Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

Lecture 15
Repeated games

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Previously on game theory

- **Multistage games**: a specific type of dynamic games, which consist in a sequence of stage games
- If there is a single NE in the last stage, the SPE of a multistage game is to just play a NE at each stage
- If there are "stick" and "carrot" NE in the last stage, it is possible to enforce strategic connections where non-NE are played in previous stages
  - This is done by threatening the other player to play the "stick" in the last stage
  - Such threats are credible only if players care enough about the future (δ large enough) and if there is a substantial difference in payoff between "stick" and "carrot" NE

Repeated games

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- A repeated game $\mathbb{G}(T, \delta)$ is a dynamic game where a static game $\mathbb{G}$ is played as a stage game for T times with discount factor δ
- Payoffs are accumulated in a (discounted) sum as in a normal multistage game
- We distinguish between
  - Finitely repeated games (finite horizon, $T = 1, 2, \cdots < \infty$)
  - Infinitely repeated games (infinite horizon, $T = \infty$)
- For infinitely repeated games, we always must have $\delta < 1$ (otherwise payoffs become divergent sums)

Two-stage prisoner's dilemma

- Consider the prisoner's dilemma repeated twice
- No discounting ($\delta = 1$)
- Does the second stage have a NE for every outcome of the first stage?

1st stage (prisoner's dilemma)

		B: m	B: f
A	M	4, 4	0, 5
A	F	5, 0	1, 1

2nd stage (also prisoner's dilemma)

		B: m	B: f
A	M	4, 4	0, 5
A	F	5, 0	1, 1

Two-stage prisoner's dilemma

- Remember theorem from previous lecture: a SPE must always include a NE in the last stage
- In the prisoner's dilemma, there is only one NE $\rightarrow$ No incentive for cooperation!
- Only SPE: (F,f) repeated twice

1st stage (prisoner's dilemma)

A \ B	m	f
M	4, 4	0, 5
F	5, 0	1, 1

2nd stage (also prisoner's dilemma)

A \ B	m	f
M	4, 4	0, 5
F	5, 0	1, 1

Finitely repeated games

- As a consequence of multi-stage games:
  - **Theorem**: The outcome of the last stage is a NE
  - **Theorem**: If stage game $\mathbb{G}$ only has NE p^*, then $\mathbb{G}(T, \delta)$ has a unique subgame-perfect equilibrium, which is one where players play p^* in every stage
  - Therefore, repetitions of stage games with a single NE are not very interesting

- Consider now this modified version of the game
- (F,f) and (H,h) are both NE
- Players know that they will have to choose one of them in the last stages (assuming they play a SPE)
- Are there any possible strategic connections?

Player B

Player A		m	f	h
	M	4, 4	0, 5	0, 0
	F	5, 0	1, 1	0, 0
	H	0, 0	0, 0	3, 3

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- Again, we have "stick" (F,f) and "carrot" (H,h): it is possible to build a SPE if δ is large enough
- *Remark*: In general, when we have multiple NE in the stage game, the repeated game may have a SPE where no NE is played for $t < T$

Player B

Player A		m	f	h
	M	4, 4	0, 5	0, 0
	F	5, 0	1, 1	0, 0
	H	0, 0	0, 0	3, 3

A remark on cooperation

- Repeated games tend to favor cooperation (although to a limited extent)
- For finitely repeated games
  - The last stage is always played “egoistically”
  - Collaboration in previous stages are possible only if there are multiple NE (“egoistic” choices) in the stage game
- The main influence to the game is the credibility of threats/promises about the future

Infinitely repeated games

Infinitely repeated games

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- An infinitely repeated game with stage game $\mathbb{G}$ and discount factor δ is denoted with $\mathbb{G}(\infty, \delta)$
  - remember that we need to have discount $\delta < 1$ in order for the game to be meaningful
- In infinitely repeated games we cannot apply backward induction (no "last" stage)
- Surprisingly, this leads to even more powerful results compared to finite-horizon games
  - We do not need the punishment and reward to be NE

*

$x = 1 + \delta + \delta^2 + \dots$

$x = 1 + \delta\underbrace{(1 + \delta + \dots)}_{x}$

$x = 1 + \delta x$

$(1-\delta)x = 1$

$x = \dfrac{1}{1-\delta}$

Grim trigger

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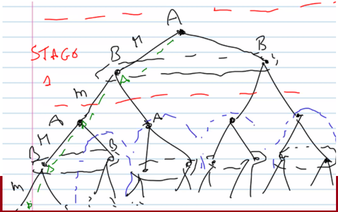
	M	F
M	4,4	-1,5
F	5,-1	1,1

- There may be SPE of $\mathbb{G}(\infty, \delta)$ in which no stage outcome is a NE of $\mathbb{G}$
- The argument can be shown again using the prisoner's dilemma
- Define a **grim trigger** strategy (GrT) as:
  - Start playing M at stage 1
  - At stage $t > 1$, play M only if outcome of *all* $t-1$ previous stages was (M, m), otherwise play F



