Fair Allocations COMP4418–Assignment 3

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Question 1 (20 marks) Consider a fair division instance $\langle N, M, v \rangle$ with n agents and m items. Prove or disprove the following:

1. **(5 marks) Any Pareto Optimal allocation must also be Leximin Optimal.**

   Answer: **Disprove**. Not every Pareto optimal allocation is Leximin optimal.

   According to the definition of PO and Leximin Optimal:

Pareto Optimal (PO): An allocation A is PO if there is NO other allocation A' s.t. $v_i(A_i) \leq v_i(A'_i)$ and for some i^*, $v_i(A_{i^*}) < v_i(A'_{i^*})$.
◦ An allocation that is not Pareto dominated by another.

Leximin Domination: Allocation A *leximin dominates* A' if there exist $t \in [n]$ s.t. $L_A[t] > L_{A'}[t]$ and for all $t' < t, L_A[t'] = L_{A'}[t']$.

Leximin Optimal: An allocation that is not lexmin dominated.

Pareto optimality and Leximin optimality are related but distinct concepts. An allocation is Pareto optimal (PO) if there is no other allocation that can make at least one agent better off without making any other agent worse off. An allocation is Leximin optimal if it maximizes the minimal utility among all agents, and subject to that, maximizes the next minimal utility, and so on.

While every Leximin optimal allocation is Pareto optimal, the converse is not necessarily true. A Pareto optimal allocation may not be Leximin optimal because it might not maximize the minimal utility.

Counterexample:

Consider an instance with three agents A_1, A_2, A_3 and three items x, y, z. Their valuations are:

- A_1: $v_1(x) = 3, v_1(y) = 1, v_1(z) = 0$
- A_2: $v_2(x) = 3, v_2(y) = 2, v_2(z) = 1$
- A_3: $v_3(x) = 0, v_3(y) = 0, v_3(z) = 3$

Two allocations:

1. Allocation S:
   - A_1 gets x
   - A_2 gets y
   - A_3 gets z
   - Utilities: (3,2,3), sorted: (2,3,3)
2. Allocation T:
   - A_1 gets y
   - A_2 gets x
   - A_3 gets z

- Utilities: $(1,3,3)$, sorted: $(1,3,3)$

Both allocations are Pareto optimal, but allocation S leximin dominates T because it has a higher minimal utility (2 vs. 1). Therefore, not all Pareto optimal allocations are Leximin optimal.

2. **(5 marks) Given any two allocations, one must pareto dominate the other.**

Answer: **Disprove**. There exist allocations where neither Pareto dominates the other.

Pareto dominance is a partial order, meaning that for some pairs of allocations, neither allocation Pareto dominates the other. To illustrate this, let's consider two agents A_1, A_2 and two allocations with utilities $(2,3)$ and $(1,4)$.

Allocation A: utilities $(2,3)$

Allocation B: utilities $(1,4)$

Analyzing Pareto Dominance:

- From Allocation A to Allocation B:
  - A_1's utility decreases from 2 to 1 (worse off).
  - A_2's utility increases from 3 to 4 (better off).
  - Since A_1 is worse off and A_2 is better off, moving from Allocation A to Allocation B is not a Pareto improvement.
- From Allocation B to Allocation A:
  - A_1's utility increases from 1 to 2 (better off).
  - A_2's utility decreases from 4 to 3 (worse off).
  - Since A_2 is worse off and A_1 is better off, moving from Allocation B to Allocation A is not a Pareto improvement.

Conclusion:

- Neither allocation Pareto dominates the other:
  - Allocation A does not Pareto dominate Allocation B.
  - Allocation B does not Pareto dominate Allocation A.

Therefore, the statement is **false**.

3. **(5 marks) For $n = 2$, any allocation that satisfies PROP is also EF.**

Answer: **Prove**. In the case of two agents, proportionality implies envy-freeness. For two agents with additive valuations, an allocation is **proportional (PROP)** if each agent receives at least half of the total value of all items according to their own valuation. An allocation is **envy-free (EF)** if no agent prefers the bundle of another agent over their own.

Suppose an allocation is proportional, so each agent gets at least $\frac{1}{2}$ of their total valuation of all items.

