Università
degli Studi
di Padova

Dipartimento
di Ingegneria
dell'Informazione

Lecture 05
Constitutions

Thomas Marchioro

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Previously on game theory

- In a n-player game $\mathbb{G} = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, strategies $(s_1^*, \ldots, s_n^*)$ are a **Nash equilibrium** if, for all i, s_i^* is a best response to $s_{-i}^* = (s_1^*, \ldots s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$
- Meaning that, $\forall s_i \in S_i$

$$\boxed{u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)}$$

or, equivalently,

$$\boxed{s_i^* = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}^*)}$$

NE as absence of regret

- Remember that we are considering static (one-shot) games
- A NE can be seen as a joint strategy in which no player has regrets on their choice
- In other words, if a NE is played, none of the players would want to unilaterally change their strategy even if they had the possibility to do so

Self-assessment

- What is a NE?
- Consider NE $(s_1, \ldots, s_n)$. Suppose player i replaces the current strategy s_i with s_i'. Can this still be a NE?
- If a strategy is ruled out by IESDS, can it be a NE?
- Compute the PoA for the Prisoner's dilemma using $C(s) = -\sum_j u_j(s)$

Homework

- A (crazy) professor decides your grade in the exam he teaches will be decided by a game:
  - You are paired with a random classmate
  - You secretly choose an integer between 18 and 30, and so does the classmate
  - If you choose the same number, that is the score that you both get
  - If the numbers are different, who proposes the lowest score L gets a grade of $L + R$, while the other gets $L - R$ (score < 18 means the exam is failed, $>$30 means 30L and gives payoff 31)
- Play the game with $R = 1$, $R = 2$, and $R = 10$.
- How do the NE change?

Howework (Solution)

- The matrix looks like this for a generic R

	18	**19**	$\cdots$	**30**
18	18, 18	18 + R, 0	$\cdots$	18 + R, 0
19	0, 18 + R	19, 19	$\cdots$	19 + R, 19 - R
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
30	0, 18 + R	19 - R, 19 + R	$\cdots$	30, 30

- However, when $L - R < 18$ the payoff always becomes 0 and when $L + R > 30$ the payoff always becomes 31

- For $R = 1$
- Matrix is simplified due to space constraints

	18	**19**	**20**	**21**
18	18, 18	19, 0	19, 0	19, 0
19	0, 19	19, 19	20, 18	20, 18
20	0, 19	18, 20	20, 20	21, 19
21	0, 19	18, 20	19, 21	21, 21

Howework (Solution)

- Best responses for student 1 are highlighted in blue, best responses for student 2 are highlighted in red

	18	**19**	**20**	**21**
18	18, 18	19, 0	19, 0	19, 0
19	0, 19	19, 19	20, 18	20, 18
20	0, 19	18, 20	20, 20	21, 19
21	0, 19	18, 20	19, 21	21, 21

- You can easily see that any joint strategy where students choose the same grade is a NE: (18, 18), (19, 19), . . . , (30, 30)
- This is a coordination game, similar to the battle of the sexes

- For $R = 2$
- Matrix is (again) simplified due to space constraints

	18	**19**	**20**	**21**
18	18, 18	20, 0	20, 0	20, 0
19	0, 20	19, 19	21, 0	21, 0
20	0, 20	0, 21	20, 20	22, 18
21	0, 20	0, 21	18, 22	21, 21

- This is a prisoner's dilemma-like game, where the only NE is a Pareto-dominated strategy
- Notice that this is true for any $R > 1$

Howework (Solution)

- In the real world, cooperative behaviors may arise, even though they are not NE
  - Criticism against rationality of players
- Usually, a high R (for example, $R = 10$) dampens the cooperation

Constitutions

Constitutions

- Let $R(A)$ be a set of rational preferences on A
  - remember: rational preference means you can write $a_1 \succeq \cdots \succeq a_n$ using each and every element of A exactly once
- A **constitution**, or **social welfare function**, is a map

$$\begin{aligned} f : & R(A)^n \longrightarrow R(A) \\ & (\succeq_1, \ldots, \succeq_n) \overset{f}{\longmapsto} f(\succeq_1, \ldots, \succeq_n) \end{aligned}$$

- A constitution maps a profile of n rational preferences $\succeq_{(i)} = (\succeq_1, \ldots, \succeq_n)$ into a unique *rational* social preference $\succeq = f(\succeq_{(i)})$

- Notation: We use $\succeq|Y$ to mean "restricting preference $\succeq$ to $Y \subseteq A$" (formally, $\succeq|Y$ is equivalent to $\succeq \cap (Y \times Y)$)
- A constitution f satisfies the **Independence of Irrelevant Alternatives** (IIA) if for all pairs $(\succeq_{(i)}), (\succeq'_{(i)})$

$$\forall i, \succeq_i |\{a,b\} = \forall i, \succeq'_i |\{a,b\}$$
$$\Longrightarrow f(\succeq_{(i)})|\{a,b\} = f(\succeq'_{(i)})|\{a,b\}$$

