Università degli Studi di Padova

Dipartimento di Ingegneria dell'Informazione

Lecture 12
Nash equilibria in dynamic games

Thomas Marchioro

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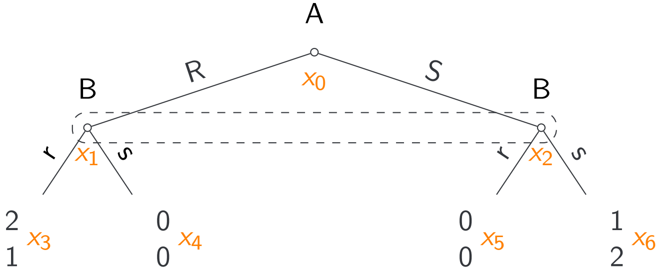
Previously on game theory

- **Dynamic game**: introduce a temporal dimension with subsequent moves
  - As opposed to **static games**, where players move at the same time
  - When unspecified, we always mean dynamic games **of complete information**
- Can be represented through the **extensive form** (whose graphical representation is a decision tree)
- Player's knowledge of their position in the game is represented by **information sets**
- A player cannot distinguish between nodes in the same information set

Previously on game theory

- **Imperfect information**: Simultaneous moves and/or random outcomes
- Consequence: players are sometimes uncertain about their position in the tree, some information sets include more than one node (circled together with a dashed line)



Previously on game theory

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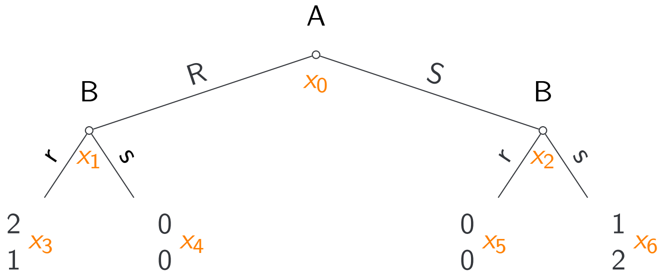
- **Perfect information**: No simultaneous moves or random outcomes
- Consequence: players are fully aware of their position in the tree, all information sets are singleton



Previously on game theory

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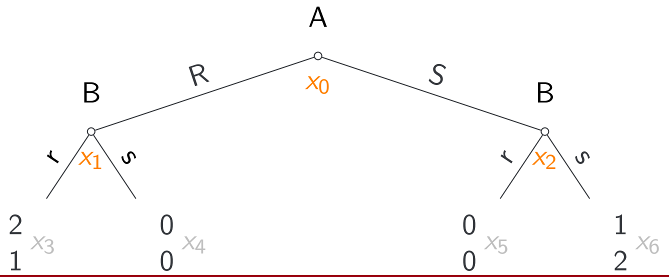
- **Strategies**: In dynamic games, a pure strategy for player i is an m_i-tuple of moves where m_i is the number of information sets where i plays (so we ignore leaves)
- A plays only at $h_A = \{x_0\} \Rightarrow$ A's strategy is just a single move (R or S)
- B plays at $h_{B,1} = \{x_1\}$ and $h_{B,2} = \{x_2\} \Rightarrow$ B's strategies are pairs (rr, rs, sr, ss)



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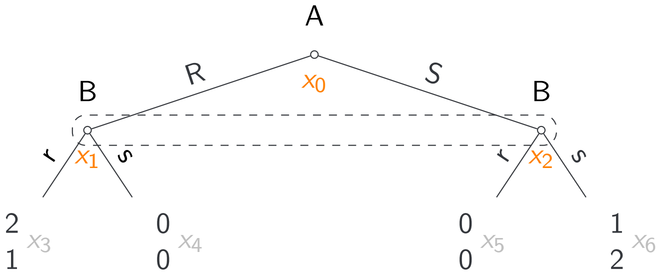
- Here, A plays at information set $h_A = \{x_0\}$, B plays at $h_B = \{x_1, x_2\}$ (cannot distinguish between the two nodes)
- Both players have only one move in their strategy (R or S for A; r or s for B)
- Matter of fact, this is the original (static) battle of the sexes



