Game Theory

COMP4418–Assignment 1

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Question 1 (20 marks) Consider the following normal-form game, which is parameterized by a value $\alpha \in \mathbb{R}$.

	x	y
a	1, -1	-3, 3
b	-2, α	4, -4

a) **For which value of α is the game zero-sum?**

According to the definition of zero-sum:

A zero-sum game is a two-player normal-form game such that $u_1(a) + u_2(a) = 0$ for all action profiles $a \in A$.

Therefore, $-2 + \alpha = 0$, $\alpha = 2$.

b) **For which values of α is the outcome (−2,α) Pareto-optimal?**

According to the definition of Pareto-optimal:

Outcome $(u_1(a), \ldots, u_n(a))$ Pareto-dominates outcome $(u_1(b), \ldots, u_n(b))$ if $u_i(a) \geq u_i(b)$ for all $i \in N$ and $u_i(a) > u_i(b)$ for at least one $i \in N$.

An outcome is Pareto-optimal if it is not Pareto-dominated by any other outcome.

For outcome **(1, -1)**, $u_1(a)$ = 1 which is larger than -2 in outcome $(-2, \alpha)$. α must be larger than -1 to avoid being Pareto-dominated by outcome (1, -1)

So $\boldsymbol{\alpha > -1}$.

For outcome **(4, -4)**, $u_1(a)$ = 4 which is larger than -2 in outcome $(-2, \alpha)$. α must be larger than -4 to avoid being Pareto-dominated by outcome (4, -4)

So $\boldsymbol{\alpha > -4}$.

For outcome **(-3, 3)**, $u_1(a)$ = 3 which is smaller than -2 in outcome $(-2, \alpha)$. α can be any number here.

So $\boldsymbol{\alpha \in R}$.

Therefore, we got three results $\begin{cases} \alpha > -1 \\ \alpha > -4 \\ \alpha \in R \end{cases}$ To ensure that all the equations hold,

the answer is $\boldsymbol{\alpha > -1.}$

c) **For which values of α can the game be solved by iterated strict dominance?**

According to the definition:

Action a_i **dominates** action b_i if $u_i(a_i, a_{-i}) > u_i(b_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.

- Action a_i dominates action b_i if, regardless of what the other agents do, a_i yields more utility than b_i.

Since iterated strict dominance is order-independent, firstly if we want to eliminate action $\boldsymbol{a}$, we need to make sure **a** is dominated by **b**, which is $\begin{cases} \mathbf{1 < -2}(\boldsymbol{impossible}) \\ \mathbf{-3 < 4} \end{cases}$.

The same case for action $\boldsymbol{b}$, $\begin{cases} \mathbf{-2 < 1} \\ \mathbf{4 < -3}(\boldsymbol{impossible}) \end{cases}$. So, we can't eliminate dominated actions by comparing the actions of the row players.

For column players, since $3 > -1$, if we want to make sure x is dominated by y (eliminate x), we need that $\alpha < -4$, which is $\begin{cases} -1 < 3 \\ \alpha < -4 \end{cases}$. The answer is $\boldsymbol{\alpha < -4}$.

d) **For which value of α is it the maximin strategy of the column player to play x with probability $\frac{1}{2}$?**

For the column player, consider each of their strategies and what the minimum payoff would be if the row player takes their action.
In this specific example, we calculated:

Since the column player to play **x** with probability $\frac{1}{2}$ then, play **y** with probability $1 - \frac{1}{2} = \frac{1}{2}$.

The expected payoff when the row player chooses a:

$$E(a) = -1 * \frac{1}{2} + 3 * \frac{1}{2} = 1$$

The expected payoff when the row player chooses b

$$E(b) = \frac{1}{2} * \alpha + (-4 * \frac{1}{2}) = \frac{1}{2}\alpha - 2$$

The column player wants to ensure that their worst-case payoff is as high as possible. After calculating the payoffs for each strategy, they will look for a strategy that maximizes the worst-case outcomes. To achieve this. We set

$$E(a) = E(b)$$

$$1 = \frac{1}{2}\alpha - 2$$

$\boldsymbol{\alpha = 6}$.

e) **For which value of α will the row player play a with probability $\frac{3}{4}$ in a Nash equilibrium?**

Indifference principle: The row player must randomize such that the column player is indifferent between all actions in his support.

the row player plays a with probability $\frac{3}{4}$ in a Nash equilibrium, then he plays b with probability $\frac{1}{4}$.

