**Game Theory**

**COMP4418–Assignment 1**

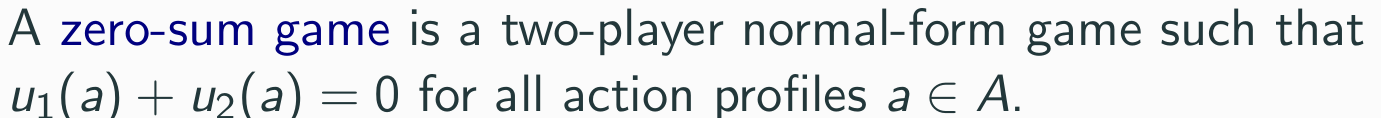
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1. **For which value of α is the game zero-sum?**

According to the definition of zero-sum:



Therefore, , .

1. **For which values of α is the outcome (−2,α) Pareto-optimal?**

According to the definition of Pareto-optimal:

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For outcome **(1, -1)**, = 1 which is larger than -2 in outcome. must be larger than -1 to avoid being Pareto-dominated by outcome (1, -1)

So .

For outcome **(4, -4)**, = 4 which is larger than -2 in outcome. must be larger than -4 to avoid being Pareto-dominated by outcome (4, -4)

So .

For outcome **(-3, 3)**, = 3 which is smaller than -2 in outcome. can be any number here.

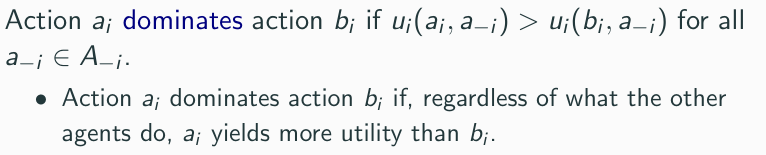
So .

Therefore, we got three results To ensure that all the equations hold,

the answer is **.**

1. **For which values of α can the game be solved by iterated strict dominance?**

According to the definition:

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Since iterated strict dominance is order-independent, firstly if we want to eliminate action , we need to make sure **a** is dominated by **b**, which is .

The same case for action , . So, we can’t eliminate dominated actions by comparing the actions of the row players.

For column players, since , if we want to make sure x is dominated by y (eliminate x), we need that , which is . The answer is .

1. **For which value of α is it the maximin strategy of the column player to play x with probability ?**

For the column player, consider each of their strategies and what the minimum payoff would be if the row player takes their action.

In this specific example, we calculated:

Since the column player to play **x** with probability then, play **y** with probability

.

The expected payoff when the row player chooses a:

The expected payoff when the row player chooses b

The column player wants to ensure that their worst-case payoff is as high as possible. After calculating the payoffs for each strategy, they will look for a strategy that maximizes the worst-case outcomes. To achieve this. We set

.

1. **For which value of α will the row player play a with probability in a Nash equilibrium?**

Indifference principle: The row player must randomize such that the column player is indifferent between all actions in his support.

the row player plays with probability in a Nash equilibrium, then he plays with probability .

Since no action is dominated, it must hold that

Therefore**, .**

**A diagram of a triangle with red and blue dots

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1. **Compute the subgame-perfect Nash equilibrium.**

For finding a subgame-perfect Nash equilibrium, we can use backwards induction.

By comparing the corresponding utilities at each stage, we can determine the equilibrium:A diagram of a graph

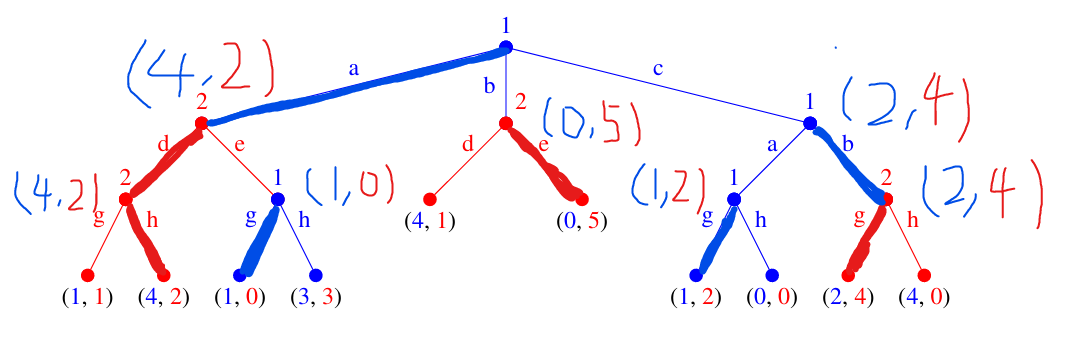
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1. **Is there a pure Nash equilibrium where** **player 1 has a utility of 4? Explain your answer!**