SPE in infinitely repeated games

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- Is the joint strategy "both players play GrT" a SPE?
- Yes, if δ is close enough to 1
  - *Proof intuition*: How to prove it? We cannot use backward induction, since there is no last stage
  - We must use the definition of SPE $\rightarrow$ Need to show that GrT is a NE in all (infinite) subgames
  - We show that GrT is a best response to itself, and therefore a NE, for the overall game for δ large enough
  - In other subgames, we have two cases, subgames where players have deviated and subgames where they keep playing (M, m)
  - In subgames where players have chosen to deviate, they always play a NE at each stage
  - Subgames where they choose to keep playing (M, m), are identical to the overall game
  - Therefore, a NE is played in all subgames! $\square$

Player B

Player A		m	f
	M	4, 4	0, 5
	F	5, 0	1, 1

- When is (M, M) a NE in the overall game?
- Since at each stage payoffs are the same, we need to only check if deviation is convenient at stage 1
- In other words, we need to compare two options:
  1. Cooperate and keep playing (M, m) forever
  2. Defect at stage 1 and keep playing (F, f) forever in later stages

SPE in infinitely repeated games

- Option 1 (cooperate) yields total payoff

$$u = 4 + 4\delta + 4\delta^2 + \cdots = \frac{4}{1-\delta}$$

- Option 2 (defect) yields total payoff

$$u = 5 + \delta + \delta^2 + \cdots = 5 + \frac{\delta}{1-\delta}$$

- So the GrT is a NE (and a SPE) only if cooperating is more convenient than defecting, i.e.

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta}$$

which is true for $\delta > 1/4$

SPE in infinitely repeated games

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- In the infinitely repeated prisoner's dilemma, it is possible to enforce cooperation using a grim-trigger strategy, even without "stick" and "carrot" NE
- How does this generalize?
- General result: Friedman theorem (a.k.a. "folk theorem")

Feasible payoff

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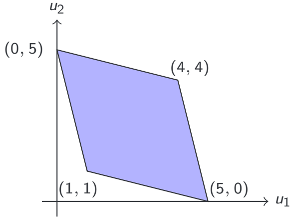
- A **feasible payoff** for game $\mathbb{G}$ is any convex combination

$$\alpha_1 u(s_1) + \alpha_2 u(s_2) + \cdots + \alpha_L u(s_L), \quad \text{with} \sum_{i=1}^{L} \alpha_i = 1$$

of pure-strategy payoffs ($L = |S_1| \cdot |S_2| \cdots |S_n|$total number of joint pure strategies)



Friedman theorem

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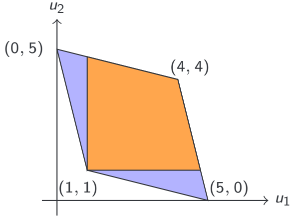
- **Theorem**: Let $\mathbb{G}$ be a finite static game of complete information. Let $(e_1, e_2, \ldots, e_n)$ be the payoffs of a NE of $\mathbb{G}$ and $(x_1, x_2, \ldots, x_n)$ be a feasible payoffs for $\mathbb{G}$. Suppose $\forall$ NE and $\forall j$, $x_j > e_j$. Then, for δ close enough to 1, $\mathbb{G}(\infty, \delta)$ has a SPE with payoffs $(x_1, \ldots, x_j)$.
- Proof follows the same process used for the prisoner's dilemma



Tit for tat

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- It may be unnecessary to keep punishment forever (holding a grudge)
  - Assume the stage game has two actions (cooperate or defect) $\rightarrow$ GrT can be replaced by "Tit for Tat"
- **Tit for tat (TFT)**: "At stage t, play what the other player chose at stage $t-1$
- Tit-for-tat strategies immediately punish deviation but are also forgiving (1-step history)
- Behavioral analogous: "eye for an eye", "live and let live"

Tit for tat

- Even though Tit for Tat is often effective, it may be "unstable" under certain conditions
- Two unsynchronized TFT players trigger a "death spiral", where each of them punishes the other in turns
- Therefore, the NE achieved by TFT is not subgame-perfect (the death spiral case is not a NE)

Self-assessment

- Consider the following game $\mathbb{G}$:

		Player B: g	Player B: w
Player A	G	5, 3	0, 4
Player A	W	6, 0	1, 1

1. Is it possible to find a SPE for $\mathbb{G}(4)$ that involves playing (G,g) at each stage?
2. Is it possible to find a NE for $\mathbb{G}(\infty)$ where (G,g) is played at each stage using a grim-trigger strategy? If so, for what δ? Is that a SPE?
3. Is it possible to find a NE for $\mathbb{G}(\infty)$ where (G,g) is played at each stage using a tit-for-tat strategy? If so, for what δ? Is that a SPE?

Exercise

- Carl (C) and Diana (D) are two university students. Every night they go to the department library, but they do not coordinate or plan any action together. Upon their arrival, they independently decide whether to: (S) study or (M) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for C and 18 for D. However, if they both choose M, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(\text{S,M}) = u_D(\text{M,S}) = 0$. Call $\mathbb{G}$ this game, and consider it in a repeated version $\mathbb{G}(T)$. Individual payoffs are summed with discount factor δ.

Exercise

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a "Grim Trigger" strategy where each player ends up in always choosing S?
3. Consider an *extended* game where a punishment strategy P is also available to both players. When either player P, payoffs are -10 for *both* players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game $\mathbb{G}'$. If you see a SPE of $\mathbb{G}'(2)$ where players may play S, state at which round do they play it, and what value of δ do you need to obtain it.

Sorry, gotta bounce!
Send me questions via e-mail