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Lecture 04
Nash equilibrium

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Terminology recap

- **Static game of complete information**: simplest type of game, played in one shot; players move unbeknownst of each other's actions but fully aware of everyone's payoffs.
  - Examples: Rock-paper-scissors, battle of the sexes, prisoner's dilemma.
- Static games of complete information are fully defined by actions, outcomes, and utilities
- In this type of games, **pure strategies** = actions (e.g., pure strategy: "I will play rock", action: playing rock)

Terminology recap

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- **Normal form of a game**: $\mathbb{G} = \{S_1, S_2, \ldots, S_n; u_1, u_2 \ldots, u_n\}$
- This is one possible way of representing the prisoner's dilemma in normal form
  - $\mathbb{G} = \{S_A, S_B; u_A; u_B\}$
  - $S_A = S_B = \{M, F\}$
  - $u_A(M, M) = u_B(M, M) = -1$, $u_A(F, F) = u_B(F, F) = -6$, $u_A(M, F) = u_B(F, M) = -9$, $u_A(F, M) = u_B(M, F) = 0$
- However, this is not very convenient to analyze. Therefore, we often prefer the graphical representation.

Terminology recap

- Graphical representation of the prisoner's dilemma normal form

		Player B	
		M	F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

Terminology recap

- Pure strategy: $s_i \in S_i$
- **Joint strategy**: $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$
- In static games of complete information, joint strategy = outcome
- Examples:
  - $s_B = F$ is a pure strategy
  - $s = (M, F)$ is a joint strategy
  - (M, F) is an outcome

Terminology recap

- **Pareto dominance**: property of joint strategies (concerns all the players)
  - A joint strategy s is Pareto dominated by another strategy s' if for all players $u_i(s) \leq u_i(s')$ (and for some the inequality is strict)
  - In the prisoner's dilemma (F, F) is Pareto dominated by (M, M)
- **Strict dominance**: property of pure strategies (concerns only one player at a time)
  - A strategy s_i of player i is strictly dominated by another strategy s_i' if, regardless of what strategy is adopted by other players, s_i' gives a higher payoff to i
  - In the prisoner's dilemma, M is strictly dominated by F for both players

Best responses and beliefs

Single-player vs multi-player

- For single-player problems, once the setup is known, the solution can be found directly
- That is not the case for multi-player games
  - The solution depends on other players
  - Sometimes rationality can help (e.g., we identify a dominated strategy and we decide not to play it)
  - We can extend this reasoning by assuming that other players are also rational, which leads to IESDS
  - Still, in most cases this does not allow to find a solution for the game

Best response

- Strategy $s_i \in S_i$ is player i's best response to moves $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ if

  $$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

  for all $s_i' \in S_i$
- *Notation*: $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$ is often shortened to "$s_{-i} \in S_{-i}$"
- This way we can simply write: $s_i \in S_i$ is best response to $s_{-i} \in S_{-i}$ if

  $$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \forall s_i' \in S_i$$

Best response

- There may be more than one best response
  - Of course, all with the same value of $u_i(s_i, s_{-i})$

Player B

Player A		L	C	R
	U	3, 3	5, 1	6, 2
	M	4, 1	8, 4	3, 6
	D	4, 0	9, 6	6, 8

- Here, U and D are both best responses to player B's strategy to play R.
- Self-assessment: What are player A's best responses to strategies L and C?

Best response

- **Claim**: A rational player who believes that others are playing $s_{-i} \in S_{-i}$ will always choose the best response to s_{-i}. (This follows from players wanting to maximize their payoffs).
- **Theorem**: If $s_i \in S_i$ is strictly dominated by some other strategy, then it is no best response to any $s_{-i} \in S_{-i}$.
  - *Proof*: There is some strategy $s_i' \in S_i$ that dominates s_i.
  - By definition, $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$
  - Therefore, s_i always yields a lower payoff than s_i' and cannot be a best response $\square$

Beliefs

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- A **belief** of player i is a possible profile of the other players' strategies, i.e., an element of the set S_{-i}
  - Beliefs are connected to best responses
- We define a best-response correspondence $\mathrm{BR} : S_{-i} \rightarrow 2^{S_i}$ that maps $s_{-i} \in S_{-i}$ to a subset of S_i such that each $s_i \in \mathrm{BR}(s_{-i})$ is a best response to s_{-i}
  - The symbol 2^{S_i} is used to denote the power set of S_i, i.e., the set of all the possible subsets of S_i
  - BR is not a function, as it maps values to sets
  - However $\mathrm{BR}(s_{-i})$ can be a singleton if there is a unique best response to s_{-i}