Yes. In considering a pure Nash equilibrium, Player 1 can choose the strategy (1, 0) instead of (3, 3). We need to compare the outcomes of (1, 0) and (4, 2). Since the utility of 2 is greater than 0, (4, 2) will be selected. Next, we compare (4, 2) with (0, 5) and (2, 4); here, the utilities yield the order of 4 > 2 > 0. Consequently, (4, 2) will be chosen, resulting in Player 1 obtaining a utility of 4.

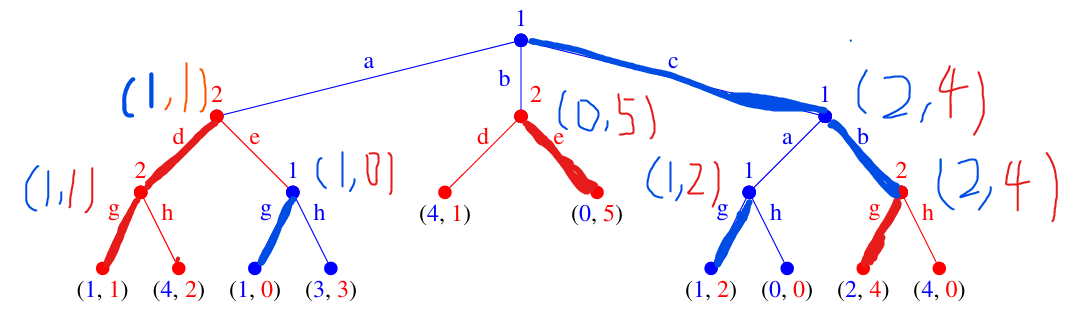
In this case, the strategies selected by the players remain fixed (with (1, 0) chosen), and under this specific strategy profile, each player's choice is the best response to the strategies of the other players. Under the current strategy combination, players have no incentive to change their choices, which establishes that this is a pure Nash equilibrium.

The following picture:

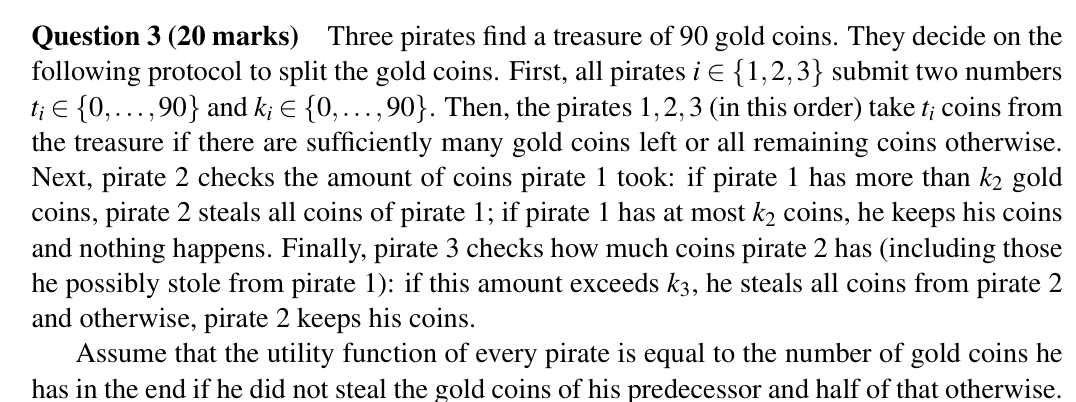
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1. **What is the maximum utility that player 2 can obtain in a pure Nash equilibrium?**

For player 2, we firstly assume player 2 can obtain utility of 5, the outcome is (0,5), it is impossible that this will be chosen because player 1’s utility is 0 which is the smallest. Then we assume player 2 can obtain utility of 4. **(2,4)** In the following picture we can see it is possible that player 2 can obtain utility of 4 in a pure Nash equilibrium.



