Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

Lecture 10
Exercise set #1

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- **Theorem**: Consider game $\mathbb{G} = (S_1, \ldots, S_n; u_1, \ldots, u_i, \ldots, u_n)$. If $p^* = (p_1^*, \ldots, p_n^*)$ is a NE for $\mathbb{G}$, then it is also a NE for $\mathbb{G}' = (S_1, \ldots, S_n; u_1, \ldots, u_i', \ldots, u_n)$, with

$$u_i'(s) = \alpha u_i(s) + c, \ \forall s \in S_1 \times \cdots \times S_n$$

and $a > 0$, $c \in \mathbb{R}$.

- In other words, you can apply any *positive* linear transformation (positive scaling + shift) to *all* payoffs of one or multiple players in the game, and preserve the NE.

Invariance of NE to linear transf.

- *Proof*: Suppose p^* is a NE for $\mathbb{G}$. Then, the following must hold for player i:

$$u_i(p_i^*, p_{-i}^*) \geq u_i(p_i, p_{-i}^*), \;\; \forall p_i \forall p_{-i}^*$$

- We can expand both sides into linear combinations

$$\sum_{s_i \in S_i} p_i^*(s_i) u_i(s_i, p_{-i}^*) \geq \sum_{s_i \in S_i} p_i(s_i) u_i(s_i, p_{-i}^*)$$

- We multiply both sides by $\alpha > 0$ and add $c \in \mathbb{R}$

$$\alpha \left(\sum_{s_i \in S_i} p_i^*(s_i) u_i(s_i, p_{-i}^*) \right) + c \geq \alpha \left(\sum_{s_i \in S_i} p_i(s_i) u_i(s_i, p_{-i}^*) \right) + c$$

Invariance of NE to linear transf.

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- Finally, we apply the associative property

$$\sum_{s_i \in S_i} p_i^*(s_i)(\underbrace{\alpha u_i(s_i, p_{-i}^*) + c}_{u_i'(s_i, p^*)}) \geq \sum_{s_i \in S_i} p_i(s_i)(\underbrace{\alpha u_i(s_i, p_{-i}^*) + c}_{u_i'(s_i, p^*)})$$

- Q.E.D. $\square$
- Actually, to have a "complete" proof, one should also expand p_{-i}^* so as to get $u_i'(s)$ in the sum; however, the procedure is still the same, since it is still a linear combination of values and associative property can be applied

Exercise 1a

- Find all NE (pure and mixed)

A \ B	M	N
F	2, 4	0, 1
G	1, 6	3, 5

Exercise 1a (sol.)

- N is strictly dominated by M ($4 > 1$; $6 > 5$)

		B	
		M	N
A	F	2, 4	0, 1
	G	1, 6	3, 5

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- Now G is strictly dominated by F

A \ B	M	N
F	2, 4	0, 1
G	1, 6	3, 5

- (F, M) is the only NE

Exercise 1b

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- Find all NE (pure and mixed)

		B: M	B: N
A	F	0, 4	3, 0
A	G	6, 0	0, 5

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- No NE in pure strategies (Nash theorem: there must be one in mixed strategies)

B

A		M	N
	F	0, 4	3, 0
	G	6, 0	0, 5

- It must be that $\mathrm{supp}(\mathbf{p}_A) = \{\mathrm{F}, \mathrm{G}\}$ and $\mathrm{supp}(\mathbf{p}_B) = \{\mathrm{M}, \mathrm{N}\}$
- $\mathbf{p}_A = (q, 1-q)$, $\mathbf{p}_B = (r, 1-r)$

Exercise 1b (sol.)

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- A plays F with probability q and G with $1 - q$
- B plays M with probability r and N with $1 - r$
- Apply the characterization theorem:

$$\begin{cases} \underbrace{4q}_{u_B(q,M)} & = \underbrace{5(1-q)}_{u_B(q,N)} \\ \underbrace{3(1-r)}_{u_A(F,r)} & = \underbrace{6r}_{u_A(G,r)} \end{cases}$$

- Mixed NE: $\mathbf{p}_A = (5/9, 4/9)$, $\mathbf{p}_B = (1/3, 2/3)$

Exercise 1c

- Find all NE (pure and mixed)

		B: M	B: N
A	F	9, 3	2, 2
A	G	0, 0	3, 9

Exercise 1c (sol.)

- Battle-of-the-sexes-like game
- Two "opposite" NE in pure strategies
- There must be one "in the middle"

		B: M	B: N
A	F	9, 3	2, 2
A	G	0, 0	3, 9

Exercise 1c (sol.)

