Game Theory

COMP 4418 – Assignment 1

Sample Solution
Total Marks: 100
Late Penalty: 10 marks per day

Worth: 15% of the course

For all questions, proofs, possibly in the form of calculations, need to be given in order to obtain points.

Question 1 (20 marks) Consider the following normal-form game, which is parameterized by a value $\alpha \in \mathbb{R}$.

	x	y
a	1, -1	-3, 3
b	-2, α	4, -4

a) For which value of α is the game zero-sum?

Solution: $\alpha = 2$ because then all $u_1(a) = -u_2(a)$ for all $a \in A$.

b) For which values of α is the outcome $(-2, \alpha)$ Pareto-optimal?

Solution: $\alpha > -1$. If $\alpha > -1$, the column player does not prefer the outcomes for (a,x) or (b,y) to (b,x). The row player never prefers (b,x) to (a,y).

c) For which values of α can the game be solved by iterated (strict) dominance?

Solution: $\alpha < -4$. If $\alpha < -4$, then y dominates x. In a second step, the row player will eliminate a, leaving (b,y) as the outcome. If $\alpha \geq -4$, no action is strictly dominated.

d) For which value of α is it the maximin strategy of the column player to play x with probability $\frac{1}{2}$?

Solution: $\alpha = 6$. We need to solve the equation

$$\begin{aligned}
-s(x) + 3 \cdot s(y) &= \alpha \cdot s(x) - 4 \cdot s(y) \\
-s(x) + 3 \cdot (1 - s(x)) &= \alpha \cdot s(x) - 4 \cdot (1 - s(x)) \\
3 - 4s(x) &= \alpha \cdot s(x) - 4 + 4s(x) \\
7 - 8s(x) &= \alpha \cdot s(x)
\end{aligned}$$

Using that $s(x) = \frac{1}{2}$, we get that

$$\begin{aligned} 7 - 8s(x) &= \alpha \cdot s(x) \\ 3 &= \frac{\alpha}{2} \\ \alpha &= 6. \end{aligned}$$

e) For which value fo α plays the row player a with probability $\frac{3}{4}$ in a Nash equilibrium?

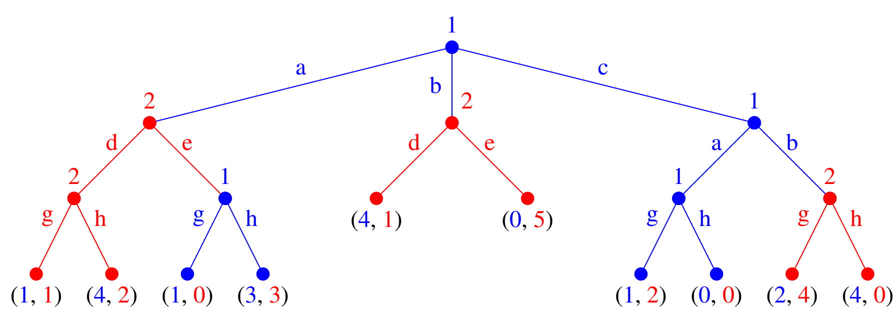
Solution: $\alpha = 8$. We need to solve the equation

$$\begin{aligned} -s(a) + \alpha \cdot s(b) &= 3s(a) - 4s(b) \\ -s(a) + \alpha \cdot (1 - s(a)) &= 3s(a) - 4(1 - s(a)) \\ \alpha \cdot (1 - s(a)) &= -4 + 8s(a) \end{aligned}$$

Using that $s(a) = \frac{3}{4}$, we have that

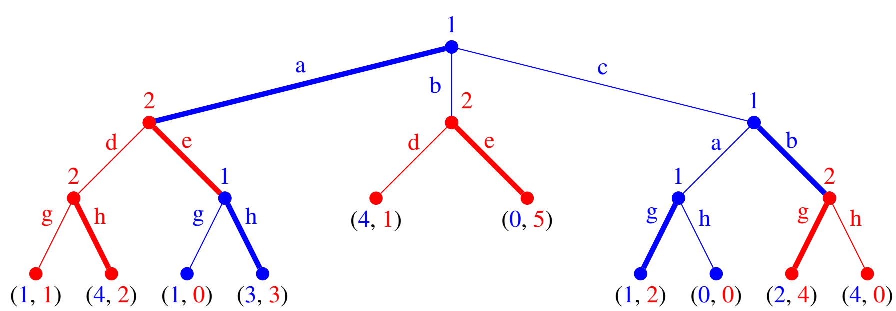
$$\begin{aligned} \alpha \cdot (1 - \frac{3}{4}) &= -4 + 8 \cdot \frac{3}{4} \\ \frac{\alpha}{4} &= 2 \\ \alpha &= 8 \end{aligned}$$

Question 2 (20 marks) Consider the following extensive-form game.



