Minimax in mixed strategies

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These notes are meant to be a complement to the lecture on minimax. They do not cover all theory from scratch, they simply contain clarifications and practical tips on how to compute minimax and maximin in mixed strategies.

1 Recap on maximin and minimax

We have defined maximin and minimax in pure strategies for player i (playing against $-i$) as

$$\text{maximin}_i = \overbrace{\max_{s_i \in S_i}}^{i \text{ plays first}} \overbrace{\min_{s_{-i} \in S_{-i}}}^{-i \text{ plays last}} u_i(s_i, s_{-i}) \tag{1}$$

$$\text{minimax}_i = \overbrace{\min_{s_{-i} \in S_{-i}}}^{-i \text{ plays first}} \overbrace{\max_{s_i \in S_i}}^{i \text{ plays last}} u_i(s_i, s_{-i}) \tag{2}$$

Intuitively, the maximin is the payoff that i obtains if i plays first trying to maximize its payoff and $-i$ plays last trying to minimize i's payoff. The minimax is the same, except i plays last. We also defined maximin^p and minimax^p as their natural extensions in mixed strategies:

$$\text{maximin}_i^p = \max_{p_i \in \Delta S_i} \min_{p_{-i} \in \Delta S_{-i}} u_i(p_i, p_{-i}) \tag{3}$$

$$\text{minimax}_i^p = \min_{p_{-i} \in \Delta S_{-i}} \max_{p_i \in \Delta S_i} u_i(p_i, p_{-i}). \tag{4}$$

To get an intuition of maximin$_i^p$ and minimax$_i^p$, you can imagine that one player starts by announcing its mixed strategy and the other responds accordingly. This means that, to evaluate maximin^p and minimax^p, we can focus exclusively on pure strategies for the player who plays last:

$$\text{maximin}_i^p = \max_{p_i \in \Delta S_i} \min_{s_{-i} \in S_{-i}} u_i(p_i, s_{-i}) \tag{5}$$

$$\text{minimax}_i^p = \min_{p_{-i} \in \Delta S_{-i}} \max_{s_i \in S_i} u_i(s_i, p_{-i}). \tag{6}$$

The reason is that the last player is playing a best response. And we know that if p_i is best response to p_{-i}, then any pure strategy $s_i \in \text{supp}(p_i)$ is a best response to p_{-i} (remember the characterization theorem of mixed NE!).

2 Minimax theorem

The minimax theorem states that in a zero-sum game between i and $-i$, we always have

$$\text{maximin}_i \leq \text{maximin}_i^p = \text{minimax}_i^p \leq \text{minimax}_i \tag{7}$$

$$\text{maximin}_{-i} \leq \text{maximin}_{-i}^p = \text{minimax}_{-i}^p \leq \text{minimax}_{-i} \tag{8}$$

The central equality states that in a zero-sum game maximin^p and minimax^p are the same, meaning that they can be used interchangeably. Moreover, the value of minimax$_i^p$ (and thus, also of maximin$_i^p$) is the payoff of i at the mixed Nash equilibria of the game.

Remark. If maximin$_i$=minimax$_i$ and maximin$_{-i}$=minimax$_{-i}$ in pure strategies, then pure-strategy minimax and minimax^p have the same value. In that event, minimax^p can be achieved through pure strategies, implying that the game has at least one pure-strategy Nash equilibrium.

3 Linear programming formulation

Finding the minimax^p for a player i can be modeled as a linear program. Suppose player i has pure strategies $S_i = \{A_1, \ldots, A_K\}$ and player $-i$ has pure strategies $S_{-i} = \{B_1, \ldots, B_L\}$. Also, let $v_{k\ell} = u_i(A_k, B_\ell)$ be the utility of player i for joint strategy (A_k, B_ℓ). The linear programming formulations of minimax^p are

Formulation 1 (maximin version):

$$\begin{aligned}&\max w\\ &\text{subject to} \sum_{k=1}^{K} a_k v_{k\ell} \geq w \ \forall \ell = 1, \ldots, L\\ &\sum_{k=1}^{K} a_k = 1\\ &a_k \geq 0 \ \forall k = 1, \ldots K\end{aligned}$$

Formulation 2 (minimax version):

$$\begin{aligned}&\min w\\ &\text{subject to} \sum_{\ell=1}^{L} b_\ell v_{k\ell} \leq w \ \forall k = 1, \ldots, K\\ &\sum_{\ell=1}^{L} b_\ell = 1\\ &b_\ell \geq 0 \ \forall \ell = 1, \ldots L\end{aligned}$$