Nash equilibrium

Nash equilibrium

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- We want to strengthen the dominated strategy concept with this idea in mind:
  - game theory should make predictions about the outcome of games played by a rational players
  - a prediction is correct if the players are **willing** to play their predicted strategy
- That is, players choose their **best response** to the predicted strategy of the others (i.e, the best response to their belief about other players' strategy)
  - A player's belief "makes sense" only if other players are also playing a best response
- If the (reasonable) beliefs of all players match, then no one regrets their strategy

Nash equilibrium: intuition

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- A Nash equilibrium is what is played if players beliefs match
- Let us mark in blue player A's best responses, and in red player B's best responses
- Suppose A's belief is that B will play S
  - Then, A's best response is to play S
- Suppose B's belief matches A (i.e., B believes that A will play S)
  - Then, B's best response is to play S
- This is a Nash equilibrium, since none of them regrets their strategy

		B: R	B: S
A	R	2, 1	0, 0
A	S	0, 0	1, 2

Back to the Prisoner's dilemma

- It does not make sense for A to believe that B will play M, since M is never a best response

		Player B	
		M	F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

- The NE is also the only survivor of IESDS

Nash equilibrium (formal definition)

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- In a n-player game $\mathbb{G} = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, strategies $(s_1^*, \ldots, s_n^*)$ are a **Nash equilibrium** if, for all i, s_i^* is a best response to $s_{-i}^* = (s_1^*, \ldots s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$
- In other terms, $\forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i^*, s_{-i}^*)$$

or, equivalently,

$$s_i^* = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}^*)$$

- This is consistent with player's rationality, that requires all of them to maximize their utility function

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- Take a possible combination $(s'_1, \ldots, s'_n)$
- If this is *not* a Nash equilibrium, there must be some player i such that s'_i is not the best response to $(s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_n)$
- That means $\exists s''_i \in S_i$ such that

$$u_i(s'_1, \ldots, s'_{i-1}, s''_i, s'_{i+1}, \ldots, s'_n) > u_i(s'_1, \ldots, s'_{i-1}, s'_i, s'_{i+1}, \ldots, s'_n)$$

- In other words, there is an incentive for player i to deviate from the joint strategy $(s'_1, \ldots, s'_n)$

NE as absence of regret

- Remember that we are considering static (one-shot) games
- A NE can be seen as a joint strategy in which no player has regrets on their choice
- In other words, if a NE is played, none of the players would want to unilaterally change their strategy even if they had the possibility to do so

Back to Example 1

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- Joint strategy (M, R) is a Nash equilibrium

Player B

Player A		L	R
	U	8, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

- A naive way to find Nash equilibria is to brute-force search: here (M, R) is the only joint strategy that satisfies the definition
  - You can verify that the utility does not decrease when player deviate *unilaterally*

Back to Example 2

- A better way to find NE is to focus on best responses
- For player A, we find the maximum left value in each column; for player B, we find the maximum right value in each row

Player B

Player A		L	C	R
	U	0, 5	4, 0	7, 3
	M	4, 0	0, 5	7, 3
	D	3, 7	3, 7	9, 9

- (D, R) is the only NE for this game (both D and R are highlighted, meaning that they are best responses to each other)

Back to odds and evens

- Here there is no Nash equilibrium (in pure strategies)
- We will see that there actually is one Nash equilibrium but we need to "extend" the definition

		Even: 0	Even: 1
Odd	0	-4, 4	4, -4
Odd	1	4, -4	-4, 4

Back to the Battle of the Sexes

- (R, R) and (S, S) are both Nash equilibria

B

A		R	S
	R	2, 1	0, 0
	S	0, 0	1, 2

Back to the Prisoner's dilemma

- Joint strategy (F, F) is a NE

		Player B: M	Player B: F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

- The NE is also the only survivor of IESDS

NE and IESDS

- **Theorem**: If $(s_1^*, \dots, s_n^*)$ is the only joint strategy left after applying IESDS, then it is a Nash equilibrium
- **Lemma**: A NE always survives IESDS
- Another result: IESDS order is irrelevant

