Calculating Nth roots

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The calculator is a recent invention in history and is quite useful when running through long calculations. But when the calculator was not around and everyone had to do these calculations by hand, they came up with ways to do the more tedious calculations with great accuracy.

A technical way to calculate the nth root of a number is by using a formula based off the binomial expansion. By using Pascal's Triangle to find the coefficient values of the expanded series of term you want to break down, you can derive a formula to find the nth roots respectively. From the Pascal's Triangle, we can see where they had derived the function that we can use for each root we want to find.

	1
	1 1
$\sqrt{n}$: $20\Delta\beta + \beta^2$	1 2 1
$\sqrt[3]{n}$: $300\Delta^2\beta + 30\Delta\beta^2 + \beta^3$	1 3 3 1
$\sqrt[4]{n}$: $4000\Delta^3\beta + 600\Delta^2\beta^2 + 40\Delta\beta^3 + \beta^4$	1 4 6 4 1
$\sqrt[5]{n}$: $50000\Delta^4\beta + 10000\Delta^3\beta^2 + 1000\Delta^2\beta^3 + 50\Delta\beta^4 + \beta^5$	1 5 10 10 5 1
.	. .
.	. .

Now, you would use these formulas with the corresponding root that you are looking to find. Applying the formula would give you a single term in the root that you are trying to find. For example:

$$\sqrt{3150}$$

The first thing that you would do is to break your number into groups of two (since we are doing a square root) from the given decimal place.

$$\sqrt{31.50.00.00}$$

The way it is set up, we are assuming that the result will not be an integer. When we start to calculate this, we start with the left-most pair. So we want to use our formula. The Δ represents the value that we have already figured out, at the preliminary step, we will always have a zero. The β will be our guess; for our example, we need to find a β such that $20\Delta\beta + \beta^2 \leq 31$ and $\Delta = 0$ since this is our first step. So what does β need to be where, $\beta^2 \leq 31$? Well, $5^2 = 25$ and 6^2 is too big, so our first value is 5. (The key is that every value that you choose should be a single digit number.)

```
   5
 √31.50.00.00
  25
   650
```

Now that we have our first value, our $\Delta = 5$. So we need to use our function again.

What does β need to be for $20(5)\beta + \beta^2 \leq 650$? Let $\beta = 6$

```
   5  6.
 √31.50.00.00
  25
   650
   636
    1400
```

Now that we have done another iteration, our Δ is our value thus far. So $\Delta = 56$ and we need to find a new β for our divisor. So, what does β need to be so that $20(56)\beta + \beta^2 \leq 1400$? Let $\beta = 1$.

```
   5  6. 1
 √31.50.00.00
  25
   650
   636
    1400
    1121
     27900
```

We can continue this process until we get the accuracy that we want. This particular problem was not too much work, but the formula gets larger for the larger root. For each case, the numbers are larger, the formula is larger and the guess that you make for β gets a little less probable to pick on your first try. For this reason, a more user friendly formula has been created for the square and cube roots.

So let's take our same example so that we can show that you get the same results. We will set it up the same way with the breaks in the value:

$$\sqrt{3150} \Rightarrow \sqrt{31.50.00.00}$$

The process will have its similarities, but overall you do fewer calculations to find each term of the result. The preliminary step is the same; find a value to which if you square it, it will be less than or equal to 31. Again, that value is 5 and it will be our first entry.

$$\begin{array}{l} \quad 5 \\ \sqrt{31.50.00.00} \\ \underline{\ 25\qquad} \\ \quad 650 \end{array}$$

Now, we will use a simple formula to find our next entry. First, you double the entry thus far and leave a space for the 'ones place' of the value. We must consider a single digit integer to put in the ones place and to then multiply by our new value.

In other words:

$$\begin{array}{l} \quad 5 \\ \sqrt{31.50.00.00} \\ \underline{\ 25\quad} \\ \quad 650 \end{array}$$

$\underline{\ n\ }$ x (10$\underline{\ n\ }$) $\leq$ 650? 650 Since n=6 works best, we'll use it.

