ME760 Homework 3

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Problem 1

Problem Statement

Estimate the spectral radius of the following matrix. Show your work.

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & 3 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

Solution

Find the determinate of $\mathbf{B}$ if $\mathbf{B}$ is $\mathbf{A} - \mathbf{I}\lambda$ to obtain eigenvalues.

$$\begin{gathered}
(7-\lambda)[(1-\lambda)(2-\lambda)] + 3[-2(1-\lambda)] = 0 \\
(7-\lambda)[(1-\lambda)(2-\lambda)] - 6(1-\lambda) = 0 \\
(1-\lambda)[(7-\lambda)(2-\lambda) - 6] = 0 \\
\lambda = 1, \\
14 - 9\lambda + \lambda^2 - 6 = 0 \\
8 - 9\lambda + \lambda^2 = 0 \\
(\lambda - 8)(\lambda - 1) = 0 \\
\lambda = 8, 1, 1
\end{gathered}$$

The spectral radius, ρ, is the largest eigenvalue of the matrix.

$$\boxed{\rho(\mathbf{A}) = 8}$$

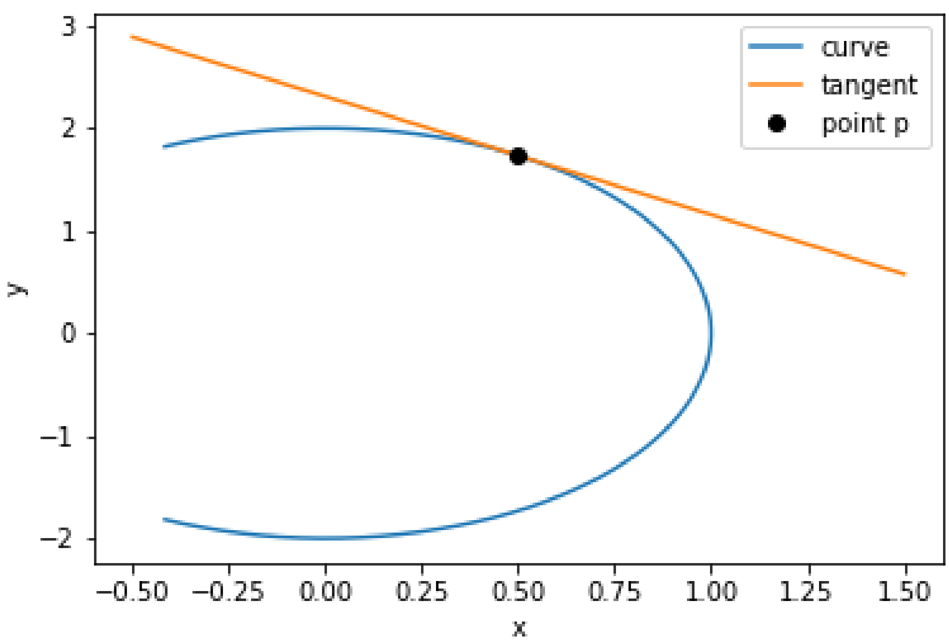
Problem 2

Problem Statement

Given the curve C: $\mathbf{r}(u) = \mathbf{i}\cos u + \mathbf{j}2\sin u$, find (a) a tangent vector $\mathbf{r}'(u)$ and the corresponding unit vector $\hat{\mathbf{r}}'(u)$, (b) $\mathbf{r}'$ and $\hat{\mathbf{r}}'$ at the point $P : (1/2, \sqrt{3}, 0)$, and (c) the equation of the line through P that is tangent to the curve. Sketch the curve and the tangent.

