COMPUTABILITY:

ANSWERS TO EXERCISES IN CHAPTER 7

7.1.4, P. 122.

1. (a) Notice that $x \in A$ iff $2x \in A \oplus B$ and $x \in B$ iff $2x+1 \in A \oplus B$. So

$$c_A(x) = c_{A\oplus B}(2x) \text{ and } c_B(x) = c_{A\oplus B}(2x+1),$$

from which it follows that, when $A \oplus B$ is recursive, so are A and B. We know that the set $\mathbf{E}$ of even numbers is recursive; (see Example 1.2 (b) above). The set $2A \equiv \{2x : x \in A\}$ can be described as all those $y \in \mathbf{E}$ such that $y/2 \equiv [y/2] \in A$. So $c_{2A}(x) = c_{\mathbf{E}}(x)c_A([x/2])$. Thus, if A is recursive, then so is $2A$. Similarly, if B is recursive, then so is $2B+1 \equiv \{2x+1 : x \in B\}$ being recursive; (note that $c_{2B+1}(x) = (1 - c_{\mathbf{E}}(x))c_B([(x-1)/2]))$. Thus, if both A and B are recursive, so is $2A \cup (2B+1) \equiv A \oplus B$; (notice that $c_{A\oplus B} = c_{2A} + c_{2B+1}$).

(b) We know that the pairing function π is a bijection from $\mathbf{N} \times \mathbf{N}$ onto $\mathbf{N}$ and the components π_1 and π_2 of its inverse π^{-1}, which satisfy the relation $\pi(\pi_1(x), \pi_2(x)) = x$, are computable; (see Exercise 2.4.16-2 on P. 41). Since A and B are assumed to be nonempty, we are allowed to pick some a in A and some b in B. Notice that $x \in A$ if and only if $\pi(x,b) \in A \otimes B$. Thus $c_A(x) = c_{A\otimes B}(\pi(x,b))$. Similarly, $c_B(y) = c_{A\otimes B}(\pi((a,y))$. These identities show that if $A \otimes B$ is recursive, then so are A and B. The reverse implication follows from the identity

$$c_{A\otimes B}(x) = c_A(\pi_1(x))c_B(\pi_2(x)),$$

in view of the fact that $x \in A\otimes B$ if and only if $\pi_1(x) \in A$ and $\pi_2(x) \in B$. (Notice that the assumption $A, B \neq \emptyset$ is essential. Otherwise, we may let A be any set and $B = \emptyset$. Then $A \otimes B \equiv \emptyset$ is recursive; but of course this cannot enforce A to be recursive.)

2. (a) The function f on $\mathbf{N}^n$ given by $f(x_1, x_2, \ldots, x_n) = p_1^{x_1}p_2^{x_2}\cdots p_n^{x_n}$ is computable, as well as the charatcteristic function c_B of B. Hence the relation

$$c_M(x_1, \ldots, x_n) = c_B(f(x_1, \ldots, x_n))$$

tells us that c_M is computable and consequently the predicate M is decidable.

(b) Notice that the function f defined above is a bijection from $\mathbf{N}^n$ onto the set of all positive natural numbers having $p_1,\ p_2,\ \ldots,\ p_n$ as only possible prime factors, which will be denoted by P_n. First we show that P_n is recursive. (Here we are actually showing part (b) of the exercise for a special case: $A = \mathbf{N}^n$. Once this is done, the rest is easy.) Recall that the functions $(y)_1,\ (y)_2,\ \ldots,\ (y)_n$ (see 2.4.15 (d) on P. 40) are computable. The predicate '$y \in P_n$' is equivalent to '$y = p_1^{(y)_1} \cdots p_n^{(y)_n}$', or '$|f((y)_1, \ldots, (y)_n) - y| = 0$', and hence is decidable. So P_n is recursive. Let

$$B = \{p_1^{x_1} \cdots p_n^{x_n} \colon (x_1, \ldots, x_n) \in A\}.$$

Using the fact that f is a bijection from $\mathbf{N}^n$ onto P_n, we can check

$$A = \{(x_1, \ldots, x_n) \colon p_1^{x_1} \cdots p_n^{x_n} \in B\}.$$

Part (a) tells us that if B is recursive, then so is A. On the other hand, the identity

$$B = \{y \in P_n \colon ((y)_1, \ldots, (y)_n) \in A\} = P_n \cap \{y \colon ((y)_1, \ldots, (y)_n) \in A\}$$

tells us that if A is recursive, then so is B.

7.2.18, P. 132.

1. Let g be the function given by

$$g(x) = \begin{cases} 1 & \text{if } x = 0; \\ \text{undefined} & \text{if } x \neq 0. \end{cases}$$

   Then g is computable. The partial chracteristic function of ${}^aW_e$ is just

$$g(|\phi_e(x) - a|),$$

   which is compuatable and hence ${}^aW_e$ is r.e. The enumeration ${}^aW_x$ $(x = 0, 1, 2, \ldots)$ includes all r.e. sets. In fact, for any r.e. set A, $a\chi_A$ is computable, where χ_A stands for the partial characteristic function of A. So $a\chi_A = \phi_n$ for some n. Clearly $A = {}^aW_n$.

2. This is because 'ϕ_x is not injective' iff '$\exists y\ \exists z\ M(x, y, z)$', where

$$M(x, y, z) \ \equiv\ \text{'}y \neq z \text{ and } \psi_U(x, z) = \psi_U(x, z)\text{'},$$

which is partially decidable. Now we can apply Theorem 2.5 on P. 124.

