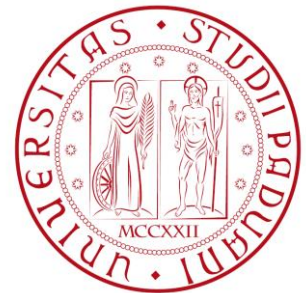


Lecture 02

Lotteries

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Game Theory 2023/24



Recap of previous lecture

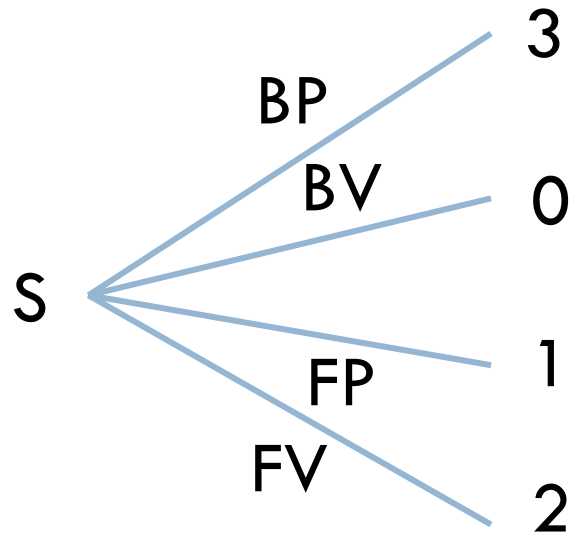
- In game theory, a game is a problem involving multiple agents (**players**) and multiple **objectives**
- **Decision problems:** actions, outcomes, preferences
- Players choose between different possible **actions**
- The **outcome** of a game is determined by the actions of all players
- Players have **preferences** about the outcome ($x \succcurlyeq x'$)
- We can use **utility functions** to obtain a quantitative representation of preferences

Recap of previous lecture

- Recall: Rational preferences satisfy
 - ▣ **Completeness:** for all $a, b \in A$ either $a \succcurlyeq b$ or $b \succcurlyeq a$
 - ▣ **Transitivity:** for all $a, b, c \in A$, $a \succcurlyeq b \wedge b \succcurlyeq c \Rightarrow a \succcurlyeq c$

- A student goes to the university cafeteria for lunch and needs to choose between:
 - Beef (B) or fish (F) for the main dish
 - Polenta (P) or vegetables (V) for the side dish
- Her preferences are:
 - Beef and polenta \succcurlyeq fish and vegetables
 - Fish and vegetables \succcurlyeq fish and polenta
 - Fish and polenta \succcurlyeq beef and vegetables
- Assign payoffs according to the preferences and draw the decision tree

- Here preferences are about combination of dishes, so each possible combination is a possible choice
- Assign $u(BV)=0$, $u(FP)=1$, $u(FV)=2$, $u(BP)=3$
- Draw a single-layered tree



LOTTERIES

- In decision problems, players are assumed to be fully aware of the consequences of their actions
- For 1-player problems actions = outcomes
- What about:
 - ▣ Incomplete information?
 - ▣ Random events?
- Can we still model problems that are affected by randomness as decision problems?

- Assume payoffs are affected by random outcomes
 - ▣ At the cafeteria, the food quality may vary
 - ▣ On one day, the fish might be rotten
 - ▣ How can we tell if beef is preferable?
- Rational players and randomness do not mix well together
- To make rational decisions involving random outcomes, we need to incorporate them into the utility function
 - ▣ How can we do that? By using the outcomes' **probability distribution**

□ Example

□ Beef gives $u(B)=6$ with 50% probability, $u(B)=4$ otherwise

□ Fish gives $u(F)=10$ with 90% probability, $u(F)=-10$ with 10% probability

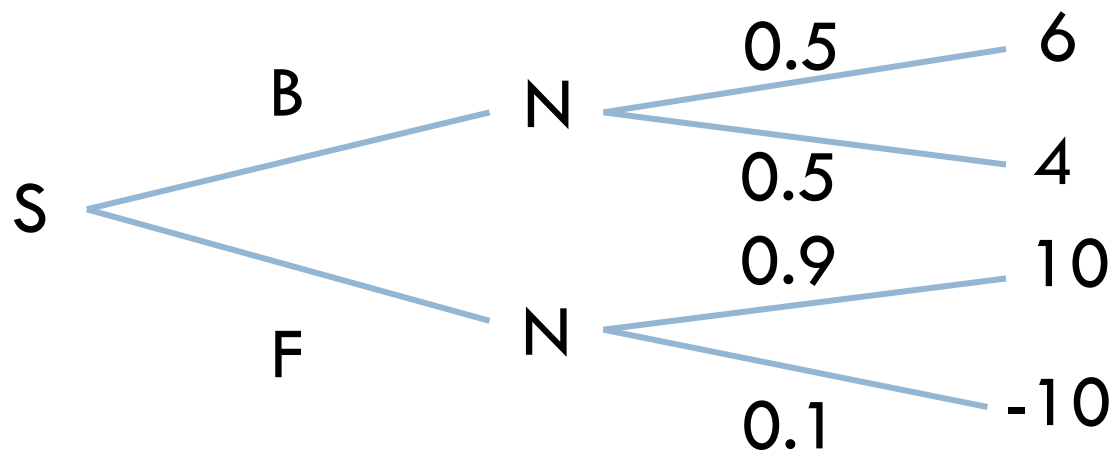
□ We can model the choice between B and F as a choice between two **lotteries**:

