

CURVES AND SURFACES

Notes for an Undergraduate Course in Computer Graphics

University of Minho

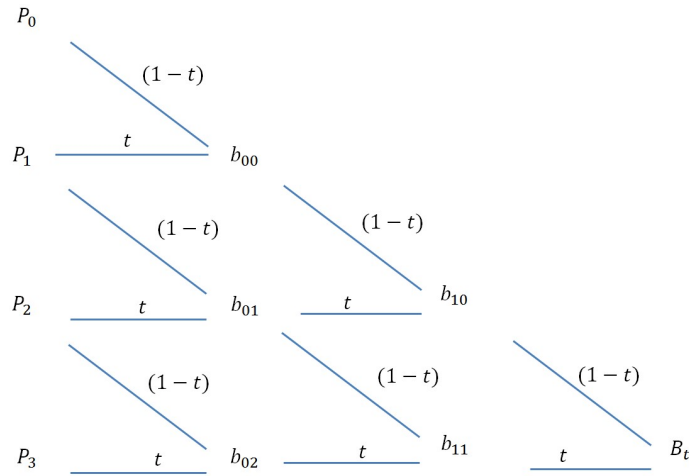
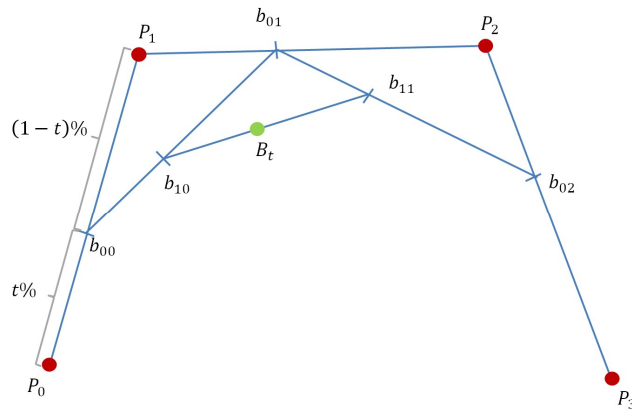
António Ramires

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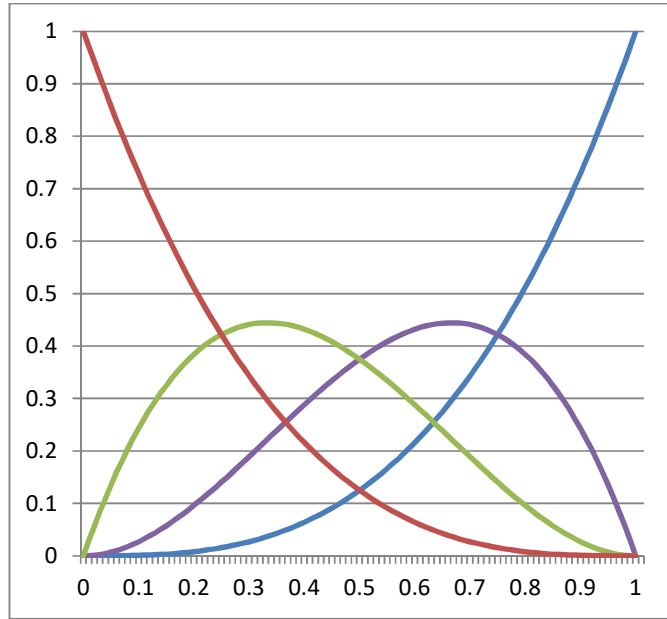
(draft 1)

1 Cubic Bezier Curves

Geometric Algorithm (De Casteljau)



$$B(t) = t^3 P_3 + 3t^2(1-t)P_2 + 3t(1-t)^2 P_1 + (1-t)^3 P_0 \quad (1)$$



$$\begin{aligned} B_{0,3}(t) &= (1-t)^3 \\ B_{1,3}(t) &= 3t(1-t)^2 \\ B_{2,3}(t) &= 3t^2(1-t) \\ B_{3,3}(t) &= t^3 \end{aligned}$$

$$\begin{aligned} B_{3,3}(t) &= t^3 \\ B_{2,3}(t) &= 3t^2(1-t) \\ B_{1,3}(t) &= 3t(1-t)^2 \\ B_{0,3}(t) &= (1-t)^3 \end{aligned} \quad (2)$$

