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Lecture 17
Stackelberg games and bargaining

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Previously on game theory

- Zero-sum games: games where players have opposite utility functions, i.e. $u_i(p) = -u_{-i}(p), \; \forall p \in \Delta S_i \times \Delta S_{-i}$
- Definitions:
  - Maximin$_i$: $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$
  - Maximin$_i^p$: $\max_{p_i} \min_{s_{-i}} u_i(p_i, s_{-i})$
  - Minimax$_i$: $\min_{s_{-i}} \max_{s_i} \; u_i(s_i, s_{-i})$
  - Minimax$_i^p$: $\min_{p_{-i}} \max_{s_i} \; u_i(s_i, p_{-i})$
- Intuitions:
  - Maximin: imagine the opponent plays last
  - Minimax: imagine the opponent plays first

Previously on game theory

- In general:

$$\boxed{\text{maximin} \leq \text{maximin}^p = \text{minimax}^p \leq \text{minimax}}$$

- Minimax theorem: In a zero-sum game with finitely many strategies
  - In pure strategies, if maximin=minimax for both players, then there is a pure NE with $u(\text{NE})$=maximin=minimax
  - In mixed strategies, we know for a fact that $\text{maximin}^p = \text{minimax}^p = u(\text{NE})$ for every NE in mixed strategies

Previously on game theory

- To find NE in zero-sum games, you can model the minimax^p/maximin^p as a linear program
- Algorithm for minimax$_i^p$:
  - Draw the utility $u_i(s_i, p_{-i})$ of player i as a function of p_{-i}, for each possible pure strategy s_i of i
  - Consider all the possible values w such that $w \geq u_i(s_i, p_{-i})$
  - Choose the minimum value of w^* that satisfies all constraints
- That is the minimax$_i^p$, and the corresponding values of p_{-i} are the possible mixed strategies of player $-i$ at the NE
- The lines/hyperplanes representing s_i that characterize the boundary at the minimax represent the support of p_i at the NE

Today on game theory

- Stackelberg games: turning static games into sequential games by making one player the "leader" and other players "followers"
  - Sequential games → SPE is guaranteed to be played if players act rationally
- Bargaining: players need to choose how to split resources

Stackelberg games

Stackelberg games

- Proposed by Heinrich von Stackelberg (1934) to model incumbent vs outsider competition
- It is a sequential version of a static game (analogous to the sequential Battle of Sexes)
- Players move one after the other
- First player 1 (**leader**), then player 2 (**follower**)
- Can be represented again with a bi-matrix
- The outcome of backward induction is also called the **Stackelberg equilibrium**

Stackelberg game: Battle of the sexes

		B	
		R	S
A	R	2, 1	0, 0
	S	0, 0	1, 2

- If A is the leader, the Stackelberg equilibrium is (R, R) with payoffs 2 and 1 for A and B
- If B is the leader, the Stackelberg equilibrium is (S, S) with payoffs 1 and 2 for A and B
- The leader has an advantage in Stackelberg games

Finding Stackelberg equilibria

			Joe	
		F	G	H
Carl	R	**2**, **2**	3, 1	0, 0
	S	1, **6**	**5**, 4	**6**, 4
	T	0, 1	4, 3	**6**, 2

- If we treat this game as a normal static game, the only NE is (R, F) yielding payoff 2 to both Carl and Joe

Finding Stackelberg equilibria

		Joe		
		F	G	H
Carl	R	2, 2	3, 1	0, 0
	S	1, 6	5, 4	6, 4
	T	0, 1	4, 3	6, 2

- If Carl is the leader, we can use backward induction
- However, we do not need to draw the tree, we can use the following algorithm
    - Step 1: maximize Joe's payoff across rows (find best responses)
    - Step 2: maximize Carl's payoff across the options selected by Joe

Finding Stackelberg equilibria

		Joe		
		F	G	H
Carl	R	2, 2	3, 1	0, 0
	S	1, 6	5, 4	6, 4
	T	0, 1	4, 3	6, 2

- The Stackelberg equilibrium is (T, G) yielding payoffs 4 and 3: an improvement over the NE
- The procedure is similar to the minimax but the outcome is different: the leader does not want to minimize the follower's payoff
    - The minimax for Joe here is 2

Finding Stackelberg equilibria

		Joe		
		F	G	H
Carl	R	2, 2	3, 1	0, 0
	S	1, 6	5, 4	**6**, 4
	T	0, 1	4, 3	**6**, 2

- If Joe is the leader, we need to do the opposite: first we maximize Carl's payoff across rows and then we maximize Joe's payoff across Carl's choices
- However, we have a tie in the last column
- In normal sequential games, we just consider both options (and their combinations) in backward induction, leading to multiple SPE

Finding Stackelberg equilibria

		Joe		
		F	G	H
Carl	R	2, 2	3, 1	0, 0
	S	1, 6	5, 4	6, 4
	T	0, 1	4, 3	6, 2

- In Stackelberg games, we would like a tie breaker to decide what players actually do in practice
- Assumption: generous follower → in case of a tie, the follower maximizes the leader's payoff
- However now, we have a tie for Joe between G and H
- Assumption: generous leader → in case of a tie, the leader maximizes the follower's payoff

Finding Stackelberg equilibria

			Joe	
		F	G	H
Carl	R	2, 2	3, 1	0, 0
	S	1, 6	5, 4	6, 4
	T	0, 1	4, 3	6, 2

- In Stackelberg games, we would like a tie breaker to decide what players actually do in practice
- Assumption: **generous follower** → in case of a tie, the follower maximizes the leader's payoff
- However now, we have a tie for Joe between G and H
- Assumption: **generous leader** → in case of a tie, the leader maximizes the follower's payoff

