Matching I

COMP4418 Knowledge Representation and Reasoning

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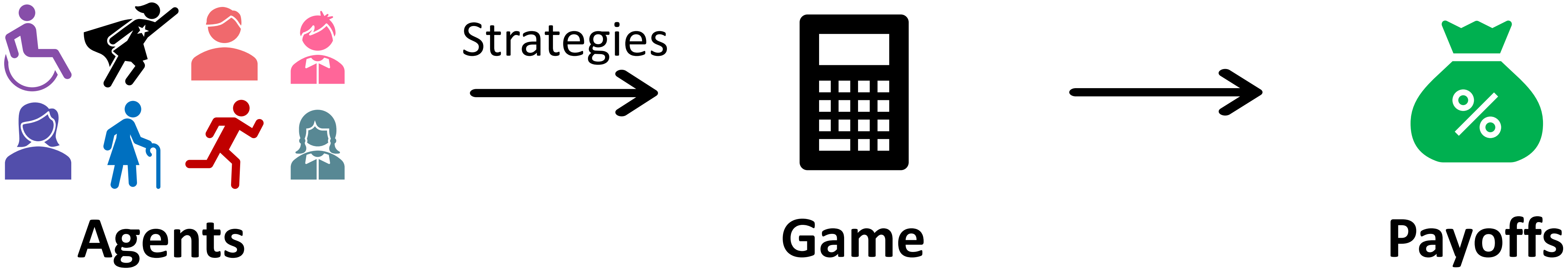
CSE, UNSW

Previously

Non-cooperative and Cooperative Games

What is a game?



Designing Games

So far:

- Games are fixed blackboxes
- Analyse the outcome

What if you could set the rules?

Mechanism Design: design the "protocol/rules" to achieve specific (desirable) outcomes.[1]

Example: King Solomon and the two mothers.

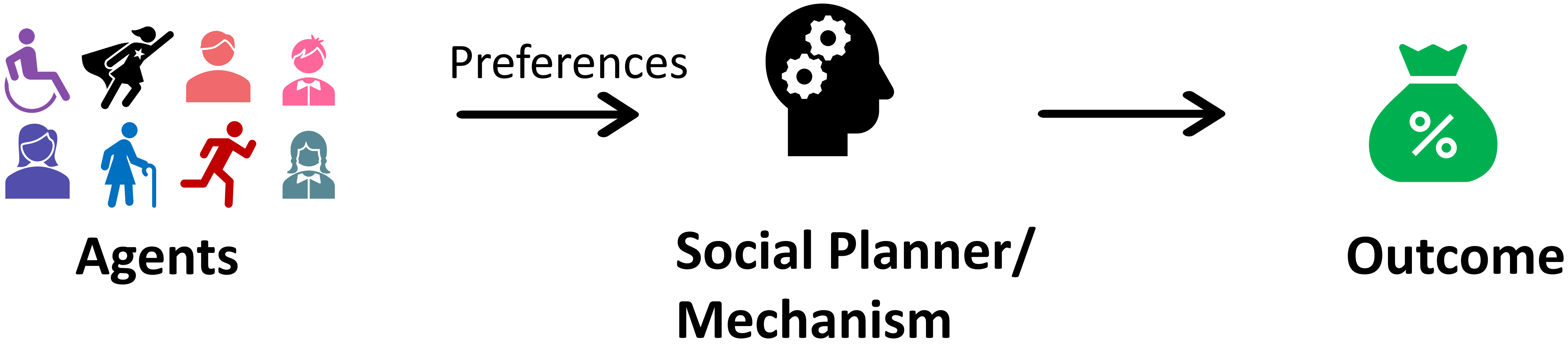
1. Y Narahari. *Game theory and mechanism design*. Vol. 4. World Scientific, 2014.

Example

Splitting a cake in half without complaints:

Cut and Choose Mechanism: Have one child cut the cake and the other choose first

Mechanism Design



Mechanism Design

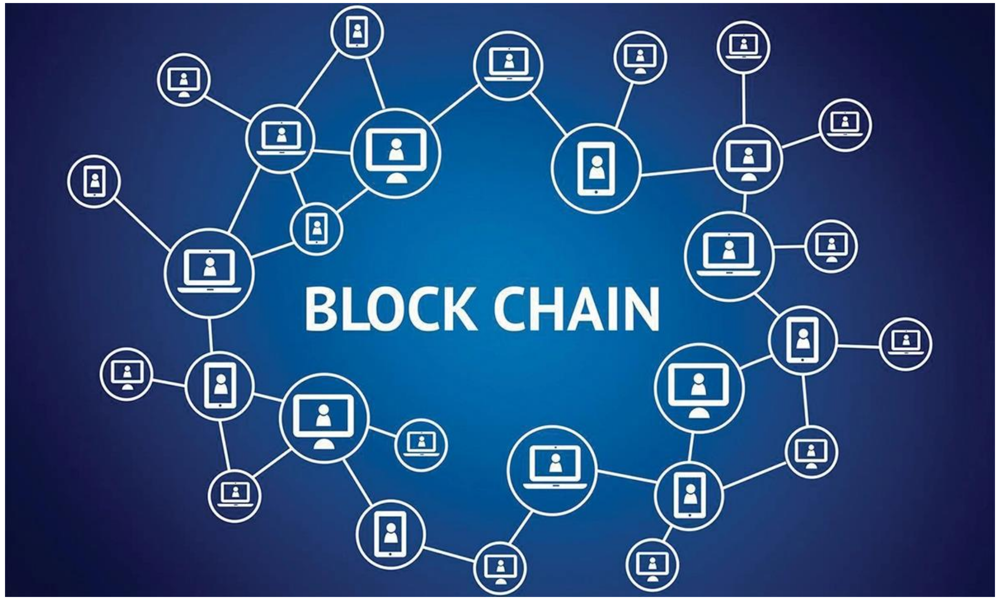
Mechanisms typically have two parts:

- **Preference Elicitation**: Ask agents for preferences
- **Preference Aggregation**: Choose an outcome based on the reported preferences.

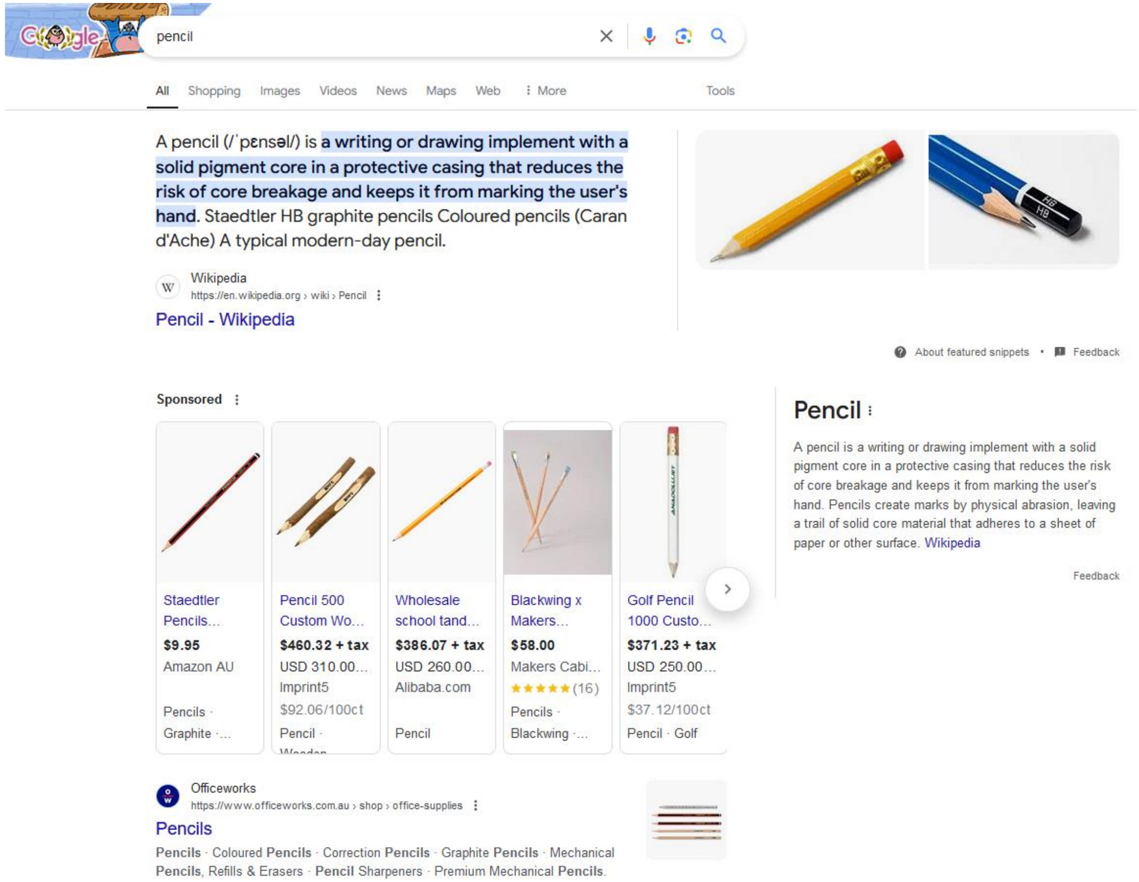
Desirable Properties:

- **Efficiency**: Pareto Optimality, Allocative Efficiency
- **Fairness**: Envy-freeness, Egalitarian Welfare
- **Strategyproofness**: All agents honestly report their preferences

Applications



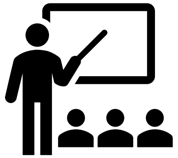
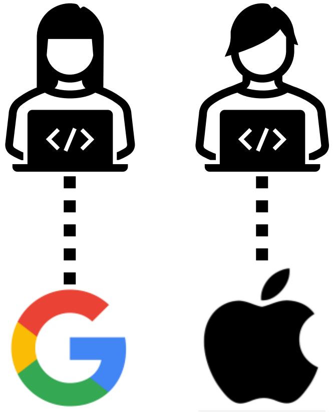
Applications

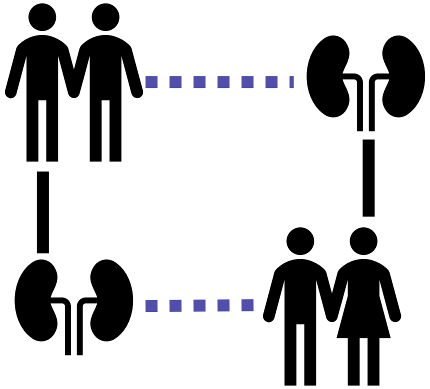


What is a Matching?

Intuition

Creating **pairs** of agents

Plus many more real life settings.

Model

Given:

Set of students S

Set of colleges C

Capacities/budgets of colleges $b = (b_c)_{c \in C}$

Need to make student-college pairs that respect the capacities

Definition

Given: students S and colleges C with budgets b, a matching $\mu \subseteq C \times S$ is such that:

i. For each student $s \in S$, μ contains at most one pair (s, c)
ii. For each college $c \in C$, μ contains at most b_c pairs (s, c).

One-one matching: $b_c = 1$ for all colleges $c \in C$

Many-to-one matching: $b_c > 1$ for some college $c \in C$

How does this relate to games and mechanism design?

Recall

Agents

Social Planner/ Mechanism

Outcomes

What are the preferences?

Preferences

Assume:

I. Each agent wants to be matched rather than unmatched

II. Agents do not care about others' matches.

III. Agent preferences are complete and transitive.

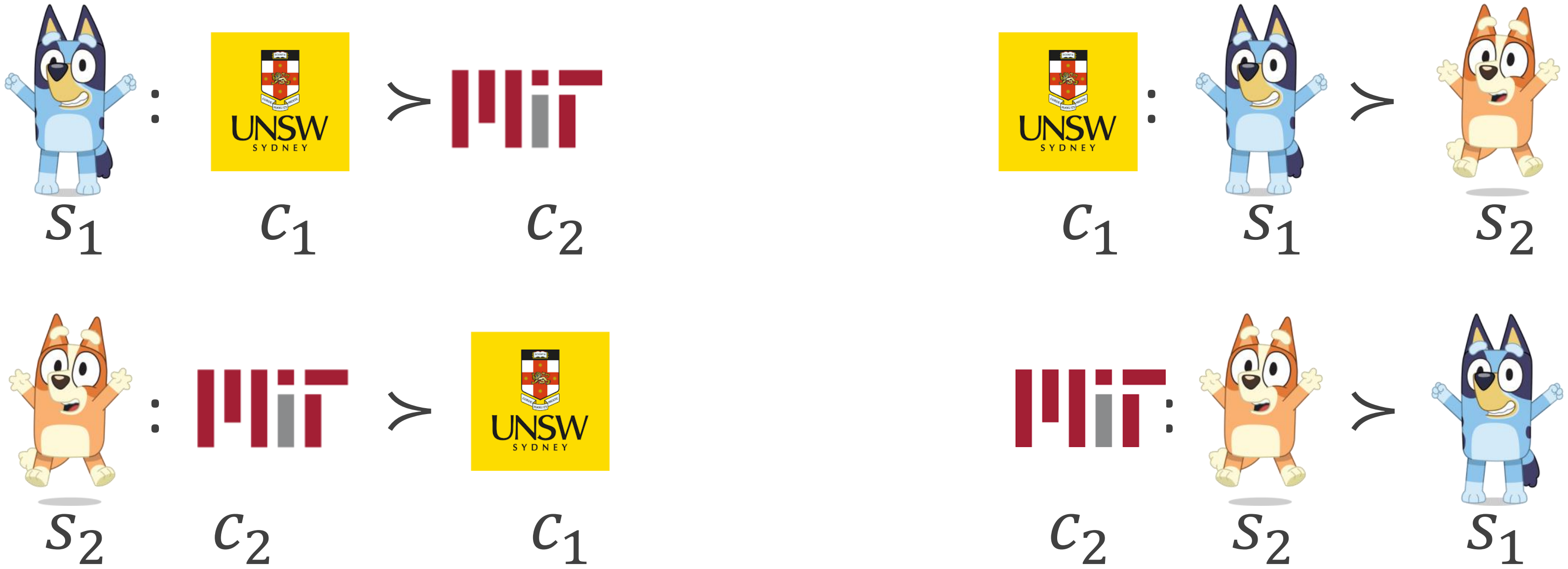
One-one matchings: Each agent has **ordinal + strict** preferences

- No ties in preferences

Many-to-one matchings: Tomorrow

Preferences

One-one matchings: Each agent has **ordinal + strict** preferences



$s_1 : c_1 \succ c_2$

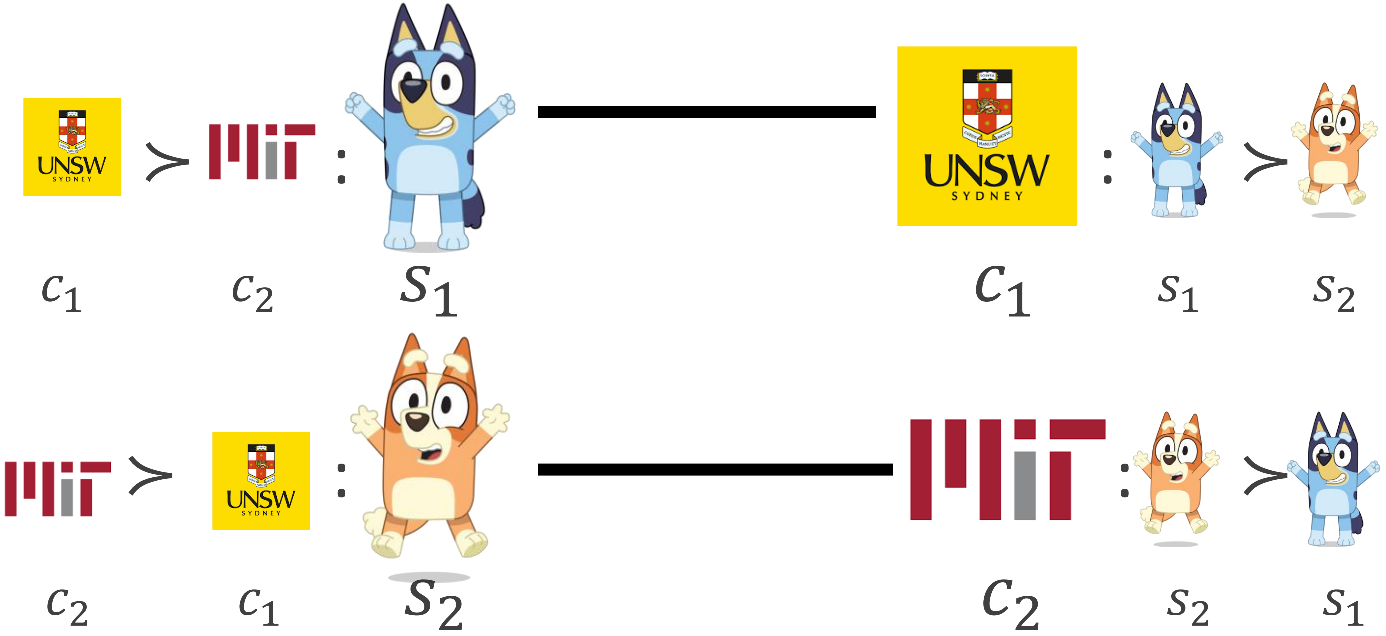
$s_2 : c_2 \succ c_1$

$c_1 : s_1 \succ s_2$

$c_2 : s_2 \succ s_1$

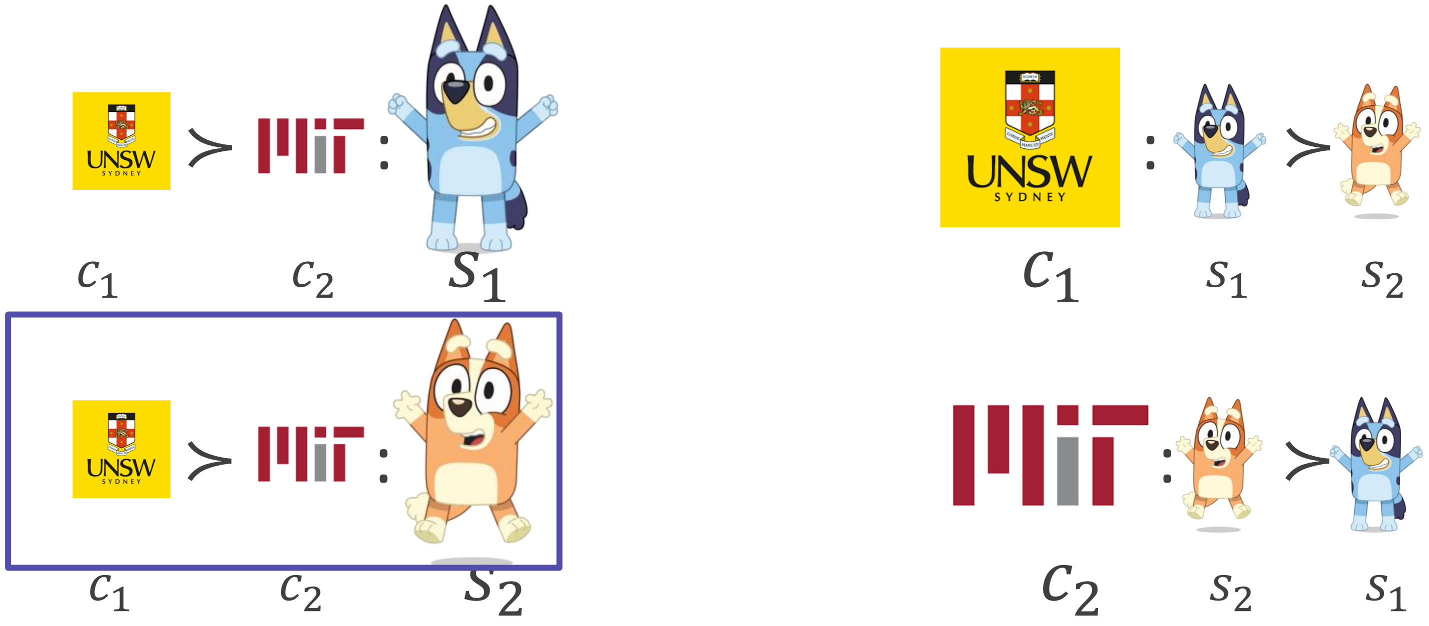
What is a Good Matching?

