Università degli Studi di Padova

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Lecture 20
Dynamic Bayesian games

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Previously on game theory

- Bayesian Nash equilibria
  - BNE can change if payoffs or priors ϕ are changed
- Signaling can affect single-person decision problems
- Specifically, in single-person decision, the decision-maker is inclined to "follow the signal"
- What about multi-person decision problems (i.e., games)?

Committee voting: recap

- Suppose a single jury member (juror) decides the fate of a defendant
- He starts with a prior estimate of the defendant being guilty $q > 1/2$ prior
- He receives a signal (e.g., evidence) saying that the defendant is guilty (t_G) or innocent (t_I)
- The received signal matches the truth with probability $p > 1/2$ posterior
  - If he receives t_G that the defendant is guilty, his posterior probability is $\Pr[G|t_G] > q$ (he is even surer)
  - If he receives t_I, his posterior probability is $\Pr[G|t_G] < q$, and may even be less than $1/2$
- In case of single-person decision, the juror tends to "follow the signal"

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- Now, we would like to check whether $p > q$ implies that (CA, CA) is a BNE in the original problem (2-person decision)
  - That would correspond to "following the signal"
- First, draw the probability of each type pair

q*p^2 prob of receiving both a guilty signal + innocent*receiving both innocent signal

Member 1 \ Member 2	t_G	t_I
t_G	$qp^2 + (1-q)(1-p)^2$	$p(1-p)$
t_I	$p(1-p)$	$q(1-p)^2 + (1-q)p^2$

- **Note**: This is not a payoff matrix, it is just a table displaying the values of probabilities $\Pr[t_1 = t_x, t_2 = t_y]$

Committee voting: 2-person decision

is following the signal a feasible outcome?

- To check whether (CA,CA) is BNE we need to ask "Is CA a best response to CA?"
  - Assume member 2 plays CA, and check if CA is best for member 1
- We do not want to write down the whole table, let us try to see if we can draw conclusions just by looking at posteriors
- With the rules of the jury, a player's choice is decisive ("**pivotal**") only if the other juror chooses C

  pivotal = makes difference
- If 2 chooses A, that is the result regardless of the 1's choice
  - If 1 believes that 2 is playing CA, *any* strategy of 1 is always a best response if the 2's type is t_I
  - In other words, if 1 thinks that 2 received signal t_I, then everything 1 does is a best response
  - So we need to check only the case $t_2 = t_G$

Committee voting: 2-person decision

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- Again, check the posterior to see the signal effect

$$\Pr[G|t_1 = t_G, t_2 = t_G] = \frac{qp^2}{qp^2 + (1-q)(1-p)^2} > q$$

both received guilty signal

- **Meaning**: if both $t_1 = t_G$ and $t_2 = t_G$: conviction is even more certain
- as before, $p > 1/2$ implies
$qp^2 + (1-q)(1-p)^2 < qp^2 + (1-q)p^2 = p^2$

$$\Pr[G|t_1 = t_I, t_2 = t_G] = \frac{qp(1-p)}{p(1-p)} = q$$

In the case one received innoocent signal and the other guilty

- **Meaning**: if they receive opposite signals, the received signal t_I is useless $\rightarrow$ posterior=prior

=> we still believe that the guy is guilty even if we receveid an innocent signal

Committee voting: 2-person decision

- Recap:
  - If player 2 is of type $t_2 = t_I$, player 1 believes that 2's move is A $\rightarrow$ 1's move does not matter
  - If player 2 is of type $t_2 = t_G$ $\rightarrow$ player 1's posterior is either q or higher
- Therefore, CA is not a best response to CA
- Actually you can prove that (CC, CC) is a BNE

In single person decision problem you're always leading to follow the signal, or maybe your posterior does it,
in n-player decision problem you're not guaranteed that your best choice is following the signal.
That's what make Bayesian game non trivial.