Therefore, the answer is **4**.

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1. **Show that there is a pure Nash equilibrium where all pirates have 30 gold coins.**

There is a pure Nash equilibrium where all pirates have 30 gold coins when

can be any number between 0 and 90 because it doesn’t affect anything here.

So, in this case, a pure Nash equilibrium exists.

For **pirate1**: if , **pirate1** will be stolen by **pirate2** since and then the utility of pirate1 will be 0. If he decreases , his utility will also decrease, if , Therefore, pirate1 has no incentive to submit different numbers. ()

For **pirate2**: if , **pirate2** will be stolen by **pirate3** since and then the utility of **pirate2** will be 0. If he decreases , his utility will also decrease, if , So, he will keep . If he changes , when , he will steal all coins from pirate1 since , and , The original is also because the utility function of every pirate is equal to the number of gold coins he has. remains unchanged. Increasing will not affect since he can’t steal **pirate1**.

Therefore, pirate2 has no incentive to submit different numbers. ().

For **pirate3**: Now , there are only coins left, so remains the same. If **pirate3** reduces , , he will steal all coins from pirate2 since and , doesn't change. Increasing will not affect since he can’t steal **pirate2**. Therefore, **pirate3** has no incentive to submit different numbers. ()

As a result, each pirate has no incentive to submit different numbers and they all have 30 gold coins, it is a pure Nash equilibrium.

1. **What is the maximal amount of gold coins that each pirate can obtain in a pure Nash equilibrium? Present for each pirate the corresponding strategy profile and reason why it is a pure Nash equilibrium.**

For **pirate1: We assume**

In this case, **pirate1** can obtain coins which is maximum possible. **pirate1** has no incentive to change numbers . Since the amount of coins left is , no matter what the value of is, **pirate2** will not obtain any coins. to make sure **pirate1** will not be stolen by **pirate2**. Since , if **pirate2** wantsto reduce to steal **pirate1,** this would lead to , which means **pirate3** will steal all coin from **pirate2,** so, remains the same. Because no coins left. **pirate3** can’t obtain coins by changing and .

First, according to , , and , Pirate 1 receives 90, Pirate 2 receives 0, and Pirate 3 receives 0 coins initially.

Next, Pirate 2 checks the coins that Pirate 1 has, which is 90 (equal to ). According to the game rules, Pirate 2 will not steal coins from Pirate 1.

Then, Pirate 3 checks the coins that Pirate 2 has, which is 0 (equal to ). As a result, Pirate 3 will not steal all the coins from Pirate 2.

Finally, **pirate1** will obtain 90 coins, **pirate2:** 0 coins, **pirate3**: 0 coins.

As a result, each pirate has no incentive to submit different numbers, it is a pure Nash equilibrium. the maximal amount of gold coins that pirate1 can obtain in a pure Nash equilibrium is **90.**

For **pirate2: We assume**

In this case, the less is, the more chance **pirate2** can steal **pirate1**, . Since that, no matter what the value of is, the amount of coins that pirate1 can obtain is 0 (if , he will be stolen by pirate2). if **pirate2** reduces will be smaller, if he increases it, he will be stolen by pirate3. (). if pirate3 reduces he will steal all coins from pirate2 which makes , the original is also 45 since . remains unchanged. Now, each pirate has no incentive to submit different numbers, it is a pure Nash equilibrium.

Let’s see if . In this case **pirate3** can increase his utility by reduce The original , if pirate3 steals all coins from pirate2,

which is larger than the original value. Then **pirate3** has incentive to report different .(reduce it) This is not a pure Nash equilibrium.

First, according to , , and , Pirate 1 receives 0, Pirate 2 receives 45, and Pirate 3 receives 45 coins initially.

Next, Pirate 2 checks the coins that Pirate 1 has, which is 0 (smaller than ). According to the game rules, Pirate 2 will not steal coins from Pirate 1.

