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Lecture 20

Dynamic Bayesian games

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- Bayesian Nash equilibria
 - BNE can change if payoffs or priors ϕ are changed
- Signaling can affect single-person decision problems
- Specifically, in single-person decision, the decision-maker is inclined to “follow the signal”
- What about multi-person decision problems (i.e., games)?

- Suppose a single jury member (juror) decides the fate of a defendant
- He starts with a prior estimate of the defendant being guilty $q > 1/2$ prior
- He receives a signal (e.g., evidence) saying that the defendant is guilty (t_G) or innocent (t_I)
- The received signal matches the truth with probability $p > 1/2$ posterior
 - If he receives t_G that the defendant is guilty, his posterior probability is $\Pr[G|t_G] > q$ (he is even surer)
 - If he receives t_I , his posterior probability is $\Pr[G|t_G] < q$, and may even be less than $1/2$
- In case of single-person decision, the juror tends to “follow the signal”

Committee voting: 2-person decision

- Now, we would like to check whether $p > q$ implies that (CA, CA) is a BNE in the original problem (2-person decision)
 - That would correspond to “following the signal”
- First, draw the probability of each type pair

$q \cdot p^2$ prob of receiving both a guilty signal + innocent * receiving both innocent signal

		Member 2	
		t_G	t_I
Member 1	t_G	$qp^2 + (1 - q)(1 - p)^2$	$p(1 - p)$
	t_I	$p(1 - p)$	$q(1 - p)^2 + (1 - q)p^2$

- **Note:** This is not a payoff matrix, it is just a table displaying the values of probabilities $\Pr[t_1 = t_x, t_2 = t_y]$

is following the signal a feasible outcome?

- To check whether (CA, CA) is BNE we need to ask “Is CA a best response to CA?”
 - Assume member 2 plays CA, and check if CA is best for member 1
- We do not want to write down the whole table, let us try to see if we can draw conclusions just by looking at posteriors
- With the rules of the jury, a player's choice is decisive (“**pivotal**”) only if the other juror chooses C pivotal = makes difference
- If 2 chooses A, that is the result regardless of the 1's choice
 - If 1 believes that 2 is playing CA, any strategy of 1 is always a best response if the 2's type is t_I
 - In other words, if 1 thinks that 2 received signal t_I , then everything 1 does is a best response
 - So we need to check only the case $t_2 = t_G$

- Again, check the posterior to see the signal effect

$$\Pr[G|t_1 = t_G, t_2 = t_G] = \frac{qp^2}{qp^2 + (1-q)(1-p)^2} > q$$

both received guilty signal

- **Meaning:** if both $t_1 = t_G$ and $t_2 = t_G$: conviction is even more certain

- as before, $p > 1/2$ implies

$$qp^2 + (1-q)(1-p)^2 < qp^2 + (1-q)p^2 = p^2$$

$$\Pr[G|t_1 = t_I, t_2 = t_G] = \frac{qp(1-p)}{p(1-p)} = q$$

In the case one received innocent signal and the other guilty

- **Meaning:** if they receive opposite signals, the received signal

t_I is useless \rightarrow posterior=prior

\Rightarrow we still believe that the guy is guilty even if we received an innocent signal

- Recap:

- If player 2 is of type $t_2 = t_I$, player 1 believes that 2's move is $A \rightarrow$ 1's move does not matter
- If player 2 is of type $t_2 = t_G \rightarrow$ player 1's posterior is either q or higher
- Therefore, CA is not a best response to CA
- Actually you can prove that (CC, CC) is a BNE

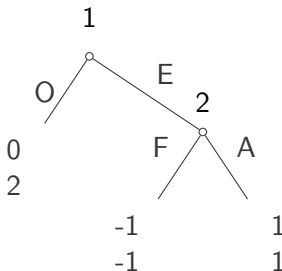
In single person decision problem you're always leading to follow the signal, or maybe your posterior does it, in n-player decision problem you're not guaranteed that your best choice is following the signal. That's what make Bayesian game non trivial.

Dynamic Bayesian games

- In static games of complete information, NE are enough
- In “static” Bayesian games, BNE are enough
 - The caveat in static Bayesian games is that strategies are type-dependent
- In dynamic games of complete information, we introduce the concept of SPE
 - Sequential rationality leads to “more rational” equilibria
 - E.g., avoid non-credible threats or irrational behaviors outside the equilibrium path
- Can we find a counterpart for dynamic Bayesian games?