Dynamic Bayesian games

Refinement of NE concepts

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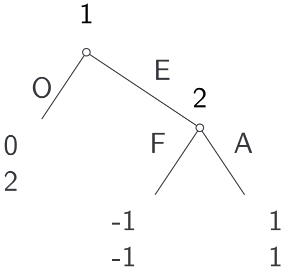
- In static games of complete information, NE are enough
- In "static" Bayesian games, BNE are enough
  - The caveat in static Bayesian games is that strategies are type-dependent
- In dynamic games of complete information, we introduce the concept of SPE
  - Sequential rationality leads to "more rational" equilibria
  - E.g., avoid non-credible threats or irrational behaviors outside the equilibrium path
- Can we find a counterpart for dynamic Bayesian games?

Example: Entry game

- Player 1 is a newcomer (e.g., in a market or network); 1 may enter (E) or stay out (O)
- Player 2 in an incumbent; 2 may fight (F) or accept (A) 1's entrance



- SPE: (E, A)
- non-SPE NE: (O, F) => non credible threats, since if we use sequential rationality, one actually knows that he has no incentive of fighting

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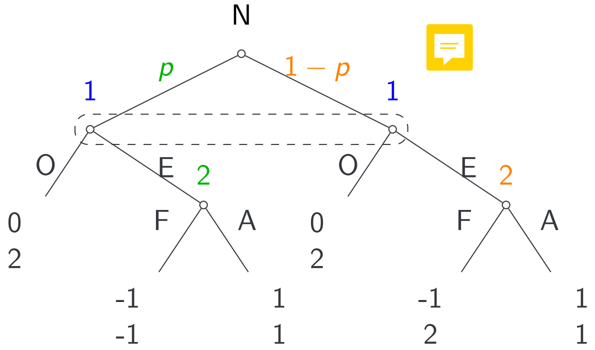
- Player 2 can be "reasonable" or "crazy" with probabilities p and $1 - p$



- For $p = 2/3$

Player 2

Player 1	AA	AF	FA	FF
O	0, **2**	0, **2**	**0**, **2**	**0**, **2**
E	**1**, 1	**1/3**, **4/3**	-1/3, -1/3	-1, 0

- NE: (E, AF), (O, FA), (O, FF)
- Here, we also have a SPE: (E, AF)
  - It is a NE in the overall game, and also in the two subgames where 2 plays as "reasonable" (choosing A), and 1 plays as "crazy" (choosing F).
- However, in most cases SPE is not be a sufficient concept for dynamic Bayesian games

Bayesian entry game 2.0

- Consider a different version of the Bayesian entry game
- This time it is the type of the newcomer (player 1) that is unknown
  - The newcomer can be either “competitive” or “weak”
  - the incumbent is always reasonable
- In case player 1 is a competitive newcomer, payoffs are the same as in the original entry game
- A weak newcomer, instead, does not have the resources to compete with the incumbent; in this case, the newcomer does not want to enter (always gets negative payoff)

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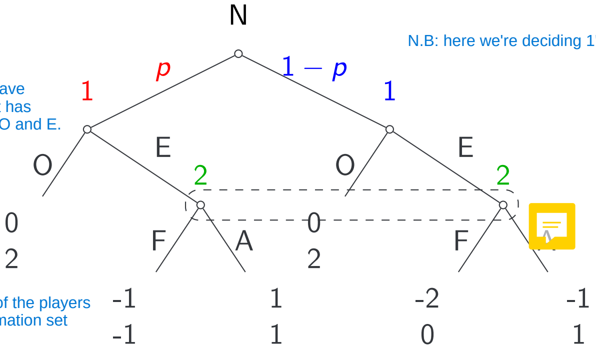
- Player 1 can be "competitive" or "weak" with probabilities p and $1 - p$

N.B: here we're deciding 1's type

For player1 here you still have to specify 2 actions since it has 2 type and he can only do O and E.

For player 2 only one.



To understand strategies of the players here we can still use information set like in dynamic games.

