Università degli Studi di Padova

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Lecture 07
Nash theorem

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Terminology recap

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- In a game $\mathbb{G} = (S_1, S_2, \ldots, S_n; u_1, u_2 \ldots, u_n)$ we can have
  - **Pure strategy**: $s_i \in S_i$
  - **Joint strategy**: $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$
- In static games of complete information, joint strategy $=$ outcome $\rightarrow$ Actually, this is a lie! (True when the joint strategy includes only pure strategies)

No Nash Equilibrium?

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- In this game, there is no NE in pure strategies

		Even: 0	Even: 1
Odd	0	-4, 4	4, -4
Odd	1	4, -4	-4, 4

- However, there is a "good" a strategy that rational players are willing to adopt
- We just need to extend the definition of strategies

Mixed strategies

Missing outcome

- Expand Odds&Evens games introducing strategy
  - 1/2:“Play 0 with probability 1/2 and 1 with probability 1/2”

		Even 0	Even 1/2	Even 1
Odd	0	-4, 4	0, 0	4, -4
Odd	1/2	0, 0	0, 0	0, 0
Odd	1	4, -4	0, 0	-4, 4

- It seems that (1/2, 1/2) is a NE. Let us formalize this.

Mixed strategies

- Remember:
  - A **probability distribution** over a non-empty discrete set A is a function $p : A \to [0, 1]$ that satisfies $\sum_{a \in A} p(a) = 1$
  - The set of possible probability distributions over A is called the *simplex* of A and denoted as ΔA
- **Mixed strategy**: In a game $\mathbb{G} = (S_1, \dots, S_n; u_1, \dots, u_n)$, a mixed strategy for player i is a probability distribution p_i over set S_i
- For player i, playing p_i means choosing strategies $S_i = (s_i^{(1)}, \dots, s_i^{(k)})$, $k = |S_i|$, with probabilities $(p_i(s_i^{(1)}), \dots, p_i(s_i^{(k)}))$
- Warning! There will be a lot of similarities with lotteries $\to$ Do not confuse the two concepts!

Expected payoff

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- Utility u_i can be extended to the expected utility, which is a real function over $\Delta S_1 \times \Delta S_2 \times \cdots \times \Delta S_n$
- If players choose mixed strategies $(p_1, \ldots, p_n)$, player i's payoff can be computed as a weighted average over p_i's

$$u_i(p_i, \ldots, p_n) = \sum_{(s_1,\ldots,s_n)\in S} \underbrace{p_1(s_1) \cdots p_n(s_n)}_{\text{probability of } (s_1,\ldots,s_n)} \cdot u_i(s_1, \ldots, s_n)$$

with $S = S_1 \times \cdots \times S_n$

- In other words, for all combinations of pure strategies:
  - fix (pure) joint strategy $s = (s_1, \ldots, s_n)$
  - compute its probability as $p_1(s_1) \cdots p_n(s_n)$
  - weigh $u_i(s_1, \ldots, s_n)$ on this probability and sum

- Consider Odds& Evens game and assume Odd decides to play 0 with probability q, while Even plays 0 with probability r
  - Conversely, 1 is played by Odd and Even with probability $1 - q$ and $1 - r$, respectively

		Even: 0 (prob r)	Even: 1 (prob $1 - r$)
Odd	0 (prob q)	-$4qr$, $4qr$	$4q(1-r)$, -$4q(1-r)$
	1 (prob $1 - q$)	$4(1-q)r$, -$4(1-q)r$	-$4(1-q)(1-r)$, $4(1-q)(1-r)$

- this is a single joint strategy $p = (p_1, p_2) = ((q, 1-q), (r, 1-r)) \rightarrow$ for compactness, we just write (q, r)

Odd \ Even	0 (prob r)	1 (prob $1-r$)
0 (prob q)	-$4qr$, $4qr$	$4q(1-r)$, -$4q(1-r)$
1 (prob $1-q$)	$4(1-q)r$, -$4(1-q)r$	-$4(1-q)(1-r)$, $4(1-q)(1-r)$