Assume, for contradiction, that agent A_1 envies agent A_2:

- $v_1(S_2) > v_1(S_1)$
- $v_1(S_1) \geq \frac{1}{2} v_1(M)$ (by proportionality)
- Total value: $v_1(S_1) + v_1(S_2) = v_1(M)$

Substituting:

- $v_1(S_2) = v_1(M) - v_1(S_1) \leq v_1(M) - \frac{1}{2}v_1(M) = \frac{1}{2}v_1(M)$

This contradicts $v_1(S_2) > v_1(S_1) \geq \frac{1}{2}v_1(M)$. Therefore, agent A_1 does not envy A_2. Similarly, using the same reasoning process, we can conclude that agent A_2 also does not envy agent A_1. Hence, the allocation is envy-free.

4. **(5 marks) Greedy round robin algorithm will return an EF1 allocation.**
Answer: **Disprove**. Greedy round robin algorithm will not always return an EF1 allocation.

Greedy Round Robin

Algorithm: (N, M, v)

Initialize $I \leftarrow N, P \leftarrow M$

Initialize empty allocation A, where $A_i \leftarrow \emptyset$ for all $i \in N$

Set $\beta_i = v_i(M)/n$

While(there exist $i \in I$, $g \in P$ s.t. $v_i(g) \geq \beta_i/2$)

- $A_i \leftarrow \{g\}$
- $I \leftarrow I \setminus \{i\}$, $P \leftarrow P \setminus \{g\}$
- For all $j \in I$, set $\beta_j = v_j(P)/|I|$

$A' \leftarrow$ **RoundRobin**(I, P, v). For all $i \in I$, $A_i \leftarrow A'_i$

Return A

Counterexample:

Consider an instance with two agents A_1, A_2 and four items x, y, z, h. Their valuations are:

- A_1: $v_1(x) = 50, v_1(y) = 49, v_1(z) = 48, v_1(h) = 47$
- A_2: $v_2(x) = 50, v_2(y) = 49, v_2(z) = 48, v_2(h) = 47$

Then, $\boldsymbol{\beta_1} = \frac{v_1(M)}{4} = \frac{50+49+48+47}{4} = 48.5$, $\frac{\boldsymbol{\beta_1}}{\boldsymbol{2}} = \boldsymbol{24.25}$

Now A_1 can get x since $v_1(x) = 50 > 24.25$

$$A_1 = \{x\}$$

We removed A_1 and x, according to the greedy round robin algorithm, A_2 will get $y, z \text{ and } h$.

$$A_2 = \{y, z, h\}$$

$\boldsymbol{v_1(A_1) = 50, v_1(A_2) = 49 + 48 + 47 = 144.}$

An allocation is **EF1** if any agent envies another agent, but this envy can be eliminated by removing at most one item from the envied agent's allocation.

We can see that, after whichever item is removed from A_2, A_1 still envies A_2.

Therefore, greedy round robin algorithm will not always return an **EF1 allocation**.

Question 2 (20 marks) Consider the following instance with $n = 4$ and $m = 8$.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
v_1	1	5	4	4	0	1	1	1
v_2	5	9	5	5	0	0	5	5
v_3	5	7	5	10	0	4	0	5
v_4	10	10	5	5	5	5	5	5

For this instance, consider running the standard round robin algorithm to find an EF1 allocation. We shall look at how different orderings over agents can lead to different allocations. For the given instance, identify:

1. **(10 marks)** The ordering over agents which leads to the following allocation: $A = (A_1, A_2, A_3, A_4)$, where $A_1 = \{g_1, g_5\}$, $A_2 = \{g_4, g_8\}$, $A_3 = \{g_3, g_7\}$ and $A_4 = \{g_2, g_6\}$.

Round Robin Algorithm

Algorithm: (N, M, v)

Initialize set of unassigned items $P \leftarrow M$

Initialize empty allocation A, where $A_i \leftarrow \emptyset$ for all $i \in N$

Choose arbitrary ordering over agents: $i_1, \cdots, i_n$

Set $j \leftarrow 1, t \leftarrow 1$

While$(P \neq \emptyset)$

- $g_t^{i_j} \leftarrow \mathrm{argmax}_{g \in P}\, v_{i_j}(g)$
- $A_{i_j} \leftarrow A_{i_j} \cup \{g_t^{i_j}\}$, $P \leftarrow P \setminus \{g_t^{i_j}\}$
- **If**$(j < n)$
  - $j \leftarrow j + 1$
- **Else**
  - $j \leftarrow 1, t \leftarrow t + 1$

Return A

According to the Round Robin Algorithm, the agent will always choose the item which is available with largest value.

