Lecture 02
Lotteries

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Game Theory 2023/24

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Recap of previous lecture

- In game theory, a game is a problem involving multiple agents (**players**) and multiple **objectives**
- **Decision problems**: actions, outcomes, preferences
- Players choose between different possible **actions**
- The **outcome** of a game is determined by the actions of all players
- Players have **preferences** about the outcome ($x \succcurlyeq x'$)
- We can use **utility functions** to obtain a quantitative representation of preferences

Recap of previous lecture

- Recall: Rational preferences satisfy
  - **Completeness:** for all $a, b \in A$ either $a \succcurlyeq b$ or $b \succcurlyeq a$
  - **Transitivity:** for all $a, b, c \in A$, $a \succcurlyeq b \wedge b \succcurlyeq c \Rightarrow a \succcurlyeq c$

“Homework”

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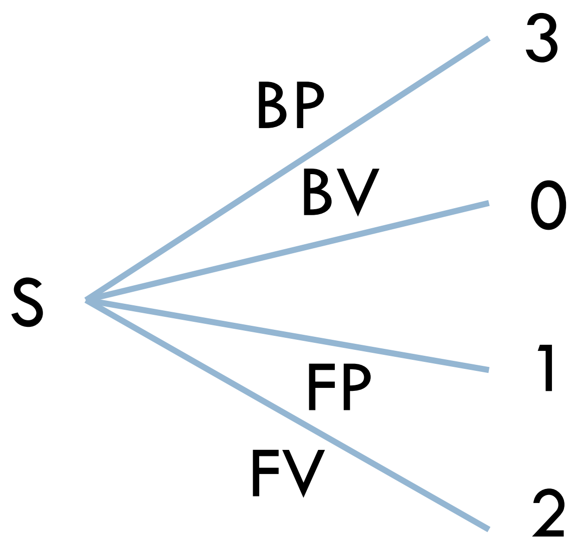
- A student goes to the university cafeteria for lunch and needs to choose between:
  - Beef (B) or fish (F) for the main dish
  - Polenta (P) or vegetables (V) for the side dish
- Her preferences are:
  - Beef and polenta $\succcurlyeq$ fish and vegetables
  - Fish and vegetables $\succcurlyeq$ fish and polenta
  - Fish and polenta $\succcurlyeq$ beef and vegetables
- Assign payoffs according to the preferences and draw the decision tree

Possible solution

- Here preferences are about combination of dishes, so each possible combination is a possible choice
- Assign u(BV)=0, u(FP)=1, u(FV)=2, u(BP)=3
- Draw a single-layered tree



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LOTTERIES

Modeling randomness

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- In decision problems, players are assumed to be fully aware of the consequences of their actions
- For 1-player problems actions = outcomes
- What about:
  - Incomplete information?
  - Random events?
- Can we still model problems that are affected by randomness as decision problems?

Random outcomes

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- Assume payoffs are affected by random outcomes
  - At the cafeteria, the food quality may vary
  - On one day, the fish might be rotten
  - How can we tell if beef is preferable?
- Rational players and randomness do not mix well together
- To make rational decisions involving random outcomes, we need to incorporate them into the utility function
  - How can we do that? By using the outcomes' **probability distribution**

Random outcomes

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- Example
  - Beef gives u(B)=6 with 50% probability, u(B)=4 otherwise
  - Fish gives u(F)=10 with 90% probability, u(F)=-10 with 10% probability
- We can model the choice between B and F as a choice between two **lotteries**:
  - (B): utility is 6 or 4 with probabilities 0.5 and 0.5
  - (F): utility is 10 or -10 with probabilities 0.9 and 0.1

Lotteries

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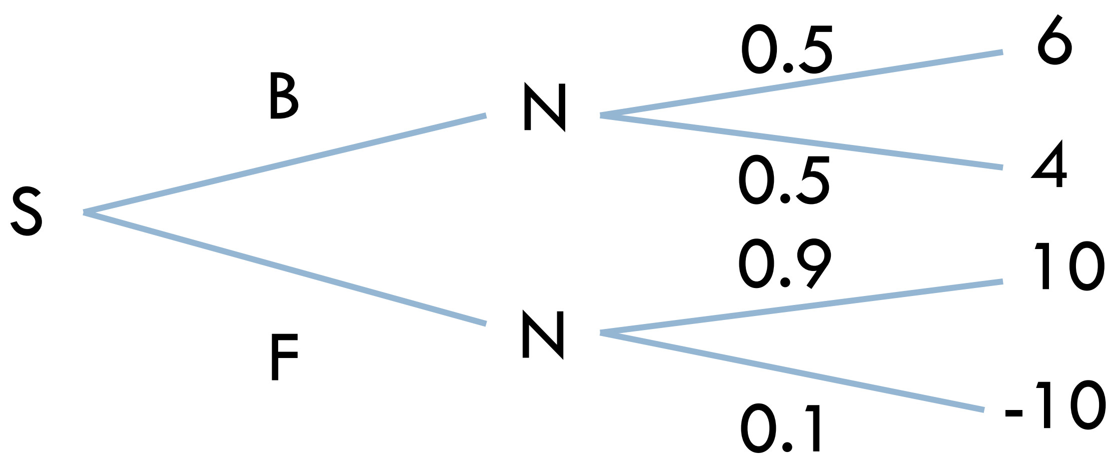
- **Definition:** A **lottery** over outcomes $X = \{x_1, \dots, x_n\}$ is a probability distribution p over X
  - $p(x_k) \geq 0, \;\; k = 1, \dots, n$
  - $\sum_{k=1}^{n} p(x_k) = 1$
- If actions are involved, p is conditional on the action
  - For $a \in A$, we consider $p(x_k|a)$ with the above properties
- A certain outcome can also be seen as a **degenerate lottery:** $p(x_k|a) = 1$ for some k and 0 for all $k' \neq k$

