



Lecture 03 Static games of complete information

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Previously on this course



- In game theory, we model problems that involve multiple agents interacting with each other
- Problems studied in game theory are called "games" and the involved agents are called "players"
- players are assumed to be rational, meaning that they have a rational set of preferences about the game's outcome and they act according to those preferences
- Preferences can be expressed numerically by using utility functions

Previously on this course



- Random outcomes are modeled as lotteries
- Lotteries are simply probability distributions describing random outcomes
- The concept of preference can be extended to lotteries
- If preferences satisfy a set of axioms (rationality, continuity, independence) we can evaluate the utility of a lottery as the expected value of the utility, averaged over the possible outcomes

Static games of complete information

Game with multiple players



- How do multiple players interact?
 - We assume they have a payoff (utility) function
- Remember: rational players move to maximize of their own payoffs
- What is the simplest interaction like this?

Static games of complete information



- **Static**: all players move together; they do not necessarily play simultaneously, but <u>without knowledge of everyone else's move</u>
- Complete information: meaning everyone's payoff function is known
 - most games covered within this class are "artificial" (theoretical models)
 - however, there are also actual games that can be modeled as static games of complete information. Examples?

Static games of complete information



- Each player i in the game simultaneously and independently chooses an action from its own set of available actions A_i
- The combination of actions chosen by the *n* players determines the outcome of the game
- Outcome $(a_1, a_2, ..., a_n)$ determines a payoff for each player through an individual utility function of player i:

$$u_i = u_i(a_1, a_2, \ldots, a_n)$$

■ 3 ingredients = actions + outcome + utility

Action versus strategy



- In decision problems we always thought in terms of actions
- In games, it is useful to think in terms of **strategies** instead
- A strategy is a plan of action
 - e.g. if these conditions are met, then my action is a, otherwise it is either a' or a''
 - this plan can even be random (we will see why)
- For the time being, let's consider only **deterministic** plans
- These are called **pure strategies**

Normal form of a game



- Each player i simultaneously chooses a strategy from a set of pure strategies S_i
- This results in a given action chosen by each of the *n* players that ultimately determines a payoff for each player
- If any player *i* plays strategy $s_i \in S_i$, the combination of moves is $(s_1, s_2, \dots, s_i, \dots, s_n)$
- Player i gets payoff $u_i(s_1, s_2, ..., s_i, ..., s_n) \in \mathbb{R}$
- The **normal form** of the game is specified by $\mathbb{G} = \{S_1, \dots, S_n; u_1, \dots, u_n\}$

Simultaneous and independent



- Simultaneous moves do not necessarily need to happen at the same time
 - they are chosen without knowledge of everyone else's actions
- These two cases are both considered simultaneous:
 - case A: two players are writing their strategy on opposite sides of a board at the same time
 - case B: player 1 is asked to write first; while player 1 writes, player 2 is blindfolded; then the board is turned and player 2 writes

Common knowledge



- We say that E is common knowledge if:
 - everyone knows E
 - everyone knows that everyone knows E
 -
- Common knowledge in games is a powerful assumption, and not an obvious one
 - it requires each player to have full knowledge not only on information pertaining themselves, but also on everyone else

Common knowledge



- "Complete information" in games means that the following information is common knowledge:
 - all possible actions of all players
 - all possible outcomes resulting from these actions
 - the individual preferences of all players about these outcomes (i.e., their utilities about them)
- Player rationality is also common knowledge
 - meaning that everyone is maximizing their own payoff and everyone knows that everyone is maximizing their payoff

Array representation



- An *n*-player game can be represented as a function in $S_1 \times S_2 \times \cdots \times S_n$ that maps tuples of strategies to tuples of payoffs
- If all S_i , i = 1, ..., n are discrete sets, we can represent a game using an n-dimensional array that contains a tuple of n values (the payoffs) in each cell/entry
 - The overall number of cells is $m_1 \times \cdots \times |S_n|$
- Typically we consider games with n=2 players, which can be represented using an ordinary matrix containing a pair of values in each cell
 - This is called a **bi-matrix**

