University of Padova – prof. Leonardo Badia
Game Theory exam – January 28, 2016

Exercise 1 Agatha (A) and Bruno (B) are two siblings. They own a model train set (T) and they usually play together with it. Their respective utilities when doing so is $u_A(\mathtt{T},\mathtt{T}) = u_B(\mathtt{T},\mathtt{T}) = 3$. Their mom (M) buys them a new toy (N), which can be either a dollhouse (D) or a set of small soldier figurines (S). The two kids must independently decide whether to keep playing with T or the new toy N. All the three players (A, B, M) involved decide their action simultaneously and unbeknownst to each other. The action set of the kids includes just T or N; the mom instead decides what new toy to buy. If the kids end up in not playing together, *all* the players (also M) get utility equal to 0. If they play together with the new toy, they get a positive utility that for A is equal to 6 and 1 for D and S, respectively; for B, the values are instead 1 and 4, respectively. The utility of M is the lump sum of the utility of the two kids.

1. Write down the normal form of this game (likely, you will need a 3D matrix: you can write two matrices instead, one per each move of player M).

2. Find all Nash equilibria of this game in pure strategies.

3. Find all additional Nash equilibria of this game in mixed strategies.

1. Normal form:

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{B} \\
 & & N & T \\
\hline
\multirow{2}{*}{A} & N & \underline{6},\underline{1},\underline{7} & 0,0,0 \\
\cline{3-4}
 & T & 0,0,0 & \underline{3},\underline{3},\underline{6} \\
\hline
\end{array}
\qquad
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{B} \\
 & & N & T \\
\hline
\multirow{2}{*}{A} & N & \underline{1},\underline{4},5 & 0,0,0 \\
\cline{3-4}
 & T & 0,0,0 & \underline{3},\underline{3},\underline{6} \\
\hline
\end{array}
$$

$\uparrow$ M plays D $\qquad\qquad$ $\uparrow$ M plays S

2. Best responses are underlined in the above matrices

Pure NE: (N,N,D), (T,T,D), (T,T,S)

3. [Hard way]

Let p = prob. A plays N

q = prob. B plays N

r = prob. M plays D

Indifference principle leads to system of eq.

$$\begin{cases} 6qr + q(1-r) = 3(1-q) & \text{(A)} \\ pr + 4p(1-r) = 3(1-p) & \text{(B)} \\ 7pq + 6(1-p)(1-q) = 5pq + 6(1-p)(1-q) & \text{(M)} \end{cases}$$

(M) $7pq + \cancel{6(1-p)(1-q)} = 5pq + \cancel{6(1-p)(1-q)}$

$\hookrightarrow$ $p = 0 \lor q = 0$

(otherwise M prefers D)

Note: BR = best response

- if $p = 0$
  $\hookrightarrow$ B's BR is $q = 0$
  M's BR is any r

- if $q = 0$
  $\hookrightarrow$ A's BR is $p = 0$
  M's BR is any r

$\boxed{\text{Any } (0, 0, r) \text{ is a mixed NE}}$

Note:
this means that M buys D or S with any probability but children keep playing with T

if $p>0 \wedge q>0$, M prefers D to S (strictly)

$\to$ check mixed NE $(p, q, 1)$

$r = 1$

$$\begin{cases} 6q = 3(1-q) \to q = \frac{1}{3} & (A) \\ p = 3(1-p) \to p = \frac{3}{4} & (B) \end{cases}$$

Note: M buys D, A and B play with new or old toy with some probabilities $\left(\frac{3}{4}, \frac{1}{3}\right)$

$\to$ Mixed NE: $\boxed{(0, 0, r) \ \forall r \in (0,1) \quad \left(\frac{3}{4}, \frac{1}{3}, 1\right)}$

3. ["Smart" way]

p = prob A plays N $\qquad$ q = prob B plays N

r = prob M plays D

if $p>0$ and $q>0$ $\to$ M prefers D $(r=1)$

$$\begin{cases} 6q = 3(1-q) \to q = \frac{1}{3} & (A) \\ p = 3(1-p) \to p = \frac{3}{4} & (B) \end{cases} \to \boxed{\left(\frac{3}{4}, \frac{1}{3}, 1\right) \text{ is mixed NE}}$$

if $p=0$ or $q=0$ $\rightarrow$ either A or B play T with prob 1

The sibling's BR is also T with prob 1.