Solution

$$\text{(a)}$$
$$\boxed{\mathbf{r}'(u) = \hat{i}(-\sin u) + \hat{j}(2\cos u)}$$
$$\hat{\mathbf{r}}'(u) = \frac{1}{\sqrt{\sin^2 u + 2\cos^2 u}}\mathbf{r}'(u)$$
$$\boxed{\hat{\mathbf{r}}'(u) = \frac{1}{\sqrt{3}}[\hat{i}(-\sin u) + \hat{j}(2\cos u)]}$$
$$\text{(b)}$$
$$\text{at } P(1/2, \sqrt{3}, 0), u = \pi/3$$
$$\boxed{\mathbf{r}'(\tfrac{\pi}{3}) = -\tfrac{\sqrt{3}}{2}\hat{i} + 1\hat{j}}$$
$$\boxed{\hat{\mathbf{r}}'(\tfrac{\pi}{3}) = -\tfrac{1}{2}\hat{i} + \tfrac{1}{\sqrt{3}}\hat{j}}$$
$$\text{(c)}$$
$$l = \mathbf{P} + \lambda\hat{\mathbf{r}}'$$
$$l = \tfrac{1}{2}(1-\lambda)\hat{i} + \sqrt{3}(1 + \tfrac{1}{3}\lambda)\ hatj$$
$$\boxed{l = \tfrac{1}{2}(1-\lambda)\hat{i} + \sqrt{3}(1 + \tfrac{1}{3}\lambda)\hat{j}}$$



Problem 3

Problem Statement

Find the length of the circular helix $\mathbf{r}(u) = \mathbf{i}a\cos u + \mathbf{j}a\sin u + \mathbf{k}cu$ from $(a, 0, 0)$ to $(a, 0, 2\pi c)$.

Solution

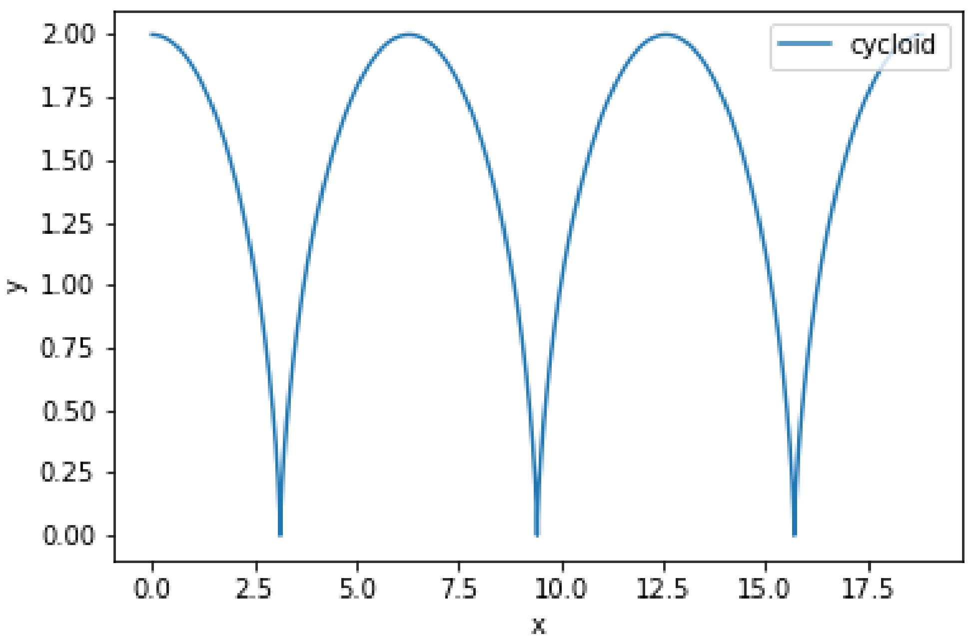
$$s = \int_a^b |\mathbf{r}'(u)| du = \int_a^b \sqrt{\mathbf{r}'(u) \cdot \mathbf{r}'(u)}$$
$$s = \int_0^{2\pi} \sqrt{a^2 \sin^2 u + a^2 \cos^2 u + c^2 du}$$
$$s = \int_0^{2\pi} \sqrt{a^2 + c^2} du$$
$$s = \sqrt{a^2 + c^2} u|_0^{2\pi}$$
$$\boxed{s = 2\pi\sqrt{a^2 + c^2}}$$

Problem 4

Problem Statement

Sketch $\mathbf{r}(t) = \mathbf{i}(R\sin\omega t + \omega Rt) + \mathbf{j}(Rcos\omega t + R)$ taking $R = 1$ and $\omega = 1$. This curve is called a cycloid and is the path of a point on the rim of a wheel of radius R that rolls without slipping along the x-axis. Find the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ at the maximum and minimum y-values of the curve.

