# COMP 4418 – Exercise Sheet: Fair Allocations I

**Exercise I: Identical Valuations**

Consider an allocation instance $\langle N, M, v \rangle$ where agents have identical valuations. That is, for any $i, j \in N$ and any $g \in M$, we have that $v_i(g) = v_j(g)$. Prove that under identical valuations:

i. All allocations have the same USW.

ii. An allocation that maximizes ESW will satisfy MMS.

iii. No allocation can have an envy graph with a cycle.

**Exercise II: Envy and Proportionality**

Give an example of an instance with indivisible items where:

i. An envy-free allocation does not exist but a proportional allocation exists.

ii. A PROP allocation exists that is not EF, even though an EF allocation exists for the instance.

iii. An allocation that is simultaneously PROP and EF1 exists.

iv. An EF1 allocation exists such that its envy-graph has a cycle.

**Exercise III: Leximin versus MMS**

Give an example of an instance with indivisible items where:

i. Every leximin optimal allocation is MMS.

ii. No leximin optimal allocation is MMS.

iii. There exists an MMS allocation that maximizes ESW but isn't leximin optimal.

iv. There is an MMS allocation that has maximum USW.

**Exercise IV: Applying Greedy Round Robin**

Consider the following instance with $n = 3$ agents and $m = 6$ items

For this instance, identify:

i. Two distinct maximum USW allocations.

ii. A PROP allocation.

iii. Three distinct $\frac{1}{2}$-MMS allocations.

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $v_1$ | 100 | 10 | 20 | 30 | 50 | 90 |
| $v_2$ | 100 | 20 | 30 | 20 | 20 | 10 |
| $v_3$ | 100 | 23 | 17 | 17 | 24 | 19 |

**Exercise V: Balanced Allocations and EFI** Consider an allocation instance $\langle N, M, v\rangle$ where $m = kn$. Suppose we add an additional constraint that each agent must receive exactly $k$ items. For this setting prove that:

i. There exists an instance where the Envy Graph Algorithm fails to produce an EF1 allocation that is balanced.

ii. Round Robin will return an EF1 allocation where each agent receives $k$ items.