Mixed strategies in dynamic games

Re-defining mixed strategies

- <u>Previous definition</u>: mixed strategies are probability distributions over the strategy set (ΔS_i for player i)
- Now $S_i =$ all possible plans of action
  $\rightarrow p_i \in \Delta S_i = \{\text{prob(plan 1), prob(plan 2), } \dots\}$
- This does not look very "dynamic"
  - Probabilities are drawn at the beginning only: player randomly chooses a plan and sticks to it
  - Wouldn't it make more sense to draw probabilities as the game unfolds?

Behavioral strategies

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- Notation: $A_i(h_i)$ the set of available actions of player i when at information set h_i
- A **behavioral strategy** specifies for any information set $h_i \in H_i$ an independent probability distribution over $A_i(h_i)$
- Denote this as $\sigma_i : H_i \to \Delta A_i(h_i)$
  - Then $\sigma_i(a_i|h_i)$ is the probability that i plays action $a_i \in A_i(h_i)$ given that he/she is at information set h_i (i.e., he/she is at any node $x_j \in h_i$)
- In other terms a behavioral strategy specifies

$$\text{Prob}(\text{action}|\text{information set})$$

Mixed $\leftrightarrow$ behavioral

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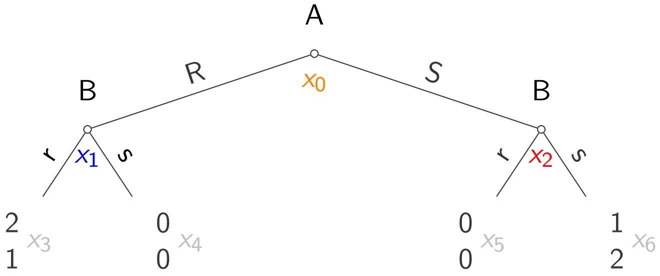
- Mixed strategy: select a random plan at the beginning and stick to it
  - E.g., in the sequential battle of sexes, B randomly selects a plan according to probabilities like $p_B(\text{rs})$
- Behavioral strategy: at each information set, select a random move from your set of available moves
  - E.g., in the sequential battle of sexes, B randomly selects an action according to probabilities like $\sigma_B(\text{r}|\{x_1\})$ and $\sigma_B(\text{s}|\{x_2\})$ which are conditioned on the information set
- Are these two descriptions equivalent? Luckily, yes (<u>under some mild conditions</u>)

- Given a mixed strategy p_B, one can always find an equivalent behavioral strategy σ_B using the law of total probability

$$\sigma_B(r|\{x_1\}) = p_B(rr) + p_B(rs)$$
$$\sigma_B(s|\{x_2\}) = p_B(rs) + p_B(ss)$$



Perfect recall

- The previous reasoning holds in general, so you can always find an equivalent behavioral strategy for each perfect strategy

$$\sigma_i(a_i|h_i) = \sum_{s_i \in S_i : s_i(h_i) = a_i} p_i(s_i)$$

- I.e., by summing the probabilities of a mixed strategy for all strategies s_i where action a_i is played at information set h_i
- And vice versa: $p_i(s_i) = \prod_{h_i} \sigma_i(a_i|h_i)$
- Condition required: perfect recall
- **Perfect recall**: players do not forget the information they have acquired
- Throughout the course, we always assume this condition to hold

Mixed $\leftrightarrow$ behavioral

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- **Takeaway**: When analyzing games, we always use mixed strategies and we ignore behavioral strategies
- However, we can interpret results in terms of behavioral strategies since the two are interchangeable

