COMPUTABILITY:

ANSWERS TO SOME EXERCISES IN CHAPTER 5 AND 6

5.1.5, P. 90.

1. This exercise is important for several reasons. For example, it tells us that taking inverses is an effect operation; see Example 5.3.1-4 on P. 94. It also supplies a piece of crucial information in the solution to Exercise 7.2.18-6 and 7 on P. 132.

   (i) By Corollary 1.3 (a) on P. 88, we know that the predicate

   $$S_1(x, z, y, t) \;\equiv\; \text{`}P_x(z) \downarrow y \text{ in } t \text{ or fewer steps}'$$

   is decidable. Thus the predicate '$y \in E_x$' is equivalent to '$\exists z\, \exists t\; S_1(x, z, y, t)$', which is equivalent to '$\exists z\; S_1(x, (z)_1, y, (z)_2)$'. We can easily check that

   $$Q(x, y, z) \equiv S_1(x, (z)_1, y, (z)_2)$$

   satisfies both conditions (a) and (b).

   (ii) Since the predicate $Q(x, y, z)$ given in part (i) is decidable, the function

   $$g(x, y) = U(\mu z\; Q(x, y, z))$$

   is computable, where U is defined by $U(z) = (z)_1$. In view of (i) (a) above, when $y \notin E_x$, the predicate '$\exists z\; Q(x, y, z)$' fails and hence $g(x, y)$ is undefined. On the other hand, suppose we have $y \in E_x$. Then it makes sense to talk about the smallest z for which $Q(x, y, z)$ holds, say $z_0 = \mu z\; Q(x, y, z)$. Thus $g(x, y)$ is defined and its value is given by $g(x, y) = U(z_0)$. Certainly $Q(x, y, z_0)$ holds. Thus, by (i) (b) we have $\phi_x(g(x, y)) = \phi_x(U(z_0)) = \phi_x((z_0)_1) = y$.

   (iii) Since f is computable, we have $f = \phi_n$ for some n. The function $g(n, y)$ (where g is the function described in part (ii) above) is computable. Also, $\phi_n(g(n, y)) = y$ for all $y \in E_n$ and $g(n, y)$ is undefined for $y \notin E_n$, where E_n is the range of $\phi_n \equiv f$. Thus $f^{-1}(y) = g(n, y)$, which is computable.

2. Since f and g are computable, $\phi_m = f$ and $\phi_n = g$ for some m and n. We have

   $$T_1(m, x, z) \equiv S_1(m, x, (z)_1, (z)_2) \;\equiv\; \text{`}P_m(x) \downarrow (z)_1 \text{ in } (z)_2 \text{ or fewer steps}'.$$

(Notice that, if the predicate $T_1(m, x.z)$ holds, then $(z)_1$ must be $f(x)$. But this does not concern us here.) The predicate $T_1(n, x, z)$ is similarly defined. Since both $T_1(m, x, z)$ and $T_1(n, x, z)$ are decidable, so is '$T_1(m, x, z)$ or $T_1(n, x, z)$'. Hence

$$\Phi(x) = U(\mu z \text{ `}T_1(m, x, z) \text{ or } T_1(n, x, z)')$$

is computable. Notice that the domain of Φ is $W_m \cup W_n$. (When $x \in W_m \cap W_n$, the value of $\Phi(x)$ is either $f(x) \equiv \phi_m(x)$ or $g(x) = \phi_n(x)$, depending on which computation, P_m(x) or $P_n(x)$, finishes first. From the identity $h(x) = \mathbf{1}(\Phi(x))$ it follows that h is computable.

5.3.2, P. 94.

1. Notice that $c_{\overline{M}} = 1 - c_M$. So it is natural to consider the compuatable function $f(x, y) = 1 - \psi_U(x, y)$. By the *s-m-n* theorem we know that there is a total computable function k such that $f(x, y) = \psi_{k(x)}(y)$. In case $\phi_e = c_M$, we have

$$c_{\overline{M}}(y) = 1 - c_M(y) = 1 - \phi_e(y) = 1 - \psi_U(e, y) = f(e, y) = \phi_{k(e)}(y).$$

   Thus $\phi_{k(e)} = c_{\overline{M}}$. Done.