Example of bi-matrix

 $|S_1| = m_1, |S_2| = m_2$

		Play	er 2	
	$s_2^{(1)}$	$s_2^{(2)}$		$s_2^{(m_2)}$
(1)	$u_1(s_1^{(1)}, s_2^{(1)}),$	$u_1(s_1^{(1)}, s_2^{(2)}),$		$u_1(s_1^{(1)}, s_2^{(m_2)}),$
$s_1^{(1)}$	$u_2(s_1^{(1)}, s_2^{(1)})$	$u_2(s_1^{(1)}, s_2^{(2)})$	•••	$u_2(s_1^{(1)}, s_2^{(m_2)})$
$s_1^{(2)}$	$u_1(s_1^{(2)}, s_2^{(1)}),$	$u_1(s_1^{(2)}, s_2^{(2)}),$		$u_1(s_1^{(2)}, s_2^{(m_2)}),$
3 ₁	$u_2(s_1^{(2)}, s_2^{(1)})$	$u_2(s_1^{(2)}, s_2^{(2)})$	• • •	$u_2(s_1^{(2)}, s_2^{(m_2)})$
:	:	:	٠.	:
(m ₁)	$u_1(s_1^{(m_1)}, s_2^{(1)}),$	$u_1(s_1^{(m_1)}, s_2^{(2)}),$		$u_1(s_1^{(m_1)}, s_2^{(m_2)})$
$s_1^{(m_1)}$	$u_2(s_1^{(m_1)}, s_2^{(1)})$	$u_2(s_1^{(m_1)}, s_2^{(2)})$	•••	$u_2(s_1^{(m_1)}, s_2^{(m_2)})$

Player 1

Example 1



- Player A has strategies $S_A = \{U, M, D\}$
- Player B has strategies $S_B = \{L, R\}$

<	U	
ıyer	M	
()	D	

L	R
8, 0	0, 5
1, 0	4, 3
0, 7	2, 0

Player B

Example 2



- Player A has strategies $S_A = \{U, M, D\}$
- Player B has strategies $S_B = \{L, C, R\}$

yer A	U
Play	M D

Player B				
L	C	R		
0, 5	4, 0	7, 3		
4, 0	0, 5	7, 3		
3, 7	3, 7	9, 9		

Example: Odds and evens



- Player Odd and Even bet 4 euros
- Player Odd has two strategies: {0,1}
- Plater Even has two strategies: $\{0,1\}$

		Even	
		0	1
рp	0	-4, 4	4, -4
0	1	4, -4	-4, 4

Example: Rock-paper-scissors



■ Both players have strategies $S_A = S_B = \{R, P, S\}$

		Player B		
		R	Р	S
Ā	R	0, 0	-4, 4	4, -4
layer	Р	4, -4	0, 0	-4, 4
Б	S	-4, 4	4, -4	0, 0

In this game we also have the possibility to draw => that will be bring to a 0 utility for both

Example: Battle of the Sexes



- Two college students, A and B, need to decide which night event to attend: rock concert (R) or science night (S).
- A prefers the concert, while B prefers the science night.
 However, both prefer to spend time with each other rather then separately
- They have not exchanged contact yet, so they are taking their decision independently and without communicating

		В	
	R	S	
R	2, 1	0, 0	
S	0, 0	1, 2	

Example: Prisoner's dilemma



- Simplified version: each player chooses between two options $S_A = S_B = \{M, F\}$
 - M: Lose 1 euro
 - F: The other player loses 20 euros

		Play	Player B	
		M	F	
A	M	-1, -1	-21, 0	
Player A	F	0, -21	-20, -20	
ă				

Example: Prisoner's dilemma



- Original version: the players are two criminals caught by the police. The police has evidence only for petty theft but not for a major crime. The two are arrested and interrogated in separate room. They can decide to either
 - Keep mum (M), i.e., to not talk
 - Fink (F), i.e., to snitch on their partner
- Payoffs represent the years of jail they get