Example: Entry game

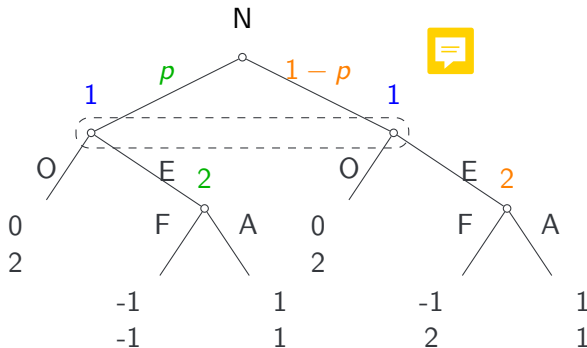
- Player 1 is a newcomer (e.g., in a market or network); 1 may enter (E) or stay out (O)
- Player 2 is an incumbent; 2 may fight (F) or accept (A) 1's entrance



- SPE: (E, A)
- non-SPE NE: (O, F) => non credible threats, since if we use sequential rationality, one actually knows that he has no incentive of fighting

Bayesian entry game

- Player 2 can be “reasonable” or “crazy” with probabilities p and $1 - p$



- For $p = 2/3$

		Player 2			
		AA	AF	FA	FF
Player 1	O	0, 2	0, 2	0, 2	0, 2
	E	1, 1	1/3, 4/3	-1/3, -1/3	-1, 0

- NE: (E, AF), (O, FA), (O, FF)
- Here, we also have a SPE: (E, AF)
 - It is a NE in the overall game, and also in the two subgames where 2 plays as “reasonable” (choosing A), and 1 plays as “crazy” (choosing F).
- However, in most cases SPE is not be a sufficient concept for dynamic Bayesian games

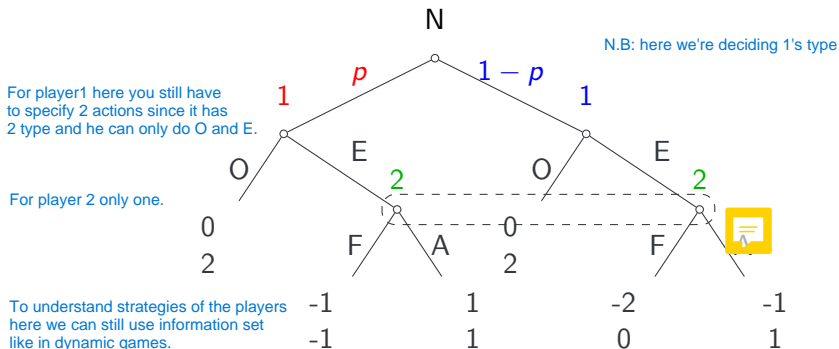
Bayesian entry game 2.0

- Consider a different version of the Bayesian entry game
- This time it is the type of the newcomer (player 1) that is unknown
 - The newcomer can be either “competitive” or “weak”
 - the incumbent is always reasonable
- In case player 1 is a competitive newcomer, payoffs are the same as in the original entry game
- A weak newcomer, instead, does not have the resources to compete with the incumbent; in this case, the newcomer does not want to enter (always gets negative payoff)

Bayesian entry game 2.0



- Player 1 can be “competitive” or “weak” with probabilities p and $1 - p$



- This time, the situation is reversed
 - Player 1 can have multiple types, while we have complete information on player 2
 - Player 1 is first to move \rightarrow we need a way to account for the game dynamics
- Player 1 has two types: 1's pure strategies are OO, OE, EO, EE
- Player 2 has only one type: 2's pure strategies are F, A (2 moves without knowing 1's type)

- *Note:* We cannot apply backward induction as the last player (player 2) does not know 1's type
- We can reduce the extensive form to yet another normal (static) form
- This time we need to average payoffs over 1's type, e.g.
 - $u_1(\text{OE}, A) = p \cdot 0 + (1 - p) \cdot -1 = p - 1$
 - $u_2(\text{OE}, A) = p \cdot 2 + (1 - p) \cdot 1 = p + 1$