Bayesian entry game 2.0

- This time, the situation is reversed
  - Player 1 can have multiple types, while we have complete information on player 2
  - Player 1 is first to move → we need a way to account for the game dynamics
- Player 1 has two types: 1's pure strategies are OO, OE, EO, EE
- Player 2 has only one type: 2's pure strategies are F, A (2 moves without knowing 1's type)

Bayesian entry game 2.0

- *Note*: We cannot apply backward induction as the last player (player 2) does not know 1's type
- We can reduce the extensive form to yet another normal (static) form
- This time we need to average payoffs over 1's type, e.g.
  - $u_1(\text{OE}, \text{A}) = p \cdot 0 + (1 - p) \cdot -1 = p - 1$
  - $u_2(\text{OE}, \text{A}) = p \cdot 2 + (1 - p) \cdot 1 = p + 1$

  We calculate it like that since we know that 2 is ALWAYS reasonable

- Let us find NE for $p = 1/2$

Player 1 \ Player 2	F	A
OO	**0**, **2**	0, **2**
OE	-1, 1	-1/2, **3/2**
EO	-1/2, 1/2	**1/2**, **3/2**
EE	-1/2, -1/2	0, **1**

- Two NE:
  - (OO,F): equilibrium where the incumbent threatens to fight
  - (EO,A): equilibrium where the incumbent accepts but only a competitive outsider enters (a weak one just stays out from the beginning)

Bayesian entry game 2.0

- (OO, F) seems to be a non-credible threat
  - Player 2 always plays F even when it would be more logical to yield (i.e. play A)
- The problem is: this game has only one subgame
- (OO, F) is technically a SPE, even though its "perfection" is questionable

  → We need to introduce a new type of equilibrium to distinguish decisions that are "perfectly rational" in dynamic Bayesian game

So we can't use EO,A is more reasonable then OO,F and we use Perfect Bayesian Equilibrium

Perfect Bayesian equilibrium

Bayesian equilibrium path

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- If we have a Bayesian NE $s^* = (s_1^*, \ldots, s_n^*)$, we say that an information set is **"on" the equilibrium path** if, given the distribution ϕ of types, it is reached with probability > 0
  - This definition applies to **Bayesian NE**
  - In the BNE given by (OO,F) the information set of node 2 is never reached $\rightarrow$ it is **"off" the equilibrium path**

- In an extensive-form Bayesian game, a **system of beliefs** μ is a probability distribution over *decision nodes* for every information set
  - In other words, it is an estimate of being at a specific node, given an information set (possibly spanning over multiple nodes)
  - It is a conditional probability Pr(node|information set)

    Clearly, this is equal to Pr(node,information set)/Pr(information set), which in turn is Pr(node)/Pr(information set)

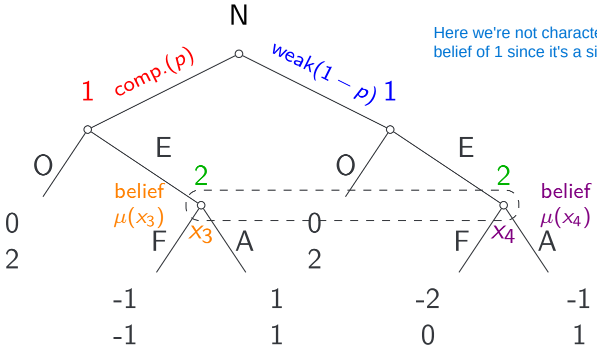
If A is a subevent of B, the prob to consider is only P(A)/P(B).

  - In our entry game, the system of beliefs of player 1 is sure, while that of player 2 depends on the types of player 1 (specifically, on its prior of 1 being competitive or weak)

Perfect Bayesian equilibrium

- We define the following requirements for sequential rationality in Bayesian games:
  1. Players must have a system of beliefs
  2. On the equilibrium path they must follow Bayes' rule on conditional probabilities
  3. Off the equilibrium path: arbitrary
  4. Given their beliefs, players are sequentially rational: i.e., they play a best response to their belief
- **Definition**: A **perfect Bayesian equilibrium** (PBE) is a pair (s^*, μ), where s^* is a Bayesian Nash equilibrium and μ is a system of beliefs satisfying 1–4.

