UNIVERSITÀ
DEGLI STUDI
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DIPARTIMENTO
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DELL'INFORMAZIONE

Lecture 16
Minimax

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Previously on game theory

- **Repeated games**: Multistage game where each stage $t = 1, \dots, T$ is an instance of the same stage game $\mathbb{G}$
- Finitely repeated games ($T < \infty$) are identical to normal multistage games, and cooperation can be enforced only if there are a "stick" NE and a "carrot" NE
  - "Cooperation" here means that in stages $t < T$ players choose an action that is better for both, but not a NE for the stage game
  - At stage T, players always play a NE
- In infinitely repeated games ($T = \infty$), instead, it is possible to enforce cooperation at every stage even if there is only one NE
  - This could be done by using a grim trigger (GrT) strategy
  - GrT: "Cooperate while the other also cooperate; play NE strategy if the other defects"

Today on game theory

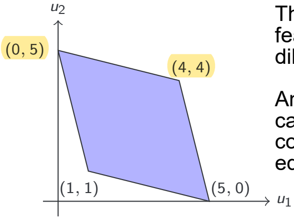
- We have seen how cooperation is possible in the case of the Prisoner's dilemma, where we have one single "bad" NE and a dominated strategy that is better for both players
- How does this generalize?
- General result: Friedman theorem (a.k.a. "folk theorem")

Feasible payoff

- A **feasible payoff** for game $\mathbb{G}$ is any convex combination of utilities

$$\alpha_1 u(s_1) + \alpha_2 u(s_2) + \cdots + \alpha_L u(s_L), \quad \text{with } \sum_{i=1}^{L} \alpha_i = 1$$

for joint strategies $s_1, \ldots, s_L$ ($L = |S_1| \cdot |S_2| \cdots |S_n|$total number of joint pure strategies)

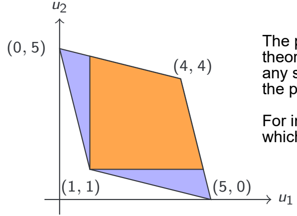


This represent all possible feasible payoffs for prisoner's dilemma.

Any point on the above edge can be reached by doing a combination of left/right above edge strategy.

Friedman theorem

- **Theorem**: Let $\mathbb{G}$ be a finite static game of complete information. Let $(e_1, e_2, \ldots, e_n)$ be the payoffs of a NE of $\mathbb{G}$ and $(x_1, x_2, \ldots, x_n)$ be a feasible payoffs for $\mathbb{G}$. Suppose $\forall$ NE and $\forall j$, $x_j > e_j$. Then, for δ close enough to 1, $\mathbb{G}(\infty, \delta)$ has a SPE with payoffs $(x_1, \ldots, \overset{\text{xn}}{\cancel{x_j}})$.
- Proof follows the same process used for the prisoner's dilemma



The payoffs supported by Friedman theorem are those obtained by playing any strategy in the orange area (where the payoff are better than the NE ones).

For instance, from this theorem any payoff which improves the (1,1) NE is a SPE.

Tit for tat

- It may be unnecessary to keep punishment forever (holding a grudge)
    - Assume the stage game has two actions (cooperate or defect) → GrT can be replaced by "Tit for Tat"
- **Tit for tat (TFT)**: "At stage t, play what the other player chose at stage $t - 1$
- Tit-for-tat strategies immediately punish deviation but are also forgiving (1-step history)
- Behavioral analogous: "eye for an eye", "live and let live"

Tit for tat

- Even though Tit for Tat is often effective, it may be "unstable" under certain conditions
- Two unsynchronized TFT players trigger a "death spiral", where each of them punishes the other in turns
- Therefore, the NE achieved by TFT is not subgame-perfect (the death spiral case is not a NE)

Self-assessment

- Consider the following game $\mathbb{G}$:

		Player B	
		g	w
Player A	G	5, 3	0, 4
	W	6, 0	1, 1

1. Is it possible to find a SPE for $\mathbb{G}(4)$ that involves playing (G,g) at each stage?
2. Is it possible to find a NE for $\mathbb{G}(\infty)$ where (G,g) is played at each stage using a grim-trigger strategy? If so, for what δ? Is that a SPE?
3. Is it possible to find a NE for $\mathbb{G}(\infty)$ where (G,g) is played at each stage using a tit-for-tat strategy? If so, for what δ? Is that a SPE?

