Fair Allocation

COMP4418 Knowledge Representation and Reasoning

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Outline

Allocation setting

Fairness with Money

Randomized assignment under ordinal preferences

Allocation of divisible items

Allocation setting

Allocation Setting

Basic Allocation Setting

- Agents $N = \{1, \ldots, n\}$
- Items $M = \{o_1, \ldots, o_m\}$
- Agents have valuation functions over bundles of items. $v_i(S)$ is the value of agent i for set of items S. In problems not involving, value is also referred to utility and function v_i is also referred to as u_i.
- Agents have preferences over bundles that are derived by the valuation functions. $\succsim = \{\succsim_1, \ldots, \succsim_n\}$ is the preference profile of agents.

An *allocation* $X = (X_1, \ldots, X_n)$ assigns $X_i \subseteq M$ to agent i.

Allocation Setting

- We will assume that $X_i \cap X_j = \emptyset$ for all $i, j \in N$ such that $i \neq j$.
- We will focus on allocations that allocate all the items: $\bigcup_{i \in N} X_i = M$.

Some notation: Preferences

- $$A \succsim_i B$$

  (agent i prefers A at least as much as B)

- $$A \succ_i B \iff A \succsim_i B \text{ and } B \not\succsim_i A$$

  (agent i strictly prefers A over B)

- $$A \sim_i B \iff A \succsim_i B \text{ and } B \succsim_i A$$

  (agent i is indifferent between A and B).

Some notation

$v_i : 2^O \longrightarrow \mathbb{R}^+$ specifies the valuation function of each agent i for bundles of items.

$$v_i(A) \geq v_i(B) \iff A \succsim_i B.$$

Allocation setting: Additive Values

Unless specified otherwise, we assume additive values:

- $v_i : M \longrightarrow \mathbb{R}^+$ specifies the utility function of each agent i.
- $v_i(M') = \sum_{o \in O'} u_i(o)$ for any $M' \subseteq M$.

Allocation setting: Additive Values

Example

	o_1	o_2	o_3	o_4
1	6	3	2	1
2	4	1	2	3

$v_1(o_1) = 6$; $v_1(o_2) = 3$; $v_1(o_3) = 2$; $v_1(o_4) = 1$.

$$v_1(\{o_1, o_2\}) > v_1(\{o_2, o_3\}).$$

$$\{o_1, o_2\} \succ_1 \{o_2, o_3\}.$$

Pareto optimality

An allocation X is *Pareto optimal* if there exists no allocation Y such that $Y_i \succsim_i X_i$ for all $i \in N$ and $Y_i \succ_i X_i$ for some $i \in N$.

Example (Not Pareto optimal)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X_1 = \{o_1, o_3, o_4\}$, $X_2 = \{o_2\}$.

Pareto optimality

Example (Pareto optimal)

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Nash Product Social Welfare

An allocation X's *Nash product welfare* is

$$\prod_{i \in N} v_i(X_i)$$

Example (Nash product welfare maximizing allocation)

	o_1	o_2	o_3	o_4
1	6	2	3	1
2	4	1	2	3

$X_1 = \{o_1, o_2\}$, $X_2 = \{o_3, o_4\}$.

EF1 Fairness

Definition (EF1 Fairness)
Given an instance $I = (N, M, v)$, an allocation X satisfies EF1 (envy-freeness up to 1 item) if for each $i, j \in N$, either $X_i \succsim_i X_j$ or there exists some item $o \in X_j$ such that

$$X_i \succsim_i X_j \setminus \{o\}.$$

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Is X EF1?

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Is X EF1?

Yes, it is.

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$X_1 = \{o_1, o_2, o_4\}$, $X_2 = \{o_3\}$.

Is X EF1?

No X is not EF1.

MNW for EF1 and PO

MNW: maximize the number of agents who get strictly more than zero and for that set of agents, maximize their Nash welfare.

Theorem
MNW is PO and EF1.

It is straightforward to show that every MNW allocation A is PO. If there were a different allocation B with $v_i(B) \geq v_i(A)$ for all $i \in N$ and $v_j(B_j) > v_j(A_j)$ for some $j \in N$ it would have strictly larger Nash welfare compared to that of A, which would contradict that A is an MNW allocation.