- Always remember that a PBE is not just a pair of strategies: there must be an associated system of beliefs μ



Here we're not characterizing the belief of 1 since it's a singleton.

$$\mu(x_3) = \Pr\{\text{player 1 is competitive} \mid E\}$$

$$1 - \mu(x_3) = \Pr\{\text{player 1 is weak} \mid E\}.$$

PBE in Bayesian entry game 2.0

- A strategy pair must be sustained by a system of beliefs: $\mu(x_3)$ and $\mu(x_4) = 1 - \mu(x_3)$ for player 2
  - e.g., if 2 believes that 1 plays EO, then $\mu(x_3) = 1$ (in other words, if 1 enters, then 2 if fully convinced that 1 is competitive)
  - this reasoning can also be applied to mixed strategies
  - consider strategy $q_C q_W$, i.e.,
    - a competitive player 1 chooses E with probability q_C (and O with $1 - q_C$)
    - a weak player 1 chooses E with probability q_W (and O with $1 - q_W$)
    - In this case, the belief of x_3 given E is

$$\mu(x_3) = \frac{\Pr(\text{node})}{\Pr(\text{information set})} = \frac{p q_C}{p q_C + (1-p) q_W}$$

mi(u4) = (1-p)qw/same denominator

PBE in Bayesian entry game 2.0

- $s^* =$(EO, A) and μ form a PBE:
  - 2 believes that only "competitive" 1 chooses to enter, so $\mu(x_3) = 1$
  - 2 playing A is a sequentially-rational response to 2's belief
- $s^* =$(OO, F) cannot form a PBE with any system of beliefs μ:
  - Bayes' rule cannot be applied since playing OO means $q_C = q_W = 0$

$$\mu(x_3) = \frac{pq_C}{pq_C + (1-p)q_W} = \frac{0}{0}$$

  - x_3 and x_4 are off-path in this case, so the beliefs are arbitrary
  - However, F is irrational in both x_3 and x_4 (A is always better for 2) and it must be either $\mu(x_3) > 0$ or $\mu(x_4) > 0$
  - That means requirement 4 is violated $\rightarrow$ not a PBE

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- Perfect Bayesian NE: (EO,A)
  - sustained by system of belief $\mu(x_1) = 1$
  - all players play in a sequentially-rational way
- Imperfect Bayesian NE: (OO,F)
  - Bayes' rule cannot be applied: $q_C = q_W = 0$
  - Whatever choice of $\mu(x_1), \mu(x_2)$ makes the choice of F irrational

Self-assessment

- What does it mean for a node x_i to be on the Bayesian equilibrium path, given BNE s^*?
- What elements do you need to characterize a PBE?
- How do you determine sustainable belief values $\mu(x_i)$ for nodes that are on the Bayesian equilibrium path?
- What values can $\mu(x_i)$ have if x_i is off the equilibrium path?

Signaling games

Signaling games

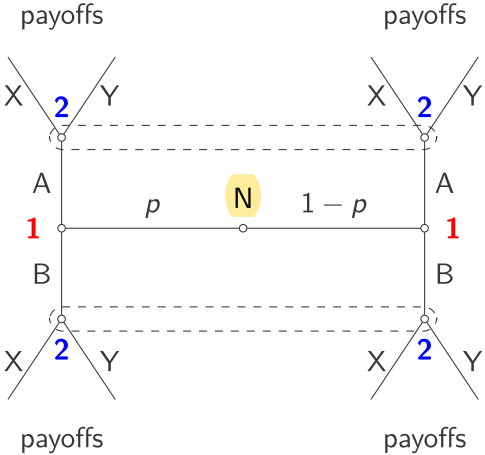
- We saw 2 Bayesian versions of the entry game where **1**=outsider/entrant and **2**=incumbent
- This can be generalized as follows:
  - **2** has multiple types, **1** has only one type: **1** moves before **2**, without any hint about **2**'s type besides the prior ϕ
    $\rightarrow$ This is called a **screening game**, SPE is enough
  - **1** has multiple types, **2** has only one type: **2**'s first move may give a hint (*signal*) about **2**'s type
    $\rightarrow$ This is called a **signaling game**, and requires PBE to achieve sequential rationality

Signaling game: definition

- A signaling game is a 2-player dynamic Bayesian game: **1** (first to move) and **2** (second to move)
  - **1**'s type is chosen among many possible types (by Nature)
  - **2** has only one type
  - **2**'s beliefs are updated after **1**'s move

- Binary case is often shown as a "butterfly"

