a cost function c, and let $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ denote the corresponding congestion game.

Exercise III: Congestion Games

- Fix some set of resources $R = \{x_1, \ldots, x_r\}$ and a cost function c, and let $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ denote the corresponding congestion game.
- We define the potential function $\phi(a) = -\sum_{x \in R} \sum_{\ell=1}^{|\{i \in N \colon x \in a_i\}|} c(x, \ell)$ for all action profiles $a \in A$.

Exercise III: Congestion Games

- Fix some set of resources $R = \{x_1, \ldots, x_r\}$ and a cost function c, and let $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ denote the corresponding congestion game.
- We define the potential function $\phi(a) = -\sum_{x \in R} \sum_{\ell=1}^{|\{i \in N \colon x \in a_i\}|} c(x, \ell)$ for all action profiles $a \in A$.
- Assume player i can deviate in an action profile a by changing to the action b_i.

Exercise III: Congestion Games

Define $a' = (b_i, a_{-i})$ and Δ_i by

$$\Delta_i = u_i(a') - u_i(a)$$

Exercise III: Congestion Games

Define $a' = (b_i, a_{-i})$ and Δ_i by

$$\begin{aligned}\Delta_i &= u_i(a') - u_i(a) \\ &= -\sum_{x \in b_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1) \\ &\quad + \sum_{x \in a_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1)\end{aligned}$$

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&\quad + \sum_{x \in a_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1) \\
&= -\sum_{x \in b_i \setminus a_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1) \\
&\quad + \sum_{x \in a_i \setminus b_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1)
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&= -\sum_{x \in b_i \setminus a_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1) \\
&\quad + \sum_{x \in a_i \setminus b_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1) \\
&= -\sum_{x \in b_i \setminus a_i} c(x, |\{j \in N : x \in a_j\}| + 1) \\
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&= -\sum_{x \in b_i \setminus a_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1) \\
&\quad + \sum_{x \in a_i \setminus b_i} c(x, |\{j \in N \setminus \{i\} : x \in a_j\}| + 1) \\
&= -\sum_{x \in b_i \setminus a_i} c(x, |\{j \in N : x \in a_j\}| + 1) \\
&\quad + \sum_{x \in a_i \setminus b_i} c(x, |\{j \in N : x \in a_j\}|) \\
&> 0
\end{aligned}$$

Exercise III: Congestion Games

$\phi(a') - \phi(a)$

Exercise III: Congestion Games

$$\phi(a') - \phi(a) = -\sum_{x \in R} \sum_{\ell=1}^{|\{j \in N\colon x \in a'_j\}|} c(x, \ell) + \sum_{x \in R} \sum_{\ell=1}^{|\{i \in N\colon x \in a_j\}|} c(x, \ell)$$

Exercise III: Congestion Games

$$
\begin{aligned}
\phi(a') - \phi(a) &= -\sum_{x \in R} \sum_{\ell=1}^{|\{j \in N\colon x \in a'_j\}|} c(x,\ell) + \sum_{x \in R} \sum_{\ell=1}^{|\{i \in N\colon x \in a_j\}|} c(x,\ell) \\
&= -\sum_{x \in b_i \setminus a_i} \sum_{\ell=1}^{|\{j \in N\colon x \in a'_j\}|} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N\colon x \in a_j\}|} c(x,\ell) \\
&\quad -\sum_{x \in a_i \setminus b_i} \sum_{\ell=1}^{|\{j \in N\colon x \in a'_j\}|} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N\colon x \in a_j\}|} c(x,\ell)
\end{aligned}
$$

Exercise III: Congestion Games

$$
\begin{aligned}
\phi(a') - \phi(a) &= -\sum_{x \in R} \sum_{\ell=1}^{|\{j \in N:\, x \in a'_j\}|} c(x,\ell) + \sum_{x \in R} \sum_{\ell=1}^{|\{i \in N:\, x \in a_j\}|} c(x,\ell) \\
&= -\sum_{x \in b_i \setminus a_i} \sum_{\ell=1}^{|\{j \in N:\, x \in a'_j\}|} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N:\, x \in a_j\}|} c(x,\ell) \\
&\quad -\sum_{x \in a_i \setminus b_i} \sum_{\ell=1}^{|\{j \in N:\, x \in a'_j\}|} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N:\, x \in a_j\}|} c(x,\ell) \\
&= -\sum_{x \in b_i \setminus a_i} \sum_{\ell=1}^{|\{j \in N:\, x \in a_j\}+1|} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N:\, x \in a_j\}|} c(x,\ell) \\
&\quad -\sum_{x \in a_i \setminus b_i} \sum_{\ell=1}^{|\{j \in N:\, x \in a_j\}|-1} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N:\, x \in a_j\}|} c(x,\ell)
\end{aligned}
$$

Exercise III: Congestion Games

$$\phi(a') - \phi(a) = -\sum_{x \in b_i \setminus a_i} \sum_{\ell=1}^{|\{j \in N \colon x \in a_j\}+1|} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N \colon x \in a_j\}|} c(x,\ell)$$

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Exercise III: Congestion Games

$$
\begin{aligned}
\phi(a') - \phi(a) &= - \sum_{x \in b_i \setminus a_i} \sum_{\ell=1}^{|\{j \in N \colon x \in a_j\}+1|} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N \colon x \in a_j\}|} c(x,\ell) \\
&\quad - \sum_{x \in a_i \setminus b_i} \sum_{\ell=1}^{|\{j \in N \colon x \in a_j\}|-1} c(x,\ell) + \sum_{\ell=1}^{|\{i \in N \colon x \in a_j\}|} c(x,\ell) \\
&= - \sum_{x \in b_i \setminus a_i} c(x, |\{j \in N \colon x \in a_j\}| + 1) \\
&\quad + \sum_{x \in a_i \setminus b_i} c(x, |\{i \in N \colon x \in a_j\}|)
\end{aligned}
$$