To show that A is EF1, suppose to the contrary that there exist agents i and j such that $v_i(A_i) < v_i(A_j \setminus \{g\})$ for every $g \in A_j$. We will show that there exists another allocation A' with Nash

MNW for EF1 and PO

welfare that is strictly larger than that of A. Since agent i envies agent j, there exists a $g^* = \arg\min_{g \in A_j, v_i(g) > 0} \left\{ \frac{v_j(g)}{v_i(g)} \right\}$. Using this, we define the allocation A' with $A'_\ell = A_\ell$ for each $\ell \in N \setminus \{i, j\}$, $A'_i = A_i \cup \{g^*\}$ and $A'_j = A_j \setminus \{g^*\}$. To show that the Nash welfare of A' is strictly larger, it suffices to show that

$$v_i(A'_i) \cdot v_j(A'_j) > v_i(A_i) \cdot v_j(A_j).$$

By the definition of g^* we have $\frac{v_j(g^*)}{v_i(g^*)} \leq \frac{v_j(A_j)}{v_i(A_j)}$, which implies

$$\frac{v_j(g^*)}{v_j(A_j)} \leq \frac{v_i(g^*)}{v_i(A_j)} < \frac{v_i(g^*)}{v_i(A_i) + v_i(g^*)}, \tag{1}$$

MNW for EF1 and PO

where the last inequality follows from the fact that agent i envies agent j even after the removal of any item in A_j. Therefore, we have

$$
\begin{aligned}
& v_i(A_i') \cdot v_j(A_j') \\
& = (v_i(A_i) + v_i(g^*)) \cdot (v_j(A_j) - v_j(g^*)) \\
& = \left(1 + \frac{v_i(g^*)}{v_i(A_i)}\right) \cdot \left(1 - \frac{v_j(g^*)}{v_j(A_j)}\right) \cdot v_i(A_i) \cdot v_j(A_j) \\
& > \frac{v_i(A_i) + v_i(g^*)}{v_i(A_i)} \cdot \left(1 - \frac{v_i(g^*)}{v_i(A_i) + v_i(g^*)}\right) \cdot v_i(A_i) \cdot v_j(A_j) \\
& = \frac{v_i(A_i) + v_i(g^*)}{v_i(A_i)} \cdot \left(\frac{v_i(A_i) + v_i(g^*) - v_i(g^*)}{v_i(A_i) + v_i(g^*)}\right) \cdot v_i(A_i) \cdot v_j(A_j) \\
& = v_i(A_i) \cdot v_j(A_j),
\end{aligned}
$$

MNW for EF1 and PO

where the first inequality follows from Inequality (1). Hence, the Nash welfare of A' is strictly larger than that of A, a contradiction.

Fairness with Money

EF with money

An outcome is a pair (X, p) where $X = (X_1, \dots X_n)$ is the allocation that specifies bundle $X_i \subseteq M$ for agent i and p specifies the payment p_i made to each agent $i \in N$.

An agent i's **utility** for a bundle-payment pair (X_j, p_j) is $u_i(X_j, p_j) = v_i(X_j) + p_j$. In other words, we assume **quasi-linear utilities**.

Example
Consider an allocation A such that $v_1(A_1) = 3$ and $p_1 = 2$. Then $u_1 = v_1(A_1) + p_1 = 3 + 2 = 5$

EF with money

An outcome (X, p) is **envy-free** if for all $i, j \in N$, it holds that $u_i(X_i, p_i) \geq u_i(X_j, p_j)$.

An allocation X is **envy-freeable** if there are payments $(p_1, \ldots, p_n)$ for agents such that for all $i, j \in N$:

$$v_i(X_i) + p_i \geq v_i(X_j) + p_j.$$

EF with money

	g_1
1	10
2	5

$X_1 = \{\}$, $X_2 = \{g_1\}$

1 is envious of 2.

Can 1 be made non-envious?

