Cooperative Game Theory and Matchings

COMP 4418 – Assignment 2

Due 25 Oct. 2024, 16:00
Total Marks: 100

Question 1 (20 marks) Consider a Student Proposing Deferred Acceptance (SPDA) algorithm on an one-one matching instance $\langle S, C \prec \rangle$ where $|S| = |C| = n$. Prove or disprove the following:

1. **(5 marks)** For any instance, at least one student always makes multiple proposals.
2. **(5 marks)** There is an instance where the number of proposals made is $\frac{n(n+1)}{2}$.
3. **(5 marks)** For any instance, there is always one college that receives exactly one proposal.
4. **(5 marks)** For an instance n students and n colleges, the maximum number of proposals that can be made is $n(n-1)+1$.

Question 2 (10 marks) Consider the following one-one matching instance with $n = 5$.

$$
\begin{array}{ll}
s_1 : c_2 \succ c_1 \succ c_3 \succ c_4 \succ c_5 & \qquad c_1 : s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \\
s_2 : c_1 \succ c_2 \succ c_5 \succ c_3 \succ c_4 & \qquad c_2 : s_2 \succ s_1 \succ s_4 \succ s_5 \succ s_3 \\
s_3 : c_3 \succ c_4 \succ c_5 \succ c_2 \succ c_1 & \qquad c_3 : s_4 \succ s_5 \succ s_3 \succ s_2 \succ s_1 \\
s_4 : c_4 \succ c_5 \succ c_3 \succ c_1 \succ c_2 & \qquad c_4 : s_5 \succ s_3 \succ s_4 \succ s_1 \succ s_2 \\
s_5 : c_1 \succ c_2 \succ c_5 \succ c_3 \succ c_4 & \qquad c_5 : s_3 \succ s_4 \succ s_5 \succ s_1 \succ s_2
\end{array}
$$

Identify all colleges who can manipulate under the SPDA and explain why. Give an inconspicuous optimal manipulation for each, if any. Analogously identify all students who can manipulate under the CPDA and give an inconspicuous optimal manipulation for each, if any.

Question 3 (10 marks) Give a polynomial time algorithm to check if a given one-one matching instance has a unique stable matching. Prove its correctness and running time.

Question 4 (10 marks) Given a one-one matchings instance $\langle S, C \succ \rangle$, let μ and μ' be two distinct stable matchings. Suppose for each student $s \in S$ if $\mu(s) \neq \mu'(s)$ then $\mu(s) \succ_s \mu'(s)$. Prove that for each college if $\mu(c) \neq \mu'(c)$ then $\mu(c) \prec \mu'(c)$.

Question 5 (10 marks) Consider a Shapley-Scarf housing market with a set of agents $N = \{0,1,2,3,4\}$, a set of items $O = \{o_0, o_1, o_2, o_3, o_4\}$, and an endowment function $\omega : N \to 2^O$ such that $\omega(i) = \{o_i\}$. The preferences of the agents are as follows from left to right in decreasing order of preference.

$$\begin{aligned}
0 &: o_0, o_4, o_2, o_1, o_3 \\
1 &: o_0, o_2, o_4, o_1, o_3 \\
2 &: o_3, o_0, o_2, o_4, o_1 \\
3 &: o_0, o_2, o_3, o_1, o_4 \\
4 &: o_3, o_2, o_1, o_4, o_0
\end{aligned}$$

Find the outcome of the TTC (top trading cycles) algorithm. Can agent 4 misreport her preference to get a more preferred allocation? Prove or disprove that the outcome is individually rational.

Question 6 (20 marks) Consider Shapley-Scarf housing markets in which we are only allowed to obtain allocations in which at most two agents are a part of a trading cycle and each agent can be a part of at most of one trading cycle. (a) Design a polynomial-time algorithm for the problem and that is individually rational and Pareto optimal among feasible outcomes. Prove its properties. (b) Design a polynomial-time algorithm for the problem and that is strategyproof and Pareto optimal among feasible outcomes. Prove its properties.

Question 7 (20 points) Consider the coalitional game (N, v) such that $N = \{1,2,3\}$ and v is defined as follows:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	4	3	2	4	3	2	12

For this game, compute the nucleolus and explain why it is the nucleolus.
Also compute the Shapley value and explain how you computed it.

SUBMISSION

- Submit your solution directly via Moodle in the assessment hub at the end of the Moodle page. Please make sure that your manuscript contains your name and zID.
- Your answers are to be submitted in a single PDF file. We will not accept any other file formats. Please make sure that your solutions are clearly readable.
- The deadline for this submission is 25 October 2024, 16:00.

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