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# Lecture 04

## Nash equilibrium

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- **Static game of complete information:** simplest type of game, played in one shot; players move unbeknownst of each other's actions but fully aware of everyone's payoffs.
  - Examples: Rock-paper-scissors, battle of the sexes, prisoner's dilemma.
- Static games of complete information are fully defined by actions, outcomes, and utilities
- In this type of games, **pure strategies** = actions (e.g., pure strategy: "I will play rock", action: playing rock)

- **Normal form of a game:**  $\mathbb{G} = \{S_1, S_2, \dots, S_n; u_1, u_2, \dots, u_n\}$
- This is one possible way of representing the prisoner's dilemma in normal form
  - $\mathbb{G} = \{S_A, S_B; u_A; u_B\}$
  - $S_A = S_B = \{M, F\}$
  - $u_A(M, M) = u_B(M, M) = -1, u_A(F, F) = u_B(F, F) = -6,$   
 $u_A(M, F) = u_B(F, M) = -9, u_A(F, M) = u_B(M, F) = 0$
- However, this is not very convenient to analyze. Therefore, we often prefer the graphical representation.

- Graphical representation of the prisoner's dilemma normal form

		Player B	
		M	F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

- Pure strategy:  $s_i \in S_i$
- **Joint strategy**:  $s = (s_1, \dots, s_n) \in S_1 \times \dots \times S_n$
- In static games of complete information, joint strategy = outcome
- Examples:
  - $s_B = F$  is a pure strategy
  - $s = (M, F)$  is a joint strategy
  - $(M, F)$  is an outcome

- **Pareto dominance:** property of joint strategies (concerns all the players)
  - A joint strategy  $s$  is Pareto dominated by another strategy  $s'$  if for all players  $u_i(s) \leq u_i(s')$  (and for some the inequality is strict)
  - In the prisoner's dilemma  $(F, F)$  is Pareto dominated by  $(M, M)$
- **Strict dominance:** property of pure strategies (concerns only one player at a time)
  - A strategy  $s_i$  of player  $i$  is strictly dominated by another strategy  $s'_i$  if, regardless of what strategy is adopted by other players,  $s'_i$  gives a higher payoff to  $i$
  - In the prisoner's dilemma,  $M$  is strictly dominated by  $F$  for both players

## Best responses and beliefs

- For single-player problems, once the setup is known, the solution can be found directly
- That is not the case for multi-player games
  - The solution depends on other players
  - Sometimes rationality can help (e.g., we identify a dominated strategy and we decide not to play it)
  - We can extend this reasoning by assuming that other players are also rational, which leads to IESDS
  - Still, in most cases this does not allow to find a solution for the game



# Best response


If i know that the other player is gonna play a certain strategy, my best response to this strategy is simply the one that maximize my utility.

- Strategy  $s_i \in S_i$  is player  $i$ 's best response to moves  $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$  if

$$u_i(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_n) \geq u_i(s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$$

for all  $s'_i \in S_i$

- *Notation:*

$(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) \in S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_n$   
is often shortened to " $s_{-i} \in S_{-i}$ " 

- This way we can simply write:  $s_i \in S_i$  is best response to  $s_{-i} \in S_{-i}$  if

$$\underline{u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \forall s'_i \in S_i}$$

- There may be more than one best response
  - Of course, all with the same value of  $u_i(s_i, s_{-i})$

		Player B		
		L	C	R
Player A	U	3, 3	5, 1	6, 2
	M	4, 1	8, 4	3, 6
	D	4, 0	9, 6	6, 8

- Here, U and D are both best responses to player B's strategy to play R.
- Self-assessment: What are player A's best responses to strategies L and C?

- **Claim:** A rational player who believes that others are playing  $s_{-i} \in S_{-i}$  will always choose the best response to  $s_{-i}$ . (This follows from players wanting to maximize their payoffs).
- **Theorem:** If  $s_i \in S_i$  is strictly dominated by some other strategy, then it is no best response to any  $s_{-i} \in S_{-i}$ .
  - *Proof:* There is some strategy  $s'_i \in S_i$  that dominates  $s_i$ .
  - By definition,  $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$  for all  $s_{-i} \in S_{-i}$
  - Therefore,  $s_i$  always yields a lower payoff than  $s'_i$  and cannot be a best response  $\square$