Finding Good Matchings

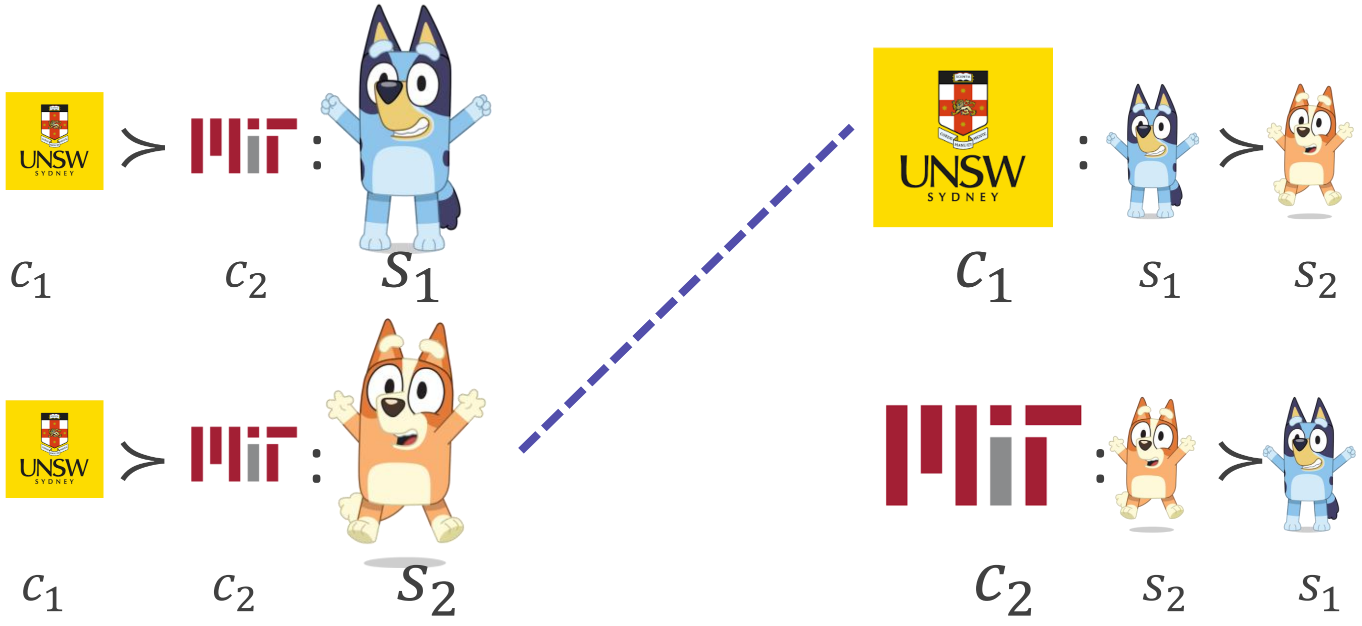


What if preferences are less than ideal?

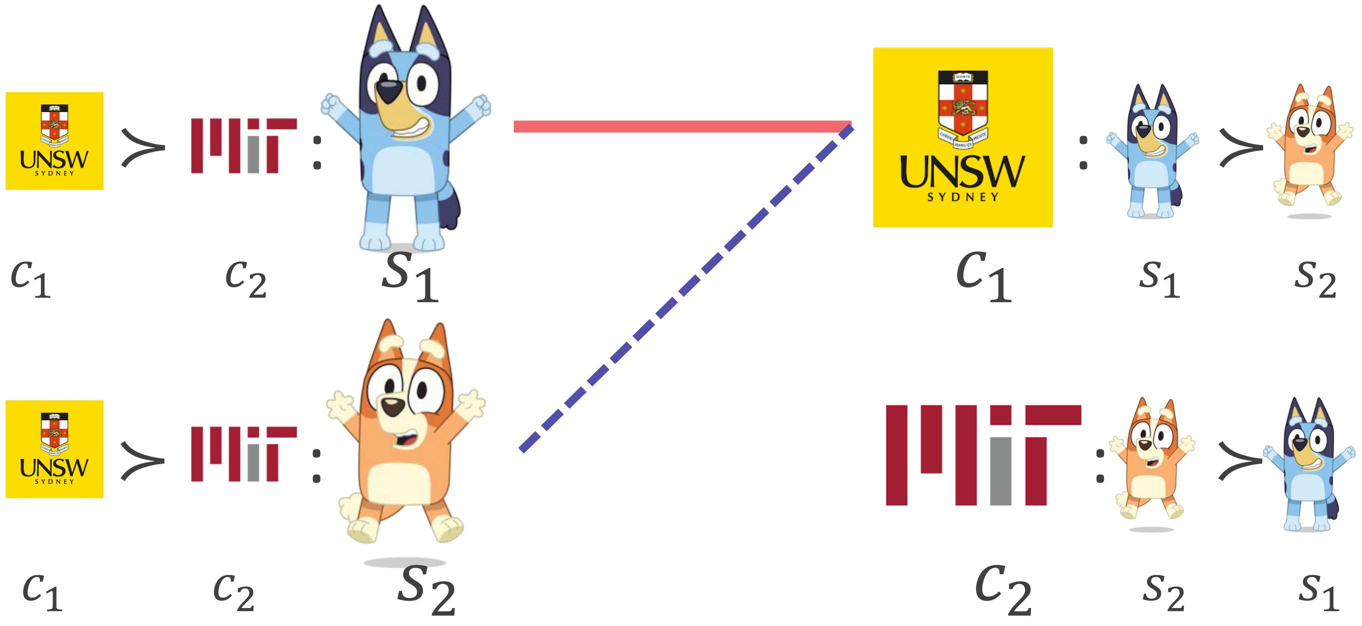
Finding Good Matchings



Finding Good Matchings

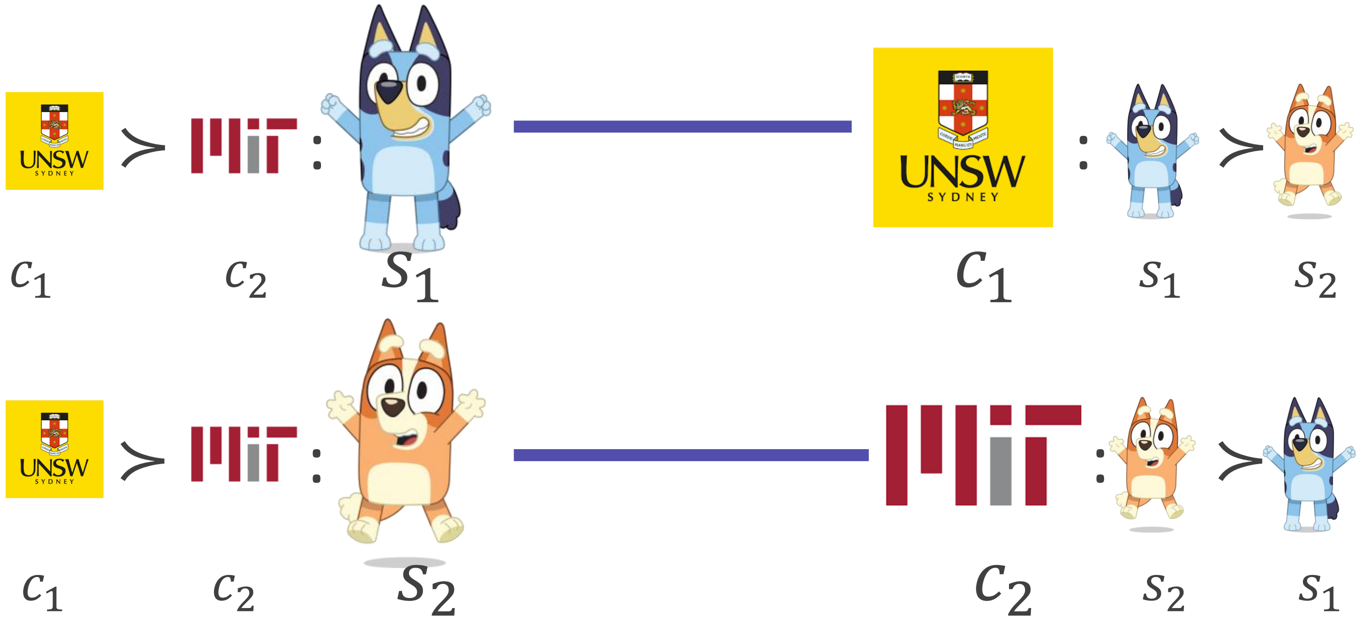


Finding Good Matchings



Does this look familiar?

Finding Good Matchings



Does this look familiar?

Blocking pair: (s, c) block μ if they prefer each other to μ.

Stable Matchings

Gale and Shapley introduced stable matchings

Definition: Matching μ is stable for $I = \langle S, C, \succ \rangle$

if it has **no blocking pairs**.

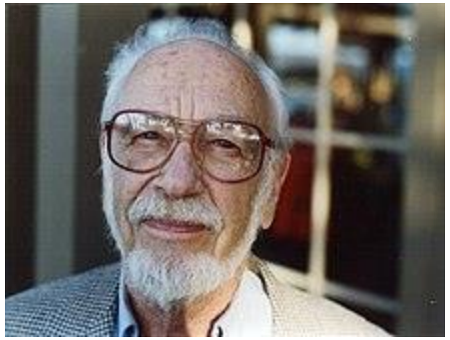
JOURNAL ARTICLE

College Admissions and the Stability of Marriage

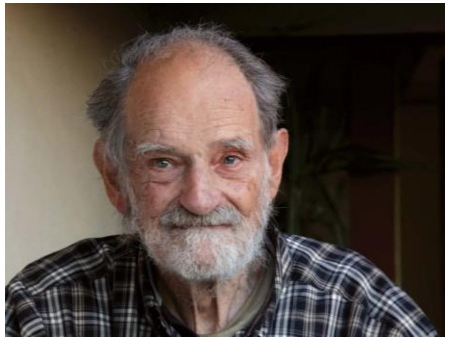
D. Gale, L. S. Shapley

The American Mathematical Monthly, Vol. 69, No. 1 (Jan., 1962), pp. 9-15 (7 pages)

https://doi.org/10.2307/2312726 • https://www.jstor.org/stable/2312726



David Gale



Lloyd Shapley

Stable Matchings

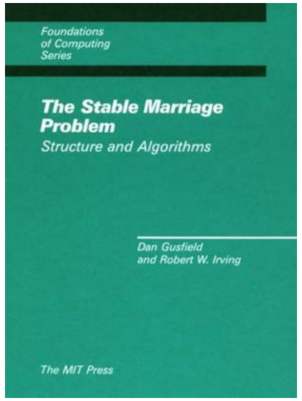
© The Nobel Foundation. Photo: U. Montan

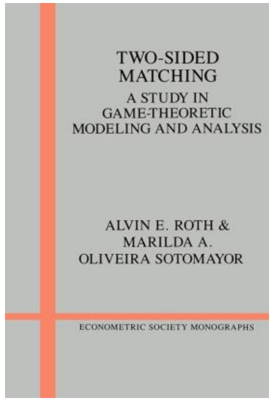
Alvin E. Roth

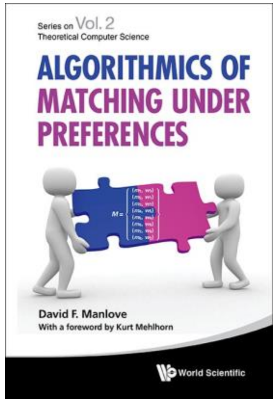
© The Nobel Foundation. Photo: U. Montan

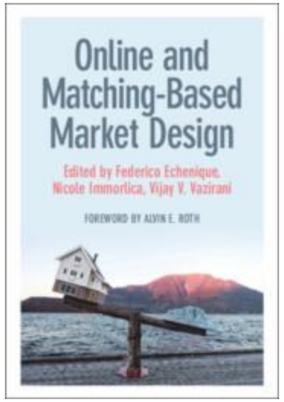
Lloyd S. Shapley

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley "for the theory of stable allocations and the practice of market design"







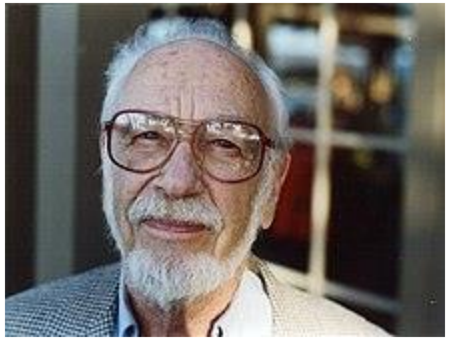


Computing Stable Matchings

Do stable matchings always exist?

One-one matchings: Yes always.

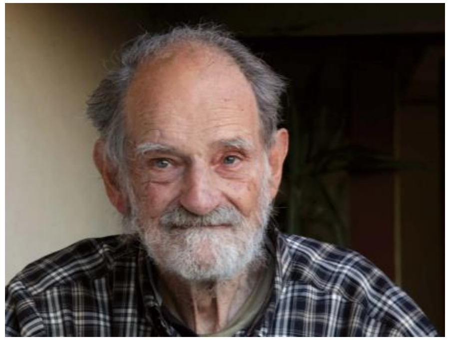
- Called "stable marriage" problem for one-one setting.



David Gale

Gale and Shapley gave the Deferred Acceptance algorithm

- Always returns a stable matching
- Runs in polynomial time



Lloyd Shapley

Deferred Acceptance Algorithm

Fix a proposing side (S) and accepting side (C).

In each round:

– Each unmatched $s \in S$ proposes to most preferred college $c \in C$ which hasn't rejected s yet

– c accepts a proposal if:

a. Unmatched
b. Prefers s to current partner

Notation

SPDA: Student Proposing Deferred Acceptance

- Always assume students propose unless stated otherwise

CPDA: College Proposing Deferred Acceptance

$\succ_a$: preference relation of $a \in S \cup C$

$\mu(a)$: partner of a under matching μ

Student Proposing Deferred Acceptance

Given: $< S, C, \succ >$

Initialize $\mu \leftarrow \emptyset$

Initialize $R_s \leftarrow \emptyset$ for each $s \in S$

While(exists unmatched $s \in S$ s.t. $R_s \neq C$)

  Let c be most preferred in $C \setminus R_s$ under $\succ_s$

  If($s \succ_c \mu(c)$)

    $\mu \leftarrow \big(\mu \setminus (\mu(c), c)\big) \cup \{(s, c)\}$

  Else $R_s \leftarrow R_s \cup \{c\}$

Return μ

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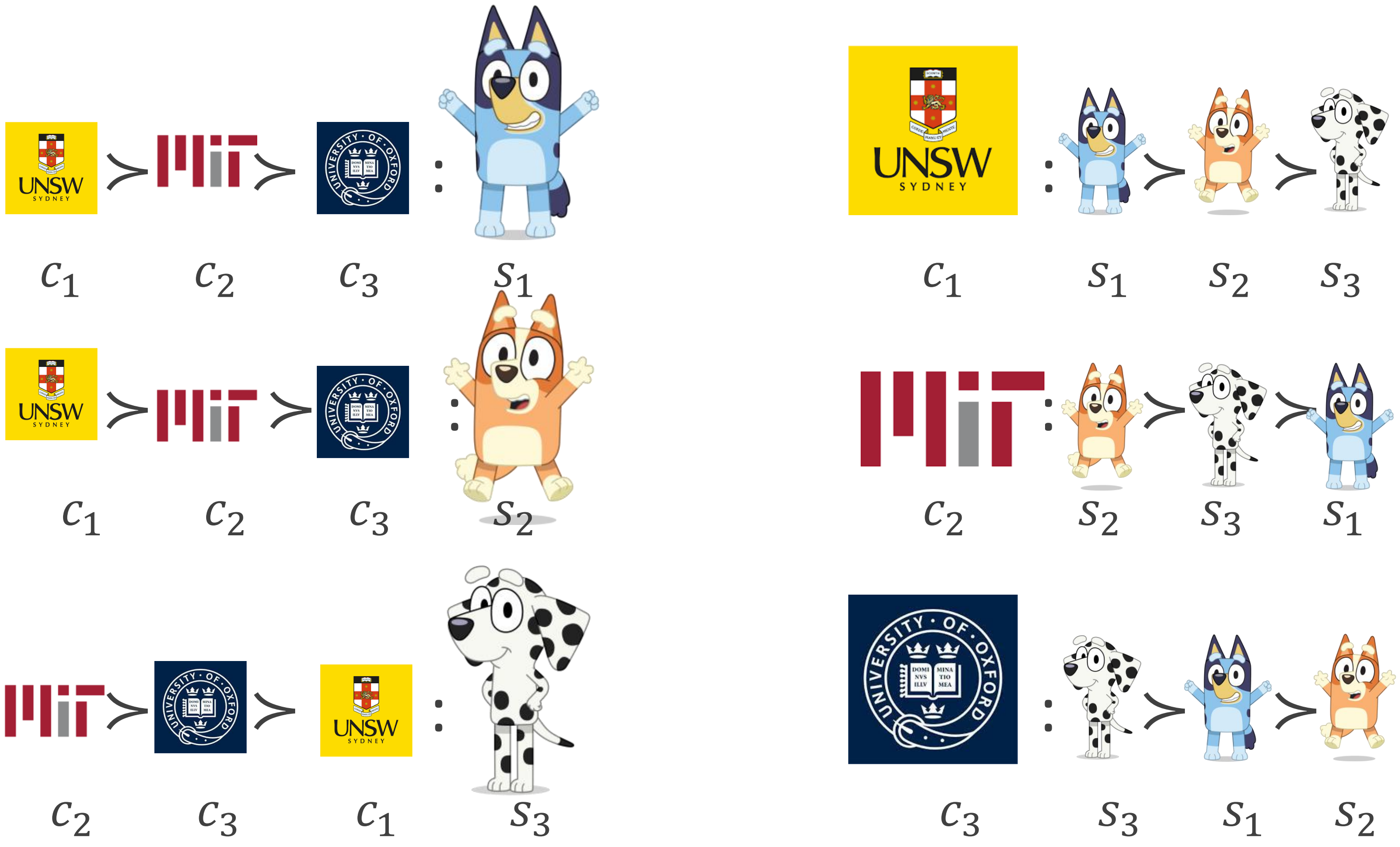