2. Since ψ_U is computable, the function $f(x, y) = \mathbf{1}(\psi_U(x, y))y$ $(= \mathbf{1}(\phi_x(y))y)$ is also computable. Explicitly,

$$f_x(y) \equiv f(x, y) = \begin{cases} y & \text{if } y \in W_x; \\ \text{undefined} & \text{otherwise} \end{cases}$$

   with $\mathrm{Dom}(f_x) = \mathrm{Range}(f_x) = W_x$. The *s-m-n* theorem tells us that there exists some total computable function k such that $\phi_{k(x)} = f_x$. Thus $E_{k(x)} = \mathrm{Range}(\phi_{k(x)}) = \mathrm{Range}(f_x) = W_x$.

3. From Example 5.3.1-3 on P. 93 we know that there is a total cumputable function $r(x, y)$ such that $W_x \cup W_y = W_{r(x,y)}$. The previous exercise tells us that there is a total computable $k(x)$ such that $E_{k(x)} = W_x$. If we can find a total computable function $h(x)$ such that $W_{h(x)} = E_x$, then we will get

$$E_x \cup E_y = W_{h(x)} \cup W_{h(y)} = W_{r(h(x),h(y))} = E_{k(r(h(x),h(y)))} = W_{s(x,y)},$$

   where $s(x, y) = k \circ r(h(x), h(y))$ is computable. It remains to find such h. To do so, we need the help of Exercise 5.1.5-1 (ii) on P. 90: there is a computable function

$g(x, y)$ such that $g(x, y)$ is defined iff $y \in E_x$. By the *s-m-n* theorem, we know that there is a total computable function h such that $g(x, y) = \phi_{h(x)}(y)$. Now

$$y \in W_{h(x)} \quad \Leftrightarrow \quad \phi_{h(x)}(y) \equiv g(x, y) \text{ is defined} \quad \Leftrightarrow \quad y \in E_x$$

and hence $W_{h(x)} = E_x$. Done.

4. This is easy. Notice that '$y \in f^{-1}(W_x)$' means that '$f(y) \in W_x$', that is, '$\phi_x(f(y))$ is defined'. So it is only natural to consider the cumputable function

$$h(x, y) = \phi_x(f(y)) \equiv \psi_U(x, f(y)).$$

Applying the *s-m-n* theorem, we can get a total computable function k such that $\phi_{k(x)}(y) = h(x, y) = \phi_x(f(y))$. Thus $W_{k(x)} = \mathrm{Dom}(\phi_{k(x)}) = \mathrm{Dom}(\phi_x \circ f) = f^{-1}(W_x)$.

5. (a) The main theorem in the present chapter, namely Theorem 1.2 on P. 86, tells us that the functions $\psi_U^{(m)}(e, y_1, \ldots, y_m)$ and $\psi^{(n)}(e_k, z_1, z_2, \ldots z_n)$ $(1 \leq k \leq m)$ are computable. Hence the (partial) function F on $\mathbf{N}^{m+1} \times \mathbf{N}^n \equiv \mathbf{N}^{m+1+n}$ defined by

$$\begin{aligned} &F((e, e_1, \ldots, e_m),\, (z_1, \ldots, z_n)) \\ &\qquad = \psi_U^{(m)}(e, \psi_U^{(n)}(e_1, z_1, \ldots, z_n),\, \ldots, \psi_U^{(n)}(e_m, z_1, \ldots, z_n)) \\ &\qquad \equiv \mathrm{Sub}(\phi_e^{(m)};\, \phi_{e_1}^{(n)},\, \ldots,\, \phi_{e_m}^{(n)})(z_1,\, \ldots,\, z_n) \end{aligned}$$

is computable. Applying the *s-m-n* theorem to this function F, we get a total computable function $s(e, e_1, \ldots, e_m)$ such that

$$F((e, e_1, \ldots, e_m),\, (z_1, \ldots, z_n)) = \phi_{s(e, e_1, \ldots, e_m)}(z_1, \ldots, z_n).$$

Done.

(b) We know that the function $f(e, \mathbf{x}) = \mu y\, (\phi_e^{n+1}(\mathbf{x}, y) = 0)$ is computable and hence there is a total computable function k such that $\phi_{k(e)}^{(n)}(\mathbf{x}) = f(e, \mathbf{x})$.

6.1.8, P. 106.

1. First we pick out those parts for which we can apply Rice's Theorem and finish them quick. (d) Let $\mathcal{B} = \{\mathbf{0}\}$. (f) Let $\mathcal{B} = \{f \in \mathcal{C}_1 : f \text{ is total and constant}\}$. (g) Let $\mathcal{B} = \{f_\emptyset\}$. (h) Let $\mathcal{B} = \{f \in \mathcal{C}_1 : \mathrm{Range}(f) \text{ is infinite}\}$. (i) Let $\mathcal{B} = \{g\}$. (In part (g) and part (i), $\mathcal{B}$ is a singleton set.)