Nature

- In game theory jargon, random events are the consequences of the choices made by another player, called "Nature" (N)
  - Nature chooses between outcomes $x_1, \ldots, x_n$ according to a lottery p
  - This can be represented in the decision tree as follows



Continuous lotteries

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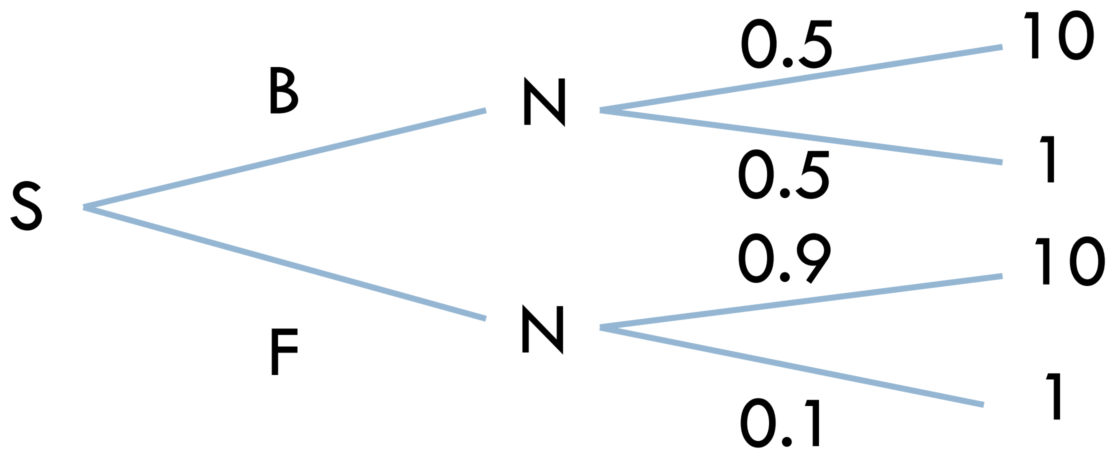
- Lotteries can also describe probabilities over a continuous space of events
- Probability of each specific outcome is zero
- Probability mass distribution → Probability density
- Still possible to represent it using the decision framework, however it become a bit scuffed
  - Nature's choice cannot be represented in the decision tree

Evaluating random outcomes

- Simplified problem: assume food can only be "tasty" or "not tasty" with u(tasty)=10 and u(not tasty)=1



- In this case, the obvious choice for rational players is fish, since they have higher chances to get 10

Evaluating random outcomes

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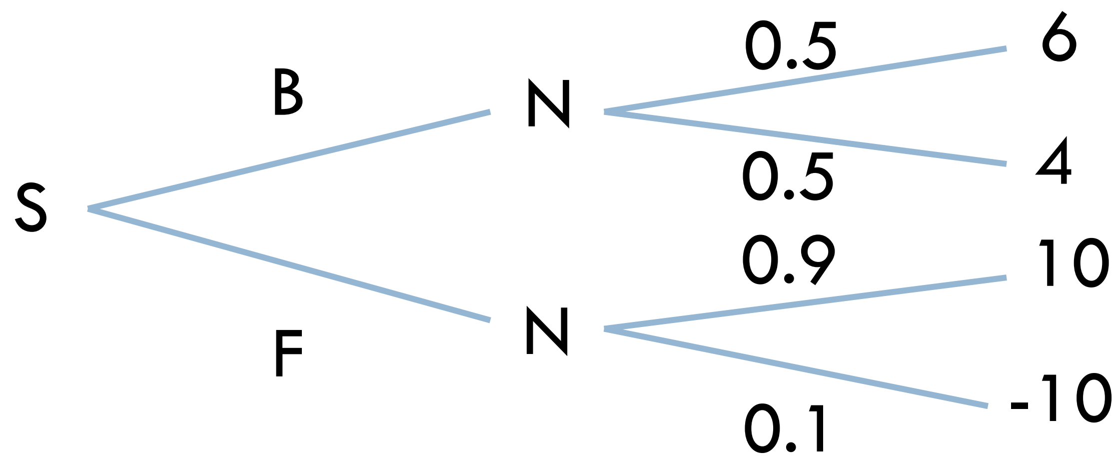
- However, with different numbers the result is not clear
- What is better? B or F?



- B: fifty-fifty chances of getting either 6 or 4
- F: a high probability of getting 10 with a small chance of getting -10

Expected utility

- Usual way of comparing random outcomes: taking the expectation
  - Also works for "degenerate" lotteries (1 outcome with 100% probability)
  - "Expected utility theory" by von Neumann and Morgenstern
  - **Intuition:** if you repeat the same choice for N trials, for N→ ∞ average utility = expectation
- Expected payoff for lottery p
  - $\mathbb{E}_{x\sim p}[u(x)] = \sum_{k=1}^{n} p(x_k) \cdot u(x_k)$

N.B: Nature = Continuous variable = Random = Continuous Lotterie → you have to find the average expectation

Expected utility

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- Von Neumann – Morgenstern (VNM) framework to define preferences among lotteries
- We write $p \succcurlyeq q$ to say "lottery p is preferred to q"
- Under VNM framework, preferences must satisfy:
  - Rationality (completeness and transitivity)
  - Continuity axiom
  - Independence axiom

Continuity axiom

- For lotteries p, q, r over action space A the following sets must be **closed**:
  - $\{a \in [0,1]: ap + (1-a)q \succcurlyeq r\}$
  - $\{a \in [0,1]: r \succcurlyeq ap + (1-a)q\}$
- This means that arbitrarily small variations in the gamble does not change preferred lotteries
  - If I prefer fish which is 100% not rotten to beef, I will still prefer fish if it has an arbitrarily small probability $\varepsilon > 0$ of being rotten