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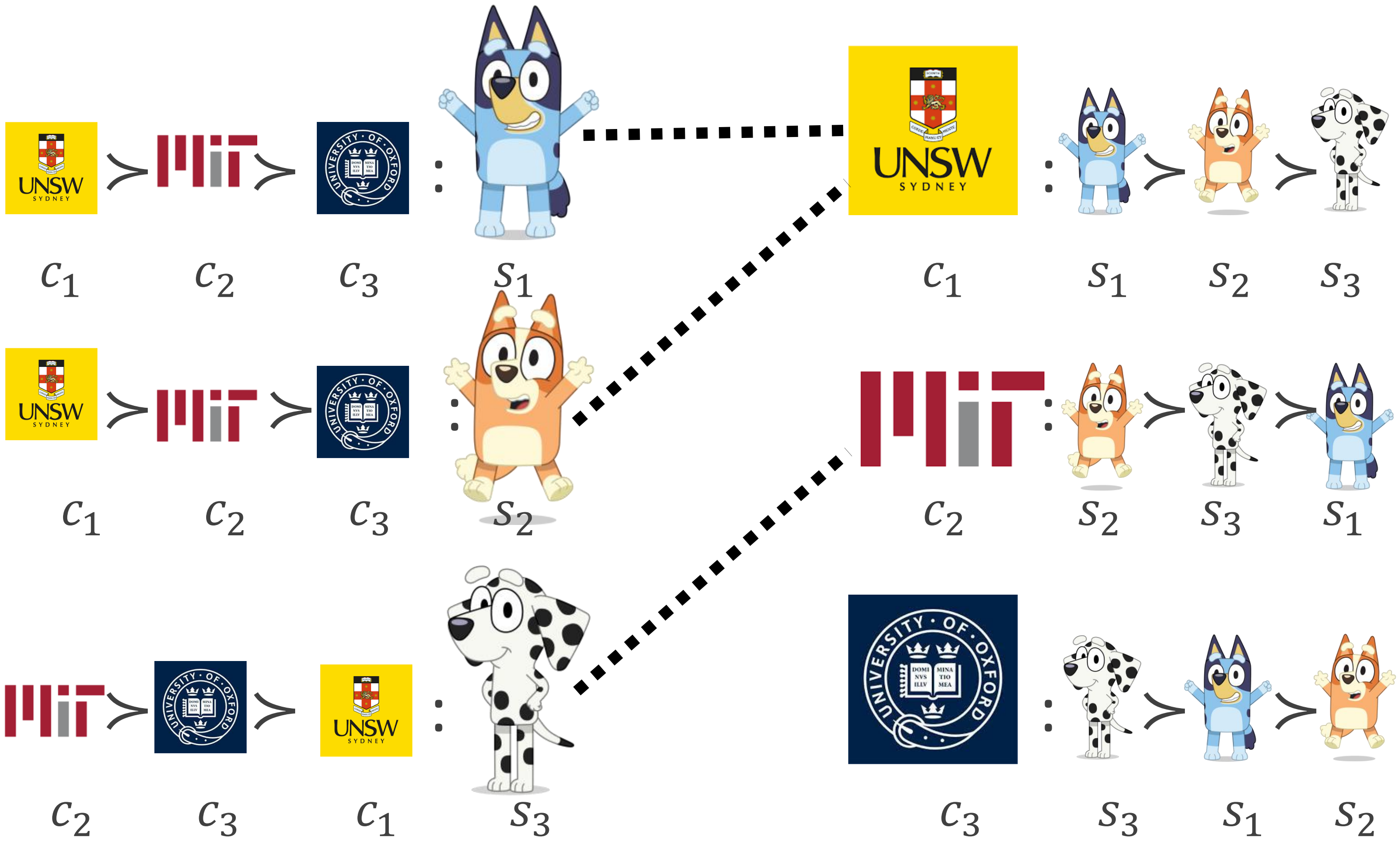

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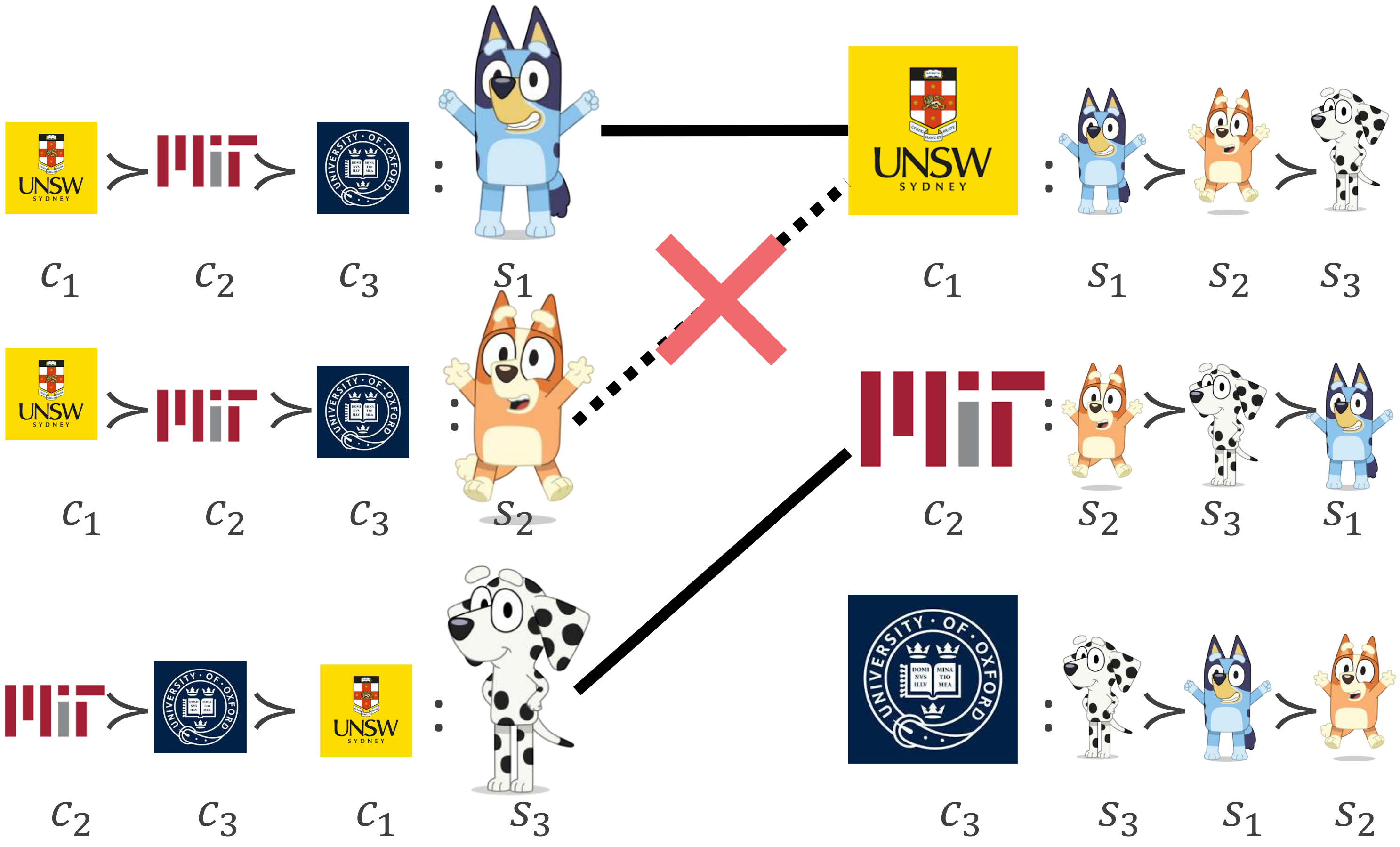

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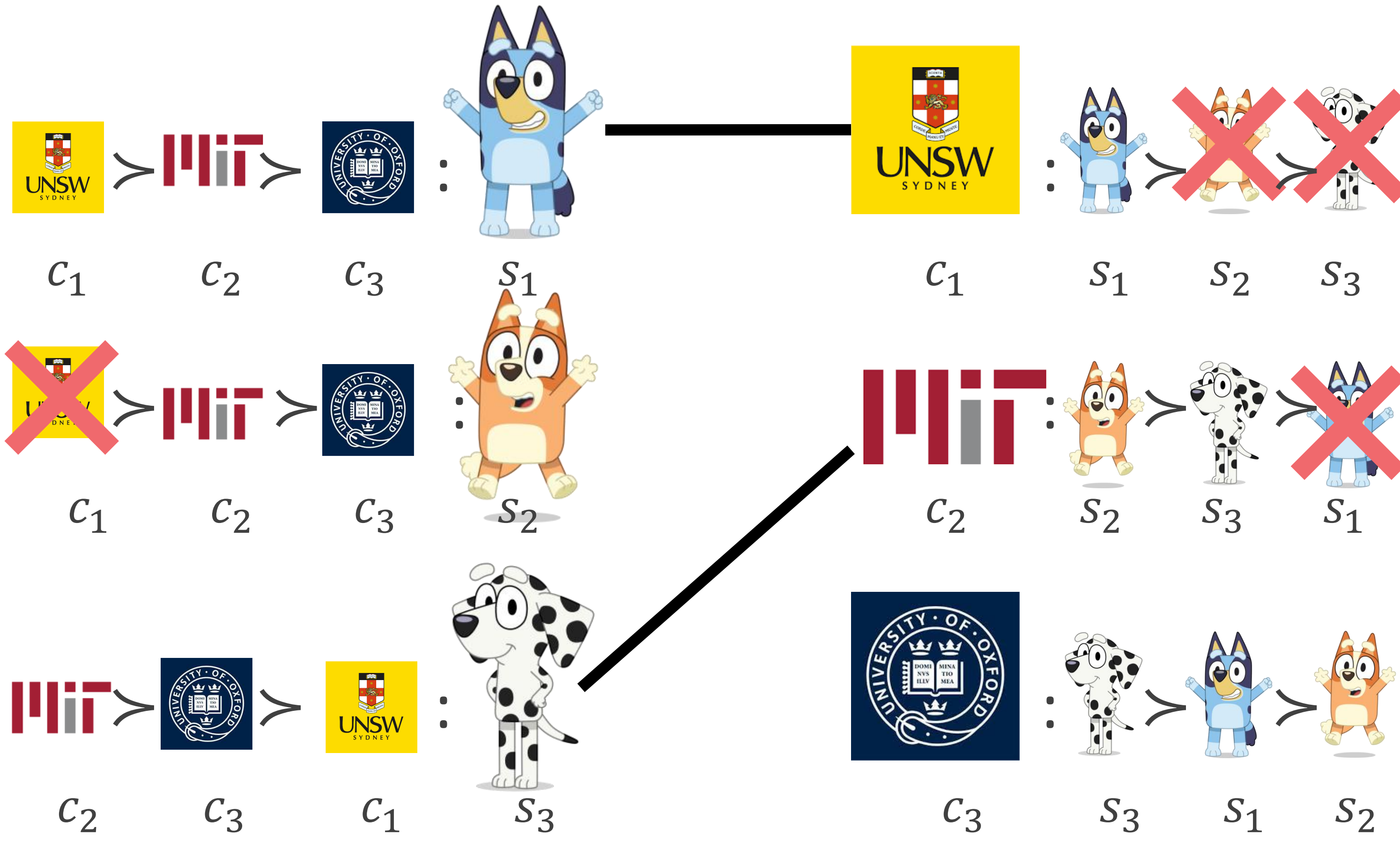

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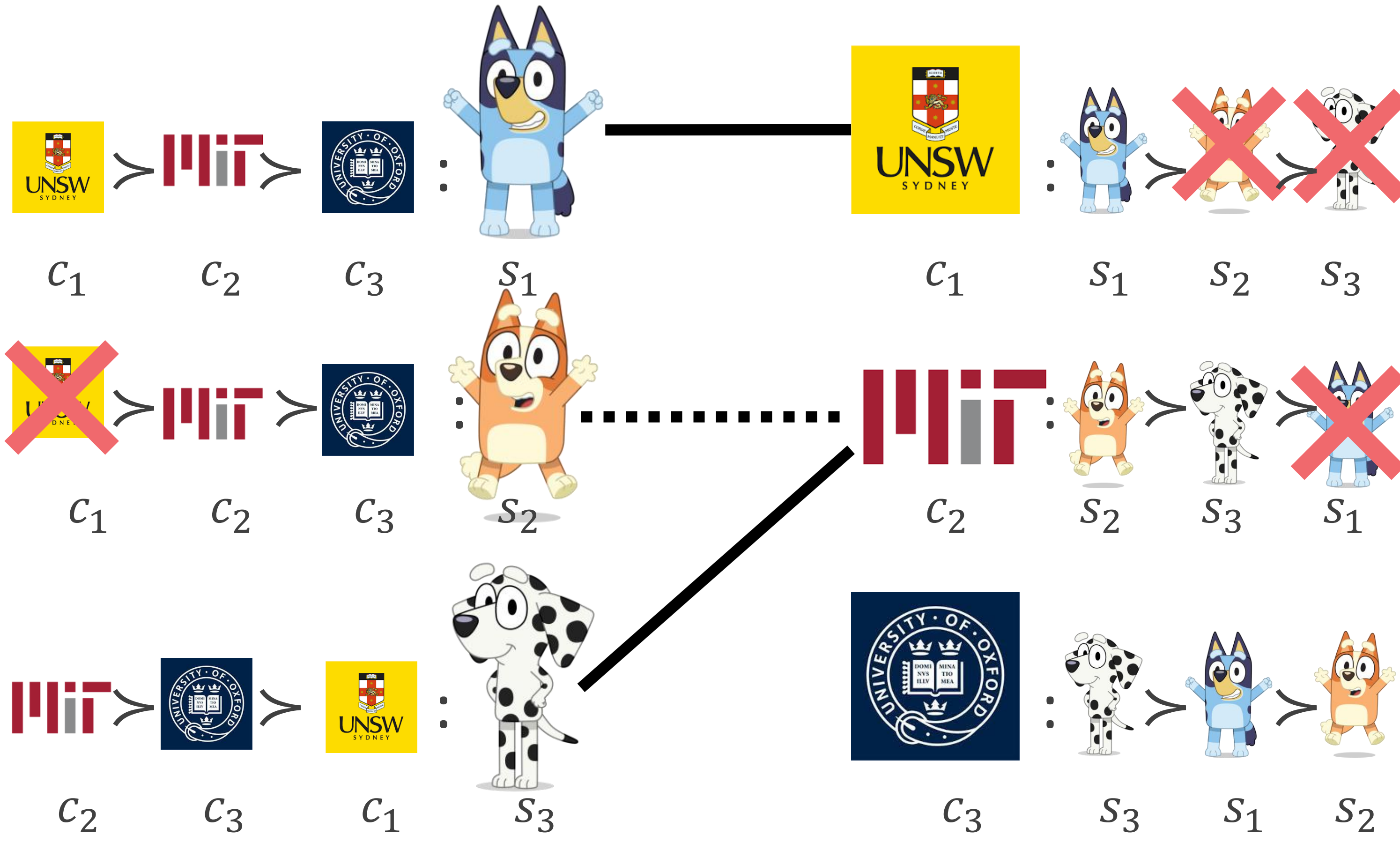

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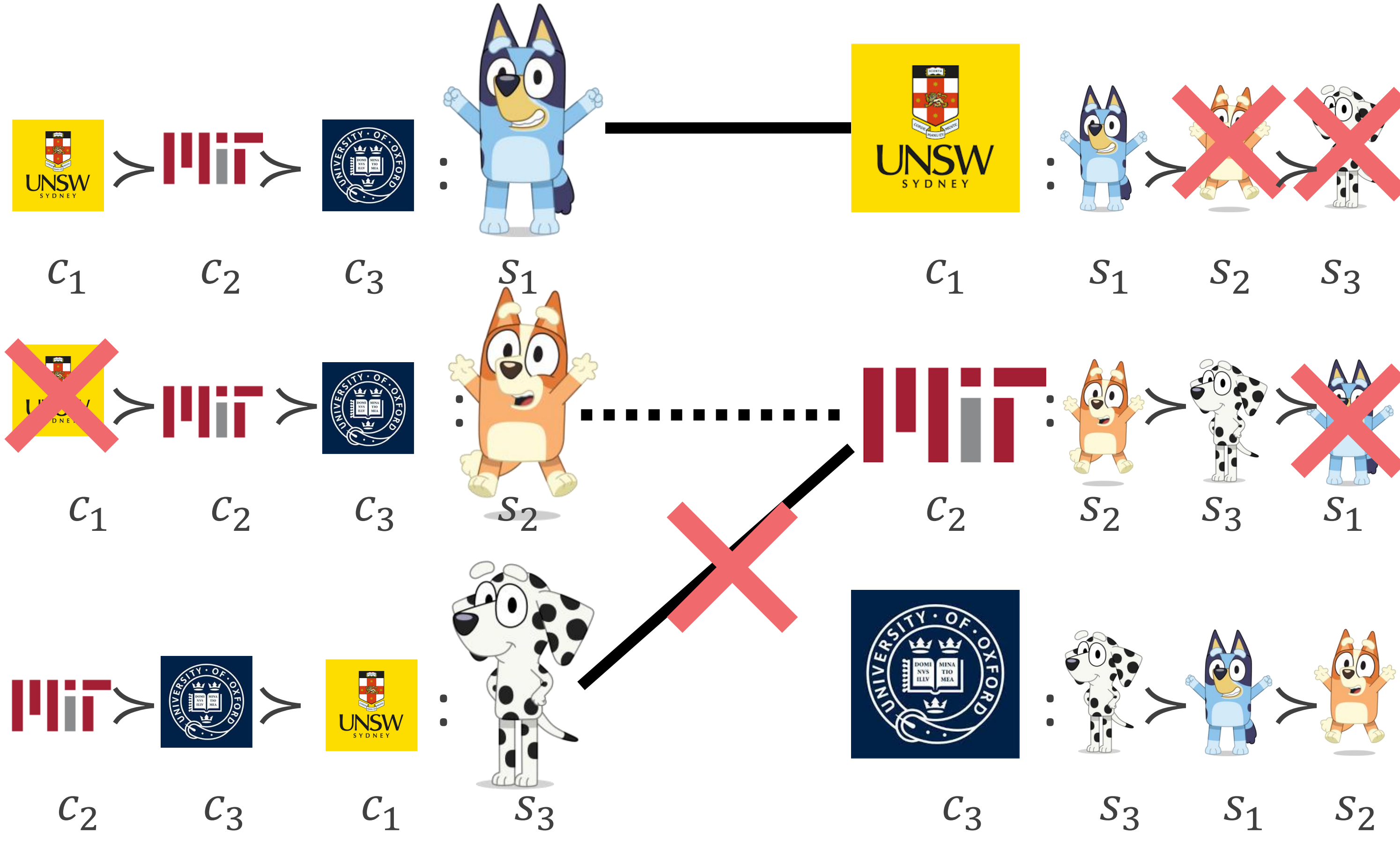