(e) From Theorem 1.6 (b) on P. 104 we know that $N(y) \equiv$ '$0 \in E_y$' is undecidable. Suppose the contrary that $M(x, y) \equiv$ '$x \in E_y$' is decidable. Then its characteristic function $c_M(x, y)$ is computable. Consequently $c_M(0, y)$ is computable. But $c_M(0, y)$ is just $c_N(y)$, the characteristic function of $N(y)$. So this contradicts the fact that c_N is not computable.

(c) (This one is a bit tricky.) Suppose the contrary that $M(x) \equiv$ '$\phi_x(x) = 0$' is decidable. [Notice that the opposite of $M(x)$ is

$$\overline{M}(x) \equiv \text{`either } x \notin W_x, \text{ or } x \in W_x \text{ but } \phi_x(x) \neq 0.' \text{]}$$

Then its characteristic function c_M is computable. Hence $c_M = \phi_n$ for some n. Since c_M is a total function, so is ϕ_n and hence $W_n = \mathbf{N}$. We check that both $\phi_n(n) = 0$ and $\phi_n(n) \neq 0$ will lead to a contradiction. If $\phi_n(n) = 0$, then M holds for n and hence $c_M(n) = 1$, contradicting $\phi_n(n) = c_M(n)$. If $\phi_n(n) \neq 0$, then M fails for n and hence $c_M(n) = 0$, again contradicting $\phi_n(n) = c_M(n)$.

(b) Applying Rice's theorem to $\mathcal{B} = \{f \in \mathcal{C}_1 : f \text{ is total}\}$ we see that the predicate $M(x) \equiv$ 'ϕ_x is total' is undecidable. Hence its characteristic function $c_M(x)$ is not computable. Take any computable function g which is total (e.g. $g = \mathbf{0}$) and let n be the integer such that $\phi_n = g$. Let $h(x, y)$ be the characteristic function of the predicate '$W_x = W_y$'. Then $f(x, n)$ is the characteristic function of '$W_x = W_n$', or '$W_x = \mathbf{N}$'. Thus $f(x, n)$ is just the characteristic function of the predicate $M(x)$. So $f(x, n) = c_M(x)$ is not computable. Hence $f(x, y)$ is not computable; (otherwise substituting y by $\mathbf{n}$ in $f(x, y)$ would give us a computable function $f(x, n)$). Therefore '$W_x = W_y$' is undecidable.

(a) First proof: use a diagonal argument. Suppose the contrary that $x \in E_x$ is decidable. Then the function

$$g(x) = \begin{cases} x & \text{if } x \notin E_x \\ \text{undefined} & \text{otherwise;} \end{cases}$$

is computable. Hence $g = \phi_n$ for some n. Notice that

$$\text{Dom}(g) = \text{Range}(g) = \{x \colon x \notin E_x\}.$$

Suppose $n \in \text{Dom}(g) = \text{Dom}(\phi_n) = W_n$. Then $n \in \text{Range}(g) = \text{Range}(\phi_n) = E_n$. On the other hand, $n \in \text{Dom}(g) = \{x \colon x \notin E_x\}$ and hence $n \notin E_n$, a contradiction. Next suppose $n \notin \text{Dom}(g)$. Then $n \notin \text{Range}(g) = \text{Range}(\phi_n) = E_n$. On the other hand, $n \notin \text{Dom}(g) = \{x \colon x \notin E_x\}$ and hence $n \in E_n$, a contradiction again.

Second proof: use reduction. Let f and k be those functions in the proof of Theorem 1.6 on P. 104. Then

$$W_{k(x)} = E_{k(x)} = \mathbf{N} \text{ iff } x \in W_x \quad \text{and} \quad W_{k(x)} = E_{k(x)} = \emptyset \text{ iff } x \notin W_x,$$

from which we deduce that $k(x) \in E_{k(x)}$ iff $x \in W_x$. Since k is a total computable function and '$x \in W_x$' is undecidable, we see that '$y \in E_y$' is also undecidable.