Extensive form $\leftrightarrow$ normal form

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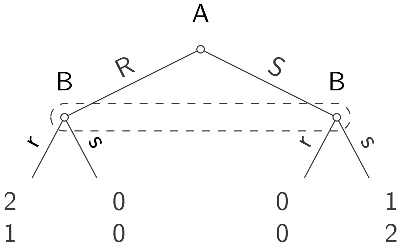
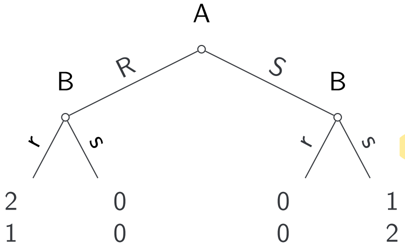
- We have already said that this extensive form represents the original (static) battle of the sexes



A \ B	r	s
R	2, 1	0, 0
S	0, 0	1, 2

- It is possible to switch from extensive to normal form (and vice versa) also for games that are actually dynamic



		B			
		rr	rs	sr	ss
A	R	2, 1	2, 1	0, 0	0, 0
	S	0, 0	1, 2	0, 0	1, 2

NE in dynamic games

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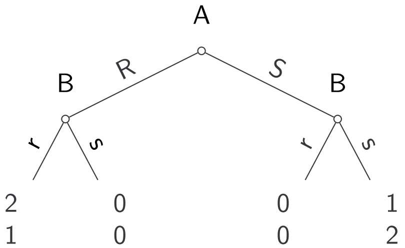
- Since dynamic game admit a normal form representation, we can find Nash equilibria



		B			
		rr	rs	sr	ss
A	R	2, 1	2, 1	0, 0	0, 0
	S	0, 0	1, 2	0, 0	1, 2

- Problem: the normal form representation is inherently static!
- Does it make sense for rational players to play NE in dynamic games?
- In the sequential version of the "battle of the sexes", we have 3 NE
  - (R, rr): A plays R, B always plays r
  - (R, rs): A plays R, B does what A does
  - (S, ss): A plays S, B always plays s
- Remember that these strategies are chosen by both players before the game starts

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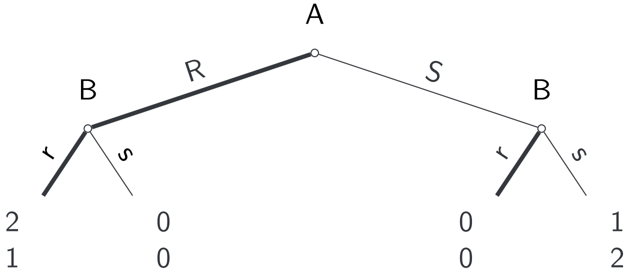
- Remember: NE mean absence of regrets; players would not like to unilaterally change their strategy in a NE

		B: rr	rs	sr	ss
A	R	2, 1	2, 1	0, 0	0, 0
	S	0, 0	1, 2	0, 0	1, 2

- "(R, rr): A plays R, B always plays r" is a NE
  - A changing to S would lead to (S, rr) with payoffs (0, 0)
  - B changing to rs would leave the payoffs unchanged; B changing to sr or ss would yield (0, 0)

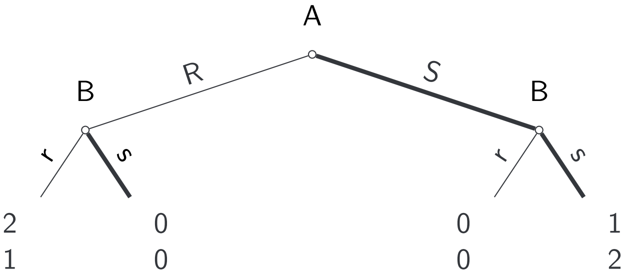
NE in dynamic games

- However, by playing rr, B's strategy is to choose r even if A chooses S
- That does not seem very rational