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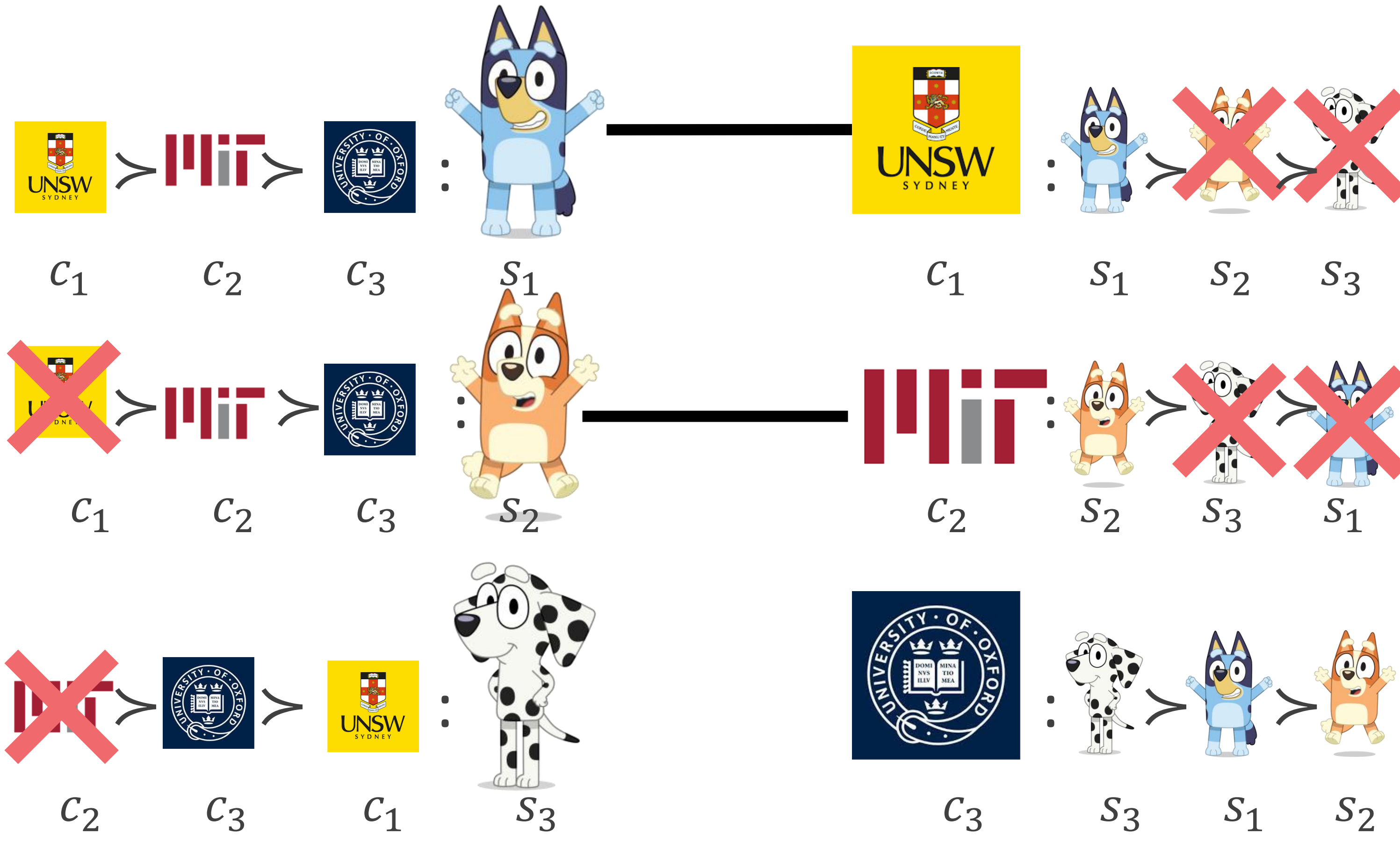

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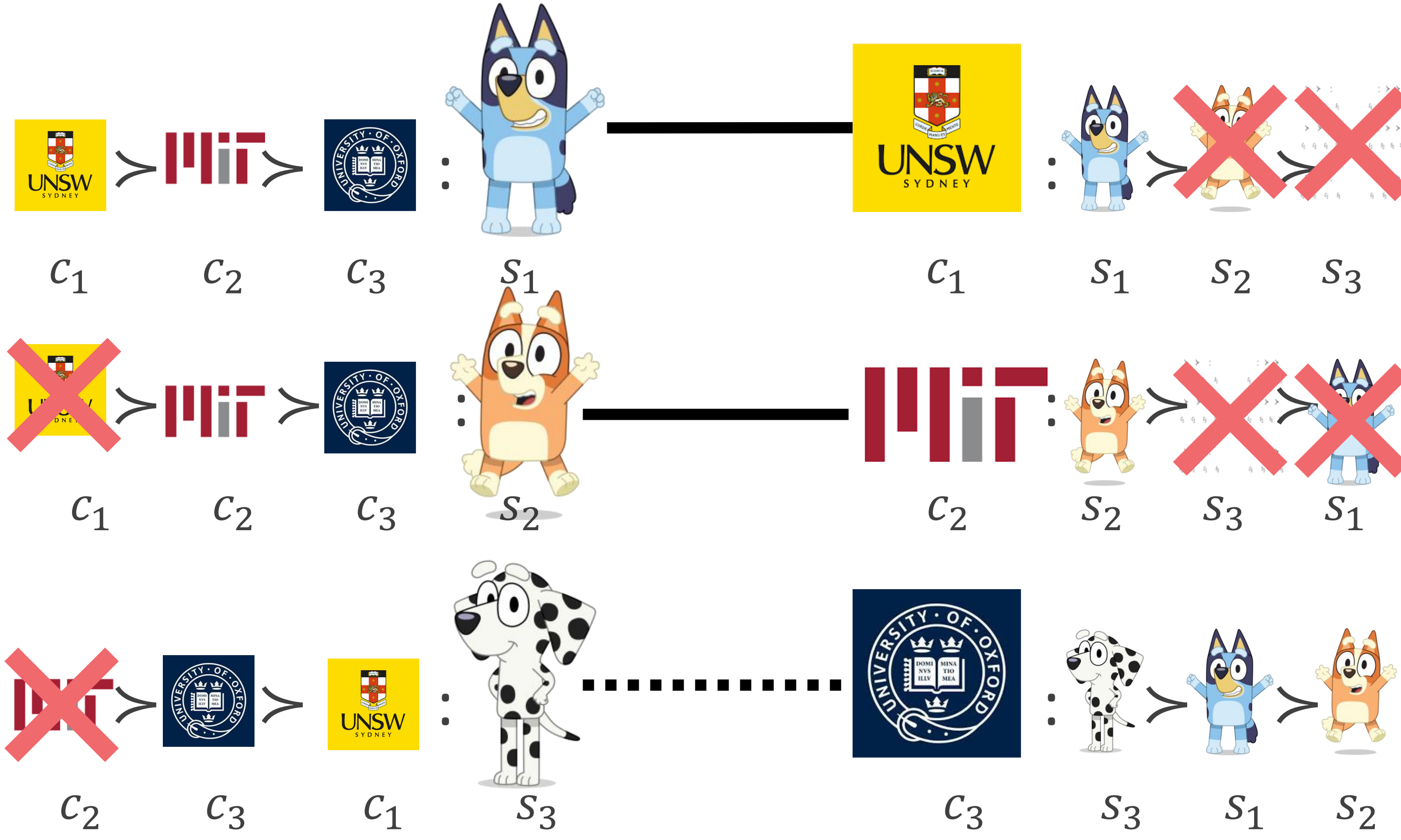

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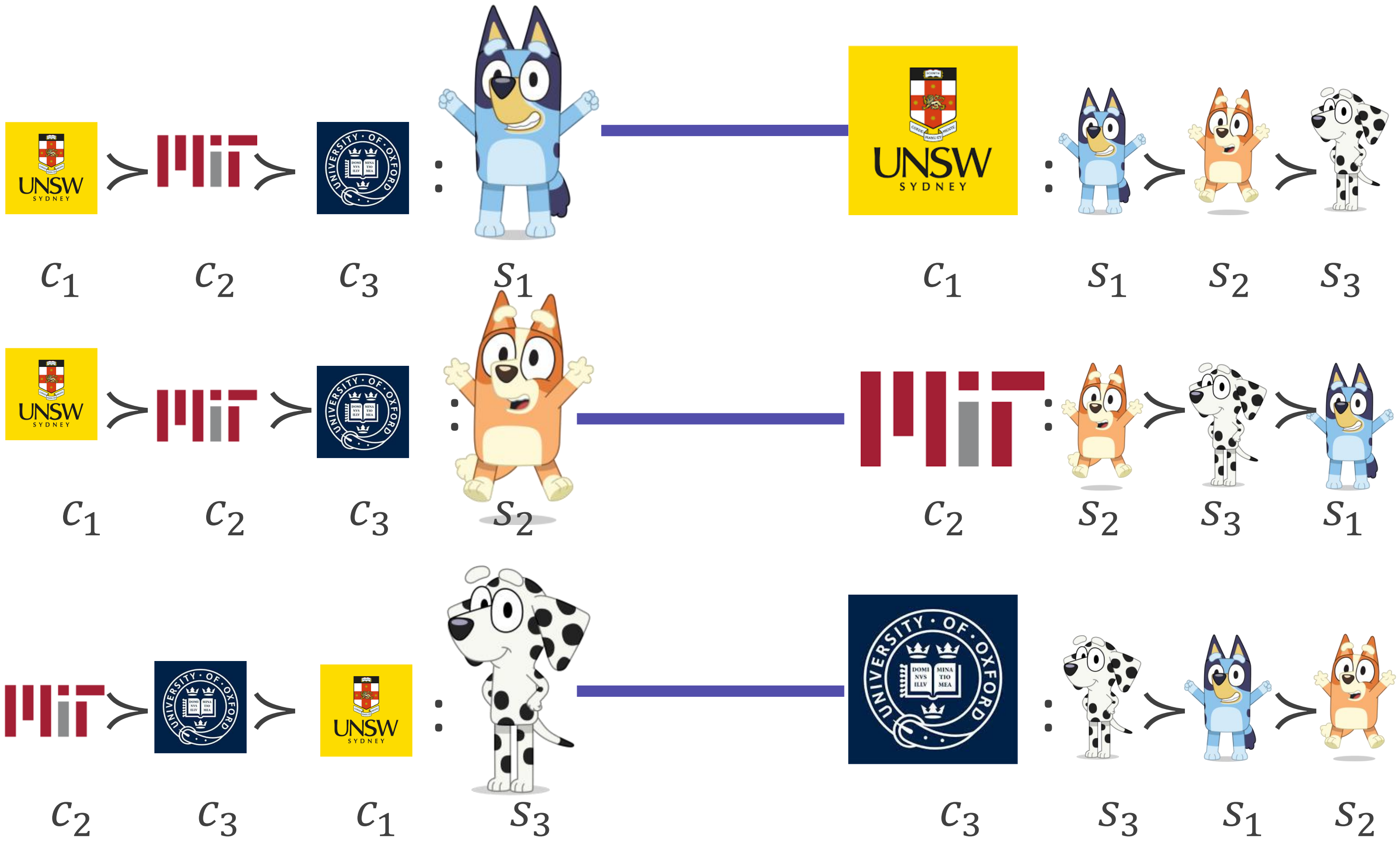


Properties of SPDA

Let $n = |S| = |C|$.

Question: Does SPDA terminate?

Running Time: Every $s \in S$ can be rejected by at most n colleges.

$\Rightarrow$ DA runs in time $O(n^2)$

Observations:

- No college once matched becomes unmatched.
- Students’ keep worsening
- Colleges’ keep improving

Question: Does SPDA always match all agents?

Properties of SPDA

Claim: No unmatched agents at the end of SPDA.

As $|S| = |C| = n$, unmatched student $\Leftrightarrow$ unmatched college.

A college c is unmatched only if no student proposed.

SPDA terminates only when no more proposals.

Therefore, unmatched agent implies there exist s and c, s.t. s has not proposed to c.

Stability

Theorem. Deferred Acceptance returns a stable matching.

WLOG*, assume SPDA. Let μ be matching returned.

Pick any $(s, c) \in S \times C$.

Case 1: $c \prec_s \mu(s)$ or $\mu(s) = c$. Clearly not blocking pair.

Case 2: $c \succ_s \mu(s)$.

s would have proposed to c before $\mu(s)$. Thus, $s \prec_c \mu(c)$.

Hence, (s,c) cannot be a blocking pair. μ is stable.

*WLOG: without loss of generality. Used when making an assumption that does not affect correctness.

Applications of Stable Matchings

Many college admissions and job settings.

THE MATCH
NATIONAL RESIDENT MATCHING PROGRAM®

Great overview of NRMP by Alvin Roth: https://vimeo.com/863432136

Are there Good vs Bad Stable Matchings?

Properties of Stable Matchings

Can an instance have multiple stable matchings?

- Yes.

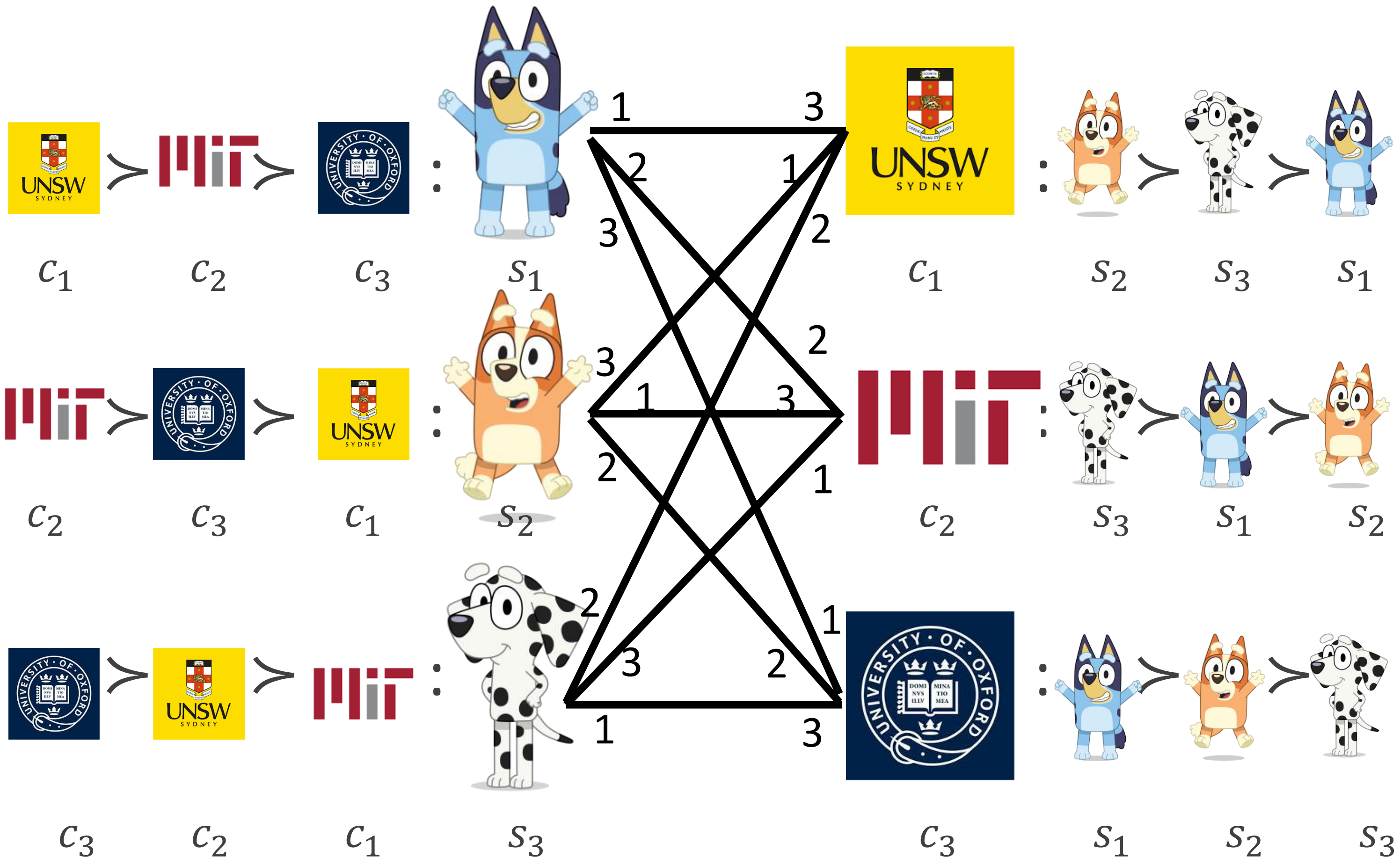
Can there be stable matchings where some agents are unmatched?

- Recall $|S| = |C| = n$ and all agents would rather be matched
- No, unmatched agents would form a blocking pair.

Can some stable matchings be better than others?

- Yes…

Structure of Stable Matchings



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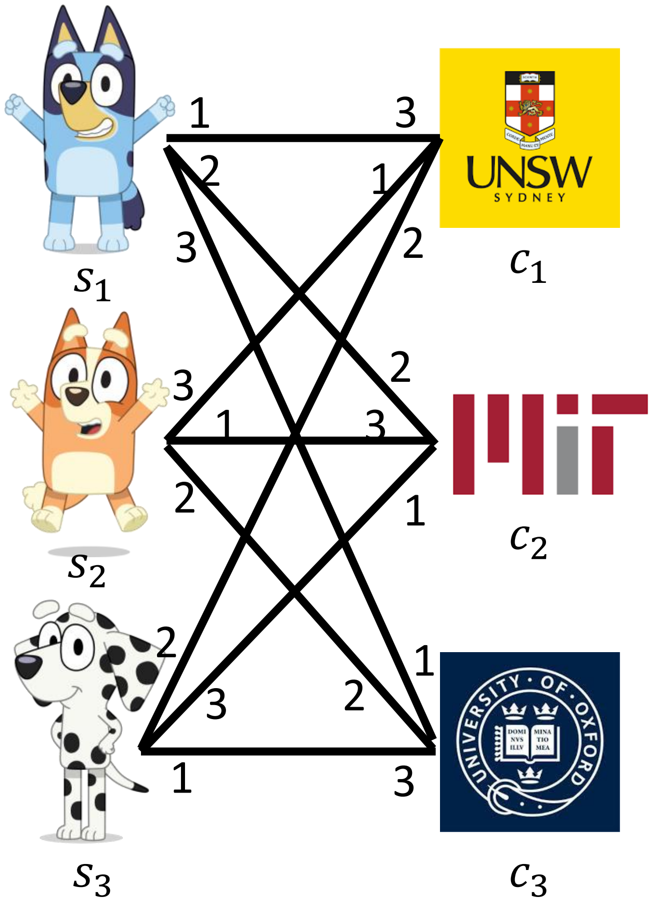
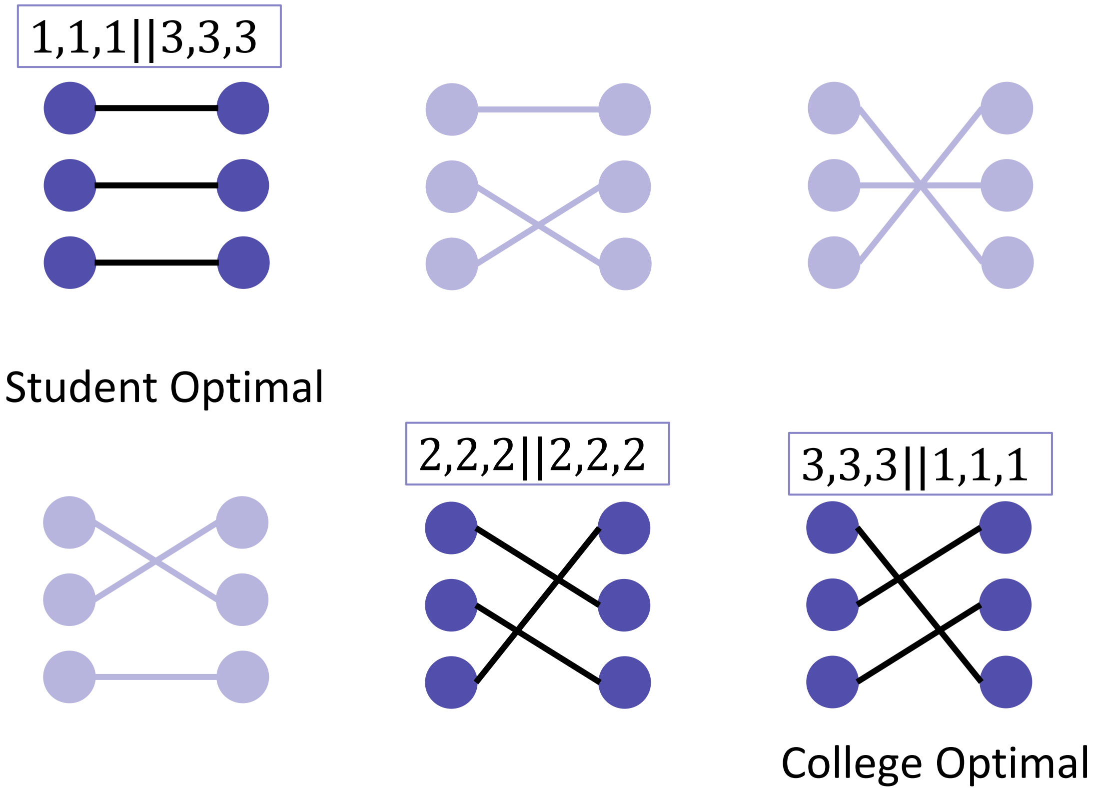




Do Student and College Optimal Stable Matchings ALWAYS Exist?