Independence axiom

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- For lotteries $p, q, r, \forall a \in [0,1]$
  - $p \succcurlyeq q \Rightarrow (1-a)p + ar \succcurlyeq (1-a)q + ar$
- This means that if we mix the same amount of another lottery into two lotteries, the preference remains unchanged
  - If I like betting on soccer more than betting on horse races, then I prefer the lottery "if heads bet on soccer, if tails play roulette" to "if heads bet on horse races, if tails play roulette"

VNM utility theorem

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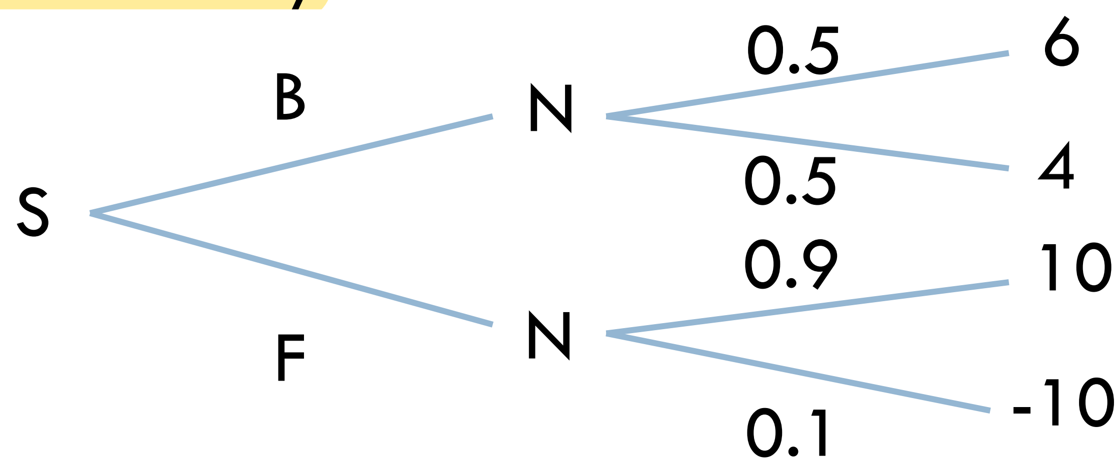
- **Theorem:** If $\succcurlyeq$ satisfies the rationality, continuity and independence axioms, it can be mapped to u such that

$$p \succcurlyeq q \Rightarrow \mathbb{E}_{x\sim p}[u(x)] \geq \mathbb{E}_{x\sim q}[u(x)]$$

- Remark: If u is a suitable utility function to describe the preference $\succcurlyeq$, any affine (linear) transformation of u is also suitable

Expected utility

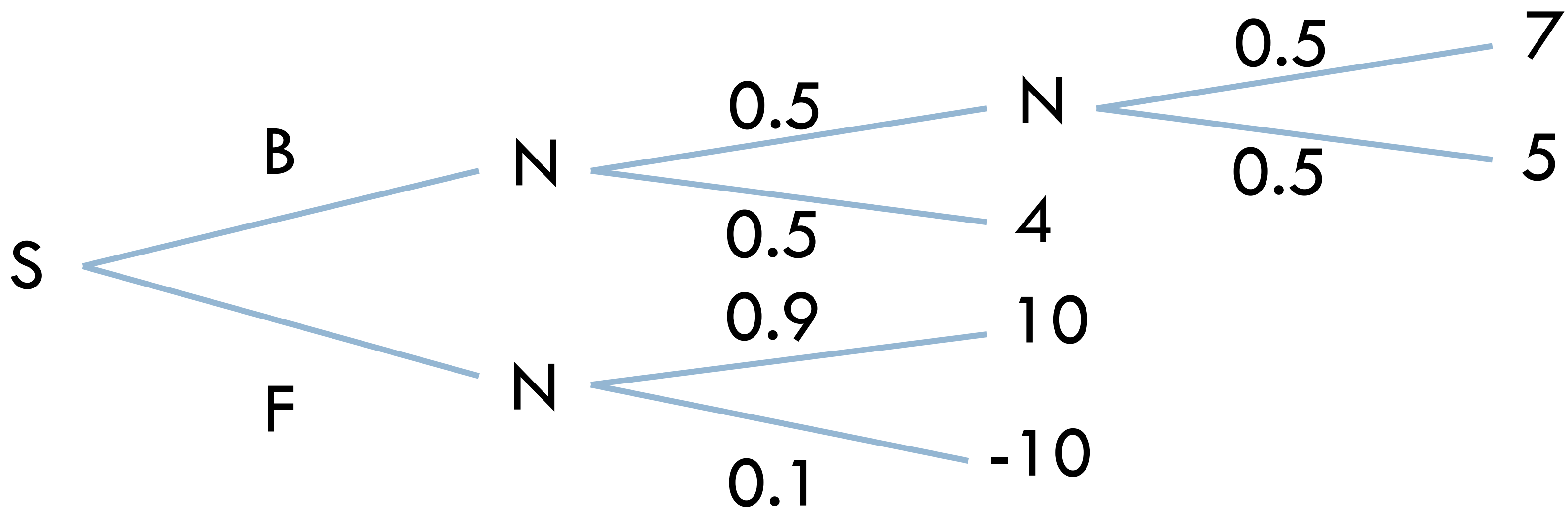
- Now we can compare the fish and beef lotteries using expected utility



- $\mathbb{E}[u(\text{B})]=0.5\times6+0.5\times4=5$
- $\mathbb{E}[u(\text{F})]=0.9\times10+0.1\times(-10)=9-1=8$
- So, fish is rationally preferable to beef

Compound lotteries

- How to account for subsequent choices of Nature?