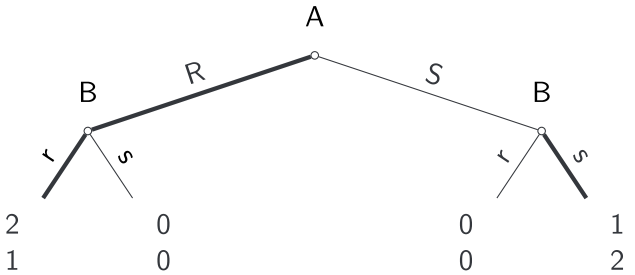


NE in dynamic games

- "(S, ss): A plays S, B always plays s" does not seem very rational either
- B plays ss, meaning that B goes to s even if A chooses R
- Also, A is not taking advantage of moving first



- “(R, rs): A plays R, B does what A does” seems to be the only really rational NE
- If A chooses R, B chooses r; if A chooses S, B chooses s
- Knowing that, A chooses R, maximizing his/her payoff



NE in dynamic games

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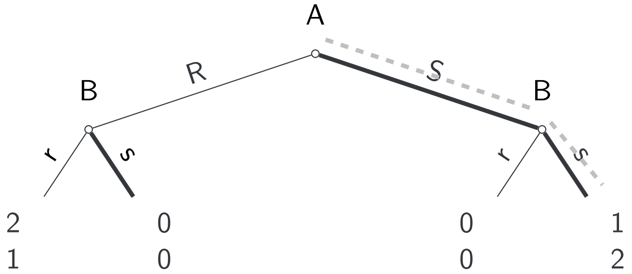
- The reason why some NE seem to be "irrational" is that NE are analyzing the game in a static way
- In dynamic games, NE do not guarantee that players always choose the best move at every stage
- We might need a new concept of equilibrium

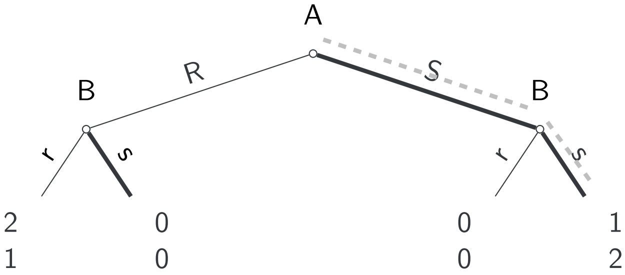
Equilibrium path

- Given a joint mixed strategy $p^* = (p_1^*, \ldots, p_n^*)$ that is a NE, its equilibrium path contains the decision tree nodes that are reached with probability > 0
- In dynamic games of perfect information, the equilibrium path of a pure NE is simply a tree-path
- E.g., for NE (S, ss), the equilibrium path is represented by the dashed lines below



Equilibrium path

- In (S, ss), the irrationality happens **off** the equilibrium path
- B playing ss means that, if A chooses, B “punishes” A by choosing S
- Is this threat credible? Rational players should always maximize their payoff



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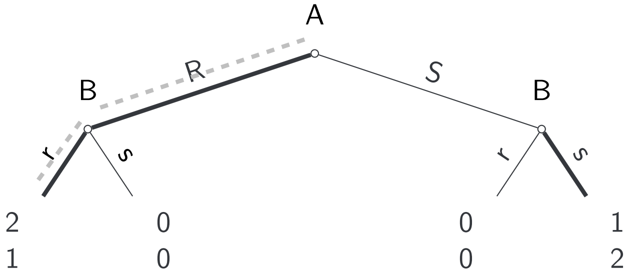
- "(R, rs): A plays R, B does what A does" seems to be a better predictor of what rational players would do
  - A knows that B's payoff is maximized when B copies A's move
  - Knowing that, A chooses R
- This NE is "more rational" than the other, since B plays a rational choice also off the equilibrium path



Solving games of perfect inf.