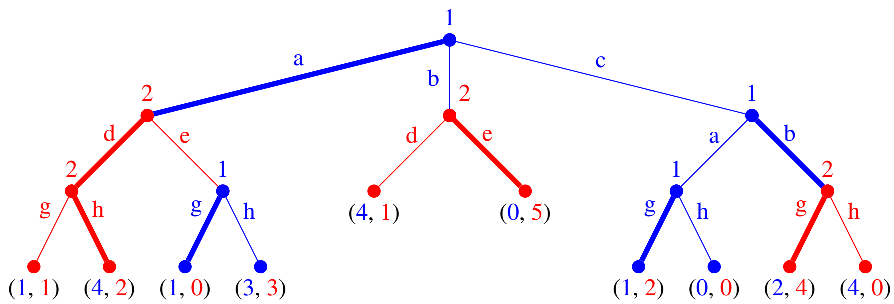
a) Compute the subgame-perfect Nash equilibrium. No explanation is required for this task.

Solution: The subgame-perfect Nash equilibrium is given by the highlighted actions.



b) Is there a pure Nash equilibrium where player 1 has a utility of 4? Explain your answer!

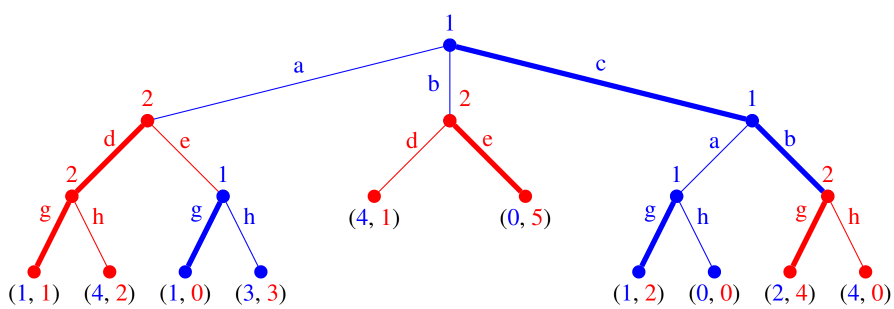
Solution: Yes, consider the following strategy.



Clearly, player 1 has no incentive to deviate as 4 is his maximum utility. Given the strategy of player 1, player 2 can only obtain a utility of 2 regardless of his strategy, so he also has no incentive to deviate.

c) What is the maximum utility that player 2 can obtain in a pure Nash equilibrium?

Solution: The maximum utility player 2 can ensure in a pure Nash equilibrium is 4. Consider the following strategies.



Player 1 has no incentive to deviate. If he plays in his first move a, he will end up with a utility of 1, and if he plays b, he gets a utility of 0. Finally, if player 1 plays c and as the second move a, he again gets a utility of 1. On the other hand, given the strategy of player 1, player 2 can only choose between getting a utility of 4 or 0.

Lastly, we note that player 2 cannot get a utility of 5 in a pure Nash equilibrium as this guarantees player 1 only a utilty of 0. However player 1 can always play c, then a, then g to obtain a utility of 1, so the outcome $(0,5)$ cannot be played in a pure Nash equilibrium.

Question 3 (20 marks) Three pirates find a treasure of 90 gold coins. They decide on the following protocol to split the gold coins. First, all pirates $i \in \{1,2,3\}$ submit two numbers $t_i \in \{0,\dots,90\}$ and $k_i \in \{0,\dots,90\}$. Then, the pirates 1, 2, 3 (in this order) take t_i coins from the treasure if there are sufficiently many gold coins left or all remaining coins otherwise. Next, pirate 2 checks the amount of coins pirate 1 took: if pirate 1 has more than k_2 gold coins, pirate 2 steals all coins of pirate 1; if pirate 1 has at most k_2 coins, he keeps his coins and nothing happens. Finally, pirate 3 checks how much coins pirate 2 has (including those he possibly stole from pirate 1): if this amount exceeds k_3, he steals all coins from pirate 2 and otherwise, pirate 2 keeps his coins.

