# COMPUTABILITY (04/12/2023)

## * Recursively enumerable sets and reducibility

Given $A, B \subseteq \mathbb{N}$ and $A \leq_m B$

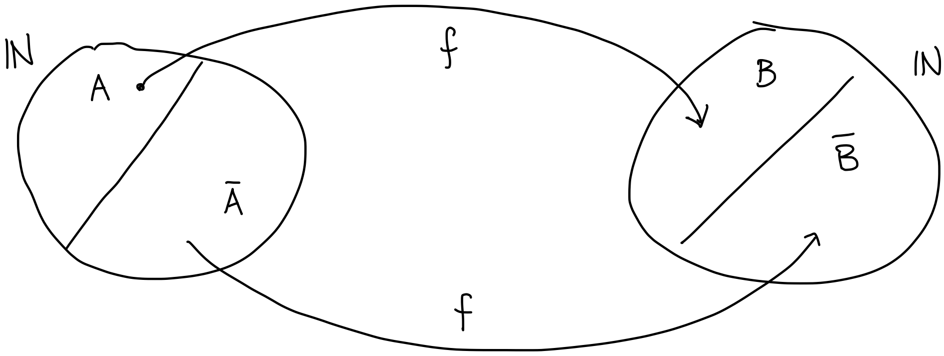
(1) if $B$ is r.e. then $A$ is r.e.

(2) if $A$ is not r.e. then $B$ is not r.e.

proof

let $A \leq_m B$ i.e. there is $f: \mathbb{N} \to \mathbb{N}$ total computable

$\forall x \quad x \in A \quad \text{iff} \quad f(x) \in B$




(1) let $B$ is r.e.

$$SC_B(x) = \begin{cases} 1 & \text{if } x \in B \\ \uparrow & \text{if } x \notin B \end{cases}$$ computable (our assumption)

then

$$SC_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{if } x \notin A \end{cases} = \underbrace{\underset{\text{computable}}{SC_B}\,(\underset{\text{computable}}{f(x)})}_{\text{composition computable}}$$

hence $SC_A$ computable

↳ $A$ is r.e.

(2) equivalent to (1)

$\square$

* <u>Why recursively enumerable?</u>

enumerable / countable $\qquad |A| \leq |\mathbb{N}|$

i.e. there is $\quad f : \mathbb{N} \to A \quad$ surjective

$$\underbrace{f(0) \quad f(1) \quad f(2) \quad f(3) \quad - - - -}_{\text{enumeration of } A}$$

recursively enumerable $\rightsquigarrow$ enumerable by a computable function

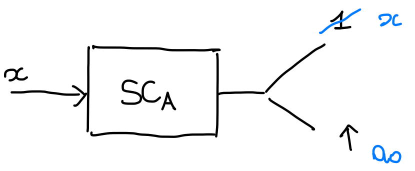
<u>Proposition</u>: Let $A \subseteq \mathbb{N}$ be a set

A r.e. iff $\left( A = \emptyset \quad \text{or} \quad \left( A = img(f) \quad \text{with } f : \mathbb{N} \to \mathbb{N} \text{ total computable} \right) \right)$

<u>proof</u>

($\Rightarrow$) let $A \subseteq \mathbb{N}$ be r.e., i.e.

$$SC_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases} \qquad \text{computable}$$




if $x \in A$

if $x \notin A$

$f(x) = x * SC_A(x) \qquad$ computable

$img(f) = \{ f(x) \mid x \in \mathbb{N} \} = A$

<u>NOT</u> total

Assume $A \neq \emptyset$, fix $a_0 \in A$

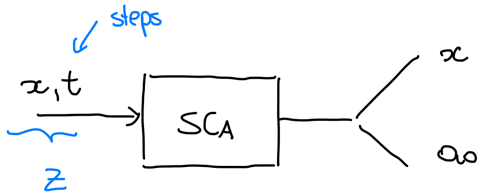
$$f(x) = \begin{cases} x & \text{if } x \in A \\ a_0 & \text{otherwise} \end{cases}$$

total

NOT COMPUTABLE

$img(f) = A$

We proceed as follows: fix $e \in \mathbb{N}$ s.t. $\varphi_e = SC_A$




if $\varphi_e(x)\downarrow$ in $t$ steps

otherwise

$(z)_1 = x$ $\quad (z)_2 = t$

$$f(z) = \begin{cases} (z)_1 & \text{if } H(e, (z)_1, (z)_2) \\ a_0 & \text{otherwise} \end{cases}$$

$$= (z)_1 \cdot \chi_H(e, (z)_1, (z)_2) + a_0 \cdot \chi_{\neg H}(e, (z)_1, (z)_2)$$

