Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

Lecture 14
Multistage games (part 2)

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Previously on game theory

- **Multistage games**: a specific type of dynamic games, which consist in a sequence of smaller games, called "stages"
- Payoffs in a multistage game are just the sum of all stages, possibly with exponential discount δ^{t-1} for stage t

$$u = u^{(1)} + \delta u^{(2)} + \cdots + \delta^{T-1} u^{(T)}$$

Example: Prisoner-Revenge

- A and B play the Prisoner's dilemma
- After that, they eventually go out of jail and can either gather a gang and fight the other (G), or leave the other alone (L)
  - If they both choose to leave the other alone, they never meet again $\rightarrow$ payoff is 0 for both
  - If they both gather a gang, they get into a harsh fight that leaves both quite beaten $\rightarrow$ payoff is -3 for both
  - If one gathers a gang and the other stays alone, the loner gets beaten to a pulp ($\rightarrow$ payoff -4), but also the other gets some bruises ($\rightarrow$ payoff -1)

Prisoner-Revenge

- The normal-form representation of each stage is as follows

First stage (Prisoner's dilemma)

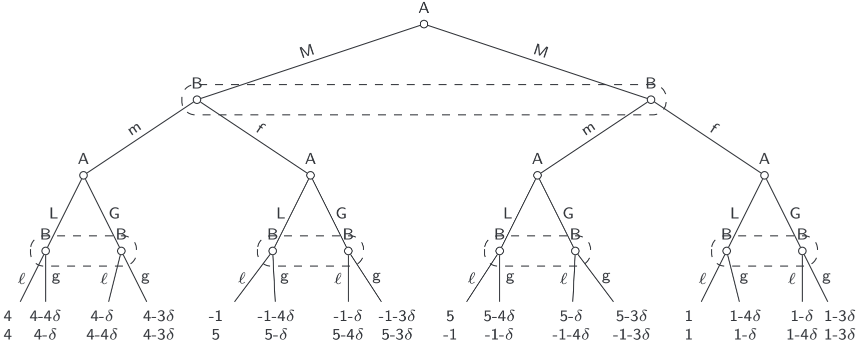
A \ B	m	f
M	4, 4	-1, 5
F	5, -1	1, 1

Second stage (Revenge)

A \ B	ℓ	g
L	0, 0	-4, -1
G	-1, -4	-3, -3

- Suppose the payoffs are aggregated with discount δ
- E.g., if the outcome of the first stage is (F, m) and the outcome of the second stage is (G, g), the payoffs are $u_A = 5 - 3\delta$ for player A, and $u_B = -1 - 3\delta$ for player B

- Extensive form



Strategies in multistage games

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- A strategy for each player must specify
  - what to do in the first stage (just one action)
  - what to do in the subsequent game(s) depending on the outcome of the previous game(s)
- The Prisoner-Revenge game has already 32 possible strategies (already complex enough)
- Strategies can be thought of as "I play X at stage 1, then I play Y if the outcome of stage 1 is O"

SPE in multistage games

- Remember that a SPE is a joint strategy that is a NE in every subgame
- In multistage games, each stage is independent of the others, therefore:
- **Theorem**: Suppose $s^{(t)} = (s_1^{(t)}, \dots, s_n^{(t)})$ is a NE for stage t of multistage game $\mathbb{G}$; then there is a SPE of $\mathbb{G}$ with equilibrium path $s^{(1)}, \dots, s^{(T)}$
- Specifically, a strategy where each players chooses the same NE strategy at each stage (ignoring the outcome of other stages) is always a SPE

SPE in Prisoner-Revenge

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- In the Prisoner-Revenge game, (F,f) is a NE of the first stage; (G,g) and (L,ℓ) are NE of the second stage
- This means that (FLLLL,f$\ell\ell\ell\ell$) and (FGGGG,fgggg) are SPE for the multistage game, since a NE is played in both the first and second stage
- However, these strategies ignore any possible strategic link between stages
  - The stage games are played independently
  - Are there other SPE that exploit the connection between stages?