- Just take the compound expectation
- Remember expectation is linear
- $\mathbb{E}[u(\mathrm{B})]=0.5\times0.5\times7+0.5\times0.5\times5+0.5\times4=5$

Continuous lotteries

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- Same as discrete case (only the diagram is harder)
- **Example:** We are digging a dwell and need to decide how deep should it be (d=dwell's depth). Digging has a cost of $d^2/$(2 meters) and the amount of extracted water is $W(d) \sim \mathcal{U}([0,\ 20d])$.
- Utility = extracted water – cost.
- $\mathbb{E}[u(d)] = \mathbb{E}\left[W(d) - \frac{d^2}{2\ \mathrm{m}}\right] = 10d - \frac{d^2}{2\ \mathrm{m}}$
- $\mathbb{E}[u(5\ \mathrm{m})] = 10 \times 5\ \mathrm{m}\ - \frac{100\ \mathrm{m}^2}{2\ \mathrm{m}} = 50 - 50 = 0$
- Best choice: $d = 10\ \mathrm{m}$ with $\mathbb{E}[u(10\ \mathrm{m})] = 50$

Ordinal vs absolute

- When randomness is not involved, the payoff values don't matter as long as they reflect preferences
  - If we have A$\succcurlyeq$B, then we can set u(A)=1 and u(B)=0 or u(A)=100 and u(B)=-π
- However, changing payoffs in lotteries may affect the preferred lottery
  - In the cafeteria example, suppose we assign -100 to the rotten fish instead of -10

Risk attitude

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- Consider the following lotteries, where the possible outcomes are to with 0, 1, or 20 euros
  - $p_A = (0, 1, 0)$, i.e., we receive 1 euro 100% guaranteed
  - $p_B = (0.95, 0, 0.05)$, i.e., with 95% probability we get nothing but with 5% probability we get 20 euros
- u(1 euro) or 0.95× u(0 euros)+0.05× u(20 euros)?
- Depends on how much a player values gaining X euros

Risk attitude

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- For a **risk-neutral** player, lotteries p_A and p_B are interchangeable
- For a **risk-averse** player $p_A \succcurlyeq p_B$ (prefers 1 euro guaranteed)
- For a **risk-loving** player $p_B \succcurlyeq p_A$ (prefers a 5% chance to get 20 euros)

Risk attitude

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- **Remark:** Monotonic utility functions such as $u(x) = x$, $u(x) = x^2$, and $u(x) = \log x$ do not affect preferences but they do affect risk attitude
  - Linear utility → risk-neutral
  - Concave utility → risk-averse
  - Convex utility → risk-loving

Risk attitude

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- **Tl;dr:** Be careful! Expected utility theory does not say that getting 1 euro is the same as gambling 2 euros with a 50/50 probability
  - That becomes true if we use a linear utility function $u(x) = ax + b,\ a > 0$
  - You may use other utility function to model different risk attitudes

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BACKWARD INDUCTION

Decisions over time

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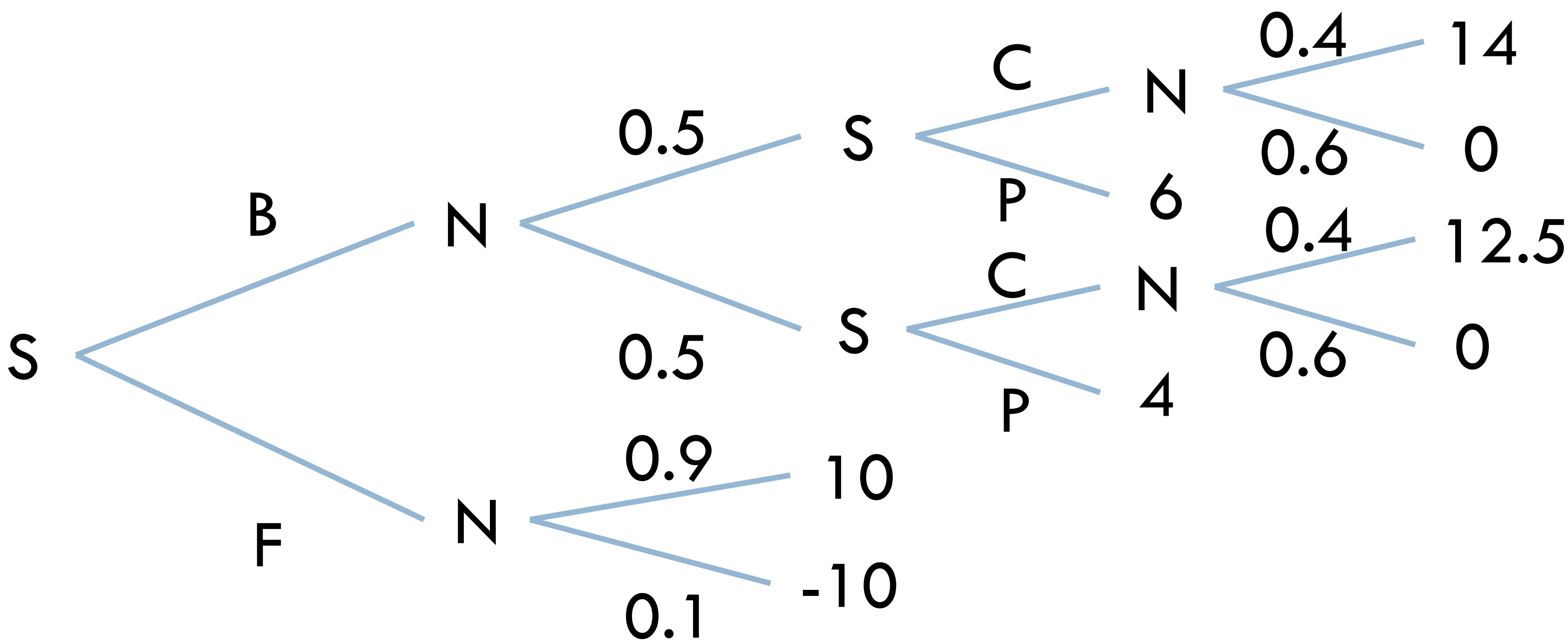
- **Example:** Consider once again the cafeteria example with the fish and beef lotteries as before
- This time, the student is also given the option to add the chef's sauce (C) to the beef or leave it plain (P)
- However, she does not know if she will like the sauce
- Assume the sauce is good with probability 0.4
- Good sauce increases yields u=14 for tasty beef and u=12.5 for bland beef
- Bad sauce always yields u=0

Decision over time

- How do we "solve" this tree?



