Matching II

COMP4418 Knowledge Representation and Reasoning

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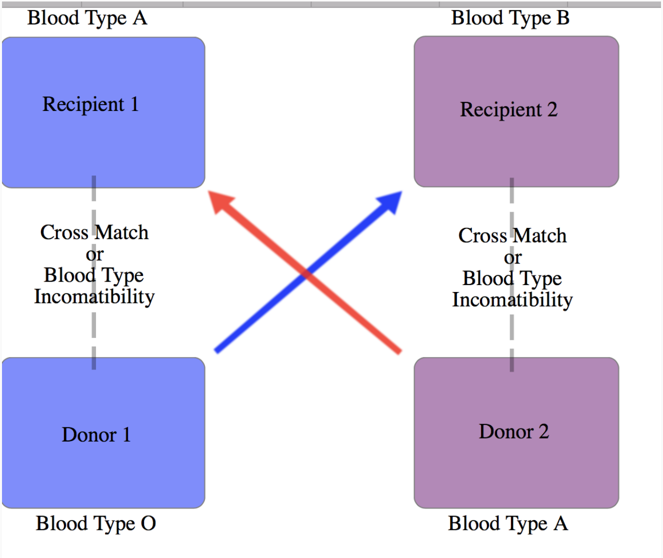
How to assign donated kidneys?



How to allocate tasks to drones?

How to match employers to employees?

Allocation of indivisible items with endowments

(Shapley-Scarf) Housing market: simple model with endowments

$(N, O, \succ, e)$

- N is the set of agents
- O is the set of items (also sometimes called houses)
- $|N| = |O|$
- $e_i = \{o\}$ iff o is owned by $i \in N$.
- Agents have strict preferences over items
- Each agent owns and is allocated one item.

(Shapley-Scarf) Housing market: simple model with endowments

A matching X matches each to each agent i its match which is denoted by X_i or $X(i)$ (both notation are used in the literature).

(Shapley-Scarf) Housing market: simple model with endowments

Example
Housing market $(N, O, e, \succ)$ such that

- $N = \{1, \ldots, 5\}$, $O = \{o_1, \ldots, o_5\}$,
- $e_i = \{o_i\}$ for all $i \in \{1, \ldots, 5\}$
- and preferences $\succ$ are defined as follows:

agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5

Individual rationality

An allocation X is *individually rational* if no agent minds participating in the allocation procedure:

$$\forall i \in N : \quad X_i \succsim_i e_i$$

If an agent does not have any endowment, her allocation is *individually rational* if her allocation is acceptable (at least as preferred as the empty allocation).

An agent can express an allocation or an item as unacceptable by simply not lising it in the preference list.

Allocation with endowments: Core

An allocation X is *core stable* if there exists no is 'blocking coalition' $S \subseteq N$ such that there exists an allocation Y of the items in $\bigcup_{i \in S} e_i$ to the agents in S such that

$$\forall i \in S : \quad Y_i \succ_i X_i$$

Fact
A core stable allocation is individually rational.

Allocation with endowments: Pareto Optimality

An allocation x is *Pareto optimal* if there exists no other allocation y such that $y(i) \succsim_i x(i)$ for all $i \in N$ and $y(i) \succ_i x(i)$ for some $i \in N$.

Under strict preferences, core stability implies Pareto optimality.

Argument: Suppose some allocation is not Pareto optimal. If the allocation is not individually rational, then it is also not core stable so we assume it is individually rational. If an agent gets the same house as its owned one, we ignore it. For other agents, let us call the set N'. Each agent in N' gets a strictly better house. Hence the agents in N' forms a core blocking coalition.

Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

Input : Housing Market Instance

Output: Allocation X

1: Construct the corresponding directed graph $G(\succsim) = (V, E)$ where $V = N \cup O$ and E is specified as follows: each house points to its owner and each agent points to the most preferred house in the graph.

2: **while** G is not empty **do**

3: Start from an agent and walk arbitrarily along the edges until a cycle is completed.

Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

Remove the cycle is removed from $G(\succsim)$. Within the removed cycle, each agent gets the house he was pointing to in G.

The graph $G(\succsim)$ is *adjusted* so that the remaining agents point to the most preferred houses among the remaining houses.

end while

Return X.