EF with money

	g_1
1	10
2	5

$X_1 = \{\}$, $X_2 = \{g_1\}$

1 is envious of 2.

Can 1 be made non-envious?

If we give \$10 to agent 1, agent 1 is not envious of agent 2!

But then agent 2 is envious of agent 1 .

For this allocation X no monetary payments can make both agents non-envious.

EF with money

	g_1
1	10
2	5

$X_1 = \{g_1\}$, $X_2 = \{\}$

2 is envious of 1.

What is the payment needed to make outcome envy-free?

EF with money

	g_1
1	10
2	5

$X_1 = \{g_1\}$, $X_2 = \{\}$

2 is envious of 1.

What is the payment needed to make outcome envy-free?

$p_1 = 0$, $p_2 = 5$.

Envy-graph

For any given allocation X, the corresponding **envy-graph** is a complete directed graph with vertex set N. For any pair of agents $i, j \in N$ the weight of arc (i, j) is the envy agent i has for agent j under the allocation X: $w(i, j) = v_i(X_j) - v_i(X_i)$. For any path or cycle C in the graph, the weight of C is the sum of weights of arcs along C.

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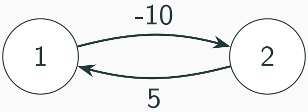
	g_1
1	10
2	5

$X_1 = \{g_1\}$, $X_2 = \{\}$



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	g_1	g_2	g_3
1	7	2	1
2	6	3	1
3	5	3	2

Envy-graph

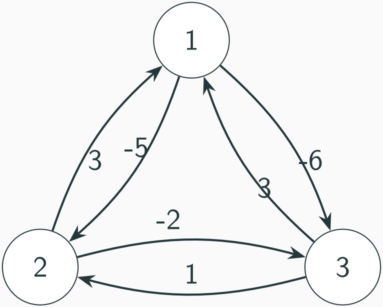
$X_1 = \{g_1\}$, $X_2 = \{g_2\}$, $X_3 = \{g_3\}$.

3 is envious of 1 and 2.

2 is envious of 1.

- $w(1,2) = v_1(X_2) - v_1(X_1) = 2 - 7 = -5$
- $w(1,3) = v_1(X_3) - v_1(X_1) = 1 - 7 = -6$
- $w(2,1) = v_2(X_1) - v_2(X_2) = 6 - 3 = 3$
- $w(2,3) = v_2(X_3) - v_2(X_2) = 1 - 3 = -2$
- $w(3,1) = v_3(X_1) - v_3(X_3) = 5 - 2 = 3$
- $w(3,2) = v_3(X_2) - v_3(X_3) = 3 - 2 = 1$

Envy-graph



Reassignment-stable

We say that an allocation X is **reassignment-stable** if

$$\sum_{i \in N} v_i(X_i) \geq \sum_{i \in N} v_i(X_{\pi(i)}).$$

for all permutations π of N.

How can we check reassignment stability? For an allocation X create a complete bipartite graph G with vertices $(N \cup \{X_1, \ldots, X_n\}))$. One side is the set of agents. The other side is the bundles in X. The weight $w(i, X_j)$ of edge (i, X_j) is equal to $v_i(X_j)$.

Then, X is reassignment stable if and only if the maximum weight matching of G is not greater than USW of X.

Theorem (Characterization of envy-freeability)

Under positive additive utilities, the following conditions are equivalent for a given allocation:

1. *the allocation is envy-freeable*
2. *the allocation is reassignment-stable*
3. *for the allocation, there is no positive weight cycle in the corresponding envy-graph*

$(1) \implies (2)$
Suppose X is envy-freeable. Then, there exists a payment vector p such that for all agents i, j $v_i(X_i) + p_i \geq v_i(X_j) + p_j$. Equivalently, $v_i(X_j) - v_i(X_i) \leq p_i - p_j$ Consider any permutation π of $[n]$. Then $\sum_{i \in N}(v_i(X_{\pi(i)}) - v_i(X_i)) \leq \sum_{i \in N}(p_i - p_{\pi(i)}) = 0$. The last entry

is zero as all the payments are considered twice and they cancel out each other. Hence X is reassignment stable.