Backward induction

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- Begin with nodes (not leaves) at the last level of the tree
- If it is Nature's move, replace the node with a leaf containing the average payoff
- If it is the player's move, replace the node with the payoff of the best choice (i.e., the payoff yielding highest utility)
- Repeat the process for the "pruned" tree

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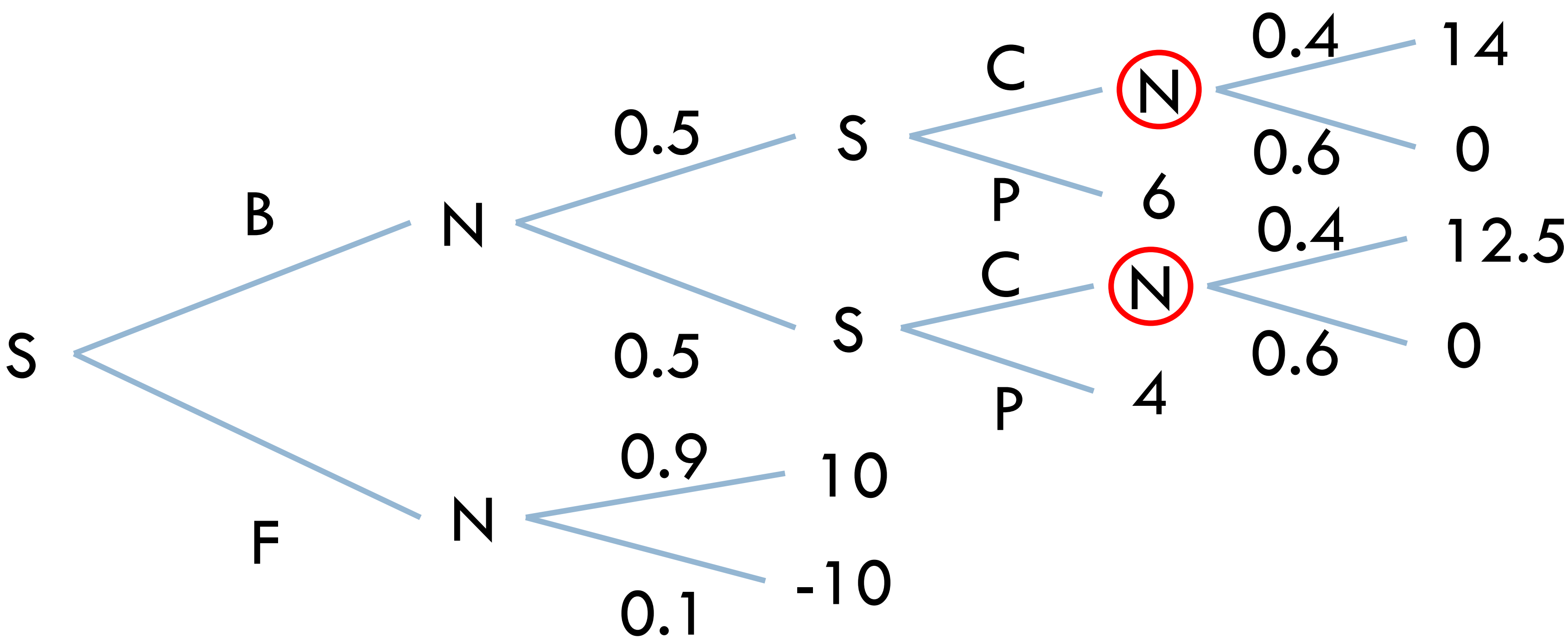
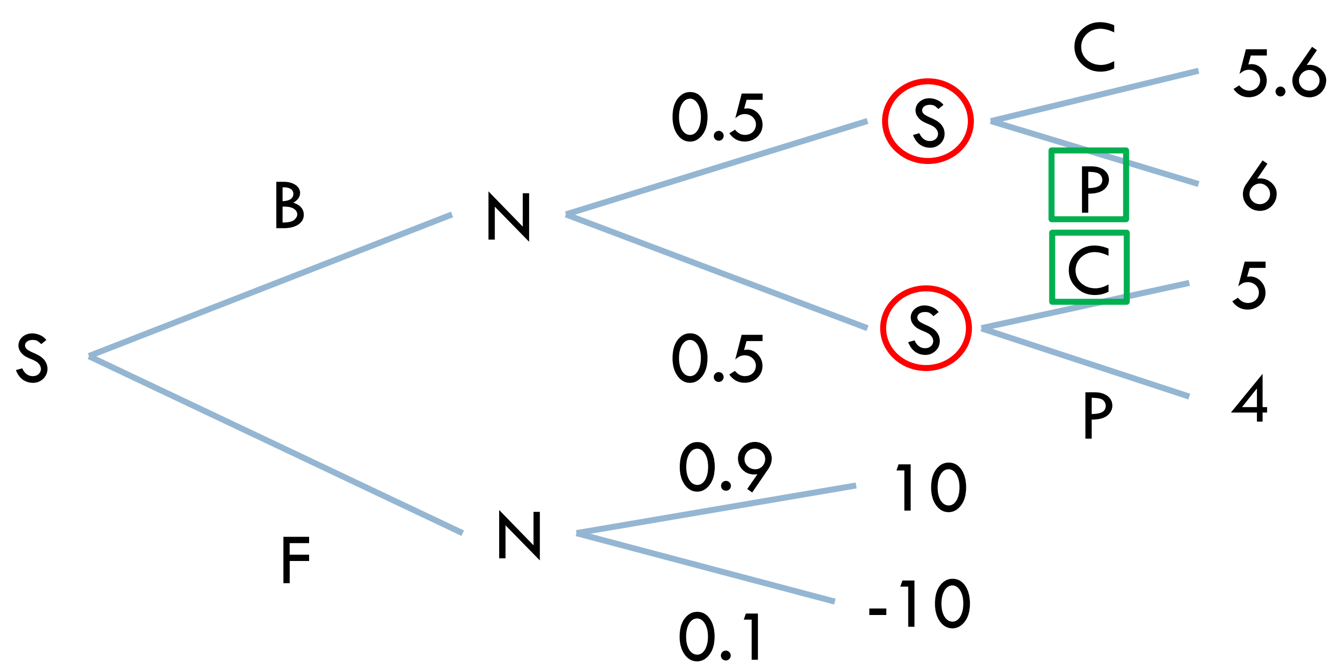
- Nature's move: replace with expected utility