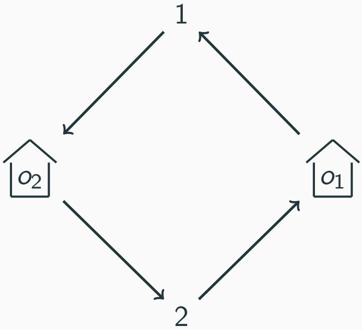
Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

- Each item points to its owner.
- Each agent points to her most preferred item in the graph.
- Find a cycle, allocate to each agent in the cycle the item she was pointing to. Remove the agents and items in the cycle. Adjust the graph so the agents in the graph point to their most preferred item in the graph.
- Repeat until the graph is empty.

Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

agents	1	2
item owned	o_1	o_2

agents	1	2
preferences	o_2	o_1
	o_1	o_2



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item owned	o_1	o_2
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preferences	$\underline{o_2}$	$\underline{o_1}$
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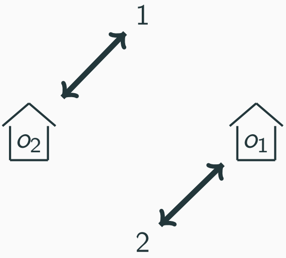

Housing Market Example

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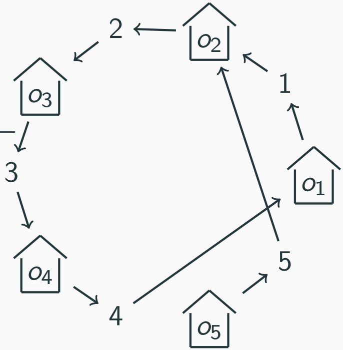
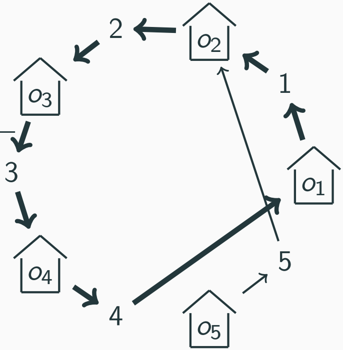


Example: TTC

agent	1	2	3	4	5
preferences	o_2	o_3	o_4	o_1	o_2
	o_1	o_2	o_3	o_5	o_4
				o_4	o_5



Housing Markets: Gale's Top Trading Cycles (TTC) Algorithm

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preferences	$\underline{o_2}$	$\underline{o_3}$	$\underline{o_4}$	$\underline{o_1}$	o_2
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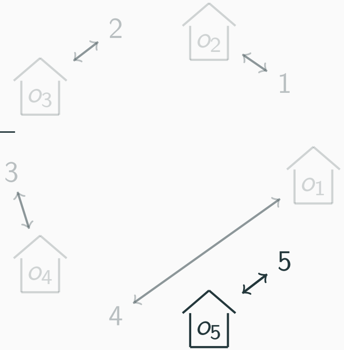

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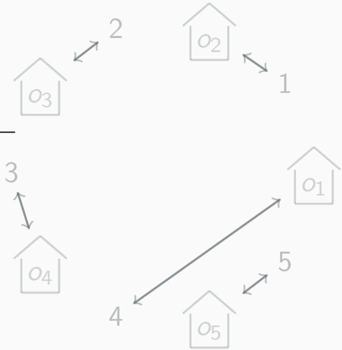

TTC (Top Trading Cycles)

Theorem (Shapley and Scarf [1974] and Roth and Postlewaite [1977])
For housing markets (with strict preferences), TTC is strategyproof, individually rational, Pareto optimal and core stable.

Theorem (Ma [1994])
For housing markets (with strict preferences), a mechanism is strategyproof, individually rational and Pareto optimal iff it is TTC.

TTC (Top Trading Cycles)

Core: We prove by induction on the cycles computed $C_1, C_2, \ldots$ that no agent in the cycles forms a core deviation. In C_1, each agent gets their most preferred house. None of them will deviate and their houses are not allocated to any other agents outside C_1. The same argument is repeated for the next cycle.

Strategyproofness:

Fix some agent i.

Suppose agent i were to misreport and create a cycle by pointing to their kth-favorite house h_k in the current graph for some $k > 1$. The algorithm would then give them that house. We need to argue they will do at least as well by reporting truthfully.

TTC (Top Trading Cycles)

When i gets h_k, we know that there is a path from h_k to i to form a cycle. Consider the path P in the current graph going from h_k to i. Until i is matched, this path never goes away. That's because the only time that an agent changes their out-edge is when the endpoint of their out-edge has disappeared.