Summing up

- Two requirements must be satisfied in order for a NE to be played:
  - Everyone plays a best response to their beliefs
  - Everyone's beliefs are **correct**
- The first requirement is quite logical and is simply the consequence of the rationality assumption
  - If I am a rational player and I believe other player are gonna act in a certain way, I will always play a best response to it
- Actually the first requirement is quite logical and consequent from rationality, while the second requirement is quite demanding
  - Beliefs may be inferred via some external reasoning (e.g., one player being particularly "influential")

More definitions of dominance and efficiency

Strict and weak dominance

- **Strict dominance**: s_i' strictly dominates s_i if
  - $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$
- **Weak dominance**: s_i' weakly dominates s_i if
  - $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$
  - $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for some $s_{-i} \in S_{-i}$

Dominance and Nash equilibrium

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- A strategy that strictly (resp., weakly) dominates every other strategy of a user is said to be **strictly (resp., weakly) dominant**
- **Lemma**: A joint strategy $(s_1^*, \dots, s_n^*)$ in which everyone plays a dominant strategy is a Nash equilibrium.
- It directly follows from the definition of NE.
- The reverse statement is false (sufficient but not necessary condition)

Do not eliminate weakly dominated s_i's

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- Extend the Odds-or-evens game with a third option: "Punch the opponent" (P)
- Both players receive negative payoff (one gets beaten, the other gets punished)
- P is weakly dominated, yet it is a NE
- If we delete it, we miss a NE

		Even: 0	Even: 1	Even: P
Odd	0	-5, 5	5, -5	-5, -5
Odd	1	5, -5	-5, 5	-5, -5
Odd	P	-5, -5	-5, -5	-5, -5

Pareto efficiency vs NE

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- Pareto efficiency and NE are different concepts
  - Pareto efficiency: you cannot improve one player's payoff without worsening the payoff of another player
  - Nash equilibrium: no player can improve their own payoff via unilateral change (i.e., keeping the other players' choice fixed)
- The outcome of the Prisoner's dilemma is not Pareto efficient!

Player A \ Player B	M	F
M	-1, -1	-9, 0
F	0, -9	-6, -6

Pareto efficiency vs NE

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- Pareto efficient strategies: (M, M), (M, F), (F, M)
- NE: (F, F), which is Pareto dominated by (M, M)

		Player B: M	Player B: F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

Pareto efficiency vs NE

- Pareto inefficient Nash equilibria arise as we assume players are only driven by the desire to maximize their own payoff
- To estimate the inefficiency of being selfish (or distributed) one can compare Nash equilibria with Pareto efficient strategies
- To this end, we can assume that a joint strategy s has social cost $C(s)$, e.g.

$$C(s) = -\sum_j u(s_j) \text{ or } C(s) = -\max_j u(s_j)$$

Price of anarchy

- The **price of anarchy** is the ratio between the social costs in the *worst* NE s^* and in the *best* Pareto efficient strategy (i.e., social optimum)

$$\text{PoA} = \frac{C(s^*)}{\min_s C(s)}$$

- In some cases, one may consider the *best* NE: in that case we call the ratio price of stability
- For certain classes of problems, there are theoretical results on the price of anarchy

Self-assessment

- What is a NE?
- Consider NE $(s_1, \ldots, s_n)$. Suppose player i replaces the current strategy s_i with s_i'. Can this still be a NE?
- If a strategy is ruled out by IESDS, can it be a NE?
- Compute the PoA for the Prisoner's dilemma using $C(s) = -\sum_j u_j(s)$

Homework

- A (crazy) professor decides your grade in the exam he teaches will be decided by a game:
  - You are paired with a random classmate
  - You secretly choose an integer between 18 and 30, and so does the classmate
  - If you choose the same number, that is the score that you both get
  - If the numbers are different, who proposes the lowest score L gets a grade of $L + R$, while the other gets $L - R$ (score < 18 means the exam is failed, $>$30 means 30L and gives payoff 31)
- Play the game with $R = 1$, $R = 2$, and $R = 10$.
- How do the NE change?

Sorry, gotta bounce!
Send me questions via e-mail