Since no action is dominated, it must hold that

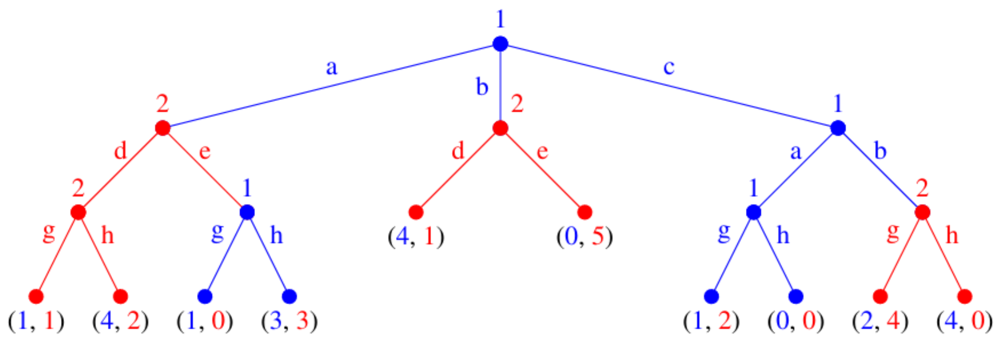
$$\boldsymbol{s(a)} * -1 + \boldsymbol{s(b)} * \alpha = 3 * \boldsymbol{s(a)} + \boldsymbol{-4} * \boldsymbol{s(b)},$$

$$\frac{\mathbf{3}}{\mathbf{4}} * -1 + \frac{\mathbf{1}}{\mathbf{4}} * \alpha = 3 * \frac{\mathbf{3}}{\mathbf{4}} + \boldsymbol{-4} * \frac{\mathbf{1}}{\mathbf{4}},$$

$$\frac{\mathbf{1}}{\mathbf{4}} * \alpha = 2$$

Therefore, $\boldsymbol{\alpha} = \mathbf{8}$.

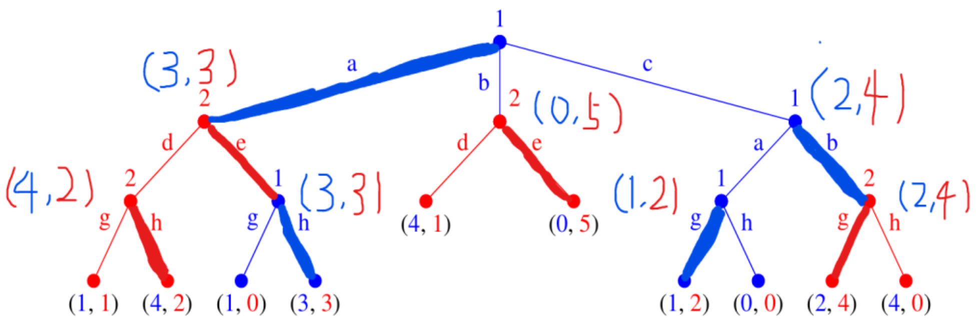
Question 2 (20 marks) Consider the following extensive-form game.



a) Compute the subgame-perfect Nash equilibrium.

For finding a subgame-perfect Nash equilibrium, we can use backwards induction.