2. Assume the contrary that such f exists. [Intuitively, using f we can solve the Halting problem 'does the computation $P_x(y)$ eventually stop or go on for ever' as follows. If the computation $P_x(y)$ stops in $f(x, y)$ or fewer steps then the answer to the Holting problem is 'yes' and otherwise 'no'. The point here is that there is no need to go beyond $f(x, y)$ steps to find out the answer to '$P_x(y) \downarrow$?'. But converting this idea to a formal proof needs some technical knowledge to handle the situation.] Recall from Corollary 5.1.3 on P. 88 that $H(e, x, t) \equiv$ '$P_e(x) \downarrow$ in t or fewer steps' is decidable. By our assumption on f, we know that

$$H_n(e, x, f(e, x)) \;\equiv\; \text{`}P_e(x) \downarrow' .$$

Notice that the characteristic function $c_M(e, x)$ of the predicate

$$M(e, x) \equiv H_n(e, x, f(e, x))$$

is just $c_{H_n}(e, x, f(e, x))$. Since H_n is decidable, $c_{H_n}(e, x, t)$ is computable. Hence $c_M(e, x) = c_{H_n}(e, x, f(e, x))$ is also computable and as a consequence the predicate $M(e, x)$ is decidable. But $M(e, x) \equiv$ '$P_e(x) \downarrow$', which is just the Halting problem. Thus we have arrived at a contradiction.

6.6.14, P.119.

1. (a) '$E_x^{(n)} \neq \emptyset$' $\equiv$ '$W_x^{(n)} \neq \emptyset$' $\equiv$ '$\exists \mathbf{y}\, \exists t\; H_n(x, \mathbf{y}, t)$', where

$$H_n(x, \mathbf{y}, t) \equiv \text{`}P_x^{(n)}(y) \downarrow \text{ in } t \text{ or fewer steps}'$$

is knwon to be decidable. So by Corollary 6.6 on P. 115, it follows that the predicate '$E_x^{(n)} \neq \emptyset$' is partially decidable.

(b) We know that $M(z) \equiv$ 'z is a perfect square' is decidable. So its partial characteristic function $\chi_M(z)$ is computable. Hence $f(x, y) = \chi_M(\phi_x(y))$ is computable.

Observe that $f(x, y)$ is just the partial characteristc function of '$\phi_x(y)$ is a perfect square'.

(c) This question is obsolete because Fermat's Last Theorem was proved in recent years by Andrew Wiles.

(d) Let $\pi = y_0 . y_1 y_2 y_3 \cdots$ $(= \sum_{k=0}^{\infty} y_k/10^k)$ be the decimal expansion of π. Then the function f defined by $f(k) = y_k$ is computable (this was known at least two thousand years ago). Hence the following predicate, denoted by $M(x, z)$,

$$f(z) \neq 7, \ f(z+1) = f(z+2) = \cdots = f(z+x) = 7 \text{ and } f(z+x+1) \neq 7,$$

is decidable. By Theorem 6.5 on P. 115, the predicate in question, which is equivalent to '$\exists z\, M(x, z)$' is partially decidable.

2. Let G be a finitely presented group. Let $A = \{a_1, \ldots, a_m\}$ be a finite set of generators and $R = \{r_1, r_2, \ldots, r_n\}$ be a finite set of defining relations. More precisely, R is a finite subset of the free group F_n generated by A and $G = F_n/N$, where N is the normal subgroup generated by R. Let $\pi\colon F_n \to G$ be the quotient map. The word problem is, given a word w, decide whether $\pi(w) = e$ (e is the unit element of G). Now $\pi(w) = e$ if and only if we can apply finitely many operations on elements z in F_n of the following types to render w to the empty word $\emptyset$: 1. multiply z by some r in R, or r^{-1}, on either side of z; 2. replace z by aza^{-1} or $a^{-1}za$ for some $a \in A$. By introducing appropriate symbols for these operations (this is similar to symbols for instructions in URM) we can effectively enumerate all these operations. Using Gödel's numbers, following the same method of Chapter 5 we can effectively enumerate all finite sequences of such operations (this is similar to enumeration of all URM programs), say S_x ($x \in \mathbf{N}$). Each S_x represents a sequence of operations converting any word w in F_n into a new word denoted by $S_x(w)$; (unlike P_x for computable functions ϕ_x, $S_x(w)$ is always defined). Also, we can effectively enumerate all words with aphabets in A, say w_x. The predicate $M(x, y) \equiv$ '$S_y(w_x) = \emptyset$' is decidable. Hence

$$`\pi(w) = e' \ \equiv \ `\exists y\, M(x, y)'$$

is partially decidable, in view of Theorem 6.5 on P. 115.

3. Omitted. The idea is similar to the last exercise.

4. The partial characteristic functions $\chi_M(\mathbf{x})$ and $\chi_N(\mathbf{x})$ of $M(\mathbf{x})$ and $N(\mathbf{x})$ are computable. Putting $P \equiv$ 'M and N' we have $\chi_P(x) = \chi_M(x)\chi_N(x)$, which is computable.

