Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

Lecture 19
Bayesian Nash equilibria

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Previously on game theory

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- **Bayesian games**: games of *incomplete information*
  - Utility function of a player is determined by their type
  - Each player knows his/her own type, but only has an estimate of the opponents' type
  - This can be seen as Nature drawing a complete-information game among many possible games according to a distribution ϕ (which is common knowledge)
    - Players have only a partial observation of Nature's choice (their own type)

Today on game theory

Bayesian Nash equilibrium

Static Bayesian game

- Static Bayesian games need:
  - set of players $\mathcal{N} = \{1, \ldots, n\}$
  - action spaces $A_1, \ldots, A_n$ (pure strategy sets)
  - type spaces T_i (for $i = 1, \ldots, n$)
  - beliefs (on types) $\phi_1, \ldots, \phi_n$
  - type-dependent payoffs $u_i(a_1, \ldots, a_n, t_i)$
- $\mathbb{G}(\mathcal{N}; A_1, \ldots, A_n; T_1, \ldots, T_n; \phi_1, \ldots, \phi_n; u_1, \ldots, u_n)$
  - where $u_i = u_i(a_1, \ldots, a_n, t_i)$
- A **pure strategy** for i can be seen as a map $s_i : T_i \to A_i$, i.e., it tells what i plays as his/her type is known
- A **mixed strategy** for i is a probability distribution over pure strategies

Strategies of Bayesian games

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- Type-contingent definition of pure/mixed strategies $\rightarrow$ similar to dynamic games
  - We can think of a general strategy as being defined before the type of i is even set
  - Player i decides a strategy $s_i : T_i \rightarrow A_i$
    - Then, if his/her type is $t_i \in T_i$, he/she will play $s_i(t_i)$
  - This is useful, since it allows **other players** to create beliefs over the strategy of a player who can be of different types

Bayesian Nash equilibrium

- A **Bayesian Nash equilibrium** is a Nash equilibrium in Bayesian games
- In $\mathbb{G}(\mathcal{N}; A_1, \ldots, A_n; T_1, \ldots, T_n; \phi_1, \ldots, \phi_n; u_1, \ldots, u_n)$, joint strategy $s^* = (s_1^*, \ldots, s_n^*)$ is a **Bayesian Nash equilibrium** if, for each player i and type $t_i \in T_i$, s_i^* maximizes the expected payoff against s_{-i}^*:

$$s_i^* = \arg\max_{s_i \in S_i} \sum_{t_{-i}} u_i(s_i, s_{-i}^*(t_{-i}), t_i)\phi_i(t_{-i})$$

Bayesian Nash equilibrium

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- This is the same as:

$$\mathbb{E}[u_i(s_i^*, s_{-i}^*(t_{-i}), t_i)|t_{-i}] \geq \mathbb{E}[u_i(s_i, s_{-i}^*(t_{-i}), t_i)|t_{-i}]$$

  for every $s_i \in S_i$
- In other words, i does not want to change strategy (at least with his/her available information)
  - player i's strategy $=$ a choice of action for each type
  - what i does not know, he/she just estimates!
- This definition can be generalized according to the type space (if continuous $\rightarrow$ integrals)

Examples of Bayesian NE

Chicken game

- The "chicken game" is a well-known anti-coordination game: two youngsters are driving on the road in opposite directions at high speed; both can chicken out (C) (i.e., steer) or keep driving straights (D)
  - chickens get nothing but shame ($u = 0$)
  - those who keep driving gain "respect" ($u = 8$)
  - if both drive, they split the respect, plus an accident happens; they receive $u = 4 - P$, where P is a punishment that depends on their parents' type
  - Parents can be the hard type (H) ($P = 16$) or the lenient type (L) ($P = 4$) with probability 0.5
  - the youngsters know their parents' type

Chicken game

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- Extensive form: trivial and left to the students as an exercise

- Normal form:

Player 2

Player 1	CC	CD	DC	DD
CC	0, 0	0, 4	0, 4	0, **8**
CD	4, 0	-1, -1	-1, **2**	-6, 1
DC	4, 0	**2**, -1	**2**, **2**	**1**, 1
DD	**8**, 0	1, -6	1, **1**	-6, -6

- Only BNE is (DC, DC)
  - Different values of P can lead to other BNE

Committee voting

- Many decisions are made by committee through majority voting
- Consider a jury with just two members deciding whether to acquit (A) or convict (C) a defendant
  - Each jury member casts a sealed vote
  - The defendant is convicted if both members vote C
- It is uncertain whether the defendant is guilty (G) or innocent (I): the prior probability is $q > 1/2$, which is common knowledge

Committee voting

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- Jury members wish to make the right decision, so their payoff is 1 if G$\rightarrow$C and I$\rightarrow$A, 0 otherwise
- If the only information is probability q, then this is a lottery:

Member 2

Member 1	A	C
A	1-q, 1-q	1-q, 1-q
C	1-q, 1-q	q, q

- and if $q > 1/2$, then (C, C) is a NE ((A, A) is always a NE)

- Assume each jury member observes the evidence and independently gets a private **signal** (his/her idea about the case) $t_i \in t_G, t_I$
- It is more likely (but not certain) ti receive signal "t_x" if the defendant status is x
- Let $p =$ Probability of receiving signal t_x given x
- $\Pr[t_G|G] = \Pr[t_I|I] = p > 1/2$ for both jurors $i = 1, 2$
- clearly, $\Pr[t_G|I] = \Pr[t_I|G] = 1 - p < 1/2$
- **Note**: These types are not about the players themselves, but about events happening in the world; still, they affect their payoffs

Committee voting: types

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- Since each player has 2 types and 2 actions
  $\rightarrow$ 4 possible strategies: AA, AC, CA, CC
  - strategy XY means t_G $\rightarrow$X and t_I $\rightarrow$Y
  - It is a coordination game, since both players have the same objective (make the right judgment)
- For the time being, consider a single-person problem where only one juror decides
  - Without the signal, he/she plays C
  - How would the signal affect their choice?

Committee voting: types

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- The signals affect the posterior probabilities of the jury members
- These can be computed using Bayes theorem

$$\Pr[G|t_G] = \frac{\Pr[G, t_G]}{\Pr[t_G]} = \frac{qp}{qp + (1-q)(1-p)} > q$$

- since $p > 1/2$, then $qp + (1-q)(1-p) < qp + (1-q)p = p$
- and, on the other hand

$$\Pr[G|t_I] = \frac{\Pr[G, t_I]}{\Pr[t_I]} = \frac{q(1-p)}{q(1-p) + (1-q)p} < q$$

$\rightarrow$ if t_G: even more confident that the defendant is guilty; if t_I: becomes doubtful

Committee voting: types

- Actually, if t_I is received, the final assessment depends on the relative values of p and q

$$\Pr[G|t_I] = \frac{q(1-p)}{q(1-p)+(1-q)p}$$

- This may be less than $1/2$, meaning that the jury member prefers to acquit rather than convict
- This happens if $p > q$
  - The reason is that the information content of the signal must be higher than the prior information
  - Instead, if $p = 1/2$, the signal given no information and posterior=prior

Committee voting: 2-person decision

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- Now, we would like to check whether $p > q$ implies that (CA, CA) is a BNE in the original problem (2-person decision)
  - That would correspond to "following the signal"
- First, draw the probability of each type pair

Member 2

Member 1	t_G	t_I
t_G	$qp^2 + (1-q)(1-p)^2$	$p(1-p)$
t_I	$p(1-p)$	$q(1-p)^2 + (1-q)p^2$

- **Note**: This is not a payoff matrix, it is just a table displaying the values of probabilities $\Pr[t_1 = t_x, t_2 = t_y]$

Committee voting: 2-person decision

- To check whether (CA,CA) is BNE we need to ask "Is CA a best response to CA?"
  - Assume member 2 plays CA, and check if CA is best for member 1
- We do not want to write down the whole table, let us try to see if we can draw conclusions just by looking at posteriors
- With the rules of the jury, a player's choice is decisive ("**pivotal**") only if the other juror chooses C
- If 2 chooses A, that is the result regardless of the 1's choice
  - If 1 believes that 2 is playing CA, *any* strategy of 1 is always a best response if the 2's type is t_I
  - In other words, if 1 thinks that 2 received signal t_I, then everything 1 does is a best response
  - So we need to check only the case $t_2 = t_G$

Committee voting: 2-person decision

- Again, check the posterior to see the signal effect

$$\Pr[G|t_1 = t_G, t_2 = t_G] = \frac{qp^2}{qp^2 + (1-q)(1-p)^2} > q$$

- **Meaning**: if both $t_1 = t_G$ and $t_2 = t_G$: conviction is even more certain
- as before, $p > 1/2$ implies $qp^2 + (1-q)(1-p)^2 < qp^2 + (1-q)p^2 = p^2$

$$\Pr[G|t_1 = t_I, t_2 = t_G] = \frac{qp(1-p)}{p(1-p)} = q$$

- **Meaning**: if they receive opposite signals, the received signal t_I is useless $\rightarrow$ posterior=prior

Committee voting: 2-person decision

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- Recap:
  - If player 2 is of type $t_2 = t_I$, player 1 believes that 2's move is A $\rightarrow$ 1's move does not matter
  - If player 2 is of type $t_2 = t_G$ $\rightarrow$ player 1's posterior is either q or higher
- Therefore, CA is not a best response to CA
- Actually you can prove that (CC, CC) is a BNE

Sorry, gotta bounce!
Send me questions via e-mail