Optimal Partners

Definition. A college c is achievable for student s if there is a stable matching μ where $\mu(s) = c$.

Optimal partner: Favourite achievable partner

Recall: agent preferences are strict

$\Rightarrow$Unique favourite achievable partner

Question: Can two students have the same optimal partner?

Optimal Partners

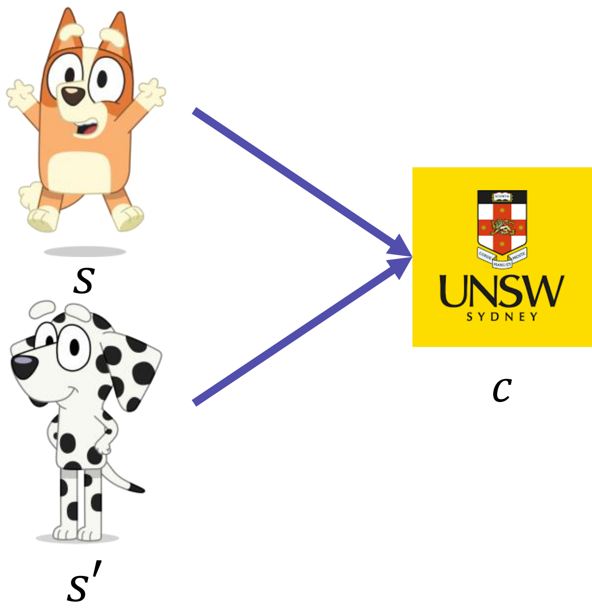
- Student-optimal mapping: each student points to fav achievable college
- College-optimal mapping: each college points to fav achievable student

Claim: Student-optimal mapping is one-one.

Proof. Suppose not.

Let c be the favourite achievable college for s and s'.

Suppose: $s \succ_c s'$



Optimal Partners

μ: stable matching where $\mu(c) = s'$

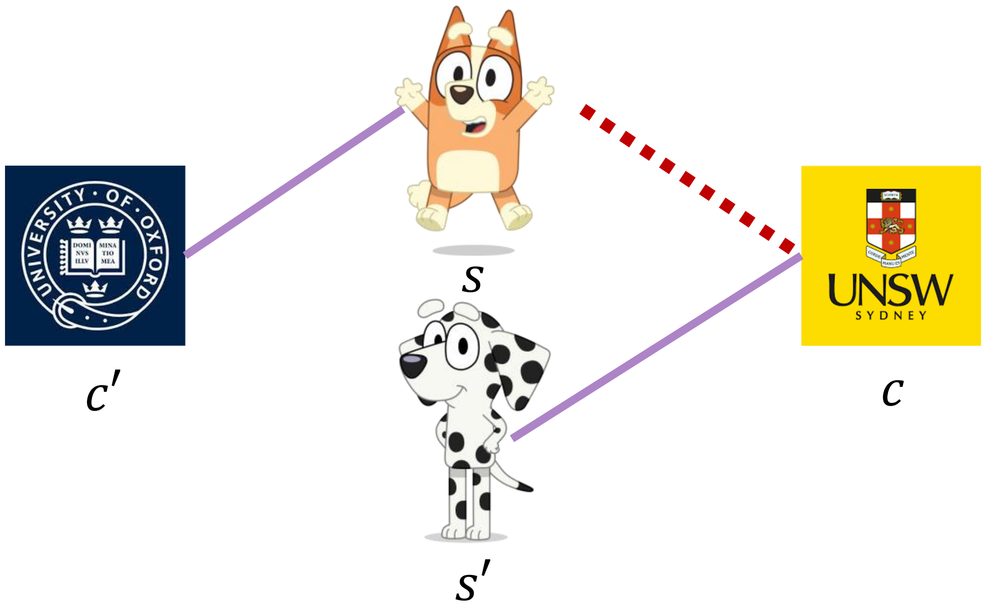
Let $\mu(s) = c'$.

Clearly, $c \succ_s c'$. Recall: $s \succ_c s'$.

$\Rightarrow$ (s, c) blocks μ.

Student optimal mapping is one-one.

- There is a student optimal stable matching.



Computing Student and College Optimal Stable Matchings

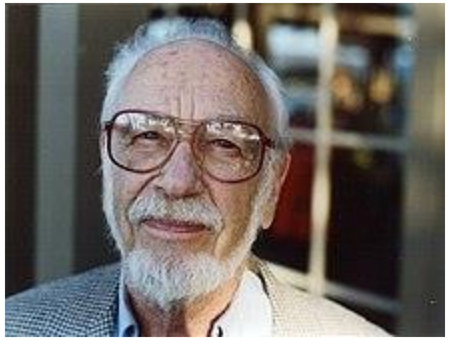
Optimal Stable Matchings

Deferred Acceptance is:

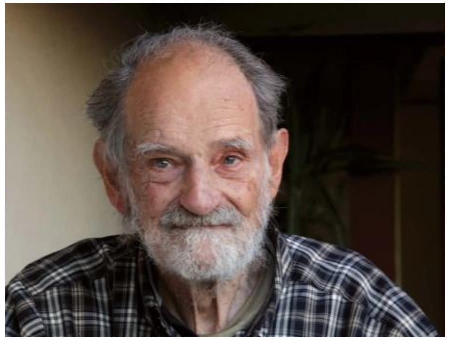
- Optimal for proposing side
- Pessimal for accepting side

Theorem.(Gale Shapley 1962) SPDA matches each student to its most preferred achievable college.

Enough to show no student is rejected by optimal college.



David Gale



Lloyd Shapley

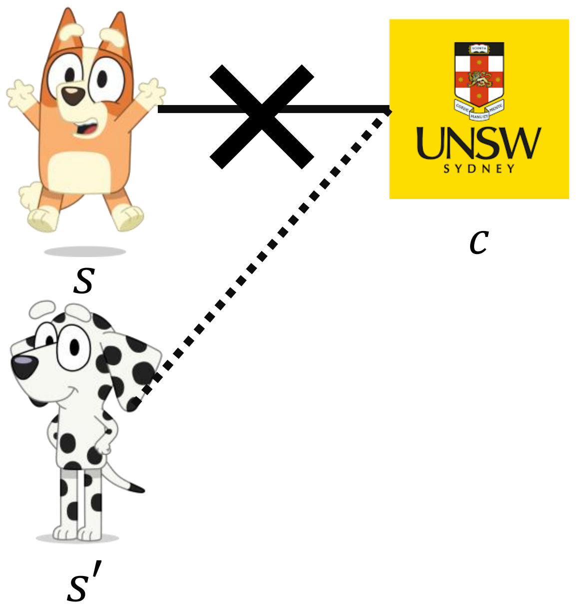
Optimal Stable Matchings

Theorem.(Gale Shapley 1962) SPDA matches each student to its optimal college.

Proof. Consider **first student** s to be rejected by optimal college c under SPDA (for s').

Thus, $s' \succ_c s$.

Consider stable matching μ where $\mu(s) = c$



Optimal Stable Matchings

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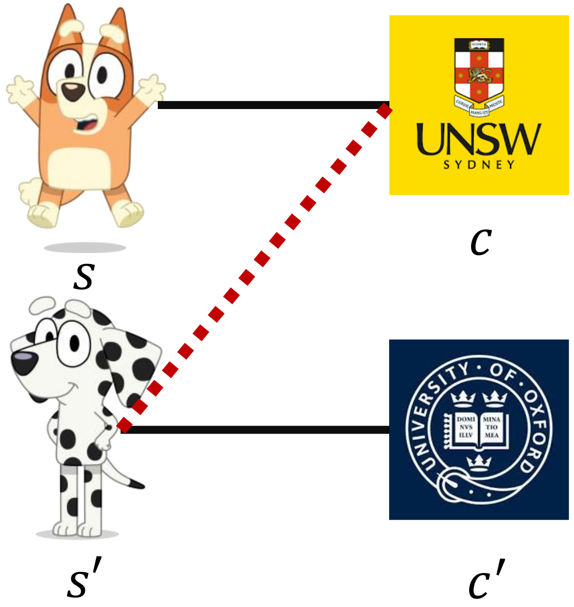
Proof. Consider **first student** s to be rejected by optimal college c under SPDA (for s').

Thus, $s' \succ_c s$.

Consider stable matching μ where $\mu(s) = c$

Let $\mu(s') = c'$.

Case 1: $c \succ_{s'} c'$. **(s', c) block μ.**



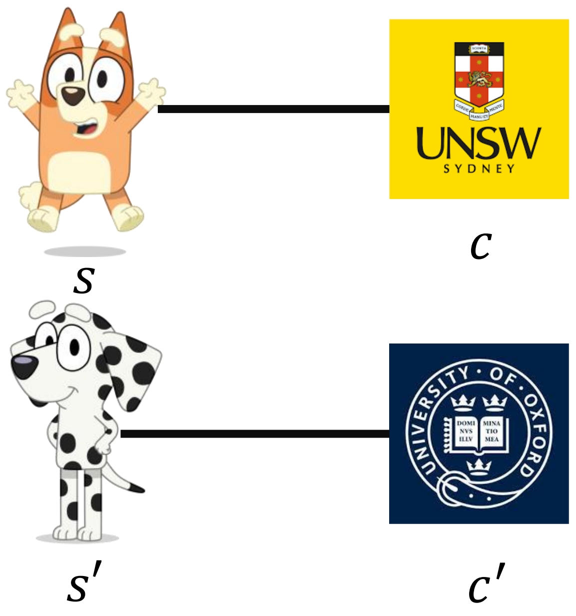
Optimal Stable Matchings

Theorem.(Gale Shapley 1962) SPDA matches each student to its optimal college.

Proof. Consider **first student** s to be rejected by optimal college c under SPDA (for s').

Let $\mu(s') = c'$.

Case 2: $c \prec_{s'} c'$. s' would propose to c' before c.



Optimal Stable Matchings

Theorem.(Gale Shapley 1962) SPDA matches each student to its optimal college.

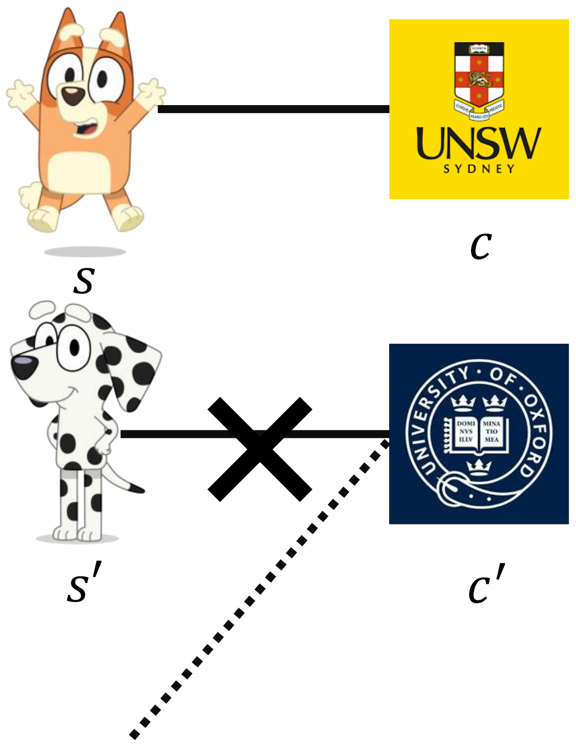
Proof. Consider **first student** s to be rejected by optimal college c under SPDA (for s').

Let $\mu(s') = c'$.

Case 2: $c \prec_{s'} c'$. s' would propose to c' before c.

s' was rejected by c' in SPDA.

Contradicts assumption.



Optimal Stable Matchings

Thus, SPDA returns student-optimal stable matching.

HW. SPDA matches each college to least preferred achievable student.

Recall: Under SPDA student get worse and colleges improve!!

Next time: Strategic behaviour and many-to-one matchings.

Previously

Introduced:

Stable Matchings

Deferred Acceptance

Student and College optimal matchings

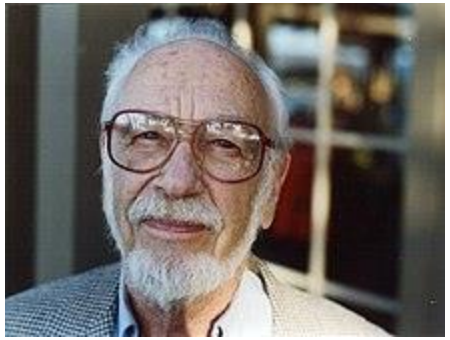
JOURNAL ARTICLE

College Admissions and the Stability of Marriage

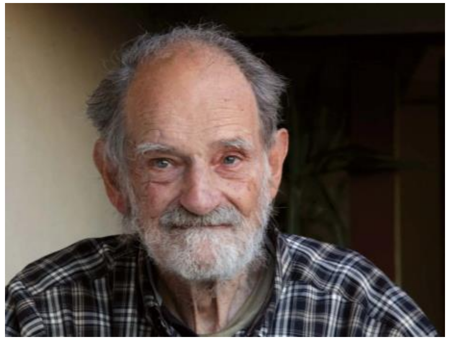
D. Gale, L. S. Shapley

The American Mathematical Monthly, Vol. 69, No. 1 (Jan., 1962), pp. 9-15 (7 pages)

https://doi.org/10.2307/2312726 • https://www.jstor.org/stable/2312726



David Gale



Lloyd Shapley

Today

Two objectives:

- Strategically reporting preferences: algorithms, some proofs
- Many-to-one matchings: Extending one-one ideas and some proof ideas

On the way: several examples of DA.

Deferred Acceptance Algorithm

Fix a proposing side (S) and accepting side (C).

In each round:

– Each unmatched $s \in S$ proposes to most preferred college $c \in C$ which hasn't rejected s yet

– c accepts a proposal if:

a. Unmatched
b. Prefers s to current partner

--- current partner gets rejected

Observations

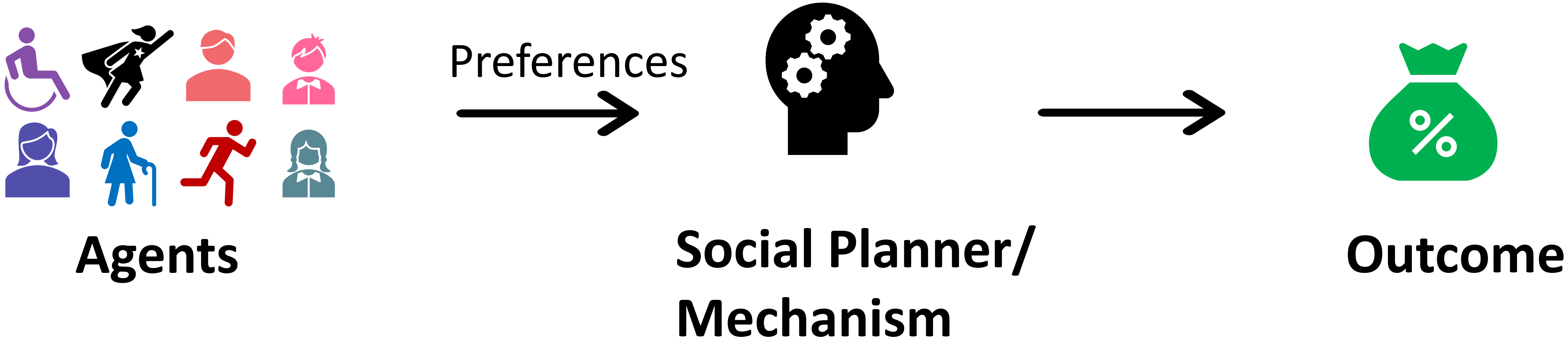
Student s can only get rejected from a college that is matched

Student never proposes to the same college twice.

Students' prospective partner only gets worse

If (s, c) are each other's first preference, (s, c) is contained in EVERY stable matching.

Recall



Under deferred acceptance, what can an agent be strategic about?

Preferences.

Can being dishonest help?

Example

$|S| = |C| = 3$

True Preferences:

$s_1 : c_2 \succ c_1 \succ c_3$

$s_2 : c_1 \succ c_2 \succ c_3$

$s_3 : c_1 \succ c_2 \succ c_3$

$c_1 : s_1 \succ s_2 \succ s_3$

$c_2 : s_2 \succ s_1 \succ s_3$

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$|S| = |C| = 3$

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SPDA is college pessimal

Can a college improve by misreporting?

Example

Suppose c_1 lies

$s_1: c_2 \succ c_1 \succ c_3$

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$s_3: c_1 \succ c_2 \succ c_3$

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True Pref:

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Beneficial manipulation: $\succ'$ is beneficial for c if the result of SPDA on $\succ'$ matches c to a better student than on $\succ_c$

Strategyproof

A mechanism is strategyproof if all agents prefer the outcome under true preferences over any outcome by misreporting.

Roth, 1982: No stable matching mechanism is strategyproof for both sides.

- See previous example

Alvin Roth

Thm. SPDA is strategyproof for **students**.

MATHEMATICS OF OPERATIONS RESEARCH
Vol. 7, No. 4, November 1982
Printed in U.S.A.

