Exposition of $Gal_L(T)$

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1 Introduction

The purspose of this writeup is to construct $Gal_L(T)$ without making any arbitrary choices, such as selecting a large saturated model.

2 Bounded, Aut-Invariant Equivalences

In this section, I will construct the Lascar equivalence relation in terms of types.

Definition 2.1. Let T be a complete theory. An n-ary *relation present in* T is a subset of $S_n(T/\varnothing)$. For any R a relation present in T, $a_1, \ldots, a_n$ in an appropriate context, say that $R(a_1, \ldots, a_n)$ holds just if $\mathrm{tp}(a_1, \ldots, a_n) \in R$.

This definition corresponds to the usual definition of an automorphism-invariant relation.

Proposition 2.2. *Let $\mathcal{U}$ be a sufficiently large model. An n-ary relation R on this model is automorphism-invariant iff it arises as a relation present in T.*

Proof. Let R be a relation present in T, $f \in Aut(\mathcal{U})$, and $a \in \mathcal{U}^n$. Since f is an automorphism, $\mathrm{tp}(a) = \mathrm{tp}(f(a))$. Therefore $R(a)$ iff $R(f(a))$, so R is automorphism-invariant.

Let R be an automorphism-invariant relation on $\mathcal{U}^n$, and let $a, b \in \mathcal{U}^n$ have the same type. Since $\mathcal{U}$ is sufficiently homogeneous, there is an automorphism f with $f(a) = b$. Since R is automorphism-invariant, R agrees on a and b. Therefore whether R holds of a tuple depends only on the tuple's type, so R arises as a type present in T. $\square$

We can define reflexivity, symmetry, and transitivity of a binary relation present in T without referencing any models.

Definition 2.3. Let $R(x, y)$ be a binary relation present in T. R is *reflexive* if $[x = y] \subseteq R$, where $[x = y]$ is the filter generated by the given formula, viewed as a (closed) set of types.

Proposition 2.4. *Let $R(x, y)$ be a binary relation present in T. R is reflexive iff it yields a reflexive relation in all models, iff it yields a reflexive relation in a sufficiently large model.*

Proof. Let $R(x, y)$ be a reflexive relation present in T, $M \models T$, and $a \in M$. Since $\mathrm{tp}(a, a) \in [x = y] \subseteq R$, $R(a, a)$. Therefore R yields a reflexive relation in all models.

Let $R(x, y)$ be a relation present in T which is not reflexive. Let $\mathcal{U} \models T$ be sufficiently large. Since R is not reflexive, take $p(x, y) \in [x = y]$ with $p \notin R$. Since $\mathcal{U}$ is sufficiently saturated, take $a, b \in \mathcal{U}$ with $\mathcal{U} \models p(a, b)$. Since $p \in [x = y]$, $a = b$. However, since $p \notin R$, $\neg R(a, b)$. Therefore R does not yield a reflexive relation in $\mathcal{U}$. $\square$

Definition 2.5. Let $R(x, y)$ be a binary relation present in T. R is *symmetric* if, for all $p(x, y) \in R$, $p(y, x) \in R$.

Proposition 2.6. *Let $R(x, y)$ be a binary relation present in T. R is symmetric iff it yields a symmetric relation in all models, iff it yields a symmetric relation in a sufficiently large model.*

Proof. Let $R(x, y)$ be a symmetric relation present in T, $M \models T$, and $a, b \in M$ with $R(a, b)$. Set $p(x, y) = \mathrm{tp}(a, b)$. Since $\mathrm{tp}(b, a) = p(y, x) \in R$, $R(b, a)$. Therefore R yields a symmetric relation in M.