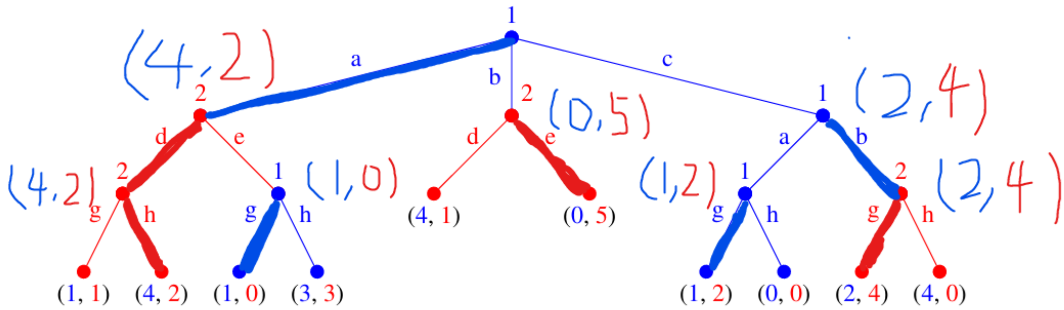
By comparing the corresponding utilities at each stage, we can determine the equilibrium:



b) Is there a pure Nash equilibrium where player 1 has a utility of 4? Explain your answer!

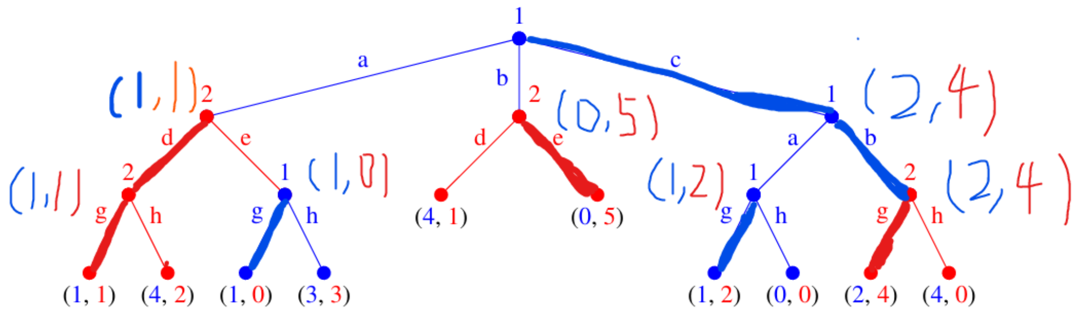
Yes. In considering a pure Nash equilibrium, Player 1 can choose the strategy (1, 0) instead of (3, 3). We need to compare the outcomes of (1, 0) and (4, 2). Since the utility of 2 is greater than 0, (4, 2) will be selected. Next, we compare (4, 2) with (0, 5) and (2, 4); here, the utilities yield the order of 4 > 2 > 0. Consequently, (4, 2) will be chosen, resulting in Player 1 obtaining a utility of 4.

In this case, the strategies selected by the players remain fixed (with (1, 0) chosen), and under this specific strategy profile, each player's choice is the best response to the strategies of the other players. Under the current strategy combination, players have no incentive to change their choices, which establishes that this is a pure Nash equilibrium.
The following picture:



c) What is the maximum utility that player 2 can obtain in a pure Nash equilibrium?

For player 2, we firstly assume player 2 can obtain utility of 5, the outcome is (0,5), it is impossible that this will be chosen because player 1's utility is 0 which is the smallest. Then we assume player 2 can obtain utility of 4. **(2,4)** In the following picture we can see it is possible that player 2 can obtain utility of 4 in a pure Nash equilibrium.



Therefore, the answer is **4**.

Question 3 (20 marks) Three pirates find a treasure of 90 gold coins. They decide on the following protocol to split the gold coins. First, all pirates $i \in \{1,2,3\}$ submit two numbers $t_i \in \{0,\dots,90\}$ and $k_i \in \{0,\dots,90\}$. Then, the pirates 1, 2, 3 (in this order) take t_i coins from the treasure if there are sufficiently many gold coins left or all remaining coins otherwise. Next, pirate 2 checks the amount of coins pirate 1 took: if pirate 1 has more than k_2 gold coins, pirate 2 steals all coins of pirate 1; if pirate 1 has at most k_2 coins, he keeps his coins and nothing happens. Finally, pirate 3 checks how much coins pirate 2 has (including those he possibly stole from pirate 1): if this amount exceeds k_3, he steals all coins from pirate 2 and otherwise, pirate 2 keeps his coins.