- Odd's payoff:

$$
\begin{aligned}
u_1(q,r) &= -4qr + 4q(1-r) + 4(1-q)r - 4(1-q)(1-r) \\
&= -4qr + 4q - 4qr + 4r - 4rq - 4 + 4q + 4r - 4qr \\
&= -16qr + 8q + 8r - 4 = -4(2q-1)(2r-1)
\end{aligned}
$$

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- We can see these as "intermediate" strategies between 0 and 1

$$
\begin{array}{cc|c|c|c|}
 & & & \text{Even} & \\
 & & 0 & r & 1 \\
\hline
 & 0 & & & \\
\hline
\text{Odd} & q & & \begin{array}{c} -16qr + 8q + 8r - 4 \\ 16qr - 8q - 8r + 4 \end{array} & \\
\hline
 & 1 & & & \\
\hline
\end{array}
$$

Pure strategies as mixed strategies

- Given a mixed strategy $p_i \in \Delta S_i$, we define the **support** of p_i as $\text{supp}(p_i) = s_i \in S_i : p_i(s_i) > 0$
- Each pure strategy $s_i \in S_i$ can be seen as a mixed strategy $p \in \Delta S_i$ such that $p(s_i) = 1$
  - meaning that $p(s_i') = 0$ for any other $s_i' \in S_i, s_i' \neq s_i$
- Every definition or result that applies to mixed strategies applies also to pure strategies, seen as degenerate mixed strategies

Strict/weak dominance

- Consider game $\mathbb{G} = (S_1, S_2, \dots, S_n; u_1, u_2 \dots, u_n)$
  - Notation: $p_{-i} = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n) \in \Delta S_1 \times \cdots \times \Delta S_{i-1} \times \Delta S_{i+1} \times \Delta S_n$ (note that there are infinite many tuples in this set!)
- Given two mixed strategies $p_i, p_i' \in \Delta S_i$, we say that p_i' **strictly dominates** p_i

$$u_i(p_i', p_{-i}) > u_i(p_i, p_{-i}), \quad \text{for all } p_{-i}$$

- We say that p_i' **weakly dominates** p_i if

$$u_i(p_i', p_{-i}) \geq u_i(p_i, p_{-i}), \quad \text{for all } p_{-i}$$
$$u_i(p_i', p_{-i}) > u_i(p_i, p_{-i}), \quad \text{for some } p_{-i}$$

Strict/weak dominance

- Infinite many possibilities for p_{-i}. How to prove that a mixed strategy dominates another one? Luckily, we can leverage some useful properties:
- p_i' **strictly dominates** p_i iff (iff = if and only if)

$$u_i(p_i', s_{-i}) > u_i(p_i, s_{-i}), \quad \text{for all } s_{-i} \in S_{-i}$$

we compare the mixed strategies pi' and pi with all other pure strategies!

- p_i' **weakly dominates** p_i iff

$$u_i(p_i', s_{-i}) \geq u_i(p_i, s_{-i}), \quad \text{for all } s_{-i} \in S_{-i}$$
$$u_i(p_i', s_{-i}) > u_i(p_i, s_{-i}), \quad \text{for some } s_{-i} \in S_{-i}$$

- In other words, we can limit our search to other players' pure strategies

- Consider game $\mathbb{G} = (S_1, S_2, \ldots, S_n; u_1, u_2 \ldots, u_n)$
- A joint mixed strategy $p^* = (p_1^*, \ldots, p_n^*) \in \Delta S_1 \times \cdots \times \Delta S_n$ is a **Nash equilibrium** if for all i:

$$\boxed{u_i(p_i^*, p_{-i}^*) \geq u_i(p_i', p_{-i}^*) \text{ for all } p_i' \in \Delta S_i}$$

- Generalization of the NE in pure strategies: no player has incentive to change his/her move (which is a mixed strategy now)
- The concept of "best response" generalizes in an analogous manner