Then, Pirate 3 checks the coins that Pirate 2 has, which is 45 (equal to ). As a result, Pirate 3 will not steal all the coins from Pirate 2.

Therefore, **pirate1** :0 coins, **pirate2:** 45 coins, **pirate3**: 45 coins.

The maximal amount of gold coins that pirate2 can obtain in a pure Nash equilibrium is **45.**

For **pirate3: We assume**

In this case, no matter what the value of is, the amount of coins that **pirate1** can obtain is 0 since . Also, no matter what the value of is, the amount of coins that **pirate2** can obtain is 0 since . Finally,**pirate3** will steal all coins from pirate2.

First, according to , , and , Pirate 1 receives 90, Pirate 2 receives 0, and Pirate 3 receives 0 coins initially. (For pirate 2 and 3, they claim 90 but only gets 0 since there are not sufficient coins left for him.)

Next, Pirate 2 checks the coins that Pirate 1 has, which is 90 (greater than ). According to the game rules, Pirate 2 will steal coins from Pirate 1.

Then, Pirate 3 checks the coins that Pirate 2 has, which is 90 (greater than ). As a result, Pirate 3 steals all the coins from Pirate 2.

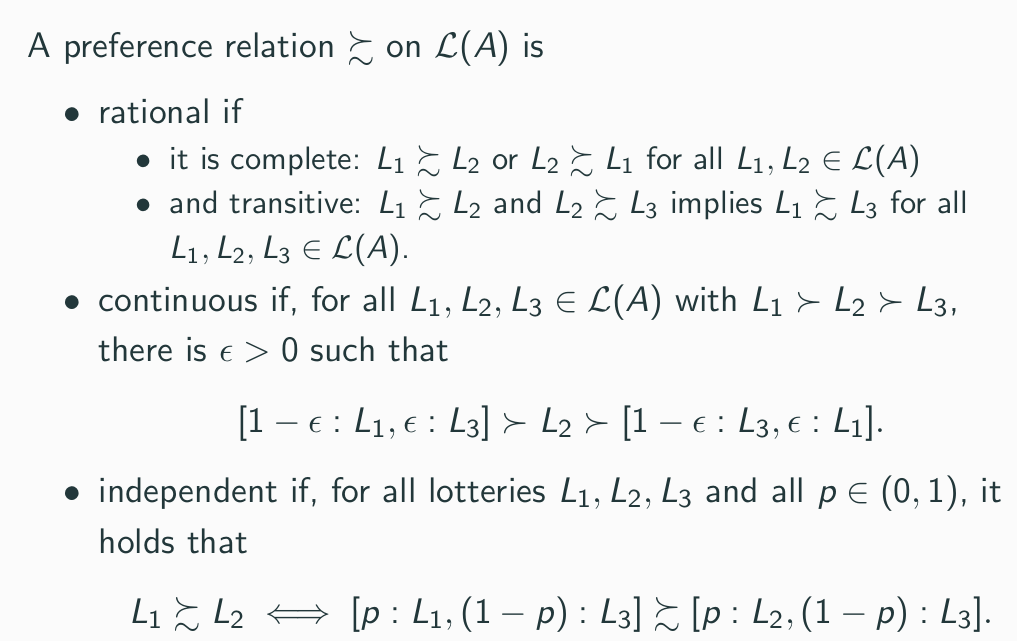
Finally, **pirate1** :0 coins, **pirate2:** 0 coins, **pirate3**: 90 coins.

The maximal amount of gold coins that pirate3 can obtain in a pure Nash equilibrium is **90.**

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The definition:



Let for , .

Now we assume .