□ (B): utility is 6 or 4 with probabilities 0.5 and 0.5

□ (F): utility is 10 or -10 with probabilities 0.9 and 0.1

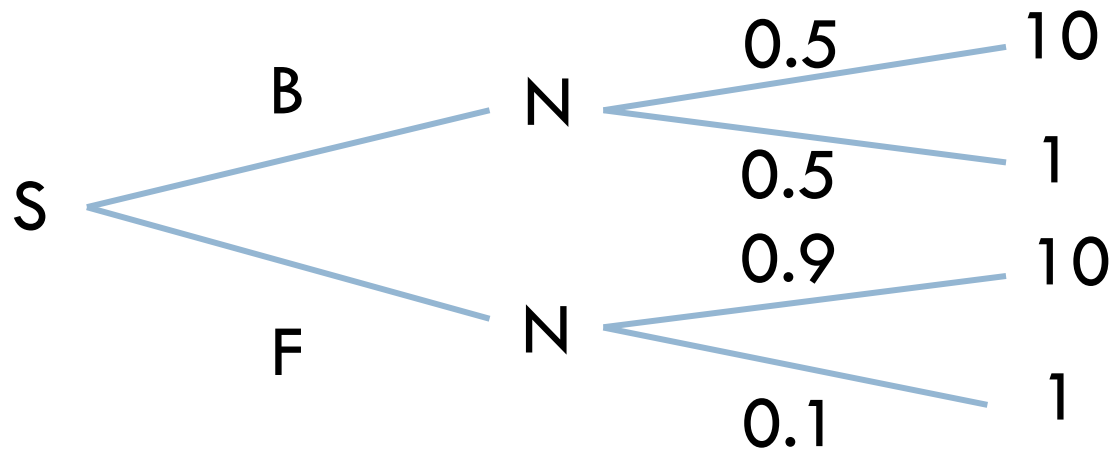
- **Definition:** A **lottery** over outcomes $X = \{x_1, \dots, x_n\}$ is a probability distribution p over X
 - $p(x_k) \geq 0, k = 1, \dots, n$
 - $\sum_{k=1}^n p(x_k) = 1$
- If actions are involved, p is conditional on the action
 - For $a \in A$, we consider $p(x_k|a)$ with the above properties
- A certain outcome can also be seen as a **degenerate lottery**: $p(x_k|a) = 1$ for some k and 0 for all $k' \neq k$

- In game theory jargon, random events are the consequences of the choices made by another player, called “Nature” (N)
- ▣ Nature chooses between outcomes x_1, \dots, x_n according to a lottery p
- ▣ This can be represented in the decision tree as follows



- Lotteries can also describe probabilities over a continuous space of events
- Probability of each specific outcome is zero
- Probability mass distribution \rightarrow Probability density
- Still possible to represent it using the decision framework, however it become a bit scuffed
 - ▣ Nature's choice cannot be represented in the decision tree

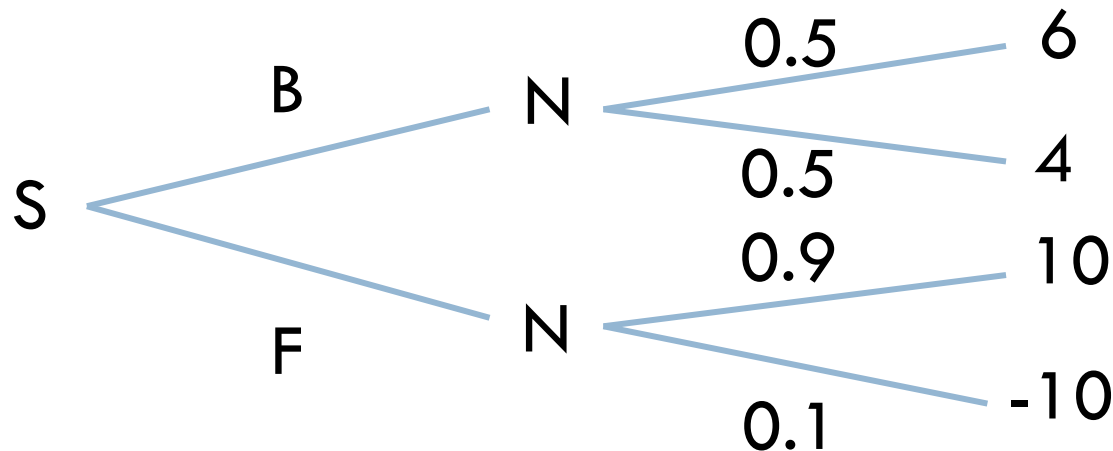
- Simplified problem: assume food can only be “tasty” or “not tasty” with $u(\text{tasty})=10$ and $u(\text{not tasty})=1$



- In this case, the obvious choice for rational players is fish, since they have higher chances to get 10

Evaluating random outcomes

- However, with different numbers the result is not clear
- What is better? B or F?




- B: fifty-fifty chances of getting either 6 or 4
- F: a high probability of getting 10 with a small chance of getting -10

- Usual way of comparing random outcomes: taking the expectation
 - Also works for “degenerate” lotteries (1 outcome with 100% probability)
 - “Expected utility theory” by von Neumann and Morgenstern
 - **Intuition:** if you repeat the same choice for N trials, for $N \rightarrow \infty$ average utility = expectation
- Expected payoff for lottery p
 - $\mathbb{E}_{x \sim p}[u(x)] = \sum_{k=1}^n p(x_k) \cdot u(x_k)$

N.B: Nature = Continuous variable = Random = Continuous Lotterie \rightarrow you have to find the average expectation

- Von Neumann – Morgenstern (VNM) framework to define preferences among lotteries
- We write $p \succcurlyeq q$ to say “lottery p is preferred to q ”
- Under VNM framework, preferences must satisfy:
 - Rationality (completeness and transitivity)
 - Continuity axiom
 - Independence axiom

- For lotteries p, q, r over action space A the following sets must be **closed**:
 - $\{a \in [0, 1]: ap + (1 - a)q \succcurlyeq r\}$ 
 - $\{a \in [0, 1]: r \succcurlyeq ap + (1 - a)q\}$
- This means that arbitrarily small variations in the gamble does not change preferred lotteries
 - If I prefer fish which is 100% not rotten to beef, I will still prefer fish if it has an arbitrarily small probability $\varepsilon > 0$ of being rotten