Preferred representation of signaling games



Equilibria of signaling games

These equlibrias are based on observation of a move and then update the beliefs

- **Separating equilibria**: each type of **1** chooses a different action; thus revealing the type to **2**
- **Pooling equilibria**: all types of **1** choose the same action; thus, **2** gets no signal about **1**'s type
- **Intermediate cases**: **1**'s action does not fully define **1**'s type, but still provides some information
  - Beliefs are updated according to Bayes' rule
  - This type of equilibria is also called "semi-separating" or "partially pooling"

Example: a coffee for Brooke

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- Ann and Brooke are dating; Brooke is invited by a colleague, Zoe, to get a coffee
- Ann is a typed player: her types are
  - Jealous with probability 0.8
  - Easygoing with probability 0.2

  (all this information is common knowledge)
- Ann can send a signal to either stay silent (S) about this proposal or to trash Zoe out (T)
- Brooke observes the signal, and decides whether to accept the coffee invitation (C) or to politely decline (D)

Example: a coffee for Brooke

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- Payoffs:
  - Jealous Ann is deeply hurt if Brooke accepts ($u_A = -6$)
  - Easygoing Ann is just not-so-angry, but still not fond of the idea ($u_A = -1$)
  - Ann prefers to stay silent ($u_A = 3$) rather than trash Zoe out ($u_A = 1$), only in case Brook declines
  - Brooke likes to go to the coffee if that is okay for Ann ($u_B = 2$)
  - If Ann is hurt, Brook prefers declining the invitation ($u_B = 0$) rather than accepting it ($u_B = -3$)

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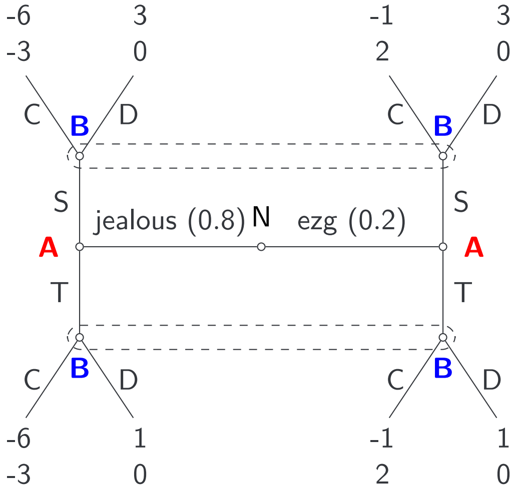
- Extensive form



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- Both players have 4 strategies but for different reasons
  - Ann because of her type: strategy is (what to do if jealous, what to do if easygoing)
  - Brooke does not have a type but observes Ann's move: strategy is (what to do if Ann plays S, what to do if Ann plays T)
  - e.g., (TS,CD) means that Ann trashes Zoe if she is jealous and remains silent if she is easygoing (separating); Brooke just "follows the signal", going to the coffee if Ann stays silent, and declining if Ann starts trashing Zoe

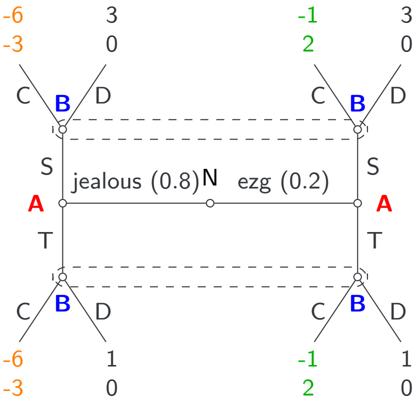
- **Warning!** A's pair is left/right, but B's pair is *B's reaction to A's move*
- First row (SS): only consider B's 1st move (reaction to S)
- Last row (CC): only consider B's 2nd move (reaction to T)

Brooke

Ann	CC	CD	DC	DD
SS	B plays C	B plays C	B plays D	B plays D
ST	B plays C			B plays D
TS	B plays C	!!!	!!!	B plays D
TT	B plays C	B plays D	B plays C	B plays D

- If B plays C, utility is always

$$u_A = 0.8\cdot(-6)+0.2\cdot(-1) = -5, \quad u_B = 0.8\cdot(-3)+0.2\cdot(2) = -2$$