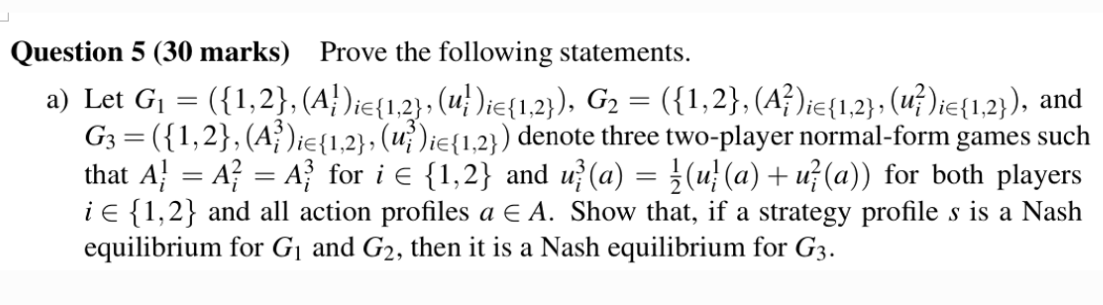
By assumption, and ,

By **independence**, since .

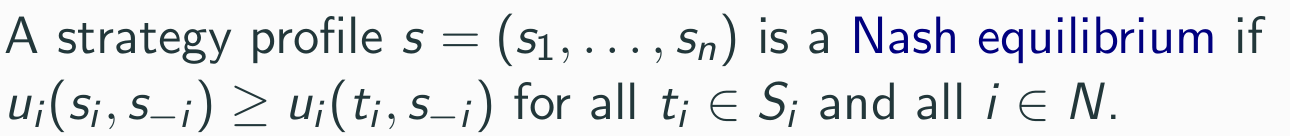
Then, so ,*.*

Since , , we can get

Therefore, by **independence**, we can prove **,**



According to the definition of Nash equilibrium:



If a strategy profile is a Nash equilibrium for and ,

For , ,

For , ,

Then we add these two inequalities and multiply both sides by , we can get

Since and , we can get

Which follows the **Nash equilibrium theorem.**

**Therefore,** **it is a Nash equilibrium for** .

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We know that and denote two Nash equilibria for such that

.

So, the strategy

Given by

Since and denote two Nash equilibria, the expect utilities of them are the same.

Assume represents the expect utility of the player with action .

**For player1** in ,

Since,

**For player2** in ,

Since

For both players, the expected utility of is the same as that of ,and .

Therefore, it is also **a Nash equilibrium for** .

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Since is a **zero-sum** game,

And ,

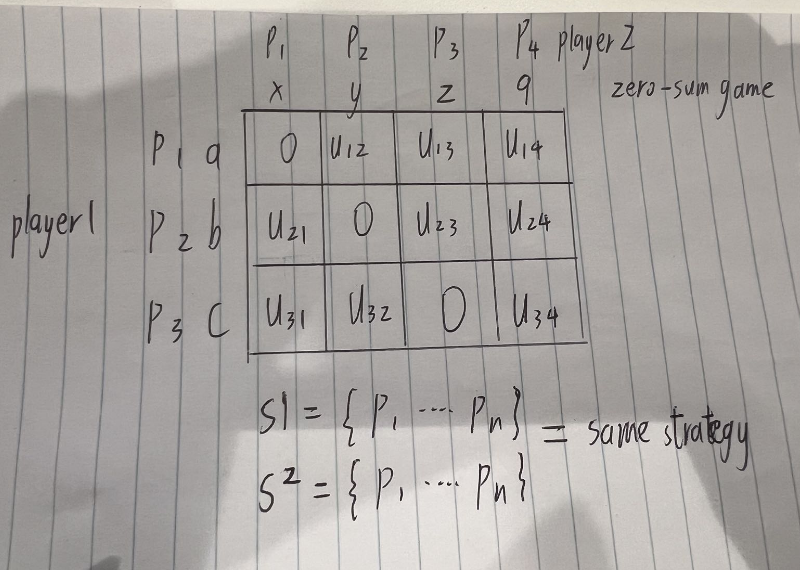
then, = -, = -.

We can let the strategy for player1 be same as the strategy for player2

So, for every strategy of player 2, player 1 just uses the same strategy to guarantee him an expected utility of at least 0.

Now we can assume = {} and = {}( use the same strategy).

The following picture shows the game.



Then, for player1

Then we know = - since = -.

When we add them together, for example (+

Because =0. ()(the same case for the other equations)

Therefore, the sum of them (+ will be 0.

The value of is **0.**