NE as absence of regrets

- In pure strategies, we could see NE as joint strategies in which no one regrets the outcome
- In mixed strategies, this is a bit more subtle: players may play the best response to other players' strategies and still regret the result
  - E.g., in Odds&Evens both players choose 0 and 1 with 50% probability
  - One of them will end up losing (hence regretting the outcome), yet they both played a best response
- In mixed NE, there is no regret about the chosen strategy, even though players may not like the final result

Back to Odds&Evens

- In the Odds&Evens game, the payoff for Odd is $-4(2q-1)(2r-1)$, while the payoff for Even is the opposite
- If $q = 1/2$ or $r = 1/2$, *both* players get payoff 0
- If $q = r = 1/2$, no player has incentive to change

Odd \ Even	0	1/2	1
0		0, 0 0, 0	
1/2	0, 0 0, 0	0, 0	0, 0 0, 0
1		0, 0 0, 0	

Self-assessment

- As an exercise, prove that $(1/2, 1/2)$ is the **only** Nash equilibrium of the Odds&Evens game
- How to proceed
  - Consider 3 cases: those where Odd's payoff is > 0, < 0, or $= 0$ (but joint strategy is not $q = r = 1/2$)
  - Show that in each case there is a player who has incentive to deviate
  - As a consequence, none of these strategies is a NE $\Rightarrow q = r = 1/2$ is the only NE $\square$

IESDS and mixed strategies

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- (Abuse of) notation: we use qL+$(1-q)$C to denote the mixed strategy "play L with probability q and C with probability $1-q$"

Player B

Player A	L	C	R
T	7, 4	5, 0	8, 1
D	6, 0	3, 4	9, 1

- R is not dominated by L or C. However, mixed strategy $p = \frac{1}{2}L + \frac{1}{2}C$ yields payoff $u_B = 2$ regardless of A's choice
- Pure strategy R is strictly dominated by p
  - R can be eliminated
  - Further eliminations are possible

Player A \ Player B	L	C	R
T	7, 4	5, 0	8, 1
D	6, 0	3, 4	9, 1

- Joint strategy (T, L) is the only survivor of IESDS $\rightarrow$ only NE of the game

IESDS and mixed strategies

- Similar results to the pure strategy case hold for IESDS in mixed strategies
  - **Theorem**: NE survive IESDS
  - **Theorem**: The order of IESDS is irrelevant
- **Remember**: Use strict (not weak) dominance! A weakly dominated strategy can be part of a NE (or belong to the support of a strategy that is part of a NE)

Characterization of mixed NE

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- **Theorem**: Consider game $\mathbb{G} = (S_1, S_2, \ldots, S_n; u_1, u_2 \ldots, u_n)$ and a joint mixed strategy $p^* = (p_1^*, \ldots, p_n^*)$ in $\mathbb{G}$. The following statements are equivalent
  1. Joint mixed strategy p^* is a Nash equilibrium
  2. For each i:

$$\boxed{\begin{aligned} u_i(p_i^*, p_{-i}^*) &= u_i(s_i, p_{-i}^*) \text{ for all } s_i \in \operatorname{supp}(p_i^*) \\ u_i(p_i^*, p_{-i}^*) &\geq u_i(s_i, p_{-i}^*) \text{ for all } s_i \notin \operatorname{supp}(p_i^*) \end{aligned}}$$

- Simply put, fixing strategy p_{-i}^*, player i receives the same payoff for all pure strategies $s_i \in \operatorname{supp}(p_i)^*$
- Clearly, this is also equal to the payoff yielded by p_i^*, being a convex combination of those pure strategies