Assume that the utility function of every pirate is equal to the number of gold coins he has in the end if he did not steal the gold coins of his predecessor and half of that otherwise.

a) Show that there is a pure Nash equilibrium where all pirates have 30 gold coins.

Solution: Let $t_i = k_i = 30$ for every pirate $i \in \{1,2,3\}$. With these strategies, every pirate ends up with 30 gold coins as no other pirate steals the coins of his predecessor. Clearly, no pirate has an incentive to take less gold coins as this only makes him worse off. On the other hand, if pirate 1 reports $t_1' > t_1$, pirate 2 will steal his coins, leaving him with 0. Since the value k_1 does not matter, this shows that pirate 1 has no incentive to deviate. Next, if pirate 2 plays a strategy (t_i', k_i') that leads to him having more than 30 gold coins before pirate 3 checks his coins, he will lose all of them, leaving him with a utility of 0. Hence, the maximum utility of pirate 2 given the strategy of pirate 3 is 30, which means that he can also not deviate. Finally, if pirate 3 reports $t_3' > t_3$, this will not affect the process because pirates 1 and 2 only leave 30 gold coins. Similarly, reporting $k_3' > k_3$ does not change the outcome. As the last point, if pirate 3 reports $k_3' < k_3$, he steals the gold coins of pirate 2, which results in him having 60 gold coins. However, as his utility is half of the gold coins if he steals the coins from his predecessor, this does not make him better off. This shows that also pirate 3 has no incentive to deviate, so our strategy profile is indeed a pure Nash equilibrium.

b) What is the maximal amount of gold coins that each pirate can obtain in a pure Nash equilibrium? Present for each pirate the corresponding strategy profile and reason why it is a pure Nash equilibrium.

Solution: There are multiple correct solutions here, but they all follow the same patterns. Central values that are shared between solutions are marked in bold.

- Pirate 1 can obtain all 90 gold coins in a pure Nash equilibrium. Assume that $\mathbf{t_1 = 90}$, $k_1 = 0$, $t_2 = 90$, $\mathbf{k_2 = 90}$, $t_3 = 90$, $\mathbf{k_3 = 0}$. Given these strategies, pirate 1 will take all coins. Pirate 2 is not willing to steal his coins as $k_2 = 90$, so he ends up with 0 coins. Lastly, pirate 3 ends up with 0 coins as he can steal none of pirate 2 nor get any from the treasure. In this profile, pirate 1 clearly has no incentive to deviate as he gets his maximum possible utility. Pirate 2 also has no incentive to deviate: if he gets even just a single coin, pirate 3 will steal it, leaving pirate 2 again with a utility of 0. Finally, regardless of the strategy of pirate 3, he will end up with no gold coins, so he also has no incentive to deviate.
- Pirate 3 can obtain all 90 gold coins in a pure Nash equilibrium. Assume that $t_1 = 90$, $k_1 = 0$, $t_2 = 90$, $\mathbf{k_2 = 0}$, $t_3 = 90$, $\mathbf{k_3 = 0}$. In this game, pirate 1 takes all coins, the pirate 2 steals his coins, and then pirate 3 takes all coins of pirate 3. Moreover, no agent has an incentive to deviate: regardless of what pirates 1 and 2 report, they will end up with 0 gold coins as their successor steals their coins. On the other hand, pirate 3 cannot benefit by deviating: his only other option is to not steal the coins of pirate 2, leaving him with a utility of 0. (Note that

pirate 3 does not obtain his maximal possible utility as he steals the coins from player 2 rather than taking them from the treasure!)