- For lotteries $p, q, r, \forall a \in [0, 1]$
 - $p \succcurlyeq q \Rightarrow (1 - a)p + ar \succcurlyeq (1 - a)q + ar$
- This means that if we mix the same amount of another lottery into two lotteries, the preference remains unchanged
 - If I like betting on soccer more than betting on horse races, then I prefer the lottery “if heads bet on soccer, if tails play roulette” to “if heads bet on horse races, if tails play roulette”

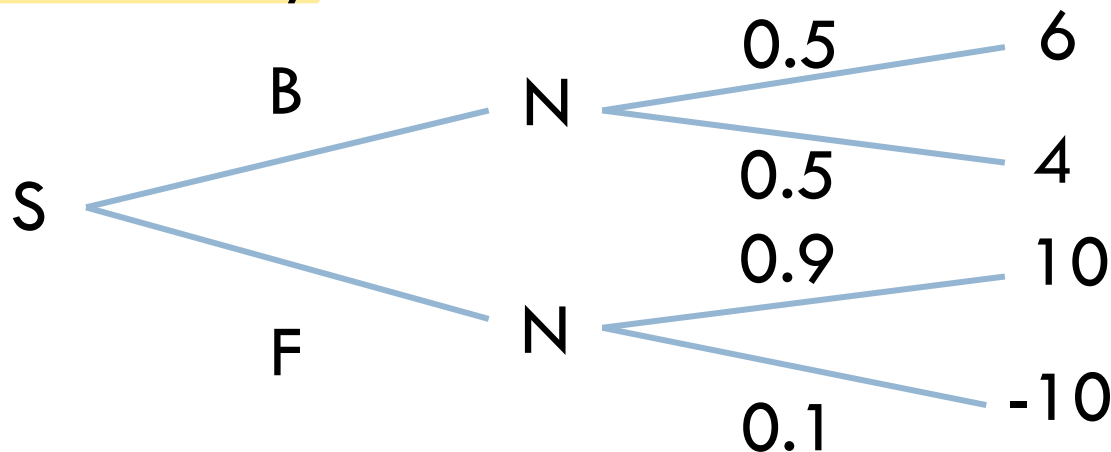
- **Theorem:** If \succsim satisfies the rationality, continuity and independence axioms, it can be mapped to u such that

$$p \succsim q \Rightarrow \mathbb{E}_{x \sim p}[u(x)] \geq \mathbb{E}_{x \sim q}[u(x)]$$

- **Remark:** If u is a suitable utility function to describe the preference \succsim , any affine (linear) transformation of u is also suitable

Expected utility

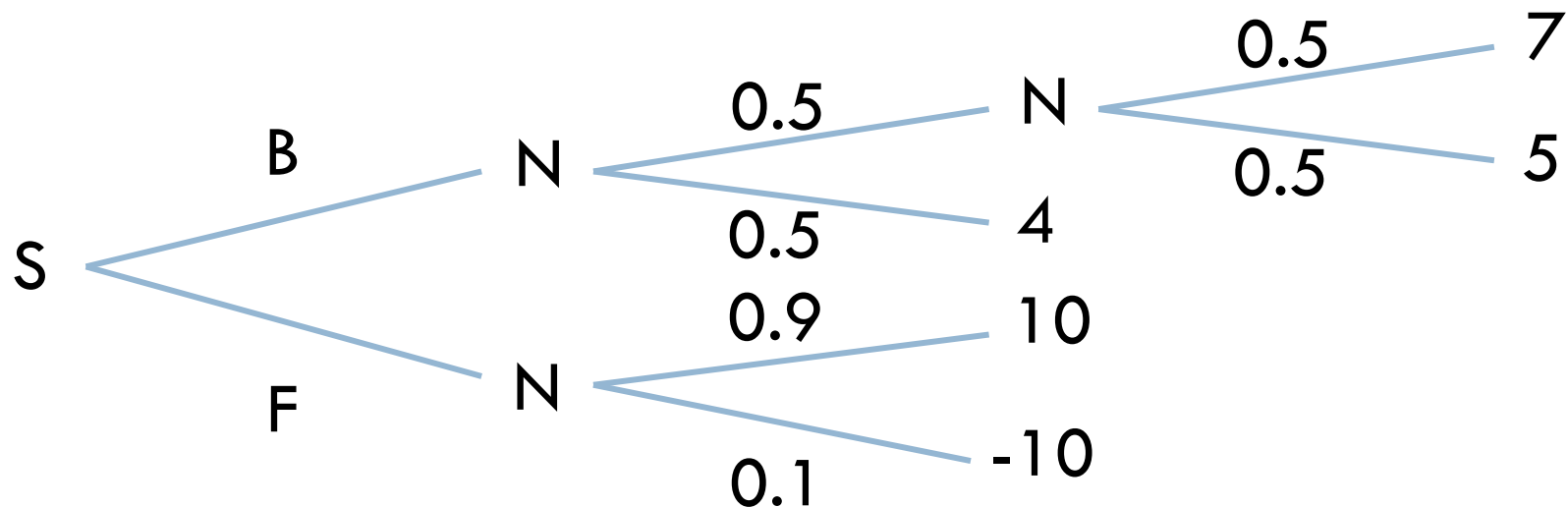
- Now we can compare the fish and beef lotteries using expected utility



- $\mathbb{E}[u(B)] = 0.5 \times 6 + 0.5 \times 4 = 5$
- $\mathbb{E}[u(F)] = 0.9 \times 10 + 0.1 \times (-10) = 9 - 1 = 8$
- So, fish is rationally preferable to beef

Compound lotteries

- How to account for subsequent choices of Nature?



