University of Padova – prof. Leonardo Badia
Game Theory exercises

Exercise 1 Agatha (A) and Bruno (B) are two siblings. They own a model train set (T) and they usually play together with it. Their respective utilities when doing so is $u_A(\mathtt{T},\mathtt{T}) = u_B(\mathtt{T},\mathtt{T}) = 3$. Their mom (M) buys them a new toy (N), which can be either a dollhouse (D) or a set of small soldier figurines (S). The two kids must independently decide whether to keep playing with T or the new toy N. All the three players (A, B, M) involved decide their action simultaneously and unbeknownst to each other. The action set of the kids includes just T or N; the mom instead decides what new toy to buy. If the kids end up in not playing together, *all* the players (also M) get utility equal to 0. If they play together with the new toy, they get a positive utility that for A is equal to 6 and 1 for D and S, respectively; for B, the values are instead 1 and 4, respectively. The utility of M is the lump sum of the utility of the two kids.

1. Write down the normal form of this game (likely, you will need a 3D matrix: you can write two matrices instead, one per each move of player M).
2. Find all Nash equilibria of this game in pure strategies.
3. Find all additional Nash equilibria of this game in mixed strategies.

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Exercise 2 Carl (C) and Diana (D) are two university students that have found that the department library is unoccupied overnight. It is a really good place to study and has a very fast Internet connection. So, they go there every night, but they do not coordinate or plan any action together. Upon their arrival every night, they independently decide whether: (S) study or (M) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for C and 18 for D. However, if they both choose M, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(\mathtt{S},\mathtt{M}) = u_D(\mathtt{M},\mathtt{S}) = 0$ (C is written as the first player). Call $\mathbb{G}$ this game, and consider it in a repeated version $\mathbb{G}(T)$, where $\mathbb{G}$ is played every night for T nights. Individual payoffs are cumulated with discount factor δ. Finally, consider an *extended* game where a punishment strategy P is also available to both players. When either player chooses P, payoffs are -10 for *both* players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game $\mathbb{G}'$. Note: despite P being weakly dominated, (P,P) is a NE for $\mathbb{G}'$.

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$.
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a "Grim Trigger" strategy where each player ends up in always choosing S?
3. If you see an SPE of $\mathbb{G}'(2)$ where players may play S, state at which round do they play it, and what value of δ do you need to obtain it.

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Exercise 3 A gang of pirates has n ranks from 1 (the ship's boy) to n (the captain). After a raid, they share a treasure. Pirate with rank $k+1$ keeps an eye on pirate k to see whether he gets a bigger share than he should. The game starts when pirate $k=1$ (the ship's boy) realizes that the treasure contains an extremely valuable pearl that has fallen far from the stash: he considers whether to take the pearl for himself (P) hiding it in its pocket or do nothing (N). Doing nothing ends the game with the pearl being unnoticed and unassigned. However, if pirate k takes the pearl, pirate $k+1$ will notice it; now, pirate $k+1$ may consider to kill him and keep the pearl for himself (P), or do nothing (N). If pirate $k+1$ does nothing, pirate k is left alive with the pearl – a very good outcome. If pirate $k+1$ kills pirate k and takes the pearl instead, this is spotted by pirate $k+2$ that now faces the same choice: whether to kill pirate $k+1$ and keep the pearl for himself (P), or to do nothing (N). This means that k is replaced with $k+1$ and the game continues up to the captain. For every pirate, the top preference is to stay alive and have the pearl; after that, they all prefer being alive without the pearl than to be killed.

1. Consider $n=5$. Choose appropriate utility values for the outcomes and draw the extensive form of the game.

2. Consider $n=5$. Solve the game by finding its subgame-perfect outcome.

3. Consider $n=8$. Does the subgame-perfect outcome change, and why?

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Exercise 4 The mayor of a big city is to be selected among four candidates: (A)Amanda Amour; (B)Bruno Bravery; (C)Claire Constitution; (D)Dave Democracy. Using symbol $\succ$ to denote "preferred to", polls indicate that:

- (A) has 42% of supporters. Also, for them (B)$\succ$(C)$\succ$(D).
- (B) has 11% of supporters. Also, for them (A)$\succ$(C)$\succ$(D).
- (C) has 27% of supporters. Also, for them (B)$\succ$(D)$\succ$(A).
- (D) has 20% of supporters. Also, for them (C)$\succ$(B)$\succ$(A).

1. The election is being held as a two-round run-off (i.e., with a ballot). What is the outcome under *sincere voting*? Denote the winner as W.
2. Assume that the supporters of (D) can identify this outcome and plan a strategy. What is the best *strategic voting* that they can enact?
3. Discuss the identity of the winner W' under strategic vote of (D)'s supporters. What kind of choice is W'? Can the supporters of W prevent this outcome by counteracting strategic vote of (D)'s supporters, with a strategic vote of their own?

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Exercise 5 Jane's bank account contains 10000 euros. Jane (J) can keep (K) the whole sum in the bank account, or buy a utility car (C) that costs 8000 euros. The bank B is also a player, who plays simultaneously to Jane, with options being either (L) to leave Jane's money untouched in the account, or (S) to buy some subprime bonds for Jane, investing every euro that Jane has left in the account (either 2000 or 10000). After one year, Jane's account is increased with some interest $\mathcal{G}$. If the bank played L, $\mathcal{G}$ is equal to 5% of the initial amount. In case the bank played S, $\mathcal{G}$ depends on whether the bank is a "good" bank or a "bad" bank. Jane estimates the probability of "good" as p and this is a common prior. However, the bank is also fully aware of its type. A good bank playing S gives Jane her money back, plus $\mathcal{G} = 15\%$ of yearly interest rate. A bad bank steals every euro that Jane put in the bank account ($\mathcal{G} = -100\%$). The final payoff are computed after 1 year as:
— **for Jane**: the grand total of the euros in the account (i.e., the initial value increased by $\mathcal{G}$); plus the value of the car, if she owns one: a one-year old car is worth 5000 euros.
— **for the bank**: if $\mathcal{G}$ is positive, then their payoff is $\mathcal{G}$; otherwise it is $-\mathcal{G}/5$.

1. Represent this Bayesian game with J and B in extensive form.

2. Represent this Bayesian game in normal form, with a type-agent representation of player B (i.e., its strategies are n-tuples of actions, one per type).

3. Find the Bayesian equilibria of the game if $p = 0.2$ and discuss whether they are (subgame) perfect.