We calculate it like that since we know that 2 is ALWAYS reasonable

Bayesian entry game 2.0

- Let us find NE for $p = 1/2$

		Player 2	
		F	A
Player 1	OO	0, 2	0, 2
	OE	-1, 1	-1/2, 3/2
	EO	-1/2, 1/2	1/2, 3/2
	EE	-1/2, -1/2	0, 1

- Two NE:
 - (OO,F): equilibrium where the incumbent threatens to fight
 - (EO,A): equilibrium where the incumbent accepts but only a competitive outsider enters (a weak one just stays out from the beginning)

- (OO, F) seems to be a non-credible threat
 - Player 2 always plays F even when it would be more logical to yield (i.e. play A)
- The problem is: this game has only one subgame
- (OO, F) is technically a SPE, even though its “perfection” is questionable
 - We need to introduce a new type of equilibrium to distinguish decisions that are “perfectly rational” in dynamic Bayesian game

So we can't use EO, A is more reasonable than OO, F and we use Perfect Bayesian Equilibrium

Perfect Bayesian equilibrium

- If we have a Bayesian NE $s^* = (s_1^*, \dots, s_n^*)$, we say that an information set is **“on” the equilibrium path** if, given the distribution ϕ of types, it is reached with probability > 0
 - This definition applies to **Bayesian NE**
 - In the BNE given by (OO, F) the information set of node 2 is never reached \rightarrow it is **“off” the equilibrium path**

- In an extensive-form Bayesian game, a **system of beliefs** μ is a probability distribution over *decision nodes* for every information set
 - In other words, it is an estimate of being at a specific node, given an information set (possibly spanning over multiple nodes)
 - It is a conditional probability $\Pr(\text{node}|\text{information set})$

Clearly, this is equal to

$\Pr(\text{node}, \text{information set}) / \Pr(\text{information set})$, which in turn is $\Pr(\text{node}) / \Pr(\text{information set})$

If A is a subevent of B,
the prob to consider is only
 $P(A)/P(B)$.

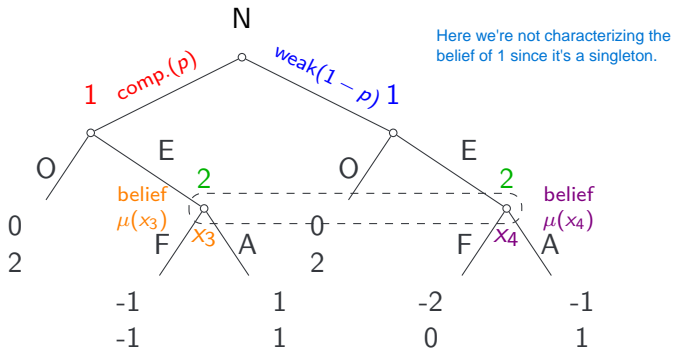


- In our entry game, the system of beliefs of player 1 is sure, while that of player 2 depends on the types of player 1 (specifically, on its prior of 1 being competitive or weak)

- We define the following requirements for sequential rationality in Bayesian games:
 - 1 Players must have a system of beliefs
 - 2 On the equilibrium path they must follow Bayes' rule on conditional probabilities
 - 3 Off the equilibrium path: arbitrary
 - 4 Given their beliefs, players are sequentially rational: i.e., they play a best response to their belief
- **Definition:** A **perfect Bayesian equilibrium** (PBE) is a pair (s^*, μ) , where s^* is a Bayesian Nash equilibrium and μ is a system of beliefs satisfying 1–4.

PBE in Bayesian entry game 2.0

- Always remember that a PBE is not just a pair of strategies: there must be an associated system of beliefs μ



$$\mu(x_3) = \Pr\{\text{player 1 is competitive} \mid E\}$$

$$1 - \mu(x_3) = \Pr\{\text{player 1 is weak} \mid E\}.$$

- A strategy pair must be sustained by a system of beliefs:
 $\mu(x_3)$ and $\mu(x_4) = 1 - \mu(x_3)$ for player 2
 - e.g., if 2 believes that 1 plays EO, then $\mu(x_3) = 1$ (in other words, if 1 enters, then 2 is fully convinced that 1 is competitive)
 - this reasoning can also be applied to mixed strategies
 - consider strategy $q_C q_W$, i.e.,
 - a competitive player 1 chooses E with probability q_C (and O with $1 - q_C$)
 - a weak player 1 chooses E with probability q_W (and O with $1 - q_W$)
 - In this case, the belief of x_3 given E is

$$\mu(x_3) = \frac{\Pr(\text{node})}{\Pr(\text{information set})} = \frac{p q_C}{p q_C + (1 - p) q_W}$$