- A **belief** of player  $i$  is a possible profile of the other players' strategies, i.e., an element of the set  $S_{-i}$ 
  - Beliefs are connected to best responses
- We define a **best-response correspondence**  $BR : S_{-i} \rightarrow 2^{S_i}$  that maps  $s_{-i} \in S_{-i}$  to a subset of  $S_i$  such that each  $s_i \in BR(s_{-i})$  is a best response to  $s_{-i}$ 
  - The symbol  $2^{S_i}$  is used to denote the power set of  $S_i$ , i.e., the set of all the possible subsets of  $S_i$
  - $BR$  is not a function, as it maps values to sets
  - However  $BR(s_{-i})$  can be a singleton if there is a unique best response to  $s_{-i}$

# Nash equilibrium

- We want to strengthen the dominated strategy concept with this idea in mind:
  - game theory should make predictions about the outcome of games played by a rational players
  - a prediction is correct if the players are **willing** to play their predicted strategy
- That is, players choose their **best response** to the predicted strategy of the others (i.e, the best response to their belief about other players' strategy)
  - A player's belief "makes sense" only if other players are also playing a best response
- If the (reasonable) beliefs of all players match, then no one regrets their strategy

# Nash equilibrium: intuition

- A Nash equilibrium is what is played if players beliefs match
- Let us mark in **blue** player A's best responses, and in **red** player B's best responses
- Suppose A's belief is that B will play S
  - Then, A's best response is to play S
- Suppose B's belief matches A (i.e., B believes that A will play S)
  - Then, B's best response is to play S
- This is a Nash equilibrium, since none of them regrets their strategy

		B	
		R	S
A	R	2, 1	0, 0
	S	0, 0	1, 2

# Back to the Prisoner's dilemma

- It does not make sense for A to believe that B will play M, since M is never a best response

		Player B	
		M	F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

- The NE is also the only survivor of IESDS



- In a  $n$ -player game  $\mathbb{G} = \{S_1, \dots, S_n; u_1, \dots, u_n\}$ , strategies  $(s_1^*, \dots, s_n^*)$  are a **Nash equilibrium** if, for all  $i$ ,  $s_i^*$  is a best response to  $s_{-i}^* = (s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*)$
- In other terms,  $\forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

or, equivalently,

$$s_i^* = \arg \max_{s_i \in S_i} u_i(s_i, s_{-i}^*)$$

- This is consistent with player's rationality, that requires all of them to maximize their utility function

- Take a possible combination  $(s'_1, \dots, s'_n)$
- If this is *not* a Nash equilibrium, there must be some player  $i$  such that  $s'_i$  is not the best response to  $(s'_1, \dots, s'_{i-1}, s'_{i+1}, \dots, s'_n)$
- That means  $\exists s''_i \in S_i$  such that

$$u_i(s'_1, \dots, s'_{i-1}, s''_i, s'_{i+1}, \dots, s'_n) > u_i(s'_1, \dots, s'_{i-1}, s'_i, s'_{i+1}, \dots, s'_n)$$

- In other words, there is an incentive for player  $i$  to deviate from the joint strategy  $(s'_1, \dots, s'_n)$

- Remember that we are considering static (one-shot) games
- A NE can be seen as a joint strategy in which no player has regrets on their choice
- In other words, if a NE is played, none of the players would want to unilaterally change their strategy even if they had the possibility to do so

# Back to Example 1

- Joint strategy (M, R) is a Nash equilibrium

		Player B	
		L	R
Player A	U	8, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

- A naive way to find Nash equilibria is to brute-force search:  
here (M, R) is the only joint strategy that satisfies the definition
  - You can verify that the utility does not decrease when player deviate *unilaterally*

# Back to Example 2

- A better way to find NE is to focus on best responses
- For player A, we find the maximum left value in each column; for player B, we find the maximum right value in each row

		Player B		
		L	C	R
Player A	U	0, 5	4, 0	7, 3
	M	4, 0	0, 5	7, 3
	D	3, 7	3, 7	9, 9

- (D, R) is the only NE for this game (both D and R are highlighted, meaning that they are best responses to each other)

# Back to odds and evens

- Here there is no Nash equilibrium (in pure strategies)
- We will see that there actually is one Nash equilibrium but we need to “extend” the definition

		Even	
		0	1
Odd	0	-4, 4	4, -4
	1	4, -4	-4, 4

- $(R, R)$  and  $(S, S)$  are both Nash equilibria

		B	
		R	S
A	R	2, 1	0, 0
	S	0, 0	1, 2

# Back to the Prisoner's dilemma

- Joint strategy (F, F) is a NE

		Player B	
		M	F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

- The NE is also the only survivor of IESDS



- **Theorem:** If  $(s_1^*, \dots, s_n^*)$  is the only joint strategy left after applying IESDS, then it is a Nash equilibrium
- **Lemma:** A NE always survives IESDS
- Another result: IESDS order is irrelevant