3. Because $\psi_U(x,y)$ is computable, its domain $\{(x,y) \in \mathbf{N}^2 : y \in W_x\}$ is r.e. In other words, the predicate '$y \in W_x$' is partially decidable. So the function f given by

$$f(x,y) = \begin{cases} y & \text{if } y \in W_x, \\ \text{undefined} & \text{otherwise} \end{cases} \tag{$*$}$$

is computable. Putting $f_x(y) = f(x,y)$, we see that $\text{Dom}(f_x) = \text{Range}(f_x) = W_x$. The $s-m-n$ theorem tells us that there is a total computable function k such that $f(x,y) = \phi_{k(x)}(y)$, or $f_x = \phi_{k(x)}$. Thus

$$E_{k(x)} = \text{Range}(\phi_{k(x)}) = \text{Range}(f_x) = W_x.$$

Also, according to Theorem 2.5 on P. 124, the $\text{Range}(\psi_U) \equiv \{(x,y) \in \mathbf{N}^2 \colon y \in E_x\}$, is also r.e. Thus the function $g(x,y)$ defined in the same way as $f(x,y)$ in ($*$), with $y \in W_x$ replaced by $y \in E_x$, is computable. The same argument gives us a total recursive function ℓ such that $g(x,y) = \phi_{\ell(x)}(y)$ and $W_{\ell(x)} = \text{Dom}(\phi_{\ell(x)}) = \text{Dom}(g_x) = E_x$.

4. In the last exercise we have seen that '$y \in W_x$' is partially decidable. So

$$\text{`}y \in \bigcup_{x\in A} W_x\text{'} \quad \equiv \quad \text{`}\exists x \ (x \in A \text{ and } y \in W_x)\text{'}$$

is also partially decidable. Hence $\bigcup_{x\in A} W_x$ is r.e. That $\bigcup_{x\in A} E_x$ is r.e. follows from

$$\text{`}y \in \bigcup_{x\in A} E_x\text{'} \quad \equiv \quad \exists x \ \exists z \ (x \in A \text{ and } \psi_U(x,z) = y).$$

(That '$x \in A$ and $\psi_U(x,z) = y$' is partially decidable follows from the facts that A is r.e. and ψ_U is computable.) Note that

$$\text{`}x \in K\text{'} \equiv \text{`}x \in W_x\text{'} \equiv \text{`}P_x(x)\downarrow\text{'} \equiv \text{`}\exists t \ P_x(x)\downarrow \text{ in } t \text{ steps'} \equiv \text{`}\exists t \ x \in K_t\text{'}.$$

So $K = \bigcup_{t\in\mathbf{N}} K_t$. As we know, $\overline{K} = \bigcap_{t\in\mathbf{N}} \overline{K}_t$ is not r.e.; (see Example 2.2.1 on P. 123). But from Corollary 1.3 (b) on P. 88 we can deduce that K_t, and hence $\overline{K}_t$, is recursive.

5. Suppose that A is r.e. Then its partial characteristic function χ_A is computable. Clearly we have $g(x) = f(x)\chi_A(x)$, and hence g is computable. Conversely, suppose that g is computable. Then $\text{Dom}(g)$ is r.e. But, since $A \subseteq \text{Dom}(f)$, $\text{Dom}(g)$ is nothing but A.

6. Clearly, if f is computable, then so is $g(x) = 2^x 3^{f(x)}$ and hence its range is r.e. But the range of g is just the set

$$G = \{2^x 3^{f(x)} \colon x \in \mathrm{Dom}(f)\}.$$

(The set G here plays the role of "the graph of f".) Conversely, suppose that G is r.e. By Theorem 2.7, there is a total computable function h such that $\mathrm{Range}(h) = G$. Define the functions h_1 and h_2 by putting $h_1(z) = (h(z))_1$ and $h_2(z) = (h(z))_2$. Then both h_1 and h_2 are total computable functions. It is easy to check that $\mathrm{Range}(h_1) = \mathrm{Dom}(f)$ and $\mathrm{Range}(h_2) = \mathrm{Range}(f)$. From Exercise 5.1.5-1 on P. 90 we see tha there is a computable function g with $\mathrm{Dom}(g) = \mathrm{Range}(h_1) \equiv \mathrm{Dom}(f)$ such that $h_1(g(x)) = x$. The function $h_2(g(x))$ is computable with $\mathrm{Dom}(h_2 \circ g) = \mathrm{Dom}(g) = \mathrm{Dom}(f)$ (because h_2 is a total function). So it remains to check that $h_2(g(x)) = f(x)$ for all $x \in \mathrm{Dom}(f)$. Let $x \in \mathrm{Dom}(f)$. Since $h(g(x)) \in G$ (because $\mathrm{Range}(h) = G$), we have $h(g(x)) = 2^\xi 3^{f(\xi)}$ for some $\xi \in \mathrm{Dom}(f)$. Thus

$$\xi = (2^\xi 3^{f(\xi)})_1 = h_1(g(x)) = x \quad \text{and} \quad f(x) = f(\xi) = (2^\xi 3^{f(\xi)})_2 = h_2(g(x)).$$

Thus $f(x) = h_2(g(x))$, which is computable.

7. By Theorem 2.7 on P. 125, $A = \mathrm{Range}(f)$ for some total computable function f. By Exercise 5.1.5-1 on P. 90 we see that there is a computable function g such that

$$\mathrm{Dom}(g) = \mathrm{range}(f) \equiv A \quad \text{and} \quad f(g(y)) = y \text{ for all } y \in A. \tag{$*$}$$

Then the function $g(f(x))$ is total and computable. Hence the set

$$C = \{x \colon g(f(x)) = x\}$$

is decidable. By Theorem 2.14 on P. 129, we see that there is an injective, total and computable function h such that $\mathrm{Range}(h) = C$. Let $k(z) = f(h(z))$. Then clearly k is a total computable function. We have to check that k is injective and $\mathrm{Range}(k) = A$. Suppose $k(x_1) = k(x_2)$, or $f(h(x_1)) = f(h(x_2))$. Then $g(f(h(x_1))) = g(f(h(x_2)))$, or $h(x_1) = h(x_2)$ by $(*)$. Since h is injecticve, $x_1 = x_2$. Next, $\mathrm{Range}(k) = \mathrm{Range}(f \circ h) \subseteq \mathrm{range}(f) = A$. Suppose $y \in A$. Then $y \in \mathrm{Dom}(g)$ and hence we may consider $z = g(y)$. Now $g(f(z)) = g(f(g(y)) = g(y) = z$ in view of $(*)$ agian. So $z \in C$. Hence there exists x such that $h(x) = z$, Thus $k(x) = f(h(x)) = f(z) = f(g(y)) = y$. This shows $\mathrm{Range}(k) = A$. Done.