Hence '$M(\mathbf{x})$ and $N(\mathbf{x})$' is partially decidable. The other part, showing that '$M(\mathbf{x})$ or $N(\mathbf{x})$' is partially decidable, is more difficult. This part is essentially the same as is Exercise 5.1.5-2 on P. 90. Just replace unary functions f and g by n-ary functions $\chi_M(\mathbf{x})$ and $\chi_N(\mathbf{x})$, and replace T_1 by T_n. Then proceed in the same manner. Notice that, if $M(x) \equiv$ '$x \in W_x$', then $M(x)$ is partially decidable, but ǹot $M(x)' \equiv$ '$x \notin W_x$' is not partially decidable.

5. (a) The predicate $N(\mathbf{x}, y, z) \equiv$ '$y < z$ and $M(\mathbf{x}, y)$' is partially decidable. So, by Theorem 6.5 on P. 115, the predicate

$$`\exists y < z \; N(\mathbf{x}, y, z)' \;\equiv\; `\exists y \; (y < z \text{ and } M(\mathbf{x}, y)'$$

is also partially decidable.

(b) Let $f(\mathbf{x}, y)$ be the partial characteristic function of $M(\mathbf{x}, y)$. Since $M(\mathbf{x}, y)$ is partially decidable, $f(\mathbf{x}, y)$ is computable. By Theorem 2.4.10 on P. 38, we know that the function

$$F(\mathbf{x}, z) = \Pi_{y<z} \, f(\mathbf{x}, y)$$

is computable. Since $F(\mathbf{x}, z)$ is nothing but the partial characteristic function of the predicate '$\forall y < z \, M(\mathbf{x}, y)$', hence the last predicate is computable.

(c) We know that the predicate '$x \notin W_x$' is not partially decidable. But we have

$$`x \notin W_x{}' \;\equiv\; `\forall y \; (x \neq y \text{ and } y \in W_x)'$$

where the predicate '$x \neq y$ and $y \in W_x$' is partially decidable, because '$x \neq y$' is decidable, while '$y \in W_x$' $\equiv$ '$(x, y) \in \mathrm{Dom}(\psi_U)'$ is partially decidable.

6. (a) Indeed, 'x is even' $\equiv$ '$\exists y \; (x - 2y = 0)$'.

(b) Indeed, 'x divies y' $\equiv$ '$\exists z \; (zx - y = 0)$'.

7. (a) Another way to say "$x \in W_x$ iff $M(k(x))$ does not hold" is "$x \notin W_x$ iff $M(k(x))$ holds". Hence the predicate '$x \notin K_x$', which is known to be not partially decidable, is reducible to $M(x)$. Hence $M(x)$ is not partially decidable.

(b) The computable function k constructed in the proof of Theorem 1.6 on P. 104 has the property that $x \in W_x$ iff $\phi_{k(x)}$ is total. It follows from part (a) above that the opposite of 'ϕ_x is total' is not partially decidable.

(c) Since the predicate '$P_x(x)$ does not converge in y or fewer steps' is decidable, the function $f(x,y)$ is computable. Put $f_x(y) = f(x,y)$. Then it is clear that f_x is total if and only if $x \notin W_x$. By the s-m-n theorem we know that there is a total computable function s such that $f(x,y) = \phi_{s(x)}(y)$. Thus we have: $x \notin W_x$ iff $\phi_{s(x)}$ is total. As we know, the predicate '$x \notin W_x$' is not partially decidable. Therefore 'ϕ_x is total' is also not partially decidable.

8. Assume that the predicate '$f(x) = y$' is partially decidable. By Theorem 6.4 on P. 114, we get a decidable predicate $R(x,y,z)$ such that

$$`f(x) = y' \;\equiv\; \exists z\; R(x,y,z).$$

   We can check that

$$f(x) = U(\mu z\; R(x,(z)_1,(z_2))), \quad \text{where } U(w) = (w)_1.$$

   Hence f is computable.

9. As usual, denote by $\chi_M(x_1,\ldots,x_n)$ and $\chi_N(\mathbf{y})$ the partial characteristic functions of $M(x_1,\ldots,x_n)$ and $N(\mathbf{y})$ respectively. Then we can check that

$$\chi_N(y) = \chi_M(g_1(\mathbf{y}),\ldots,g_n(\mathbf{y})),$$

   which is computable.

From C.K. Fong; August 2, 2000.