Università
degli Studi
di Padova

Dipartimento
di Ingegneria
dell'Informazione

Lecture 03
Static games of complete information

Thomas Marchioro

October 11, 2023

Previously on this course

- In game theory, we model problems that involve multiple agents interacting with each other
- Problems studied in game theory are called "games" and the involved agents are called "players"
- players are assumed to be *rational*, meaning that they have a rational set of preferences about the game's outcome and they act according to those preferences
- Preferences can be expressed numerically by using utility functions

Previously on this course

DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

- Random outcomes are modeled as lotteries
- Lotteries are simply probability distributions describing random outcomes
- The concept of preference can be extended to lotteries
- If preferences satisfy a set of axioms (rationality, continuity, independence) we can evaluate the utility of a lottery as the expected value of the utility, averaged over the possible outcomes

Static games of complete information

Game with multiple players

- How do multiple players interact?
  - We assume they have a payoff (utility) function
- Remember: rational players move to maximize of their own payoffs
- What is the simplest interaction like this?

Static games of complete information

DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE

- **Static**: all players move together; they do not necessarily play simultaneously, but without knowledge of everyone else's move
- **Complete information**: meaning everyone's payoff function is known
    - most games covered within this class are "artificial" (theoretical models)
    - however, there are also actual games that can be modeled as static games of complete information. **Examples?**

Static games of complete information

- Each player i in the game simultaneously and independently chooses an action from its own set of available actions A_i
- The combination of actions chosen by the n players determines the outcome of the game
- Outcome $(a_1, a_2, \ldots, a_n)$ determines a payoff for each player through an individual utility function of player i:

$$\boxed{u_i = u_i(a_1, a_2, \ldots, a_n)}$$

- 3 ingredients $=$ actions $+$ outcome $+$ utility

Action versus strategy

- In decision problems we always thought in terms of actions
- In games, it is useful to think in terms of **strategies** instead
- A strategy is a **plan of action**
  - e.g. if these conditions are met, then my action is a, otherwise it is either a' or a''
  - this plan can even be random (we will see why)
- For the time being, let's consider only **deterministic** plans
- These are called **pure strategies**

Normal form of a game

- Each player i simultaneously chooses a strategy from a set of pure strategies S_i
- This results in a given action chosen by each of the n players that ultimately determines a payoff for each player
- If any player i plays strategy $s_i \in S_i$, the combination of moves is $(s_1, s_2, \ldots, s_i, \ldots, s_n)$
- Player i gets payoff $u_i(s_1, s_2, \ldots, s_i, \ldots, s_n) \in \mathbb{R}$
- The **normal form** of the game is specified by $\mathbb{G} = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$

Simultaneous and independent

- Simultaneous moves do not necessarily need to happen at the same time
  - they are chosen without knowledge of everyone else's actions
- These two cases are both considered simultaneous:
  - case A: two players are writing their strategy on opposite sides of a board at the same time
  - case B: player 1 is asked to write first; while player 1 writes, player 2 is blindfolded; then the board is turned and player 2 writes

Common knowledge

- We say that E is common knowledge if:
  - everyone knows E
  - everyone knows that everyone knows E
  - ...
- Common knowledge in games is a powerful assumption, and not an obvious one
  - it requires each player to have full knowledge not only on information pertaining themselves, but also on everyone else

Common knowledge

- "Complete information" in games means that the following information is common knowledge:
  - all possible actions of all players
  - all possible outcomes resulting from these actions
  - the individual preferences of all players about these outcomes (i.e., their utilities about them)
- Player rationality is also common knowledge
  - meaning that everyone is maximizing their own payoff and everyone knows that everyone is maximizing their payoff

Array representation

DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

- An n-player game can be represented as a function in $S_1 \times S_2 \times \cdots \times S_n$ that maps tuples of strategies to tuples of payoffs
- If all $S_i, i = 1, \ldots, n$ are discrete sets, we can represent a game using an n-dimensional array that contains a tuple of n values (the payoffs) in each cell/entry
  - The overall number of cells is $m_1 \times \cdots \times |S_n|$
- Typically we consider games with $n = 2$ players, which can be represented using an ordinary matrix containing a pair of values in each cell
  - This is called a **bi-matrix**