- Pirate 2 can obtain at most 45 gold coins in a pure Nash equilibrium. First, to show that he can obtain 45 gold coins, consider the following strategy profile: $t_1 = 0$, $k_1 = 0$, $t_2 = 45$, $\mathbf{k_2 = 0}$, $\mathbf{t_3 = 45}$, $\mathbf{k_3 = 45}$. In this game, no player steals the coins of his predecessor, so pirate 1 ends with 0 coins, pirate 2 with 45, and pirate 3 with 45. Moreover, no agent has an incentive to deviate: pirate 1 will end up with 0 coins regardless of his strategy as pirate 2 will steal his coins if he takes some. Pirate 2 cannot take more than 45 coins as otherwise pirate 3 steals his coins. Finally, pirate 3 can only increase his coins by stealing those of pirate 3. However, while this leaves him 90 coins, his utility is only 45, so he did not get better off. This shows that no pirate can deviate.
  Next, we show that pirate 2 cannot have more gold coins in a pure Nash equilibrium. In particular, if pirate 2 ends up with $x > 45$ coins in a game, then pirate 3 did not steal his coins and has $y < 45$ games in the end. This means that the utility of pirate 3 is y. On the other hand, if pirate 3 changes his strategy to steal the coins of pirate 2 (i.e., $k_3 = 0$), then he will end up with $x + y$ coins, giving him a utility of $\frac{x+y}{2} > y$. Hence, in any such action profile, player 3 has an incentive to deviate.

Question 4 (10 marks) Let $A = \{a, b, c\}$ and $\succsim$ denote a rational preference relation over $\mathscr{L}(A)$ that is independent and satisfies that $[1 : a] \succ [1 : b]$ and $[\frac{1}{2} : a, \frac{1}{2} : c] \sim [1 : b]$. Show that $[1 : b] \succ [1 : c]$.

Solution: We will first show that $[1 : a] \succ [1 : c]$. To this end, we observe that, by transitivity, $[1 : a] \succ [\frac{1}{2} : a, \frac{1}{2} : c]$. Independence then implies that $[1 : a] \succ [1 : c]$. By independence, it then follows that $[\frac{1}{2} : a, \frac{1}{2} : c] \succ [1 : c]$. We hence have that $[1 : b] \sim [\frac{1}{2} : a, \frac{1}{2} : c] \succ [1 : c]$, so transitivity implies that $[1 : b] \succ [1 : c]$.

Question 5 (30 marks) Prove the following statements.

a) Let $G_1 = (\{1, 2\}, (A^1_i)_{i\in\{1,2\}}, (u^1_i)_{i\in\{1,2\}})$, $G_2 = (\{1, 2\}, (A^2_i)_{i\in\{1,2\}}, (u^2_i)_{i\in\{1,2\}})$, and $G_3 = (\{1, 2\}, (A^3_i)_{i\in\{1,2\}}, (u^3_i)_{i\in\{1,2\}})$ denote three two-player normal-form games such that $A^1_i = A^2_i = A^3_i$ for $i \in \{1, 2\}$ and $u^3_i(a) = \frac{1}{2}(u^1_i(a) + u^2_i(a))$ for both players $i \in \{1, 2\}$ and all action profiles $a \in A$. Show that, if a strategy profile s is a Nash equilibrium for G_1 and G_2, then it is a Nash equilibrium for G_3.

   Solution: Fix three games G_1, G_2, and G_3 as given in the statement and assume that s is a Nash equilibrium for both games. Now, assume for contradiction that s is not a Nash equilibrium for G_3. This means that one of the players can deviate. Without loss of generality, assume that player 1 increases his expected utility by deviating from s to (t, s_2). This means that $u^3_1(t, s_2) > u^3_1(s)$. Using the definition of u^3_1, it follows that

$$\frac{1}{2}u^1_1(t, s_2) + \frac{1}{2}u^2_1(t, s_2) > \frac{1}{2}u^1_1(s) + \frac{1}{2}u^2_1(s).$$

   This means that either $u^1_1(t, s_2) > u^1_1(s)$ or $u^2_1(t, s_2) > u^2_1(s)$, thus showing that player 1 can beneficially deviate from s in G_1 or G_2. This contradicts the assumption that

s is a Nash equilibrium in both of these games, so the assumption that s is no Nash equilibrium in G_3 must be wrong.

b) Let $G = (\{1,2\}, (A_i)_{i\in\{1,2\}}, (u_i)_{i\in\{1,2\}})$ denote a two-player normal-form game and let s^1 and s^2 denote two Nash equilibria for G such that $s_i^1(a_i) > 0$ if and only if $s_i^2(a_i) > 0$ for both players $i \in \{1,2\}$ and all actions $a_i \in A_i$. Show that the strategy profile s^3 given by $s_1^3(a_1) = \frac{1}{2}(s_1^1(a_1) + s_1^2(a_1))$ for all $a_1 \in A_1$ and $s_2^3(a_2) = s_2^1(a_2)$ for all $a_2 \in A_2$ is also a Nash equilibrium for G.