THE ECONOMICS OF MATCHING: STABILITY AND INCENTIVES*†

ALVIN E. ROTH

Manipulating Deferred Acceptance

Multiple manipulations may work

True Preferences:

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Multiple manipulations may work

Beneficial manipulation for $\boldsymbol{c_1}$:

s_1: $c_2 \succ c_1 \succ c_3 \succ c_4$

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$s_1\colon \boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

$s_2\colon \cancel{c_3} \succ \boxed{c_1} \succ c_2 \succ c_4$

$s_3\colon \cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

$s_4\colon \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1\colon \boxed{s_2} \succ \boxed{s_4} \succ s_1 \succ \cancel{s_3}}$

$c_2\colon s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

$c_3\colon \boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

$c_4\colon s_4 \succ s_3 \succ s_2 \succ s_1$

True:
$c_1\colon s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Beneficial manipulation for c_1:

$s_1: \boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

$s_2: \cancel{c_3} \succ \boxed{c_1} \succ c_2 \succ c_4$

$s_3: \cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

$s_4: \cancel{c_1} \succ c_4 \succ c_2 \succ c_3$

$c_1: \boxed{s_2} \succ \cancel{s_4} \succ s_1 \succ \cancel{s_3}$

$c_2: s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

$c_3: \boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

$c_4: s_4 \succ s_3 \succ s_2 \succ s_1$

True:
$c_1: s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Beneficial manipulation for c_1:

$s_1: \boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

$s_2: \cancel{c_3} \succ \boxed{c_1} \succ c_2 \succ c_4$

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$s_4: \cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

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$c_2: s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

$c_3: \boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

$c_4: \boxed{s_4} \succ s_3 \succ s_2 \succ s_1$

True:
$c_1: s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Beneficial manipulation for c_1:

s_1: $\boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

s_2: $c_3 \succ \boxed{c_1} \succ c_2 \succ c_4$

s_3: $c_1 \succ \boxed{c_3} \succ c_2 \succ c_4$

s_4: $c_1 \succ \boxed{c_4} \succ c_2 \succ c_3$

$\boldsymbol{c_1}$: $\boxed{\boldsymbol{s_2}} \succ \boldsymbol{s_4} \succ \boldsymbol{s_1} \succ \boldsymbol{s_3}$

c_2: $s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

c_3: $\boxed{s_3} \succ s_2 \succ s_1 \succ s_4$

c_4: $\boxed{s_4} \succ s_3 \succ s_2 \succ s_1$

True:
c_1: $s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

$s_1: c_2 \succ c_1 \succ c_3 \succ c_4$

$s_2: c_3 \succ c_1 \succ c_2 \succ c_4$

$s_3: c_1 \succ c_3 \succ c_2 \succ c_4$

$s_4: c_1 \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1: s_1 \succ s_4 \succ s_2 \succ s_3}$

$c_2: s_2 \succ s_1 \succ s_3 \succ s_4$

$c_3: s_3 \succ s_2 \succ s_1 \succ s_4$

$c_4: s_4 \succ s_3 \succ s_2 \succ s_1$

True:
$c_1: s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

True: $c_1 : s_1 \succ s_2 \succ s_3 \succ s_4$

$s_1 : \boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

$s_2 : \boxed{c_3} \succ c_1 \succ c_2 \succ c_4$

$s_3 : \boxed{c_1} \succ c_3 \succ c_2 \succ c_4$

$s_4 : \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1 : s_1 \succ \boxed{s_4} \succ s_2 \succ \boxed{s_3}}$

$c_2 : s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

$c_3 : s_3 \succ \boxed{s_2} \succ s_1 \succ s_4$

$c_4 : s_4 \succ s_3 \succ s_2 \succ s_1$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

True:
$c_1: s_1 \succ s_2 \succ s_3 \succ s_4$

$s_1: \boxed{c_2} \succ c_1 \succ c_3 \succ c_4$	$c_1: s_1 \succ \boxed{s_4} \succ s_2 \succ \cancel{s_3}$
$s_2: \boxed{c_3} \succ c_1 \succ c_2 \succ c_4$	$c_2: s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$
$s_3: \cancel{c_1} \succ c_3 \succ c_2 \succ c_4$	$c_3: s_3 \succ \boxed{s_2} \succ s_1 \succ s_4$
$s_4: \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$	$c_4: s_4 \succ s_3 \succ s_2 \succ s_1$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

s_1: $\boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

s_2: $\boxed{c_3} \succ c_1 \succ c_2 \succ c_4$

s_3: $\cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

s_4: $\boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1: s_1 \succ \boxed{s_4} \succ s_2 \succ \cancel{s_3}}$

True:
$c_1: s_1 \succ s_2 \succ s_3 \succ s_4$

$c_2: s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

$c_3: \boxed{s_3} \succ \boxed{s_2} \succ s_1 \succ s_4$

$c_4: s_4 \succ s_3 \succ s_2 \succ s_1$

Manipulating Deferred Acceptance

Multiple manipulations may work

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s_1: $\boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

s_2: $\cancel{c_3} \succ c_1 \succ c_2 \succ c_4$

s_3: $\cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

s_4: $\boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1: s_1 \succ \boxed{s_4} \succ s_2 \succ \cancel{s_3}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

$c_3: \boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

$c_4: s_4 \succ s_3 \succ s_2 \succ s_1$

True:

$c_1: s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

s_1: $\boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

s_2: $\cancel{c_3} \succ \boxed{c_1} \succ c_2 \succ c_4$

s_3: $\cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

s_4: $\boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1 : s_1 \succ \boxed{s_4} \succ \boxed{s_2} \succ \cancel{s_3}}$

c_2: $s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

c_3: $\boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

c_4: $s_4 \succ s_3 \succ s_2 \succ s_1$

True:
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Manipulating Deferred Acceptance

Multiple manipulations may work

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s_3: $\cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

s_4: $\boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1: s_1 \succ \boxed{s_4} \succ \cancel{s_2} \succ \cancel{s_3}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_3 \succ s_4$

$c_3: \boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

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Manipulating Deferred Acceptance

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$s_3 : \cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

$s_4 : \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1 : s_1 \succ \boxed{s_4} \succ \cancel{s_2} \succ \cancel{s_3}}$

$c_2 : \boxed{s_2} \succ \boxed{s_1} \succ s_3 \succ s_4$

$c_3 : \boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

$c_4 : s_4 \succ s_3 \succ s_2 \succ s_1$

True:
$c_1 : s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

s_1: $\cancel{c_2} \succ c_1 \succ c_3 \succ c_4$

s_2: $\cancel{c_3} \succ \cancel{c_1} \succ \boxed{c_2} \succ c_4$

s_3: $\cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

s_4: $\boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1: s_1 \succ \boxed{s_4} \succ \cancel{s_2} \succ \cancel{s_3}}$

c_2: $\boxed{s_2} \succ \cancel{s_1} \succ s_3 \succ s_4$

c_3: $\boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

c_4: $s_4 \succ s_3 \succ s_2 \succ s_1$

True:
$c_1 : s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

$s_1 \colon \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_2 \colon \cancel{c_3} \succ \cancel{c_1} \succ \boxed{c_2} \succ c_4$

$s_3 \colon \cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

$s_4 \colon \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1} \colon \boxed{\boldsymbol{s_1}} \succ \boxed{\boldsymbol{s_4}} \succ \cancel{\boldsymbol{s_2}} \succ \cancel{\boldsymbol{s_3}}$

$c_2 \colon \boxed{s_2} \succ \cancel{s_1} \succ s_3 \succ s_4$

$c_3 \colon \boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

$c_4 \colon s_4 \succ s_3 \succ s_2 \succ s_1$

True:

$c_1 \colon s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

s_1: $\cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

s_2: $\cancel{c_3} \succ \cancel{c_1} \succ \boxed{c_2} \succ c_4$

s_3: $\cancel{c_1} \succ \boxed{c_3} \succ c_2 \succ c_4$

s_4: $\cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

c_1: $\boxed{s_1} \succ \cancel{s_4} \succ \cancel{s_2} \succ \cancel{s_3}$

c_2: $\boxed{s_2} \succ \cancel{s_1} \succ s_3 \succ s_4$

c_3: $\boxed{s_3} \succ \cancel{s_2} \succ s_1 \succ s_4$

c_4: $\boxed{s_4} \succ s_3 \succ s_2 \succ s_1$

True:
c_1: $s_1 \succ s_2 \succ s_3 \succ s_4$

Manipulating Deferred Acceptance

Multiple manipulations may work

Optimal manipulation for c_1:

$s_1: c_2 \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_2: c_3 \succ c_1 \succ \boxed{c_2} \succ c_4$

$s_3: c_1 \succ \boxed{c_3} \succ c_2 \succ c_4$

$s_4: c_1 \succ \boxed{c_4} \succ c_2 \succ c_3$

$\boldsymbol{c_1: \boxed{s_1} \succ s_4 \succ s_2 \succ s_3}$

$c_2: \boxed{s_2} \succ s_1 \succ s_3 \succ s_4$

$c_3: \boxed{s_3} \succ s_2 \succ s_1 \succ s_4$

$c_4: \boxed{s_4} \succ s_3 \succ s_2 \succ s_1$

True:

$c_1: s_1 \succ s_2 \succ s_3 \succ s_4$

Computing Optimal Manipulations

Optimal Manipulation: Manipulation s.t. under SPDA, manipulating college c is matched to best possible partner for **ALL** preferences.

Thm: An optimal manipulation can be computed in $O(n^3)$ time.

Home > Management Science > Vol. 47, No. 9 >

Gale-Shapley Stable Marriage Problem Revisited: Strategic Issues and Applications

Chung-Piaw Teo, Jay Sethuraman, Wee-Peng Tan

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Computing Optimal Manipulations

Algorithm: $(S, C, \succ, c_j)$

Run SPDA with $\succ_{c_j}$, let μ be returned

Let S_j be the set of students who propose to c_j under $\succ_{c_j}$

Potential partners $P \leftarrow \left\{\left(s, \succ_{c_j}\right) \middle| s \in S_j\right\}$, exhausted $E \leftarrow \left\{\left(\mu(c_j), \succ_{c_j}\right)\right\}$

While$(P \setminus E \neq \emptyset)$

- Choose any $(s, \succ') \in P \setminus E$. Let $\succ^s$ be $\succ'$with s as most preferred student
- Run SPDA with $\succ^s$, let S_s be students who propose to c_j under $\succ_s$
- $P \leftarrow P \cup \{(s', \succ^s) | s' \in S_s \setminus P\}, E \leftarrow E \cup \{(s, \succ_s)\}$

Return $\succ^s$ for best $(s, \succ') \in P$

Computing Optimal Manipulations

True Preferences:

$s_1 : c_2 \succ c_3 \succ c_4 \succ c_1$

$s_2 : c_4 \succ c_1 \succ c_3 \succ c_2$

$s_3 : c_2 \succ c_1 \succ c_3 \succ c_4$

$s_4 : c_1 \succ c_4 \succ c_2 \succ c_3$

$c_1 : s_1 \succ s_2 \succ s_4 \succ s_3$

$c_2 : s_2 \succ s_1 \succ s_4 \succ s_3$

$c_3 : s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4 : s_4 \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

True Preferences:

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $\boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

s_3: $\boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

s_4: $\boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

c_1: $s_1 \succ s_2 \succ s_4 \succ s_3$

c_2: $s_2 \succ s_1 \succ s_4 \succ s_3$

c_3: $s_3 \succ s_4 \succ s_1 \succ s_2$

c_4: $s_4 \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

True Preferences:

$s_1 \colon \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2 \colon \boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3 \colon \boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

$s_4 \colon \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$c_1 \colon s_1 \succ s_2 \succ \boxed{s_4} \succ s_3$

$c_2 \colon s_2 \succ \boxed{s_1} \succ s_4 \succ \boxed{s_3}$

$c_3 \colon s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4 \colon \boxed{s_4} \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

True Preferences:

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$s_2 \colon \boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3 \colon \cancel{c_2} \succ c_1 \succ c_3 \succ c_4$

$s_4 \colon \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$c_1 \colon s_1 \succ s_2 \succ \boxed{s_4} \succ s_3$

$c_2 \colon s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3 \colon s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4 \colon \boxed{s_4} \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

True Preferences:

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$s_2: \boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3: \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_4: \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$c_1: s_1 \succ s_2 \succ \boxed{s_4} \succ \boxed{s_3}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: \boxed{s_4} \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

True Preferences:

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

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c_1: $s_1 \succ s_2 \succ \boxed{s_4} \succ s_3$

c_2: $s_2 \succ \boxed{s_1} \succ s_4 \succ s_3$

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- Choose any $(s, \succ') \in P \setminus E$. Let $\succ^s$ be $\succ'$ with s as most preferred student
- Run SPDA with $\succ^s$, let S_s be students who propose to c_j under $\succ_s$
- $P \leftarrow P \cup \{(s', \succ^s) | s' \in S_s \setminus P\}$, $E \leftarrow E \cup \{(s, \succ_s)\}$

Return $\succ^s$ for best $(s, \succ') \in P$

Computing Optimal Manipulations

$P = \{(s_4, \succ_{c_1})(s_3, \succ_{c_1})\}$

$E = \{(s_4, \succ_{c_1})\}$

For s_3

$s_1 : c_2 \succ c_3 \succ c_4 \succ c_1$

$s_2 : c_4 \succ c_1 \succ c_3 \succ c_2$

$s_3 : c_2 \succ c_1 \succ c_3 \succ c_4$

$s_4 : c_1 \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1 : s_3 \succ s_1 \succ s_2 \succ s_4}$

$c_2 : s_2 \succ s_1 \succ s_4 \succ s_3$

$c_3 : s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4 : s_4 \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

$P = \{(s_4, \succ_{c_1})(s_3, \succ_{c_1})\}$

$E = \{(s_4, \succ_{c_1})\}$

For s_3

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $\boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

s_3: $\boxed{c_2} \succ c_1 \succ c_3 \succ c_4$

s_4: $\boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1: s_3 \succ s_1 \succ s_2 \succ \boxed{s_4}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \boxed{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: s_4 \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

$$\boldsymbol{P} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})(\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$$

$$\boldsymbol{E} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})\}$$

For s_3

$s_1 \colon \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2 \colon \boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3 \colon \cancel{c_2} \succ c_1 \succ c_3 \succ c_4$

$s_4 \colon \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1} \colon \boldsymbol{s_3} \succ \boldsymbol{s_1} \succ \boldsymbol{s_2} \succ \boxed{\boldsymbol{s_4}}$

$c_2 \colon s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3 \colon s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4 \colon s_4 \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

$\boldsymbol{P} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})(\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$

$\boldsymbol{E} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})\}$

For s_3

$s_1: \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2: \boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3: \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_4: \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1}: \boxed{\boldsymbol{s_3}} \succ \boldsymbol{s_1} \succ \boldsymbol{s_2} \succ \boxed{\boldsymbol{s_4}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: s_4 \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

$$\boldsymbol{P = \{(s_4, \succ_{c_1})(s_3, \succ_{c_1})\}}$$

$$\boldsymbol{E = \{(s_4, \succ_{c_1})\}}$$

For s_3

$s_1: \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2: \boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3: \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_4: \cancel{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1: \boxed{s_3} \succ s_1 \succ s_2 \succ \cancel{s_4}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: s_4 \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

$$P = \{(s_4, \succ_{c_1})(s_3, \succ_{c_1})\}$$

$$E = \{(s_4, \succ_{c_1})\}$$

For s_3

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $\boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

s_3: $\cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

s_4: $\cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

c_1: $\boxed{s_3} \succ s_1 \succ s_2 \succ \cancel{s_4}$

c_2: $s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

c_3: $s_3 \succ s_4 \succ s_1 \succ s_2$

c_4: $\boxed{s_4} \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

$$\boldsymbol{P = \{(s_4, \succ_{c_1})(s_3, \succ_{c_1})\}}$$

$$\boldsymbol{E = \{(s_4, \succ_{c_1})\}}$$

For s_3

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $\cancel{c_4} \succ c_1 \succ c_3 \succ c_2$

s_3: $\cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

s_4: $\cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

$\boldsymbol{c_1}$: $\boldsymbol{\boxed{s_3} \succ s_1 \succ s_2 \succ \cancel{s_4}}$

c_2: $s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

c_3: $s_3 \succ s_4 \succ s_1 \succ s_2$

c_4: $\boxed{s_4} \succ s_1 \succ \cancel{s_2} \succ s_3$

Computing Optimal Manipulations

$\boldsymbol{P} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})(\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$

$\boldsymbol{E} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})\}$

For s_3

$s_1: \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2: \cancel{c_4} \succ \boxed{c_1} \succ c_3 \succ c_2$

$s_3: \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_4: \cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

$\boldsymbol{c_1}: \boxed{\boldsymbol{s_3}} \succ \boldsymbol{s_1} \succ \boxed{\boldsymbol{s_2}} \succ \cancel{\boldsymbol{s_4}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: \boxed{s_4} \succ s_1 \succ \cancel{s_2} \succ s_3$

Computing Optimal Manipulations

$\boldsymbol{P} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})(\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$

$\boldsymbol{E} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})\}$

For s_3

$s_1: \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2: \cancel{c_4} \succ \boxed{c_1} \succ c_3 \succ c_2$

$s_3: \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_4: \cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

$\boldsymbol{c_1}: \boxed{\boldsymbol{s_3}} \succ \boldsymbol{s_1} \succ \boxed{\boldsymbol{s_2}} \succ \cancel{\boldsymbol{s_4}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: \boxed{s_4} \succ s_1 \succ \cancel{s_2} \succ s_3$

Computing Optimal Manipulations

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $\cancel{c_4} \succ \cancel{c_1} \succ c_3 \succ c_2$

s_3: $\cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

s_4: $\cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

$\boldsymbol{P = \{(s_4, \succ_{c_1})(s_3, \succ_{c_1})\}}$

$\boldsymbol{E = \{(s_4, \succ_{c_1})\}}$

For s_3

$\boldsymbol{c_1}$: $\boxed{\boldsymbol{s_3}} \succ \boldsymbol{s_1} \succ \cancel{s_2} \succ \cancel{s_4}$

c_2: $s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

c_3: $s_3 \succ s_4 \succ s_1 \succ s_2$

c_4: $\boxed{s_4} \succ s_1 \succ \cancel{s_2} \succ s_3$

Computing Optimal Manipulations

$\boldsymbol{P} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})(\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$

$\boldsymbol{E} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}})\}$

For s_3

$s_1: \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2: c_4 \succ c_1 \succ \boxed{c_3} \succ c_2$

$s_3: c_2 \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_4: c_1 \succ \boxed{c_4} \succ c_2 \succ c_3$

$\boldsymbol{c_1}: \boxed{\boldsymbol{s_3}} \succ \boldsymbol{s_1} \succ \cancel{\boldsymbol{s_2}} \succ \cancel{\boldsymbol{s_4}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ s_3$

$c_3: s_3 \succ s_4 \succ s_1 \succ \boxed{s_2}$

$c_4: \boxed{s_4} \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

$$\boldsymbol{P = \{(s_2, \succ_{s_3}), (s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}}$$

$$\boldsymbol{E = \{(s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}}$$ For s_2

$s_1: \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2: \boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3: \cancel{c_2} \succ c_1 \succ c_3 \succ c_4$

$s_4: \boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1: s_2 \succ s_3 \succ s_1 \succ \boxed{s_4}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: s_4 \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

$$\boldsymbol{P} = \{(\boldsymbol{s_2}, \succ_{\boldsymbol{s_3}}), (\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}}), (\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$$

$$\boldsymbol{E} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}}), (\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$$

For s_2

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $\boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

s_3: $\cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

s_4: $\boxed{c_1} \succ c_4 \succ c_2 \succ c_3$

$\boldsymbol{c_1}$: $\boldsymbol{s_2} \succ \boxed{\boldsymbol{s_3}} \succ \boldsymbol{s_1} \succ \boxed{\boldsymbol{s_4}}$

c_2: $s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

c_3: $s_3 \succ s_4 \succ s_1 \succ s_2$

c_4: $s_4 \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

$$P = \{(s_2, \succ_{s_3}), (s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}$$

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For s_2

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$s_2: \boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3: \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_4: \cancel{c_1} \succ c_4 \succ c_2 \succ c_3$

$c_1: s_2 \succ \boxed{s_3} \succ s_1 \succ \cancel{s_4}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: s_4 \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $\boxed{c_4} \succ c_1 \succ c_3 \succ c_2$

s_3: $\cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

s_4: $\cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

$$\boldsymbol{P} = \{(\boldsymbol{s_2}, \succ_{\boldsymbol{s_3}}), (\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}}), (\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$$

$$\boldsymbol{E} = \{(\boldsymbol{s_4}, \succ_{\boldsymbol{c_1}}), (\boldsymbol{s_3}, \succ_{\boldsymbol{c_1}})\}$$