- Just take the compound expectation
- Remember expectation is linear
- $\mathbb{E}[u(B)] = 0.5 \times 0.5 \times 7 + 0.5 \times 0.5 \times 5 + 0.5 \times 4 = 5$

- Same as discrete case (only the diagram is harder)
- **Example:** We are digging a dwell and need to decide how deep should it be (d =dwell's depth). Digging has a cost of $d^2/(2 \text{ meters})$ and the amount of extracted water is $W(d) \sim \mathcal{U}([0, 20d])$.
- Utility = extracted water – cost.
- $\mathbb{E}[u(d)] = \mathbb{E}\left[W(d) - \frac{d^2}{2 \text{ m}}\right] = 10d - \frac{d^2}{2 \text{ m}}$
- $\mathbb{E}[u(5 \text{ m})] = 10 \times 5 \text{ m} - \frac{100 \text{ m}^2}{2 \text{ m}} = 50 - 50 = 0$
- Best choice: $d = 10 \text{ m}$ with $\mathbb{E}[u(10 \text{ m})] = 50$

- When randomness is not involved, the payoff values don't matter as long as they reflect preferences
 - ▣ If we have $A \succcurlyeq B$, then we can set $u(A)=1$ and $u(B)=0$ or $u(A)=100$ and $u(B)=-\pi$
- However, changing payoffs in lotteries may affect the preferred lottery
 - ▣ In the cafeteria example, suppose we assign -100 to the rotten fish instead of -10

- Consider the following lotteries, where the possible outcomes are to win 0, 1, or 20 euros
 - $p_A = (0, 1, 0)$, i.e., we receive 1 euro 100% guaranteed
 - $p_B = (0.95, 0, 0.05)$, i.e., with 95% probability we get nothing but with 5% probability we get 20 euros
- $u(1 \text{ euro})$ or $0.95 \times u(0 \text{ euros}) + 0.05 \times u(20 \text{ euros})$?
- Depends on how much a player values gaining X euros

- For a **risk-neutral** player, lotteries p_A and p_B are interchangeable
- For a **risk-averse** player $p_A \succcurlyeq p_B$ (prefers 1 euro guaranteed)
- For a **risk-loving** player $p_B \succcurlyeq p_A$ (prefers a 5% chance to get 20 euros)

- **Remark:** Monotonic utility functions such as $u(x) = x$, $u(x) = x^2$, and $u(x) = \log x$ do not affect preferences but they do affect risk attitude
 - Linear utility \rightarrow risk-neutral
 - Concave utility \rightarrow risk-averse
 - Convex utility \rightarrow risk-loving

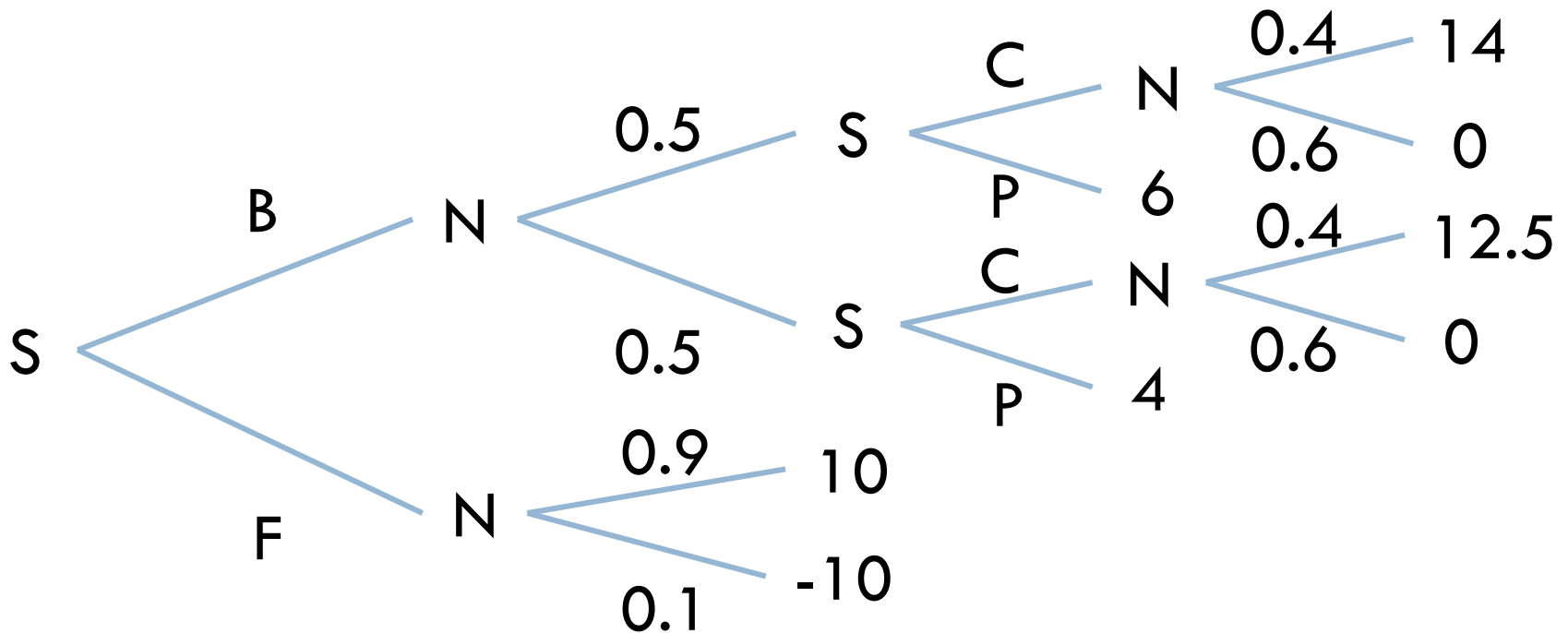
- **Tldr:** Be careful! Expected utility theory does not say that getting 1 euro is the same as gambling 2 euros with a 50/50 probability
 - ▣ That becomes true if we use a linear utility function
$$u(x) = ax + b, a > 0$$
 - ▣ You may use other utility function to model different risk attitudes