  - i.e., adding or removing elements to the set of alternatives does not change the output of a constitution for the pair $\{a,b\}$

Properties of constitution

- Constitution f is **Pareto-efficient** if for all profiles $(\succeq_{(i)})$, for all $a, b \in A$

$$\forall i, a \succeq_i b \Longrightarrow a \succeq b, \text{ where } \succeq = f(\succeq_{(i)})$$

  i.e., if everyone prefers a over b, that becomes also the preference in the constitution

- Pareto efficiency relates to the concept of "being better for everybody"

Properties of constitutions

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- f is a **dictatorship** if $\exists i$ such that

$$a \succeq_i b \implies a \succeq b, \text{ where } \succeq = f(\succeq_{(i)})$$

i.e., if the constitution simply mimics i's preferences

- f is **monotonic** if a single individual ranking higher $a \in A$ never causes a rank lower in the constitution
- f satisfies **non-imposition** if all rational preferences can be outputs (formally, it is surjective)

Arrow's impossibility theorem

- **Theorem** (Arrow, 1951)
  There is no constitution f for which all these properties hold at the same time
  - f is not a dictatorship
  - f is monotonic
  - f satisfies IIA and non-imposition
- **Theorem** (Arrow, 1963), also known as Arrow's impossibility theorem
  If constitution f
  - is Pareto-efficient
  - satisfies IIA

  then f is a dictatorship!

Elections and paradoxes

Elections and democracy

- What is **democracy**?
- Usually we immediately connect democracy with elections, as well as with "majority rule"
- What does majority mean?
- Things get complicated when we have multiple choices

Election and democracy

- Say we have 3 voters and 2 candidates
- The preference are as follows

voter	**1**	**2**	**3**
best	A	A	B
worst	B	B	A

- A beats B by majority rule
- A democratic society should choose A

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- Say we have 3 voters and 3 candidates
- The preference are as follows

voter	**1**	**2**	**3**
best	A	A	B
	B	C	C
worst	C	B	A

- According to majority, $A > B$, $B > C$, $A > C$. A beats all the other candidates
- A democratic society should choose A

Election and democracy

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- Say we have 3 voters and 3 candidates
- The preference are as follows

voter	**1**	**2**	**3**
best	A	C	B
	B	A	C
worst	C	B	A

- According to majority, $A > B$, $B > C$, $C > A$. There is no "best" candidate.
- What should a democratic societety choose
  - In democratic elections, cycles lead to paradoxes!

Terminology

- A candidate that beats (majority-wise) all the others is called the **Condorcet winner**
- If there is no winner, then there must be a cycle, formally called a **Condorcet cycle**
- Also mixed cases are possible for > 3 candidates (e.g., a winner, and a cycle among the 3 remaining candidates)

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- The cases with three candidates directly originate from the two-candidate case

voter	**1**	**2**	**3**
best	A		
	B	A	B
		B	
worst			A

- It all depends on where we put C between A and B!

Remark 1

voter	1	2	3
best	A	C	C
	B	A	B
	C	B	
worst			A

- In this case, C is the Condorcet winner

Remark 1

-

voter	1	2	3
best	A		
	B	A	B
	C	B	C
worst		C	A

- In this case, C is the worst of all candidates (“Condorcet loser”)

Remark 1

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voter	**1**	**2**	**3**
best	A	C	
	B	A	B
	C	B	C
worst			A

- In this case, we have a Condorcet cycle

Remark 2

- Condorcet cycles cannot occur when only two alternatives are present
- With ≥ 3 alternatives there may be cycles
- The probability of Condorcet cycles grows with the number of candidates
- If preferences are sufficiently randomized, for a large $(n \to \infty)$ number of candidates, Condorcet cycles are sure to occur

- Probability of having at least one cycle (given uniformly random preferences)

voters $\rightarrow$ choices $\downarrow$	3	5	7	9	∞
3	5.6%	6.9%	7.5%	7.8%	8.8%
5	16.0%	20.0%	21.5%	23.0%	25.1%
7	23.9%	29.9%	30.5%	34.2%	36.9%
∞	100.0%	100.0%	100.0%	100.0%	100.0%

Remark 3

- Even though we speak of candidates and elections, the same concepts could apply to:
  - Network scheduling: think of candidates A, B, C as users/packets/objects and of voters' preferences as criteria to choose between them
  - Optimization: think of candidates A, B, C as possible solutions and of voters' preferences 1, 2, 3 as different objective/utility functions