Let $R(x, y)$ be a binary relation present in T which is not symmetric. Let $\mathcal{U}$ be a sufficiently large model. Since R is not symmetric, take $p(x, y) \in R$ with $p(y, x) \notin R$. Since $\mathcal{U}$ is sufficiently saturated, take $a, b \in \mathcal{U}$ with $\mathrm{tp}(a, b) = p(x, y)$, so $R(a, b)$. However, $\mathrm{tp}(b, a) = p(y, x) \notin R$, so $\neg R(b, a)$. Therefore R does not yield a symmetric relation in $\mathcal{U}$. $\square$

Definition 2.7. Let $R(x, y)$ be a binary relation present in T. R is *transitive* if, for all $p(x, y), q(x, y) \in R$, $[p(x, z), q(z, y)] \restriction_{x,y} \subseteq R$, where $[p(x, z), q(z, y)] \restriction_{x,y}$ is the filter in x, y obtained by restricting the filter in x, y, z generated by the formulas in $p(x, z)$ and $q(z, y)$ to those formulas only mentioning x and y.

Proposition 2.8. *Let $R(x, y)$ be a binary relation present in T. R is transitive iff it yields a transitive relation in all models, iff it yields a transitive relation in a sufficiently large model.*

Proof. Let $R(x, y)$ be a transitive relation present in T, $M \models T$, $a, b, c \in M$ with $R(a, b)$ and $R(b, c)$. Set $p(x, y) = \mathrm{tp}(a, b)$ and $q(x, y) = \mathrm{tp}(b, c)$. We know that $p, q \in R$. Furthermore, $r(x, z, y) := \mathrm{tp}(a, b, c) \in [p(x, z), q(z, y)]$, so $\mathrm{tp}(a, c) = r \restriction_{x,y} \in R$, and therefore $R(a, c)$. Therefore R yields a transitive relation in M.

Let $R(x, y)$ be a binary relation present in T which is not transitive. Let $\mathcal{U}$ be a sufficiently large model. Since R is not transitive, take $p(x, y), q(x, y) \in R$, $r(x, z, y) \in [p(x, z), q(z, y)]$, with $r \restriction_{x,y} \notin R$. Since $\mathcal{U}$ is sufficiently saturated, take $a, b, c \in \mathcal{U}$ with $\mathrm{tp}(a, b, c) = r$. This implies that $\mathrm{tp}(a, b) = p$ and $\mathrm{tp}(b, c) = q$, so $R(a, b)$ and $R(b, c)$. However, since $\mathrm{tp}(a, c) = r \restriction_{x,y}$, $\mathrm{tp}(a, c) \notin R$ and $\neg R(a, c)$. Therefore R does not yield a transitive relation in $\mathcal{U}$. $\square$

For an equivalence relation present in T, there is a purely combinatorial definition of boundedness. We want to say that an equivalence relation is bounded if there aren't too many equivalence classes. Since an equivalence relation present in T is just a set of types over the empty set, the ways in which elements can be in different equivalence classes correspond to those types *not* in the relation. By Ramsey theory, if there are too many inequivalent elements, there are infinitely many elements inequivalent by the same type. By compactness, it is therefore possible to have arbitrarily many elements inequivalent by this type. This characterization is formalized below.

Definition 2.9. Let $p(x, y) \in S_2(T)$. Then $\mathrm{sym}(p)$, the *symmetric closure* of p, is $\{p(x, y), p(y, x)\}$. Since $\mathrm{sym}(p) \subseteq S_2(T)$, $\mathrm{sym}(p)$ is a (symmetric) relation present in T. Additionally, since $\mathrm{sym}(p)$ is a finite set of points, it is closed, and therefore corresponds to a filter of formulas.

Definition 2.10. Let $p(x, y) \in S_2(T)$. $\mathrm{sym}(p)$ is *graph-infinite* if

$$[\mathrm{sym}(p)(x_i, x_j), x_i \neq x_j : i \neq j],$$

a filter in the variables $\{x_i\}_{i \in \omega}$, is consistent.