8. (a) The predicate '$x \in E_x$' is undecidable; (see Exercise 1.8.11 (a) on P. 106). However, since $\psi_U(x, z)$ is computable, the predicate '$\psi_U(x, z) = y$' is partially decidable and hence so is '$\exists z\ (\psi(x, z) = y)$', which is equivalent to '$y \in E_x$'. So

$$\{(x, x) \colon x \in E_x\} \equiv \{(x, y) \colon x = y\} \cap \{(x, y) \colon y \in E_x\}$$

is r.e. Hence $\{x : x \in E_x\}$ is r.e. The complement of $\{x : x \in E_x\}$ cannot be r.e.; otherwise, in view of Theorem 6.6.11 on P. 117, '$x \in E_x$' would be decidable.

(b) This set is recursive because the predicate 'x is a perfect square' is decidable.

(c) From Exercise 2 above we know that this set is r.e. But both itself and its complement $\{x : \phi_x \text{ is not injective}\}$ are not recursive. To prove this, apply Rice's Theorem to $\mathcal{B}_1 = \{f \in \mathcal{C}_1 \colon f \text{ is injective}\}$ and $\mathcal{B}_2 = \{f \in \mathcal{C}_2 \colon f \text{ is not injective}\}$.

(d) The set here is either $\{0, 1, \ldots, n\}$ for some n or $\mathbf{N}$. At the present stage we have no knowledge about the exact nature of this set. But no matter which one occurs to be, it is recursive and so is its complement.

(e) This is an ambiguous question. The answer really depends on our knowledge about the program P_m. If P_m is a program for computing a total computable function, then the set $\{x \colon P_m(x) \uparrow\}$ is void and hence is decidable. On the other hand, take any partially decidable but undecidable predicate $M(x)$ (such as '$x \in W_x$') and let f be its partial characteristic function. Then 'not $M(x)$' is not partially decidable. Since f is computable, $f = \phi_m$ for some m. Now P_m is the program for computing $\phi_m = f$ and hence

$$\{x \colon P_m(x) \uparrow\} = \{x \colon f(x) \text{ is undefined}\} = \{x \colon \text{ not } M(x)\}$$

is not r.e. So the answer depends on which P_m is given.

9. (a) The same proof as that for Exercise 1.4-2 (a) on P. 123 works, with characteristic functions replaced by partial characteristic functions.

(b) Suppose that A is r.e. Then the function

$$g(x_1, \ldots, x_n) = \begin{cases} p_1^{x_1} \cdots p_n^{x_n} & \text{if } (x_1, \ldots, x_n) \in A, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

is computable. So the set

$$\text{Range}(g) = B \equiv \{p_1^{x_1} \cdots p_n^{x_n} \colon \ (x_1, \ldots, x_n) \in A\}$$

is r.e. Conversely, suppose that B is r.e. Then, from part (a) above and the identity (used in the answer to Exercise 1.4-2 on P. 123)

$$A = \{(x_1, \dots, x_n) \colon\ p_1^{x_1} \cdots p_n^{x_n} \in B\},$$

we see that A is r.e.

(c) Suppse that A is r.e. and $A \neq \emptyset$. Then A is the range of some total computable function, say $h \colon \mathbf{N} \to \mathbf{N}$. Let $f_k(x) = (h(x))_k$ $(1 \leq k \leq n)$. Then f_k is computable and A is the range of the total computable function $f = (f_1, \dots, f_n)$. Conversely, suppose that $A = \text{Range}(f)$ for some computable $f = (f_1, \dots, f_n)$. Then $g \circ f$ is computable; (g is the function given in part (a) above). So Range$(g \circ f)$ is r.e. But Range$(g \circ f)$ is just B. So, by part (b), we see that A is r.e.

10. By assumption, c_A and χ_B are computable. That $f^{-1}(A)$ is recursive and $f^{-1}(B)$ is r.e. follow from the identities

$$c_{f^{-1}(A)}(x) = c_A(f(x)) \quad \text{and} \quad \chi_{f^{-1}(B)}(x) = \chi_B(f(x)).$$

By Exercise 5, we know that $f|B$ is computable and hence its range is r.e. But the range of $f|B$ is just $f(B)$; (recall that f is total here). So $f(B)$ is recursive. In case f is bijective, A is recursive (r.e.) iff $f(A)$ is recursive (r.e.).

11. It is enough to check that, in each part of the exercise, the following condition fails for the relevant set $\mathcal{A}$ of unary functions:

$$f \in \mathcal{A} \ \text{ iff } \ \text{there is a finite function } \theta \in \mathcal{A} \text{ such that } \theta \subseteq f.$$

(a) $\mathcal{A} = \{f_\emptyset\}$, that is, $\mathcal{A}$ consists of the single element $f_\emptyset$. The above condition certainly fails since the inclusion $f_\emptyset \subseteq f$ holds for any f!

(b) $\mathcal{A} = \{f \in \mathcal{C}_1 : \text{Dom}(f) \text{ is finite}\}$. Take any total computable function f and any finite set B in $\mathbf{N}$ and let $\theta = f|B$. Then $f \notin \mathcal{A}$, but $\theta \in \mathcal{A}$ and $\theta \subseteq f$.

(c) Let $\mathcal{A} = \{f \in \mathcal{C}_1 : \text{Dom}(f) \text{ is infinite}\}$. Then $\mathcal{A}$ does not contain any finite function. *A fortiori*, for any $f \in \mathcal{A}$, there is no finite function θ in $\mathcal{A}$ such that $\theta \subseteq f$.

(d) Let $\mathcal{A} = \{\mathbf{0}\}$. Again, $\mathcal{A}$ does not have any finite function and the same argument as the previous one works.