BACKWARD INDUCTION

- **Example:** Consider once again the cafeteria example with the fish and beef lotteries as before
- This time, the student is also given the option to add the chef's sauce (C) to the beef or leave it plain (P)
- However, she does not know if she will like the sauce
- Assume the sauce is good with probability 0.4
- Good sauce increases yields $u=14$ for tasty beef and $u=12.5$ for bland beef
- Bad sauce always yields $u=0$

Decision over time

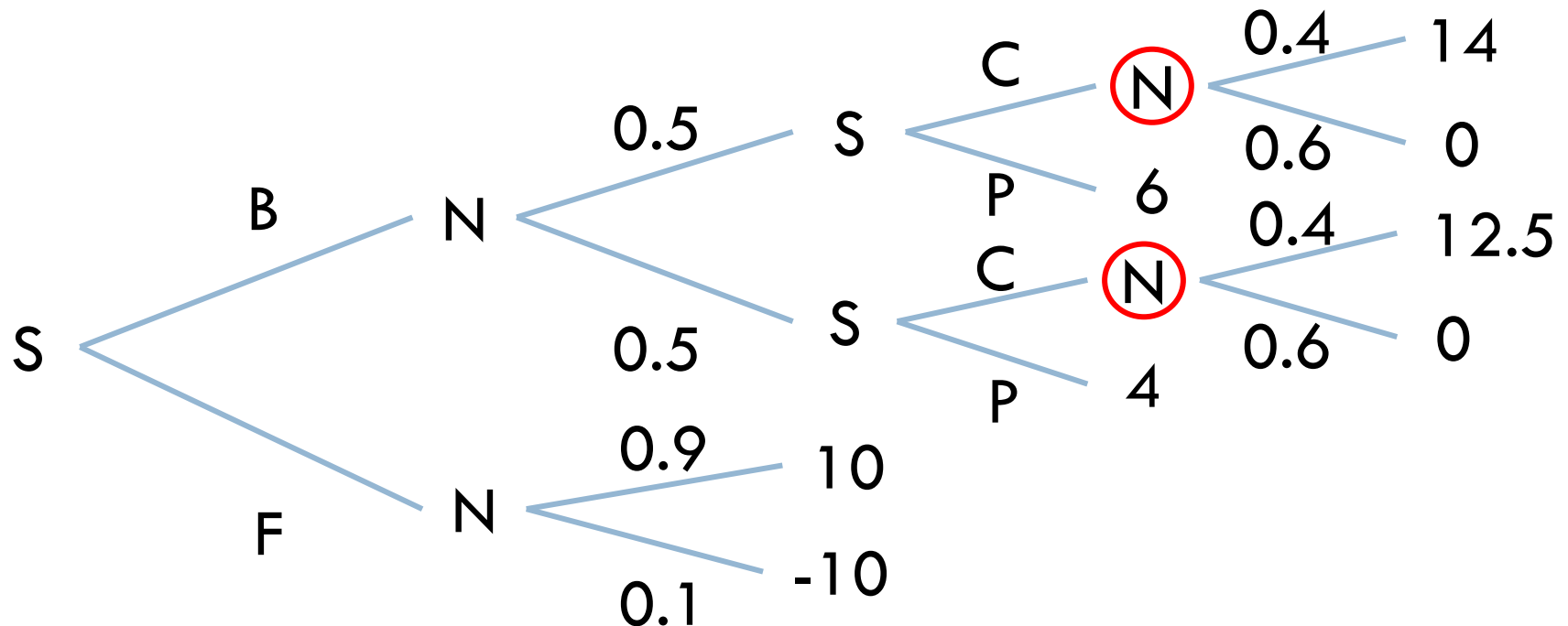
□ How do we “solve” this tree?



- Begin with nodes (not leaves) at the last level of the tree
- If it is Nature's move, replace the node with a leaf containing the average payoff
- If it is the player's move, replace the node with the payoff of the best choice (i.e., the payoff yielding highest utility)
- Repeat the process for the “pruned” tree