For s_2

$\boldsymbol{c_1}$: $\boldsymbol{s_2} \succ \boxed{\boldsymbol{s_3}} \succ \boldsymbol{s_1} \succ \cancel{\boldsymbol{s_4}}$

c_2: $s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

c_3: $s_3 \succ s_4 \succ s_1 \succ s_2$

c_4: $\boxed{s_4} \succ s_1 \succ \boxed{s_2} \succ s_3$

Computing Optimal Manipulations

$s_1: \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

$s_2: \cancel{c_4} \succ c_1 \succ c_3 \succ c_2$

$s_3: \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$

$s_4: \cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

$\boldsymbol{P = \{(s_2, \succ_{s_3}), (s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}}$

$\boldsymbol{E = \{(s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}}$ For s_2

$\boldsymbol{c_1: s_2 \succ \boxed{s_3} \succ s_1 \succ \cancel{s_4}}$

$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: \boxed{s_4} \succ s_1 \succ \cancel{s_2} \succ s_3$

Computing Optimal Manipulations

$$s_1: \boxed{c_2} \succ c_3 \succ c_4 \succ c_1$$
$$s_2: \cancel{c_4} \succ \boxed{c_1} \succ c_3 \succ c_2$$
$$s_3: \cancel{c_2} \succ \boxed{c_1} \succ c_3 \succ c_4$$
$$s_4: \cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$$

$$\boldsymbol{P} = \{(\boldsymbol{s_2}, \succ_{s_3}), (\boldsymbol{s_4}, \succ_{c_1}), (\boldsymbol{s_3}, \succ_{c_1})\}$$

$$\boldsymbol{E} = \{(\boldsymbol{s_4}, \succ_{c_1}), (\boldsymbol{s_3}, \succ_{c_1})\}$$

For s_2

$$\boldsymbol{c_1}: \boxed{\boldsymbol{s_2}} \succ \boxed{\boldsymbol{s_3}} \succ \boldsymbol{s_1} \succ \cancel{\boldsymbol{s_4}}$$
$$c_2: s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$$
$$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$$
$$c_4: \boxed{s_4} \succ s_1 \succ \cancel{s_2} \succ s_3$$

Computing Optimal Manipulations

$$P = \{(s_2, \succ_{s_3}), (s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}$$

$$E = \{(s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}$$

For s_2

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $c_4 \succ \boxed{c_1} \succ c_3 \succ c_2$

s_3: $\cancel{c_2} \succ \cancel{c_1} \succ c_3 \succ c_4$

s_4: $\cancel{c_1} \succ \boxed{c_4} \succ c_2 \succ c_3$

c_1: $\boxed{s_2} \succ \cancel{s_3} \succ s_1 \succ \cancel{s_4}$

c_2: $s_2 \succ \boxed{s_1} \succ s_4 \succ \cancel{s_3}$

c_3: $s_3 \succ s_4 \succ s_1 \succ s_2$

c_4: $\boxed{s_4} \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

$$P = \{(s_2, \succ_{s_3}), (s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}$$

$$E = \{(s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}$$ For s_2

s_1: $\boxed{c_2} \succ c_3 \succ c_4 \succ c_1$

s_2: $c_4 \succ \boxed{c_1} \succ c_3 \succ c_2$

s_3: $c_2 \succ c_1 \succ \boxed{c_3} \succ c_4$

s_4: $c_1 \succ \boxed{c_4} \succ c_2 \succ c_3$

c_1: $\boxed{s_2} \succ \cancel{s_3} \succ s_1 \succ \cancel{s_4}$

c_2: $s_2 \succ \boxed{s_1} \succ s_4 \succ s_3$

c_3: $\boxed{s_3} \succ s_4 \succ s_1 \succ s_2$

c_4: $\boxed{s_4} \succ s_1 \succ s_2 \succ s_3$

Computing Optimal Manipulations

$$\boldsymbol{P = \{(s_2, \succ_{s_3}), (s_4, \succ_{c_1}), (s_3, \succ_{c_1})\}}$$

$$\boldsymbol{E = \{(s_4, \succ_{c_1}), (s_3, \succ_{c_1}), (s_2, \succ_{s_3})\}}$$

$s_1: c_2 \succ c_3 \succ c_4 \succ c_1$

$s_2: c_4 \succ c_1 \succ c_3 \succ c_2$

$s_3: c_2 \succ c_1 \succ c_3 \succ c_4$

$s_4: c_1 \succ c_4 \succ c_2 \succ c_3$

$c_1: s_1 \succ s_2 \succ s_4 \succ s_3$

$c_2: s_2 \succ s_1 \succ s_4 \succ s_3$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: s_4 \succ s_1 \succ s_2 \succ s_3$

No new potential partner, so stop.

Optimal manipulation for $c_1: s_2 \succ s_3 \succ s_1 \succ s_4$

Inconspicuous Optimal Manipulation

Optimal manipulation returned may look very different from true preferences

- May cause suspicion

Inconspicuous manipulation: Misreport a preference that is identical to the true preference, except the location of one student.

Vaish and Garg, 2017. Inconspicuous optimal manipulations always exist. Can be found in polynomial time


Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence (IJCAI-17)

Manipulating Gale-Shapley Algorithm: Preserving Stability and Remaining Inconspicuous

Rohit Vaish **Dinesh Garg**

Inconspicuous Optimal Manipulation

Vaish and Garg:

Find optimal manipulation $s, \succ^s$.

Let s' be the second best student to propose to c_j under $\succ^s$.

Let $\succ'$ be the same as $\succ_{c_j}$ with s' moved to the right of s.

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Manipulating Gale-Shapley Algorithm: Preserving Stability and Remaining Inconspicuous

Rohit Vaish **Dinesh Garg**

Inconspicuous Optimal Manipulation

Recall previous example:

$s_1: c_2 \succ c_3 \succ c_4 \succ c_1$

$s_2: c_4 \succ c_1 \succ c_3 \succ c_2$

$s_3: c_2 \succ c_1 \succ c_3 \succ c_4$

$s_4: c_1 \succ c_4 \succ c_2 \succ c_3$

$c_1: s_1 \succ s_2 \succ s_4 \succ s_3$

$c_2: s_2 \succ s_1 \succ s_4 \succ s_3$

$c_3: s_3 \succ s_4 \succ s_1 \succ s_2$

$c_4: s_4 \succ s_1 \succ s_2 \succ s_3$

Optimal manipulation for c_1: $s_2 \succ s_3 \succ s_1 \succ s_4$

IOM for c_1: $s_1 \succ s_2 \succ s_3 \succ s_4$

Stable Manipulations

Would the result of a manipulation be stable w.r.t true preferences?

- Not for suboptimal manipulations

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Stable Manipulations

True Preferences:

$s_1: c_2 \succ c_1 \succ c_3 \succ c_4$

$s_2: c_3 \succ c_1 \succ c_2 \succ c_4$

$s_3: c_1 \succ c_2 \succ c_3 \succ c_4$

$s_4: c_1 \succ c_4 \succ c_3 \succ c_2$

$c_1: s_1 \succ s_2 \succ \boxed{s_3} \succ s_4$

$c_2: s_3 \succ \boxed{s_1} \succ s_2 \succ s_4$

$c_3: s_1 \succ \boxed{s_2} \succ s_3 \succ s_4$

$c_4: s_1 \succ s_2 \succ s_3 \succ \boxed{s_4}$

Suboptimal Manipulation:

$\boldsymbol{c_1: \boxed{s_2} \succ s_4 \succ s_3 \succ s_1}$

$c_2: \boxed{s_3} \succ s_1 \succ s_2 \succ s_4$

$c_3: \boxed{s_1} \succ s_2 \succ s_3 \succ s_4$

$c_4: s_1 \succ s_2 \succ s_3 \succ \boxed{s_4}$

(s_1, c_1) blocks under true preferences.

Stable Manipulations

Would the result of a manipulation be stable w.r.t true preferences?

- Not for suboptimal manipulations
- **Yes, for optimal manipulations**

Thm. Under an optimal manipulation, result of SPDA is stable under true preferences

- If $\succ$ and $\succ'$ only differ in the position of one student then, all students who propose to c under $\succ$ propose under $\succ'$
- Any blocking pair for optimal manipulation $\succ^*$ must be with c
- If (s, c) blocks, s must have proposed to c under SPDA on $\succ^*$
- Then $\succ^s$ would be an even better manipulation. **Contradiction**.

Recap

So far for **one-one matchings**, we showed:

- Stable matchings exist
- Can be found in polynomial time

When $|S| = |C|$ then all agents are matched

Further,

- Stable matchings are manipulable
- Optimal manipulations are stable

What about Many-to-One Matchings?

Many-to-one Matchings

Now, given $< S, C, \boldsymbol{b}, \succ>$ with $m = |S| \geq |C| = n$.

Given: students S and colleges C with budgets b, a matching $\mu \subseteq C \times S$ is such that:

i. For each student $s \in S$, μ contains at most one pair (s, c)
ii. For each college $c \in C$, μ contains at most b_c pairs (s, c).

So $\mu(s) \in C \cup \emptyset$ and

$\mu(c) \in 2^S$ s.t. $|\mu(c)| \leq b_c$

Stable Many-to-One Matchings

Stable Matching: μ is stable if there is no blocking pair.

Blocking Pair: (s, c) block μ if both

- $c \succ_s \mu(s)$ AND
- c wants to match with s rather than follow μ

---- *What does this mean?*

Need to define agent preferences.

Agent Preferences

Student preferences: ordering over colleges C

What do college preferences look like?

For one-one matchings: ordering over students S

For many-to-one matchings: orderings over all subsets of students S, s.t. for any s, s' and $T \subset S \setminus \{s, s'\}$,

$$T \cup \{s\} \succ T \cup \{s'\} \Leftrightarrow \{s\} \succ \{s'\}$$

Responsive preferences.

Responsive Preferences

Help extend the set of preferences over individual students

May not be complete:

If $s_1 \succ s_2 \succ s_3 \succ s_4$,

No restriction on who should be preferred between: $\{s_1, s_4\}$ vs $\{s_2, s_3\}$

Can we now define stable matchings?

Stable Many-to-One Matchings

Stable Matching: μ is stable if there is no blocking pair.

Blocking Pair: (s, c) block μ if both

- $c \succ_s \mu(s)$ AND
- Either: $|\mu(c)| < b_c$ OR
- There exists $s' \in \mu(c)$ s.t. $s \succ_c s'$.

How do we find a stable matching?

- Reduce a many-to-one instance to a one-one instance:
- $$< S, C, \succ, b > \rightarrow < S', C', \succ' >$$

Canonical Reduction

Agents:

- For each $s_i \in S$ create $s_i' \in S'$
- For each $c_j \in C$ create $c_j^1, \cdots, c_j^{b_{c_j}}$

Preferences:

- For each $c_j \in C$ and each $t \in [b_{c_j}]$, $\succ'_{c_j^t}$ is the same as $\succ_{c_j}$ on individual students.
- For each $s_i \in S$ and $c_j \neq c_{j'}$ for any $t \in [b_{c_j}]$ and $t' \in [b_{c_{j'}}]$
  $c_j^t \succ'_{s_i'} c_{j'}^{t'} \Leftrightarrow c_j \succ_{s_i} c_{j'}$
- For each $s_i \in S$ and $c_j \in C$ for any $t < t' \in [b_{c_j}]$, $c_j^t \succ'_{s_i'} c_j^{t'}$

Canonical Reduction

Is the reduction complete?

$|S'| = |S|$ and $|C'| = \sum_{c \in C} b_c$

If $|S| < \sum_{c \in C} b_c$: Add dummy students who all colleges in $|C'|$ like less than all students in $|S|$

If $|S| > \sum_{c \in C} b_c$: Add dummy colleges who all students in $|S'|$ like less than all colleges in $|C|$

Can we now use this to find stable matchings?

Canonical Reduction

Thm. There is a one-one correspondence between stable matchings of a many-to-one instance and its canonical one-one instance.

Proof. HW

ELSEVIER

Discrete Applied Mathematics

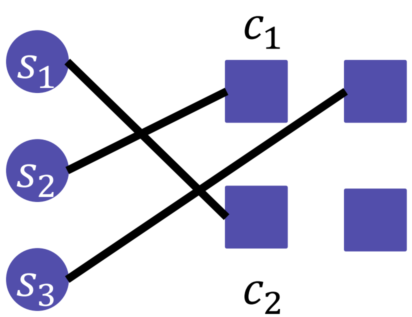
Volume 11, Issue 3, July 1985, Pages 223-232

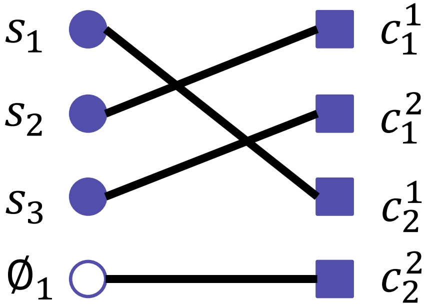
Some remarks on the stable matching problem

David Gale *, Marilda Sotomayor **

Canonical Reduction

Thm. There is a one-one correspondence between stable matchings of a many-to-one instance and its canonical one-one instance.





Example:

- $s_1 \quad : c_2 \succ c_1$
- $s_2, s_3 : c_1 \succ c_2$
- $c_1 : \{s_1, s_2, s_3\} \succ \{s_1, s_2\} \succ \{s_1, s_3\} \succ \{s_1\} \succ \{s_2, s_3\} \succ \{s_2\} \succ \{s_3\}$ $\quad \boldsymbol{b_{c_1} = 2}$
- $c_2 : \{s_1, s_2, s_3\} \succ \{s_2, s_3\} \succ \{s_1, s_3\} \succ \{s_3\} \succ \{s_1, s_2\} \succ \{s_2\} \succ \{s_1\}$ $\quad \boldsymbol{b_{c_1} = 2}$

Canonical Reduction: Implications

Thm. There is a one-one correspondence between stable matchings of a many-to-one instance and its canonical one-one instance.

Consequences:

Stable matchings always exist

Deferred Acceptance for many-to-one matchings

- **SPDA**: College c tentatively accepts top b_c students
- **CPDA**: College c proposes to top b_c students

HW: Student optimal and college optimal many-to-one matchings

Manipulating Many-to-One Matchings

Manipulation for Proposing Side

Under SPDA

One-one case: no beneficial manipulation for students

Many-to-one case: no beneficial manipulation for students
- Else SPDA wouldn't be strategyproof for one-one

Under CPDA

Colleges manipulating implies manipulation by all its copies.
- Can be helpful

Alvin Roth

ELSEVIER

Journal of Economic Theory

Volume 36, Issue 2, August 1985, Pages 277-288

The college admissions problem is not equivalent to the marriage problem ☆

Alvin E Roth

Example of Manipulation

Example by Roth:

$s_1 \colon c_3 \succ c_1 \succ c_2$

$s_2 \colon c_2 \succ c_1 \succ c_3$

$s_3 \colon c_1 \succ c_3 \succ c_2$

$s_4 \colon c_1 \succ c_2 \succ c_3$

$c_1 \colon s_1 \succ s_2 \succ s_3 \succ s_4 \quad b_{c_1} = 2$

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$s_1: c_3 \succ \boxed{c_1} \succ \boxed{c_2}$

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Example of Manipulation

Example by Roth:

$s_1: c_3 \succ \boxed{c_1} \succ \cancel{c_2}$

$s_2: c_2 \succ \boxed{c_1} \succ c_3$

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Let c_1 misreport

True: $c_1 : s_1 \succ s_2 \succ s_3 \succ s_4$

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Better for c_1!!!

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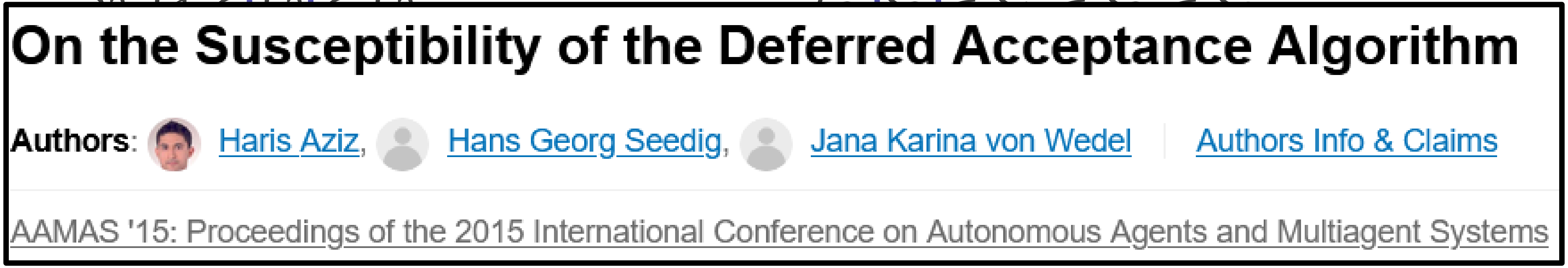

On the Susceptibility of the Deferred Acceptance Algorithm

Authors: Haris Aziz, Hans Georg Seedig, Jana Karina von Wedel | Authors Info & Claims

AAMAS '15: Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems

What else?

We have shown:

- Stable Many-to-one Matchings exist
- Can be found in polynomial time
- Can be manipulated
  - Even by the proposing side
- Can have unmatched agents

Can an agent be matched in one stable matching, unmatched in the other?

Matching “Rural” Hospitals

Rural Hospitals Theorem

Thm. Given a many-to-one matching instance $< S, C, b, >>$, for any college $c \in C$ and two stable matchings μ and μ'

1. Size of c's stable set is the same: $|\mu(c)| = |\mu'(c)|$

2. If $|\mu(c)| < b_c$ then $\mu(c) = \mu'(c)$

Alvin Roth

On the Allocation of Residents to Rural Hospitals: A General Property of Two-Sided Matching Markets

Alvin E. Roth

Econometrica, Vol. 54, No. 2 (Mar., 1986), pp. 425-427 (3 pages)

Proof Sketch

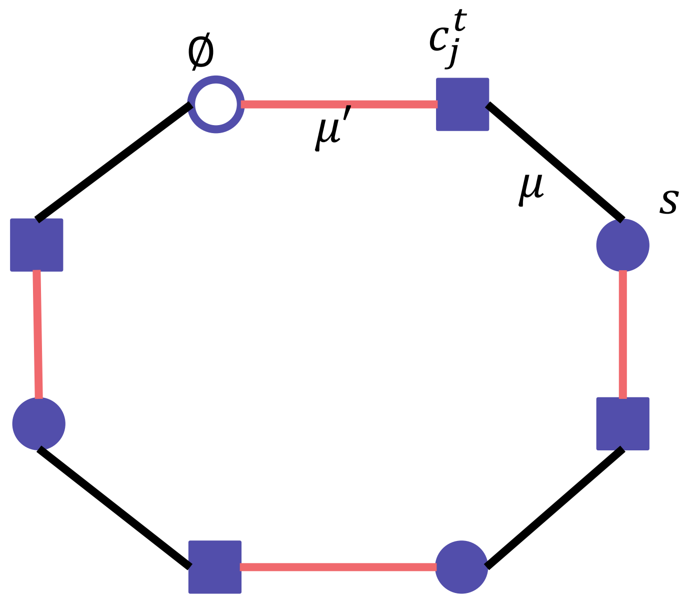
Part 1. Size of c's stable set is the same: $|\mu(c)| = |\mu'(c)|$

Suppose not, consider the canonical reduction.