(e) Let $\mathcal{A}$ be the set of all nonzero computable functions:

$$\mathcal{A} \ = \ \{f \in \mathcal{C}_1 \colon\ f \neq \{\mathbf{0}\}\} \ = \ \mathcal{C}_1 \setminus \{\mathbf{0}\}.$$

For any finite set B, the finite function $\theta = \mathbf{0}|B$ is in $\mathcal{A}$ and $\theta \subseteq \mathbf{0}$, but $\{\mathbf{0}\} \notin \mathcal{A}$.

12. Suppose that $\phi_x \in \mathcal{B}$ is decidable. Hence both $\phi_x \in \mathcal{B}$ and $\phi_x \notin \mathcal{B}$ are partially decidable. By Rice-Shapiro's theorem, the condition

$$\text{“}f \in \mathcal{A} \text{ iff } \theta \subseteq f \text{ for some finite } \theta \text{ in } \mathcal{A}\text{”}$$

is satisfied for both $\mathcal{A} = \mathcal{B}$ and $\mathcal{A} = \overline{\mathcal{B}}$. In case $f_\emptyset \in \mathcal{B}$, from $f_\emptyset \subseteq f$ for all $f \in \mathcal{C}_1$ we get $f \in \mathcal{B}$ for all $f \in \mathcal{C}_1$, that is, $\mathcal{B} = \mathcal{C}_1$. In case $f_\emptyset \notin \mathcal{B}$, we have $f_\emptyset \in \overline{\mathcal{B}}$ and the same argument shows that $\overline{\mathcal{B}} = \mathcal{C}_1$, which tells us that $\mathcal{B} = \emptyset$.

13. (a) (This is a very interesting Exercise.) Suppose the contrary that the sets

$$K_0 = \{x\colon \phi_x(x) = 0\} \text{ and } K_1 = \{x\colon \phi_x(x) = 1\}$$

are recursively separable: there is a recursive set R such that $K_0 \subseteq R$ and $K_1 \subseteq \overline{R}$. Then c_R is computable and hence $c_R = \phi_m$ for some m. If $m \in R$, then $\phi_m(m) = c_R(m) = 1$ and hence $m \in K_1$. This is impossible, since R and K_1 do not intersect. If $m \notin R$, then $m \in \overline{R}$ and hence $\phi_m(m) = c_R(m) = 0$, that is, $m \in K_0$. This is impossible because $K_0 \subseteq R$.

(b) For each x, since ϕ_x is computable, its domain W_x is r.e. So, in case $W_a \cap W_b = \emptyset$ and $W_a \cup W_b = \mathbf{N}$, it follows from Theorem 6.6.11 on P. 117 that W_a and W_b are recursive. Thus the “only if” part is clear. To show that “if” part, we suppose the contrary that A and B are recursively separable: there is a recursive set R such that $A \subseteq R$ and $B \subseteq \overline{R}$. Then χ_R and $\chi_{\overline{R}}$ are computable and hence there exist a and b in $\mathbf{N}$ such that $\phi_a = \chi_R$ and $\phi_b = \chi_{\overline{R}}$. Now $W_a = R$, which includes A and $W_b = \overline{R}$, which includes B.

7.3.13, P. 138.

1. In view of Theorem 3.4 on P. 135, it is enough to check that in each part of the present exercise, the relevant set $\mathcal{B}$ of computable functions satisfies the condition $\mathcal{B} \neq \mathcal{C}_1$ and $f_\emptyset \in \mathcal{B}$. This is easy to do:

   (a) $\mathcal{A} = \{f \in \mathcal{C}_1 \colon \mathrm{Dom}(f) \text{ is finite}\}$.

   (b) $\mathcal{A} = \{f \in \mathcal{C}_1 \colon f \text{ is not surjective}\}$.

   (c) $\mathcal{A} = \{f \in \mathcal{C}_1 \colon f \text{ is injective}\}$.

(d) $\mathcal{A} = \{f \in \mathcal{C}_1 \colon f$ is not a polynomial function$\}$.

2. For part (b) and part (c), according to Theorem 3.8 on P. 136, it is enough to check that $A_b \equiv \{x \colon E_x^{(n)} \neq \emptyset\}$ and $A_c \equiv \{x \colon \phi_x$ is not injective$\}$ are r.e. and $A_b,\ A_c \neq \emptyset,\ \mathbf{N}$. Note that

$$`x \in A_b\text{'} \ \equiv\ `\exists y_1\ \cdots\ \exists y_n\ \exists z\ (\psi_U^{(n)}(x, y_1, \ldots, y_n) = z)\text{'},$$

which is a partially decidable predicate. So A_b is r. e. From Exercise 2.18-2 on P. 132 we know that A_c is r.e. That $A_b,\ A_c \neq \emptyset,\ \mathbf{N}$ is obvious. Thus we have established part (b) and part (c).

(a) We know that $A_a = \{y \colon y \in E_y\}$ is r.e. (see the answer to Exercise 2-18-8 (a) on P. 132). On the other hand, we know that there is a total computable function k such that $x \in W_x$ iff $k(x) \in E_{k(x)}$; see the second solution to Exercise 6.1.8-1 (a) on P. 106. Then, $x \notin W_x$ iff $k(x) \notin E_{k(x)}$, or

$$x \in \overline{K} \ \text{ iff } \ k(x) \in \mathbf{N} \setminus \{y \in E_y\} = \overline{A_a}.$$

Since $\overline{K}$ is productive, so is $\overline{A_a}$, in view of Theorem 3,2 on P. 134. Therefore A_a is creative.

(d) First we show that the set $B = \{x \colon\ \phi_x(x) \in A\}$ is r.e. The function $f(x) = \psi_U(x, x) \equiv \phi_x(x)$ (with $\{x \colon x \in W_x\}$ as its domain) is computable. Hence $\chi_B(x) = \chi_A(f(x))$ is also computable. Next we show that $\overline{B} \equiv \{x \colon\ \phi_x(x) \notin A\}$ is productive; (here $\phi_x(x) \notin A$ means either $x \notin W_x$ or $x \in W_x$ but $\phi_x(x)$ is not in A).