$\mu(x_4) = (1 - p) q_W / \text{same denominator}$



- $s^* = (EO, A)$ and μ form a PBE:
 - 2 believes that only “competitive” 1 chooses to enter, so $\mu(x_3) = 1$
 - 2 playing A is a sequentially-rational response to 2's belief
- $s^* = (OO, F)$ cannot form a PBE with any system of beliefs μ :
 - Bayes' rule cannot be applied since playing OO means $q_C = q_W = 0$

$$\mu(x_3) = \frac{p q_C}{p q_C + (1 - p) q_W} = \frac{0}{0}$$

- x_3 and x_4 are off-path in this case, so the beliefs are arbitrary
- However, F is irrational in both x_3 and x_4 (A is always better for 2) and it must be either $\mu(x_3) > 0$ or $\mu(x_4) > 0$
- That means requirement 4 is violated \rightarrow not a PBE

- Perfect Bayesian NE: (EO,A)
 - sustained by system of belief $\mu(x_1) = 1$
 - all players play in a sequentially-rational way
- Imperfect Bayesian NE: (OO,F)
 - Bayes' rule cannot be applied: $q_C = q_W = 0$
 - Whatever choice of $\mu(x_1), \mu(x_2)$ makes the choice of F irrational

- What does it mean for a node x_i to be on the Bayesian equilibrium path, given BNE s^* ?
- What elements do you need to characterize a PBE?
- How do you determine sustainable belief values $\mu(x_i)$ for nodes that are on the Bayesian equilibrium path?
- What values can $\mu(x_i)$ have if x_i is off the equilibrium path?

Signaling games

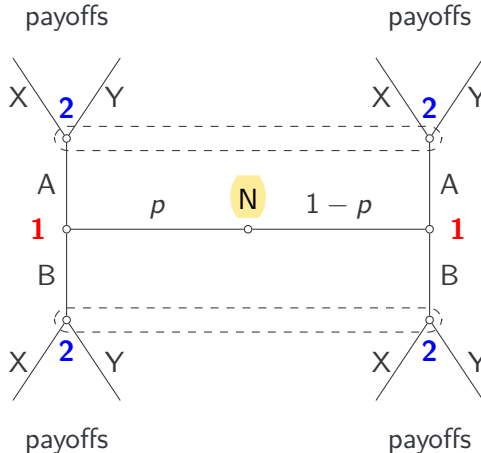


- We saw 2 Bayesian versions of the entry game where **1**=outsider/entrant and **2**=incumbent
- This can be generalized as follows:
 - **2** has multiple types, **1** has only one type: **1** moves before **2**, without any hint about **2**'s type besides the prior ϕ
→ This is called a **screening game**, SPE is enough
 - **1** has multiple types, **2** has only one type: **2**'s first move may give a hint (*signal*) about **2**'s type
→ This is called a **signaling game**, and requires PBE to achieve sequential rationality

- A signaling game is a 2-player dynamic Bayesian game: **1** (first to move) and **2** (second to move)
 - **1**'s type is chosen among many possible types (by Nature)
 - **2** has only one type
 - **2**'s beliefs are updated after **1**'s move

- Binary case is often shown as a “butterfly”

Preferred representation
of signaling games



These equilibria are based on observation of a move and then update the beliefs

- **Separating equilibria:** each type of 1 chooses a different action; thus revealing the type to 2
- **Pooling equilibria:** all types of 1 choose the same action; thus, 2 gets no signal about 1's type
- **Intermediate cases:** 1's action does not fully define 1's type, but still provides some information
 - Beliefs are updated according to Bayes' rule
 - This type of equilibria is also called “semi-separating” or “partially pooling”