This means that agent i can defer pointing to h_k until there is no better house to point to (which is what the algorithm does on their behalf when they report truthfully). QED.

Size of profile $\succsim$: $\sum_{i \in N} |h \succsim_i o_i|$.
[Count for the agents the number of items that are at least as preferred as the endowed item]

TTC (Top Trading Cycles)

Theorem: For the housing market problem, there is at most one strategy-proof mechanism that is individually rational and Pareto efficient.

Proof

Let f and g be two distinct mechanisms with the properties.

Claim: Let $\succ$ be a profile of minimal size where $f(\succ) \neq g(\succ)$ Then each agent in $\succ$ has exactly two acceptable houses.

Proof of Claim: If someone has exactly one acceptable house, the profile is not minimal as that agent and endowment can be removed. Suppose for contradiction, one agent i has more than 2 acceptable houses. Denote by $f_i^{\succ}$ the house given to agent i under

TTC (Top Trading Cycles)

$f(\succ)$. If $f_i^{\succ} = g_i^{\succ}$, we can modify the profile and only make that allocated house acceptable. Therefore, suppose $f_i^{\succ} \neq g_i^{\succ}$ and that $f_i^{\succ} \succ_i g_i^{\succ}$. Modify $\succ$ to $\succ'$ by only making $f_i^{\succ}$ and o_i acceptable to i.

By strategyproofness of f, we know that $f_i^{\succ} = f_i^{\succ'}$. By strategyproofness of g, we get that $g_i^{\succ'} = o_i$. If this were not the case and $g_i^{\succ'} = f_i^{\succ}$, then g is not strategyproof as i can manipulate and get a better outcome under rule g and preference $\succ$ by reporting $\succ'$. We have proven that $\succ'$ is a smaller profile in which f and g disagree. Hence, the minimal profile one in which each agent has exactly two acceptable houses. The claim is proved. $\square$

TTC (Top Trading Cycles)

We have established the claim. We now consider profiles $\succ$ where each agent has exactly two acceptable houses and $f(\succ) \neq g(\succ)$. Let N_f be the agents who strictly prefer outcome under f to g. Let N_g be the agents who strictly prefer outcome under g to f. Then for very agent in N_f, her favourite house in the endowment of another agent in N_f. Same for agents in N_g. By definition, $N_f \cap N_g = \emptyset$. Since f and g differ on $\succ$, at least one of N_f or N_g is non-empty. Without loss of generality, suppose $N_f \neq \emptyset$. In that case, consider the outcome which N_f trade among themselves to get their most preferred houses and N_g trade among themselves to get their most preferred houses. The outcome Pareto dominates $g(\succ)$ which is a contradiction.QED.

Restrictions on Cycle Size

Restrictions on Cycle Size

L-allocations: We consider allocations that are a result of disjoint exchange cycles such that there are at most L agents in each cycle.

Theorem
If $L = 3$, agents have ties/indifferences in their preferences, checking whether there exists an L-allocation in which each agent gets maximally preferred item is NP-complete

Proof:

The Cycle Cover Problem is defined as follows: given a directed graph $G = (V, E)$, determine whether there exists a collection $\{C_1, , C_2, \ldots, C_k\}$, where each C_i is a simple directed cycle in G (i.e., a cycle with no repeated vertices), such that for every vertex v in G, there is a unique cycle C_i containing it.

Restrictions on Cycle Size

The 3-Cycle Cover Problem is defined identically to the Cycle Cover Problem except for the additional constraint that each cycle must have length at most three. The 3-Cycle Cover Problem is NP-complete. Reduce the problem where each vertex corresponds to an agent and an agent i only finds only those items acceptable that are owned by agents in $N_i = \{j \mid (i, j) \in E\}$. $\square$

Multiple endowments

General Exchange Markets

Lexicographic Preferences:

$\{a_1, \ldots, a_k\} \succ_i \{b_1, \ldots, b_k\}$ iff for the smallest j such that $a_j \not\sim_i b_j$, it holds that $a_j \succ_i b_j$. For sets of unequal sizes, we can pad the smaller set with dummy items to make the comparison.

Example: $i : a \succ_i b \succ_i c$

$\{a, c\} \succ_i^{lex} \{b, c\}$; $\{b, c\} \succ_i^{lex} \{c\}$.

Theorem
Under the lexicographic preference domain, there exists no exchange rule that satisfies strategyproofness, individual rationality, and Pareto efficiency.