SPE in multistage games

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- We need to start from the end of the game
  - as we did in backward induction
- **Theorem**: Any NE s^* (even if it is not a SPE) of multistage game $\mathbb{G} = (\mathbb{G}_1, \ldots, \mathbb{G}_T)$ requires that a NE is played in the last stage $\mathbb{G}_T$
- Intuition: In the last stage, players do not have future stages that influence their payoff: they just play the best responses for that stage
- **Theorem**: If stage games $\mathbb{G}_1, \ldots, \mathbb{G}_T$ all have a unique NE, then $\mathbb{G} = (\mathbb{G}_1, \ldots, \mathbb{G}_T)$ has a unique SPE

Strategic connections

- These last two theorems imply that if the last stage of a multistage game has a single NE, that is what rational players will play
  - Not much of a surprise, and nothing we can do about it
- However, what if the last stage has **multiple** NE?
  - Surprisingly, that enables non-NE strategies to be played in previous stages
  - In other words, it is possible to find SPE where players do not play the NE strategy in some of the stages

Strategic conn. in Prisoner-Revenge

- In Prisoner-Revenge (M,m) is not a NE for the first stage
  - in fact, strategy M is strictly dominated for both players
- However, in the second stage we have two NE: a "good" NE (L,ℓ) and a "bad" NE (G, g)
- If the discount factor δ is *large enough*, we can leverage the NE in the second stage to enforce collaboration in the first stage
  - Remember: a larger δ means that players care more about future payoffs

Strategic conn. in Prisoner-Revenge

- Set strategies s_A =(M,L,G,G,G) and s_B =(m,ℓ,g,g,g)
- In other words, the strategy for both players is "Play M in the 1st stage; in the 2nd stage, play L if the outcome was (M, m) in stage 1, otherwise play G"
- That means each prisoner will leave the other alone if they both keep mum in the first stage, otherwise they will join a gang and pick a fight
- We need to verify if this strategy is **sustainable**
  - Do players have an incentive to deviate from such strategy?
  - Remember: a "deviation" means that one player changes strategy while the other keeps the current one

Strategic conn. in Prisoner-Revenge

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s_A =(M,L,G,G,G) and s_B =(m,ℓ,g,g,g) => new strategy

First stage (Prisoner's dilemma)

A \ B	m	f
M	4, 4	-1, 5
F	5, -1	1, 1

Second stage (Revenge)

A \ B	ℓ	g
L	0, 0	-4, -1
G	-1, -4	-3, -3

- In the last stage, no player has incentive to deviate unilaterally, since the strategy always involves playing a NE in it
- We need to check in stage 1 if s_A =(M,L,G,G,G) is a best response to s_B =(m,ℓ,g,g,g)
  - $u_A(s_A, s_B) = 4 + 0\delta$
  - $u_A((F,L,G,G,G), s_B) = 5 - 3\delta$ (change only the strategy at stage 1)
- The strategy is *sustainable if* $4 \geq 5 - 3\delta$, i.e. for $\delta \geq 1/3$

Strategic connections: comment

- Strategic connection is possible if the last stage has multiple NE that are different in terms of payoff: a "stick" and a "carrot"
- So, the SPE is created as follows:
  - play desired non-NE action in the first stage
  - reward the other player(s) with the "carrot" if they collaborate
  - otherwise, punish them with the "stick"
- The discount factor δ should be high enough for the difference between stick and carrot to have an impact

Strategic connections: comment

- The value of δ relates to the credibility of a threat
- If $\delta \to 0$, players do not care about future; therefore, threatening punishment with the "stick" is not credible
- The punishment is effective only if short-term gains are not worth compared to long-term losses
  - Note that the latter is weighted on δ

Strategic conn. in Prisoner-Revenge

- The carrot-and-stick strategy can be used to create a SPE with any joint outcome in the first stage (as long as it is sustainable)
- For example, we can create a SPE that supports (F, m) as first moves (the rest of the strategy is identical: friendly NE if all players comply, gang NE otherwise)
- This joint strategies is denoted as (s_1, s_2) with $s_1 =$(F,L,G,G,G) and $s_2 =$(m,ℓ,g,g,g)
- Always sustainable for player A (best response played in both stages)
- Sustainable for player B if $u_B(s_1, (\text{m},\ell,\text{g},\text{g},\text{g})) \geq u_B(s_1, (\text{f},\ell,\text{g},\text{g},\text{g}))$

Strategic conn. in Prisoner-Revenge

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GENERAL RULE:
u(cooperation) + δu(carrot) >= u(unilateral deviation) + δu(stick)