Proposition 2.11. *Let $p(x, y) \in S_2(T)$. Then the following are equivalent:*

- $\mathrm{sym}(p)$ *is graph-infinite;*
- *for some infinite κ, there is a model with κ many elements pairwise related by* $\mathrm{sym}(p)$*;*
- *for all infinite κ, there is a model with κ many elements pairwise related by* $\mathrm{sym}(p)$*; and*
- *for any κ-saturated model $\mathcal{U}$, there are at least κ many elements of $\mathcal{U}$ pairwise related by* $\mathrm{sym}(p)$.

Proof. This is an easy consequence of compactness. $\square$

Definition 2.12. Let xEy be an equivalence relation present in T. E is *bounded* if no $p(x, y) \notin E$ is graph-infinite.

This definition aligns with the traditional definition of boundedness.

Proposition 2.13. *Let xEy be an equivalence relation present in T. E is bounded iff, in a sufficiently large model $\mathcal{U}$, the number of equivalence classes is strictly less than κ, where κ is the saturation of $\mathcal{U}$.*

Proof. ($\Leftarrow$) Assume that E is not bounded. Then take $p(x, y) \notin E$ graph-infinite. By Proposition 2.11, there are κ many elements of $\mathcal{U}$ pairwise related by $\mathrm{sym}(p)$. Since $p \notin E$, and since E is symmetric, $\mathrm{sym}(p)$ is disjoint from E. Therefore we have found κ many pairwise inequivalent elements of $\mathcal{U}$.

($\Rightarrow$) Let $\tau = |S_2(T)|$. By Ramsey theory, there is a least cardinal λ such that, for any coloring of the complete 2-graph with λ vertices and τ colors, there is a complete, infinite, homogeneous subgraph. Since $\mathcal{U}$ is sufficiently large, in particular $\kappa \geq \lambda$.

Assume that there are at least κ equivalence classes in $\mathcal{U}$. Pick one representative from each of the $\geq \kappa$ classes. Consider each pair of representatives $\{a, b\}$ "colored" by $\mathrm{sym}(a, b) = \mathrm{sym}(b, a)$. Since $\kappa \geq \lambda$, there is an complete, infinite, homogeneous subgraph. That is, there is some graph-infinite $\mathrm{sym}(p)$ with $p \notin E$. Therefore E is not bounded. □

Corollary 2.13.1. *Let xEy be a bounded equivalence relation. Then there is a cardinal λ such that all models have strictly fewer than λ equivalence classes.*

Proof. This is just the λ from the proof of Proposition 2.13. □

The definition of the Lascar equivalence relation, E_L, of a theory proceeds as usual.

Definition 2.14. Let T be a complete theory. The *Lascar equivalence relation*, E_L, for that theory is the intersection (as sets of types) of all bounded equivalence relations.

Remark 2.15. It is clear that intersection of relations as sets of types corresponds with the usual intersection of relations in any model.

Proposition 2.16. *For any theory, E_L is a bounded equivalence relation.*

Proof. From Remark 2.15, since a relation present in a theory is an equivalence relation iff it yields an equivalence relation in a sufficiently saturated model, and since the intersection of (usual) equivalence relations is an equivalence relation, E_L is an equivalence relation. It remains to show that E_L is bounded.

Assume that E_L is not bounded. Then there is some $p \notin E_L$ such that $\mathrm{sym}(p)$ is graph-infinite. Since E_L is the intersection of bounded equivalence relations, $p \notin E$ for some bounded equivalence relation E. However, this contradicts $\mathrm{sym}(p)$ being graph-infinite. □

Finally, this is all just an alternate exposition of the usual definition.

Theorem 2.17. *The Lascar equivalence relation, as defined in Definition 2.14, induces the usual Lascar equivalence relation in any sufficiently large model $\mathcal{U}$.*

Proof. This is a direct combination of the preceding results. □

3 Maximally Distinct Parameter Sets

In this section, let T be a complete theory and xEy be an arbitrary bounded equivalence relation. (We will focus later on the special case of E_L.) We will discuss abstract parameter sets without those parameters being in any specific model. Since we will nevertheless have enough to talk about types of these parameters, E will yield an equivalence relation on a given parameter set. We will be interested in the case when these parameters are all from distinct equivalence classes. The boundedness of E will allow us to extend elementarily any such parameter set such into a maximal one.