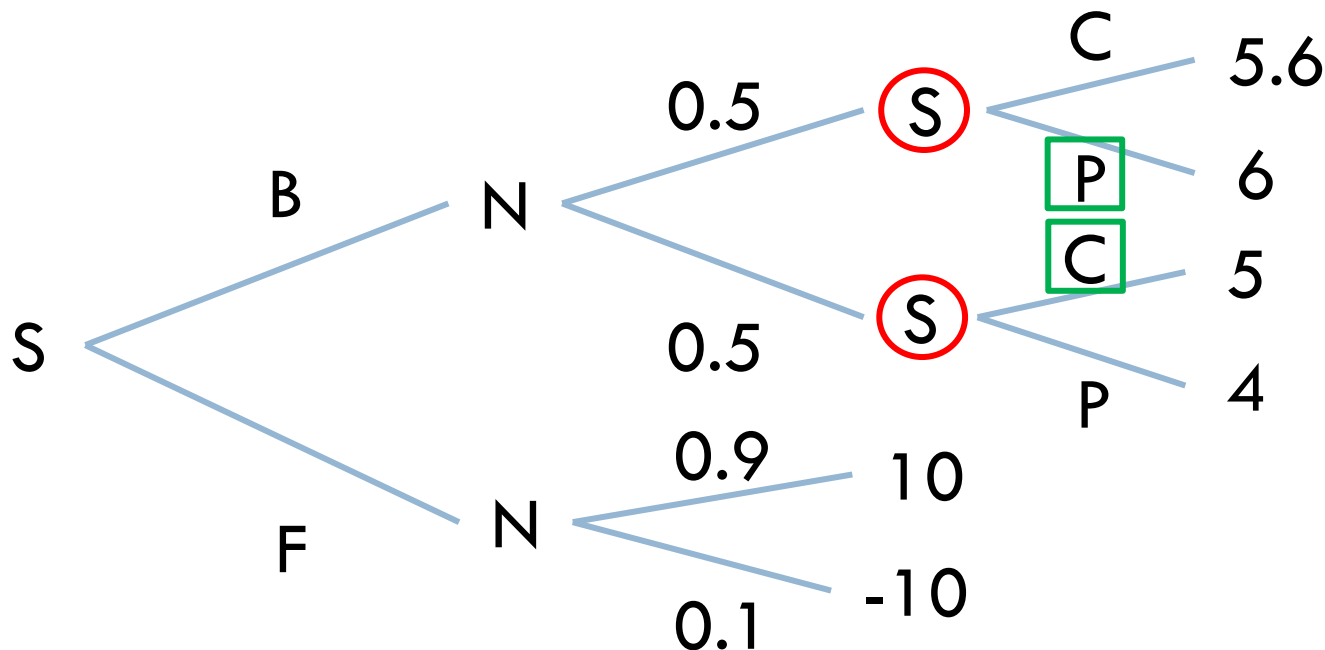
Decision over time

- Nature's move: replace with expected utility



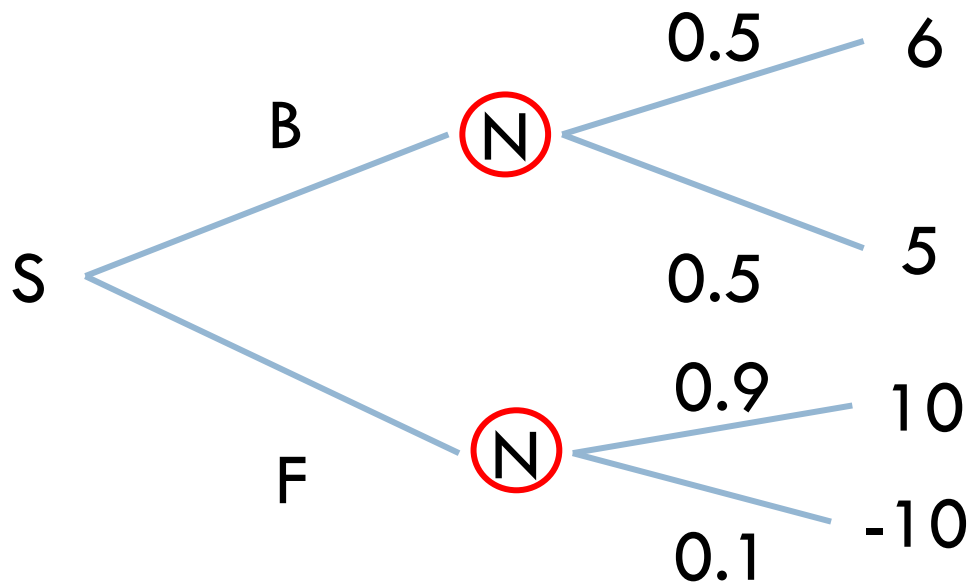
Decision over time

- Player's move: choose best option



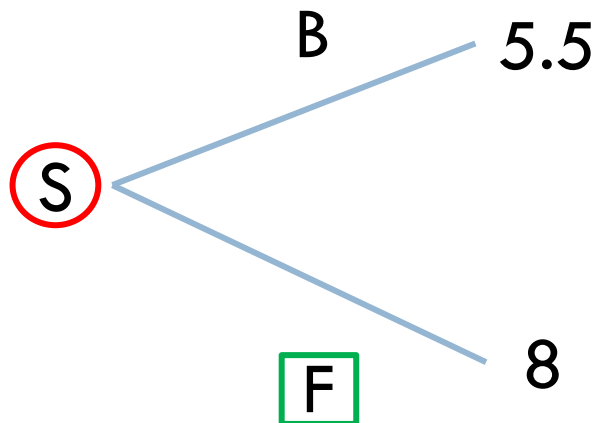
Decision over time

- Nature's move: replace with expected utility

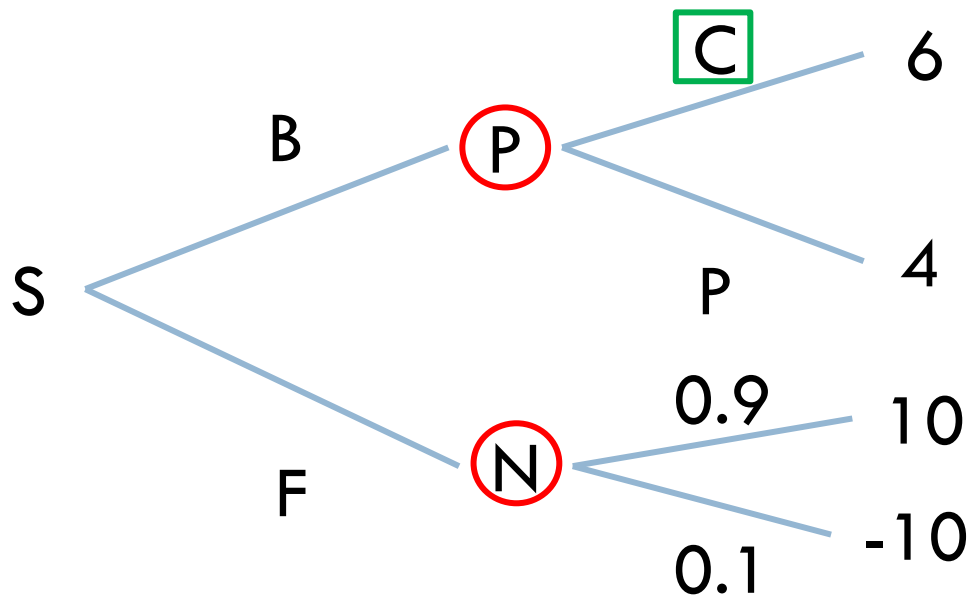


Decision over time

- Player's move: choose best option
- In conclusion, the player's best choice is to still take the fish

