Novel Alsgorithm for ‘N^{th} Root of Number’ using Multinomial Expansion

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ABSTRACT

This paper gives general algorithm to extract n^{th} root of any real number . Algorithm uses multinomial expansion and intelligent guessing to compute digits of root. Paper also discusses various novel approaches to expand multinomial.

KEYWORD

Algorithm, N^{th} Root, Multinomial Expansion , Intelligent Guessing , Integer Partition

PREREQUISITE

1) Binomial and multinomial theorem

GENERAL METHOD

Different method or approaches, based on multinomial expansion, can be used to extract n^{th} root of any real number. But first we restrict ourselves only with computing n^{th} root of *perfect n^{th} power*. Afterword general algorithm is given to extract n^{th} root of any number.

1) BACKWARD APPROACH

Example $(149721291)^{\frac{1}{3}} = ?$ (Given 149721291 perfect cube)

➡ To calculate cube root, first divide given cube into‘3 digit group’ from right to left.

i.e. Given cube = <u>149</u> <u>721</u> <u>291</u>

$\therefore$ No. of digit = No. of groups = 3

Let integer a,b,c be unit , tens and hundred place of 3 digit cube root respectively.

$\Rightarrow \quad 0 \le a,b \le 9 \ , 1 \le c \le 9$

Now expanding cube of $cba = a + 10b + 100c$ in power of 10’s , we get

i.e. $(a+10b+100c)^3 = (a^3 + 3a^2b*10 + (3ab^2 + 3a^2c)10^2 + ...$

Let higher terms in RHS that are multiple of 10^3 be 1000Y, then

$(a+10b+100c)^3 = a^3 + 3a^2b*10 + (3ab^2 + 3a^2c)10^2 + 1000Y$

Now given cube = 150568768 $\Rightarrow a^3 + 3a^2b*10 + (3ab^2 + 3a^2c)10^2 + 1000Y = 150568768$

$a^3 + 3a^2b*10 + (3ab^2 + 3a^2c)10^2 + 1000Y = 149721291$

$\Rightarrow 3a^2b*10 + (3ab^2 + 3a^2c)10^2 + 1000Y = 149721291 - a^3$ -------------(1)

As LHS is multiple of 10, So RHS $149721291 - a^3$ must be divisible by 10

$\Rightarrow$ $149721291 - a^3$ must ends with 0 i.e. a^3 must ends with 1.

Now there is only one integer i.e. 1 between 0 to 9 whose cube end with 1 , $\therefore$ a=1

Substituting a=1 in above equation (1)

$3a^2b*10 + (3ab^2 + 3a^2c)10^2 + 1000Y = 149721291 - 1 = 149721290$

$3a^2b + (3ab^2 + 3a^2c)10 + 100Y = 14972129$ (Dividing both side by 10)

Similarly,

$(3ab^2 + 3a^2c)10 + 100Y = 14972129 - 3a^2b$

LHS is multiple of 10 $\Rightarrow 3a^2b$ =3b must ends with 9 $\Rightarrow$ b=3

$(3ab^2 + 3a^2c)10 + 100Y = 14972129 - 3*3 = 14972120$

$\therefore$ $(3ab^2 + 3a^2c) + 10Y = 1497212$

$\Rightarrow 10Y = 1497212 - (3ab^2 + 3a^2c) = 1497212 - (3*1*3^2 + 3*1^2*c)$

$10Y = 1497185 - 3c$

LHS divisible by 10 $\Rightarrow 3c$ must ends with 5 and $1 \le c \le 9 \Rightarrow c = 5$

$\therefore$ Cube root= cba= 531

In short,

1) a^3 ends with 1 (End digit of 149721291)

$\Rightarrow$ a = 1 $\therefore a^3 = 1$,Subtracting it from 149721291 and eliminating last digit i.e. $14972129\cancel{0} = 14972129$

$$\begin{array}{r} 149721291 \\ -1 \\ \hline 14972129\cancel{0} \end{array}$$

2) $3a^2b = 3b$ ends with 9 (End digit of 14972129)