Example of bi-matrix

DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

- $|S_1| = m_1, |S_2| = m_2$

Player 2

Player 1

	$s_2^{(1)}$	$s_2^{(2)}$	$\cdots$	$s_2^{(m_2)}$
$s_1^{(1)}$	$u_1(s_1^{(1)}, s_2^{(1)}),$ $u_2(s_1^{(1)}, s_2^{(1)})$	$u_1(s_1^{(1)}, s_2^{(2)}),$ $u_2(s_1^{(1)}, s_2^{(2)})$	$\cdots$	$u_1(s_1^{(1)}, s_2^{(m_2)}),$ $u_2(s_1^{(1)}, s_2^{(m_2)})$
$s_1^{(2)}$	$u_1(s_1^{(2)}, s_2^{(1)}),$ $u_2(s_1^{(2)}, s_2^{(1)})$	$u_1(s_1^{(2)}, s_2^{(2)}),$ $u_2(s_1^{(2)}, s_2^{(2)})$	$\cdots$	$u_1(s_1^{(2)}, s_2^{(m_2)}),$ $u_2(s_1^{(2)}, s_2^{(m_2)})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$s_1^{(m_1)}$	$u_1(s_1^{(m_1)}, s_2^{(1)}),$ $u_2(s_1^{(m_1)}, s_2^{(1)})$	$u_1(s_1^{(m_1)}, s_2^{(2)}),$ $u_2(s_1^{(m_1)}, s_2^{(2)})$	$\cdots$	$u_1(s_1^{(m_1)}, s_2^{(m_2)}),$ $u_2(s_1^{(m_1)}, s_2^{(m_2)})$

Example 1

- Player A has strategies $S_A = \{\text{U, M, D}\}$
- Player B has strategies $S_B = \{\text{L, R}\}$

		Player B	
		L	R
Player A	U	8, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

Example 2

- Player A has strategies $S_A = \{\text{U, M, D}\}$
- Player B has strategies $S_B = \{\text{L, C, R}\}$

Player A	Player B	L	C	R
	U	0, 5	4, 0	7, 3
	M	4, 0	0, 5	7, 3
	D	3, 7	3, 7	9, 9

Example: Odds and evens

- Player Odd and Even bet 4 euros
- Player Odd has two strategies: $\{0, 1\}$
- Plater Even has two strategies: $\{0, 1\}$

		Even: 0	Even: 1
Odd	0	-4, 4	4, -4
	1	4, -4	-4, 4

Example: Rock-paper-scissors

- Both players have strategies $S_A = S_B = \{R, P, S\}$

Player A \ Player B	R	P	S
R	0, 0	-4, 4	4, -4
P	4, -4	0, 0	-4, 4
S	-4, 4	4, -4	0, 0

In this game we also have the possibility to draw
=> that will be bring to a 0 utility for both

Example: Battle of the Sexes

- Two college students, A and B, need to decide which night event to attend: rock concert (R) or science night (S).
- A prefers the concert, while B prefers the science night. However, both prefer to spend time with each other rather then separately
- They have not exchanged contact yet, so they are taking their decision independently and without communicating

		B: R	B: S
A	R	2, 1	0, 0
A	S	0, 0	1, 2

Example: Prisoner's dilemma

- Simplified version: each player chooses between two options $S_A = S_B = \{\text{M, F}\}$
  - M: Lose 1 euro
  - F: The other player loses 20 euros

		Player B	
		M	F
Player A	M	-1, -1	-21, 0
	F	0, -21	-20, -20

Example: Prisoner's dilemma

DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE

- Original version: the players are two criminals caught by the police. The police has evidence only for petty theft but not for a major crime. The two are arrested and interrogated in separate room. They can decide to either
  - Keep mum (M), i.e., to not talk
  - Fink (F), i.e., to snitch on their partner
- Payoffs represent the years of jail they get