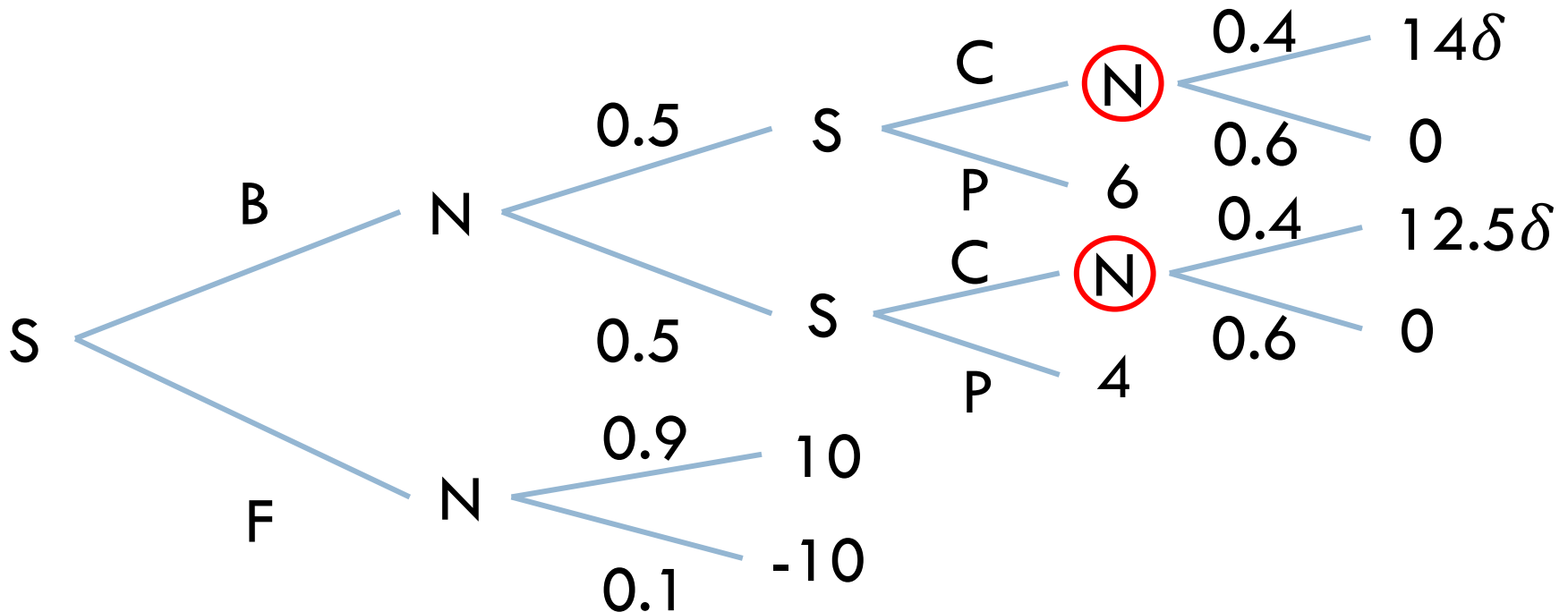


- **Remark:** It is possible to have Nature's and player's moves at the same tree level



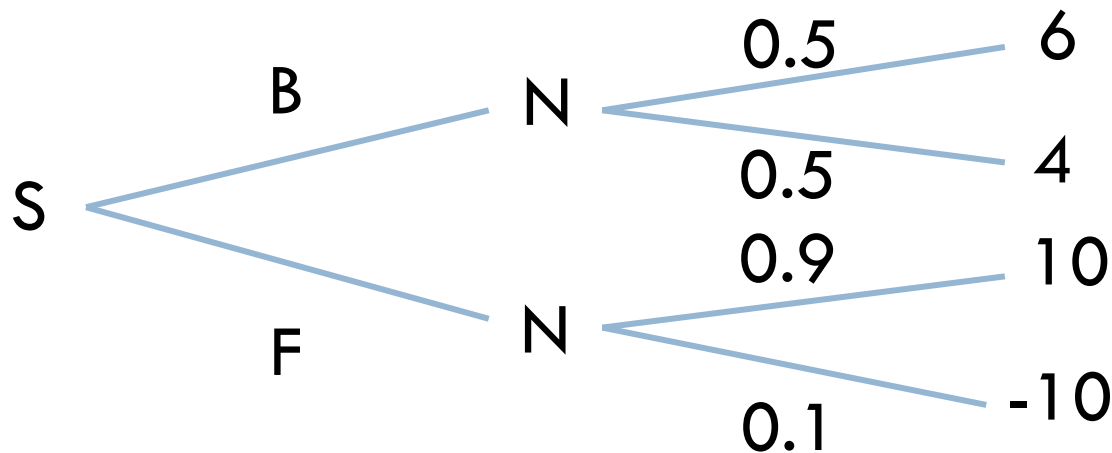
Discount for future payoffs

- If the player's decisions are made far apart, we may include a discount factor $0 < \delta < 1$
 - ▣ Likely, that's not the case for adding the chef's sauce



- Expected utility implies that a rational player chooses its actions so as to make the right choice **on average**
- Suppose the player has the possibility to see Nature's choice in advance: how much is this information worth?

