COMPUTABILITY (14/11/2023)

OBSERVATION: A function which is total and not computable

$$f(x) = \begin{cases} \boxed{\varphi_x(x) + 1} & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \psi_U(x,x) + 1 & \boxed{\begin{array}{l}\text{if } \varphi_x(x)\downarrow \\ \text{otherwise}\end{array}} \\ 0 & \end{cases}$$

HALTING PROBLEM : Show that the predicate below is UNDECIDABLE

$$Halt(x) = \begin{cases} \text{true} & \text{if } \varphi_x(x)\downarrow \quad (\text{i.e. } x \in W_x) \\ \text{false} & \text{if } \varphi_x(x)\uparrow \quad (\text{i.e. } x \notin W_x) \end{cases}$$

idea: by contradiction: we show that assuming $Halt(x)$ decidable we can prove f computable

$$f(x) = \begin{cases} \varphi_x(x)+1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \psi_U(x,x)+1 & \text{if } Halt(x) \\ 0 & \text{otherwise} \end{cases}$$

$$\cancel{\ } \quad (\psi_U(x,x)+1) \cdot \chi_{Halt}(x) \quad - \quad \begin{cases} 1 & \text{if } Halt(x) \\ 0 & \text{otherwise} \end{cases}$$

$\uparrow$

when $\varphi_x(x)\uparrow$ then $\psi_U(x,x)+1\uparrow$ $\Rightarrow$ the expression is $\uparrow$

instead

$$f(x) \overset{|||}{=} \Big(\mu(t,y,z) . \big(S(x,x,y,t) \wedge z=y+1 \wedge Halt(x)\big) \vee \big(z=0 \wedge \neg Halt(x)\big)\Big)_z \overset{|||}{}$$

$$= \Big(\mu\,\omega . \big(S(x,x,(\omega)_2,(\omega)_1) \wedge (\omega)_3 = (\omega)_2+1 \wedge Halt(x)\big) \vee \big((\omega)_3 = 0 \wedge \neg Halt(x)\big)\Big)_3$$

$(\omega)_1 = t$

$(\omega)_2 = y$

$(\omega)_3 = z$

if you call

$$Q(x,\omega) \equiv (S(x, x, (\omega)_2, (\omega)_1) \wedge (\omega)_3 = (\omega)_2 + 1 \wedge Halt(x)) \vee$$
$$((\omega)_3 = 0 \wedge \neg Halt(x))$$

decidable

$$= (\mu\omega . \ |\chi_Q(x,\omega) - 1|)_3$$

computable as it arises as minimalisation of computable functions

$\Rightarrow$ contradiction

$\Rightarrow$ $Halt(x)$ not decidable

$\square$

<u>EXERCISE</u>: Let $Q(x)$ decidable predicate

$f_1, f_2 : \mathbb{N} \to \mathbb{N}$ computable

and define

$$f(x) = \begin{cases} f_1(x) & \text{if } Q(x) \\ f_2(x) & \text{if } \neg Q(x) \end{cases} \qquad \text{computable}$$

<u>proof</u>

If f_1, f_2 total

$$f(x) = f_1(x) \cdot \chi_Q(x) + f_2(x) \cdot \chi_{\neg Q}(x)$$

if $Q(x)$	1	0
if $\neg Q(x)$	0	1

$\Rightarrow$ f computable

In general, let $e_1, e_2 \in \mathbb{N}$ s.t. $\varphi_{e_1} = f_1$ and $\varphi_{e_2} = f_2$

$$f(x) = \Big(\mu(t,y). \ (S(e_1, x, y, t) \wedge Q(x)) \vee (S(e_2, x, y, t) \wedge \neg Q(x))\Big)_{\to y}$$

$$= \Big(\mu \omega . \; \underbrace{\big(S(e_1, x, (\omega)_2, (\omega)_1) \wedge Q(x) \big) \vee \big(S(e_2, x, (\omega)_2, (\omega)_1) \wedge \neg Q(x) \big)}_{\text{decidable}} \Big)_2$$

$(\omega)_1 = t$

$(\omega)_2 = y$

$\Rightarrow$ f is computable $\square$

EXERCISE :

- $f(x) = \begin{cases} 0 & \varphi_x(x)\uparrow \\ \uparrow & \varphi_x(x)\downarrow \end{cases}$ not computable

- if $Halt(x)$ is decidable then f is computable

EXERCISE : TOTALITY

$Tot(x) \equiv$ " φ_x is total " $\equiv$ " φ_x is terminating on every input "

is undecidable

In fact

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } Tot(x) \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ f is total $\begin{cases} Tot(x) & \Rightarrow f(x) = \varphi_x(x) + 1 \downarrow \\ \neg Tot(x) & \Rightarrow f(x) = 0 \end{cases}$