- Two requirements must be satisfied in order for a NE to be played:
  - Everyone plays a best response to their beliefs
  - Everyone's beliefs are **correct**
- The first requirement is quite logical and is simply the consequence of the rationality assumption
  - If I am a rational player and I believe other player are gonna act in a certain way, I will always play a best response to it
- Actually the first requirement is quite logical and consequent from rationality, while the second requirement is quite demanding
  - Beliefs may be inferred via some external reasoning (e.g., one player being particularly “influential”)

## More definitions of dominance and efficiency

- **Strict dominance:**  $s'_i$  strictly dominates  $s_i$  if
  - $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$  for all  $s_{-i} \in S_{-i}$
- **Weak dominance:**  $s'_i$  weakly dominates  $s_i$  if
  - $u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$  for all  $s_{-i} \in S_{-i}$
  - $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$  for some  $s_{-i} \in S_{-i}$

- A strategy that strictly (resp., weakly) dominates every other strategy of a user is said to be **strictly (resp., weakly) dominant**
- **Lemma:** A joint strategy  $(s_1^*, \dots, s_n^*)$  in which everyone plays a dominant strategy is a Nash equilibrium.
- It directly follows from the definition of NE.
- The reverse statement is false (sufficient but not necessary condition)

# Do not eliminate weakly dominated $s_i$ 's

- Extend the Odds-or-evens game with a third option: “Punch the opponent” (P)
- Both players receive negative payoff (one gets beaten, the other gets punished)
- P is weakly dominated, yet it is a NE
- If we delete it, we miss a NE

		Even		
		0	1	P
Odd	0	-5, 5	5, -5	-5, -5
	1	5, -5	-5, 5	-5, -5
	P	-5, -5	-5, -5	-5, -5

- Pareto efficiency and NE are different concepts
  - Pareto efficiency: you cannot improve one player's payoff without worsening the payoff of another player
  - Nash equilibrium: no player can improve their own payoff via unilateral change (i.e., keeping the other players' choice fixed)
- The outcome of the Prisoner's dilemma is not Pareto efficient!

		Player B	
		M	F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

# Pareto efficiency vs NE

- Pareto efficient strategies: (M, M), (M, F), (F, M)
- NE: (F, F), which is Pareto dominated by (M, M)

		Player B	
		M	F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6



- Pareto inefficient Nash equilibria arise as we assume players are only driven by the desire to maximize their own payoff
- To estimate the inefficiency of being selfish (or distributed) one can compare Nash equilibria with Pareto efficient strategies
- To this end, we can assume that a joint strategy  $s$  has social cost  $C(s)$ , e.g.

$$C(s) = - \sum_j u(s_j) \text{ or } C(s) = - \max_j u(s_j)$$

Sum of the outcomes from the worst NE.

In Prisoner's Dilemma case it's MM so it's -2.

- The **price of anarchy** is the ratio between the social costs in the *worst* NE  $s^*$  and in the *best* Pareto efficient strategy (i.e., social optimum)

$$\text{PoA} = \frac{C(s^*)}{\min_s C(s)}$$

- In some cases, one may consider the *best* NE: in that case we call the ratio price of stability
- For certain classes of problems, there are theoretical results on the price of anarchy

- What is a NE?
- Consider NE  $(s_1, \dots, s_n)$ . Suppose player  $i$  replaces the current strategy  $s_i$  with  $s'_i$ . Can this still be a NE?
- If a strategy is ruled out by IESDS, can it be a NE?
- Compute the PoA for the Prisoner's dilemma using  $C(s) = -\sum_j u_j(s)$

- A (crazy) professor decides your grade in the exam he teaches will be decided by a game:
  - You are paired with a random classmate
  - You secretly choose an integer between 18 and 30, and so does the classmate
  - If you choose the same number, that is the score that you both get
  - If the numbers are different, who proposes the lowest score  $L$  gets a grade of  $L + R$ , while the other gets  $L - R$  (score  $< 18$  means the exam is failed,  $> 30$  means 30L and gives payoff 31)
- Play the game with  $R = 1$ ,  $R = 2$ , and  $R = 10$ .
- How do the NE change?

Sorry, gotta bounce!  
Send me questions via e-mail