Assume that the utility function of every pirate is equal to the number of gold coins he has in the end if he did not steal the gold coins of his predecessor and half of that otherwise.

a) Show that there is a pure Nash equilibrium where all pirates have 30 gold coins.

There is a pure Nash equilibrium where all pirates have 30 gold coins when

$$t_1 = 30 \;\; k_1 \in \{0,\dots,90\}$$
$$t_2 = 30 \;\; k_2 = 30$$
$$t_3 = 30 \;\; k_3 = 30$$

k_1 can be any number between 0 and 90 because it doesn't affect anything here.
So, in this case, a pure Nash equilibrium exists.
For **pirate1**: if $t_1 > 30$, **pirate1** will be stolen by **pirate2** since $k_2 = 30$ and $t_1 > k_2$ then the utility of pirate1 will be 0. If he decreases t_1, his utility will also decrease, if $t_1 < 30$, $u_1 < 30$**.** Therefore, pirate1 has no incentive to submit different numbers. ($t_1 = 30 \;\; k_1 \in \{0,\dots,90\}$)

For **pirate2**: if $t_2 > 30$, **pirate2** will be stolen by **pirate3** since $k_3 = 30$ and $t_2 > k_3$ then the utility of **pirate2** will be 0. If he decreases t_2, his utility will also decrease, if $t_2 < 30$, $u_2 < 30$**.** So, he will keep t_2. If he changes k_2, when $k_2 < 30$, he will steal all coins from pirate1 since $k_2 < t_1$, and $u_2 = (30 + 30)/2 = 30$, The original u_2 is also 30 because the utility function of every pirate is equal to the number of gold coins he has. u_2 remains unchanged. Increasing k_2 will not affect u_2 since he can't steal **pirate1**.
Therefore, pirate2 has no incentive to submit different numbers. ($t_2 = 30 \;\; k_2 = 30$).

For **pirate3**: Now $t_1 = t_2 = 30$, there are only $90 - 30 - 30 = 30$ coins left, so t_3 remains the same. If **pirate3** reduces k_3 , $k_2 < 30$, he will steal all coins from pirate2 since $k_3 < t_2$ and $u_3 = (30 + 30)/2 = 30$, u_3 doesn't change. Increasing k_3 will not affect u_3 since he can't steal **pirate2**. Therefore, **pirate3** has no incentive to submit different numbers. ($t_3 = 30 \;\; k_3 = 30$)
As a result, each pirate has no incentive to submit different numbers and they all have 30 gold coins, it is a pure Nash equilibrium.

b) What is the maximal amount of gold coins that each pirate can obtain in a pure Nash equilibrium? Present for each pirate the corresponding strategy profile and reason why it is a pure Nash equilibrium.

For **pirate1: We assume**

$$t_1 = 90 \;\; k_1 \in \{0,\dots,90\}$$
$$t_2 = 0 \;\; k_2 = 90$$
$$t_3 = 90 \;\; k_3 = 0$$

In this case, **pirate1** can obtain 90 coins which is maximum possible. **pirate1** has no incentive to change numbers $u_1 = 90$. Since the amount of coins left is 0, no matter what the value of t_2 is, **pirate2** will not obtain any coins. $k_2 = 90$ to make sure **pirate1** will not be stolen by **pirate2**. Since $k_3 = 0$, if **pirate2** wants to reduce k_2 to steal **pirate1,** this would lead to $k_3 > t_2$, which means **pirate3** will steal all coin from **pirate2,** so, k_2 remains the same. Because no coins left. **pirate3** can't obtain coins by changing t_3 and k_3.
First, according to t_1, t_2, and t_3, Pirate 1 receives 90, Pirate 2 receives 0, and Pirate 3 receives 0 coins initially.
Next, Pirate 2 checks the coins that Pirate 1 has, which is 90 (equal to k_2). According to the

game rules, Pirate 2 will not steal coins from Pirate 1.
Then, Pirate 3 checks the coins that Pirate 2 has, which is 0 (equal to k_3). As a result, Pirate 3 will not steal all the coins from Pirate 2.
Finally, **pirate1** will obtain 90 coins, **pirate2:** 0 coins, **pirate3**: 0 coins.