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- A plays F with probability q and G with $1 - q$
- B plays M with probability r and N with $1 - r$
- Apply the characterization theorem:

$$\begin{cases} 3q = 2q + 9(1-q) \\ 9r + 2(1-r) = 3(1-r) \end{cases}$$

- Mixed NE: $\mathbf{p}_A = (9/10, 1/10)$, $\mathbf{p}_B = (1/10, 9/10)$

Exercise 1d

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- Find all NE (pure and mixed)

A \ B	M	N
F	2, 2	0, 6
G	6, 0	1, 1

Exercise 1d (sol.)

- Prisoner's-dilemma-like game

A \ B	M	N
F	2, 2	0, 6
G	6, 0	1, 1

- Only NE is (G, N)
- If we try to find a mixed NE, we get

$$2q = 6q + (1-q) \Rightarrow q = -1/5 \Rightarrow \text{Impossible!}$$

Exercise 2

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		B: J	K	L	M
A	X	6, 7	5, 5	3, 8	8, 1
A	Y	4, 9	9, 2	0, 4	2, 3
A	Z	8, 4	2, 8	4, 2	3, 6

1. Show that there is no NE in pure strategies
2. Show that $(\mathbf{p}_A, \mathbf{p}_B)$ with $\mathbf{p}_A = (2/3, 0, 1/3)$ and $\mathbf{p}_B = (5/11, 4/11, 2/11, 0)$ is a mixed NE
3. List all joint pure strategies that are Pareto optimal (i.e., Pareto efficient)

Exercise 2 (sol.)

1. Show that there is no NE in pure strategies

A \ B	J	K	L	M
X	6, 7	5, 5	3, 8	8, 1
Y	4, 9	9, 2	0, 4	2, 3
Z	8, 4	2, 8	4, 2	3, 6

Exercise 2 (sol.)

2. Show that $(\mathbf{p}_A, \mathbf{p}_B)$ with $\mathbf{p}_A = (2/3, 0, 1/3)$ and $\mathbf{p}_B = (5/11, 4/11, 2/11, 0)$ is a mixed NE

- Use characterization theorem: for each player i

$$u_i(s_i, \mathbf{p}_{-i}) = u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \text{ for each } s_i \in \mathrm{supp}(\mathbf{p}_i)$$
$$u_i(s_i, \mathbf{p}_{-i}) \leq u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \text{ for each } s_i \notin \mathrm{supp}(\mathbf{p}_i)$$

Exercise 2 (sol.)

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- For player A

$$u_A(X, \mathbf{p}_B) = 6 \cdot \frac{5}{11} + 5 \cdot \frac{4}{11} + 3 \cdot \frac{2}{11} = \frac{56}{11}$$
$$u_A(Y, \mathbf{p}_B) = 4 \cdot \frac{5}{11} + 9 \cdot \frac{4}{11} + 0 \cdot \frac{2}{11} = \frac{56}{11}$$
$$u_A(Z, \mathbf{p}_B) = 8 \cdot \frac{5}{11} + 2 \cdot \frac{4}{11} + 4 \cdot \frac{2}{11} = \frac{56}{11}$$

- All pure strategies yield equal payoff, so $\mathbf{p}_B$ is a sustainable NE strategy

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- For player B

$$
\begin{aligned}
u_B(\mathbf{p}_A, J) &= 7 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = \frac{18}{3} = 6 \\
u_B(\mathbf{p}_A, K) &= 5 \cdot \frac{2}{3} + 8 \cdot \frac{1}{3} = \frac{18}{3} = 6 \\
u_B(\mathbf{p}_A, L) &= 8 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{18}{3} = 6 \\
u_B(\mathbf{p}_A, M) &= 1 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = \frac{8}{3} \leq 6
\end{aligned}
$$

- M does not belong to the support and yields lower payoff, so $\mathbf{p}_A$ is a sustainable NE strategy

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3. List all joint pure strategies that are Pareto optimal (i.e., Pareto efficient)

- Easier to delete Pareto dominated strategies

A \ B	J	K	L	M
X	6, 7	~~5, 5~~	~~3, 8~~	~~8, 1~~
Y	4, 9	9, 2	~~0, 4~~	~~2, 3~~
Z	8, 4	~~2, 8~~	~~4, 2~~	~~3, 6~~

- Remaining ones are Pareto optimal: (X, J), (Y, J), (Z, J), (Y, K).