$(2) \implies (3)$

Suppose some allocation X has a corresponding envy-graph with a cycle C of strictly positive weight. Then consider a permutation π such that $\pi(i) = i$ if $i \notin C$ and if $i \in C$, then $\pi(i)$ is the agent that i points to C. In that case

$$\sum_{i \in N} v_i(X_i) < \sum_{i \in N} v_i(X_{\pi(i)}).$$

which means that x is not reassignment stable.

$(3) \implies (1)$ Suppose (3) holds. Let $\ell(i)$ be the maximum weight of any path in the envy graph that starts from i. Let each agent i's payment be $p_i = \ell(i)$. Then

$p_i = \ell(i) \geq w_{i,j} + \ell(j) = v_i(X_j) - v_i(X_i) + p_j$. Thus, $v_i(X_i) + p_i \geq v_i(X_j) + p_j$ This implies that (X, p) is envy free and hence X is envy-freeable. $\square$

The chracterization gives us mathematical insights to compute envy-free outcomes.

First compute an envy-freeable allocation. Then use the payment scheme in ($(3) \implies (1)$) to compute the payments.

Reassignment-stable

Compute the utilitarian welfare maximizing allocation.

Compute a Pareto optimal welfare maximizing allocation.

Bundle all items together and give it to the agent who has the maximum value for M.

Computing longest paths

How to compute a maximum weight path from i to j in a graph $G = (V, E, w)$ that has no positive weight cycle?

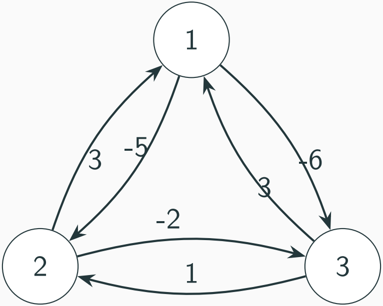
Equivalent to computing the minimum weight path from i to j in graph $G' = (V, E, w')$ where $w'(e) = -w(e)$ for all $e \in E$. Such as graph has no negative weight cycle.

This can be done by the Bellman Ford Algorithm.

Computing longest paths

```
function BellmanFord(list vertices, list edges, vertex source) is

    // This implementation takes in a graph, represented as
    // lists of vertices (represented as integers [0..n-1]) and
edges,
    // and fills two arrays (distance and predecessor) holding
    // the shortest path from the source to each vertex

    distance := list of size n
    predecessor := list of size n

    // Step 1: initialize graph
    for each vertex v in vertices do
        // Initialize the distance to all vertices to infinity
        distance[v] := inf
        // And having a null predecessor
        predecessor[v] := null

    // The distance from the source to itself is, of course, zero
    distance[source] := 0

    // Step 2: relax edges repeatedly
    repeat |V|-1 times:
        for each edge (u, v) with weight w in edges do
            if distance[u] + w < distance[v] then
                distance[v] := distance[u] + w
                predecessor[v] := u
```

Computing payments



	g_1	g_2	g_3
1	7	2	1
2	6	3	1
3	5	3	2

Computing payments

$$p_3 = 4,\ p_2 = 3,\ p_1 = 0$$

Agent 3 is not envious of Agent 1 because

$$v_3(X_3) + p_3 = 2 + 4 \geq 5 + 0 = v_3(X_1) + p_1$$

Agent 3 is not envious of Agent 2 because

$$v_3(X_3) + p_3 = 2 + 4 \geq 3 + 3 = v_3(X_2) + p_2$$

Agent 2 is not envious of Agent 1 because

$$v_2(X_2) + p_2 = 3 + 3 \geq 6 + 0 = v_2(X_1) + p_1$$

Agent 2 is not envious of Agent 3 because

$$v_2(X_2) + p_2 = 3 + 3 \geq 1 + 4 = v_2(X_3) + p_3$$

Agent 1 is not envious of Agent 2 because

$$v_1(X_1) + p_1 = 7 + 0 \geq 2 + 3 = v_1(X_2) + p_2$$

Computing payments

Agent 1 is not envious of Agent 3 because

$$v_1(X_1) + p_1 = 7 + 0 \geq 1 + 4 = v_1(X_3) + p_3$$

Computing an allocation that is both EF1 and envy-freeable

Algorithm of Brustle et al (2019)