$\Rightarrow$ b = 3 $\therefore 3a^2b = 9$,Subtracting it from 14972129 and eliminating last digit i.e. $1497212\cancel{0} = 1497212$

$$\begin{array}{r} 14972129 \\ -9 \\ \hline 1497212 \end{array}$$

3) $3ab^2 + 3a^2c = 27 + 3c$ ends with 2 (End digit of 1497212)

$\Rightarrow 3c$ ends with 5 $\Rightarrow$ c = 5 $\therefore 3ab^2 + 3a^2c = 42$.

$$\begin{array}{r} 1497212 \\ -42 \\ \hline 149717 \end{array}$$

$\therefore$ Cube Root(150568768) = $cba = 531$

Applicability

Above method is convenient to extract n^{th} root of perfect n^{th} power that satisfies

$$GCD(10, n * U^{n-1}) = 1$$

$\Rightarrow GCD(10, n) = 1$ and GCD(10, U) =1 Where U= Unit digit of n^{th} root .
i.e. Unit digits of " n and n^{th} root " must be relative prime to 10 (base).
(Why ? Think).

2) FORWARD APPROACH

Example 1) $(20352513413376)^{\frac{1}{4}} = ?$ (Given number is perfect 4^{th} power)

Steps

1) Arrange given perfect n^{th} power into *n digit group* from right to left.

Here n = 4.

20 3525 1241 3376

$\therefore$ No. of groups = 4 $\Rightarrow$ No. of digit in root = 4

Let root = $1000a + 100b + 10c + d$

2) Except leftmost group , separate digits of all groups

i.e. $\quad \big|\ 20 \quad 3 \quad 5 \quad 2 \quad 5 \quad 1 \quad 2 \quad 4 \quad 1 \quad 3 \quad 3 \quad 7 \quad 6$

3) To extract 4 digit of perfect 4^{th} power, Only most significant '4+1' term are sufficient. Expanding $(1000a+100b+10c+d)^4$ according to descending power of ten using multinomial theorem, we get most significant 5 terms of expansion as

$$
\begin{aligned}
(1000a+100b+10c+d)^4 &= (a^4)*10^{12}+(\frac{4!}{3!1!}a^3b)*10^{11}+(\frac{4!}{2!2!}a^2b^2+\frac{4!}{3!1!}a^3c)*10^{10} \\
&\quad + (\frac{4!}{1!3!}a^1b^3+\frac{4!}{2!1!1!}a^2bc+\frac{4!}{3!1!}a^3d+)*10^9+\ldots\ldots\ldots \\
&= (a^4)*10^{12}+(4a^3b)*10^{11}+(6a^2b^2+4a^3c)*10^{10} \\
&\quad + (4a^1b^3+12a^2bc+4a^3d)*10^9+\ldots.
\end{aligned}
$$

4) Now highest a^4 less than or equal to 20 is $2^4=16 \quad \therefore a=2.$

Subtract a^4 from leftmost group and attach subtraction before next digit to get next gross dividend.

$$
\begin{array}{l|l}
 & 2 \\
\text{i.e.} & 20 \quad {}_4 3 \quad 5 \quad 2 \quad 5 \quad 1 \quad 2 \quad 4 \quad 1 \quad 3 \quad 3 \quad 7 \quad 6
\end{array}
$$

Why This Works ?

Let N=20 3525 1241 3376

x is 4^{th} root of N

$f(x) = x^4 - 203525\ 1241\ 3376$

($f(x)$ has only one real positive root -According to Decarte's rule)

(Decartes Rule- Maximum no. of positive root of

$f(x)$ = No. of sign (coefficient sign) changes in $f(x)$)

$2^4 \le 20 < 3^4 \Rightarrow 2^4*10^{12} \le 20*10^{12} < 3^4*10^{12}$

$2000^4 \le N < 3000^4$

$\Rightarrow 2000^4 - N \le 0$ and $3000^4 - N > 0$

i.e. $f(2000) = -ve$ & $f(3000) = +ve$

Between two opposite signs of polynomial there always exist at least one real root.