Pareto efficiency



A joint strategy s is **Pareto-dominated** by another strategy s' if

$$u_i(s') \ge u_i(s)$$
 for each player i
 $u_i(s') > u_i(s)$ for some player i

- A joint strategy *s* that is not Pareto-dominated by any other joint strategy *s'* is called **Pareto-efficient**
- There may be more than one Pareto-efficient strategy, none of which dominates the others

Strict dominance

Strictly dominated strategy



- Consider game $\mathbb{G} = \{S_1, \dots, S_n; u_1, \dots, u_n\}$
- If $s_i, s_i' \in S_i$, we say that s_i is <u>strictly dominated</u> by s_i' if i's payoff when playing s_i' is always greater than when playing s_i for any possible choice of moves by the other players
- Formally

$$u_i(s_1,\ldots,s_i',\ldots,s_n)>u_i(s_1,\ldots,s_i,\ldots,s_n)$$

$$\forall (s_1,\ldots,s_{i-1},\ldots,s_{i+1},\ldots,s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \ldots S_n$$

■ Rational players do not play strictly dominated strategies



■ Can you find any strictly dominated strategy?

Player	В
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⋖	U
ıyer	Μ
<u>B</u>	D

L	R
8, 0	0, 5
1, 0	4, 3
0, 7	2, 0



- Strategy D is dominated by M for player A
 - $u_A(M, L) = 1 > u_A(D, L) = 0$
 - $u_A(M,R) = 4 > u_A(D,R) = 2$

		Player B		
		L	R	
A	U	8, 0	0, 5	
layer ,	M	1, 0	4, 3	
<u> </u>	D	0, 7	2, 0	



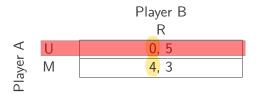
- Now strategy L is dominated by R for player B
 - $u_B(U, R) = 5 > u_B(U, L) = 0$
 - $u_B(U,R) = 3 > u_B(D,R) = 0$

		Player B		
		L	R	
A	U	8, 0	0, 5	
layer A	M	1, 0	4, 3	



■ Now strategy U is strictly dominated by M for A

$$u_A(M,R) = 4 > u_A(U,R) = 0$$

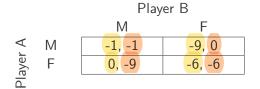


■ Rational players end up playing (M, R) with payoffs (4, 3)

Back to the prisoner's dilemma



Are there any strictly dominated strategies?



Back to the prisoner's dilemma



- For both players, M is strictly dominated by F
 - $u_A(F, M) = 0 > u_A(M, M) = -1$
 - $u_A(F,F) = -6 > u_A(F,M) = -9$

		Player B		
		M	F	
Α,	M	-1, -1	-9, 0	
layer /	F	0, -9	-6, -6	
Ä				

- Rational players end up playing (F, F)
- This is not a Pareto-efficient joint strategy: (F, F) is Pareto-dominated by (M, M)
- The final outcome is "bad", hence the dilemma

Solving problems via IESDS



- This procedure is called "iterated elimination of strictly dominated strategies" (IESDS)
- Sometimes, it allows to obtain a reduced version of a game by relying on common knowledge
 - All players know that a certain strategy is dominated for one player, so they rule it out
- Unfortunately, in many cases it does not lead to a solution for the game

Example 2



- In this game there is no strictly dominated strategy
- However, (D, R) seems to be a good choice for the players

		Player B		
		L	C	R
A	U	0, 5	4, 0	7, 3
layer A	M	4, 0	0, 5	7, 3
Ę	D	3, 7	3, 7	9, 9

Back to Odds and evens



■ No strategy seems to be better than the other

рp	0
Ō	1

0	1
-4, 4	4, -4
4, -4	-4, 4

Even

Example: Battle of the Sexes



■ There are two joint strategies that are "good" for rational players: (R, R) and (S, S)

		В	
		R	S
_	R	2, 1	0, 0
_	S	0, 0	1, 2

Questions?