UNIVERSITÀ
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# Lecture 03
# Static games of complete information

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October 11, 2023

# Previously on this course



- In game theory, we model problems that involve multiple agents interacting with each other
- Problems studied in game theory are called "games" and the involved agents are called "players"
- players are assumed to be *rational*, meaning that they have a rational set of preferences about the game's outcome and they act according to those preferences
- Preferences can be expressed numerically by using utility functions

# Previously on this course



- Random outcomes are modeled as lotteries
- Lotteries are simply probability distributions describing random outcomes
- The concept of preference can be extended to lotteries
- If preferences satisfy a set of axioms (rationality, continuity, independence) we can evaluate the utility of a lottery as the expected value of the utility, averaged over the possible outcomes

# Static games of complete information

# Game with multiple players



- How do multiple players interact?
  - We assume they have a payoff (utility) function
- Remember: rational players move to maximize of their own payoffs
- What is the simplest interaction like this?

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- **Static**: all players move together; they do not necessarily play simultaneously, but without knowledge of everyone else's move
- **Complete information**: meaning everyone's payoff function is known
  - most games covered within this class are "artificial" (theoretical models)
  - however, there are also actual games that can be modeled as static games of complete information. **Examples?**

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- Each player $i$ in the game simultaneously and independently chooses an action from its own set of available actions $A_i$
- The combination of actions chosen by the $n$ players determines the outcome of the game
- Outcome $(a_1, a_2, \ldots, a_n)$ determines a payoff for each player through an individual utility function of player $i$:

$$u_i = u_i(a_1, a_2, \ldots, a_n)$$

- 3 ingredients = actions + outcome + utility

# Action versus strategy



- In decision problems we always thought in terms of actions
- In games, it is useful to think in terms of **strategies** instead
- A strategy is a **plan of action**
  - e.g. if these conditions are met, then my action is $a$, otherwise it is either $a'$ or $a''$
  - this plan can even be random (we will see why)
- For the time being, let's consider only **deterministic** plans
- These are called **pure strategies**

# Normal form of a game

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- Each player $i$ simultaneously chooses a strategy from a set of pure strategies $S_i$
- This results in a given action chosen by each of the $n$ players that ultimately determines a payoff for each player
- If any player $i$ plays strategy $s_i \in S_i$, the combination of moves is $(s_1, s_2, \dots, s_i, \dots, s_n)$
- Player $i$ gets payoff $u_i(s_1, s_2, \dots, s_i, \dots, s_n) \in \mathbb{R}$
- The **normal form** of the game is specified by $\mathbb{G} = \{S_1, \dots, S_n; u_1, \dots, u_n\}$

# Simultaneous and independent

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- Simultaneous moves do not necessarily need to happen at the same time
  - they are chosen without knowledge of everyone else's actions
- These two cases are both considered simultaneous:
  - case A: two players are writing their strategy on opposite sides of a board at the same time
  - case B: player 1 is asked to write first; while player 1 writes, player 2 is blindfolded; then the board is turned and player 2 writes

# Common knowledge



- We say that E is common knowledge if:
  - everyone knows E
  - everyone knows that everyone knows E
  - ...
- Common knowledge in games is a powerful assumption, and not an obvious one
  - it requires each player to have full knowledge not only on information pertaining themselves, but also on everyone else

# Common knowledge

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- "Complete information" in games means that the following information is common knowledge:
  - all possible actions of all players
  - all possible outcomes resulting from these actions
  - the individual preferences of all players about these outcomes (i.e., their utilities about them)
- Player rationality is also common knowledge
  - meaning that everyone is maximizing their own payoff and everyone knows that everyone is maximizing their payoff

# Array representation

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- An $n$-player game can be represented as a function in $S_1 \times S_2 \times \cdots \times S_n$ that maps tuples of strategies to tuples of payoffs
- If all $S_i, i = 1, \ldots, n$ are discrete sets, we can represent a game using an $n$-dimensional array that contains a tuple of $n$ values (the payoffs) in each cell/entry
  - The overall number of cells is $m_1 \times \cdots \times |S_n|$
- Typically we consider games with $n = 2$ players, which can be represented using an ordinary matrix containing a pair of values in each cell
  - This is called a **bi-matrix**