$\therefore$ Root (x) must lies between 2000 and 3000

Therefore most significant digit of root is 1

Similar reasoning can be given for subsequent steps of this method.

Aryabhata's square root method and many other method indirectly use this concept for intelligent guessing.

5) Now look up for highest integer b such that $(4a^3b) = 4*8*b = 32b$ less than or equal to 43

i.e. $32b \le 43 \Rightarrow$ b ≤ 1 $\therefore$ *Highest* b=1

subtracting $4a^3b = 32$ from 43 , we get 11. Attach 11 before next digit 5 to get next gross dividend 115.

$$\begin{array}{l|l} & 2 \quad 1 \\ \hline \text{i.e.} & 20 \;\; {}_{4}3 \;\; {}_{11}5 \;\; 2 \;\; 5 \;\; 1 \;\; 2 \;\; 4 \;\; 1 \;\; 3 \;\; 3 \;\; 7 \;\; 6 \end{array}$$

6) Look for highest integer c such that $(6a^2b^2 + 4a^3c)$ less than or equal to gross dividend 115.

i.e. $(6a^2b^2 + 4a^3c) \le 115 \Rightarrow 24 + 32c \le 115$

$32c \le 91 \Rightarrow$ c=2

Subtracting $(6a^2b^2 + 4a^3c)$ from 115, we get $91-64 = 27$, attaching it before next digit 2 to get next gross dividend.

$$\begin{array}{l|l} & 2 \quad 1 \quad 2 \\ \hline \text{i.e.} & 20 \;\; {}_{4}3 \;\; {}_{11}5 \;\; {}_{27}2 \;\; 5 \;\; 1 \;\; 2 \;\; 4 \;\; 1 \;\; 3 \;\; 3 \;\; 7 \;\; 6 \end{array}$$

7) Similarly, $4a^1b^3 + 12a^2bc + 4a^3d \le 272$

$\Rightarrow 8 + 96 + 32d \le 272 \Rightarrow 32d \le 168$

$\Rightarrow d = 4$

Subtracting $(4a^1b^3 + 12a^2bc + 4a^3d)$ from 272, we get $168-128 = 40$, attaching it before next digit 5 to get next gross dividend.

$$\begin{array}{l|l} & 2 \quad 1 \quad 2 \quad 4 \\ \hline \text{i.e.} & 20 \;\; {}_{4}3 \;\; {}_{11}5 \;\; {}_{27}2 \;\; {}_{40}5 \;\; 1 \;\; 2 \;\; 4 \;\; 1 \;\; 3 \;\; 3 \;\; 7 \;\; 6 \end{array}$$

$\therefore (20352513413376)^{\frac{1}{4}} = 2124$

Applicability

Above method is applicable to extract n^{th} root of any number.

3) FORWARD –BACKWARD APPROACH

1) $(19477056866406726525123713030989808121)^{\frac{1}{7}} = ?$ (Given number is perfect 7^{th} power)

Arranging number into 7 digit group from right to left

194 7705686 6406726 5251237 1303098 9808121

$\therefore$ No. of digit in 7^{th} root = No. of groups = 6.

Let abcdef = f + 10e + 100d+ 1000c + 10000b + 100000a be root.