After we make our new entry, we will continue to use this process until we get the accuracy desired.

$$\begin{array}{l} \quad 5 \quad 6. \\ \sqrt{31.50.00.00} \\ \underline{\ 25\quad} \\ \qquad\qquad 650 \\ \quad \underline{636\quad} \\ \quad\ 1400 \end{array}$$

$\underline{n}$ x (112$\underline{\ n\ }$) $\leq$ 1400? 1400 Since n=1 works best, we'll use it.

$$\begin{array}{l} \quad 5 \quad 6.\ 1 \\ \sqrt{31.50.00.00} \\ \underline{25\quad} \\ \quad 650 \\ \quad \underline{636\quad} \\ \quad\quad 1400 \\ \quad\quad \underline{1121\quad} \\ \quad\quad\quad 27900 \end{array}$$

$\underline{\ n\ }$ x (1122$\underline{\ n\ }$) $\leq$ 27900? Since n=2 works best, we'll use it.

As you can see, the formula takes over and the steps are repeated to the users' extent. If we wanted to find the cube root of a certain number, then there is a rule that is similar to the one that we just used, but a little more extensive. So, let's take the cube root of a rather large value that is a little hard to approximate:

$$\sqrt[3]{55742968}$$

The first thing that we will do is to break up the number in groups of three since we are taking the third root. I will also include a few place holders assuming that the result will not be an integer.

$$\sqrt[3]{55.742.968.000.000}$$

The first step is pretty straight forward; we need a value that, if cubed, will be less than or equal to 55. Since 4^3 is too large, we will use 3.

$$\begin{array}{l} \quad 3 \\ \sqrt[3]{55.742.968.000.000} \\ \quad \underline{27\quad\ } \\ \quad 28742 \end{array}$$

Now, to find the following values of our result, our formula will be more intricate to compensate a larger binomial expansion. First, we square the result thus far and multiply it by 3. We will then add two zero place holders after that value. Then, we take the product of 30, the value that we have so far, and a number that we will select. We then sum those to values with our selected number squared. This result must be multiplied by our selected number to be less than or equal to 28742. So, here's the formula:

$$
\begin{array}{ll}
\quad 3 \text{ x } \Delta^2 [00] & \Delta = \text{value so far} \\
+\ 30 \text{ x } \Delta \text{ x } n & n = \text{our guess} \\
+\qquad\qquad n^2 & \Sigma = \text{the sum of the three terms} \\
\hline
\qquad n \text{ x } \Sigma \leq 28742 &
\end{array}
$$

So, as we fill in the values accordingly and make a guess for n, we can find the next value of our answer. Let n = 8.

$$
\begin{array}{ll}
 & \quad 3 \quad 8 \\
 & \sqrt[3]{55.742.968.000.000} \\
\quad 3 \text{ x } (3)^2 [00] = 2700 & \underline{27} \\
+\ 30 \text{ x } (3) \text{ x } (8) = 720 & 28742 \\
+\qquad\quad (8)^2 = 64 & \underline{27872} \\
\hline
 & \quad 870968
\end{array}
$$

$$(8) \text{ x } 3484 = 27872 \leq 28742$$

To find the next value, we will reapply the formula to the current conditions. Let n = 2.

$$
\begin{array}{ll}
 & \quad 3 \quad 8 \quad 2. \\
 & \sqrt[3]{55.742.986.000.000} \\
 & \underline{27} \\
\quad 3 \text{ x } (38)^2 [00] = 433200 & 28742 \\
+\ 30 \text{ x } (38) \text{ x } (2) = 2280 & \underline{27872} \\
+\qquad\quad (2)^2 = 4 & \quad 870968 \\
\hline
 & \quad \underline{870968} \\
(2) \text{ x } 435484 = 870968 \leq 870968 & \qquad 0
\end{array}
$$

Since our last iteration left us with no remainder, we can conclude that this value was a perfect cube and that there will be only zeros after the decimal place.

These formulas are very useful for the common root. Of course, if we wanted a more uncommon root, then a more advanced formula would be necessary and it could be based off of the binomial expansion that was presented earlier. These are just a few methods to finding these values and with a little more work, we could develop patterns for the larger roots.