Exercise (of previous exam)

- Carl (C) and Diana (D) are two university students. Every night they go to the department library, but they do not coordinate or plan any action together. Upon their arrival, they independently decide whether to: (S) study or (M) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for C and 18 for D. However, if they both choose M, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(\text{S,M}) = u_D(\text{M,S}) = 0$. Call $\mathbb{G}$ this game, and consider it in a repeated version $\mathbb{G}(T)$. Individual payoffs are summed with discount factor δ.

Exercise

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a "Grim Trigger" strategy where each player ends up in always choosing S?
3. Consider an *extended* game where a punishment strategy P is also available to both players. When either player P, payoffs are -10 for *both* players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game $\mathbb{G}'$. If you see a SPE of $\mathbb{G}'(2)$ where players may play S, state at which round do they play it, and what value of δ do you need to obtain it.

Minimax

Maximin

- Consider a 2-player game and strategies $s_i \in S_i$, $s_{-i} \in S_{-i}$
- **Maximin** (or maxmin):

$$w_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

- Interpretation:
  - Define function $f_i : S_i \to \mathbb{R}$, $f_i(s_i) = \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$
  - Find **security strategy** $s_i^* = \arg\max_{s_i} f_i(s_i)$ i.e., i's best response to $-i$'s worst strategy
  - Evaluate $w_i = f_i(s_i^*)$
- A security strategy is a conservative approach allowing i to achieve the highest payoff against the worst move by $-i$

Minimax

- **Minimax** (or minmax):

$$z_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$$

- Interpretation:
  - Define function $F_i : S_{-i} \to \mathbb{R}$, $F_i(s_i) = \max_{s_i \in S_i} u_i(s_i, s_{-i})$ which is essentially the payoff of i's best responses
  - Find $s^*_{-i} = \arg\min_{s_i} F_i(s_i)$ i.e., i's best response to $-i$'s worst strategy
  - Evaluate $z_i = F_i(s^*_{-i})$
- In simple terms, z_i is the minimum payoff that i would obtain if i could move after $-i$

Example

In this game we are playing as A

		Player B			
		L	C	R	f_A (min)
Player A	T	5, -	3, -	4, -	3
	D	2, -	6, -	1, -	1
F_A (max)		5	6	4	

- Maximin of A: 3 (safe strategy for A yields payoff 3 if we assume B plays the worst possible response)
- Minimax of A: 4 (if A moves after B, A is guaranteed to get at least 4)

Example

- We can prove that:
  1. For every player i, $\text{maximin}_i \leq \text{minimax}_i$
  2. If joint strategy s is a Nash equilibrium, then for every player $\text{minimax}_i \leq u_i(s)$
- So, $\text{maximin}_i \leq \text{minimax}_i \leq u_i(\text{NE})$
- The first inequality is obvious
- The second one follows from players not wanting to unilaterally deviate from NE
  - Minimax_i can also be seen as i's "worst" best response

Example

f = maximin
F = minimax

Player B

		L	C	R	f_A (min)
Player A	T	**5**, **6**	3, 2	**4**, 1	3
	D	2, 0	**6**, **8**	1, 2	1
F_A (max)		5	6	4	

- As observed before, $\text{maximin}_A < \text{minimax}_A$
- Also there are two NE:
  - (T,L) giving A a payoff of 5
  - (D,C) giving A a payoff of 6
- Both better than minimax_A
- Check for B

Another example

Player B

Player A	L	C	R	$f_A \downarrow$, $F_B \uparrow$
T	3, **4**	5, 0	3, 1	3, 4
D	**5**, **4**	**6**, 2	**7**, 2	5, 4
$F_A \uparrow$, $f_B \downarrow$	5, 4	6, 0	7, 1	