For A_1, g_2 has the largest value (5) but $A_1 = \{g_1, g_5\}$, so A_1 is not the first agent.

For A_2, g_2 has the largest value (9) but $A_2 = \{g_4, g_8\}$, so A_2 is not the first agent.

For A_3, g_4 has the largest value (10) but $A_3 = \{g_3, g_7\}$, so A_3 is not the first agent.

For A_4, g_2 and g_1 has the largest value (10) and $A_4 = \{g_2, g_6\}$, **so A_4 is the first agent.**

Since A_4 is the first agent, item g_2 is removed:

For A_1, g_3 and g_4 has the largest value (4) but $A_1 = \{g_1, g_5\}$, so A_1 is not the second agent.

For A_3, g_4 has the largest value (10) but $A_3 = \{g_3, g_7\}$, so A_3 is not the second agent.

So, $\boldsymbol{A_2}$ **is the second agent.**

Now, $\boldsymbol{g_2}$ and $\boldsymbol{g_4}$ are removed:

For A_1, g_3 has the largest value (4) but $A_1 = \{g_1, g_5\}$, so A_1 is not the third agent.

So, $\boldsymbol{A_3}$ **is the third agent.**

Therefore, the ordering is $\boldsymbol{A_4 - A_2 - A_3 - A_1}$**.**

2. **(5 marks)** An alternate EF1 allocation that can result from the same ordering which would Pareto dominate A.

We use the same ordering:

For A_4, we can choose g_1 instead of g_2, then A_2 will choose g_2 (largest value). A_3 will choose g_4 and A_3 will choose g_3.

$$A_4 = \{g_1\}$$
$$A_2 = \{g_2\}$$
$$A_3 = \{g_4\}$$
$$A_1 = \{g_3\}$$

Second round:

A_4 will choose g_5, then A_2 will choose g_7. A_3 will choose g_8 and A_1 will choose g_6.

$$A_4 = \{g_1, g_5\}$$
$$A_2 = \{g_2, g_7\}$$
$$A_3 = \{g_4, g_8\}$$
$$A_1 = \{g_3, g_6\}$$

For allocation A:

$$\boldsymbol{v_4(A_4) = 10 + 5 = 15}$$
$$\boldsymbol{v_2(A_2) = 5 + 5 = 10}$$
$$\boldsymbol{v_3(A_3) = 0 + 5 = 5}$$
$$\boldsymbol{v_1(A_1) = 0 + 1 = 1}$$

For allocation A':

$$\boldsymbol{v_4(A_4) = 10 + 5 = 15 = 15}$$
$$\boldsymbol{v_2(A_2) = 9 + 5 = 14 > 10}$$
$$\boldsymbol{v_3(A_3) = 10 + 5 = 15 > 5}$$
$$\boldsymbol{v_1(A_1) = 5 + 5 = 10 > 1}$$

Therefore, every value in A' is equal or greater than that in allocation A, A' pareto dominates A.

3. **(5 marks)** An alternate ordering for the standard round robin algorithm that would result in the allocation identified in the previous part.

To result in the allocation identified in the previous part, we can use this ordering:

$$\boldsymbol{A_4 - A_3 - A_2 - A_1.}$$

Process:

For A_4, we can choose g_1, then A_3 will choose g_4 (largest value). A_2 will choose g_2 and A_1 will choose g_3.

$$A_4 = \{g_1\}$$
$$A_3 = \{g_4\}$$
$$A_2 = \{g_2\}$$
$$A_1 = \{g_3\}$$

Second round:

A_4 will choose g_5, then A_3 will choose g_6. A_2 will choose g_7 and A_1 will choose g_8.

$$A_4 = \{g_1, g_5\}$$
$$A_3 = \{g_4, g_6\}$$
$$A_2 = \{g_2, g_7\}$$
$$A_1 = \{g_3, g_8\}$$

Compare the value:

$$\boldsymbol{v_4(A_4) = 10 + 5 = 15 = 15}$$
$$\boldsymbol{v_2(A_2) = 9 + 5 = 14 > 10}$$
$$\boldsymbol{v_3(A_3) = 10 + 4 = 14 > 5}$$
$$\boldsymbol{v_1(A_1) = 4 + 1 = 5 > 1}$$

Therefore, the allocation generated by ordering $\boldsymbol{A_4 - A_3 - A_2 - A_1}$ pareto dominates A.