$\rightarrow$ f is different from all total computable functions

(if φ_x is total $\Rightarrow$ $f(x) = \varphi_x(x) + 1 \neq \varphi_x(x)$)

$\Downarrow$ f not computable

If we assume that $Tot(x)$ is decidable we derive f computable

$\rightsquigarrow$ contradiction

In fact

$$f(x) = \begin{cases} f_1(x) & \text{if } Tot(x) \\ f_2(x) & \text{if } \neg Tot(x) \end{cases}$$

where $f_1, f_2 : \mathbb{N} \to \mathbb{N}$

$$f_1(x) = \varphi_x(x) + 1 = \Psi_U(x,x) + 1 \quad \forall x$$

$$f_2(x) = 0 \quad \forall x$$

computable

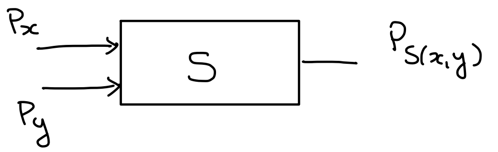
$\Rightarrow$ by the previous exercise f computable, absurd.

$\Rightarrow$ $Tot(x)$ not decidable.

* EFFECTIVE OPERATIONS ON COMPUTABLE FUNCTIONS

① there exists a total computable function $s: \mathbb{N}^2 \to \mathbb{N}$

$$\forall x,y \qquad \varphi_{s(x,y)}(z) = \varphi_x(z) * \varphi_y(z) \qquad \forall z$$



def $P_{s(x,y)}(z)$:
    $v_x = P_x(z)$
    $v_y = P_y(z)$
    return $v_x * v_y$

define $g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x,y,z) = \varphi_x(z) * \varphi_y(z)$$

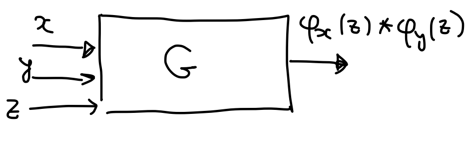
$$= \Psi_U(x,z) * \Psi_U(y,z)$$

g is computable (composition of computable functions)

Hence by (corollary of) smn theorem there is $s: \mathbb{N}^2 \to \mathbb{N}$ total computable such that

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \varphi_x(z) * \varphi_y(z)$$

(



function s takes G and x,y and "hard code" the value of x,y into G and gives back the resulting program

)

EXERCISE : Effectiveness of inverting a function

There is a total computable function $K : \mathbb{N} \to \mathbb{N}$

s.t. $\forall x$ if φ_x is injective then $\varphi_{K(x)} = (\varphi_x)^{-1}$

$$P_x \longrightarrow \boxed{K} \longrightarrow P_{K(x)}$$

$(\varphi_x)^{-1}(y)$

$z=0 \quad P_x(0) \overset{?}{=} y$

$z=1 \quad P_x(1) = y$

$\vdots$

define

$$g(x,y) = (\varphi_x)^{-1}(y) = \begin{cases} z & \text{s.t. } \varphi_x(z) = y \quad \text{if it exists} \\ \uparrow & \text{otherwise} \end{cases}$$

(if φ_x is injective)

$$= \big(\mu(z,t).\ S(x,z,y,t)\big)_z$$

$$= \big(\mu w.\ S(x,(w)_1,y,(w)_2)\big)_1$$

$$= \big(\mu w.\ |\chi_S(x,(w)_1,y,(w)_2) - 1|\big)_1$$

computable

Hence, by smn theorem, there is $K : \mathbb{N} \to \mathbb{N}$ total computable

s.t.

$$\varphi_{K(x)}(y) = g(x,y) = (\varphi_x)^{-1}(y) \qquad \text{if } \varphi_x \text{ injective}$$

What do we get when φ_x is not injective?

$\varphi_{K(x)}(y)$ is one of the counter images of y

QUESTION: Given $f : \mathbb{N} \to \mathbb{N}$ computable. Define

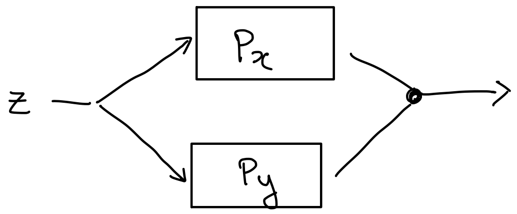
$$g(y) = \begin{cases} \min\{x \mid f(x) = y\} & \text{if } \exists x. \text{ s.t. } f(x) = y \\ \uparrow & \text{otherwise} \end{cases}$$

Is g computable?

EXERCISE There is a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that

$$W_{s(x,y)} = W_x \cup W_y$$

$$\varphi_{s(x,y)}(z)\downarrow \quad \text{iff} \quad \varphi_x(z)\downarrow \quad \text{or} \quad \varphi_y(z)\downarrow$$



$g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x,y,z) = \begin{cases} \downarrow \ 1 & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}\left(\mu t . \ H(x,z,t) \vee H(y,z,t)\right)$$

where $\mathbb{1}(x) = 1 \quad \forall x$

g is computable and thus by smn theorem $\exists\, s : \mathbb{N}^2 \to \mathbb{N}$ total computable s.t.