$$u_1 = 90, u_2 = 0, u_2 = 0$$

As a result, each pirate has no incentive to submit different numbers, it is a pure Nash equilibrium. the maximal amount of gold coins that pirate1 can obtain in a pure Nash equilibrium is **90.**

For **pirate2: We assume**

$$t_1 = 0 \ \ k_1 \ \in \{0,\ldots,90\}$$
$$t_2 = 45 \ \ k_2 \ = 0$$
$$t_3 = 45 \ \ k_3 \ = 45$$

In this case, the less k_2 is, the more chance **pirate2** can steal **pirate1**, $k_2 = 0$. Since that, no matter what the value of t_1 is, the amount of coins that pirate1 can obtain is 0 (if $t_1 > 0$, he will be stolen by pirate2). if **pirate2** reduces t_2, u_2 will be smaller, if he increases it, he will be stolen by pirate3. ($u_2 = 0$). if pirate3 reduces k_3, he will steal all coins from pirate2 which makes $u_3 = (45 + 45)/2 = 45$, the original u_3 is also 45 since $u_3 = t_3 = 45$. u_3 remains unchanged. Now, each pirate has no incentive to submit different numbers, it is a pure Nash equilibrium.
Let's see if $t_2 = 46, k_2 = 0, t_3 = 44 \ \ k_3 = 46$. In this case **pirate3** can increase his utility u_3 by reduce k_3**.** The original $u_3 = 44$, if pirate3 steals all coins from pirate2, $u_3 = (44 + 46)/2$ $u_3 = 45$ which is larger than the original value. Then **pirate3** has incentive to report different k_3.(reduce it) This is not a pure Nash equilibrium.
First, according to t_1, t_2, and t_3, Pirate 1 receives 0, Pirate 2 receives 45, and Pirate 3 receives 45 coins initially.
Next, Pirate 2 checks the coins that Pirate 1 has, which is 0 (smaller than k_2). According to the game rules, Pirate 2 will not steal coins from Pirate 1.
Then, Pirate 3 checks the coins that Pirate 2 has, which is 45 (equal to k_3). As a result, Pirate 3 will not steal all the coins from Pirate 2.
Therefore, **pirate1** :0 coins, **pirate2:** 45 coins, **pirate3**: 45 coins.

$$u_1 = 0, u_2 = 45, u_2 = 45$$

The maximal amount of gold coins that pirate2 can obtain in a pure Nash equilibrium is **45.**

For **pirate3: We assume**

$$t_1 = 90 \ \ k_1 \ \in \{0,\ldots,90\}$$
$$t_2 = 90 \ \ k_2 \ = 0$$
$$t_3 = 90 \ \ k_3 \ = 0$$

In this case, no matter what the value of t_1 is, the amount of coins that **pirate1** can obtain is 0 since $k_2 = 0$. Also, no matter what the value of t_2 is, the amount of coins that **pirate2** can obtain is 0 since $k_3 = 0$. Finally, **pirate3** will steal all coins from pirate2.

First, according to t_1, t_2, and t_3, Pirate 1 receives 90, Pirate 2 receives 0, and Pirate 3 receives 0 coins initially. (For pirate 2 and 3, they claim 90 but only gets 0 since there are not sufficient coins left for him.)
Next, Pirate 2 checks the coins that Pirate 1 has, which is 90 (greater than k_2). According to the game rules, Pirate 2 will steal coins from Pirate 1.
Then, Pirate 3 checks the coins that Pirate 2 has, which is 90 (greater than k_3). As a result, Pirate 3 steals all the coins from Pirate 2.
Finally, **pirate1** :0 coins, **pirate2:** 0 coins, **pirate3**: 90 coins.

$$u_1 = 0, u_2 = 0, u_2 = 90/2 = 45$$

The maximal amount of gold coins that pirate3 can obtain in a pure Nash equilibrium is **90.**

Question 4 (10 marks) Let $A = \{a,b,c\}$ and $\succsim$ denote a rational preference relation over $\mathscr{L}(A)$ that is independent and satisfies that $[1 : a] \succ [1 : b]$ and $[\frac{1}{2} : a, \frac{1}{2} : c] \sim [1 : b]$. Show that $[1 : b] \succ [1 : c]$.