Finding Stackelberg equilibria

		Joe		
		F	G	H
Carl	R	2, 2	3, 1	0, 0
	S	1, 6	5, 4	6, 4
	T	0, 1	4, 3	6, 2

- In this case, the Stackelberg equilibrium is (S, H), with payoffs 6 and 4 → an even better outcome
- In general: at the Stackelberg equilibrium, both the leader and the follower are never worse compared to the Nash equilibrium
    - Main idea: the follower plays a best response, and the leader anticipates that

Stackelberg zero-sum games

		Even	
		0	1
Odd	0	-4, 4	4, -4
	1	4, -4	-4, 4

- In this game, if Odd is the leader and declares their move, that's an automatic loss
- Assumption: the leader has the option to not reveal their strategy (or to reveal a mixed strategy)
- Here, Stackelberg equilibrium = Nash equilibrium

Comments on Stackelberg

- The leader has the "first-move advantage"
  - His/her payoff in Stackelberg equilibrium $\geq$ payoff in NE of the static game
- The follower is not necessarily worse off in Stackelberg equilibrium
  - In general, his/her payoff $\geq$ minimax

		B	
		R	S
A	R	2, 1	0, 0
	S	0, 0	1, 2

Comments on Stackelberg

- However, in adversarial/competitive setups (specifically, in zero-sum games), the leader being better off implies that the follower is worse off
- That might seem strange: the follower has more information
  - in this case, more information → lower payoff
  - consequence of rationality: player 1 can anticipate player's 2 knowledge and therefore his/her response

Dynamic bargain

Bargain

- Bargain = negotiation of resource sharing
- Assume two players need to split a given amount of resources
  - Player 1 gets a fraction x, player 2 gets $1 - x$
- Two main approaches
  - Nash bargaining (axiomatic, static)
  - Modeled as a dynamic game with alternate stages where players 1 and 2 switch proposer/responder roles

Dynamic bargain

- At stage $t = 1$: the proposer (P) is player 1, the responder (R) is 2
  - P proposes split $(x, 1 - x)$ and R can decide to accept or refuse. If R accepts, the game ends, otherwise they go to stage 2.
- At stage $t = 2$: P is 2, the R is 1.
  - As before, P proposes $(x', 1 - x')$ and R decides whether to accept or not. If R refuses, they go to stage 3.

$\vdots$

- At a generic stage t: P is player 1 if t is odd, otherwise P is 2.
  - R accepts $\Rightarrow$ game ends; R refuses $\Rightarrow$ go to stage $t + 1$.
- Assumption: if disagreement persists after a deadline T, then they both get payoff 0.

Dynamic bargain

- If the game ends at stage $1 < t < T$, both players get discounted payoff with δ^{t-1}
  - Intuition: for a same split $(x, 1-x)$, players prefer to reach an agreement first
- If the deadline is $T = 1$ (either they agree immediately or the resources are wasted), this is called the **Ultimatum game**
  - All joint strategies with P proposing $(x, 1-x)$ and R accepting and are NE
  - P proposing $(1, 0)$ and R accepting is the only SPE

SPE of dynamic bargain

- The Ultimatum game can be used to deduce the outcome of a generic dynamic bargain game
- This can be done via backward induction
  - Suppose that the deadline is at stage T, with T odd
  - Then 1 is the last proposer and knows that 2 is going to accept any split. If stage T is reached, the game ends with payoffs $u_1 = \delta^{T-1}$; $u_2 = 0$
  - At round $T-1$, 2 is the proposer and can anticipate that by offering $x \geq \delta$. Of course, being rational, 2 chooses $x = \delta$: the game ends with payoffs $u_1 = \delta \cdot \delta^{T-2}$, $u_2 = (1-\delta) \cdot \delta^{T-2}$.
  - By iterating this reasoning, they can reach an agreement at stage 1 with payoffs

$$\boxed{u_1 = \frac{1+\delta^T}{1+\delta} \qquad u_2 = \frac{\delta-\delta^T}{1+\delta}}$$

example on professor notes

SPE of dynamic bargain

- **Proposition**: Any SPE of the dynamic bargaining game must have the players reaching an agreement in the first round
  - Simply a consequence of backward induction
  - Iterating the game: (i) wastes reward because of the discount; (ii) sends the players to another round of proposer-responder, which rational players want to avoid
- *Note*: this is not a repeated game because of the termination option (in a multistage game, players always play *all* stages, which must give independent payoffs)

Infinite dynamic bargain

- Interestingly, this reasoning applies even to infinite horizon
  - Backward induction does not work, but player still have incentive not to waste resources
- For $T \to \infty$,

$$u_1 = \frac{1}{1+\delta} \qquad u_2 = \frac{\delta}{1+\delta}$$

  which for $\delta \to 1$ approaches an equal split
- Also in the infinite-horizon case, we can prove that any SPE requires players to reach an agreement in the first round

Infinite dynamic bargain

- Still, we need to prove that the SPE is unique (without resorting to backward induction)
- *Intuition*: this can be proven by contradiction
    - Assume that there is more than one SPE
    - We know that in all SPE players agree on the first round, so the difference must be in the payoffs
    - Suppose that the best for 1 yields payoff v_1 and the worst yields payoff w_1
    - Player 2 gets the remaining part, so either $1 - v_1$ or $1 - w_1$
    - If stage 2 is reached, 2 can either get $v_2 = \delta v_1$ or $w_2 = \delta w_1$ (same infinite game, but with reversed roles)
    - That means that the split proposed by player 1 at stage 1 should be $1 - v_1 = \delta v_1$ or $1 - w_1 = \delta w_1$
    - In both cases, that leads to $v_1 = w_1 = \frac{1}{1+\delta}$

Send me questions via e-mail