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \begin{cases} 1 & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

s is the desired function

$z \in W_{s(x,y)}$ iff $\underset{g(x,y,z)}{\underset{\parallel}{\varphi_{s(x,y)}(z)}}\downarrow$ iff $\varphi_x(z)\downarrow$ or $\varphi_y(z)\downarrow$

iff $z \in W_x$ or $z \in W_y$

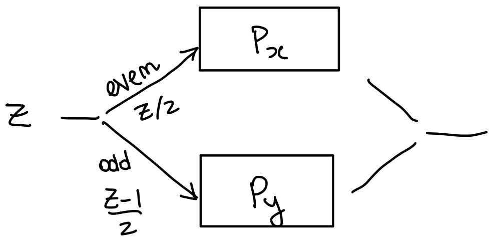
iff $z \in W_x \cup W_y$

EXERCISE There is a total computable function $s: \mathbb{N}^2 \to \mathbb{N}$ such that

$$E_{s(x,y)} = E_x \cup E_y$$

($P_{s(x,y)}$ produces as outputs all values produced by P_x and P_y)

	0	1	2	3	...
P_x	0	3	4	$\uparrow$	...
P_y	1	5	$\uparrow$	2	



$$g(x,y,z) = \begin{cases} \varphi_x\left(z/2\right) & z \text{ even} \\ \varphi_y\left(\frac{z-1}{2}\right) & z \text{ odd} \end{cases}$$

$$\cancel{=} \cancel{\varphi_U(x, qt(2,z)) * \overline{sg}(rm(2,z)) +}$$
$$\cancel{\varphi_U(y, qt(2,z)) * rm(2,z)}$$

$$= \left(\mu(n,t).\ (S(x, z/2, n, t) \wedge z \text{ even}) \vee (S(y, \tfrac{z-1}{2}, n, t) \wedge z \text{ odd})\right)_n$$

$$= \Big(\mu w.\ \underbrace{(S(x, qt(2,z), (w)_1, (w)_2) \wedge z \text{ even}) \vee (S(y, qt(2,z), (w)_1, (w)_2) \wedge z \text{ odd})}_{\text{decidable}}\Big)_1$$

computable

By smn theorem $\exists s: \mathbb{N}^2 \to \mathbb{N}$ total computable s.t.

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \begin{cases} \varphi_x\left(z/2\right) & z \text{ even} \\ \varphi_y\left(\frac{z-1}{2}\right) & z \text{ odd} \end{cases}$$

I claim that s is the desired function, i.e. $E_{s(x,y)} = E_x \cup E_y$

$(\subseteq)$ $\quad v \in E_{S(x,y)}$

$\exists z.$ s.t. $\quad \varphi_{S(x,y)}(z) = v$

$$\varphi_{S(x,y)} = g(x,y,z)$$

hence two possibilities

- $v = \varphi_x\left(\frac{z}{2}\right) \quad \rightsquigarrow \quad v \in E_x$
- $v = \varphi_y\left(\frac{z-1}{2}\right) \quad \rightsquigarrow \quad v \in E_y$

$\Big\} \rightarrow v \in E_x \cup E_y$

$(\supseteq)$ $\quad v \in E_x \cup E_y \quad \rightsquigarrow \quad v \in E_{S(x,y)}$

i.e. ① $v \in E_x \quad \rightsquigarrow \quad v \in E_{S(x,y)}$

② $v \in E_y \quad \rightsquigarrow \quad v \in E_{S(x,y)}$

① $\quad v \in E_x \quad$ i.e. $\quad \exists z \quad$ s.t. $\quad \varphi_x(z) = v$

therefore $\quad \varphi_{S(x,y)}(\underset{\uparrow \text{even}}{2z}) = \varphi_x\left(\frac{\cancel{2}z}{\cancel{2}}\right) = \varphi_x(z) = v$

$\rightsquigarrow \quad v \in E_{S(x,y)}$

② identical

$\square$

* <u>EXERCISE</u> : variant of URM machine

URM^P

$Z(n)$
$\cancel{S(n)}$ $\quad P(n) \quad r_n \leftarrow r_n \dot{-} 1$
$T(m,n)$
$J(m,n,t)$

$\mathcal{C}_P$ $\quad URM^P$-computable function

$$\mathcal{C}_P \overset{?}{=} \mathcal{C}$$

* <u>EXERCISE</u> : Are there $f, g : \mathbb{N} \to \mathbb{N}$ functions s.t.

① f computable, g not computable $\quad f \circ g$ computable

② f not computable, g not computable $\quad f \circ g$ computable