Player B

Player A		M	F
	M	-1, -1	-9, 0
	F	0, -9	-6, -6

Pareto efficiency

DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE

- A joint strategy s is **Pareto-dominated** by another strategy s' if

$$u_i(s') \geq u_i(s) \text{ for each player } i$$
$$u_i(s') > u_i(s) \text{ for some player } i$$

- A joint strategy s that is not Pareto-dominated by any other joint strategy s' is called **Pareto-efficient**
- There may be more than one Pareto-efficient strategy, none of which dominates the others

Strict dominance

Strictly dominated strategy

DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

- Consider game $\mathbb{G} = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$
- If $s_i, s_i' \in S_i$, we say that s_i is *strictly dominated* by s_i' if $i's$ payoff when playing s_i' is always greater than when playing s_i for any possible choice of moves by the other players
- Formally

$$\boxed{u_i(s_1, \ldots, s_i', \ldots, s_n) > u_i(s_1, \ldots, s_i, \ldots, s_n)}$$

$$\forall (s_1, \ldots, s_{i-1}, \ldots, s_{i+1}, \ldots, s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \ldots S_n$$

- Rational players do not play strictly dominated strategies

Back to example 1

DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE

- Can you find any strictly dominated strategy?

Player B

Player A		L	R
	U	8, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

- Strategy D is dominated by M for player A
  - $u_A(M, L) = 1 > u_A(D, L) = 0$
  - $u_A(M, R) = 4 > u_A(D, R) = 2$

Player B

Player A		L	R
	U	8, 0	0, 5
	M	1, 0	4, 3
	D	0, 7	2, 0

Back to example 1

- Now strategy L is dominated by R for player B
  - $u_B(U, R) = 5 > u_B(U, L) = 0$
  - $u_B(U, R) = 3 > u_B(D, R) = 0$

Player B

Player A	L	R
U	8, 0	0, 5
M	1, 0	4, 3

Back to example 1

- Now strategy U is strictly dominated by M for A
  - $u_A(M, R) = 4 > u_A(U, R) = 0$

Player B

Player A	R
U	0, 5
M	4, 3

- Rational players end up playing (M, R) with payoffs (4, 3)

Back to the prisoner's dilemma

- Are there any strictly dominated strategies?

		Player B: M	Player B: F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

Back to the prisoner’s dilemma

- For both players, M is strictly dominated by F
  - $u_A(F, M) = 0 > u_A(M, M) = -1$
  - $u_A(F, F) = -6 > u_A(F, M) = -9$

		Player B	
		M	F
Player A	M	-1, -1	-9, 0
	F	0, -9	-6, -6

- Rational players end up playing (F, F)
- This is not a Pareto-efficient joint strategy: (F, F) is Pareto-dominated by (M, M)
- The final outcome is “bad”, hence the dilemma

Solving problems via IESDS

- This procedure is called "iterated elimination of strictly dominated strategies" (IESDS)
- Sometimes, it allows to obtain a reduced version of a game by relying on common knowledge
  - All players know that a certain strategy is dominated for one player, so they rule it out
- Unfortunately, in many cases it does not lead to a solution for the game

Example 2

- In this game there is no strictly dominated strategy
- However, (D, R) seems to be a good choice for the players

		Player B		
		L	C	R
Player A	U	0, 5	4, 0	7, 3
	M	4, 0	0, 5	7, 3
	D	3, 7	3, 7	9, 9

Back to Odds and evens

- No strategy seems to be better than the other

		Even 0	Even 1
Odd	0	-4, 4	4, -4
Odd	1	4, -4	-4, 4

Example: Battle of the Sexes

- There are two joint strategies that are "good" for rational players: (R, R) and (S, S)

		B: R	B: S
A	R	2, 1	0, 0
A	S	0, 0	1, 2

Questions?