Question 3 (20 marks) Consider an indivisible item setting with $m > n$ where agents are indifferent between the items. That is, for any $i \in N$ and $g \neq g' \in M$, we have that $v_i(g) = v_i(g') > 0$. However, agent valuations are not (guaranteed to be) identical. That is, there may be $i \neq j$ and $g \in M$, s.t. $v_i(g) \neq v_j(g)$. For this setting:

1. **(5 marks) Show that an MMS allocation always exists.**

Definition. A is maximin fair (**MMS**) if for all $i \in N$, $v_i(A_i) \geq MMS_i$

Yes, an MMS allocation always exists in this setting. Since agents value all items equally, each agent $i's$ maximin share is $MMS_i = v_i \times \lfloor \frac{m}{n} \rfloor$. To construct an MMS allocation, we can proceed as follows:

- Divide the m items into n bundles, each containing at least $\lfloor \frac{m}{n} \rfloor$ items.
- Assign one bundle to each agent arbitrarily.

Since $m > n$, each agent receives at least one item, ensuring that their utility is at least MMS_i. Any remaining items can be distributed arbitrarily, possibly increasing agents' utilities above their MMS values $(v_i(A_i) \geq MMS_i)$. Thus, an MMS allocation exists.

2. (5 marks) Show that an EF1 allocation will always be MMS.

Envy-free up to one item: An allocation A is EF1 if for any two agents $i, j \in N$, there exists $g \in A_j$ s.t. $v_i(A_i) \geq v_i(A_j \setminus g)$.

Yes, in this setting, **an EF1 allocation will always be MMS**. In an EF1 allocation, for any pair of agents i and j, agent i does not envy agent j after removing any single item from $j's$ bundle. Since agents value all items equally, their utilities are proportional to the number of items they receive.
For agent i not to envy agent j up to one item, it must be that: $u_i \geq u_j - v_i$
This condition ensures that no agent has significantly more items than another **(differing by at most one item)**. As a result, each agent receives at least $\lfloor \frac{m}{n} \rfloor$ items, ensuring they get at least their MMS value $(MMS_i = v_i \times \lfloor \frac{m}{n} \rfloor)$. Therefore, an EF1 allocation will always be MMS allocation in this setting.

3. **(10 marks)** Give examples of instances in this setting such that:

a. Any allocation with maximum ESW is not MMS.

Egalitarian Social Welfare (ESW): ESW of an allocation is the lowest value received by an agent from it.

$$\mathrm{ESW}(A) = \min_{i \in N} v_i(A_i)$$

Yes, here is an example:

- **Instance:**
  - $m = 4$ items, $n = 2$ agents.
  - Agent 1 values each item at $v_1 = 1$.
  - Agent 2 values each item at $v_2 = 200$.
- **Agents' MMS values:**
  - $MMS_1 = 1 \times \lfloor 4/2 \rfloor = 2$.
  - $MMS_2 = 200 \times 2 = 400$.
- **Allocation maximizing ESW:**
  - Allocate 3 items to Agent 1: $u_1 = 3$.
  - Allocate 1 item to Agent 2: $u_2 = 200$.

In this allocation, Agent 2 receives less than her MMS value $(u_2 = 200 < MMS_2 = 400)$. Therefore, this allocation that maximizes ESW results in Agent 2 not receiving her MMS, as giving more items to Agent 1 increases the minimum utility but deprives Agent 2 of her MMS.

b. There is at least one allocation with maximum USW which is $\frac{1}{2}$−MMS in this instance.

Utilitarian Social Welfare (USW): Sum of agents' valuations for the allocation

$$\mathrm{USW}(A) = \sum_{i \in N} v_i(A_i)$$

We assume that we have 2 agents A_1, A_2 and 3 items.

Valuations:

- assume both Agent 1 and Agent 2 value each item equally at $v1 = v2 = 1$. That is, all agents value all items the same.

Since all items have the same value, we can allocate 2 items to A_1 and 1 item to A_2 to achieve **maximum USW**, which is $2 + 1 = 3$.

Utility of Agent 1:

$$\boldsymbol{u_1 = 2 > \frac{1}{2} \times MMS_1 = 0.5}$$

Agent 1's utility is greater than half of their MMS value.