Intuition. In formulation 1, the constraints $\sum_{k=1}^{K} a_k v_{k\ell} \geq w$ are dictating that the achievable values of a slack variable w should be below $u(a, B_\ell)$, for any mixed strategy a of player i and each pure strategy B_ℓ of player $-i$. This represents the minimization step $f_i(p_i) = \min_{s_{-i}} u(p_i, s_{-i})$ of the maximin^p (with $p_i = a$ and $s_{-i} = B_\ell$). Then, the actual maximin^p value is found by maximizing the slack variable w over all possible choices of mixed strategies $p_i = a$ for player i. This last part represents the maximization step $\max f_i(p_i)$.
Formulation 2 operates in a dual manner, first setting the constraints over each pure strategy of player $-i$ to mimic the maximization step of minimax $F(p_{-i}) = \max_{s_i} u(s_i, p_{-i})$ (meaning $u(s_i, p_{-i}) \leq w$ with $p_{-i} = b$ and $s_i = A_k$) and then minimizing the slack variable w.

Example. Consider a zero-sum game between player 1 and player 2, with $S_1 = \{A_1, A_2\}$ and $S_2 = \{B_1, B_2\}$. Player 1's payoffs are

Player 1	Player 2: B_1	B_2
A_1	v_{11}	v_{12}
A_2	v_{21}	v_{22}

meaning that $u_1(A_k, B_\ell) = v_{k\ell}$ (and, therefore, $u_2(A_k, B_\ell) = -v_{k\ell}$). We can compute the minimax of player 1 in mixed strategies[1] according to formulation 1. First, we set the constraints over mixed strategies of player 1 for each pure strategy of player 2:

$$\begin{cases} a_1 v_{11} + a_2 v_{21} \geq w & \text{(constraint dictated by } B_1) \\ a_1 v_{12} + a_2 v_{22} \geq w & \text{(constraint dictated by } B_2) \end{cases} \tag{9}$$

The value of minimax^p for player 1 is the maximum value of w satisfying the constraints. Clearly, it should also be $a_1 + a_2 = 1$ and $a_1 \geq 0, a_2 \geq 0$, since a_1 and a_2 represent the probability values for a mixed strategy of player 1. Therefore, we may as well write $a_1 = 1 - \alpha$ and $a_2 = \alpha$.

$$\begin{cases} (1-\alpha) v_{11} + \alpha v_{21} \geq w & \text{(constraint dictated by } B_1) \\ (1-\alpha) v_{12} + \alpha v_{22} \geq w & \text{(constraint dictated by } B_2) \end{cases} \tag{10}$$

[1] To be nitpicky, that would be the maximin. However, in mixed strategies they are the same, so the two terms can be used interchangeably!

Alternatively, we may use formulation 2, which leads to constraints

$$\begin{cases} b_1 v_{11} + b_2 v_{12} \le w & \text{(constraint dictated by } A_1) \\ b_1 v_{12} + b_2 v_{12} \le w & \text{(constraint dictated by } A_2) \end{cases} \tag{11}$$

and maximize w. Also in this case, we can write $b_1 = 1 - \beta$ and $b_2 = \beta$

$$\begin{cases} (1-\beta) v_{11} + \beta v_{12} \le w & \text{(constraint dictated by } A_1) \\ (1-\beta) v_{12} + \beta v_{12} \le w & \text{(constraint dictated by } A_2) \end{cases} \tag{12}$$

Let's put some numbers and see how this is done in practice. Suppose player 1's payoffs are as follows:

		Player 2	
		B_1	B_2
Player 1	A_1	3	0
	A_2	1	2

Formulation 1 yields constraints

$$\begin{cases} 3(1-\alpha) + \alpha \ge w & \text{(constraint dictated by } B_1) \\ 2\alpha \ge w & \text{(constraint dictated by } B_2) \end{cases} \tag{13}$$

which are shown in figure 2. You can make sense of the plot by thinking as follows: first player 1 chooses a mixed strategy and then player 2 replies to minimize 1's payoff. Player 2 can play B_1 and keep 1's payoff below the line $(1-\alpha)3 + \alpha$; or 2 can play B_2 and keep 1's payoff below the line 2α. Player 2 plays a best response, so the achievable payoff values for player 1 are below *both* lines. Player 1 chooses a mixed strategy achieving the highest feasible payoff that satisfies all constraints. By observing figure 1, you can see that this value is found at the intersection of the two lines

$$3(1-\alpha) + \alpha = 3\alpha \tag{14}$$

i.e., at $\alpha = 3/4$, yielding $w^* = 1.5$. That implies that player 1's strategy at the Nash equilibrium is $p_1 = \frac{1}{4}A_1 + \frac{3}{4}A_2$ ("play A_1 with probability 1/4, play A_2 with probability 3/4").