Definition 3.1. A *parameter set* is a set C of fresh constant symbols, and an associated extension of T to a complete theory in the language with these new symbols added.

Remark 3.2. If c is an n-tuple in some parameter set C, the set of formulas $\phi(x)$ in the original language which hold of c in the extended theory is a type in $S_n(T)$. Therefore any relation present in T yields a relation on any parameter set.

Definition 3.3. Let C, D be parameter sets. Then $C \preceq D$, that is, D is an *elementary extension* of C, if $C \subseteq D$ and the complete theory associated with D extends the complete theory associated with C.

Remark 3.4. It is clear that elementary extension of parameter sets works very much like elementary extension of models. In particular, it defines a partial order, and every chain has a least upper bound.

Definition 3.5. A parameter set is *E-distinct* if every element of the parameter set is in a distinct equivalence class.

Proposition 3.6. *Let C be an E-distinct parameter set. Then C extends elementarily into an (not necessarily canonical) E-distinct parameter set D that is maximal such.*

Proof. By Corollary 2.13.1, we can assume without loss of generality that all parameter sets that matter have an underlying set that is a subset of some fixed cardinal λ. In particular, under this assumption, there is a set containing all parameter sets that matter. We already have, from Remark 3.4, that every chain has a least upper bound. The least upper bound of a chain of E-distinct parameter sets must also be E-distinct, since any pair of parameters occurs already somewhere in the chain. Therefore Zorn's Lemma provides the desired result. $\square$

4 The Galois Groupoid

In this section, T will again be a complete theory and xEy a bounded equivalence relation present in T. There is a natural connected groupoid associated with this equivalence relation. The hom-sets of this groupoid arise as projections of Stone spaces, and the automorphism group of any object is isomorphic to the usual Galois group associated with the relation.

Definition 4.1. The *Galois groupoid* associated with E, $Gal_E(T)$, has as objects the maximal E-distinct parameter sets, and has hom-sets and compositions defined as follows. For C, D objects, let $S(T \cup C \cup D)$ denote the space of completions of the theories associated with C and D (with C and D viewed as disjoint sets of parameters). By Lemma 4.1.1, under any theory in $S(T \cup C \cup D)$, $E \cap (C \times D)$ is a bijection from C to D. Define $Hom(C, D)$ to be the bijections that arise in this way. By Lemma 4.1.2, the composition of two such bijections is in the relevant hom-set. By Lemma 4.1.3 and Lemma 4.1.4, respectively, $Gal_E(T)$ has identities and inverses, and so is a groupoid. By Lemma 4.1.5, $Gal_E(T)$ is connected.

Lemma 4.1.1. *Let C, D be maximal E-distinct parameter sets, and fix a complete theory in $S(T \cup C \cup D)$. Then, under this theory, $E \cap (C \times D)$ is a bijection from C to D.*

Proof. Let $c \in C$. It must be that cEd for at least one $d \in D$; otherwise, D would not be maximally E-distinct. However, since no two $d_1, d_2 \in D$ are equivalent, then cEd for *exactly* one $d \in D$. Since this situation is symmetric in C and D, $E \cap (C \times D)$ is a bijection. □

Lemma 4.1.2. *Let B, C, D be maximal E-distinct parameter sets, and let $f \in Hom(B, C)$, $g \in Hom(C, D)$. Then $gf \in Hom(B, D)$.*