1: Initialize allocation $X = (X_1, X_2, \ldots, X_n)$ with $X_i = \emptyset$ for all $i \in [n]$.
2: Construct a weighted complete bipartite graph $G = (N, M, E, w)$ where $w(i, o) = v_i(o)$.
3: **while** G has some item vertices **do**
4:     Compute a maximum weight matching M of in G (has size at most $|N|$)
5:     $X_i \longleftarrow X_i \cup M(i)$ for all $i \in N$
6: **end while**
7: Return X.

Computing an allocation that is both EF1 and envy-freeable

Theorem
The returned allocation is both EF1 and envy-freeable.

Achieving EF1

Sequential Allocation: Agents come in in some order and pick their most preferred available item.

recursively balanced sequence (no agent has a lead of 2 turns over another) Only need a **recursively balanced** sequence (no agent has a lead of 2 turns over another)

i	j	i	j	i	$\cdots$	j	i
o'_0	o_1	o'_1	o_2	o'_2	$\cdots$	o_k	o'_k

j may be envious of i.

Achieving EF1

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recursively balanced sequence (no agent has a lead of 2 turns over another) Only need a **recursively balanced** sequence (no agent has a lead of 2 turns over another)

	j	i	j	i	$\cdots$	j	i
	o_1	o_1'	o_2	o_2'	$\cdots$	o_k	o_k'

Achieving EF1

Let M^t be the maximum matching found in round t. It suffices to show that no directed cycle in the envy graph corresponding to the final allocation X has positive weight.

There is no positive weight cycle corresponding to each M^t. Therefore there is no weight cycle corresponding to combining all M^ts (allocation X). Consequently, by characterization of envy-freeability the allocation produced by the algorithm is envy-freeable.

Randomized assignment under ordinal preferences

Setting: Random Assignment Problem

An random assignment problem is a tuple $(N, M, \succ)$ where

- $N = \{1, \ldots, n\}$ is the set of agents
- $M = \{o_1, \ldots, o_n\}$ is the set of items
- $\succ_i$ is the strict and transitive preference of agent $i \in N$ over M
- agents may have private cardinal utilities

Setting: Random Assignment Problem

$$\succ_1, \succ_2\colon \quad o_1, o_2, o_3, o_4$$

$$\succ_3, \succ_4\colon \quad o_2, o_1, o_4, o_3$$

Feasible Outcome:

$$p = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}.$$

where $p(i)(o_j)$ is the probability of agent i getting o_j. It is the entry in the i-th row and j-th column of the matrix p.

$p(i)$ is the **allocation** of agent i.

Serial Dictatorship

For the assignment problem $(N, M, \succ)$ where $|N| = |M|$, Serial Dictatorship with respect to permutation π over N: agents get one turn each in the order of the permutation. They sequentially take their most preferred item that has not yet been allocated.

Example

$$\succ_1, \succ_2\colon \quad o_1, o_2, o_3, o_4 \qquad\qquad \succ_3, \succ_4\colon \quad o_2, o_1, o_4, o_3$$

$\pi = 1234.$

$$\text{SerialDictator}((N, M, \succ), \pi) = (\{o_1\}, \{o_2\}, \{o_4\}, \{o_3\}).$$

Serial Dictatorship

Non-bossiness: an agent cannot change her preference so that she gets the same allocation but some other agent gets a different allocation.

Neutral: the allocation does not depend on the names of the items.

Theorem (Svensson [1999])
For housing allocation problems, a mechanism is strategyproof, non-bossy and neutral if and only if it is a serial dictatorship.

Theorem (Abdulkadiroğlu and Sönmez [1998])
For housing allocation problems, an allocation is Pareto optimal iff it is a result of serial dictatorship.

RSD (Random Serial Dictatorship)

For an assignment problem $(N, M, \succ)$, takes a permutation π uniformly at random and then applies serial dictatorship with respect to it.

