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Lecture 19

Bayesian Nash equilibria

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- **Bayesian games:** games of *incomplete information*
 - Utility function of a player is determined by their type
 - Each player knows his/her own type, but only has an estimate of the opponents' type
 - This can be seen as Nature drawing a complete-information game among many possible games according to a distribution ϕ (which is common knowledge)
 - Players have only a partial observation of Nature's choice (their own type)

Today on game theory



Bayesian Nash equilibrium

- **Static Bayesian** games need:
 - set of players $\mathcal{N} = \{1, \dots, n\}$
 - action spaces A_1, \dots, A_n (pure strategy sets)
 - type spaces T_i (for $i = 1, \dots, n$)
 - beliefs (on types) ϕ_1, \dots, ϕ_n
 - type-dependent payoffs $u_i(a_1, \dots, a_n, t_i)$
- $\mathbb{G}(\mathcal{N}; A_1, \dots, A_n; T_1, \dots, T_n; \phi_1, \dots, \phi_n; u_1, \dots, u_n)$
 - where $u_i = u_i(a_1, \dots, a_n, t_i)$
- A **pure strategy** for i can be seen as a map $s_i : T_i \rightarrow A_i$, i.e., it tells what i plays as his/her type is known
- A **mixed strategy** for i is a probability distribution over pure strategies

- Type-contingent definition of pure/mixed strategies → similar to dynamic games
 - We can think of a general strategy as being defined before the type of i is even set
 - Player i decides a strategy $s_i : T_i \rightarrow A_i$
 - Then, if his/her type is $t_i \in T_i$, he/she will play $s_i(t_i)$
 - This is useful, since it allows **other players** to create beliefs over the strategy of a player who can be of different types

- A **Bayesian Nash equilibrium** is a Nash equilibrium in Bayesian games
- In $\mathbb{G}(\mathcal{N}; A_1, \dots, A_n; T_1, \dots, T_n; \phi_1, \dots, \phi_n; u_1, \dots, u_n)$, joint strategy $s^* = (s_1^*, \dots, s_n^*)$ is a **Bayesian Nash equilibrium** if, for each player i and type $t_i \in T_i$, s_i^* maximizes the expected payoff against s_{-i}^* :

$$s_i^* = \arg \max_{s_i \in S_i} \sum_{t_{-i}} u_i(s_i, s_{-i}^*(t_{-i}), t_i) \phi_i(t_{-i})$$



- This is the same as:

$$\mathbb{E}[u_i(s_i^*, s_{-i}^*(t_{-i}), t_i) | t_{-i}] \geq \mathbb{E}[u_i(s_i, s_{-i}^*(t_{-i}), t_i) | t_{-i}]$$

for every $s_i \in S_i$

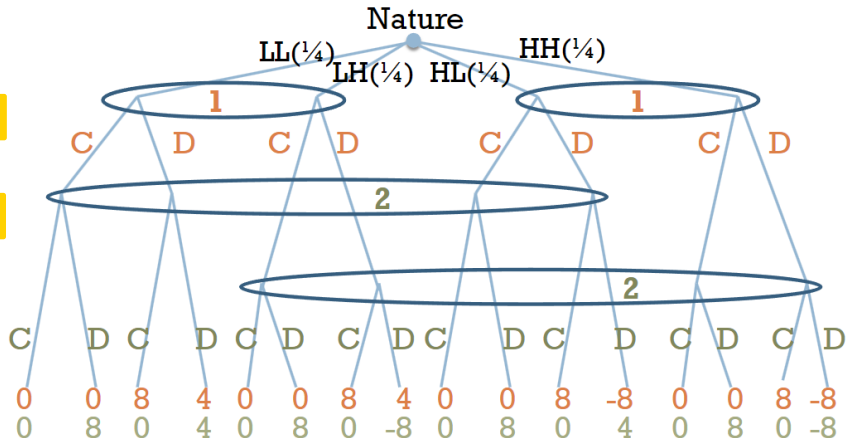
- In other words, i does not want to change strategy (at least with his/her available information)
 - player i 's strategy = a choice of action for each type
 - what i does not know, he/she just estimates!
- This definition can be generalized according to the type space (if continuous \rightarrow integrals)

Examples of Bayesian NE

- The “chicken game” is a well-known anti-coordination game: two youngsters are driving on the road in opposite directions at high speed; both can chicken out (C) (i.e., steer) or keep driving straights (D)
 - chickens get nothing but shame ($u = 0$)
 - those who keep driving gain “respect” ($u = 8$)
 - if both drive, they split the respect, plus an accident happens; they receive $u = 4 - P$, where P is a punishment that depends on their parents’ type
 - Parents can be the hard type (H) ($P = 16$) or the lenient type (L) ($P = 4$) with probability 0.5
 - the youngsters know their parents’ type

Chicken game

EXTENSIVE FORM



Chicken game

- Normal form:

		Player 2			
		CC	CD	DC	DD
Player 1	CC	0, 0	0, 4	0, 4	0, 8
	CD	4, 0	-1, -1	-1, 2	-6, 1
	DC	4, 0	2 , -1	2 , 2	1 , 1
	DD	8 , 0	1, -6	1, 1	-6, -6