Here we can extract 4 least significant digit cdef by backward approach as below

$$(f+10e+10^2d+10^3c+10^4b+10^5a)^7 = f^7+(7f^6e)*10+(21f^5e^2+7f^6d)*10^2$$
$$+(28f^4e^3+21f^5ed+7f^6d)*10^3+.......+a^7*10^{35}$$

1) f^7 ends with 1 (End digit of 19477056866406726525123713030989808121)

$\Rightarrow$ f = 1 $\therefore f^7=1$,Subtracting it from ...08121, and eliminating last digit

$$\begin{array}{r} ......08121 \\ -1 \\ \hline ....0812\cancel{0} \end{array}$$

2) $7f^6e=7e$ ends with 2 (End digit of ...812)

$\Rightarrow e=6$ $\therefore 7f^6e=42$,Subtracting it from ...812 and eliminating last digit .

$$\begin{array}{r} ......0812 \\ -42 \\ \hline .....077\cancel{0} \end{array}$$

3) $21f^5e^2+7f^6d=756+7d$ ends with 7 (End digit of ...77)

$\Rightarrow 7d$ ends with 7-6=1 $\Rightarrow$ d = 3 $\therefore$ 756+7d = 777, Subtracting it from ...8077 and eliminating last digit .

$$\begin{array}{r} ......8077 \\ -777 \\ \hline .....370\cancel{0} \end{array}$$

4) $28f^4e^3+21f^5ed+7f^6c=28*216+21*18+7d$ ends with 0 (End digit of ...370)

$\Rightarrow 7c$ ends with =4 $\Rightarrow$ c = 2

$\therefore$ cdef = 2361 --------------------------(1)

Now we extract most significant 2 digit i.e. ab by forward approach as below

$$(10^5a+10^4b+10^3c+10^2d+10e+f)^7 = (a^7)*10^{35}+(7a^6b)*10^{34}+(21a^5b^2+7a^6c)*10^{33}+.......+e^7$$

$$\text{i.e.} \quad \begin{array}{l|l} & 2 \quad 1 \quad 3 \\ \hline & 194 \quad {}_{66}7 \quad {}_{219}7 \quad {}_{181}0 \quad 5 \quad 6 \quad 8 \quad .... \quad 8 \quad 1 \quad 2 \quad 1 \end{array}$$

i) $a^7 \le 194 \quad \Rightarrow \quad a = 2$

$194 - a^7 = 194 - 128 = 66$

ii) $7a^6 b \le 667 \quad \Rightarrow b \le [\frac{667}{7 * 2^6}] \quad \Rightarrow b = 1$

$\Rightarrow \; 667 - 7a^6 b = 667 - 448 = 219$

iii) $21a^5 b^2 + 7a^6 c \le 2197$

$\Rightarrow 21 * 32 * 1 + 448c \le 2197 \qquad \Rightarrow 448c \le (2197 - 672)$

$\Rightarrow c \le [\frac{1525}{448}] \qquad \Rightarrow c = 3$

$2197 - 21a^5 b^2 + 7a^6 c = 1525 - 448 * 3$

$= \text{Remainder}(\frac{1525}{448}) = 181$

$0 <$ (c =3) $< 10 \;\Rightarrow$ value of b is correct.

$\therefore \quad ab = 21$ ----------------------(2)

$\therefore \; (19477056866406726525123713030989808121)^{\frac{1}{7}} = abcdef = 212361$

(From equation (1) and (2))

GENERIC ALGORITHM

I. Initialization

1) $N =$ Number to extract n^{th} root

$= D_{mn+n-1} D_{mn+n-2} D_{mn+n-3} \ldots\ldots D_2 D_1 D_0$

(Number in decimal notation)

$= G_m G_{m-1} G_{m-2} \ldots G_1 G_0$

(Arranging digits of number N into 'n digit group' from right to left)

2) Assume R $= (N)^{\frac{1}{n}} = a_m a_{m-1} a_{m-2} \ldots a_1 a_0 = (10^m a_m + 10^{m-1} a_{m-2} + \ldots. + 10^2 a_2 + 10 a_1 + a_0)$

3) MSB (R) $= a_m$ = Highest n^{th} power of integer that is less than or equal to G_m