To accomplish this, we use one of the most abominable tricks in this book, which is given in Example 3.7 (c). You should study this example carefully, at least five times, to get the secret behind this trick. [To be honest, without the guidance of this example, I don't have the slightest idea how to do this exercise.]

Since A is r.e. and nonempty, there is a total computable function g such that $A = \text{Range}(g)$. Applying the *s-m-n* theorem to the computable function $g(\phi_x(y)) \equiv g(\psi_U(x, y))$, we get a total computable function k such that

$$g(\phi_x(y)) = \phi_{k(x)}(y).$$

Notice that $y \in W_x$ iff $y \in W_{k(x)}$ and $\phi_{k(x)}(y) \in A$. This we have

$$k(x) \in W_x \quad \Leftrightarrow \quad k(x) \in W_{k(x)} \textit{ and } \phi_{k(x)}(k(x)) \quad \Leftrightarrow \quad k(x) \in B.$$

From this we see that, in case $W_x \subseteq \overline{B}$, we must have $k(x) \in \overline{B} \setminus W_x$. [If $k(x) \in W_x$, then $k(x) \in B$, contradicting $W_x \subseteq \overline{B}$. If $k(x) \notin \overline{B}$, then $k(x) \in B$ and hence $k(x) \in W_x$, again contradicting $W_x \subseteq \overline{B}$.] Hence k is a productive function for $\overline{B}$.

(e) First we show the easy part: $A \equiv \{x \colon \phi_x(x) = f(x)\}$ is r.e. Let H be the singleton set $\{0\}$ and h be its partial characteristic function. Then h is computable and hence so is $g(x) = h(|\psi_U(x,x) - f(x)|)$. The set A is just the domain of g and hence is r.e. Now we deal with the hard part: $\overline{A}$ is productive. Applying the s-m-n theorem, we can find a total computable function k such that

$$\phi_{k(x)}(y) = \mathbf{0}(\phi_x(y)) + f(x).$$

Then we have: $y \in W_x$ iff $y \in W_{k(x)}$ and $\phi_{k(x)}(y) = f(x)$. Thus

$$k(x) \in W_x \quad \Leftrightarrow \quad k(x) \in W_{k(x)} \text{ and } \phi_{k(x)}(k(x)) = f(x) \quad \Leftrightarrow \quad k(x) \in A.$$

From this we can deduce that k is a productive function for A.

3. (Tricky.) Let f be a productive function for $A \cap B$: whenever $W_x \subseteq A \cap B$, we have $f(y) \in A \cap B \setminus W_y$. [Motivation for the next step: To find a production function for A, we have to consider the situation $W_x \subseteq A$, which gives $W_x \cap B \subseteq A \cap B$. Hence it is desirable to put $W_x \cap B$ in the form of W_y.] Since B is r.e., its partial characteristic function χ_B is computable and consequently the function $\psi_U(x,z)\chi_B(z)$ is computable. By the s-m-n theorem, we know that there is a total computable function k such that $\phi_{k(x)}(z) = \psi_U(x,z)\chi_B(z) = \phi_x(z)\chi_B(z)$. Notice that

$$W_{k(x)} = \mathrm{Dom}(\phi_{k(x)}) = \mathrm{Dom}(\phi_x) \cap \mathrm{Dom}(\chi_B) = W_x \cap B.$$

Define a total computable function g by putting $g(x) = f(k(x))$. Then, when $W_x \subseteq A$, we have $W_{k(x)} \equiv W_x \cap B \subseteq A \cap B$ and consequently

$$g(x) = f(k(x)) \in A \cap B \setminus W_{k(x)} = A \cap B \setminus W_x \cap B = (A \setminus W_x) \cap B \subseteq A \setminus W_x.$$

Done.

4. (Another tricky question.) Let f be a productive function for $\overline{C}$: $f(y) \in \overline{C} \setminus W_y$ whenever $W_y \subseteq \overline{C}$. Since both C and A are r.e., so is $C \cup A$. Thus, all we have to do is to show that $\overline{C \cup A} \equiv \overline{A} \cap \overline{C}$ is productive.

[Motivation for the following argument: To find a productive function for $\overline{C \cup A}$, we consider the situation $W_x \subseteq \overline{C \cup A}$. We have three disjoint sets C, A and W_x, and we try to find $g(x)$ in none of them. Certainly $f(x)$ is in neither C nor W_x, but we cannot avoid $f(x)$ to fall into A. How can we get some $g(x)$ to avoid this to happen? Here is the trick: incorporate A into W_x to get a "new W_x".]

The s-m-n theorem provides a total computable function k such that $W_x \cup A = W_{k(x)}$. The function g defined by $g(x) = f(k(x))$ is total and computable. Suppose that $W_x \subseteq \overline{C \cup A} \equiv \overline{C} \cap \overline{A}$. Then $W_x \cup A \subseteq \overline{C}$, or $W_{k(x)} \subseteq \overline{C}$. Thus

$$g(x) = f(k(x)) \in \overline{C} \setminus W_{k(x)} = \overline{C} \setminus A \cup W_x = \overline{C} \cap \overline{A} \setminus W_x.$$

Done.

5. This is the direct consequence of Theorem 3.11 on P. 137 and Theorem 2.15 on P. 130.

6. Using the argument in the answer to Exercise 1.4.1 on P. 122, using partial characteristic functions instead of characteristc functions, we can show that both $A \oplus B$ and $A \otimes B$ are r.e.

   (a) First we show that $A \oplus B$ is creative under the assumption that A is creative. We know that both $2A$ and $2B + 1$ are r. e. Now $A \oplus B$ is just the union of disjoint sets $2A$ and $2B + 1$. So, according to Exercise 4, it is enough to show that $2A$ is creative. Since $2A$ is r.e., it is enough to show that $\overline{2A}$ is productive. Since $\overline{2A} \cap \mathbf{E} = 2\overline{A}$ ($\mathbf{E}$ is the set of even numbers), according to Exercise 3, it is enough to show that $2\overline{A}$ is productive. Let $f(x) = 2x$. Then $x \in \overline{A}$ iff $f(x) \in 2\overline{A}$. Hence it follows from Theorem 3.2 on P. 134 that $2\overline{A}$ is productive.