$f$ is

- computable
- total
- $\boxed{img(f) = A}$

$(\subseteq)$ let $x \in img(f) \overset{?}{\rightsquigarrow} x \in A$

$\downarrow$ there is $z$ s.t. $x = f(z)$, hence there are two possibilities

- $x = f(z) = (z)_1$ with $H(e, (z)_1, (z)_2)$

  hence $\varphi_e((z)_1)\downarrow$, thus $SC_A((z)_1)\downarrow 1$

  therefore $x = (z)_1 \in A$

- $x = f(z) = a_0 \in A$

$(\supseteq)$ let $x \in A \overset{?}{\rightsquigarrow} x \in img(f)$

$\downarrow$ $SC_A(x) = 1\downarrow$ and thus $\varphi_e(x)\downarrow$ for a suitable number of steps $t$

i.e. $H(e, x, t)$ is true

Therefore if we take $z \in \mathbb{N}$ st. $(z)_1 = x$, $(z)_2 = t$

(e.g. $z = 2^x \cdot 3^t \cdot \ldots$ )

$$f(z) = (z)_1 = x$$

thus $x \in img(f)$

$(\Leftarrow)$

- if $A = \emptyset$ then $A$ is r.e. ( since $\emptyset$ is finite hence recursive )

- if $A = img(f)$ $f$ total computable

  $x \in A$ iff there exists $z \in \mathbb{N}$ s.t. $f(z) = x$

then

$$SC_A(x) = \mathbb{1}\Big(\underbrace{\mu z.\ |f(z) - x|}_{\substack{1\ \cancel{\downarrow} \quad \text{if } x \in img(f) = A \\ \uparrow \quad \text{otherwise}}}\Big)$$

computable

$\Downarrow$

$A$ is r.e.

$\square$

<u>OBSERVATION</u>: let $A \subseteq \mathbb{N}$

$A$ is r.e. iff $A = dom(f)$ $f$ computable

(hence

$W_0, W_1, W_2, \text{------}$ enumeration of r.e. sets )

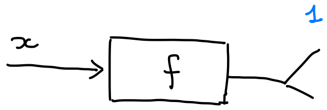
<u>proof</u>

$(\Rightarrow)$ let $A \subseteq \mathbb{N}$ be r.e., i.e.

$$SC_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases} \quad \text{computable}$$

then $A = dom(SC_A)$, as desired.

$(\Leftarrow)$ let $A = dom(f)$ with $f$ computable




$1$ $f(x) \in \mathbb{N}$ if $x \in A$

$\uparrow$ otherwise

$SC_A(x) = \mathbb{1}(f(x))$ computable

hence $A$ r.e.

$\square$

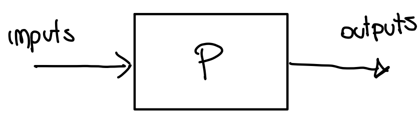
EXERCISE : Let $A \subseteq \mathbb{N}$

$A$ r.e. iff $A = img(f)$ $f$ computable

## * Rice - Shapiro's theorem

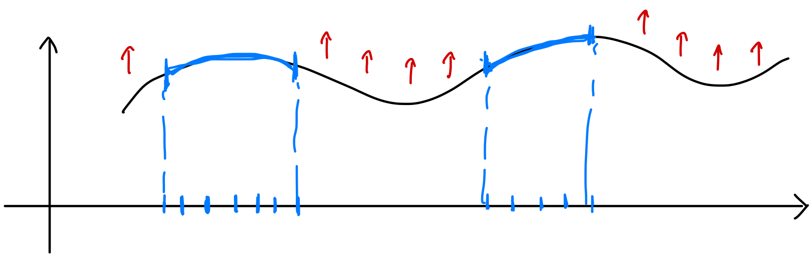
The only properties of the behaviour of programs which can be semi-decidable are the "finitary properties"

properties which depends on the behaviour on a finite number of inputs




Examples :