The definition:

A preference relation $\succsim$ on $\mathcal{L}(A)$ is

- rational if
  - it is complete: $L_1 \succsim L_2$ or $L_2 \succsim L_1$ for all $L_1, L_2 \in \mathcal{L}(A)$
  - and transitive: $L_1 \succsim L_2$ and $L_2 \succsim L_3$ implies $L_1 \succsim L_3$ for all $L_1, L_2, L_3 \in \mathcal{L}(A)$.
- continuous if, for all $L_1, L_2, L_3 \in \mathcal{L}(A)$ with $L_1 \succ L_2 \succ L_3$, there is $\epsilon > 0$ such that

$$[1-\epsilon : L_1, \epsilon : L_3] \succ L_2 \succ [1-\epsilon : L_3, \epsilon : L_1].$$

- independent if, for all lotteries L_1, L_2, L_3 and all $p \in (0,1)$, it holds that

$$L_1 \succsim L_2 \iff [p : L_1, (1-p) : L_3] \succsim [p : L_2, (1-p) : L_3].$$

Let $L_x = [1{:}x]$ for $x \in \{a,b,c\}$, $L_1 = [\frac{1}{2} : a,\ \frac{1}{2} : c]$.

Now we assume $L_2 = [\frac{1}{2} : b,\ \frac{1}{2} : c]$.

By assumption, $L_1 \sim L_b$ and $L_a \succ L_b$,

By **independence**, $[\frac{1}{2} : a,\ \frac{1}{2} : c] \succ [\frac{1}{2} : b,\ \frac{1}{2} : c]$ since $L_a \succ L_b$.

Then, $L_1 \sim L_b$ so $L_b \succ [\frac{1}{2} : b,\ \frac{1}{2} : c]$, $L_b \succ L_2$.

Since $[1 : b] \succ [\frac{1}{2} : b,\ \frac{1}{2} : c]$, $[\frac{1}{2} : b,\ \frac{1}{2} : c] = [\frac{1}{2} : c,\ \frac{1}{2} : b]$, we can get

$$[\frac{1}{2} : b, \frac{1}{2} : b] \succ [\frac{1}{2} : c, \frac{1}{2} : b]$$

Therefore, by **independence**, we can prove $\boldsymbol{L_b} \succ \boldsymbol{L_c}$, $[1 : b] \succ [1 : c]$

Question 5 (30 marks) Prove the following statements.

a) Let $G_1 = (\{1,2\}, (A_i^1)_{i\in\{1,2\}}, (u_i^1)_{i\in\{1,2\}})$, $G_2 = (\{1,2\}, (A_i^2)_{i\in\{1,2\}}, (u_i^2)_{i\in\{1,2\}})$, and $G_3 = (\{1,2\}, (A_i^3)_{i\in\{1,2\}}, (u_i^3)_{i\in\{1,2\}})$ denote three two-player normal-form games such that $A_i^1 = A_i^2 = A_i^3$ for $i \in \{1,2\}$ and $u_i^3(a) = \frac{1}{2}(u_i^1(a) + u_i^2(a))$ for both players $i \in \{1,2\}$ and all action profiles $a \in A$. Show that, if a strategy profile s is a Nash equilibrium for G_1 and G_2, then it is a Nash equilibrium for G_3.

According to the definition of Nash equilibrium:

A strategy profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium if $u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i})$ for all $t_i \in S_i$ and all $i \in N$.