M's choice does not matter

$\rightarrow (0,0,r)$ is mixed NE $\forall r \in (0,1)$

$$\boxed{\begin{array}{c}\text{Mixed NE: } (0,0,r) \ \forall r \in (0,1) \\ \left(\frac{3}{4}, \frac{1}{3}, 1\right)\end{array}}$$

University of Padova – prof. Leonardo Badia
Game Theory exercises

Exercise 1 A gang of pirates has n ranks from 1 (the ship's boy) to n (the captain). After a raid, they share a treasure. Pirate with rank $k+1$ keeps an eye on pirate k to see whether he gets a bigger share than he should. The game starts when pirate $k=1$ (the ship's boy) realizes that the treasure contains an extremely valuable pearl that has fallen far from the stash: he considers whether to take the pearl for himself (P) hiding it in his pocket or do nothing (N). Doing nothing ends the game with the pearl being unnoticed and unassigned. However, if pirate k takes the pearl, pirate $k+1$ will notice it; now, pirate $k+1$ may consider to kill him and keep the pearl for himself (P), or do nothing (N). If pirate $k+1$ does nothing, pirate k is left alive with the pearl – a very good outcome. If pirate $k+1$ kills pirate k and takes the pearl instead, this is spotted by pirate $k+2$ that now faces the same choice: whether to kill pirate $k+1$ and keep the pearl for himself (P), or to do nothing (N). This means that k is replaced with $k+1$ and the game continues up to the captain. For every pirate, the top preference is to stay alive and have the pearl; after that, they all prefer being alive without the pearl than to be killed.

1. Consider $n=5$. Choose appropriate utility values for the outcomes and draw the extensive form of the game.

2. Consider $n=5$. Solve the game by finding its subgame-perfect outcome.

3. Consider $n=8$. Does the subgame-perfect outcome change, and why?

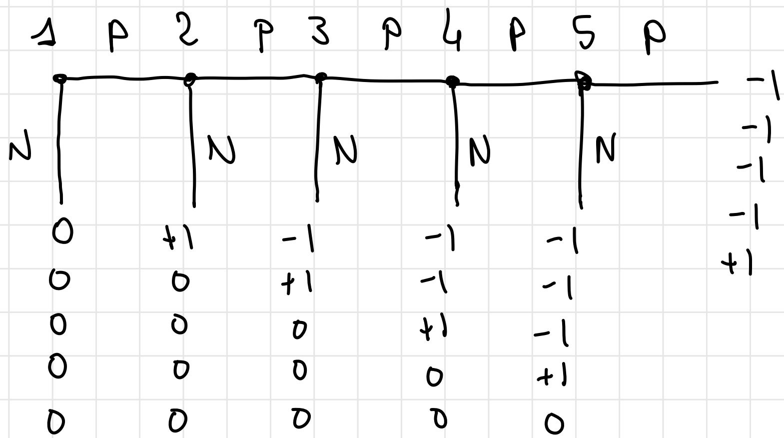
1. For each pirate: alive + pearl $\succ$ alive $\succ$ dead

$u = +1$ (alive + pearl), $u = 0$ (alive), $u = -1$ (dead)

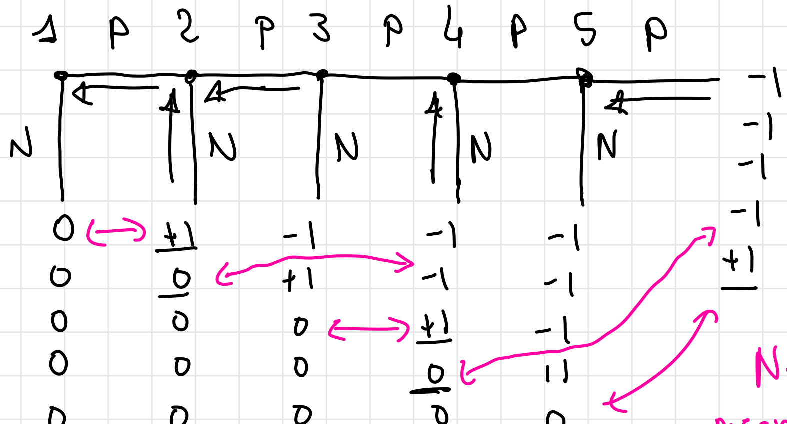
↳ Note: any other choice of payoffs that preserves preferences works too