Bernstein polynomials

$$B_{i,3} = \binom{3}{i} t^i (1-t)^{3-i} \quad (3)$$

Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k(n-k)!} \quad (4)$$

$$B(t) = \sum_{i=0}^3 P_i B_{i,3}(t) \quad (5)$$

$$B(t) = t^3 P_3 + (-3t^3 + 3t^2) P_2 + (3t^3 - 6t^2 + 3t) P_1 + (-t^3 + 3t^2 - 3t + 1) P_0 \quad (6)$$

$$B(t) = [-t^3 + 3t^2 - 3t + 1, \quad 3t^3 - 6t^2 + 3t, \quad -3t^3 + 3t^2, \quad t^3] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (7)$$

$$B(t) = [B_{03}(t) \quad B_{13}(t) \quad B_{23}(t) \quad B_{33}(t)] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (8)$$

$$B(t) = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (9)$$

Derivative – Tangent to the curve

$$B'(t) = [3t^2 \quad 2t \quad 1 \quad 0] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (10)$$

2 Other Cubic Curves

Cubic Polynomial:

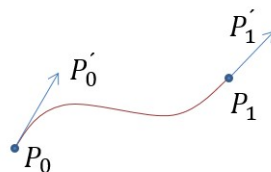
$$p(t) = at^3 + bt^2 + ct + d = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad (11)$$

$$p'(t) = 3at^2 + 2bt + c \quad (12)$$

Three cubic polynomials, one for each coordinate (x, y, z):

$$[x(u) \quad y(u) \quad z(u)] = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = TA \quad (13)$$

2.1 Hermite



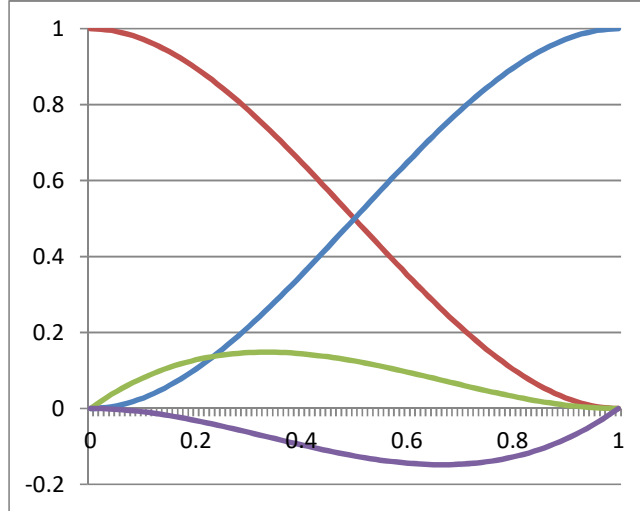
$$\begin{aligned}
p(0) &= d \\
p(1) &= a + b + c + d \\
p'(0) &= c \\
p'(1) &= 3a + 2b + c
\end{aligned} \tag{14}$$

$$P = \begin{bmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = C \times A \tag{15}$$

$$A = C^{-1}P \tag{16}$$

$$[x(u) \quad y(u) \quad z(u)] = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{bmatrix} \tag{17}$$

$$[t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \tag{18}$$



$$\begin{aligned}
&2u^3 - 3u^2 + 1 \\
&-2u^3 + 3u^2 \\
&u^3 - 2u^2 + u \\
&u^3 - u^2
\end{aligned}$$

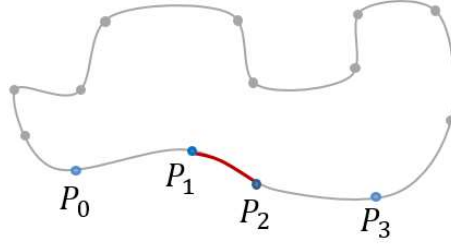
2.2 Catmull-Rom

$$\begin{aligned}
\frac{P_2 - P_0}{2} &= P'_1 = P'(0) = c \\
P_1 &= P(0) = d \\
P_2 &= P(1) = a + b + c + d \\
\frac{P_3 - P_1}{2} &= P'_2 = P'(1) = 3a + 2b + c
\end{aligned} \tag{19}$$

$$\begin{aligned}
P_0 &= a + b - c + d \\
P_1 &= P(0) = d \\
P_2 &= P(1) = a + b + c + d \\
P_3 &= 6a + 4b + 2c + 1
\end{aligned} \tag{20}$$