Proof. Take $p \in S(T \cup B \cup C)$, $q \in S(T \cup C \cup D)$ yielding f and g, respectively. Consider the theory $[p, q] \subseteq S(T \cup B \cup C \cup D)$. Note that both p and q restrict to the same theory in the language with constant symbols only from C (that is, the completion of T associated with the parameter set C). The theory $[p, q]$ is therefore the extension of that theory the new sentences from p and q, each mentioning disjoint sets of new constants. Since p and q are individually consistent, $[p, q]$ is consistent. Therefore there is some $r \in S(B \cup C \cup D)$ extending both p and q. In this theory, $E \cap (B \times D)$ is the composition of $E \cap (C \times D)$ with $E \cap (B \times C)$. Therefore the composition gf arises from $r \restriction_{B \cup D}$, so $gf \in Hom(B, D)$. □

Lemma 4.1.3. *Let C be a maximal E-distinct parameter set. The identity function is in $Hom(C, C)$.*

Proof. Take $\{c_{1,\alpha}\}$ and $\{c_{2,\alpha}\}$ the same enumeration of C_1 and C_2, two disjoint copies of C. Consider the theory $[c_{1,\alpha} = c_{2,\alpha}] \subseteq S(T \cup C_1 \cup C_2)$. This is clearly a complete, consistent theory which yields the identity function. □

Lemma 4.1.4. *Let C, D be maximal E-distinct parameter sets, and let $f \in Hom(C, D)$. Then $f^{-1} \in Hom(D, C)$.*

Proof. Take $p \in S(T \cup C \cup D) = S(T \cup D \cup C)$ such that, under p, $E \cap (C \times D)$ is f. Since E is symmetric, also under p, $E \cap (D \times C)$ is $f{-}1$. □

Lemma 4.1.5. *Let C, D be maximal E-distinct parameter sets. Then $Hom(C, D)$ is not empty.*

Proof. Note that $S(T \cup C \cup D)$ is the extension of T by consistent theories mentioning disjoint sets of new parameters. $S(T \cup C \cup D)$ is therefore consistent, so contains at least one point. This point yields a morphism from C to D. □

This concludes the combinatorial construction of the Galois groupoid for any bounded equivalence relation present in a theory. The construction is completely canonical in that it does not rely on any arbitrary choices, such as the choice of a sufficiently large model. As such, for example, there should be no hindrance to converting this construction into a functor between appropriate categories. Furthermore, the construction makes explicit the relationship between equivalence relations, the groupoids, and relevant Stone spaces, and should facilitate investigations into topological structure.

Since the automorphism groups of objects in connected groupoid are all isomorphic, and since $Gal_E(T)$ is nonempty, $Gal_E(T)$ determines a unique isomorphism class of groups. The only remaining business in this exposition is to show that this isomorphism class is the same as the one obtained in the traditional, non-canonical construction.

Theorem 4.2. *Let E be a bounded equivalence relation present in a complete theory T, and let $\mathcal{U} \models T$ be sufficiently large. Let C be a maximal E-distinct parameter set. Then there is a surjective group homomorphism from $Aut(\mathcal{U})$ to $Aut(C)$ whose kernel is exactly $Fix(E)$, the automorphisms fixing E-equivalence classes.*

Proof. Since $\mathcal{U}$ is sufficiently saturated, we may take an elementary embedding of C into $\mathcal{U}$, and thus view C as a set of parameters in $\mathcal{U}$. For any $f \in Aut(\mathcal{U})$, by identifying C with C_1 and $f(C)$ with C_2, two disjoint copies of C, we may view $\text{tp}(C, f(C)) \in S(T \cup C_1 \cup C_2)$. Define the bijection arising from this type (in the sense of Lemma 4.1.1) to be $\Phi(f)$.

Since Φ is really a proxy for viewing how elements of $Aut(\mathcal{U})$ permute equivalence classes, it is immediate that Φ is a group homomorphism and that $\ker(\Phi) = Fix(E)$. Φ is surjective because $\mathcal{U}$ is sufficiently homogeneous. $\square$