$z_A = 5$ $z_B = 4$

$w_A = 5$ $w_B = 4$

- Here, maximin$_A$= u_A(NE)= 5 and maximin$_B$= u_B(NE)= 4 so it must be

$$\text{minimax}_i = \text{maximin}_i = u_i(\text{NE})$$

for both players

Yet another example

Player B

		L	C	R	$f_A \downarrow$, $F_B \uparrow$
Player A	T	**4**, 0	3, **1**	**3**, 0	3, 1
	M	3, 0	**4**, 0	2, **1**	4, 1
	D	2, **0**	1, **0**	0, **0**	2, 0
$F_A \uparrow$, $f_B \downarrow$		4, 0	4, 0	3, 0	

$z_A = 3$ $z_B = 0$

$w_A = 3$ $w_B = 0$

- However, $\text{minimax}_i = \text{maximin}_i = c_i$ does not mean $u(\text{NE}) = c_i$
- Here, for example, there is no NE in pure strategies

Yet another example

Player B

		L	C	R	$f_A \downarrow, F_B \uparrow$
Player A	T	**4**, 0	3, **1**	**3**, 0	3, 1
	M	3, 0	**4**, 0	2, **1**	4, 1
	D	2, **0**	1, **0**	0, **0**	2, 0
$F_A \uparrow, f_B \downarrow$		4, 0	4, 0	3, 0	

$z_A = 3$ $z_B = 0$

$w_A = 3$ $w_B = 0$

- There is only one NE in mixed strategies (you can apply IESDS to eliminate D and L)
  - $\mathbf{p}_A$: "Play T with 1/2 and M with 1/2"
  - $\mathbf{p}_B$: "Play C with 1/2 and R with 1/2"
- Payoffs at NE are 3 ($=$ minimax$_A$) and 1/2 ($>$ minimax$_B$)

Zero-sum games

Zero-sum game

- We speak of **zero-sum game** if $u_i(s) = -u_{-i}(s)$

		Player B		
		L	C	R
Player A	T	-9, 9	8, -8	5, -5
	M	-2, 2	6, -6	2, -2
	D	-1, 1	3, -3	4, -4

- Examples: odds&evens, rock-paper-scissors, chess, . . .

Adversarial games

- **Adversarial games** are a more general class of games where two players i and $-i$ are adversaries and have utilities s.t.

$$u_i \uparrow \iff u_{-i} \downarrow$$

- Many adversarial games can be framed as zero-sum game (e.g., in a game where the player with most point wins, you can use $u_A = \text{points}_A - \text{points}_B$, and $u_B = -u_A$)

Minimax theorem

- In general, $\text{maximin}_i \leq \text{minimax}_i \leq u_i(\text{NE})$ and $\text{maximin}_i = \text{minimax}_i$ does not guarantee the existence of a pure NE with utility minimax_i
- However, in the case of zero-sum games we can apply the minimax theorem
- **Theorem**: Let $\mathbb{G}$ be a zero-sum game with finite number of strategies. Then,
  1. $\mathbb{G}$ has a pure NE $\iff$ $\text{maximin}_i = \text{minimax}_i$ for each player i
  2. All NE yield the same payoffs (minimax_i,-minimax_i)
  3. In all NE, every player is playing a security strategy

Minimax theorem: Example

f = maximin => select the minimum and take the maximum
F = minimax => select the maximum and take the minimum

		Player B			
		L	C	R	fA(min) FB(max)
Player A	T	-9, **9**	**8**, -8	**5**, -5	-9, 9
	M	-2, **2**	6, -6	2, -2	-2, 6
	D	**-1**, **1**	3, -3	4, -4	-1, 1
	FA(max) fB(min)	-1, 1	8, -8	5, -5	

- For player A:
  - $\text{maximin}_A = -1$
  - $\text{minimax}_A = -1$
  - Only NE is (D,L), with utility $u_A(\text{D,L}) = -1$
- For player B: $\text{maximin}_B = \text{minimax}_A = u_B(\text{D,L}) = 1$

Minimax theorem: intuition

- In zero-sum games, you can represent the game using a regular matrix instead of a bi-matrix ($-i$'s payoffs are implied)

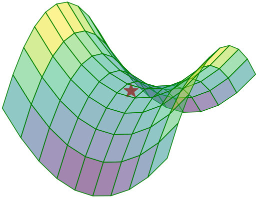
		Player B		
		L	C	R
Player A	T	-9	8	5
	M	-2	6	2
	D	-1	3	4