Some “real world” examples

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- Fiscal politics of governments

	economic left	**anti-deficit**	**economic right**
best	+ Taxes + Spending	+ Taxes − Spending	− Taxes − Spending
	− Taxes − Spending	+ Taxes + Spending	+ Taxes − Spending
worst	+ Taxes − Spending	− Taxes − Spending	+ Taxes + Spending

Some "real world" examples

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- Quality of Service

	"well-behaved"	**high delay**	**high losses**
best	Voice over IP	Video Streaming	Best Effort Data
	Video Streaming	Best Effort Data	Voice over IP
worst	Best Effort Data	Voice Over IP	Video Streaming

Electoral systems

Setting the agenda

- Assume 3 competitors A, B, and C: we choose between A and B in a first round, then the winner goes up against C
- Seems fair? Not in a Condorcet cycle!
- Assume the cycle is A<B<C<A: C wins with this setup but would lose with another
- For example, consider the following system: "choose between C and B first, then the winner goes up against A" $\rightarrow$ A wins

Other methods

- There are actually many electoral systems (which work also as selection rules in allocation problems), such as
  - ~~Plurality voting~~
  - Two-phase run-off
  - Borda counting
  - Approval voting
  - Instant run-off

Plurality voting

NOT ASKED IN THE EXAM

- Let each voter sort the candidates in order of personal preference
    - Some candidates will get "first place" by some voters
- In the "plurality voting" criterion, the winner is whoever has most first places among the voters
- Is this mechanism immune to paradoxes?

Plurality voting

NOT ASKED IN THE EXAM

- Assume we have 9 voters

voter	1–4 (4 voters)	5–7 (3 voters)	8–9 (2 voters)
best	A	B	C
	B	C	B
worst	C	A	A

- A wins (4 voters vs 3 voters of B and 2 of C)
- However majority prefers B to A
- Also, majority prefers C to A
- There is even a Condorcet winner (B), since B>C

Two-phase run-off

- We have a two-round voting
- First round: we select the two candidates with highest amount of votes
- Second round: run-off between those candidates

Two-phase run-off

- Consider again the 9-voter case

voter	**1–4 (4 voters)**	**5–7 (3 voters)**	**8–9 (2 voters)**
best	A	B	C
	C	C	B
worst	B	A	A

- A and B go to the run-off, B wins 5-4
- However, majority prefers C>A and C>B
- C is the Condorcet winner, but C does not even make it to the run-off

Borda count

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- Plurality voting and Two-phase run-off favor "polarized" solutions over "compromises"
- A strong candidate in a (large) minority wins over a weak one who is appreciated by many
- Borda count tries to overcome this issue:
    - Suppose we have M candidates, and voters need to sort them by preference
    - For each person, M-1 point go to their favorite candidate, M-2 go to the next one, and so on until the last-favorite candidates, who gets 0 points
- Is this fairer?

Borda count

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- Consider again the 9-voter case

voter	**1–5 (5 voters)**	**6–8 (3 voters)**	**9 (1 voter)**
best	A	B	C
	B	C	B
worst	C	A	A

- A scores 10 points, B scores 12, and C scores 5 (so B wins)
- However, A is the Condorcet winner, since A>B, and A>C

Borda count with dropout

- Borda-like counts are used in sports (e.g., to decide who is the MVP in baseball)

voter	**1–5 (5 voters)**	**6–7 (2 voters)**	**8–9 (2 voters)**
best	D	A	A
	C	D	B
	B	B	D
worst	A	C	C

- Total points: A gets 12 points, B gets 11 points, C gets 10 points, and D gets 21 points
- So, D gets gold medal, A gets silver, and B gets bronze

Borda count with dropout

- Suppose D withdraws from the competition (e.g., anti-doping or naked pictures)

voter	**1–5 (5 voters)**	**6–7 (2 voters)**	**8–9 (2 voters)**
best		A	A
	C		B
	B	B	
worst	A	C	C

- Total points: A gets 8 points, B gets 9 points, and C gets 10
- So, C gets gold, B gets silver, and A gets bronze
- D's withdrawal completely reverses the order

Approval voting

- Each voter can give more than one preference
- 1 preference = 1 point
- The number N of preferences is a fixed number between 1 and M (M = number of candidates)
- For N=1 we fall back into plurality voting

Approval voting

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- Consider again the 9-voter case

voter	1–3 (3 voters)	4–6 (3 voters)	7–8 (2 voters)	9
best	A	D	B	A
	C	B	D	B
	D	C	C	C
worst	B	A	A	D

- Top N=2 approvals: A gets 4, B gets 6, C gets 3, and D gets 5 $\rightarrow$ B wins
- Top N=3 approvals: A gets 4, B gets 6, C gets 9, and D gets 8 $\rightarrow$ C wins