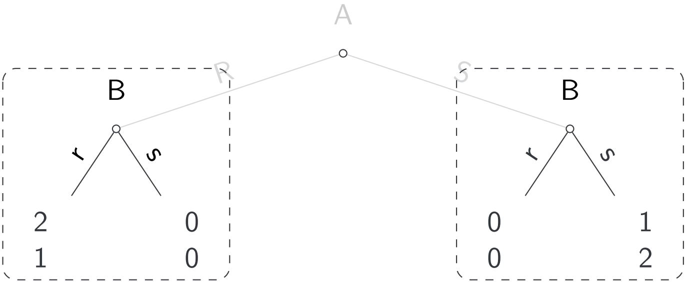
- A dynamic game of perfect information is a **sequential game** and can be represented using a regular decision tree (all information sets are singletons)
- Players move one after the other; players that move later have full information on previous players' choices and can exploit it
- Conversely, players that more first know the payoffs of next players and choose their move accordingly
- Sequential games can be solved starting from the last players' choices and proceeding backwards

Backward induction

- Sequential games can be "solved" via backward induction
- "Solving" a sequential game means finding an equilibrium path where players exploit the dynamic aspect of the game
- **Algorithm**:
  1. Start from the leaves of the tree
  2. Partition them according to their parent node (which is a move of some player i)
  3. For each set in the partition, find the leaf x_j with maximum payoff for player i
  4. Add the corresponding branch to player i's strategy
  5. Replace the parent node with the payoff of x_j
  6. Repeat until the root is reached

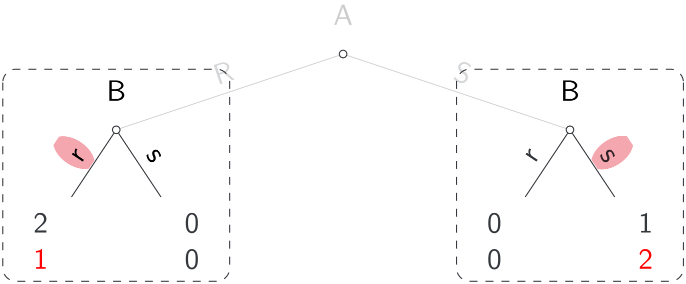
Backward induction

- $s_A = ?, s_B = ??$
  1. Start from the leaves
  2. Partition them according to their parent node (B's move)



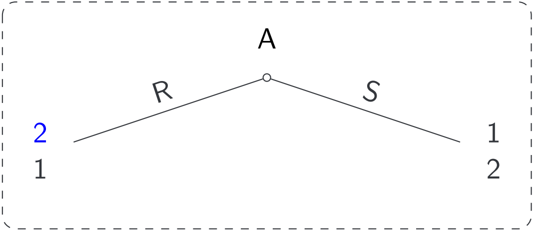
3. For each set in the partition, find the leaf with maximum payoff for B
4. Add the corresponding branch to player B's strategy

- $s_A = ?, s_B = $rs



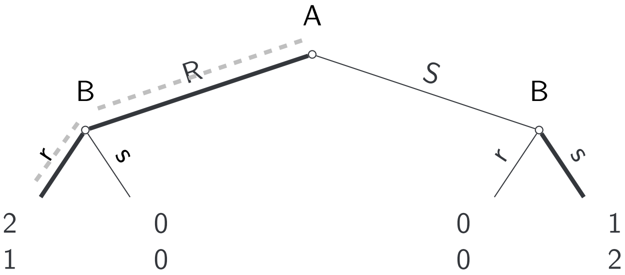
Backward induction

5. Replace the parent node with the payoffs of the chosen leaf
6. Repeat until the root is reached



- $\boxed{s_A = \mathrm{R}, s_B = \mathrm{rs}}$

- Using backward induction not only we find a "sequentially rational" equilibrium path
- We also find rational moves off the equilibrium path
- **Question**: Why is that important?