# Example of bi-matrix

- $|S_1| = m_1, |S_2| = m_2$

Player 2

Player 1

| | $s_2^{(1)}$ | $s_2^{(2)}$ | $\cdots$ | $s_2^{(m_2)}$ |
|---|---|---|---|---|
| $s_1^{(1)}$ | $u_1(s_1^{(1)}, s_2^{(1)})$, $u_2(s_1^{(1)}, s_2^{(1)})$ | $u_1(s_1^{(1)}, s_2^{(2)})$, $u_2(s_1^{(1)}, s_2^{(2)})$ | $\cdots$ | $u_1(s_1^{(1)}, s_2^{(m_2)})$, $u_2(s_1^{(1)}, s_2^{(m_2)})$ |
| $s_1^{(2)}$ | $u_1(s_1^{(2)}, s_2^{(1)})$, $u_2(s_1^{(2)}, s_2^{(1)})$ | $u_1(s_1^{(2)}, s_2^{(2)})$, $u_2(s_1^{(2)}, s_2^{(2)})$ | $\cdots$ | $u_1(s_1^{(2)}, s_2^{(m_2)})$, $u_2(s_1^{(2)}, s_2^{(m_2)})$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $s_1^{(m_1)}$ | $u_1(s_1^{(m_1)}, s_2^{(1)})$, $u_2(s_1^{(m_1)}, s_2^{(1)})$ | $u_1(s_1^{(m_1)}, s_2^{(2)})$, $u_2(s_1^{(m_1)}, s_2^{(2)})$ | $\cdots$ | $u_1(s_1^{(m_1)}, s_2^{(m_2)})$, $u_2(s_1^{(m_1)}, s_2^{(m_2)})$ |

# Example 1

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- Player A has strategies $S_A = \{\text{U, M, D}\}$
- Player B has strategies $S_B = \{\text{L, R}\}$

| Player A \ Player B | L | R |
|---|---|---|
| U | 8, 0 | 0, 5 |
| M | 1, 0 | 4, 3 |
| D | 0, 7 | 2, 0 |

# Example 2



- Player A has strategies $S_A = \{\text{U, M, D}\}$
- Player B has strategies $S_B = \{\text{L, C, R}\}$

| Player A | Player B | L | C | R |
|---|---|---|---|---|
| | U | 0, 5 | 4, 0 | 7, 3 |
| | M | 4, 0 | 0, 5 | 7, 3 |
| | D | 3, 7 | 3, 7 | 9, 9 |

# Example: Odds and evens



- Player Odd and Even bet 4 euros
- Player Odd has two strategies: $\{0, 1\}$
- Plater Even has two strategies: $\{0, 1\}$

| | | Even: 0 | Even: 1 |
|---|---|---|---|
| Odd | 0 | -4, 4 | 4, -4 |
| | 1 | 4, -4 | -4, 4 |

# Example: Rock-paper-scissors

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- Both players have strategies $S_A = S_B = \{\text{R, P, S}\}$

Player B

| Player A | | R | P | S |
|---|---|---|---|---|
| | R | 0, 0 | -4, 4 | 4, -4 |
| | P | 4, -4 | 0, 0 | -4, 4 |
| | S | -4, 4 | 4, -4 | 0, 0 |

# Example: Battle of the Sexes



- Two college students, A and B, need to decide which night event to attend: rock concert (R) or science night (S).
- A prefers the concert, while B prefers the science night. However, both prefer to spend time with each other rather then separately
- They have not exchanged contact yet, so they are taking their decision independently and without communicating

| | | B: R | B: S |
|---|---|---|---|
| A | R | 2, 1 | 0, 0 |
| A | S | 0, 0 | 1, 2 |

- Simplified version: each player chooses between two options $S_A = S_B = \{\text{M}, \text{F}\}$
  - M: Lose 1 euro
  - F: The other player loses 20 euros

| | | Player B | |
|---|---|---|---|
| | | M | F |
| Player A | M | -1, -1 | -21, 0 |
| | F | 0, -21 | -20, -20 |

# Example: Prisoner's dilemma



- Original version: the players are two criminals caught by the police. The police has evidence only for petty theft but not for a major crime. The two are arrested and interrogated in separate room. They can decide to either
  - Keep mum (M), i.e., to not talk
  - Fink (F), i.e., to snitch on their partner
- Payoffs represent the years of jail they get

| | | Player B | |
|---|---|---|---|
| | | M | F |
| Player A | M | -1, -1 | -9, 0 |
| | F | 0, -9 | -6, -6 |