Utility of Agent 2:

$$\boldsymbol{u_2 = 1 = \frac{1}{2} \times MMS_2 = 1}$$

Agent 2's utility is equal to half of their MMS value.

Conclusion: In this allocation that maximizes USW, agent 1's utility is greater than half of their MMS value and agent 2's utility is exactly equal to half of their MMS value. Thus, it satisfies the requirement of the problem: there exists an allocation that maximizes USW, in which at least one agent's utility is equal to half of their MMS value.

Question 4 (20 marks) Consider the random assignment problem with 3 agents with the following preferences over 3 items.

$$\succ_1: \quad g_1 \succ_1 g_2 \succ_1 g_3$$
$$\succ_2: \quad g_1 \succ_2 g_2 \succ_2 g_3$$
$$\succ_3: \quad g_2 \succ_3 g_1 \succ_3 g_3$$

Find the random assignment as a result of the following rules.

1. **(10 marks) probabilistic serial (PS)**

PS (Probabilistic Serial):

Agent 1	$\frac{1}{2} g_1$	$\frac{1}{6} g_2$	$\frac{1}{3} g_3$
Agent 2	$\frac{1}{2} g_1$	$\frac{1}{6} g_2$	$\frac{1}{3} g_3$
Agent 3	$\frac{1}{2} g_2$	$\frac{1}{6} g_2$	$\frac{1}{3} g_3$

$$\text{PS}(\succ_1, \succ_2, \succ_3) = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

2. (10 marks) random serial dictator (RSD)

RSD (Random Serial Dictatorship)

For an assignment problem $(N, M, \succ)$, takes a permutation π uniformly at random and then applies serial dictatorship with respect to it.

Possible Orders (Permutations):

There are $3! = 6$ possible orders:

1. (1,2,3)
2. (1,3,2)
3. (2,1,3)
4. (2,3,1)
5. (3,1,2)
6. (3,2,1)

Allocations for Each Order:

1. **Order (1,2,3):**
   - Agent 1 picks $g1$.
   - Agent 2 picks $g2$.
   - Agent 3 picks $g3$.
2. **Order (1,3,2):**
   - Agent 1 picks $g1$.
   - Agent 3 picks $g2$.
   - Agent 2 picks $g3$.
3. **Order (2,1,3):**
   - Agent 2 picks $g1$.
   - Agent 1 picks $g2$.
   - Agent 3 picks $g3$.
4. **Order (2,3,1):**
   - Agent 2 picks $g1$.
   - Agent 3 picks $g2$.
   - Agent 1 picks $g3$.
5. **Order (3,1,2):**
   - Agent 3 picks $g2$.
   - Agent 1 picks $g1$.
   - Agent 2 picks $g3$.
6. **Order (3,2,1):**

- Agent 3 picks g2.
- Agent 2 picks g1.
- Agent 1 picks g3.

Counting Allocations:

- **Agent 1:**
  - Gets $g1$ in orders 1, 2, 5 (3 out of 6 times).
  - Gets $g2$ in order 3 (1 out of 6 times).
  - Gets $g3$ in orders 4, 6 (2 out of 6 times).
- **Agent 2:**
  - Gets $g1$ in orders 3, 4, 6 (3 out of 6 times).
  - Gets $g2$ in order 1 (1 out of 6 times).
  - Gets $g3$ in orders 2, 5 (2 out of 6 times).
- **Agent 3:**
  - Gets $g2$ in orders 2, 4, 5, 6 (4 out of 6 times).
  - Gets $g3$ in orders 1, 3 (2 out of 6 times).

$$\boldsymbol{RSD(N, M, \succ)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Question 5 (20 marks) Consider the following instance with $n = 4$ and $m = 8$.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
v_1	1	5	4	4	0	1	1	1
v_2	5	9	5	5	0	0	5	5
v_3	5	7	5	10	0	4	0	5
v_4	10	10	5	5	5	4	1	1

Consider the allocation A in which $A_1 = \{g_1, g_2\}$, $A_2 = \{g_3, g_4\}$, $A_3 = \{g_5, g_6\}$, and $A_4 = \{g_7, g_8\}$.

1. (5 marks) Prove or disprove that the allocation is envy-free.

From the graph and the Allocation A, we know that $A_1 = \{g1, g2\}, A_4 = \{g7, g8\}$ and $v_1 = 1 + 5 = 6$, $v_4 = 1 + 1 = 2$ so 4 is envious of 1.