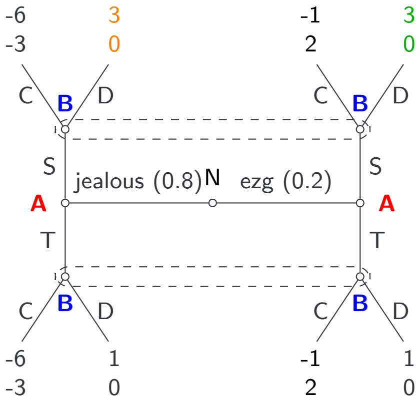
Example: a coffee for Brooke

- When B plays D, we need to distinguish between Ann's 4 possible moves (her payoff changes, B's is always 0)

Ann \ Brooke	CC	CD	DC	DD
SS	-5, -2	-5, -2	B plays D	B plays D
ST	-5, -2			B plays D
TS	-5, -2			B plays D
TT	-5, -2	B plays D	-5, -2	B plays D

- If B plays D and A plays S, i.e., (SS, D*)

$$u_A = 0.8 \cdot (3) + 0.2 \cdot (3) = 3$$



Example: a coffee for Brooke

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- Likewise, if B plays D and A plays T, i.e., (TT,*D), then $u_A = 1$

Ann \ Brooke	CC	CD	DC	DD
SS	-5, -2	-5, -2	3, 0	3, 0
ST	-5, -2			B plays D
TS	-5, -2			B plays D
TT	-5, -2	1, 0	-5, 2	1, 0

- What about intermediate cases (ST,DD) and (TS, DD)?

Example: a coffee for Brooke

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- For (ST,DD) and (TS, DD), you just average the payoffs: in the first case S is played with probability 0.8 and T with probability 0.2; the second case is the opposite

Ann \ Brooke	CC	CD	DC	DD
SS	-5, -2	-5, -2	3, 0	3, 0
ST	-5, -2	?, ?	?, ?	2.6, 0
TS	-5, -2	?, ?	?, ?	1.4, 0
TT	-5, -2	1, 0	-5, 2	1, 0

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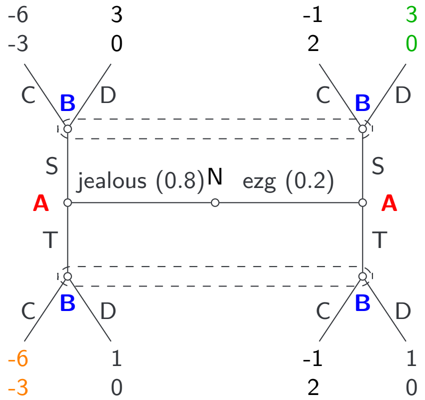
- E.g., for (TS,DC) (remember: D is answer to S and C is answer to T)

$$u_A = 0.8 \cdot (-6) + 0.2 \cdot (3) = -4.2$$



Example: a coffee for Brooke

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		Brooke			
		CC	CD	DC	DD
Ann	SS	-**5**, -2	-5, -2	**3**, **0**	**3**, **0**
	ST	-**5**, -2	-4.6, -2.4	2.2, **0.4**	2.6, 0
	TS	-**5**, -2	0.6, **1.6**	-4.2, -2.4	1.4, 0
	TT	-**5**, -2	**1**, **0**	-5, 2	1, **0**

- 3 pure NE: (SS,DC), (SS,DD), (TT,CD)
- 2 mixed NE: (TT, 1/2CD+1/2DD), (1/6SS+5/6TS,2/9CD+7/9DD)

Example: a coffee for Brooke

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- So far, we have only found NE, **now we need to classify them**! Are they PBE?
- To verify that, we need to construct systems of beliefs μ for Brooke
  - i.e., $\mu =$ is Brooke's belief that Ann is *jealous*
  - One belief for each possible observed move by Ann: μ_S if she stays silent; μ_T if she trashes Zoe out

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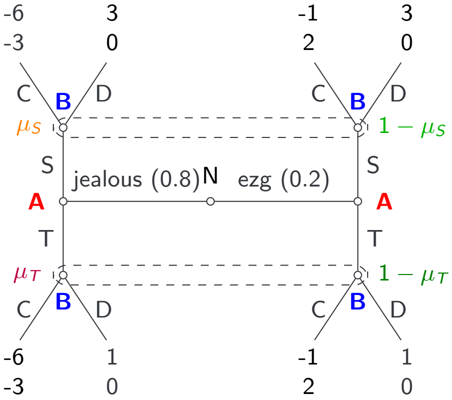
- Beliefs are easy to compute for separating strategies like ST
  - *Note*: We do not need to do that in this exercise, it is just an example