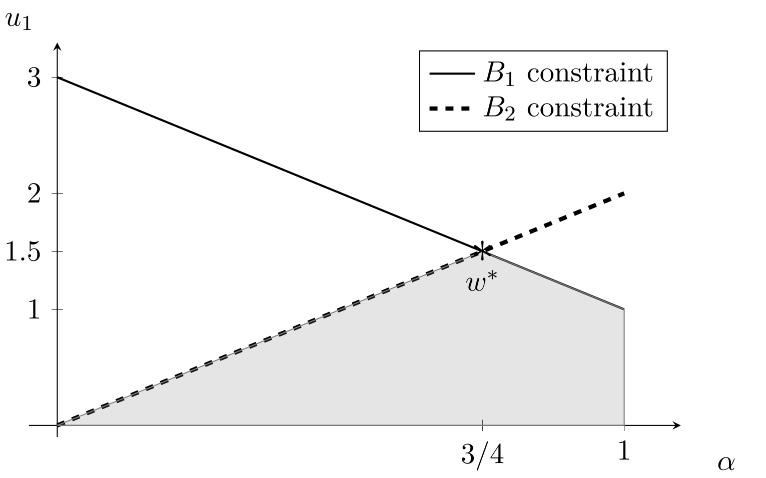

Figure 1: Representation of the constraints obtained using formulation 1. The gray area is the set of feasible values of w. The minimax^p value is found at the intersection of the two lines, on the y-axis. The corresponding value on the x-axis is the value of α that characterizes player 1's mixed strategy at the Nash equilibrium, i.e. $p_1 = \frac{1}{4}A_1 + \frac{3}{4}A_2$.

Formulation 2 yields constraints

$$\begin{cases} 3(1-\beta) \le w & \text{(constraint dictated by } A_1) \\ (1-\beta) + 2\beta \le w & \text{(constraint dictated by } A_2) \end{cases} \tag{15}$$

and the value of w^* is found at the intersection of the two lines

$$3(1-\beta) = (1-\beta) + 2\beta \tag{16}$$

i.e., at $\beta = 1/2$, yielding again $w^* = 1.5$.

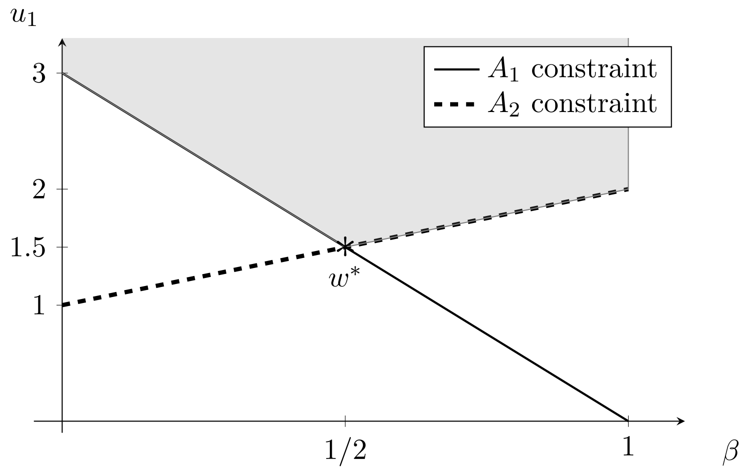

Figure 2: Representation of the constraints obtained using formulation 2. The minimax^p value is again found at the intersection of the two lines, on the y-axis. The corresponding value on the x-axis is the value of α that characterizes player 2's mixed strategy at the Nash equilibrium, i.e. $p_2 = \frac{1}{2}B_1 + \frac{1}{2}B_2$.

FAQ

Are constraints always lines? Constraints are always "linear", but how they look depends on the number of strategies for each player. If players have 2 strategies each, constraints are lines. Otherwise, they could be planes, or hyperplanes. Also, in terms of representation, it is always more convenient to see the game from the perspective of the player who has less. If player 1 has 4 strategies and player 2 has 2 strategies, then formulation 2 is more convenient to find player 1's minimax. With formulation 1, you would need to represent the constraints as two 3-dimensional hyperplanes (not easy to do on a piece of paper); with formulation 2, you just need to draw four lines.

When can we assume $w^* = 0$? Having the value of w^* (i.e., the maximin^p/minimax^p) of a zero-sum game equal to zero means that the game is balanced and no player has an advantage over the other. Games in which players have the same available strategies $S_1 = S_2 = \{s_1, s_2, \ldots, s_L\}$ and symmetric payoffs (i.e., $u_1(s_i, s_j) = u_2(s_j, s_i)$) are always balanced. This is the case, for games such as Rock-Paper-Scissors and Odds&Evens. There are also other types of zero-sum games that are balanced in which players have different strategies. However, in those cases, determining in advance that $w^* = 0$ is not straightforward.