# Pareto efficiency



- A joint strategy $s$ is **Pareto-dominated** by another strategy $s'$ if

$$u_i(s') \geq u_i(s) \text{ for each player } i$$
$$u_i(s') > u_i(s) \text{ for some player } i$$

- A joint strategy $s$ that is not Pareto-dominated by any other joint strategy $s'$ is called **Pareto-efficient**
- There may be more than one Pareto-efficient strategy, none of which dominates the others

# Strict dominance

# Strictly dominated strategy



- Consider game $\mathbb{G} = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$
- If $s_i, s_i' \in S_i$, we say that $s_i$ is *strictly dominated* by $s_i'$ if $i's$ payoff when playing $s_i'$ is always greater than when playing $s_i$ for any possible choice of moves by the other players
- Formally

$$u_i(s_1, \ldots, s_i', \ldots, s_n) > u_i(s_1, \ldots, s_i, \ldots, s_n)$$

$$\forall (s_1, \ldots, s_{i-1}, \ldots, s_{i+1}, \ldots, s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \ldots S_n$$

- Rational players do not play strictly dominated strategies

# Back to example 1

- Can you find any strictly dominated strategy?

| Player A | | Player B: L | Player B: R |
|---|---|---|---|
| | U | 8, 0 | 0, 5 |
| | M | 1, 0 | 4, 3 |
| | D | 0, 7 | 2, 0 |

# Back to example 1

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- Strategy D is dominated by M for player A
  - $u_A(M, L) = 1 > u_A(D, L) = 0$
  - $u_A(M, R) = 4 > u_A(D, R) = 2$

Player B

| Player A | | L | R |
|---|---|---|---|
| | U | 8, 0 | 0, 5 |
| | M | 1, 0 | 4, 3 |
| | D | 0, 7 | 2, 0 |

# Back to example 1

- Now strategy L is dominated by R for player B
  - $u_B(U, R) = 5 > u_B(U, L) = 0$
  - $u_B(U, R) = 3 > u_B(D, R) = 0$

Player B

| Player A | | L | R |
|---|---|---|---|
| | U | 8, 0 | 0, 5 |
| | M | 1, 0 | 4, 3 |

# Back to example 1

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- Now strategy U is strictly dominated by M for A
  - $u_A(M, R) = 4 > u_A(U, R) = 0$

Player B

| Player A | R |
|---|---|
| U | 0, 5 |
| M | 4, 3 |

- Rational players end up playing (M, R) with payoffs (4, 3)

# Back to the prisoner's dilemma

- Are there any strictly dominated strategies?

| | | Player B: M | Player B: F |
|---|---|---|---|
| Player A | M | -1, -1 | -9, 0 |
| Player A | F | 0, -9 | -6, -6 |

# Back to the prisoner's dilemma

- For both players, M is strictly dominated by F
  - $u_A(F, M) = 0 > u_A(M, M) = -1$
  - $u_A(F, F) = -6 > u_A(F, M) = -9$

| | | Player B | |
|---|---|---|---|
| | | M | F |
| Player A | M | -1, -1 | -9, 0 |
| | F | 0, -9 | -6, -6 |

- Rational players end up playing (F, F)
- This is not a Pareto-efficient joint strategy: (F, F) is Pareto-dominated by (M, M)
- The final outcome is "bad", hence the dilemma

# Solving problems via IESDS



- This procedure is called "iterated elimination of strictly dominated strategies" (IESDS)
- Sometimes, it allows to obtain a reduced version of a game by relying on common knowledge
  - All players know that a certain strategy is dominated for one player, so they rule it out
- Unfortunately, in many cases it does not lead to a solution for the game

# Example 2

- In this game there is no strictly dominated strategy
- However, (D, R) seems to be a good choice for the players

| | | Player B | | |
|---|---|---|---|---|
| | | L | C | R |
| Player A | U | 0, 5 | 4, 0 | 7, 3 |
| | M | 4, 0 | 0, 5 | 7, 3 |
| | D | 3, 7 | 3, 7 | 9, 9 |

# Back to Odds and evens



- No strategy seems to be better than the other

| Odd \ Even | 0 | 1 |
|---|---|---|
| 0 | -4, 4 | 4, -4 |
| 1 | 4, -4 | -4, 4 |

# Example: Battle of the Sexes



- There are two joint strategies that are "good" for rational players: (R, R) and (S, S)

| | | B: R | B: S |
|---|---|---|---|
| A | R | 2, 1 | 0, 0 |
| A | S | 0, 0 | 1, 2 |

Questions?