Example: a coffee for Brooke

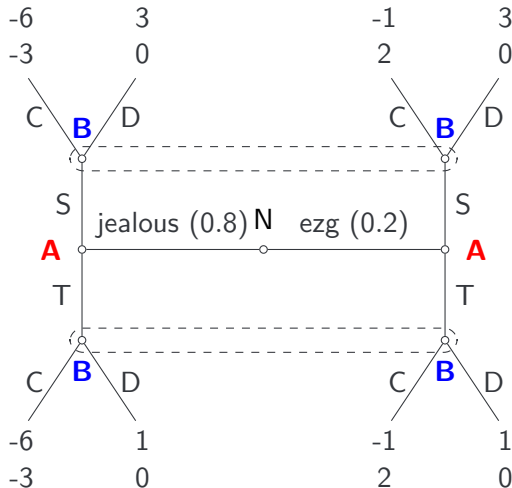
- Ann and Brooke are dating; Brooke is invited by a colleague, Zoe, to get a coffee
- Ann is a typed player: her types are
 - Jealous with probability 0.8
 - Easygoing with probability 0.2(all this information is common knowledge)
- Ann can send a signal to either stay silent (S) about this proposal or to trash Zoe out (T)
- Brooke observes the signal, and decides whether to accept the coffee invitation (C) or to politely decline (D)

Example: a coffee for Brooke

- Payoffs:
 - Jealous Ann is deeply hurt if Brooke accepts ($u_A = -6$)
 - Easygoing Ann is just not-so-angry, but still not fond of the idea ($u_A = -1$)
 - Ann prefers to stay silent ($u_A = 3$) rather than trash Zoe out ($u_A = 1$), only in case Brook declines
 - Brooke likes to go to the coffee if that is okay for Ann ($u_B = 2$)
 - If Ann is hurt, Brook prefers declining the invitation ($u_B = 0$) rather than accepting it ($u_B = -3$)

Example: a coffee for Brooke

■ Extensive form



Example: a coffee for Brooke

- Both players have 4 strategies but for different reasons
 - Ann because of her type: strategy is (what to do if jealous, what to do if easygoing)
 - Brooke does not have a type but observes Ann's move: strategy is (what to do if Ann plays S, what to do if Ann plays T)
 - e.g., (TS,CD) means that Ann trashes Zoe if she is jealous and remains silent if she is easygoing (separating); Brooke just “follows the signal”, going to the coffee if Ann stays silent, and declining if Ann starts trashing Zoe

Example: a coffee for Brooke

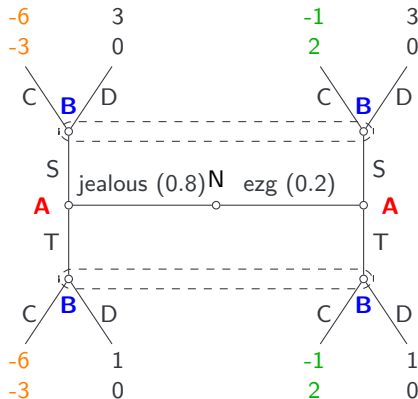
- **Warning!** A's pair is left/right, but B's pair is *B's reaction to A's move*
- First row (SS): only consider B's 1st move (reaction to S)
- Last row (CC): only consider B's 2nd move (reaction to T)

		Brooke			
		CC	CD	DC	DD
Ann	SS	B plays C	B plays C	B plays D	B plays D
	ST	B plays C			B plays D
	TS	B plays C	!!!	!!!	B plays D
	TT	B plays C	B plays D	B plays C	B plays D

Example: a coffee for Brooke

- If B plays C, utility is always

$$u_A = 0.8 \cdot (-6) + 0.2 \cdot (-1) = -5, \quad u_B = 0.8 \cdot (-3) + 0.2 \cdot (2) = -2$$



Example: a coffee for Brooke

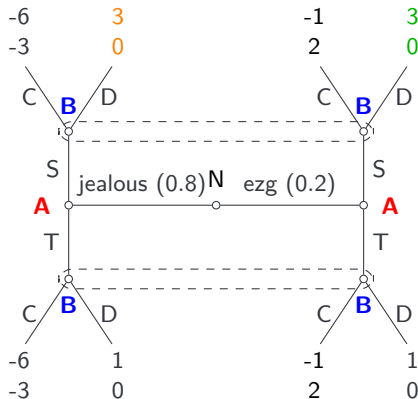
- When B plays D, we need to distinguish between Ann's 4 possible moves (her payoff changes, B's is always 0)

		Brooke			
		CC	CD	DC	DD
Ann	SS	-5, -2	-5, -2	B plays D	B plays D
	ST	-5, -2			B plays D
	TS	-5, -2			B plays D
	TT	-5, -2	B plays D	-5, -2	B plays D