General Exchange Markets

Suppose $N = \{1, 2\}$; $H = \{a, b, c\}$; $\omega_1 = \{a, b\}$ and $\omega_2 = \{c\}$.
The agent's true preferences are:

$$\begin{aligned} 1: &\quad c, a, b \\ 2: &\quad a, b, c \end{aligned}$$

If both report truthfully, the allocations that are Pareto optimal and IR are (ac, b), (bc, a), (c, ab).

If the outcome is (ac, b), then 2 can misreport

$$2': \quad a, c, b$$

Then the Pareto optimal outcomes are (bc, a), (c, ab) both of which give 2 a better outcome according to 2's true preferences.

If the outcomes are (bc, a) or (c, ab), agent 1 can misreport

$$1' : \quad a, c, b$$

The only Pareto optimal outcome is (ac, b) which improves agent 1's outcome.

Exchange Markets with Multiple Endowments

For any exchange problem, the following are compatible:

- strategyproofness and individual rationality: no exchange
- individual rationality, and Pareto efficiency: Get Pareto improvements over an individually rational outcome
- Pareto optimality and strategyproofness: Apply 'serial dictator'. Agents come in an order and get their best bundle from the remaining items.

General Exchange Market with Dichotomous Preferences

General Exchange Market with Dichotomous Preferences

An exchange market is a tuple $I = (N, O, e, D)$ where $N = \{1, \ldots, n\}$ be a set of n agents and O be the set of items.

- The vector $e = (e_1, \ldots, e_n)$ specifies the endowment $e_i \subset O$ of each agent $i \in N$. Agents have disjoint endowments.
- Each agent has a demand set $D_i \subset 2^{2^O}$.
  [D_i is a set of subsets of O]

  Each element of D_i is a bundle of items that satisfies agent i.
  [Can be viewed as a disjunction of goals]

General Exchange Market with Dichotomous Preferences

An exchange market is a tuple $I = (N, O, e, D)$ where
$N = \{1, \ldots, n\}$ be a set of n agents and O be the set of items.

- Any allocation $X = (X_1, \ldots, X_n)$ specifies the allocated bundle $X_i \subset O$ of each agent $i \in N$.

  We say that allocation X *satisfies* agent i if $X_i \supseteq d$ for some $d \in D_i$.

Example
$D_i = \{\{a\}, \{b, c\}\}$

$X_i = \{a, b\}$ satisfies agent i.

Generality of the Model

- If $|e_i| = 1$ for each $i \in N$ and $|d| = 1$ for each $d \in D_i$, the market can model a kidney exchange market
- If $|e_i| = 2$ for each $i \in N$ and $|d| = 2$ for each $d \in D_i$, the market can model a lung exchange market
- It can also model a multi-organ exchange market
- Can model altruistic donors

Strongly Individually Rationality

An allocation X is *strongly individually rational (S-IR)* if $X_i \supseteq e_i$ or $X_i \supseteq d$, for some $d \in D_i$.

An agent will only give away some of her possession
if she is satisfied in return.

Conditional Utilitarian Priority Mechanism

If an agent i is satisfied, her utility $u_i(X)$ is 1. Otherwise, it is 0.

$\rho(I)$ is the set of *feasible allocations*.

Take any permutation π of N.

$$CUP(I, \rho, \pi) = \text{argmax}_{X \in \rho(I)}\left(\sum_{i \in N} u_i(X), u_{\pi(1)}(X), \ldots, u_{\pi(n)}(X)\right).$$

CUP lexicographically first optimizes $\sum_{i \in N} u_i(X)$, then $u_{\pi(1)}(X)$, then $u_{\pi(2)}(X)$ and so on.

Conditional Utilitarian Priority Mechanism

For any feasibility restriction on the set of S-IR allocations, CUP is core stable.

Argument: Suppose that the outcome X of CUP admits core blocking coalition S for . This means that none of the agents in S got a satisfying outcome. Since the outcome is S-IR, these agents keep their endowment. They can all exchange their endowments among themselves to get a satisfying outcome. But this contradicts that X is an outcome of CUP.

Theorem (Aziz [2020])

For any feasibility restriction on the set of S-IR allocations, CUP is strategyproof.

Further Reading

T Sonmez, and U. Unver: Matching, allocation, and exchange of discrete resources, 2011.
`http://fmwww.bc.edu/EC-P/wp717.pdf`

Contact

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