Characterization of mixed NE

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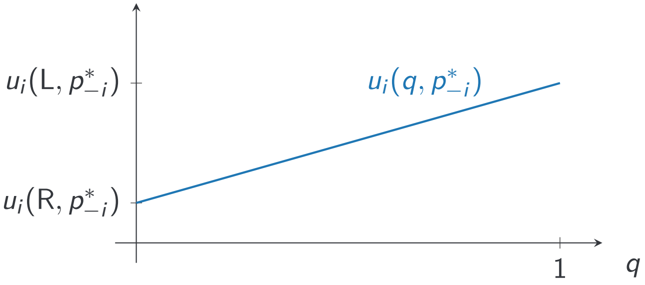
- Intuition: suppose p^*_{-i} is fixed and consider joint mixed strategy $q\mathrm{L} + (1-q)\mathrm{R}$ for player i (support $= \{\mathrm{L}, \mathrm{R}\}$)
- If $u_i(\mathrm{L}, p^*_{-i}) \neq u_i(\mathrm{R}, p^*_{-i})$ then either L or R yields lower payoff than the other $\Rightarrow$ Player i should remove it from the support to maximize u_i $\Rightarrow$ Not a NE



Back to the Battle of the Sexes

		B: R(r)	B: S(1-r)
A	R(q)	2, 1	0, 0
	S(1-q)	0, 0	1, 2

- This game has two NE in pure strategies: (R, R) and (S, S)
- We can show that there is also a mixed NE
- Player A chooses R w.p. q, player B chooses R w.p. r
- A joint mixed strategy is uniquely identified by (q, r)
  - A's payoff: $u_A(q, r) = 2 \cdot qr + 1 \cdot (1 - q)(1 - r)$
  - B's payoff: $u_B(q, r) = 1 \cdot qr + 2 \cdot (1 - q)(1 - r)$

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- q = probability A plays R, r = probability B plays R
- Assume (q^*, r^*) is a NE
  - Note: it must be $\text{supp}(q^*) = \text{supp}(r^*) = \{\text{R, S}\}$ (otherwise, we fall back to the pure-strategy NE)
- Due to the "characterization" theorem, it must be

$$u_A(q^*, r^*) = \underbrace{u_A(\text{S}, r^*) = u_A(\text{R}, r^*)}_{\text{we use this eq. to find } r^*}$$

- Plug the values $q = 0$ (for S) and $q = 1$ (for R) in $u_A(q, r) = 2qr + (1 - q)(1 - r)$ and solve for $r = r^*$
- $1 - r^* = 2r^*$
- Solution for B: $r^* = 1/3$

Back to the Battle of the Sexes

- Similarly, we impose $u_B(q^*, \text{S}) = u_B(q^*, \text{R})$
- Plug the values $r = 0$ (for S) and $r = 1$ (for R) in $u_B(q, r) = qr + 2(1 - q)(1 - r)$ and solve for $q = q^*$
- $2 - 2q^* = q^*$
- Solution for A: $q^* = 2/3$
- Mixed NE: A plays (R, S) with probabilities $(2/3, 1/3)$, B plays (R, S) with probabilities $(1/3, 2/3)$
- **Note**: A's NE strategy is found using B's utility function, and vice versa

Back to the Prisoner's dilemma

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- We have only one NE in pure strategies. What about mixed strategies?

Player B

Player A		M	F
	M	-1, -1	-9, 0
	F	0, -9	-6, -6

Nash theorem

Nash theorem (intro)

- The reasoning we used to find the third (mixed) NE of the Battle of Sexes can be generalized
- Every two-player game with two strategies has a NE in mixed strategies (although they could be degenerate mixed strategies, i.e., pure strategies)
- This is easy to prove, and part of the more general Nash theorem
- **Theorem** (Nash, 1950): Every game with finite pure-strategy sets S_i has at least one Nash equilibrium, possibly involving mixed strategies

Understanding mixed strategies

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- Mixed strategies are key for Nash Theorem
  - How do we interpret the probabilities involved in mixed strategies?
  - In the end, players play a pure strategy (i.e., take a deterministic action)
- Possible interpretations
  - Large numbers: If the game is played $M \gg 1$ times, a probability q for s_i means that s_i gets played qM times
  - Fuzzy values: Uncertain actions, players do not know
  - **Beliefs**: The probability q reflects the uncertainty that the other players have about my choice (which is actually deterministic)

