Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

Lecture 06
Applications of Nash Equilibrium

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October 19, 2023

General applications of NE

- Prediction tool
  - In many cases, there is a NE (as we will see, we can extend the definition so that there is always one)
  - However, it is not unique: we many need more information to decide which NE is the most likely outcome
- Criticism
  - NE does not guarantee a "good" (Pareto efficient) solution, since players are driven by selfishness

Duopolies

Cournot duopoly

- Cournot (1838) anticipated Nash's results in a particular context: a special duopoly model
- In Cournot's model, there are two firms (firm 1 and firm 2) producing a good in quantities q_1 and q_2. Let $Q = q_1 + q_2$.
- The cost of producing q is the same for both firms $C(q) = cq$ (with $c > 0$ constant)
- When the good is sold on the market, its price is $P(Q) = a - Q$ (with constant $a > c$)
  - More precisely, $P(Q) = (a - Q)\mathbb{1}(a - Q)$

Cournot duopoly

- Suppose the firms choose q_1 and q_2 simultaneously: can we predict their optimal production?
  - i.e., is there a NE of the game?
- Both firms $i = 1, 2$ have a single-move strategy q_i and $S_i = [0, \infty)$
  - actually, any $q_i > a$ yields negative utility, so we can set $S_i = [0, a)$
- The payoff of a firm is simply its profit (revenue minus cost):

$$u_i(q_i, q_j) = q_i \cdot (P(q_i + q_j) - c) = q_i \cdot (a - q_i - q_j - c)$$

NE of Cournot duopoly

- Is there any NE (q_1^*, q_2^*)
- For each player i, q_i^* must satisfy:

$$q_i^* = \arg\max_{q_i} u_i(q_i, q_j^*)$$

- Compute the partial derivative of u_i w.r.t. q_i and set it equal to 0

$$\begin{aligned}\frac{\partial}{\partial q_i} u_i &= \frac{\partial}{\partial q_i}[q_i \cdot (a - q_i - q_j - c)] \\ &= a - 2q_i - q_j - c = 0\end{aligned}$$

- Solution for both: $q_1 = q_2 = (a-c)/3$
- Payoff for both: $u_1 = u_2 = (a-c)^2/9$

Monopoly solution

- In the case of a single firm (monopoly) the optimal production would be

$$q_m = \arg\max_q q(a - q - c)$$

- This leads to $q_m = (a - c)/2$ and $u_m = (a - c)^2/4$
  - We denote it with q_m to differentiate it from the NE solution q^* of the Duopoly case
  - The monopolist produces less that the two firms of the Duopoly together $q_m < q_1 + q_2$
  - Lower production at higher price = profit!

Trust-based monopoly

- The two firms could compare the NE, which achieves profit $u^* = (a-c)^2/9$, with the a "joint monopoly" solution
- In other words, they could cooperate as if they were a single monopoly
- They produce half of q_m each and share $u_m = (a-c)^2/4$ (getting $(a-c)^2/8$ each)
- They produce less than in the NE case so the price is higher and the revenue is increased

Why is the monopoly not a NE?

- Each firm has an incentive to deviate from the "monopoly" strategy ($q_1 = q_m/2$ is not the best response to $q_2 = q_m/2$ and vice versa)
- When the price is high, unilaterally increasing the production level for firm i raises its revenue (and decreases j's revenue)
- At the same time, this decreases the price, causing firms to increase the quantity until there is no more incentive to do so

Bertrand duopoly

- Bertrand (1883) criticized Cournot's model by arguing that firms choose prices, not quantities
- Now, we have an entirely different game. Strategies are prices $p_i \in S_i = [0, \infty)$
- Assume people buy $q_i = a - p_i$ from the firm with cheaper price and 0 from the other (if the p_i values are equal, q_i is equally shared between them)
- Cost is $C(q) = cq$ (as in the Cournot case, $a > c$)
- Competition leads to lowering the price
- NE of this game is $p_1^* = p_2^* = c$

Bertrand duopoly

- Similarly to Cournot's, Bertrand equilibrium is clearly not the best outcome for the firms
- In fact, they could agree on a higher price and share the market (monopoly solution). The price can be pushed up to $(a + c)/2 > c$
- However, this is not a NE, as each firm has incentive to deviate (decreasing the price to conquer the market)
- This process is indefinitely repeated as long as the price is c

Bertrand duopoly

- Interestingly, both firms set price = cost
  - which means they have zero profit
- The reason behind this strange outcome is in the best response to each player's belief
  - if firm 1 believes that firm 2 will set $p_2 = c$, its profit will be 0 anyway
- Notice that profit is 0 even if firm 1 sets $p_1 > c$
  - but $(c + \epsilon, c)$ is not a NE, since not all players chose a best response (firm 1 did, firm 2 did not)

Bertrand duopoly

- Economy-wise, Bertrand equilibrium is nice for the customers. Is it realistic, though?
- Problem: it does not account for **imperfect substitutes**
- Let $q_i = a - p_i + bp_j$
- Note: this is yet another game
- b can be seen as a "substitutability" rate between goods
- It can be shown that there is a NE

$$p_1^* = p_2^* = (a+c)/(2-b)$$

Bertrand duopoly

- Or, consider a case with different production costs
  - For example, $c_1 = 1$, $c_2 = 2$ (cost advantage for firm 1)
  - For simplicity, prices are set in steps of $\epsilon = 0.01$ (discretized)
- In this scenario, there is no way firm 2 can "win"
  - Firm 1 can set $p_1 = 1.99$ and become monopolist
  - One possible Nash equilibrium is (1.99, 2.00)
  - However, if $\epsilon \to 0$ we have a problem: (2, 2) is not a NE, as payoffs are discontinuous
  - Discretizing the state space is a common trick to avoid these problems