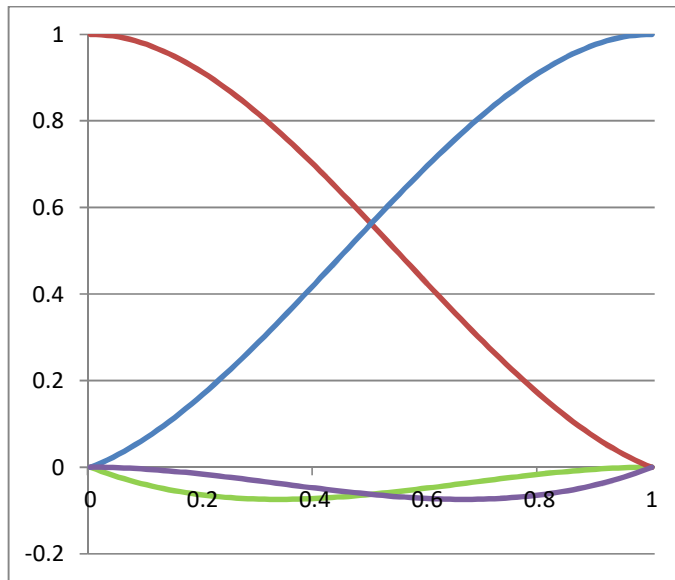
$$P = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = C \times A \tag{21}$$

$$A = C^{-1}P \tag{22}$$



$$[x(u) \quad y(u) \quad z(u)] = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -0.5 & 1.5 & -1.5 & 0.5 \\ 1 & -2.5 & 2 & -0.5 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{23}$$

$$[t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} -0.5 & 1.5 & -1.5 & 0.5 \\ 1 & -2.5 & 2 & -0.5 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.5t^3 + t^2 - 0.5t \\ 1.5t^3 - 2.5t^2 + 1 \\ -1.5t^3 + 2t^2 + 0.5t \\ 0.5t^3 - 0.5t^2 \end{bmatrix}^T \tag{24}$$



$$\begin{aligned} & -0.5t^3 + t^2 - 0.5t \\ & 1.5t^3 - 2.5t^2 + 1 \\ & -1.5t^3 + 2t^2 + 0.5t \\ & 0.5t^3 - 0.5t^2 \end{aligned}$$

3 Bezier Patches

$$B(u, v) = \sum_{j=0}^3 \sum_{i=0}^3 B_i(u) P_{ij} B_j(v) \quad (25)$$

Let $U = [u^3 \quad u^2 \quad u \quad 1]$ and $M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$$B(u, v) = UM \begin{bmatrix} P_{00} \\ P_{10} \\ P_{20} \\ P_{30} \end{bmatrix} B_0(v) + UM \begin{bmatrix} P_{01} \\ P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} B_1(v) + UM \begin{bmatrix} P_{02} \\ P_{12} \\ P_{22} \\ P_{32} \end{bmatrix} B_2(v) + UM \begin{bmatrix} P_{03} \\ P_{13} \\ P_{23} \\ P_{33} \end{bmatrix} B_3(v)$$

$$B(u, v) = UM \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} B_0(v) \\ B_1(v) \\ B_2(v) \\ B_3(v) \end{bmatrix}$$

$$B(u, v) = UM \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} (VM)^T$$

$$B(u, v) = [u^3 \quad u^2 \quad u \quad 1] M \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} M^T \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

Tangents

$$\frac{\partial B(u, v)}{\partial u} = [3u^2 \quad 2u \quad 1 \quad 0] M \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} M^T V^T$$

$$\frac{\partial B(u, v)}{\partial v} = U M \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} M^T \begin{bmatrix} 3v^2 \\ 2v \\ 1 \\ 0 \end{bmatrix}$$

Normal

The normal vector at any point of the surface is defined as the cross product of the normalized tangent vectors. Note that normalizing the result of the cross product is required in the general case.