Belief

- A **belief** of player i is a possible profile of opponents' strategies: an element of set ΔS_{-i}
  - Same definition as in pure strategies but with ΔS_{-i}
- Again, the best-response correspondence $\mathrm{BR} : \Delta S_{-i} \to 2^{\Delta S_{-i}}$ associates $p_{-i} \in \Delta S_{-i}$ with a subset of ΔS_i such that each $p_i \in \mathrm{BR}(p_{-i})$ is a best response to p_{-i}
  - Best responses are still not unique

NE as best responses

- Using beliefs, we can speak of **best response** to an opponent's (mixed) strategy
- Intuition:

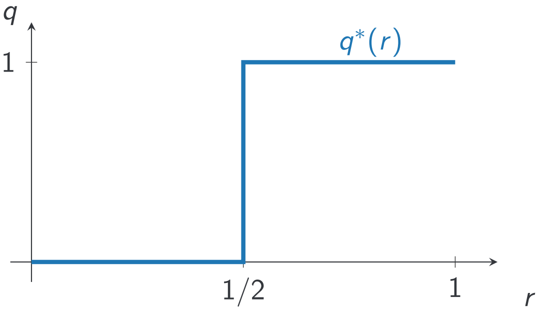
		B: F	B: G
A	U	6, 1	0, 4
A	D	2, 5	4, 0

  - B ignores what A will play
  - So B assumes $q =$ probability that A plays U
  - Likewise, A assumes $r =$ probability that B plays F
- E.g., if A's belief is that B always plays F (i.e., $r = 1$), A's best response is to play U ($q = 1$). In general?

		B: F	B: G
A	U	6, 1	0, 4
A	D	2, 5	4, 0

- It holds: $u_A(\mathrm{D}, r) = 2r + 4(1 - r), u_A(\mathrm{U}, r) = 6r$
- U is actually A's best response as long as $r > 1/2$, else it is D; if $r = 1/2$, they are equivalent
- Denote A's best response with $q^*(r)$

NE as best responses



- A's best response is either U or D, i.e. $q^*(r) = 1, 0,$ respectively:

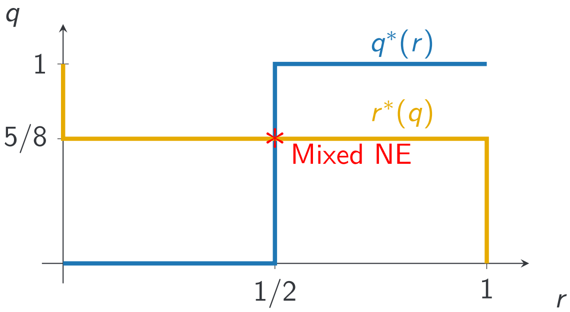
$$q^*(r) = \begin{cases} 0 & \text{if } r < 1/2 \\ 1 & \text{if } r > 1/2 \end{cases}$$

NE as best responses

		B	
		F	G
A	U	6, 1	0, 4
	D	2, 5	4, 0

- For B: $u_B(q, \mathrm{F}) = q + 5(1-q), u_B(q, \mathrm{G}) = 4q$
- B's best response $r^*(q)$ is

$$r^*(q) = \begin{cases} 1 & \text{if } q < 5/8 \\ 0 & \text{if } q > 5/8 \end{cases}$$