Exercise III: Congestion Games

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\phi(a') - \phi(a) &= -\sum_{x\in b_i\setminus a_i} \sum_{\ell=1}^{|\{j\in N\colon x\in a_j\}+1|} c(x,\ell) + \sum_{\ell=1}^{|\{i\in N\colon x\in a_j\}|} c(x,\ell) \\
&\quad - \sum_{x\in a_i\setminus b_i} \sum_{\ell=1}^{|\{j\in N\colon x\in a_j\}|-1} c(x,\ell) + \sum_{\ell=1}^{|\{i\in N\colon x\in a_j\}|} c(x,\ell) \\
&= -\sum_{x\in b_i\setminus a_i} c(x, |\{j \in N\colon x \in a_j\}| + 1) \\
&\quad + \sum_{x\in a_i\setminus b_i} c(x, |\{i \in N\colon x \in a_j\}|) \\
&= \Delta_i > 0
\end{aligned}
$$

Exercise III: Congestion Games

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- Since the potential function only can have a finite number of values, we eventually must reach a maximum!
- This maximum corresponds to a pure Nash equilibrium!

Exercise IV: The Minimax Theorem

Prove the Maximin theorem. Use the fact that Nash equilibria are guaranteed exist.

The Maximin theorem states the following: It holds for every zero-sum game $(\{1,2\},(A_1,A_2),(u_1,u_2))$ that

$$\max_{s\in S_1}\min_{t\in S_2} u_1(s,t) = -\max_{t\in S_2}\min_{s\in S_1} u_2(s,t).$$

Exercise IV: The Minimax Theorem

- Let $v_1(s) = \min_{t \in S_2} u_1(s,t)$ and $v_2(t) = \max_{s \in S_1} u_1(s,t)$.

Exercise IV: The Minimax Theorem

- Let $v_1(s) = \min_{t \in S_2} u_1(s, t)$ and $v_2(t) = \max_{s \in S_1} u_1(s, t)$.
- For every $s \in S_1, t \in S_2$, it holds that $v_1(s) \leq u_1(s, t) \leq v_2(t)$.

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- For every $s \in S_1, t \in S_2$, it holds that $v_1(s) \leq u_1(s, t) \leq v_2(t)$.
- Since this holds for every $s \in S_1$, we have for every $t \in S_2$ that $\max_{s \in S_1} v_1(s) \leq v_2(t)$.

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- For every $s \in S_1, t \in S_2$, it holds that $v_1(s) \leq u_1(s,t) \leq v_2(t)$.
- Since this holds for every $s \in S_1$, we have for every $t \in S_2$ that $\max_{s \in S_1} v_1(s) \leq v_2(t)$.
- Since this holds for every $t \in S_2$, we have that $\max_{s \in S_1} v_1(s) \leq \min_{t \in S_2} v_2(t)$.

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- For every $s \in S_1, t \in S_2$, it holds that $v_1(s) \leq u_1(s,t) \leq v_2(t)$.
- Since this holds for every $s \in S_1$, we have for every $t \in S_2$ that $\max_{s \in S_1} v_1(s) \leq v_2(t)$.
- Since this holds for every $t \in S_2$, we have that $\max_{s \in S_1} v_1(s) \leq \min_{t \in S_2} v_2(t)$.
- This shows that

$$\max_{s \in S_1} \min_{t \in S_2} u_1(s,t) \leq \min_{t \in S_2} \max_{s \in S_1} u_1(s,t) = -\max_{t \in S_2} \min_{s \in S_1} u_2(s,t).$$

Exercise IV: The Minimax Theorem

Let (s^*, t^*) be an arbitrary Nash equilibrium of the given zero-sum game.

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$$\max_{s \in S_1} \min_{t \in S_2} u_1(s, t) \geq \min_{t \in S_2} u_1(s^*, t)$$

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$$\begin{aligned}
\max_{s \in S_1} \min_{t \in S_2} u_1(s, t) &\geq \min_{t \in S_2} u_1(s^*, t) \\
&= -\max_{t \in S_2} -u_1(s^*, t)
\end{aligned}$$

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&= -\max_{t \in S_2} u_2(s^*, t)
\end{aligned}$$

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&= -\max_{t \in S_2} u_2(s^*, t) \\
&= -u_2(s^*, t^*)
\end{aligned}$$

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&= -\max_{t \in S_2} -u_1(s^*, t) \\
&= -\max_{t \in S_2} u_2(s^*, t) \\
&= -u_2(s^*, t^*) \\
&= u_1(s^*, t^*) \\
&= \max_{s \in S_1} u_1(s, t^*)
\end{aligned}
$$

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$$
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&= -\max_{t \in S_2} -u_1(s^*, t) \\
&= -\max_{t \in S_2} u_2(s^*, t) \\
&= -u_2(s^*, t^*) \\
&= u_1(s^*, t^*) \\
&= \max_{s \in S_1} u_1(s, t^*) \\
&\geq \min_{t \in S_2} \max_{s \in S_1} u_1(s, t)
\end{aligned}
$$

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&= -\max_{t \in S_2} -u_1(s^*, t) \\
&= -\max_{t \in S_2} u_2(s^*, t) \\
&= -u_2(s^*, t^*) \\
&= u_1(s^*, t^*) \\
&= \max_{s \in S_1} u_1(s, t^*) \\
&\geq \min_{t \in S_2} \max_{s \in S_1} u_1(s,t) \\
&= -\max_{t \in S_2} \min_{s \in S_1} u_2(s,t)
\end{aligned}$$