Example: a coffee for Brooke

- If B plays D and A plays S, i.e., (SS, D*)

$$u_A = 0.8 \cdot (3) + 0.2 \cdot (3) = 3$$



Example: a coffee for Brooke

- Likewise, if B plays D and A plays T, i.e., (TT,*D), then $u_A = 1$

		Brooke			
		CC	CD	DC	DD
Ann	SS	-5, -2	-5, -2	3, 0	3, 0
	ST	-5, -2			B plays D
	TS	-5, -2			B plays D
	TT	-5, -2	1, 0	-5, 2	1, 0

- What about intermediate cases (ST,DD) and (TS, DD)?

Example: a coffee for Brooke

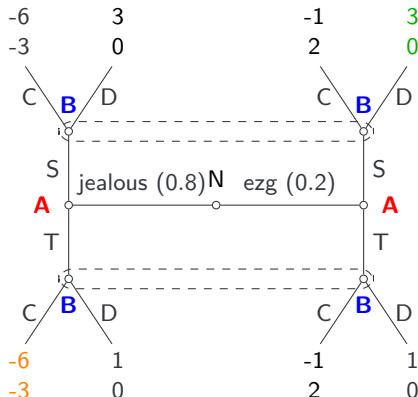
- For (ST,DD) and (TS, DD), you just average the payoffs: in the first case S is played with probability 0.8 and T with probability 0.2; the second case is the opposite

		Brooke			
		CC	CD	DC	DD
Ann	SS	-5, -2	-5, -2	3, 0	3, 0
	ST	-5, -2	?, ?	?, ?	2.6, 0
	TS	-5, -2	?, ?	?, ?	1.4, 0
	TT	-5, -2	1, 0	-5, 2	1, 0

Example: a coffee for Brooke

- E.g., for (TS,DC) (remember: D is answer to S and C is answer to T)

$$u_A = 0.8 \cdot (-6) + 0.2 \cdot (3) = -4.2$$



Example: a coffee for Brooke

		Brooke			
		CC	CD	DC	DD
Ann	SS	-5, -2	-5, -2	3, 0	3, 0
	ST	-5, -2	-4.6, -2.4	2.2, 0.4	2.6, 0
	TS	-5, -2	0.6, 1.6	-4.2, -2.4	1.4, 0
	TT	-5, -2	1, 0	-5, 2	1, 0

- 3 pure NE: (SS,DC), (SS,DD), (TT,CD)
- 2 mixed NE: (TT, $1/2\text{CD} + 1/2\text{DD}$),
($1/6\text{SS} + 5/6\text{TS}$, $2/9\text{CD} + 7/9\text{DD}$)



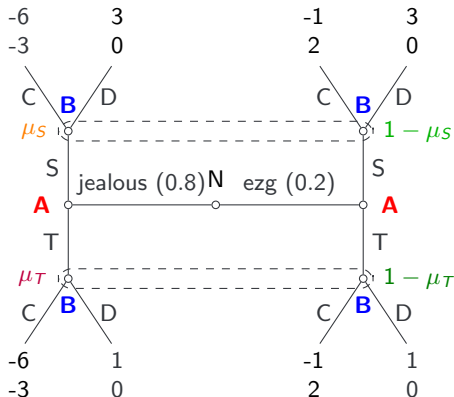
Example: a coffee for Brooke

- So far, we have only found NE, **now we need to classify them!** Are they PBE?
- To verify that, we need to construct systems of beliefs μ for Brooke
 - i.e., $\mu =$ is Brooke's belief that Ann is *jealous*
 - One belief for each possible observed move by Ann: μ_S if she stays silent; μ_T if she trashes Zoe out

Example: a coffee for Brooke

■ Beliefs are easy to compute for separating strategies like ST

- *Note:* We do not need to do that in this exercise, it is just an example

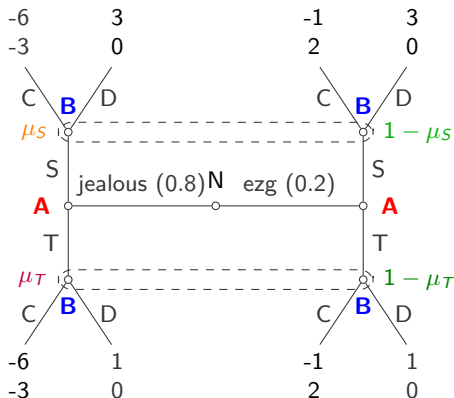


Brooke believes
 jealous Ann stays
 silent and easygo-
 ing Ann trashes
 Zoe out: $\mu_S = 1$
 (100% chance Ann
 is jealous if S is
 observed); $\mu_T = 0$
 (0% chance Ann
 is jealous if T is
 observed)