If a strategy profile s is a Nash equilibrium for G_1 and G_2,
For G_1, $u_i^1(s_i^1, s_{-i}^1) \geq u_i^1(t_i^1, s_{-i}^1)$,
For G_2, $u_i^2(s_i^2, s_{-i}^2) \geq u_i^2(t_i^2, s_{-i}^2)$,

Then we add these two inequalities and multiply both sides by $\frac{1}{2}$, we can get

$$\frac{1}{2}(u_i^1(s_i^1, s_{-i}^1) + u_i^2(s_i^2, s_{-i}^2)) \geq \frac{1}{2}(u_i^1(t_i^1, s_{-i}^1) + u_i^2(t_i^2, s_{-i}^2))$$

Since $A_i^3 = A_i^2 = A_i^1$ and $u_i^3(a) = \frac{1}{2}(u_i^1(a) + u_i^2(a))$, we can get

$$u_i^3(s_i^3, s_{-i}^3) \geq u_i^3(t_i^3, s_{-i}^3)$$

Which follows the **Nash equilibrium theorem.**

Therefore, it is a Nash equilibrium for G_3.

b) Let $G = (\{1,2\}, (A_i)_{i\in\{1,2\}}, (u_i)_{i\in\{1,2\}})$ denote a two-player normal-form game and let s^1 and s^2 denote two Nash equilibria for G such that $s_i^1(a_i) > 0$ if and only if $s_i^2(a_i) > 0$ for both players $i \in \{1,2\}$ and all actions $a_i \in A_i$. Show that the strategy profile s^3 given by $s_1^3(a_1) = \frac{1}{2}(s_1^1(a_1) + s_1^2(a_1))$ for all $a_1 \in A_1$ and $s_2^3(a_2) = s_2^1(a_2)$ for all $a_2 \in A_2$ is also a Nash equilibrium for G.

We know that s^1 and s^2 denote two Nash equilibria for G such that
$s_i^1(a_i) > 0 \; if \; and \; only \; if \; s_i^2(a_i) > 0$.

So, the strategy

$s^1 \; is \; (s_i^1(a_1), s_i^1(a_2))$

$s^2 \; is \; (s_i^2(a_1), s_i^2(a_2))$

Given by $s_1^3(a_1) = \frac{1}{2}(s_1^1(a_1) + s_1^2(a_1))$

Since s^1 and s^2 denote two Nash equilibria, the expect utilities of them are the same.
Assume $u_i(a_i)$ represents the expect utility of the player i with action a_i.

$u_1(s_i^1(a_1)) = u_1(s_i^2(a_1))$

$u_2(s_i^1(a_2)) = u_2(s_i^2(a_2))$

For player1 in s^3,

$u_1(s_1^3(a_1)) = u_1(\frac{1}{2}(s_1^1(a_1) + s_1^2(a_1)))$

Since $u_1(s_i^1(a_1)) = u_1(s_i^2(a_1))$, $u_1(s_1^3(a_1)) = \frac{1}{2}u_1(s_1^1(a_1)) + \frac{1}{2}u_1(s_1^2(a_1))$

$$u_1(s_1^3(a_1)) = u_1((s_1^1(a_1)) = u_1((s_1^2(a_1))$$

For player2 in s^3,
Since $s_2^3(a_2) = s_2^1(a_2)$

$$u_2(s_2^3(a_2)) = u_2(s_2^1(a_2)) = u_2(s_2^1(a_2))$$

For both players, the expected utility of s^3 is the same as that of s^2,and s^1.

Therefore, it is also **a Nash equilibrium for $\boldsymbol{G}$**.

c) Let $G = (\{1,2\},(A_i)_{i\in\{1,2\}},(u_i)_{i\in\{1,2\}})$ denote a two-player zero-sum game such that $A_1 = A_2 = \{a_1,\dots,a_k\}$ for $k > 1$ and $u_1(a_i,a_j) = u_2(a_j,a_i)$ for all $a_i,a_j \in A$. Show that the value of G (i.e., the security level of player 1) is 0.
Hint: Show that for every strategy of player 2, player 1 has a strategy that guarantees him an expected utility of at least 0.