   Next we show: $A \otimes B$ is creative under the assumption that A is creative and $B \neq \emptyset$. [Here we should briefly recall the the function $\pi(x, y)$ with $\pi^{-1} = (\pi_1, \pi_2)$, and $A \otimes B = \{\pi(x, y)\colon x \in A \text{ and } y \in B\}$.] Since B is nonempty, we may take an element b_0 in B. Let $B_0 = \{b_0\}$ and $B_1 = B \setminus B_0$. Then both B_0 and B_1 are r.e. and so are $A \otimes B_0$ and $A \otimes B_1$. Clearly

   $$A \otimes B = (A \otimes B_0) \cup (A \otimes B_1) \quad \text{and} \quad (A \otimes B_0) \cap (A \otimes B_1) = \emptyset.$$

   Hence, according to Exercise 4 above, it is enough to show that $A \otimes B_0$ is creative. So it remains to show that $\overline{A \otimes B_0}$ is productive. In order to apply Theorem 3.2 on

P. 134 to the total computable function $f(x) = \pi(x, b_0)$ to do this, we have to check that

$$x \in \overline{A} \quad \text{iff} \quad f(x) \in \overline{A \otimes B_0}.$$

Indeed, if $x \in \overline{A}$, then $f(x) \in \overline{A} \otimes B_0 \subseteq \overline{A \otimes B_0}$. Conversely, suppose we have $f(x) \in \overline{A \otimes B_0}$. Then we must have $x \in \overline{A}$; otherwise we would have $x \in A$ and hence $f(x) \in A \otimes B_0$. Done.

(b) Suppose that $A \oplus B$ is creative and B is recursive. Then $\bar(B)$ is also recursive. Hence $A \oplus \overline{B}$ is r.e. It is easy to check that $(A \oplus B) \cap (A \oplus \overline{B}) = \emptyset$. Hence it follows from the result of Exercise 4 that

$$(A \oplus B) \cup (A \oplus \overline{B}) \equiv A \oplus (B \cup \overline{B}) \equiv A \oplus \mathbf{N} \equiv 2A \cup (2\mathbf{N} + 1)$$

is creative. Hence $\overline{A \oplus \mathbf{N}} \equiv \overline{A} \oplus \mathbf{N} \equiv 2\overline{A} \cup (2\mathbf{N}+1)$ is productive. In view of Theorem 3.2, to show that $\overline{A}$ is productive, it is enough to find a total computable function f such that $x \in 2\overline{A} \cup (2\mathbf{N} + 1)$ iff $f(x) \in \overline{A}$. Take any $b \in \overline{A}$. The following function f will do the job:

$$f(x) = \begin{cases} [x/2] & \text{if } x \in 2\mathbf{N}, \\ b_0 & \text{if } x \in 2\mathbf{N} + 1. \end{cases}$$

Finally, assume that $A \otimes B$ is creative and B is recursive. We use the same argument to show that $A \otimes \overline{B}$ is r.e. and

$$A \otimes \mathbf{N} \equiv (A \otimes B) \cup (A \otimes \overline{B})$$

is creative. To show that A is creative, or more precisely, $\overline{A}$ is productive, in view of Theorem 3.2 on P. 134, it is enough to find a total computable function f such that $x \in \overline{A} \otimes \mathbf{N}$ iff $f(x) \in \overline{A}$. Recall the bijection $\pi \colon \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ in the definition of $A \otimes B$, with $\pi^{-1} = (\pi_1, \pi_2)$ where π_1 and π_2 are computable. The function $f = \pi_1$ will do the job.

7. [Comment: the proof of Rice-Shapiro's theorem is beautiful!] Imitating the proof of Rice-Shapiro's theorem, we define

$$f(x, y) = \begin{cases} \text{undefined} & \text{if } P(x) \downarrow \text{ in } y \text{ or fewer steps.} \\ g(y) & \text{otherwise.} \end{cases}$$

Then f is computable. By the s-m-n theorem, we can find a total computable function s such that $\phi_{s(x)}(y) = f(x, y)$. Note that $\phi_{s(x)} \subseteq f$. Also,

$$\begin{aligned} z \in W_z \quad &\Rightarrow \quad \phi_{s(x)} \text{ is finite} \quad \Rightarrow \quad \phi_{s(x)} \notin \mathcal{B}, \\ z \notin W_z \quad &\Rightarrow \quad \phi_{s(x)} = f \quad \Rightarrow \quad \phi_{s(x)} \in \mathcal{B}. \end{aligned}$$

Thus, letting $B = \{x \colon \phi_x \in \mathcal{B}\}$, we have

$$z \in \overline{K} \quad \text{iff} \quad z \notin W_z \quad \text{iff} \quad s(z) \in B.$$

Since $\overline{K}$ is productive, it follows from Theorem 3.2 on P. 134 that B is also productive.

8. This one is easy! The purpose of this exercise is of course to illustrate the usefulness of the fact in the last exercise! Let $\mathcal{B}$ be any set of total computable functions. Take any $g \in \mathcal{B}$. Then clearly any finite $\theta \subseteq g$ is not in $\mathcal{B}$. So $\{x \colon \phi_x \in \mathcal{B}\}$ is creative. For part (a), we let $\mathcal{B}$ be the set of all total computable functions. For part (b), let $\mathcal{B}$ be the set of all polynomial functions.