Exercise 3

- Two firms (F1 and F2) work on a joint project from the European Commission. They can allocate an integer number of employees on the project, from 0 to infinity. They decide independently and without consulting with each other. The outcome of the project is that, if the number of employees allocated by each firm is identical (even zero!), both firms get a funding of 290 k€ from the European Commission. If the two firms assign a different number of employees, the European Commission gives them different fundings: 700 k€ to the one with more employees, and 320 k€ to the one with fewer employees. However, assigning employees costs 200 k€ per employee. The *utility* of a firm is funding minus costs.
  1. Show that no rational firm will allocate more than 2 employees
  2. Draw the normal form of this game and find its pure NE
  3. Find the additional NE in mixed strategies.

Exercise 3 (sol.)

- Let us start by understanding the problem
- Consider 1k€= 1 payoff unit
- The firm that allocates most employees receives more money (700) but needs to pay more (-200 payoff for each employee); the other firm receives 320 and also needs to pay the employees
- Worst case scenario is when they choose the number: they only receive 290 and still need to pay their employees
- Examples:
  - if F1 allocates 2 employees and F2 allocates 1, they get $700 - 400 = 300$ and $320 - 200 = 120$, respectively
  - if they both allocate 1 employee, they get $290 - 200 = 90$ each

1. Show that no rational firm will allocate more than 2 employees

- By choosing $n_i = 0$, the firm can secure a payoff of at least 290
  - $u_i(0,0) = 290$; $u_i(0, n_{-i}) = 320$ for $n_{-i} > 0$
- with $n_i \geq 3$, a firm can get at most $u(3, n_{-i}) = 700 - n_i \cdot 200 < 290$ when $n_{-i} < n_i$
- So all $n_i \geq 3$ are strictly dominated strategies (a rational firm would never choose them)
- Notice that 2 is not strictly dominated, since a firm may play it and get $700 - 400 = 300 > 290$, in case the other chooses 1 or 0 employees

Exercise 3 (sol.)

2. Draw the normal form of this game and find its pure NE

- The set of players is $\{\text{F1}, \text{F2}\}$
- The set of strategies for both is $\mathcal{S}_1 = \mathcal{S}_2 = \{0, 1, 2\}$

F1 \ F2	**0**	**1**	**2**
0	290, 290	320, 500	320, 300
1	500, 320	90, 90	120, 300
2	300, 320	300, 120	-110, -110

Exercise 3 (sol.)

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2. Draw the normal form of this game and find its pure NE

- The set of players is $\{F1, F2\}$
- The set of strategies for both is $\mathcal{S}_1 = \mathcal{S}_2 = \{0, 1, 2\}$

F1 \ F2	**0**	**1**	**2**
0	290, 290	320, 500	320, 300
1	500, 320	90, 90	120, 300
2	300, 320	300, 120	-110, -110

Exercise 3 (sol.)

- Even though $n_i = 2$ is not dominated by 0 or 1 separately, it is strictly dominated by a combination of them
- i.e., "choose 0 with probability 0.95 and 1 with probability 0.05", which gives utility $29 \cdot 9.5 + 50 \cdot 0.5 = 300.5 > 300$ against 0 and $32 \cdot 9.5 + 9 \cdot 0.5 = 309.5 > 300$ and some positive value > -110 against 2.

		F2		
		0	**1**	**2**
F1	**0**	290, 290	320, 500	320, 300
	1	500, 320	90, 90	120, 300
	2	300, 320	300, 120	-110, -110

Exercise 3 (sol.)

- Strictly dominated strategies cannot be part of NE (pure or mixed)

F2

F1		0	1
	0	290, 290	320, 500
	1	500, 320	90, 90

- (0, 1) and (1, 0) are NE
- There must be a mixed NE as well

Exercise 3 (sol.)

3. Find the additional NE in mixed strategies.

- Symmetric payoffs: $q = r$
- Characterization theorem:
  $290q + 320(1-q) = 500q + 90(1-q) \Rightarrow q = r = 23/44$
- Mixed: NE $\mathbf{p}_1 = \mathbf{p}_2 = (23/44, 21/44)$

Alternative sol.