Example: a coffee for Brooke

- Unfortunately, in this game we only have pooling equilibria and intermediate cases

- E.g., consider pooling strategy SS



Brooke believes jealous Ann stays silent regardless of whether she is jealous easygoing: $\mu_S = 0.8$ (same as the prior); What about μ_T ?

Example: a coffee for Brooke

- We know that to sustain a PBE with pooling strategy SS, μ_S must stay 0.8
- Off the Bayesian equilibrium path, beliefs are arbitrary. However, they should still satisfy sequential rationality!
- E.g., to make $((SS, DC), (\mu_S, \mu_T))$ as PBE, C must be a best response to T for Brooke
 - That happens if

$$\mu_T u_B(C|J) + (1 - \mu_T) u_B(C|E) \geq \mu_T u_B(D|J) + (1 - \mu_T) u_B(D|E)$$

$$\mu_T(-3) + (1 - \mu_T)(2) \geq 0$$

- Meaning that any $\mu_T \leq 2/5$ is sufficient to sustain a PBE with NE (SS,DC)
- Conversely, any $\mu_T \geq 2/5$ sustains a PBE with NE (SS,DD)

Example: a coffee for Brooke

- Summary so far:
- NE1: $((SS, DC), (\mu_S, \mu_T))$ is a PBE for $(\mu_S = 0.8, \mu_T \leq 0.4)$
- NE2: $((SS, DD), (\mu_S, \mu_T))$ is a PBE for $(\mu_S = 0.8, \mu_T \geq 0.4)$
- NE3: $((TT, CD), (\mu_S, \mu_T))$ is a PBE for $(\mu_S \leq 0.4, \mu_T = 0.8)$
 - Analogous to NE1, same payoffs for Brooke
- NE4: $((TT, 1/2CD + 1/2DD), (\mu_S, \mu_T))$ is a PBE for $(\mu_S = 0.4, \mu_T = 0.8)$
 - Same as above, but this time Brooke should be indifferent between C and D against S
- NE5: $((1/6SS + 5/6TS, 2/9CD + 7/9DD), (\mu_S, \mu_T))$?

Example: a coffee for Brooke

- NE5: $((1/6SS+5/6TS, 2/9CD+7/9DD), (\mu_S, \mu_T))$
- This can be a semi-separating PBE
 - Ann is always silent if easygoing but may start badmouthing Zoe if she is jealous
 - This is because she believes that Brooke may sometimes choose C if she stays 100% silent (if she stays silent, B chooses C with probability $2/9$)
 - The description makes sense, but what about the system of beliefs? It is actually more complex and requires Bayes' rule to be used non-trivially

Example: a coffee for Brooke

- NE5: $((1/6SS+5/6TS, 2/9CD+7/9DD), (\mu_S, \mu_T))$
- Easy part: $\mu_T = 1 \rightarrow$ Brooke believes Ann chooses to trash Zoe out only if she is jealous; if she is easygoing, Ann always plays S
- Harder part: $\mu_S = ?$
- Depending on it, Brooke may prefer C or D. And to play a mixed strategy, Brooke must be indifferent between them (characterization theorem)
- We have already seen that this happens for $\mu_S = 0.4$

Example: a coffee for Brooke

- NE5: $((1/6SS+5/6TS, 2/9CD+7/9DD), (\mu_S, \mu_T))$
- Denote with q the probability that jealous Ann plays S (the probability that she plays T is $1 - q$)
- Remember:

$$\mu_S = \frac{\Pr[S, \text{jealous}]}{\Pr[S]} = \frac{pq}{pq + (1 - p) \cdot 1} = \frac{0.8 \cdot 1/6}{0.8 \cdot 1/6 + 0.2} = 0.4$$

- If we already know μ_S , we can use this formula to find q

Questions?