To make 4 be made non-envious, we need give A_4 $10 + 10 - 1 - 1 = 18$.

$$M_4 = 18$$

However, $M_1 = 1 + 1 - 1 - 1 = 0$.

And $v_1(A_1) + M_1 = 6 + 0 = 6$, $v_1(A_4) + M_4 = 2 + 18 = 20 > v_1(A_1) + M_1$

If we give A_4 18 , then 1 will be envious of 4.

An outcome (X, p) is **envy-free** if for all $i, j \in N$, it holds that $u_i(X_i, p_i) \geq u_i(X_j, p_j)$.

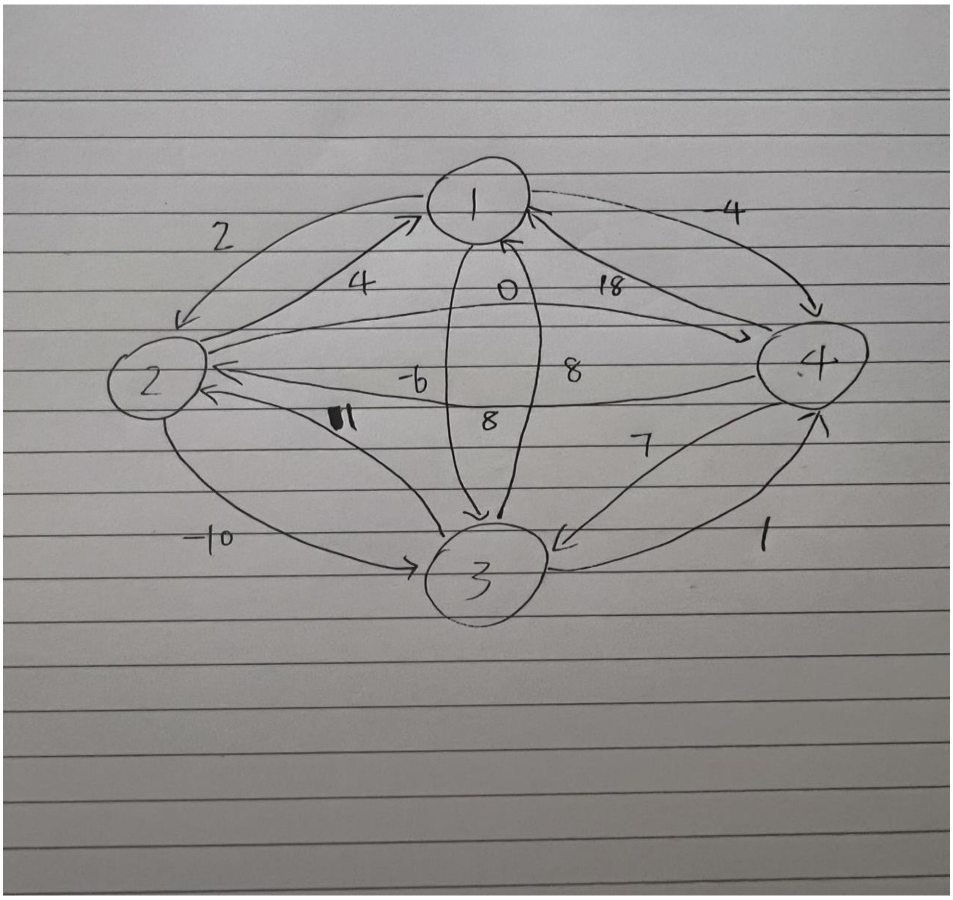
Additionally, this allocation doesn't follow the theorem, **so the allocation is not envy-free.**

2. **(5 marks)** Prove or disprove that the allocation is envy-freeable.
3. **(5 marks)** Compute the corresponding envy-graph with the amount of envy on the edge weights.
4. **(5 marks)** Find the subsidy needed to be given to each agent in order to make the allocation envy-free or show that no such subsidy exists.