First stage (Prisoner's dilemma)

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Second stage (Revenge)

A \ B	ℓ	g
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- To find δ such that the strategy is sustainable (and thus a SPE), we solve $u_B(s_1, (\text{m},\ell,\text{g},\text{g},\text{g})) \geq u_B(s_1, (\text{f},\ell,\text{g},\text{g},\text{g}))$ for δ
- We obtain $-1 + 0\delta \geq 1 - 3\delta$, meaning that (s_1, s_2) is a SPE if $\delta \geq 2/3$

One-stage deviation principle

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- Prisoner-Revenge has 2 stages
  - Deviations are possible only at stage 1
  - Stage 2 is the last: players must have a NE there
- How does this generalize when we have more stages? Maybe in a 5-stage game players are interested in deviating by changing their strategy in two stages at once
- We need to follow the **one-stage deviation principle**
- But first some definitions:
- **Optimal strategy**: A strategy s_i is optimal for player i if for each information set h_i there is no way to improve it
  - i.e., no s_i' such that $u_i(s_i'|\{h_i\}) > u_i(s_i|\{h_i\})$
- **One-stage unimprovable strategy**: A strategy s_i is one-stage unimprovable if there is no s_i' that differs in one single stage such that $u_i(s_i'|\{h_i\}) > u_i(s_i|\{h_i\})$ for some h_i

One-stage deviation principle

- Clearly, optimum $\Rightarrow$ one-stage unimprovable
- Interestingly, the opposite is also true
- **Theorem**: A one-stage unimprovable strategy must be optimal
- *Proof*: Proceed by contradiction, assuming that there is a strategy s_i that is *one-stage unimprovable* but not optimal.
  - Then, there must be a strategy s_i' that deviates from s_i by one step or more.
  - Consider the deviation on the last stage, and consider the subgame having the corresponding stage as root
  - In that game, strategy s_i' must also be better than s_i.
  - However, in that subgame there is only a one-stage deviation.
  - Therefore, s_i is *one-stage improvable* (contradiction). $\square$

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- Consider multistage game $\mathbb{G} = (\mathbb{G}_1, \mathbb{G}_2)$, with $\mathbb{G}_1$ being the first stage, and $\mathbb{G}_2$ being the second (and last) stage:
  - Is it possible (for some $\mathbb{G}_1$ and $\mathbb{G}_2$) to find a SPE for $\mathbb{G}$ where a non-NE is played in $\mathbb{G}_2$?
  - Is it possible (for some $\mathbb{G}_1$ and $\mathbb{G}_2$) to find a NE for $\mathbb{G}$ where a non-NE is played in $\mathbb{G}_2$?
  - Is it possible (for some $\mathbb{G}_1$ and $\mathbb{G}_2$) to find a SPE for $\mathbb{G}$ where a non-NE is played in $\mathbb{G}_1$?
  - Is it possible (for some $\mathbb{G}_1$ and $\mathbb{G}_2$) to find a SPE for $\mathbb{G}$ where a strictly dominated strategy is played in $\mathbb{G}_1$?
  - What is the minimum number of NE in stage game $\mathbb{G}_2$ to enable a carrot-and-stick SPE in $\mathbb{G}$? What characteristics should these NE have?

Exercise

Ashley and Brook live together. During the winter break they contemplate giving each other a nice gift (G) for Christmas or not (N). They know each other's preferences so they are able to buy a gift for 10 euros that is worth like 100 euros for the other. They make this decision independently and without telling each other. After Christmas, they also consider whether to celebrate New Year's eve downtown (D) or stay home (H).
For the New Year's eve celebration, they decide independently of each other in a coordination-game fashion. Staying home has utility of 0 for both. Going downtown has utility of 50. However, spending New Year's eve apart from each other has utility of -100 for both. The total payoff of the players is the sum of the partial payoffs in each stage with a discount factor of δ for the second stage.

Exercise

1. Write down the normal form of both stages of the multi-stage game.
2. Find a trivial subgame-perfect equilibrium of the game where the players just play a Nash equilibrium in all stages, without any strategic connection.
3. Is there a strategically connected SPE of the whole game where Ashley and Brook give gifts to each other? If so, show the minimum required discount factor value $\delta_{\min}$ for that to hold.

Questions?