Example (RSD)
Consider an assignment problem in which $N = \{1, 2, 3, 4\}$, $M = \{o_1, o_2, o_3, o_4\}$ and the preferences $\succsim$ are as follows.

$$\succ_1, \succ_2\colon \quad o_1, o_2, o_3, o_4$$

$$\succ_3, \succ_4\colon \quad o_2, o_1, o_4, o_3$$

The following is the result of RSD:

$$RSD(N, M, \succ) = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}.$$

RSD (Random Serial Dictatorship)

Theorem (Aziz et al. [2013], Saban and Sethuraman [2013])
Checking whether an agent gets a particular item with probability at least $p \in (0, 1)$ is NP-hard.

$$\succ_1, \succ_2\colon \quad o_1, o_2, o_3, o_4$$

$$\succ_3, \succ_4\colon \quad o_2, o_1, o_4, o_3$$

$$RSD(N, M, \succ) = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}$$

PS (Probabilistic Serial)

Consider an assignment problem in which $N = \{1, 2, 3, 4\}$, $M = \{o_1, o_2, o_3, o_4\}$ and the preferences $\succsim$ are as follows.

$$\begin{aligned} 1, 2 &: \quad o_1, o_2, o_3, o_4 \\ 3, 4 &: \quad o_2, o_1, o_4, o_3 \end{aligned}$$

The following is the result of PS:

$$PS(N, M, \succ) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

The probability of agent 1 getting o_3 is $1/2$.

PS (Probabilistic Serial)

For an assignment problem $(N, M, \succ)$.

- Each item is considered to have a divisible probability weight of one, and agents simultaneously and with the same speed eat their most preferred item.
- Once an item has been eaten, the agent proceeds to eat the next most preferred item that has not been completely eaten.
- The procedure terminates after all the items have been eaten.
- The allocation of an agent by PS is the amount of each item she has eaten.

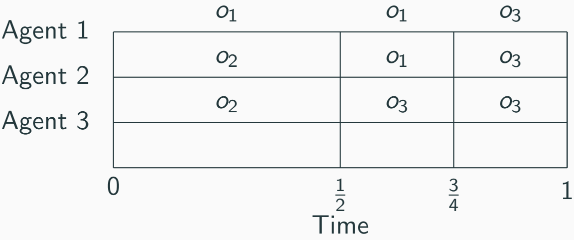
Proposed by Bogomolnaia and Moulin [2001].

PS (Probabilistic Serial)

$\succ_1$: o_1, o_2, o_3

$\succ_2$: o_2, o_1, o_3

$\succ_3$: o_2, o_3, o_1



PS (Probabilistic Serial)

$$PS(\succ_1, \succ_2, \succ_3) = \begin{pmatrix} 3/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

PS (Probabilistic Serial)

Example (PS)

Consider an assignment problem in which $N = \{1, 2, 3, 4\}$, $M = \{o_1, o_2, o_3, o_4\}$ and the preferences $\succ$ are as follows.

$$\succ_1, \succ_2\colon \quad o_1, o_2, o_3, o_4$$

$$\succ_3, \succ_4\colon \quad o_2, o_1, o_4, o_3$$

The following is the result of PS:

$$PS(N, M, \succ) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

The probability of agent 1 getting o_3 is $1/2$.

SD (Stochastic Dominance) relation between allocations

An agent **SD-prefers** one allocation over another if for each item o, the former allocation gives the agent as much probability of getting at least preferred an item as the latter allocation.

$$p(i) \succsim_i^{SD} q(i)$$

$$\iff \sum_{o_j \in \{o_k | o_k \succsim_i o\}} p(i)(o_j) \geq \sum_{o_j \in \{o_k | o_k \succsim_i o\}} q(i)(o_j) \text{ for all } o \in M.$$

Example (SD relation)

$$1 : \quad o_1, o_2, o_3, o_4$$

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 \end{pmatrix} \succ_1^{SD} \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \end{pmatrix}$$

SD (Stochastic Dominance) relation between allocations

An agent **SD-prefers** one allocation over another if for each item o, the former allocation gives the agent as much probability of getting at least preferred an item as the latter allocation.

$$p(i) \succsim_i^{SD} q(i) \iff$$

$$\sum_{o_j \in \{o_k | o_k \succsim_i o\}} p(i)(o_j) \geq \sum_{o_j \in \{o_k | o_k \succsim_i o\}} q(i)(o_j) \quad \forall o \in M.$$

Stochastic dominance implies getting at least as much utility for all utility functions consistent with the ordinal preferences.