- Since players have opposite payoffs, it is sufficient to check maximin=minimax for one player only
- Since $\max_x f(x, y) = \min_x(-f(x, y))$, it also holds

$$\text{minimax}_i = -\text{maximin}_{-i}$$
$$\text{maximin}_i = -\text{minimax}_{-i}$$

Minimax theorem: remarks

- The common value of maximin=minimax is called the **value** of the game
  - Some games with infinitely many strategies are “without value” (minimax theorem does not hold)
- A joint security strategy (if any), i.e., a NE, is a **saddle point** of the game



Minimax theorem: remarks

- The proof of the theorem is due to von Neumann (1928) and makes use of linear programming
- The criterion of minimaximizing the utility has been widely employed in artificial intelligence applications: e.g., chess, which is a zero-sum (sequential) game

Mixed minimax

Mixed minimax and maximin

- We can define the mixed counterparts of maximin and minimax as

$$\text{maximin}_i^p : \max_{p_i \in \Delta S_i} \min_{p_{-i} \in \Delta S_{-i}} u_i(p_i, p_{-i})$$

$$\text{minimax}_i^p : \min_{p_{-i} \in \Delta S_{-i}} \max_{p_i \in \Delta S_i} u_i(p_i, p_{-i})$$

  and, likewise, we can extend the definitions of f_i and F_i to mixed strategies
    - $f_i : \Delta S_i \to \mathbb{R}$, $f_i(p_i) = \min_{p_{-i} \in \Delta S_{-i}} u_i(p_i, p_{-i})$
    - $F_i : \Delta S_{-i} \to \mathbb{R}$, $F_i(p_{-i}) = \max_{p_i \in \Delta S_i} u_i(p_i, p_{-i})$
- A mixed strategy p_i^* maximizing f_i is called a **mixed security strategy** and its corresponding value in f_i is called **mixed security payoff**

Mixed security strategy

- **Note 1**: $f_i(p_i)$ can be found by minimizing $u_i(p_i, s_{-i})$, i.e., considering only pure strategies of player $-i$. Likewise, $F_i(p_{-i})$ can be found by maximizing $u_i(s_i, p_{-i})$.
  - This is due to expected utility: mixed-strategy payoffs are convex combinations of pure-strategy payoffs
  - Similar to characterization theorem for mixed NE: we consider mixed strategy of one player and the best response by the other player
- **Note 2**: mixed minimax and maximin always exist and are equal
  - $u_i(p_i, p_{-i})$ is a continuous function

Example

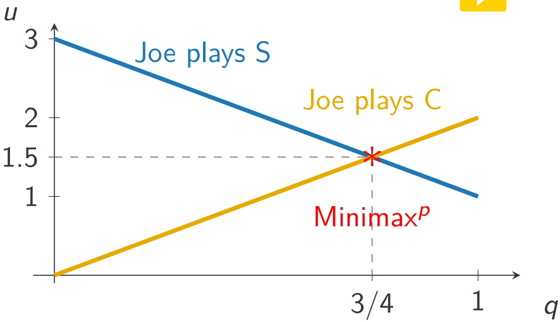
		Joe	
		S	C
Jim	T	3, -	0, -
	M	1, -	2, -

- Jim: maximin=1, minimax=2
- Jim can increase his maximin if he plays $\frac{1}{4}\text{T}+\frac{3}{4}\text{M}$
  $\rightarrow$ $\text{maximin}^p = 1.5$
- Likewise, Joe can decrease Jim's minimax by playing $\frac{1}{3}\text{S}+\frac{2}{3}\text{C}$
  $\rightarrow$ $\text{minimax}^p = 1.5$
- How do we find these values?