Brooke believes jealous Ann stays silent and easygoing Ann trashes Zoe out: $\mu_S = 1$ (100% chance Ann is jealous if S is observed); $\mu_T = 0$ (0% chance Ann is jealous if T is observed)

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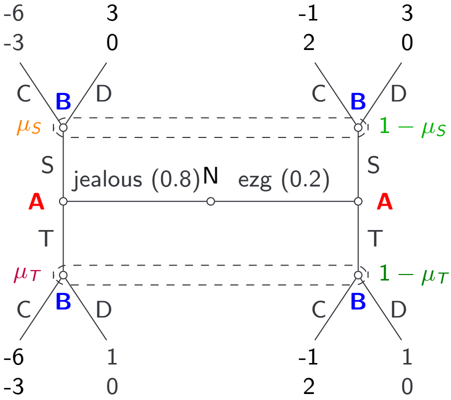
- Unfortunately, in this game we only have pooling equilibria and intermediate cases
  - E.g., consider pooling strategy SS



Brooke believes jealous Ann stays silent regardless of whether she is jealous easygoing: $\mu_S = 0.8$ (same as the prior); What about μ_T?

Example: a coffee for Brooke

- We know that to sustain a PBE with pooling strategy SS, μ_S must stay 0.8
- Off the Bayesian equilibrium path, beliefs are arbitrary. However, they should still satisfy sequential rationality!
- E.g., to make ((SS,DC),(μ_S, μ_T)) as PBE, C must be a best response to T for Brooke
  - That happens if

$$\mu_T u_B(C|J)+(1-\mu_T)u_B(C|E) \geq \mu_T u_B(D|J)+(1-\mu_T)u_B(D|E)$$

$$\mu_T(-3) + (1-\mu_T)(2) \geq 0$$

  - Meaning that any $\mu_T \leq 2/5$ is sufficient to sustain a PBE with NE (SS,DC)
  - Conversely, any $\mu_T \geq 2/5$ sustains a PBE with NE (SS,DD)

Example: a coffee for Brooke

- Summary so far:
- NE1: ((SS, DC),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.8, \mu_T \leq 0.4$)
- NE2: ((SS, DD),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.8, \mu_T \geq 0.4$)
- NE3: ((TT,CD),(μ_S,μ_T)) is a PBE for ($\mu_S \leq 0.4, \mu_T = 0.8$)
  - Analogous to NE1, same payoffs for Brooke
- NE4: ((TT, 1/2CD+1/2DD),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.4$, $\mu_T = 0.8$)
  - Same as above, but this time Brooke should be indifferent between C and D against S
- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T)) ?

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- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T))
- This can be a semi-separating PBE
  - Ann is always silent if easygoing but may start badmouthing Zoe if she is jealous
  - This is because she believes that Brooke may sometimes choose C if she stays 100% silent (if she stays silent, B chooses C with probability 2/9)
  - The description makes sense, but what about the system of beliefs? It is actually more complex and requires Bayes' rule to be used non-trivially

Example: a coffee for Brooke

- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T))
- Easy part: $\mu_T = 1 \rightarrow$ Brooke believes Ann chooses to trash Zoe out only if she is jealous; if she is easygoing, Ann always plays S
- Harder part: $\mu_S =$?
- Depending on it, Brooke may prefer C or D. And to play a mixed strategy, Brooke must be indifferent between them (characterization theorem)
- We have already seen that this happens for $\mu_S = 0.4$

- NE5: $((1/6SS+5/6TS, 2/9CD+7/9DD), (\mu_S, \mu_T))$
- Denote with q the probability that jealous Ann plays S (the probability that she plays T is $1-q$)
- Remember:

$$\mu_S = \frac{\Pr[S, \text{jealous}]}{\Pr[S]} = \frac{pq}{pq + (1-p)\cdot 1} = \frac{0.8 \cdot 1/6}{0.8 \cdot 1/6 + 0.2} = 0.4$$

- If we already know μ_S, we can use this formula to find q

Questions?