Solution



$$R = \omega = 1$$
$$\mathbf{r}(t) = \mathbf{i}(\sin t + t) + \mathbf{j}(\cos t + 1)$$
$$\mathbf{v}(t) = \tfrac{dr}{dt}$$
$$\mathbf{v}(t) = \mathbf{i}(\cos t + t) - \mathbf{j}(\sin t)$$
$$\mathbf{a}(t) = \tfrac{dv}{dt}$$
$$\mathbf{a}(t) = -\mathbf{i}(\sin t) - \mathbf{j}(\sin t)$$

$$\boxed{\begin{array}{l} \text{max: } t = 0, x = 0, y = 2 \\ \mathbf{v}(t) = 2\mathbf{i} \\ \mathbf{a}(t) = -1\mathbf{j} \\ \text{min: } t = 0\pi, x = \pi, y = 0 \\ \mathbf{v}(t) = 0 \\ \mathbf{a}(t) = 1\mathbf{j} \end{array}}$$

Problem 5

Problem Statement

The flow of heat in a temperature field takes place in the direction of the maximum decrease of temperature. For the temperature field $T(x, y, z) = z/(x^2 + y^2)$ find this direction in general and at the point $(0, 1, 2)$.

Solution

$$\text{Find } \boldsymbol{\nabla} T(x, y, z)$$

$$\frac{\partial T}{\partial x} = \frac{-2xz}{(x^2+y^2)^2}\mathbf{i}$$

$$\frac{\partial T}{\partial y} = \frac{-2yz}{(x^2+y^2)^2}\mathbf{j}$$

$$\frac{\partial T}{\partial z} = \frac{1}{(x^2+y^2)^2}\mathbf{k}$$

$$\boxed{\boldsymbol{\nabla} T(x, y, z) = \frac{-2xz}{(x^2+y^2)^2}\mathbf{i} - \frac{2yz}{(x^2+y^2)^2}\mathbf{j} + \frac{1}{(x^2+y^2)^2}\mathbf{k}}$$

$$\text{At } P(0, 1, 2):$$

$$0\mathbf{i} - \frac{2(1)(2)}{1}\mathbf{j} + 1\mathbf{k}$$

$$\boxed{-4\mathbf{j} + \mathbf{k}}$$

Problem 6

Problem Statement

Find the unit normal (a) to the surface $ax + by + cz + d = 0$ at any point P, and (b) to the surface $x^2 + y^2 + z^2 = 26$ at the point $(1, 4, 3)$.

Solution

(a)

$$\boxed{\hat{n} = \frac{\boldsymbol{\nabla} f(\mathbf{r})}{|\boldsymbol{\nabla} f(\mathbf{r})|} = \frac{a\mathbf{i}+b\mathbf{j}+c\mathbf{k}}{\sqrt{a^2+b^2+c^2}}}$$

(b)

$$\boldsymbol{\nabla} f(\mathbf{r}) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

At point P:

$$= 2\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$$

$$\text{magnitude} = \sqrt{2^2 + 8^2 + 6^2} = \sqrt{104} = 2\sqrt{26}$$

$$\boxed{\hat{n} = \frac{1}{\sqrt{26}}(1\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})}$$

Problem 7

Problem Statement

Find the divergence of $(-\mathbf{i}y + \mathbf{j}x)/(x^2 + y^2)$.

Solution

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_y}{\partial y}$$

$$\mathbf{v} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

Apply the quotient rule for derivatives.

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$$

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{v} = 0}$$

Problem 8

Problem Statement

Prove that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0$.

Solution

$$\mathbf{v} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$\boldsymbol{\nabla} \times \mathbf{v} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix}$$

$$\boldsymbol{\nabla} \times \mathbf{v} = \hat{i}(\tfrac{\partial}{\partial y} w - \tfrac{\partial}{\partial z} v) + \hat{j}(\tfrac{\partial}{\partial z} u - \tfrac{\partial}{\partial x} w) + \hat{k}(\tfrac{\partial}{\partial x} v - \tfrac{\partial}{\partial y} u)$$

Then do *div* of that quantity.

$$0 = (\tfrac{\partial^2}{\partial x \partial y} w - \tfrac{\partial^2}{\partial x \partial z} v) + (\tfrac{\partial^2}{\partial y \partial z} u - \tfrac{\partial^2}{\partial y \partial x} w) + (\tfrac{\partial^2}{\partial x \partial z} v - \tfrac{\partial^2}{\partial y \partial z} u)$$

Each term in the equation above will cancel out.