Decision over time

- Player's move: choose best option



Decision over time

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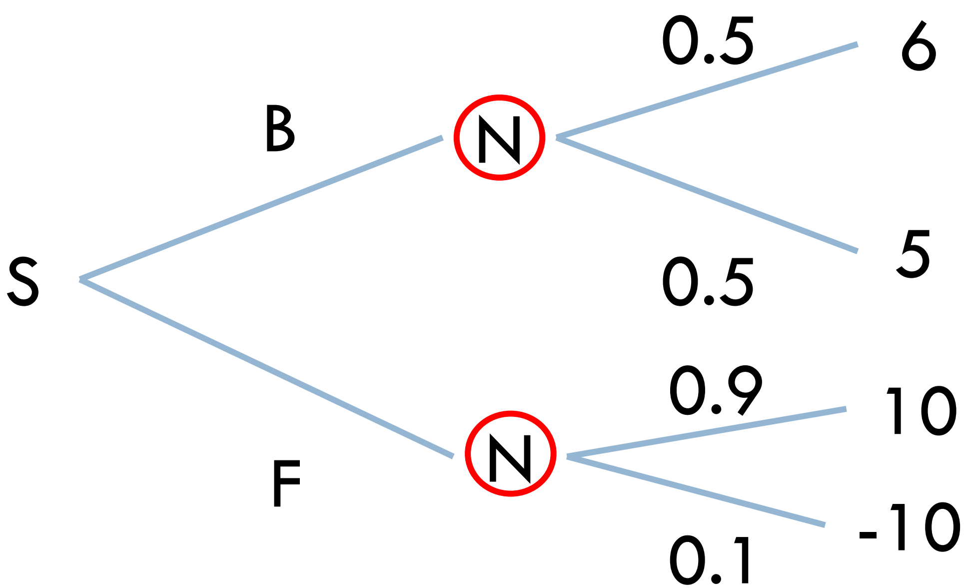
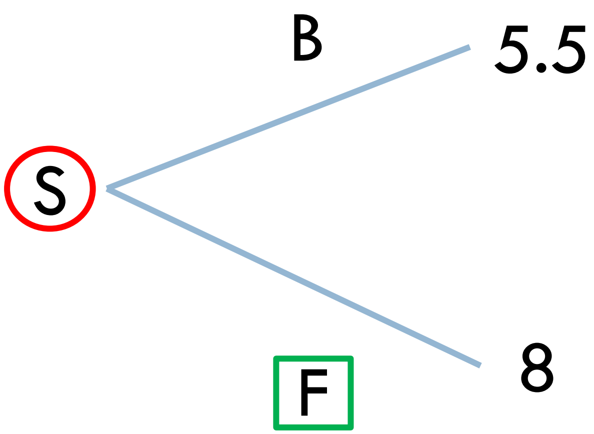
- Nature's move: replace with expected utility



Decision over time

- Player's move: choose best option
- In conclusion, the player's best choice is to still take the fish



Decision over time

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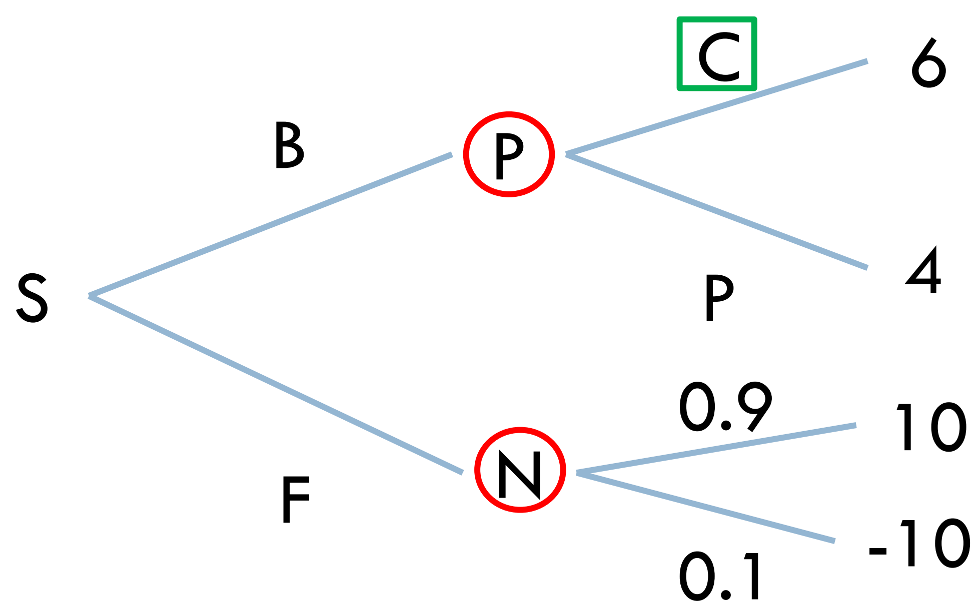
- **Remark:** It is possible to have Nature's and player's moves at the same tree level