Hotelling model

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- Hotelling (1929) proposed a model of competition, readjusted here as follows
- Two street vendors of ice-cream serve a seaside boulevard roughly 1 km long
- They need to decide where to locate their stand
  - Ice-cream cones sold by the vendors are perfect substitutes for each other ($\rightarrow$ same price)
  - People buy ice-cream by the nearer vendor
  - People distribution on the street is uniform
  - For simplicity, assume 101 possible locations (one every 10 meters): 0, 1, ... 99, 100

Hotelling model

- If vendor A chooses location 22 and vendor B chooses 35, vendor A gets people from 0 to 28, while B gets all the others
  - But A has and incentive to move right (he/she can do better by moving to 36)
- Easy to see the only NE: they both choose 50
  - Such a result has often been used as a political paradigm (median voter theorem)
  - Politicians converge to the median, where they can get votes of both the moderates and extremists

Tragedy of the commons

Tragedy of the commons

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- Many political philosophers and economists, since at least Hume (1739) have understood that, if moved only by private incentives, people will tend to misuse public resources
- Oil extraction, water consumption, environmental pollution
- This problem is commonly referred to as the "tragedy of the commons"
- There are several perspectives on it

Tragedy of the commons

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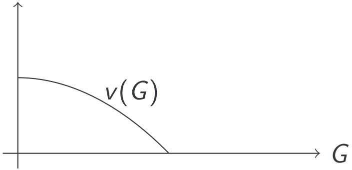
- Classic version (Hardin, 1968):
  - We have n farmers in a village, which forage their goat in a common green
  - Each farmer owns g_i goats, so the total number of goats is $G = g_1 + \cdots + g_n$
  - Each goat costs c in caring expenses
  - The use of the common green shared by G goats has value $v(G)$ per goat
  - The value decreases with G

Tragedy of the commons

- An information-theory version:
  - We have n users connected to a WiFi hotspot, accessing a shared spectrum. Each user activates g_i connections.
  - The overall network throughput has value $v(G)$ (decreasing with G)

$$v(G_{max} = 0)$$
$$v'(G) < 0$$
$$v''(G) < 0$$



Tragedy of the commons

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- Notation: $g_{-i} = (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$
- The payoff to each user is

$$u_i(g_i, g_{-i}) = g_i \underbrace{(v(G) - c)}_{\text{utility of one item}}, G = g_i + g_{-i}$$

- Let us find the NE g_i^*

$$\max_{g_i} g_i(v(g_i + g_{-i}) - c)$$

- Take the derivative $\frac{\partial}{\partial g_i} u_i(g_i, g_{-i})$ and find g_i^* where it is 0

$$v(g_i^* + g_{-i}^*) + g_i^* v'(g_i^* + g_{-i}^*) - c = 0$$

Tragedy of the commons

- Symmetry in the problem $\Rightarrow$ we can take the derivative w.r.t. g_i replace g_i^* with G^*/n

$$v(G^*) + (G^*/n) \cdot v'(G^*) - c = 0$$

- The global welfare is $G(v(G) - c)$, so we have a global optimum at G_m for which

$$v(G_m) + G_m \cdot v'(G_m) - c = 0$$

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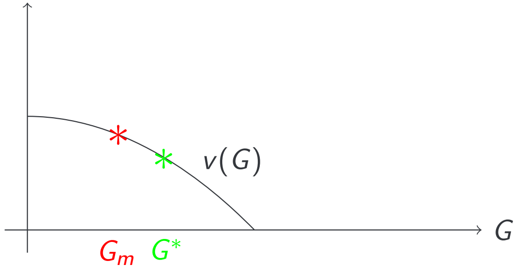
- $v(G_m) + G_m v'(G_m) = v(G^*) + (G^*/n) \cdot v'(G^*)$



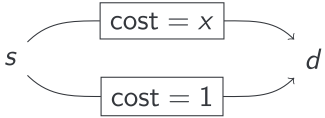
- At NE, we have $v(G^*) + (G^*/n) \cdot v'(G^*) = 0$, which can be interpreted as follows
- A user with g_i items (goat or connections) may consider getting h more items
  - the maintenance cost increases by $ch/c = c$
  - each items loses value by $(v(G+h) - v(G))/h$ (i.e., $v'(G)$ for small h), so the total value loss is $g_i v'(G)$
  - At NE all users have G^*/n
- The global viewpoint considers the loss of all users $G_m \cdot v'(G_m)$ (no $1/n$ term)

Selfish routing

A scenario with high Price of Anarchy

Selfish routing

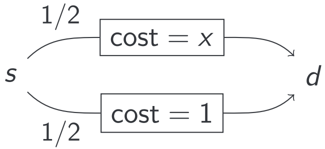
- Pigou (1920): 2 possible paths from s to d
- For the top path cost $\propto$ congestion $x \in [0, 1]$



- Say 1 unit of traffic goes from s to d
- Top path is a dominant strategy
- All traffic incurs a cost of 1

Selfish routing

- Can we do better? Split traffic 1/2 and 1/2



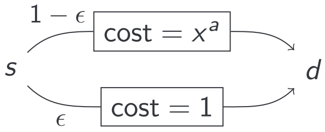
Unit cost is 1/2 on upper edge, 1 on lower

Average cost is 3/4. Overall optimum, but players have incentive to deviate

Price of Anarchy = 4/3

Selfish routing

- Even worse with non-linear cost ($a > 1$)



Top path is again dominant, total cost $= 1$

If a fraction ϵ goes to the bottom path, cost$= (1-\epsilon)^a + \epsilon$

For $a \to \infty$, optimal cost $\to 0$; Unbounded PoA

Sorry, gotta bounce!
Send me questions via e-mail