$A_1 = \{g1, g2\}, A_2 = \{g3, g4\}, A_3 = \{g5, g6\}, A_4 = \{g7, g8\}$

- $w(1,2) = v_1(A_2) - v_1(A_1) = 8 - 6 = 2$
- $w(1,3) = v_1(A_3) - v_1(A_1) = 1 - 6 = -5$
- $w(1,4) = v_1(A_4) - v_1(A_1) = 2 - 6 = -4$
- $w(2,1) = v_2(A_1) - v_2(A_2) = 14 - 10 = 4$
- $w(2,3) = v_2(A_3) - v_2(A_2) = 0 - 10 = -10$
- $w(2,4) = v_2(A_4) - v_2(A_2) = 10 - 10 = 0$
- $w(3,1) = v_3(A_1) - v_3(A_3) = 12 - 4 = 8$
- $w(3,2) = v_3(A_2) - v_3(A_3) = 15 - 4 = 11$
- $w(3,4) = v_3(A_4) - v_3(A_3) = 5 - 4 = 1$
- $w(4,1) = v_4(A_1) - v_4(A_4) = 20 - 2 = 18$
- $w(4,2) = v_4(A_2) - v_4(A_4) = 10 - 2 = 8$
- $w(4,3) = v_4(A_3) - v_4(A_4) = 9 - 2 = 7$

The Envy-graph:



Theorem (Characterization of envy-freeability)

Under positive additive utilities, the following conditions are equivalent for a given allocation:

1. *the allocation is envy-freeable*
2. *the allocation is reassignment-stable*
3. *for the allocation, there is no positive weight cycle in the corresponding envy-graph*

$(1) \implies (2)$
Suppose X is envy-freeable. Then, there exists a payment vector p such that for all agents i, j $v_i(X_i) + p_i \geq v_i(X_j) + p_j$. Equivalently, $v_i(X_j) - v_i(X_i) \leq p_i - p_j$ Consider any permutation π of $[n]$. Then $\sum_{i\in N}(v_i(X_{\pi(i)}) - v_i(X_i)) \leq \sum_{i\in N}(p_i - p_{\pi(i)}) = 0$. The last entry

According to this theorem, since there are many positive weight cycles in the envy-graph, **The allocation is not envy-freeable.**

To eliminate envy using subsidies, we need to adjust each agent's utility so that no agent envies another after subsidies are applied.
Let:

- p_I be the subsidy given to agent i.
- Adjusted utility for agent i: $U_i = v_i(A_i) + p_i$.

Constraints:

For every pair of agents i and j, the following must hold:

$$\boldsymbol{U_i \geq v_i(A_i) + p_i}$$

This ensures agent i does not envy agent j.

Set Up the Inequalities:

From Agent v$_1$:

1. $6 + p1 \geq 8 + p2$ (Does not envy v$_2$)
2. $6 + p1 \geq 1 + p3$ (Does not envy v$_3$)
3. $6 + p1 \geq 2 + p4$ (Does not envy v$_4$)

From Agent v$_2$:

4. $10 + p2 \geq 14 + p1$ (Does not envy v$_1$)
5. $10 + p2 \geq 0 + p3$ (Does not envy v$_3$)
6. $10 + p2 \geq 10 + p4$ (Does not envy v$_4$)

From Agent v$_3$:

7. $4 + p3 \geq 12 + p1$ (Does not envy v$_1$)
8. $4 + p3 \geq 15 + p2$ (Does not envy v$_2$)

9. $4 + p3 \geq 5 + p4$ (Does not envy v_4)

From Agent v_4:

10. $2 + p4 \geq 20 + p1$ (Does not envy v_1)
11. $2 + p4 \geq 10 + p2$ (Does not envy v_2)
12. $2 + p4 \geq 9 + p3$ (Does not envy v_3)

Simplify the Inequalities:

1. $p1 - p2 \geq 2$
2. $p1 - p3 \geq -5$
3. $p1 - p4 \geq -4$
4. $p2 - p1 \geq 4$
5. $p2 - p3 \geq -10$
6. $p2 - p4 \geq 0$
7. $p3 - p1 \geq 8$
8. $p3 - p2 \geq 11$
9. $p3 - p4 \geq 1$
10. $p4 - p1 \geq 18$
11. $p4 - p2 \geq 8$
12. $p4 - p3 \geq 7$

Analyze the Inequalities:

- From inequalities 1 and 4:
    - $p1 - p2 \geq 2$
    - $p2 - p1 \geq 4$
    - Adding both: $0 \geq 6$, which is impossible.

Conclusion:

The system of inequalities is inconsistent. Therefore, **no such subsidy** exists that can make the allocation envy-free.