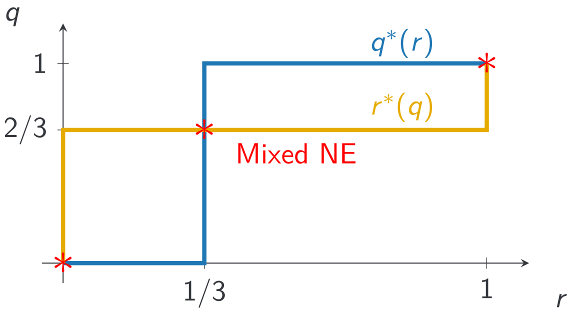
- Joint strategy $p^* = (q = 1/2, r = 5/8)$ is a NE
- NE are points where the choice of each player is best response to the other player's choice

Existence of NE

- The existence of at least one NE is guaranteed by topological reasons
- There may be more than on NE (e.g., Battle of the Sexes)

A \ B	R	S
R	2, 1	0, 0
S	0, 0	1, 2

- $u_A(\mathrm{R}, r) = 2r, u_A(\mathrm{S}, r) = 1 - r,\ q^*(r) = 1 - \mathbb{1}(r - 1/3)$
- $u_B(q, \mathrm{R}) = q, u_A(q, \mathrm{S}) = 2(1 - q),\ r^*(q) = 1 - \mathbb{1}(q - 2/3)$



- Here there are three NE
- In any event, $q^*(r)$ must intersect $r^*(q)$ at least once
- Nash theorem generalizes this idea

Nash theorem

- For game $\mathbb{G} = (S_1, S_2, \ldots, S_n; u_1, u_2 \ldots, u_n)$, define

$$\mathrm{BR}_i : \Delta S_1 \times \ldots \Delta S_{i-1} \times \Delta S_{i+1} \times \cdots \times \Delta S_n \to 2^{\Delta S_i}$$

$$\mathrm{BR}_i(p_{-i}) = \{p_i \in \Delta S_i : u_i(p_i, p_{-i}) \text{ is maximized}\}$$

- Then, define $\mathbf{BR} : \Delta S \to 2^{\Delta S}$ as

$$\mathbf{BR}(p) = \mathrm{BR}_1(p_{-1}) \times \cdots \times \mathrm{BR}_n(p_{-n})$$

- $\mathrm{BR}_i(p_{-i})$ is the set of best responses of i to other player's strategies; **BR** is their aggregate
  - p is a NE if $p \in \mathbf{BR}(p)$
  - Properties of $\mathrm{BR}_i(p_{-i})$: (1) is always non-empty; (2) always contains at least one pure strategy

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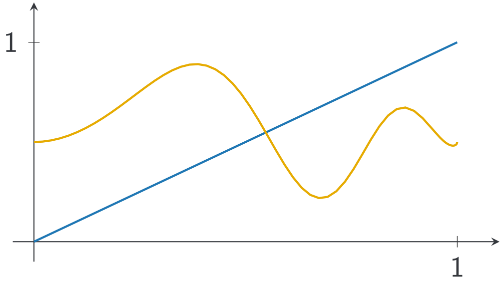
- **Brouwer's fixed point theorem**: If $f(x)$ is a continuous function $f : \mathcal{I} \to \mathcal{I}$, where $\mathcal{I} \subset \mathbb{R}$ to itself, $\exists x^* \in \mathcal{I}$ such that $f(x^*) = x^*$
- *Proof* (sketch): Consider $\mathcal{I} = [0, 1]$. If $f(0) > 0$ and $f(1) < 1$, apply Bolzano-Weierstrass theorem to $f(x) - x$



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- **Kakutani's fixed point theorem**: Consider
  - $A \subset \mathbb{R}^n$ non-empty, compact, and convex
  - correspondence $F : A \to 2^A$ such that
    - For all $x \in A$, $F(x)$ is non-empty and convex
    - If $\{x_i\}$, and $\{y_i\}$ are sequences in $\mathbb{R}^n$ converging to x and y, respectively: $y_i \in F(x_i) \Rightarrow y \in F(x)$ (F's graph is closed)
    - Then, there exists x^* $\in A$ such that x^* $\in F($$x^*$$)$
- **Nash theorem**: Nothing but Kakutani's theorem applied to the global best-response correspondence **BR**

Questions?