- Q: "What if I do not realize that strategy 2 is dominated?"
- Pure NE can be found via best-response search
- To find mixed NE, you just apply the characterization theorem for mixed strategy $\mathbf{p}_i = (q_0, q_1, q_2)$, i.e., $u_i(0, \mathbf{p}_i) = u_i(1, \mathbf{p}_i) = u_i(2, \mathbf{p}_i)$

$$\begin{cases} q_0 u_i(0,0) + q_1 u_i(0,1) + q_2 u_i(0,2) = q_0 u_i(1,0) + q_1 u_i(1,1) + q_2 u_i(1,2) \\ q_0 u_i(0,0) + q_1 u_i(0,1) + q_2 u_i(0,2) = q_0 u_i(2,0) + q_1 u_i(2,1) + q_2 u_i(2,2) \\ q_0 + q_1 + q_2 = 1 \end{cases}$$

- You get $q_2 < 0$ which is impossible $\Rightarrow$ Strategy 2 does not belong to the support!

Exercise 4

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- A strategic interaction takes place between the taxpayer T and the tax inspector I. T is supposed to pay a share S of his income so as to have a net income equal to R after paying taxes. However, T is considering two alternatives: hide part of his income (**H**) so as to pay $S - L$ instead of S (thus getting a net income of $R + L$), or pay all due taxes in full (**P**). Player I also has two options: check T for tax fraud (**C**) or not (**N**). Performing a check has a cost equal to E. If the inspector finds out that T has hidden part of the income, then the taxpayer will have to pay all due taxes plus a fine of F, and an equal amount is collected by the inspector. The probability of being caught after a tax inspection is p. Formalize this conflict in the form of a static game of complete information and find the Nash equilibria.

Exercise 4 (sol.)

- Normal form representation

I

T		C	N
	H	$R+(1-p)L-pF$, $S-(1-p)L-E+pF$	$R+L$, $S-L$
	P	R, $S-E$	R, S

- To simplify the game we can apply the invariance to linear transformations and: subtract R from I's payoffs; subtract S from T's payoffs
- Remember: All NE are preserved

		I: C	I: N
T	H	$(1-p)L - pF$, $-(1-p)L - E + pF$	L, $-L$
T	P	0, $-E$	0, 0

Exercise 4 (sol.)

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- Since all quantities E, F, L are > 0 we know that:
  - if I plays N, T's best response is H ($L > 0$)
  - If T plays P, I's best response is N ($-E < 0$)

T	I: C	I: N
H	$(1-p)L - pF$, $-(1-p)L - E + pF$	L, $-L$
P	0, $-E$	0, 0

- What if T plays H, or I plays C? It depends

Exercise 4 (sol.)

- If $-(1-p)L - E + pF < -L$, i.e., $E > p(F+L)$

		I: C	I: N
T	H	$(1-p)L - pF$, $-(1-p)L - E + pF$	L, $-L$
	P	0, $-E$	0, 0

- (H, N) is the only NE
- C is strictly dominated by N, so there cannot be NE involving C
- Interpretation: too much effort required to perform the check

Exercise 4 (sol.)

- If $-(1-p)L - E + pF > -L$, i.e., $E < p(F+L)$; and $(1-p)L - pF > 0$, i.e., $p < L/(L+F)$

I

T	C	N
H	$(1-p)L - pF$, $-E + pF - (1-p)L$	L, $-L$
P	0, $-E$	0, 0

- Only NE is (H, C): T has low probability of being caught, so he always decides to hide his income; however, checking does not require much effort, so it for the inspector

Exercise 4 (sol.)

- If $-(1-p)L - E + pF < -L$, i.e., $E < p(F+L)$; and $(1-p)L - pF < 0$, i.e., $p > L/(L+F)$

		I: C	I: N
T	H	$(1-p)L - pF$, $-E + pF - (1-p)L$	L, $-L$
T	P	0, $-E$	0, 0

- No NE in pure strategies: there must be a mixed one.

Exercise 4 (sol.)

- Mixed NE is $(\mathbf{p}_T, \mathbf{p}_I$, with $\mathbf{p}_T = (q^*, 1 - q^*)$ and $\mathbf{p}_I = (r^*, 1 - r^*)$
- The values of q^* and r^* can be found using the characterization theorem

$$r(1-p)L - rpF + (1-r)L = 0 \Rightarrow r^* = \frac{L}{p(L+F)}$$

$$-qE + qpF - q(1-p)L - (1-q)E = -qL \Rightarrow q^* = \frac{E}{p(L+F)}$$

- Interpretation: q^* could be a reasonable estimate for the prob. of T committing tax fraud: proportional to I's effort E; inv. proportional to $L + F$ and to the prob. of being caught p
- r^* is the fraction of taxpayers that the inspector should check; this is proportional L

Questions?