Discount for future payoffs

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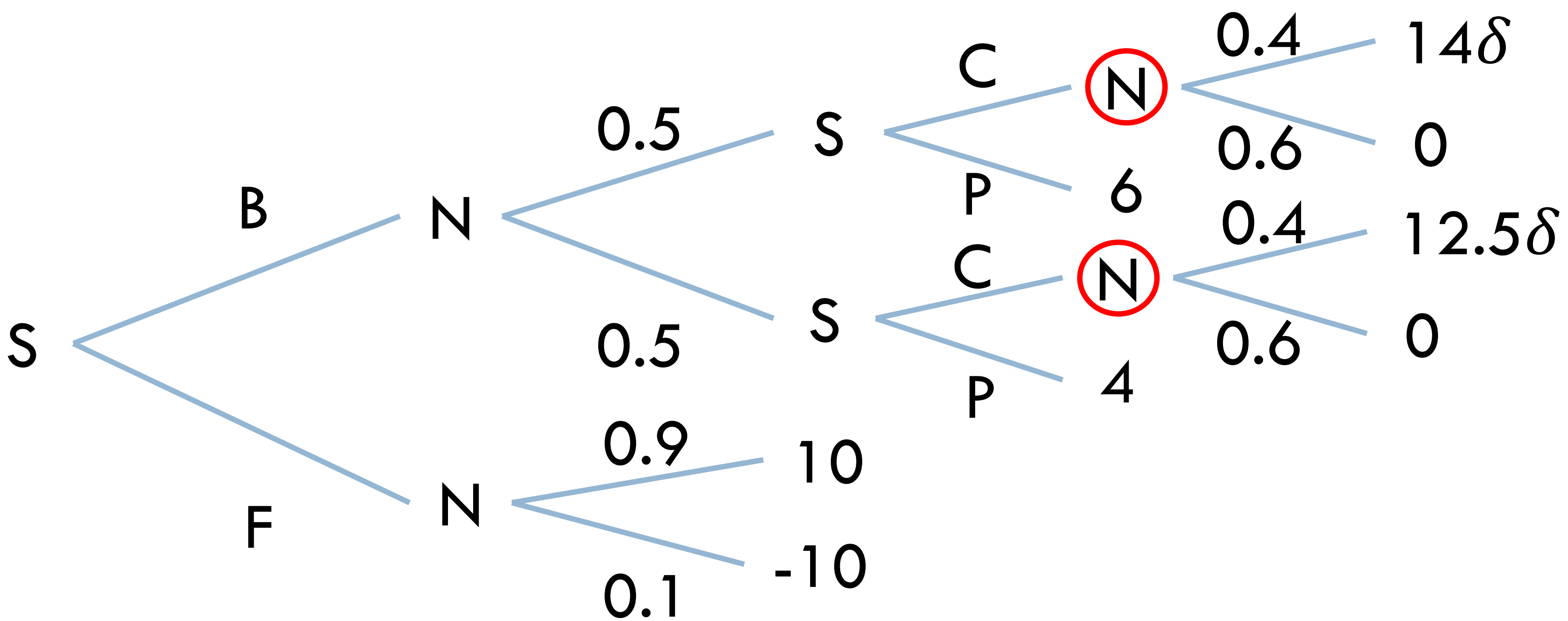
- If the player's decisions are made far apart, we may include a discount factor $0 < \delta < 1$
  - Likely, that's not the case for adding the chef's sauce



The value of information

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- Expected utility implies that a rational player chooses its actions so as to make the right choice **on average**
- Suppose the player has the possibility so see Nature's choice in advance: how much is this information worth?

The value of information

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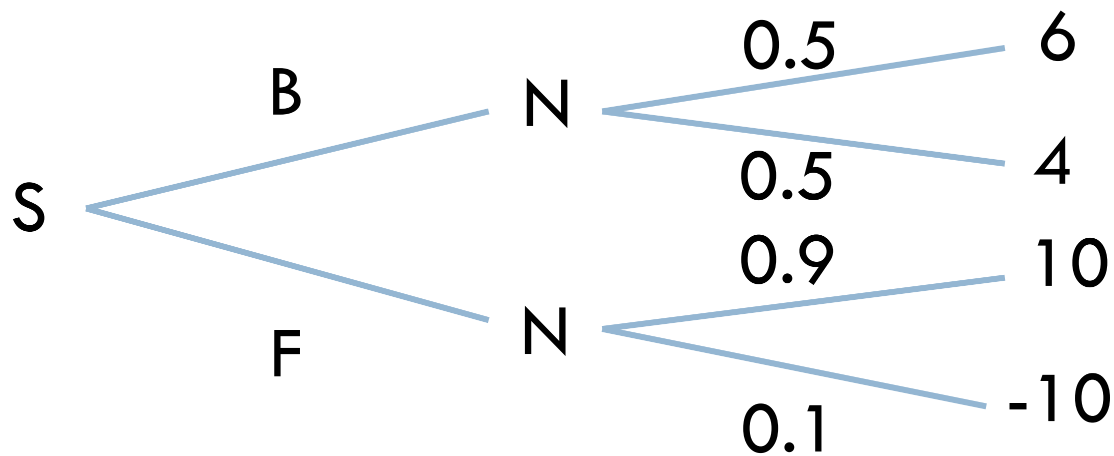

- Assume a friend of S knows how good is the cafeteria's food today and he is willing to tell her (under reasonable compensation)