- the program P on input ∅ outputs value 1 — finitary
- program P is defined on at least two inputs — finitary
- program P is defined on every input — not finitary
- program P produces infinitely many values as outputs — not finitary
- the program P computes the factorial — not finitary



→ finite function

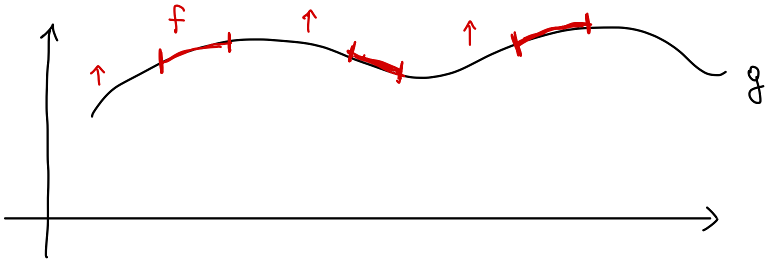
$\theta: \mathbb{N} \to \mathbb{N}$ is a finite function if $dom(\theta)$ finite

$$\theta(x) = \begin{cases} y_1 & \text{if } x = x_1 \\ y_2 & \text{if } x = x_2 \\ \vdots & \\ y_n & \text{if } x = x_n \\ \uparrow & \text{otherwise} \end{cases}$$

→ subfunction

we say that $f$ is a subfunction of $g$, written $f \subseteq g$,

if $\forall x$ if $f(x)\downarrow$ then $g(x)\downarrow$ and $f(x) = g(x)$




Theorem (Rice - Shapiro)

let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable functions.

and let $A = \{x \mid \varphi_x \in \mathcal{A}\}$

Then if $\boxed{A \text{ is r.e.}}$ then

$\Rightarrow$ $\not\Leftarrow$

$$\boxed{\forall f \quad ( f \in \mathcal{A} \quad \Leftrightarrow \quad \exists \vartheta \subseteq f, \vartheta \text{ finite s.t. } \vartheta \in \mathcal{A} )}$$

↑ property is finitary

proof (next lesson)

EXERCISE : let $f : \mathbb{N} \to \mathbb{N}$ be computable, let $g = f$ almost everywhere

(except for a finite set

$\{x \mid f(x) \neq g(x)\}$ finite

Then $g$ is computable.

proof

Assume $f$ computable

and $g(x) = f(x) \quad \forall x \neq x_0 \qquad f(x_0) \neq g(x_0)$

(1) if $g(x_0)\uparrow$ hence $f(x_0)\downarrow$

then $g(x) = f(x) + \underbrace{\mu w.\ \overline{sg}\, |x - x_0|}_{\begin{cases} 0 & \text{if } x \neq x_0 \\ \uparrow & \text{otherwise} \end{cases}}$ computable

(2) if $g(x_0) = y_0 \in \mathbb{N}$

let $e \in \mathbb{N}$ be such that $f = \varphi_e$

$$g(x) = \left( \mu w. \begin{array}{ll} ((S(e, x, (w)_1, (w)_2) & \wedge (x \neq x_0)) \\ ( ((w)_1 = y_0) & \wedge (x = x_0) ) \end{array} \right)_1$$

computable

An inductive reasoning allows to conclude in the general case.

Alternatively:

$D = \{ x \in \mathbb{N} \mid f(x) \neq g(x) \}$ finite

$$\vartheta(x) = \begin{cases} g(x) & \text{if } x \in D \\ \uparrow & \text{otherwise} \end{cases}$$ finite function $\rightsquigarrow$ computable

observe

$$g(x) = \begin{cases} f(x) & x \notin D \\ \vartheta(x) & x \in D \end{cases}$$

computable

decidable (D finite)

computable since it is defined by cases using a decidable predicate and computable function.

<u>Exercise</u>:

Define a total non-computable $f: \mathbb{N} \to \mathbb{N}$ such that

$$\text{img}(f) = \{ 2^n \mid n \in \mathbb{N} \}$$