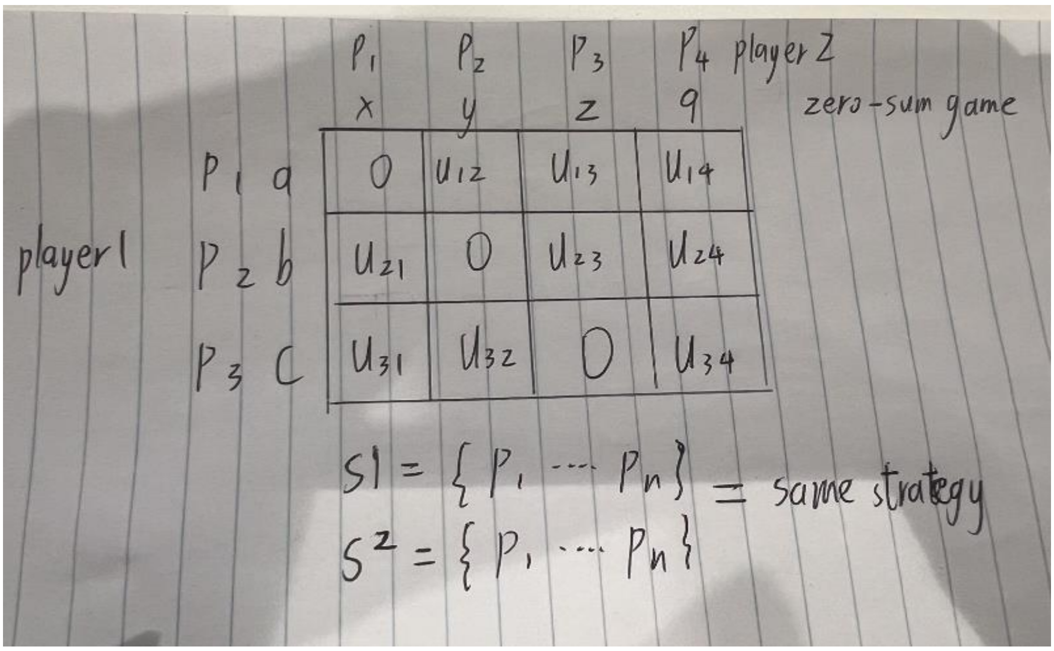
Since G is a **zero-sum** game,

$$u_1(a) + u_2(a) = 0$$

And $u_1(a_i, a_j) = u_2(a_j, a_i)$,
then, $u_1(a_i, a_j) = -u_1(a_j, a_i)$, $u_2(a_i, a_j) = -u_2(a_j, a_i)$.
We can let the strategy for player1 s_1 be same as the strategy for player2 s_2

$$s_1 = s_2$$

So, for every strategy of player 2, player 1 just uses the same strategy to guarantee him an expected utility of at least 0.
Now we can assume $s_2 = \{p_1, p_2 \dots p_n\}$ and $s_1 = \{p_1, p_2 \dots p_n\}$(use the same strategy).
The following picture shows the game.



Then, for player1 $u_1(p_1) = u_1(p_1, \mathrm{a}) = p_1 * p_1 * 0 + u_{12} * p_1 * p_2 + u_{13} * p_1 * p_3 \dots u_{1\mathrm{n}} * p_1 * p_\mathrm{n}$

$u_1(p_2) = u_1(p_2, \mathrm{b}) = p_1 * p_2 * u_{21} + 0 * p_2 * p_2 + u_{23} * p_2 * p_3 \dots u_{2\mathrm{n}} * p_2 * p_\mathrm{n}$

$u_1(p_\mathrm{n}) = u_1(p_\mathrm{n}, \mathrm{n}) = p_1 * p_\mathrm{n} * u_{\mathrm{n}1} + u_{\mathrm{n}2} * p_\mathrm{n} * p_2 + u_{\mathrm{n}3} * p_\mathrm{n} * p_3 \dots 0 * p_\mathrm{n} * p_\mathrm{n}$

Then we know u_{ij} = -u_{ji} since $u_2(a_\mathrm{i}, a_\mathrm{j})$ = -$u_2(a_\mathrm{j}, a_\mathrm{i})$.

When we add them together, for example ($u_1(p_1)$+ $u_1(p_2)\dots u_1(p_\mathrm{n})$)

Because $p_1 * p_2 * u_{12} + u_{21} * p_1 * p_2$=0. ($u_{12}$ = -u_{21})(the same case for the other equations)

Therefore, the sum of them ($u_1(p_1)$+ $u_1(p_2)\dots u_1(p_\mathrm{n})$) will be 0.

The value of G is **0.**