SD (Stochastic Dominance) relation between allocations

$$p(i) \succsim_i^{SD} q(i) \iff$$

$$\sum_{o \in M} (p(i)(o)) u_i(o) \geq \sum_{o \in M} (q(i)(o)) u_i(o) \quad \forall u_i \in \mathcal{U}(\succsim_i)$$

where

$$u_i(o) \geq u_i(o') \quad \text{if} \quad o \succsim_i o' \; \forall u_i \in \mathcal{U}(\succsim_i)$$

Quest for fairness and efficiency

- **SD envy-freeness**: Each agent SD-prefers her allocation over allocations of other agents:

$$p(i) \succsim_i^{SD} p(j) \text{ for all } i, j \in N.$$

- **SD-efficiency**: Pareto optimality with respect to the SD relation. Assignment p is SD-efficient if there exists no q such that

$$q(i) \succsim_i^{SD} p(i) \text{ for all } i \in N$$

and

$$q(i) \succ_i^{SD} p(i) \text{ for some } i \in N.$$

- f is **SD-strategyproof** if

$$f(\succsim)(i) \succsim_i^{SD} f(\succsim_i', \succsim_{-i})(i) \text{ for } i \in N.$$

PS (Probabilistic Serial)

$$\succ_1, \succ_2: \quad o_1, o_2, o_3, o_4$$

$$\succ_3, \succ_4: \quad o_2, o_1, o_4, o_3$$

$$RSD(N, M, \succ) = \begin{pmatrix} 5/12 & 1/12 & 5/12 & 1/12 \\ 5/12 & 1/12 & 5/12 & 1/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \\ 1/12 & 5/12 & 1/12 & 5/12 \end{pmatrix}.$$

$$PS(N, M, \succ) = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

Theorem (Bogomolnaia and Moulin [2001])
RSD is SD-strategyproof but not SD-efficient or SD envy-free. PS is SD-efficient and SD envy-free but not SD-strategyproof.

PS is not strategyproof

$$\succ_1: \quad o_1, o_2, o_3$$
$$\succ_2: \quad o_2, o_1, o_3$$
$$\succ_3: \quad o_2, o_3, o_1$$

$$PS(\succ_1, \succ_2, \succ_3) = \begin{pmatrix} 3/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

If agent 1 misreports her preferences as follows: $\succ_1': \quad o_2, o_1, o_3$, then

$$PS(\succ_1', \succ_2, \succ_3) = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \end{pmatrix}.$$

PS is not strategyproof

Then, if $u_1(o_1) = 7$, $u_1(o_2) = 6$, and $u_1(o_3) = 0$, then agent 1 gets more expected utility when she reports $\succ_1'$.

Allocation of divisible items

Allocation of divisible items: AW (Adjusted Winner)

Agent 1 and 2 are each given C utility points that they can use for acquiring m items. Let x_i be the number of points used by 1 on item i and y_i be the number of points used by 2 on item i. Call $\frac{x_i}{y_i}$ the *ratio* of item i.

1: Each item is assigned to the agent that values it the most. Ties are broken in favour of agent 1.
2: **while** agent 1 gets strictly more utility than agent 2 **do**
3: Consider an item with the smallest ratio that agent 1 gets partially or fully. Transfer as much of the item to agent 2 while ensuring agent 1 gets at least as much utility as agent 2.
4: **end while**
5: Return the allocation

Allocation of divisible items: AW (Adjusted Winner)

$$\underbrace{\frac{x_{k_1}}{y_{k_1}} \geq \frac{x_{k_2}}{y_{k_2}} \geq \cdots \geq \frac{x_{k_i}}{y_{k_i}} \geq}_{\text{Allocation of agent 1}} \Big| \geq \underbrace{\frac{x_{k_{i+1}}}{y_{k_{i+1}}} \geq \cdots \geq \frac{x_{k_m}}{y_{k_m}}}_{\text{Allocation of agent 2}}$$