9. This is a very hard problem. In order to facilitate our rather complicated argument, first we prove the following lemma: *if A and B are disjoint r. e. sets in $\mathbf{N}^n$, then the function*

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in A, \\ 0 & \text{if } \mathbf{x} \in B, \\ \text{undefined} & \text{if } \mathbf{x} \notin A \cup B \end{cases} \tag{$*$}$$

is computable. For r.e. sets in $\mathbf{N}^n$, see Exercise 2.18-9 on pp. 132-133. By the device described in that exercise, we can reduce the above lemma to the case $n = 1$. So we assume $n = 1$ here. We know that $A \cup B$ is also r.e. In case $A \cup B$ is empty, there is nothing to proof. Let us assume that $A \cup B \neq \emptyset$. By Theorem 2.7 on P. 125, we know that there is a total computable function h such that $A \cup B = \text{Range}(h)$. By Exercise 5.1.5-1 (ii) on P. 90, we see that there is a computable function g with $\text{Dom}(g) = \text{Range}(h) \equiv A \cup B$ such that $h(g(y)) = y$ for all $y \in A \cup B$. We can easily check that g is injective and the range of g is $G = \{x \colon g(h(x)) = x\}$. Notice that $g \circ h$ is a total computable function and hence G is a recursive set. Let $A_0 = G \cap f^{-1}(A)$ and $B_0 = G \cap f^{-1}(B)$. By Exercise 2.18-10 on P. 133, we know that $f^{-1}(A)$ and $f^{-1}(B)$ are r.e. So A_0 and B_0 are also r.e. Notice that A_0 and B_0 are disjoint and their union is G, a recursive set. So A_0 and $B_0 \cup (\mathbf{N} \setminus G)$ are disjoint r.e. sets with $A_0 \cup (B_0 \cup (\mathbf{N} \setminus G)) = \mathbf{N}$. By Theorem 2.6 on P. 124, we conclude that A_0 is recursive; (similarly, B_0 is recursive). So the characteristic function c_{A_0} of A_0 is computable. Notice that the function f given by $(*)$ is equal to $c_{A_0} \circ g$, which is computable. Done.

Returning to the present exercise, let us recall the decidable predicate $T(x, y, z) =$ '$P_x(y) \downarrow (z)_1$ in $(z)_2$ or fewer steps'. Consider the predicate

$$M(u, v, y, z) = \text{'}T(u, y, z) \text{ but not } T(v, y, z)\text{'}$$

which is also decidable. Let $X_{u,v} = \{y : \exists z\ M(u,v,y,z)\}$. Then $X_{u,v}$ is a r.e. set contained in W_u and, for all $a, b \in \mathbf{N}$, we have

$$X_{a,b} \cap X_{b,a} = \emptyset, \quad W_a \setminus W_b \subseteq X_{a,b} \quad \text{and} \quad W_b \setminus W_a \subseteq X_{b,a}.$$

Thus, in case $W_a \cap W_b = \emptyset$, we have $X_{a,b} = W_a$ and $X_{a,b} = W_b$. Let

$$m(u,v,y) = \mu z\ M(u,v,y,z) \quad \text{and} \quad m_{u,v}(y) = m(u,v,y).$$

Then $m(u,v,y)$ is a computable function, and, for all $(u,v) \in \mathbf{N}^2$, $\mathrm{Dom}(m_{u,v}) = X_{u,v}$. The *s-m-n* theorem tells us that there is a total computable function k on $\mathbf{N}^2$ such that $\phi_{k(u,v)}(y) = m(u,v,y)$. Thus $X_{u,v} = \mathrm{Dom}(m_{u,v}) = \mathrm{Dom}(\phi_{k(u,v)}) = W_{k(u,v)}$. The function $\phi(x,u,v) = \psi_U(x, k(u,v))$ is computable

$$\text{‘}x \in X_{u,v}\text{’} \quad \equiv \quad \text{‘}x \in W_{k(u,v)}\text{’} \quad \equiv \quad \text{‘}(x,u,v) \in \mathrm{Dom}(\phi)\text{’}.$$

Since ϕ is computable, its domain is r.e. Thus the sets

$$A = \{(x,u,v) \in \mathbf{N}^3 \colon x \in X_{u,v}\} \quad \text{and} \quad B = \{(x,u,v) \in \mathbf{N}^3 \colon x \in X_{v,u}\}$$

are r. e. Since $X_{u,v} \cap X_{v,u} = \emptyset$ for all (u,v), we have $A \cap B = \emptyset$. By our lemma established at the beginning, the function f given by $(*)$ is computable. By the *s-m-n* theorem, we obtain a computable function $F(u,v)$ such that $\phi_{F(u,v)}(x) = f(x,u,v)$. We can easily check

$$\phi_{F(u,v)}(x) \equiv f(u,v,x) = \begin{cases} 1 & \text{if } x \in X_{u,v}, \\ 0 & \text{if } x \in X_{v,u}, \\ \text{undefined} & \text{if } x \notin X_{u,v} \cup X_{v,u}. \end{cases}$$

Suppose we have $K_0 \equiv \{x : \phi_x(x) = 0\} \subseteq W_a$ and $K_1 \equiv \{x : \phi_x(x) = 1\} \subseteq W_b$ with $W_a \cap W_b = \emptyset$. Then $W_a = X_{a,b}$ and $W_b = X_{b,a}$. If $F(a,b) \in W_a \equiv X_{a,b}$, then $\phi_{F(a,b)}(F(a,b)) = 1$ and hence $F(a,b) \in K_1$, contradictory to the facts that $F(a,b) \in W_a$, $W_a \cap W_b = \emptyset$ and $K_1 \subseteq W_b$. Thus $F(a,b) \notin W_a$. Similarly, $F(a,b) \notin W_b$. We have shown that $F(a,b) \notin W_a \cup W_b$. Therefore K_0 and K_1 are effectively recursively inseparable. This shows part (a).

Part (b) is much easier. Let f be the total computable function described at the beginning of this exercise. Since A and B are r.e., we have $A = W_a$ and $B = W_b$ for some a and b. By Example 5.3.1-3 on P. 93, we know that there is a total computable function s on $\mathbf{N}^2$ such that $W_{s(x,y)} = W_x \cup W_y$. Note that $W_{s(x,b)} = W_x \cup B$. Let

$g(x) = f(a, s(x, b))$. Suppose $W_x \subseteq \overline{A}$. Then A and $W_x \cup B \equiv W_{s(x,b)}$ are disjoint and hence $g(x) \equiv f(a, s(x, b)) \notin W_a \cup W_{s(x,b)} = A \cup B \cup W_x$. In particular, $g(x) \in \overline{A} \setminus W_x$. This shows that g is a productive function for $\overline{A}$. Hence A is creative.