- Only BNE is (DC, DC)
 - Different values of P can lead to other BNE



- Many decisions are made by committee through majority voting
- Consider a jury with just two members deciding whether to acquit (A) or convict (C) a defendant
 - Each jury member casts a sealed vote
 - The defendant is convicted if both members vote C
- It is uncertain whether the defendant is guilty (G) or innocent (I): the prior probability is $q > 1/2$, which is common knowledge

- Jury members wish to make the right decision, so their payoff is 1 if $G \rightarrow C$ and $I \rightarrow A$, 0 otherwise
- If the only information is probability q , then this is a lottery:

		Member 2	
		A	C
Member 1	A	$1-q, 1-q$	$1-q, 1-q$
	C	$1-q, 1-q$	q, q

- and if $q > 1/2$, then (C, C) is a NE ((A, A) is always a NE)

- Assume each jury member observes the evidence and independently gets a private **signal** (his/her idea about the case) $t_i \in t_G, t_I$
- It is more likely (but not certain) t_i receive signal " t_x " if the defendant status is x
- Let p = Probability of receiving signal t_x given x
- $\Pr[t_G|G] = \Pr[t_I|I] = p > 1/2$ for both jurors $i = 1, 2$
- clearly, $\Pr[t_G|I] = \Pr[t_I|G] = 1 - p < 1/2$
- **Note:** These types are not about the players themselves, but about events happening in the world; still, they affect their payoffs

- Since each player has 2 types and 2 actions
→ 4 possible strategies: AA, AC, CA, CC
 - strategy XY means $t_G \rightarrow X$ and $t_I \rightarrow Y$
 - It is a coordination game, since both players have the same objective (make the right judgment)
- For the time being, consider a single-person problem where only one juror decides
 - Without the signal, he/she plays C
 - How would the signal affect their choice?

Committee voting: types

$$\Pr\{H|S\} = \frac{\phi(S \cap H)}{\phi(S \cap H) + \phi(S \cap [\text{not } H])} = \frac{\phi(S \cap H)}{\phi(S)}$$

- The **signals** affect the posterior probabilities of the jury members
- These can be computed using Bayes theorem

$$\Pr[G|t_G] = \frac{\Pr[G, t_G]}{\Pr[t_G]} = \frac{qp}{qp + (1-q)(1-p)} > q$$

- since $p > 1/2$, then $qp + (1-q)(1-p) < qp + (1-q)p = p$
- and, on the other hand

$$\Pr[G|t_I] = \frac{\Pr[G, t_I]}{\Pr[t_I]} = \frac{q(1-p)}{q(1-p) + (1-q)p} < q$$

→ if t_G : even more confident that the defendant is guilty; if t_I : becomes doubtful

- Actually, if t_i is received, the final assessment depends on the relative values of p and q

$$\Pr[G|t_i] = \frac{q(1-p)}{q(1-p) + (1-q)p}$$

- This may be less than $1/2$, meaning that the jury member prefers to acquit rather than convict
- This happens if $p > q$
 - The reason is that the information content of the signal must be higher than the prior information
 - Instead, if $p = 1/2$, the signal given no information and posterior=prior

- Now, we would like to check whether $p > q$ implies that (CA, CA) is a BNE in the original problem (2-person decision)
 - That would correspond to “following the signal”
- First, draw the probability of each type pair

		Member 2	
		t_G	t_I
Member 1	t_G	$qp^2 + (1 - q)(1 - p)^2$	$p(1 - p)$
	t_I	$p(1 - p)$	$q(1 - p)^2 + (1 - q)p^2$

- **Note:** This is not a payoff matrix, it is just a table displaying the values of probabilities $\Pr[t_1 = t_x, t_2 = t_y]$

- To check whether (CA, CA) is BNE we need to ask “Is CA a best response to CA?”
 - Assume member 2 plays CA, and check if CA is best for member 1
- We do not want to write down the whole table, let us try to see if we can draw conclusions just by looking at posteriors
- With the rules of the jury, a player's choice is decisive (“**pivotal**”) only if the other juror chooses C
- If 2 chooses A, that is the result regardless of the 1's choice
 - If 1 believes that 2 is playing CA, *any* strategy of 1 is always a best response if the 2's type is t_I
 - In other words, if 1 thinks that 2 received signal t_I , then everything 1 does is a best response
 - So we need to check only the case $t_2 = t_G$

- Again, check the posterior to see the signal effect

$$\Pr[G|t_1 = t_G, t_2 = t_G] = \frac{qp^2}{qp^2 + (1-q)(1-p)^2} > q$$

- **Meaning:** if both $t_1 = t_G$ and $t_2 = t_G$: conviction is even more certain
- as before, $p > 1/2$ implies
$$qp^2 + (1-q)(1-p)^2 < qp^2 + (1-q)p^2 = p^2$$

$$\Pr[G|t_1 = t_I, t_2 = t_G] = \frac{qp(1-p)}{p(1-p)} = q$$

- **Meaning:** if they receive opposite signals, the received signal t_I is useless \rightarrow posterior=prior

- Recap:
 - If player 2 is of type $t_2 = t_I$, player 1 believes that 2's move is A \rightarrow 1's move does not matter
 - If player 2 is of type $t_2 = t_G \rightarrow$ player 1's posterior is either q or higher
- Therefore, CA is not a best response to CA
- Actually you can prove that (CC, CC) is a BNE

Sorry, gotta bounce!
Send me questions via e-mail