Some c_j^t is matched in μ but not in μ'

Consider the union of μ and μ'

Results in disjoint cycles in the graph on S and C



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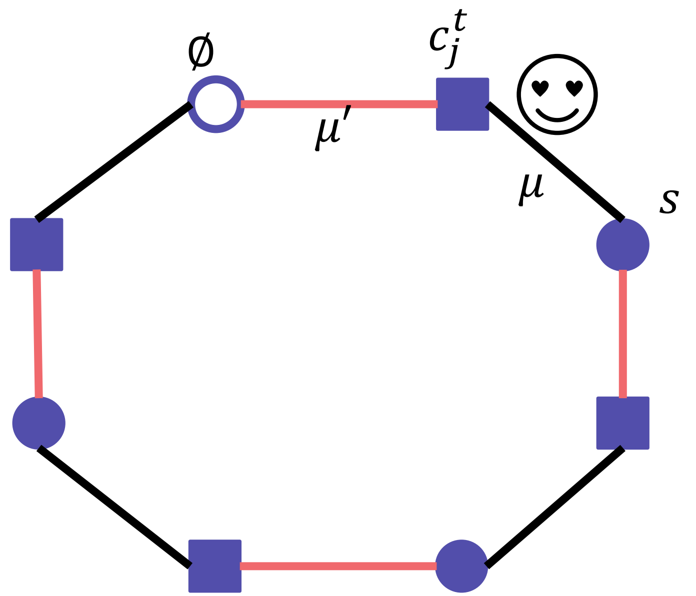

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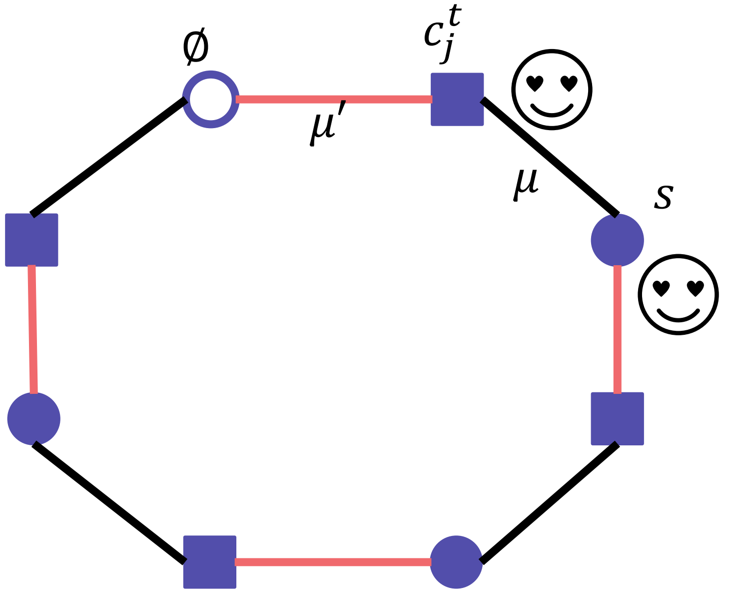

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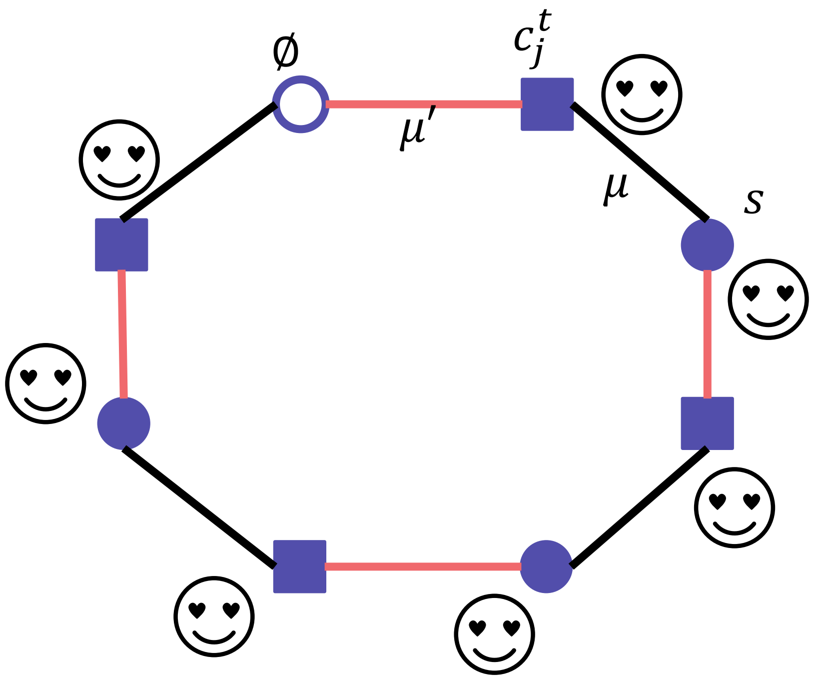

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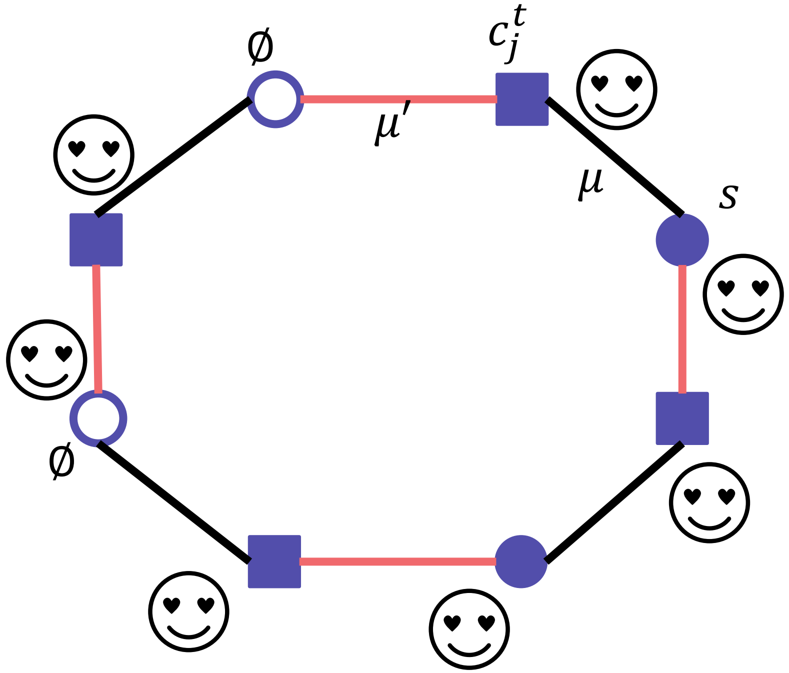

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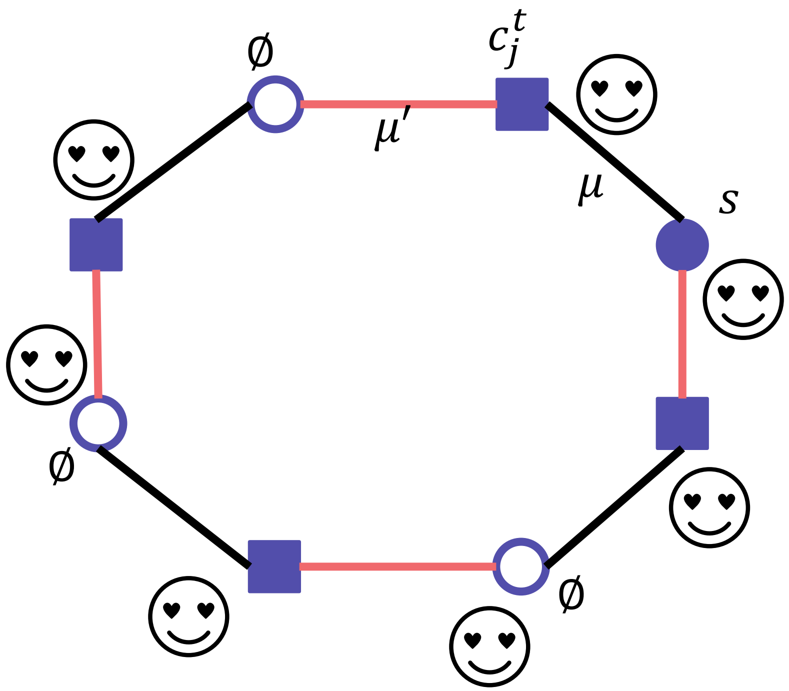

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Now clearly, $s \succ_c \emptyset$, so (s, c) block μ'.

Contradiction

So rural colleges with empty seats cannot benefit by changing matching mechanism!!

Manipulation

So far: only manipulation by preferences

What other forms of manipulation are possible?

One-one matchings: $< S, C, \succ >$

- Nothing

Many-to-one matchings: $< S, C, b, \succ >$

- **Capacity!**

Manipulation by Capacity

$s_1: \boxed{c_2} \succ c_1$ $\qquad c_1: \ldots \succ \{s_1\} \succ \boxed{\{s_2, s_3\}} \succ \{s_2\} \succ \{s_3\}$ $\qquad b_{c_1} = 2$

$s_2: \boxed{c_1} \succ c_2$ $\qquad c_2: \ldots \succ \{s_3\} \succ \{s_1, s_2\} \succ \{s_2\} \succ \boxed{\{s_1\}}$ $\qquad b_{c_2} = 2$

$s_3: \boxed{c_1} \succ c_2$

Student Proposing

Manipulation by Capacity

$s_1: c_2 \succ c_1$ $\quad$ $c_1: \ldots \succ \{s_1\} \succ \{s_2, s_3\} \succ \{s_2\} \succ \{s_3\}$ $\quad$ $b_{c_1} = 2$

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$s_3: c_1 \succ c_2$

College Proposing

Manipulation by Capacity

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$c_2 \colon \ldots \succ \cancel{\{s_3\}} \succ \{s_1, s_2\} \succ \cancel{\{s_2\}} \succ \boxed{\{s_1\}}$ $\quad b_{c_2} = 2$

College Proposing

Manipulation by Capacity

$s_1 \colon \boxed{c_2} \succ c_1$ $\qquad$ $c_1 \colon \ldots \succ \{s_1\} \succ \boxed{\{s_2, s_3\}} \succ \{s_2\} \succ \{s_3\}$ $\qquad$ $b_{c_1} = 2$

$s_2 \colon \boxed{c_1} \succ c_2$ $\qquad$ $c_2 \colon \ldots \succ \{s_3\} \succ \{s_1, s_2\} \succ \{s_2\} \succ \boxed{\{s_1\}}$ $\qquad$ $b_{c_2} = 2$

$s_3 \colon \boxed{c_1} \succ c_2$

College Proposing

Manipulation by Capacity

$s_1: c_2 \succ c_1$ $\quad$ $c_1: \ldots \succ \{s_1\} \succ \{s_2, s_3\} \succ \{s_2\} \succ \{s_3\}$ $\quad$ $b_{c_1} = 2$

$s_2: c_1 \succ c_2$ $\quad$ $c_2: \ldots \succ \{s_3\} \succ \{s_1, s_2\} \succ \{s_2\} \succ \{s_1\}$ $\quad$ $\boldsymbol{b_{c_2} = 1}$

$s_3: c_1 \succ c_2$

Suppose c_2 misreports

Manipulation by Capacity

$s_1: c_2 \succ c_1$ $\quad$ $c_1: \ldots \succ \{s_1\} \succ \{s_2, s_3\} \succ \{s_2\} \succ \{s_3\}$ $\quad$ $b_{c_1} = 2$

$s_2: c_1 \succ c_2$ $\quad$ $c_2: \ldots \succ \{s_3\} \succ \{s_1, s_2\} \succ \{s_2\} \succ \{s_1\}$ $\quad$ $\boldsymbol{b_{c_2} = 1}$

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College Proposing

Manipulation by Capacity

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$c_2 : \ldots \succ \boxed{\{s_3\}} \succ \{s_1, s_2\} \succ \{s_2\} \succ \{s_1\}$ $\quad \boldsymbol{b_{c_2} = 1}$

Suppose c_2 misreports

College Proposing

Manipulation by Capacity

$s_1: c_2 \succ \boxed{c_1}$ $\quad$ $c_1: \ldots \succ \boxed{\{s_1\}} \succ \{s_2, s_3\} \succ \boxed{\{s_2\}} \succ \{s_3\}$ $\quad$ $b_{c_1} = 2$

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$s_3: c_1 \succ \boxed{c_2}$

College Proposing

ELSEVIER

Journal of Economic Theory

Volume 77, Issue 1, November 1997, Pages 197-204

Manipulation via Capacities in Two-Sided Matching Markets

Tayfun Sönmez *

Misreporting capacities can help!

Manipulation by Capacity

$s_1: c_2 \succ \boxed{c_1}$ $\quad c_1: \ldots \succ \{\boxed{s_1}\} \succ \{s_2, s_3\} \succ \{\boxed{s_2}\} \succ \{s_3\}$ $\quad b_{c_1} = 2$

$s_2: \boxed{c_1} \succ c_2$ $\quad c_2: \ldots \succ \{\boxed{s_3}\} \succ \{s_1, s_2\} \succ \{s_2\} \succ \{s_1\}$ $\quad \boldsymbol{b_{c_2} = 1}$

$s_3: c_1 \succ \boxed{c_2}$

College Proposing

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Manipulation via Capacities in Two-Sided Matching Markets

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Misreporting capacities can help!

Capacity Modification in the Stable Matching Problem

Authors: Salil Gokhale, Samarth Singla, Shivika Narang, Rohit Vaish Authors Info & Claims

AAMAS '24: Proceedings of the 23rd International Conference on Autonomous Agents and Multiagent Systems • Pages 697 - 705

Exercise Sheet

Exercise I

Given a one-one matching instance $< S, C >>$, define the **pain** of a matching μ to be the sum of the ranks of the matched partners under μ. For an arbitrary n construct a one-one instance with n students and n colleges such that it admits a stable matching with **pain** of $n(n+1)$.

For example, in the following instance, matching $\mu = \{(s_1, c_1), (s_2, c_2)\}$ has $\text{pain}(\mu) = 5$.

$s_1 : c_1 \succ c_2$	$c_1 : s_1 \succ s_2$
$s_2 : c_1 \succ c_2$	$c_2 : s_2 \succ s_1$

Exercise I

Given a one-one matching instance $< S, C >>$, define the **pain** of a matching μ to be the sum of the ranks of the matched partners under μ. For an arbitrary n construct a one-one instance with n students and n colleges such that it admits a stable matching with **pain** of $n(n+1)$.

Solution. Consider the following instance with $|S| = |C| = n$ where all students have the same preference and all colleges have the same preference:

$$s: c_1 \succ c_2 \succ c_3 \cdots \succ c_n$$

$$c: s_1 \succ s_2 \succ s_3 \cdots \succ s_n$$

Unique stable matching $\mu = \{(s_i, c_i) | i = 1,2, \cdots, n\}$

$$\text{Pain}(\mu) = 1 + 1 + 2 + 2 + \cdots + n + n = 2(1 + 2 + \cdots + n\ ) = n(n+1)$$

Exercise II

For an arbitrary n construct a one-one matching instance where exactly $\left\lfloor \frac{n}{2} \right\rfloor$ students have a unique achievable college. For all other students, they are their optimal college's least preferred student.

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For an arbitrary n construct a one-one matching instance where exactly $\left\lfloor \frac{n}{2} \right\rfloor$ students have a unique achievable college. For all other students, they are their optimal college's least preferred student.

Solution. Consider the following instance with $|S| = |C| = n$. The first $\left\lfloor \frac{n}{2} \right\rfloor$ students and colleges have the same preference.

$$s_1, \cdots, s_{\left\lfloor \frac{n}{2} \right\rfloor}: c_1 \succ c_2 \succ \cdots \succ c_n$$

$$s_i: c_i \succ c_1 \succ c_2 \succ \cdots \succ c_{i-1} \succ c_{i+1} \succ \cdots \succ c_n \quad \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \cdots, n$$

Exercise II

For an arbitrary n construct a one-one matching instance where exactly $\left\lfloor \frac{n}{2} \right\rfloor$ students have a unique achievable college. For all other students, they are their optimal college's least preferred student.

Solution. Consider the following instance with $|S| = |C| = n$. The first $\left\lfloor \frac{n}{2} \right\rfloor$ students and colleges have the same preference.

$$c_1, \cdots, c_{\left\lfloor \frac{n}{2} \right\rfloor}: s_1 \succ s_2 \succ \cdots \succ s_n$$

$$c_i: s_1 \succ s_2 \succ \cdots \succ s_{i-1} \succ s_{i+1} \succ \cdots \succ s_n \succ s_i \quad \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \cdots, n$$

Exercise II

For an arbitrary n construct a one-one matching instance where exactly $\left\lfloor \frac{n}{2} \right\rfloor$ students have a unique achievable college. For all other students, they are their optimal college's least preferred student.

Solution. Sufficient to show that SPDA will return the matching $\mu = \{(s_i, c_i) | i = 1,2,\cdots,n\}$.

Can show that by induction for $i = 1,\cdots,\left\lfloor \frac{n}{2} \right\rfloor$ (s_i, c_i) matched under SPDA.

Each remaining c_i only gets a proposal from s_i.

Exercise III

Given a many-to-one instance $< S, C, b, \succ >$, let S_c be the top b_c most preferred students of $c \in C$. If these sets are mutually disjoint, how many proposals will occur under CPDA?

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Given a many-to-one instance $< S, C, b, \succ >$, let S_c be the top b_c most preferred students of $c \in C$. If these sets are mutually disjoint, how many proposals will occur under CPDA?

Solution.

Under CPDA, each $c \in C$ proposes to its top b_c students.

If the sets are disjoint, no student gets two proposals.

Thus, no rejections are made.

Hence, number of proposals is $\sum_{c \in C} b_c$

Exercise IV

Construct a one-one matching instance where under the SPDA the optimal manipulation of a college c would not match it to its optimal partner.

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Solution. Consider c_1 in following instance.

$s_1\colon c_2 \succ c_3 \succ c_4 \succ c_5 \succ c_1$

$s_2\colon c_3 \succ c_4 \succ c_5 \succ c_1 \succ c_2$

$s_3\colon c_5 \succ c_1 \succ c_4 \succ c_2 \succ c_3$

$s_4\colon c_3 \succ c_1 \succ c_2 \succ c_4 \succ c_5$

$s_5\colon c_1 \succ c_5 \succ c_2 \succ c_3 \succ c_4$

$c_1\colon s_1 \succ s_2 \succ s_3 \succ s_5 \succ s_4$

$c_2\colon s_2 \succ s_1 \succ s_4 \succ s_5 \succ s_3$

$c_3\colon s_3 \succ s_2 \succ s_5 \succ s_1 \succ s_4$

$c_4\colon s_4 \succ s_5 \succ s_1 \succ s_2 \succ s_3$

$c_5\colon s_5 \succ s_1 \succ s_2 \succ s_3 \succ s_4$

Exercise IV

Construct a one-one matching instance where under the SPDA the optimal manipulation of a college c would not match it to its optimal partner.

Solution.

College optimal matching $\mu_C = \{(s_1, c_1), (s_2, c_2), (s_3, c_3), (s_4, c_4), (s_5, c_5)\}$.

Optimal manipulation for c_1: $s_3 \succ s_4 \succ s_1 \succ s_2 \succ s_5$

Exercise V

Prove that in a many-to-one matching instance, under the CPDA there is no beneficial manipulation for a college c s.t $b_c = 1$.

Solution. A beneficial manipulation for college c with $b_c = 1$ implies a beneficial manipulation in canonical instance.

In canonical instance, no single college can beneficially manipulate under CPDA.

- As CPDA is strategyproof for college for one-one matchings.