7.4.4, P. 141.

1. Since A and B are r.e., so is $A \oplus B$; (this is because $A \oplus B = (A \oplus \emptyset) \cup (\emptyset \oplus B)$, a union of two r.e. sets). Since $\overline{A}$ and $\overline{B}$ are infinite, so is $\overline{A} \oplus \overline{B}$. Now $\overline{A \oplus B}$ is nothing but $\overline{A} \oplus \overline{B}$ and hence is infinite. Finally, we show that the complement of $A \oplus B$ does not contain any infinite r.e. set. Suppose the contrary that there is an infinite r.e. set C in the complement of $A \oplus B$. Let $C_e = \{x : 2x \in C\}$ and $C_o = \{x : 2x + 1 \in C\}$. Then both C_e and C_o are r. e. Clearly $C_e \subseteq \overline{A}$ and $C_o \subseteq \overline{B}$. Since A and B are creative, C_e and C_o must be finite. Consequently C is also finite, a contradiction.

2. [This exercise gives an alternative proof of Theorem 4.3 on P. 141.] Since the predicate '$y > x$ and $f(y) < f(x)$' is decidable, the predicate '$\exists y\,(y > x$ and $f(y) < f(x))\}$' is partially decidable. Hence the set A is r.e. Next we show that $\overline{A}$ is infinite. Suppose the contrary that $\overline{A}$ is finite. Then there exists some m such that $x < m$ for all $x \in \overline{A}$, or, equivalently, $x \in A$ for all $x \geq m$. Let $y_0 = m$. That $y_0 \in A$ means that there is some $y_1 > y_0 = m$ such that $f(y_1) < f(y_0)$. Since $y_1 > m$, we have $y_1 \in A$ and hence there exists $y_2 > y_1$ such that $f(y_2) > f(y_1)$. Continue in this manner. We find a sequence $y_0 < y_1 < y_2 < y_3 < \cdots$ such that

   $$f(y_0) > f(y_1) > f(y_2) > f(y_3) > \cdots \quad .$$

   This is impossible, because there is no strictly decreasing infinite sequence in $\mathbf{N}$.

   Finitely, we show that $\overline{A}$ does not contain an infinite r.e. set, which is the hardest part. Suppose the contrary that there is an infinite r.e. set B in $\overline{A}$. Take any z. Since B is infinite and f is injective, there exists some $w \in B$ such that $f(w) > z$. The problem is: how can we choose w so that it is computable as a function of z? The assumption that B is r.e. does not seem to be good enough to do this directly! If B were recursive, this would be easy. But here a weaker assumption on B is made. Well, this indicates that we need a trick. Here it is:

   Since B is r.e., there is a total computable function g such that $B = \text{Range}(g)$. Define

   $$h(z) = \mu w \text{ `} f(g(w)) > z\text{'}.$$

Then h is a total computable function; (that h is total was explained in the last paragraph). By the way h is defined, $f(g(h(z))) > z$. Since $g(h(z)) \in B$, we have $g(h(z)) \notin A$, which means that $f(y) \geq f(g(h(z)))$ for all $y > g(h(z))$. In view of $f(g(h(z))) > z$, we see that $f(y) > z$ for all $y > g(h(z))$. [This tells us that to answer the question 'is $z = f(y)$ for some y?', there is no need to consider y beyond $g(h(z))$.] Thus we can write the characteristc function of A as

$$c_A(z) = \overline{\mathrm{sg}}(k(x)), \quad \text{where} \quad k(x) = \prod_{y=0}^{g(h(z))} |f(y) - z|,$$

which is a computable function. But this tells us that $A \equiv \mathrm{Range}(f)$ is recursive, a contradiction.

3. Since A is r. e., so is $A \otimes \mathbf{N}$. Indeed, this follows from the identity $\chi_{A \otimes \mathbf{N}}(x) = \chi_A(\pi_1(x))$. Since A is not recursive, in view of Exercise 7.1.4-1 (b) on P. 122, neither is $A \otimes \mathbf{N}$. Since A is not creative (and $\mathbf{N}$ is recursive), it follows from Exercise 3.13-6 (b) on P. 139 that $A \otimes \mathbf{N}$ is not creative. Finally, take any element $b \in \overline{A}$. Then $\{b\} \otimes \mathbf{N}$ is an infinite recursive set in $\overline{A \otimes \mathbf{N}}$ and hence $A \otimes \mathbf{N}$ is not simple.

4. That $A \otimes B$ is not simple can be proved in the same way as the last exercise. Take any element d not in A. Then $\{d\} \otimes \mathbf{N}$ is an infinite recursive set in the complement of $A \otimes B$.

   The set $C = \overline{\overline{A} \otimes \overline{B}}$ can be rewritten as $C = (A \otimes \mathbf{N}) \cup (\mathbf{N} \otimes B)$. Since A and B are r.e., so are $A \otimes \mathbf{N}$ and $\mathbf{N} \otimes B$. Hence C is r.e. Since $\overline{A}$ and $\overline{B}$ are infinite sets, so is $\overline{C} \equiv \overline{A} \otimes \overline{B}$. Finally, we show that $\overline{C}$ does not contain any infinite r.e. set. Suppose that contrary that there is an infinite r.e. set D in $\overline{A} \otimes \overline{B}$. Since π_1 and π_2 are total computable functions, $D_1 = \pi_1(D)$ and $D_2 = \pi_2(D)$ are also r.e. (To see that, write $D = \mathrm{Range}(f)$ where f is a total computable function. Then $D_1 = \mathrm{Range}(\pi_1 \circ f)$ and $D_2 = \mathrm{Range}(\pi_2 \circ f)$.) Clearly $D_1 \subseteq \overline{A}$ and $D_2 \subseteq \overline{B}$. Since A and B are simple sets, D_1 and D_2 are finite. It is easy to check that $D \subseteq D_1 \otimes D_2$ and hence D must be finite, a contradiction.

From C.K. Fong; August 6, 2000.