Subgame-perfect Nash equilibria

Subgames

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- The solution found via backward induction is sequentially rational and it is also rational off the equilibrium path
- These properties characterize a new type of equilibrium, called subgame-perfect Nash equilibrium
- In order to define it, we need to introduce the concept of "subgame"

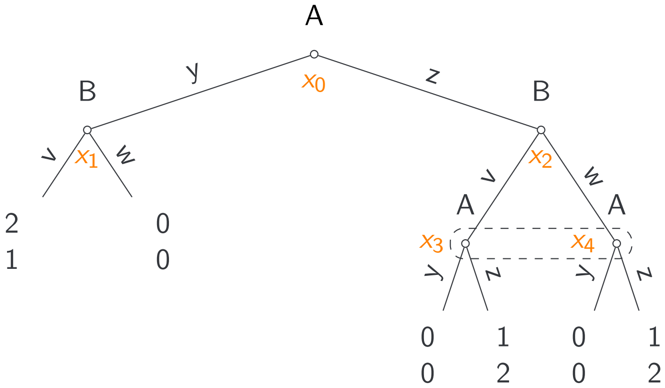
Subgames

- A (proper) **subgame** $\mathbb{G}'$ of a game $\mathbb{G}$ contains a single node of the tree and all of its descendants, with the requirement that

$$x_j \in \mathbb{G}', x_j \in h_i \Rightarrow x_k \in \mathbb{G}' \ \forall x_k \in \mathbb{G}'$$

- i.e., if a node is included in a proper subgame, all the nodes in the same information set must also be included
- A game $\mathbb{G}$ is always a proper subgame of itself
- The definition implies that for each singleton information set $\{x_j\}$, you have a proper subgame with root x_j

Subgames



- Subgames: trees with root x_0, x_1, x_2
  - Trees with root x_3, x_4 are *not* proper subgames

Subgame-perfect Nash equilibrium

- **Definition** (Subgame-perfect Nash equilibrium): A Nash equilibrium is **subgame-perfect** if the strategies chosen by the players yield a NE in every subgame
  - Subgame-perfect Nash equilibria (SPE) are a refinement of the definition of NE
  - In order to be a SPE, a joint strategy must be a NE and must also pass an additional test
- Every finite dynamic game has at least one SPE (could be more than one)
  - This means that every sequential game (tic-tac-toe, chess, go, etc.) has an optimal way to be played

Subgame-perfect Nash equilibrium

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- How to prove that a SPE is unique? For perfect information games with finite horizon (sequential games), *a SPE is the outcome of backward induction*
- If there is a unique outcome of backward induction, the SPE is unique
  - In the sequential battle of the sexes (R, rs) is the only SPE
  - (R, rr) and (S, ss) are NE but not SPE
- This can be somehow extended for other dynamic games by taking into account the **credibility of threats**
- We will see that in multistage games

Self-assessment

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- What is the difference between mixed and behavioral strategies? Under what condition are the two concepts equivalent?
- How do you find the solution(s) of a sequential game?
- What is a subgame-perfect NE? In which cases is a NE not subgame-perfect?

Exercise

- It is the discount sales season, and Lou (L) wants to go shopping. He thinks it is best to wait until the last three days of the discount sales, because prices are cheapest. On day 1, Lou asks Karen (K) to go with him. If Karen says yes (Y), they go shopping and the game ends. If Karen says no (N), Lou can either give up (G) or request (R) again the following day. On day 2, the same happens. On day 3, if K still says N, the game ends as well and does not go into a further day. If in the end K and L do not go shopping, both of their payoffs are 0. If they do, their payoffs are computed as $u_K = d$ and $u_L = 5 - 2d$, respectively, where d is the day in which they go. All of this information about the game is common knowledge among the players.

Exercise

1. Represent the game in its extensive form.
2. How many (pure) strategies do K and L have, respectively?
3. Represent the game in its normal form.
4. Find the SPE of this game.
5. Find *one* NE that is not subgame-perfect.
6. Find *all* the NE of this game (both SPE and non-SPE).

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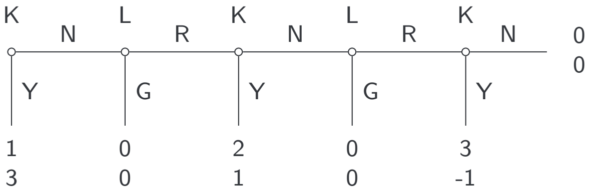
1. Extensive form:



Questions?