Instant run-off

- Once again, we ask each voter for their "order of preference"
- Only top preferences count to reach a majority
- We iteratively remove candidates with the lowest amount of top preferences

- Let us see an example with 17 votes

voter	6 voters	5 voters	4 voters	2 voters
best	A	C	B	B
	B	A	C	A
worst	C	B	A	C

No majority $\rightarrow$ candidate C is eliminated

A gains 5 votes and wins (with 11 votes)

Instant run-off

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- What if the last 2 voters preferred A over B

voter	**6 voters**	**5 voters**	**4 voters**	**2 voters**
best	A	C	B	A
	B	A	C	B
worst	C	B	A	C

- In this situation A loses!
- B is now eliminated at the first round
- A loses due to an increase in preferences

Setting the agenda

- The selection of a particular system may advantage some competitors in an almost invisible way
- This is a very subtle factor in many fields: politics, sports, sciences, everyday life
- Luckily, there is a limit to it

Setting the agenda

- A>B>C>A are in a Condorcet cycle. D is worst.

voter	**1–4 (4 voters)**	**5–7 (3 voters)**	**8–9 (2 voters)**
best	A	C	B
	B	A	C
	C	B	A
worst	D	D	D

- There is no way for D to win (A>D, B>D, C>D)
- However, if we make semifinals and finals, whoever goes against D is guaranteed to win

Setting the agenda

- A>B>C>A are in a Condorcet cycle. D is best.

voter	**1–4 (4 voters)**	**5–7 (3 voters)**	**8–9 (2 voters)**
best	D	D	D
	A	C	B
	B	A	C
worst	C	B	A

- D always wins, while the order of A, B, and C depends on the agenda setting

“Cheating” in Condorcet cycles

Cheating

- A>B>C>A are in a Condorcet cycle.

voter	1–4 (4 voters)	5–7 (3 voters)	8–9 (2 voters)
best	A	B	C
	D	A	B
	C	D	D
worst	B	C	A

- However, A is the winner is many systems (plurality, Borda count, Top-2 approval, . . .)
- Suppose we choose plurality: A wins

Cheating

- 8–9 are not happy with the outcome, as for them A is the worst
- Thus, they decide to "cheat" and indicate B as preferred choice instead of C

voter	1–4 (4 voters)	5–7 (3 voters)	8–9 (2 voters)
best	A	B	~~C~~ B
	D	A	~~B~~ C
	C	D	D
worst	B	C	A

- Now B wins (for 8–9, this is an improvement)

Cheating

- For voters 1–4, this is not a good outcome

voter	1–4 (4 voters)	5–7 (3 voters)	8–9 (2 voters)
best	A	B	~~C~~ B
	D	A	~~B~~ C
	C	D	D
worst	B	C	A

- They may protest and ask for help from 5–7
- However, 5–7 are happy, as B is their favorite candidate

Cheating

- However, 1–4 can act first and “cheat” too

voter	1–4 (4 voters)	5–7 (3 voters)	8–9 (2 voters)
best	~~A~~ C	B	~~C~~ B
	~~D~~ A	A	~~B~~ C
	~~C~~ D	D	D
worst	B	C	A

- They can change and support A (whom they prefer over C): now A wins again
- C wins with only 2 original votes

Extensions to Arrow's Theorem

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- Social function f is **strategy-proof** if for any profile $(\succeq_{(i)})$ and a certain preference $\succeq'_j$

$$f(\succeq_{(i)}) \succeq_j f(\succeq'_j, \succeq_{-j})$$

  - i.e., no one has incentive to cheat
- **Gibberard-Satterthwaite theorem** Any strategy-proof constitution that does not prevent anyone from winning is a dictatorship

Problems of electoral systems

- It seems that every system is "bad" in certain cases
- Recall Arrow's Theorem: if a constitution is Pareto efficient and satisfies IIA, then it is a dictatorship
- "Ways out"
  - some conditions are weakened
  - use free-approval voting (vote "for" or "against")
  - restrict the profile

Majority rule

- This last solution has been proposed in different settings by many economist
- Formally, a majority rule $\succeq$ can be defined as

$$a \succeq b \iff |\{i : a \succeq_i b\}| \geq |\{i : b \succeq_i a\}|$$

- Properties:
  - Pareto efficient
  - satisfies IIA
  - not a dictatorship
- ... but not a constitution either!

Majority rule

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- Majority rule satisfies completeness but not **transitivity**
- The reason is the existence of Condorcet cycles
- If we are able to somehow eliminate the existence of Condorcet cycles, majority rule becomes a constitution and possesses "nice" properties

Questions?