Allocation of divisible items: AW (Adjusted Winner)

	o_1	o_2	o_3
1	67	6	27
2	34	5	61

- Initially, agent 1 gets 73 points; Agent 2 gets 61 points
- o_2 is given from agent 1 to agent 2
- o_1 must be partially given to agent 2. Agent 2 gets $1/101$ of o_1 and agent 1 gets $100/101$ so that both get $67 \times \frac{100}{101} \approx 66.3366337$ points.

Allocation of divisible items: AW (Adjusted Winner)

	o_1	o_2	o_3
1	$^{100}/_{101}$ 67	6	27
2	$^{1}/_{101}$ 34	5	61

Allocation of divisible items: AW (Adjusted Winner)

Equitability: all agents get the same utility.

Theorem (Brams and Taylor [1996])
AW is Pareto optimal, equitable, envy-free, and proportional, and requires at most one item to be split.

Theorem (Aziz et al. [2015])
For two agents, AW is the only Pareto optimal and equitable rule that requires at most one item to be split.

Allocation of divisible items: Proportional Allocation Rule

Both agents are given equal number of points that they can allocate to the items. Let x_i be the number of points used by 1 on item i and y_i be the number of points used by 2 on item i. Then agent 1 gets $\frac{x_i}{x_i+y_i}$ of the item o_i and 2 gets $\frac{y_i}{x_i+y_i}$ of the item o_i

Theorem (Brams and Taylor [1996])
The Proportional Allocation Rule is equitable and envy-free but not necessarily Pareto optimal.

Argument for equitability:

Utility of agent 1 is $\sum_{i=1}^{m}(x_i \times \frac{x_i}{x_i+y_i})$. Utility of agent 2 is $\sum_{i=1}^{m}(y_i \times \frac{y_i}{x_i+y_i})$.

Allocation of divisible items: Proportional Allocation Rule

$$\sum_{i=1}^{m} \frac{x_i^2 - y_i^2}{x_i + y_i} = \sum_{i=1}^{m} \frac{(x_i - y_i)(x_i + y_i)}{x_i + y_i} = \sum_{i=1}^{m}(x_i - y_i) = \sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i = 0.$$

Allocation of divisible items

Theorem (Zhou [1990])
If fractional allocations are allowed and agents have additive cardinal utilities, then strategyproofness, Pareto optimality and envy-freeness are incompatible.

Note that any two of the properties are easy to achieve:

- strategyproofness and Pareto optimality: dictatorship
- strategyproofness and envy-freeness: null allocation
- envy-freeness and Pareto optimality: Nash welfare maximizing allocation.

PA (Partial Allocation) mechanism for allocation of divisible items

- Compute the Nash welfare maximizing allocation X^* based on the reported valuations.
- For each agent i, remove the agent and compute the Nash welfare maximizing allocation $X^*{}_{-i}$ that would arise when i does not exist.
- Allocate to each agent i a fraction f_i of everything i receives according to X^* where

$$f_i = \frac{\prod_{i' \neq i}[v_{i'}(X^*)]}{\prod_{i' \neq i}[, v_{i'}(X^*_{-i})]}.$$

Theorem (Cole et al. [2013])
PA is strategyproof, envy-free and each agent gets $1/e$ of the utility she would get in a Nash welfare maximizing allocation.

Survey and Further Reading

- Most relevant resource: book chapter by Bouveret et al. [2016] in the Handbook of Computational Social Choice. http://www.cse.unsw.edu.au/~haziz/comsoc.pdf
- Brandt et al. [2016] especially chapters 11-14
- Brams and Taylor [1996]
- Robertson and Webb [1998]
- Moulin [2003]
- Endriss [2010]
- Roth and Sotomayor [1990]
- Gusfield and Irving [1989]
- Manlove [2013]
- Chalkiadakis et al. [2011]

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