The value of information

- If S is notified in advance, her best option will change depending on the information received
- The possible outcomes are the same but the moves' order changes
- This situation can be modeled by making Nature move first

The value of information

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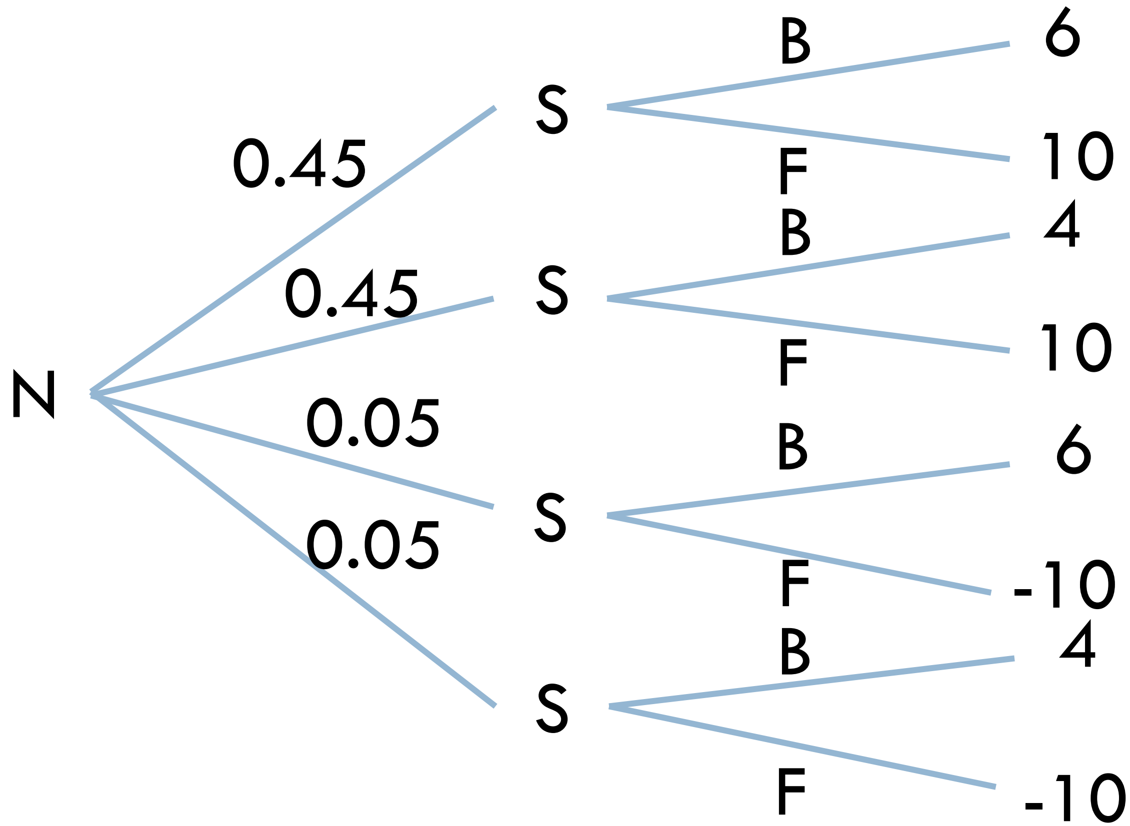

- S is always able to make the best choice, no gambling
- Expected utility: $0.9\times10+ 0.05\times6+ 0.05\times4=9.25$

The value of information

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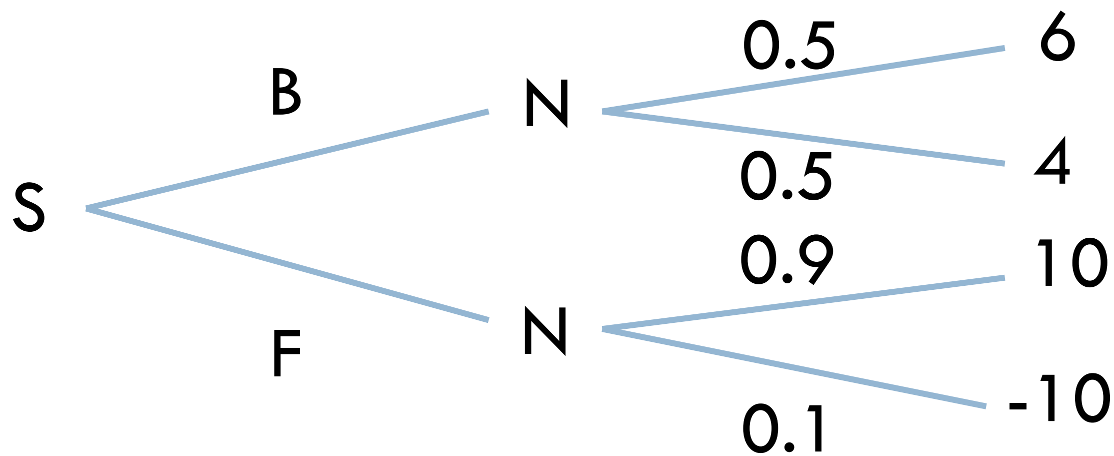

- Expected utility: 8 (choose fish with its expected payoff)
- Knowing Nature's choice is worth 1.25

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SELF-ASSESSMENT

Self-assessment

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- When is it possible to model preferences between lotteries using average payoffs?
- Which utility function can we use to model a risk-averse player? $u(x) = x^2$ or $u(x) = \log x$?
- How can we solve a decision problem involving sequential choices made by both a player and Nature?

“Homework”

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- Solve the decision problem of a student P who needs to choose whether to do a project or not for this course
- Same rules as this course:
  - 0-28 points in the written test
  - No project: 3 points by default; Project: 0-5 points
- The project must be selected before the written test
- Assign the probabilities for written test’s score and project’s score according to your own estimation