$\boxed{\text{Case } n = 5}$

ship's boy



2. Note: arrows = best choices according to backward induction



Note: $v \leftrightarrow v'$ means that "i" gets to choose between those

The outcome of backward induction is subgame-perfect:

1 plays P and 2 plays N

Note: you don't need to draw the extensive form twice, I only did it for clarity

3. For $n=8$ you can repeat the same reasoning knowing that 8 will play P (no one can kill him)

8	→	7	→	6	→	5	...	2	→	1
P		N		P		N		P		N

Subgame-perfect outcome: 1 plays N

University of Padova – prof. Leonardo Badia
Game Theory exercises

Exercise 2 The mayor of a big city is to be selected among four candidates: (A)Amanda Amour; (B)Bruno Bravery; (C)Claire Constitution; (D)Dave Democracy. Using symbol $\succ$ to denote "preferred to", polls indicate that:

- (A) has 42% of supporters. Also, for them (B)$\succ$(C)$\succ$(D).
- (B) has 11% of supporters. Also, for them (A)$\succ$(C)$\succ$(D).
- (C) has 27% of supporters. Also, for them (B)$\succ$(D)$\succ$(A).
- (D) has 20% of supporters. Also, for them (C)$\succ$(B)$\succ$(A).

2 players with most votes go to run-off

1. The election is being held as a two-round run-off (i.e., ~~with a ballot~~). What is the outcome under *sincere voting*? Denote the winner as W.
2. Assume that the supporters of (D) can identify this outcome and plan a strategy. What is the best *strategic voting* that they can enact?
3. Discuss the identity of the winner W' under strategic vote of (D)'s supporters. What kind of choice is W'? Can the supporters of W prevent this outcome by counteracting strategic vote of (D)'s supporters, with a strategic vote of their own?

1.

	V_1 (42)	V_2 (11)	V_3 (27)	V_4 (20)
best	A	B	C	D
	B	A	B	C
	C	C	D	B
worst	D	D	A	A

With sincere voting:

- run-off: A vs C

  A gets $V_1 + V_2 = 53$ $\rightarrow$ $\boxed{W = A}$

  C gets $V_3 + V_4 = 47$

2. D's supporters strategic vote: vote for B

- first round: A gets $V_1 = 42$ D gets 0

  B gets $V_2 + V_4 = 31$

  C gets $V_3 = 20$

- run-off: A vs B

  A gets $V_1 = 42$ $\rightarrow$ $\boxed{W' = B}$

  B gets $V_2 + V_3 + V_4 = 58$

3. A's supporters can split their votes

$$V_1 = 42 \begin{cases} \rightarrow 32 \text{ vote for A} \\ \rightarrow 10 \text{ vote for C} \end{cases}$$

Note: other splits are possible as long as A and C go to run-off

First round:

A gets 32

B gets 31

C gets 27+10 = 37

D gets 0

Run-off: A vs C → $\boxed{\text{A wins}}$

University of Padova – prof. Leonardo Badia
Game Theory exercises

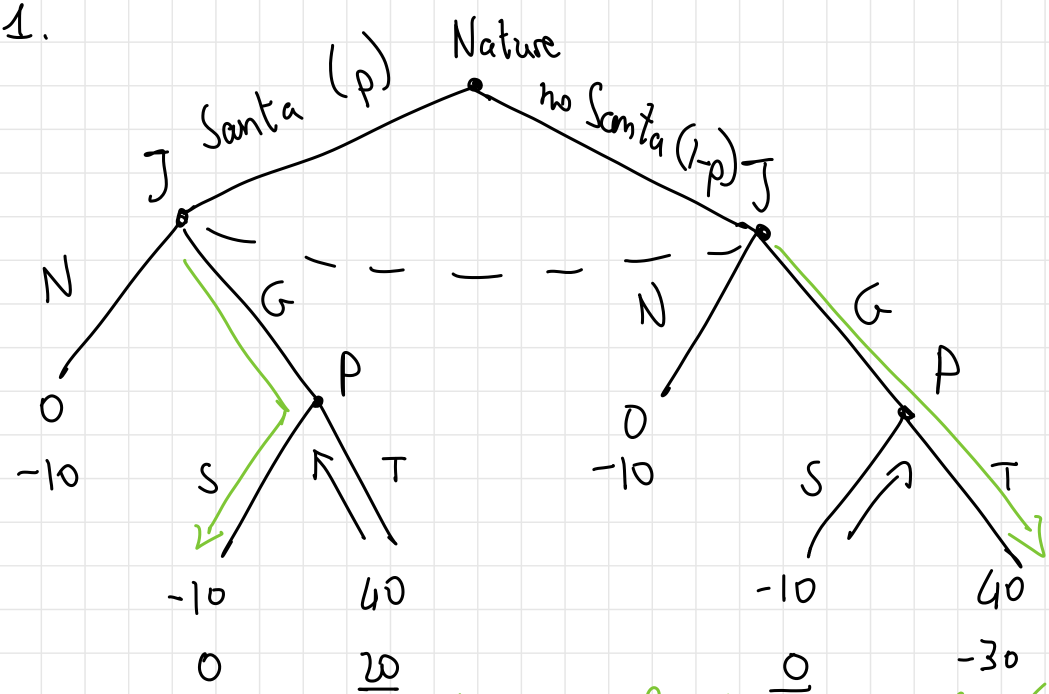
Exercise 4 Little Jimmy really hopes to receive a space train (T) for Christmas. His parents have already bought him a sweater (S) but they tell him that he might receive T if he behaves as a good boy (G) instead of a naughty one (N). Jimmy's behavior ultimately depends on whether Santa Claus exists or not, which he evaluates as having probability p. If Jimmy is naughty, he will not receive any gift, regardless of Santa Claus existing or not. Jimmy and the parents' utility will be 0 and -10, respectively. If Jimmy is good, he will either receive S or T, according to the decision made by his parents. If Santa exists, Jimmy can receive the space train at no cost for his parents. If Santa does not exist, Jimmy can receive T only if the parents pay for it. Jimmy's utility for S and T is -10 and $+40$, respectively. The parents' reward when giving S and T to their son is 0 and $+20$. However, if they have to buy the space train themselves, subtract 50 from their utility. Also note that, being adults, the parents know whether Santa exists or not; on the other hand, they also know the value of p estimated by their son (that is, the prior is common knowledge).

1. Represent this game in extensive form.

2. Represent this game in normal form, with a type-player representation of the Bayesian players.

3. What kind of equilibria would be enough to characterize this Bayesian game? Discuss for $p = 0.9$.

1.



Note: green lines show payoffs to average for (G, ST)

2.

		P: SS	P: ST	P: TS	P: TT
J	N	0, -10	0, -10	0, -10	0, -10
	G	-10, 0	$40-50p$, $30p-30$	$50p-10$, $20p$	40, $50p-30$

Note: if J plays N, payoffs are the same both for "Santa" and "no Santa"

(*)

$$p \cdot u(G,S \mid \text{santa}) + (1-p)\, u(G,S \mid \text{no santa})$$

$$= p \begin{bmatrix} -10 \\ 0 \end{bmatrix} + (1-p) \begin{bmatrix} 40 \\ -30 \end{bmatrix}$$

$$= \begin{bmatrix} 40-50p \\ 30p-30 \end{bmatrix}$$

Bayesian strategy XY for P

if "Santa" if "no Santa"

3. For $p = 0.9$

	SS	ST	TS	TT
N	$\underline{0}$, $\underline{-10}$	$\underline{0}$, $\underline{-10}$	0, $\underline{-10}$	0, $\underline{-10}$
G	-10, 0	-5, -3	$\underline{35}$, $\underline{18}$	$\underline{40}$, 15

BNE: (N, SS), (N, ST), (G, TS)

SPE: (G, TS) → sequentially rational

Screening game → SPE is enough
(J receives no signal, so no reason to set up system of beliefs)