SOS

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Aufgabe 1)

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 10^{-4} & 0 & 10^{-4} \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1 & 0 & 20'000 \\ 0 & 1 & 0 \\ 1 & 0 & -10'000 \end{pmatrix}$$

a) $\text{cond}(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$\|A\|_\infty =$ 20'001

$\text{cond}(A) = 3 \cdot 20'001 = \underline{\underline{60'003}}$

b) $\varepsilon > 0 \rightarrow \tilde{b} = \begin{pmatrix} 1 \\ 1 \\ \varepsilon \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 1-1 \\ 0-\varepsilon \end{pmatrix}$

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \leq \underbrace{\text{cond}_\infty(A)} \cdot \frac{\|b - \tilde{b}\|_\infty}{\|b\|_\infty} = 0.01$$

$$60'003 \cdot \frac{\varepsilon}{1} = 0.001$$

$$\varepsilon = \frac{0.001}{60'003}$$

$$\underline{\underline{\varepsilon = 1.667 \cdot 10^{-7}}}$$

c)

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0.0001 & 0 & 0.0001 & 0 \end{array}\right) \cdot (-0.0001) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -0.0001 & -0.0001 \end{array}\right)$$

$$x = \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix} \quad \text{✓}$$

$$x_3 = \frac{-0.0001}{-0.0001} = 1$$

$$x_2 = 1$$

$$x_1 = \frac{1 - 2 \cdot 1}{1} = -1$$

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0.0001 & 0 & 0.0001 & 1.667 \cdot 10^{-7} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -0.0001 & -9.98 \cdot 10^{-5} \end{array}\right)$$

$$\tilde{x}_3 = \frac{-9.98 \cdot 10^{-5}}{-0.0001} = 0.998$$

$$\tilde{x} = \begin{pmatrix} -0.996 \\ 1 \\ 0.998 \end{pmatrix} \quad \tilde{x}_2 = 1$$

$$\tilde{x}_1 = \frac{1 - (2 \cdot 0.998)}{1} = \underline{\underline{-0.996}}$$

$\tilde{x}_1 = 1 - (2 \cdot 0.998) = -0.996$

$$\frac{\|\tilde{x} - x\|_\infty}{\|x\|_\infty} = \frac{\begin{pmatrix} -0.996 - (-1) \\ 1 - 1 \\ 0.998 - 1 \end{pmatrix}}{1} = \begin{pmatrix} 0.004 \\ 0 \\ 0.002 \end{pmatrix} = \frac{0.004}{1} = 0.004$$

$0.004 \leq 0.01$

d) Formel:

$$\frac{\|\tilde{x} - x\|_\infty}{\|x\|_\infty} \leq \frac{\text{cond}(A)}{1 - \text{cond}(A) \cdot \frac{\|\tilde{A} - A\|_\infty}{\|A\|_\infty}} \cdot \left(\frac{\|\tilde{A} - A\|_\infty}{\|A\|_\infty} + \frac{\|\tilde{b} - b\|_\infty}{\|b\|_\infty} \right)$$

$\text{cond}(A) = 60'003$

$$\tilde{A} = \begin{pmatrix} 1+10^{-7} & 0+10^{-7} & 2+10^{-7} \\ 0+10^{-7} & 1+10^{-7} & 0+10^{-7} \\ 10^{-4}+10^{-7} & 0+10^{-7} & 10^{-4}+10^{-7} \end{pmatrix} \quad \frac{\|\tilde{A} - A\|_\infty = 3 \cdot 10^{-7}}{\|A\|_\infty = 3} = 1 \cdot 10^{-7}$$

$$\frac{60'003}{1 - 0.006} \cdot \left(1 \cdot 10^{-7} + 1.667 \cdot 10^{-7} \right)$$

$= 0.016003 \rightarrow 1.61\,\%$