4) $R_0 = G_m - (a_m)^n$

5) Divisor $=$ n * $(a_m)^{n-1}$

II. Iteration

For (k = 1 to m)

{

i) $GD_k = R_{k-1} * 10 + D_{mn-k}$

ii) $ND_k = GD_k - C_k(a_m, a_{m-1}, \ldots\ldots, a_{m-k+1})$

where $C_k(a_m, a_{m-1}, \ldots\ldots, a_{m-k+1})$ denotes total contribution of $(a_m, a_{m-1}, \ldots\ldots, a_{m-k+1})$ to coefficient of 10^{mn-k} in expansion $(10^m a_m + 10^{m-1} a_{m-1} + 10^{m-2} a_{m-2} + \ldots. + 10^2 a_2 + 10a_1 + a_0)^n$

It is given as below

$$\sum_{\substack{p_m + p_{m-1} + \ldots + p_{m-k+1} = n \\ mp_m + (m-1)p_{m-1} + (m-2)p_{m-2} + \ldots + (m-k+1)p_{m-k+1} = mn-k \\ 0 \le \text{integer } p_r \le [\frac{mn-k}{r}]}} \frac{n!}{p_m! p_{m-1}! p_{m-2}! \ldots . p_{m-k+1}!} (a_m)^{p_m} (a_{m-1})^{p_{m-1}} (a_{m-2})^{p_{m-2}} \ldots (a_{m-k+1})^{p_{m-k+1}}$$

iii) if (ND_k < 0) // Validation

{

$a_{m-k+1} = (a_{m-k+1}) - 1$ // Decrease Last computed Quotient by 1

$R_{k-1} = R_{k-1}$ + Divisor

Continue same iteration // Go to step (i) to recompute ND_k

}

iv) $a_{m-k} =$ Quotient $(\frac{ND_k}{Divisor}) = [\frac{ND_k}{Divisor}]$

$R_k =$ Remainder $(\frac{ND_k}{Divisor}) = (ND_k)$ % $Divisor$

}

FUTURE ENHANCEMENT

In above algorithm, we need to expand multinomial $(10^m a_m + 10^{m-1} a_{m-1} + 10^{m-2} a_{m-2} + \ldots. + 10^2 a_2 + 10a_1 + a_0)^n$ to compute $C_k(a_m, a_{m-1}, \ldots\ldots, a_{m-k+1})$. Multinomial can be expanded using various approach that are listed as below

1) Recursive Approach

Use binomial theorem or Pascal triangle recursively to expand multinomial.

2) Algorithmic Approach

Develop or use existing integer processing based faster algorithm to break given integer power i.e. n into at most m positive integer i.e. $(p_m, p_{m-1}, \ldots\ldots, p_0)$ such that

$$0 * P_0 + 1P_1 + 2P_2 + 3P_3 + \ldots. + (m-1)P_{m-1} + mP_m = L$$

In other word , develop efficient algorithm to determine all possible tuple

$(p_0, p_1, p_2, \ldots., p_m)$ to solve

$$p_0 + p_1 + p_2 + \ldots + p_m = n$$

$$P_1 + 2P_2 + 3P_3 + \ldots. + (m-1)P_{m-1} + mP_m = L \qquad \forall k \ 0 \le P_k \le \frac{\mathrm{L}}{\mathrm{k}}$$

After computing tuple , $C_L(a_m, a_{m-1}, \ldots\ldots, a_{m-L+1})$ can be easily obtained by using multinomial theorem. Organise tuple and use relation between two tuple to avoid repetitive factorial computation.

3) Elimination and Retention Approach

Elimination and retention approach provides elegant, systematic way to expand multinomial. This method will be explored in upcoming book *'Modern Approach to Speed Math Secrets'*. Note that all these algorithm and method for expansion of multinomial are also useful to compute any integer power of any number. Thus in future one can enhance this method by developing efficient algorithm to expand multinomial.

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