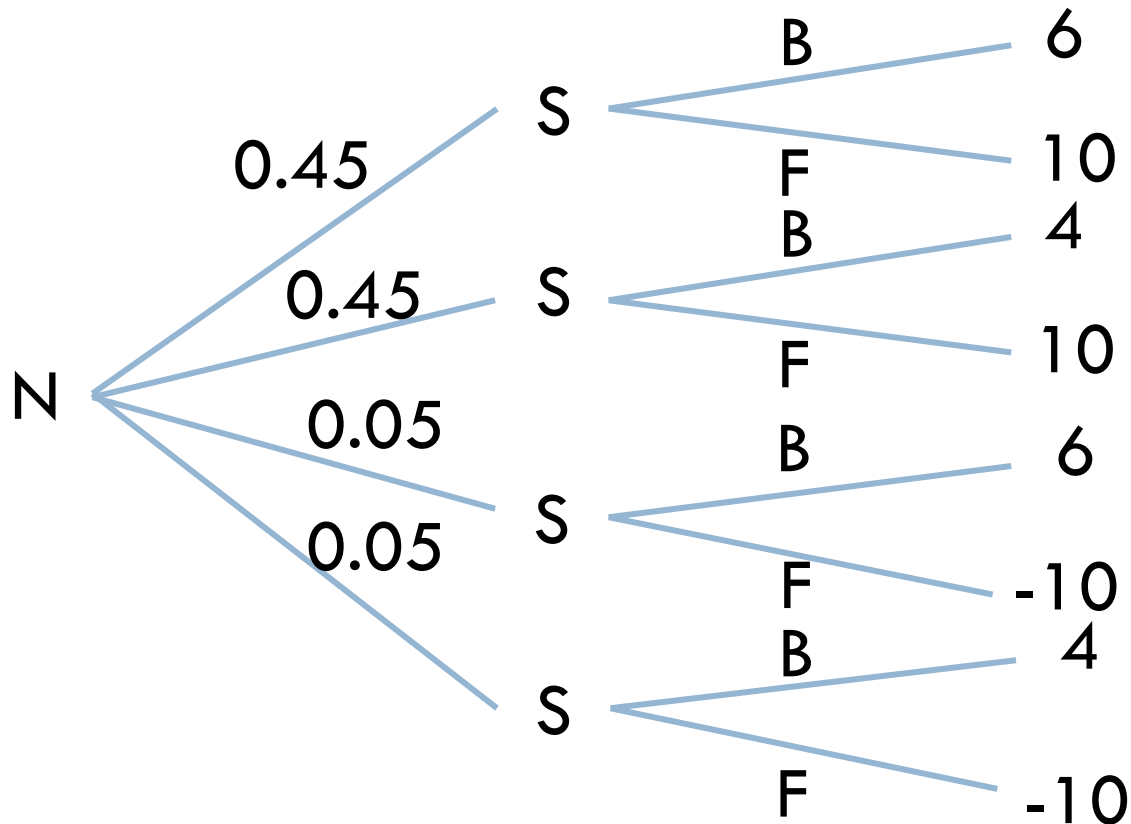
The value of information



- Assume a friend of S knows how good is the cafeteria's food today and he is willing to tell her (under reasonable compensation)

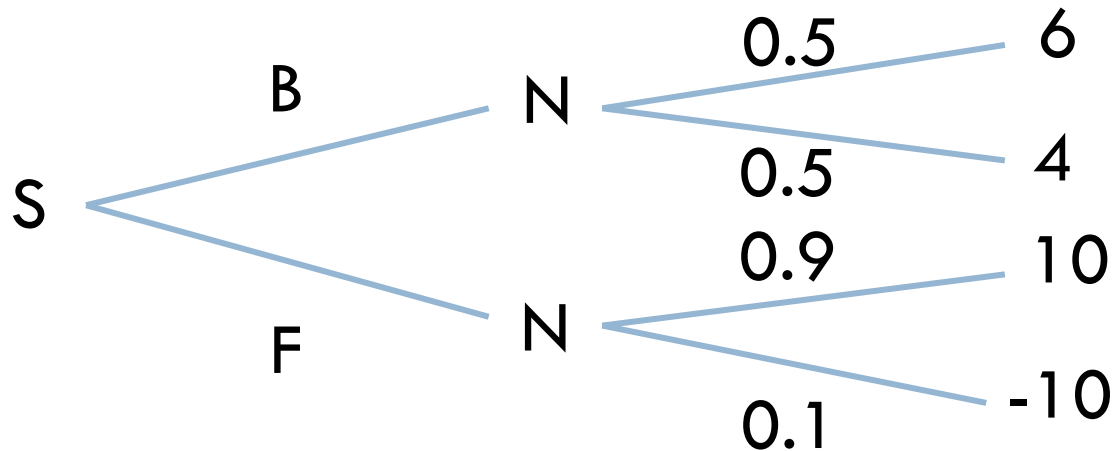
- If S is notified in advance, her best option will change depending on the information received
- The possible outcomes are the same but the moves' order changes
- This situation can be modeled by making Nature move first

The value of information



- S is always able to make the best choice, no gambling
- Expected utility: $0.9 \times 10 + 0.05 \times 6 + 0.05 \times 4 = 9.25$

The value of information



- Expected utility: 8 (choose fish with its expected payoff)
- Knowing Nature's choice is worth 1.25

SELF-ASSESSMENT

- When is it possible to model preferences between lotteries using average payoffs?
- Which utility function can we use to model a risk-averse player? $u(x) = x^2$ or $u(x) = \log x$?
- How can we solve a decision problem involving sequential choices made by both a player and Nature?

“Homework”

- Solve the decision problem of a student P who needs to choose whether to do a project or not for this course
- Same rules as this course:
 - 0-28 points in the written test
 - No project: 3 points by default; Project: 0-5 points
- The project must be selected before the written test
- Assign the probabilities for written test's score and project's score according to your own estimation