Example: Minimax^p

- Let $q = \text{Prob}(\text{Jim plays M})$
- $u(q, \text{S}) = 3(1-q) + q = 3 - 2q,\ u(q, \text{C}) = 2q$
- $3 - 2q = 2q$ for $q = 3/4$

  $\rightarrow \text{minimax}^p = u(3/4, \text{C}) = u(3/4, \text{S}) = 1.5$



Minimax^p theorem

- **Theorem**: Let $\mathbb{G}$ be a zero-sum game with finitely many strategies. Then,
  1. For every player i, $\text{maximin}_i^p = \text{minimax}_i^p = u_i(p)$ for each joint strategy p that is a Nash equilibrium
  2. All Nash equilibria in mixed strategies are security strategies for player i and yield a payoff equal to maximin_i^p
- **Note**: Again, $\text{maximin}_i^p = -\text{minimax}_{-i}^p$
- All NE yield the same payoff
- minimax^p is called the **value** of the game

Linear programming

- Linear programming (or linear optimization) is a class of constrained optimization problems that can be expressed as follows

  minimize w w.r.t. $x \in \mathbb{R}^{\ell}$ (w is a slack variable)

  subject to $Ux \leq w$ (linear constraints on slack variable)

  and $Ax \leq b$ (linear constraints on x)

  and $x \geq 0$ (all elements of x must be non-negative)

- Maximin^p and minimax^p can be framed as linear programming problems
- This often allows to solve them efficiently

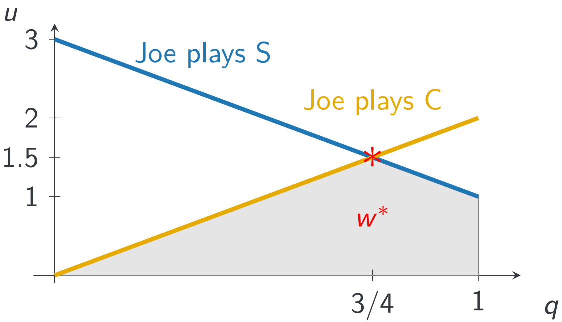
Linear programming for minimax^p

- Let $v_{jk} = u(s_i^{(j)}, s_{-i}^{(k)})$
- Let $a_j = \text{Prob}(s_i^{(j)})$ for player i
- Minimax^p:

$$\begin{aligned}&\text{maximize } w \text{ w.r.t. } a_j \ (\forall j)\\&\text{subject to } \sum_j a_j v_{jk} \geq w \ (\forall k)\\&\text{and } \sum_j a_j = 1\\&\text{and } a_j \geq 0 \ (\forall j)\end{aligned}$$

- w is a slack variable that is lower than $u(a, s^k_{-i})$ for each opponent response

Example: Minimax^p



- The gray region represents all values of w satisfying the constraints
- The maximum value w^* is equal to the minimax^p

Linear programming for maximin^p

- Let $v_{jk} = u(s_i^{(j)}, s_{-i}^{(k)})$
- Let $b_k = \text{Prob}(s_{-i}^{(k)})$ for player $-i$
- Maximin^p:

$$\begin{aligned}&\text{minimize } w \text{ w.r.t. } b_k\ (\forall k)\\&\text{subject to } \sum_k b_k v_{jk} \leq w\ (\forall j)\\&\text{and } \sum_j b_k = 1\\&\text{and } b_k \geq 0\ (\forall k)\end{aligned}$$

How to solve minimax

- LP problems can be solved via optimization
- Simplex method is widely used (CPLEX, lpsolve): (worst-case) exponential, often fast in practice
- Meta-heuristic techniques (Genetic Algorithms, Tabu search): sometimes even faster, but they do not guarantee to find the optimal solution
- **Important**: For sequential zero-sum games (chess, card games), backward induction is still preferable
    - However there are some "tricks" to avoid searching all branches in large games
    - Check out the alpha-beta pruning

Self-assessment

- What is a security strategy?
- Consider a generic game $\mathbb{G}$
    - Is maximin_i always equal to minimax_i for each player i?
    - Is maximin_i^p always equal to minimax_i^p for each player i?
    - If $\text{maximin}_i = \text{minimax}_i$ for each player i, does that mean that the game has a pure NE? Does your answer change if $\mathbb{G}$ is zero-sum?
    - If $\mathbb{G}$ is a zero-sum game between i and $-i$, is there a relationship between the minimax of i and the maximin $-i$?
- What is the *value* of a zero-sum game? Does it always exist if the game has infinitely many strategies?

Send me questions via e-mail