Solution: Fix a game G as given in the task and let s^1 and s^2 denote two Nash equilibria for G such that $s^1(a_i) > 0$ if and only if $s_i^2(a_i) > 0$ for both players $i \in \{1,2\}$ and all actions $a_i \in A_i$. Moreover, define s^3 by $s_1^3(a_1) = \frac{1}{2}(s_1^1(a_i) + s_1^2(a_i))$ and $s_2^3(a_2) = s_2^1(a_2)$. By the indifference principle, we have that $u_1(a, s_2^1) = u_1(b, s_2^1) \geq u_1(c, s_2^1)$ for all $a, b, c \in A_1$ with $s_1^1(a) > 0$, $s_1^1(b) > 0$, and $s_1^1(c) = 0$. Next, since s_1^1 and s_1^2 have the same support, we have that $s_1^3(a) > 0$ if and only if $s_1^1(a) > 0$ for all $a \in A_1$. This implies that $u_1(a, s_2^1) = u_1(b, s_2^1) \geq u_1(c, s_2^1)$ for all $a, b, c \in A_1$ with $s_1^3(a) > 0$, $s_1^3(b) > 0$, and $s_1^3(c) = 0$. Since $s_2^1 = s_2^3$, player 1 has no incentive to deviate in s^3.

Next, we can again use the indifference principle to infer that $u_2(s_1^1, a) = u_2(s_1^1, b) \geq u_2(s_1^1, c)$ for all $a, b, c \in A_2$ with $s_2^1(a) > 0$, $s_2^1(b) > 0$, and $s_2^1(c) = 0$, and that $u_2(s_1^2, a) = u_2(s_1^2, b) \geq u_2(s_1^2, c)$ for all $a, b, c \in A_2$ with $s_2^2(a) > 0$, $s_2^2(b) > 0$, and $s_2^2(c) = 0$. Because $s_2^1(a) > 0$ if and only if $s_2^2(a) > 0$, we hence conclude that

$$\frac{1}{2}u_2(s_1^1, a) + \frac{1}{2}u_2(s_1^2, a) = \frac{1}{2}u_2(s_1^1, b) + \frac{1}{2}u_2(s_1^2, b) \geq \frac{1}{2}u_2(s_1^1, c) + \frac{1}{2}u_2(s_1^2, c).$$

for all actions $a, b, c \in A_2$ with $s_2^1(a) > 0$, $s_2^1(b) > 0$, and $s_2^1(c) = 0$. Since $s_2^1 = s_2^3$, this shows that $u_2(s_1^3, a) = u_2(s_1^3, b) \geq u_2(s_1^3, c)$ for all such alternatives. Hence, player 2 can also not benefit by deviating and s^3 is indeed a Nash equilibrium.

c) Let $G = (\{1,2\}, (A_i)_{i\in\{1,2\}}, (u_i)_{i\in\{1,2\}})$ denote a two-player zero-sum game such that $A_1 = A_2 = \{a_1, \dots, a_k\}$ for $k > 1$ and $u_1(a_i, a_j) = u_2(a_j, a_i)$ for all $a_i, a_j \in A_1 = A_2$. Show that the value of G (i.e., the security level of player 1) is 0.
Hint: Show that for every strategy of player 2, player 1 has a strategy that guarantees him an expected utility of at least 0.

