COMP 4418 – Exercise Sheet: Cooperative Game Theory I

Exercise I: Computing Shapley Value

Alice, Bob, and Charles plan to make strawberry pancakes to sell at market day. Each of them contributes key ingredients: Alice brings flour worth \$2, Bob contributes sugar valued at \$1, and Charles provides strawberries worth \$3. By working together in various combinations, they can gain additional benefits. Alice and Bob can collaborate to create standard pancakes worth \$5. Alice and Charles, without Bob's sugar, can make strawberry pancakes valued at \$7. Meanwhile, Bob and Charles can sell sugar and strawberries for a total profit of \$4. If all three cooperate, they can produce highly sought-after strawberry pancakes, yielding a total profit of \$16. This scenario can be modeled as a cooperative game as follows (A: Alice, B: Bob, C: Charles, and $S \subseteq \{A,B,C\}$ represents the coalition). Please compute the Shapley Value to fairly distribute the profits among them.

S	$\emptyset$	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$v(S)$	0	2	1	3	5	7	4	16

Solution of Exercise I

Let $N = \{A,B,C\}$. The Shapley value of each player i is represented as

$$\phi_i(N,v) = \frac{1}{N!} \sum_{\pi \in \Pi_N} (v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))).$$

We consider all the permutations and compute the marginal contribution of each player in every permutation (Denote the marginal contribution of each player as "MC of the player" in the table).

π	MC of A	MC of B	MC of C
(A,B,C)	$v(\{A\}) = 2$	$v(\{A,B\}) - v(\{A\}) = 3$	$v(\{A,B,C\}) - v(\{A,B\}) = 11$
(A,C,B)	$v(\{A\}) = 2$	$v(\{A,B,C\}) - v(\{A,C\}) = 9$	$v(\{A,C\}) - v(\{A\}) = 5$
(B,A,C)	$v(\{A,B\}) - v(\{B\}) = 4$	$v(\{B\}) = 1$	$v(\{A,B,C\}) - v(\{A,B\}) = 11$
(B,C,A)	$v(\{A,B,C\}) - v(\{B,C\}) = 12$	$v(\{B\}) = 1$	$v(\{B,C\}) - v(\{B\}) = 3$
(C,A,B)	$v(\{A,C\}) - v(\{C\}) = 4$	$v(\{A,B,C\}) - v(\{A,C\}) = 9$	$v(\{C\}) = 3$
(C,B,A)	$v(\{A,B,C\}) - v(\{B,C\}) = 12$	$v(\{B,C\}) - v(\{C\}) = 1$	$v(\{C\}) = 3$

For player A: $\phi_A(N,v) = \frac{1}{6}(2+2+4+12+4+12) = 6.$
For player B: $\phi_B(N,v) = \frac{1}{6}(3+9+1+1+9+1) = 4.$
For player C: $\phi_C(N,v) = \frac{1}{6}(11+5+11+3+3+3) = 6.$

Exercise II: Core

Consider the following cooperative game (N, v) with $N = \{1,2,3\}$ and the valuation of coalitions $S \subseteq N$ as follows. Assume the valuations are non-negative and v is monotonic.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	50	70	X	100

a) Consider the case where $X = 60$. Is the core of the game empty? If so, provide a proof. If the core is not empty, give a payoff vector that lies in the core.

b) For what value of X is the core of the game non-empty?

Solution of Exercise II

a) When $X = 60$, the core is non-empty, and the solution $x_1 = 30, x_2 = 20, x_3 = 50$ is one of the solutions in the core.

b) A payoff vector $x = (x_1, x_2, \ldots, x_n)$ is in the core of a coalition game (N, v) if for all $S \subset N$, it satisfies $x(S) \geq v(S)$. In the case of the game with three players, let (x_1, x_2, x_3) represent the payoff vector. To meet the core conditions, we first have $x_1, x_2, x_3 \geq 0$ and the following inequalities must hold:

$$\begin{aligned} x_1 + x_2 &\geq 50 \\ x_1 + x_3 &\geq 70 \\ x_2 + x_3 &\geq X. \end{aligned}$$

Summing these inequalities gives $x_1 + x_2 + x_3 \geq 60 + \frac{X}{2}$. When $X > 80$, the core requires that $x_1 + x_2 + x_3 > 100$. However, since value of the grand coalition is only 100 (i.e., $x_1 + x_2 + x_3 \leq 100$), the core of the game can only be non-empty if $X \leq 80$.

Exercise III: Banzhaf Index Consider the following simple cooperative game (N, v) with $N = \{1,2,3,4\}$ and the valuation of coalitions $S \subseteq N$ as follows.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
$v(S)$	0	0	0	0	0	0	1	0	1	0	1	1	0	1	1	1

a) Compute the Shapley Value of the simple game.

b) Compute the Banzhaf Indices of the simple game.

Solution of Exercise III

a) In a simple game, the Shapley value of each player i can be represented as

$$\phi_i = \frac{\text{\# permutations in which } i \text{ has a marginal contribution of 1}}{|N|!}.$$

Player 1 has marginal contribution of 1 in permutations: $(3,1,2,4)$, $(3,1,4,2)$.

Player 2 has marginal contribution of 1 in permutations: $(3,2,1,4)$, $(3,2,4,1)$.

Player 4 has marginal contribution of 1 in permutations: $(3,4,1,2)$, $(3,4,2,1)$.

For all the other permutations, player 3 has a marginal contribution of 1.

Hence, the Shapley value: $\phi_1 = \frac{1}{12}$, $\phi_2 = \frac{1}{12}$,$\phi_3 = \frac{3}{4}$,$\phi_4 = \frac{1}{12}$.

b) To compute the Banzhaf Index, we first list all the coalitions with valuation of 1 and check which player is critical in the coalition:

$S = \{1,3\}$: both player 1 and player 3 are critical.

$S = \{2,3\}$: both player 2 and player 3 are critical.

$S = \{3,4\}$: both player 3 and player 4 are critical.

$S = \{1,2,3\}$: only player 3 is critical.

$S = \{1,3,4\}$: only player 3 is critical.

$S = \{2,3,4\}$: only player 3 is critical.

$S = \{1,2,3,4\}$: only player 3 is critical.

Then we compute the Banzhaf value η_i of each player i: $\eta_1 = 1, \eta_2 = 1, \eta_3 = 7, \eta_4 = 1$.

Finally, the Banzhaf indices can be computed as follows. $\beta_1 = \frac{1}{10}, \beta_2 = \frac{1}{10}, \beta_3 = \frac{7}{10}, \beta_4 = \frac{1}{10}$.