Solution: Before giving the general proof, we will consider the special case that $k = 2$ to give some more intuition. Thus let G denote a two-player zero-sum game that satisfies our symmetry condition and that $A_1 = A_2 = \{x, y\}$. First, by symmetry and zero-sum, it follows that $u_1(x,x) = -u_2(x,x) = -u_1(x,x)$ and $u_1(y,y) = -u_2(y,y) = u_1(y,y)$, so we derive that $u_1(x,x) = u_1(y,y) = 0$. Furthermore, our assumptions imply that $u_1(x,y) = -u_2(x,y) = -u_1(y,x)$. Now, let player 2 play an arbitrary strategy s_2. We will show that player 1 has a expected utility of 0 if he plays $s_1 = s_2$. To this end, we note that

$$\begin{aligned}
u_1(s_1, s_2) &= s_1(x) \cdot s_2(x) \cdot u_1(x,x) + s_1(x) \cdot s_2(y) \cdot u_1(x,y) + s_1(y) \cdot s_2(x) \cdot u_1(y,x) + s_1(y) \cdot s_2(y) \cdot u_1(y,y) \\
&= s_2(x) \cdot s_2(x) \cdot 0 + s_2(x) \cdot s_2(y) \cdot u_1(x,y) + s_2(y) \cdot s_2(x) \cdot (-u_1(x,y)) + s_2(y) \cdot s_2(y) \cdot 0 \\
&= 0
\end{aligned}$$

This means that player 2 has a security level of at most 0. A symmetric argument works for player 1, so we can now derive that the value of the game is 0.

Now, to generalize this to arbitrary games, ;et $[k] = \{1, 2, \dots, k\}$, fix a game G satisfying all criteria, and let s_2 denote an arbitrary strategy of player 2. We define the strategy of player s_1 of player 1 again by $s_1(a) = s_2(a)$ for all actions $a \in A_1 = A_2$, i.e., player 1 simply plays the same strategy as player 2. We can now compute that

$$\begin{aligned}
u_1(s_1, s_2) &= \sum_{i\in[k]} \sum_{j\in[k]} s_1(a_i) \cdot s_2(a_j) \cdot u_1(a_i, a_j) \\
&= \sum_{i\in[k]} \sum_{j=1}^{i-1} s_1(a_i) \cdot s_2(a_j) \cdot u_1(a_i, a_j) \\
&\quad + \sum_{i\in[k]} s_1(a_i) \cdot s_2(a_i) \cdot u_1(a_i, a_i) \\
&\quad + \sum_{i\in[k]} \sum_{j=i+1}^{k} s_1(a_i) \cdot s_2(a_j) \cdot u_1(a_i, a_j).
\end{aligned}$$

Now, we first note that $u_1(a_i, a_i) = u_2(a_i, a_i) = -u_1(a_i, a_i)$ by our assumption and the fact that our game is zero-sum. This means that $u_1(a_i, a_i) = 0$ for all $a_i \in A_1$. By a similar argument, we have that

$$\begin{aligned}
\sum_{i\in[k]} \sum_{j=i+1}^{k} s_1(a_i) \cdot s_2(a_j) u_1(a_i, a_j) &= \sum_{i\in[k]} \sum_{j=i+1}^{k} s_1(a_i) \cdot s_2(a_j) \cdot u_2(a_j, a_i) \\
&= -\sum_{i\in[k]} \sum_{j=i+1}^{k} s_1(a_i) \cdot s_2(a_j) \cdot u_1(a_j, a_i) \\
&= -\sum_{j\in[k]} \sum_{i=1}^{j-1} s_1(a_i) \cdot s_2(a_j) \cdot u_1(a_j, a_i) \\
&= -\sum_{i\in[k]} \sum_{j=1}^{i-1} s_1(a_i) \cdot s_2(a_j) \cdot u_1(a_i, a_j).
\end{aligned}$$

The second two last equation reorders our sum and the last equation renames i to j (and vice versa) and uses that $s_1(a_i) = s_2(a_i)$. We derive now that

$$u_1(s_1, s_2) = \sum_{i\in[k]} \sum_{j=1}^{i-1} s_2(a_i) \cdot s_2(a_j) u_1(a_i, a_j) - \sum_{i\in[k]} \sum_{j=1}^{i-1} s_2(a_i) \cdot s_2(a_j) u_1(a_i, a_j) = 0.$$

Since this holds for all strategies of player 2, we know that the security level v_2 of player 2 is at most 0. Conversely, a symmetric analysis shows that the security level v_1 of player 1 is at most 0. Finally, we have that $v_1 = -v_2$ by the Minimax theorem, so it follows that $v_1 = v_2 = 0$, which proves the claim.