Computability
February 2, 2024

Exercise 1

a. Provide the definition of a recursive set.

b. Provide the definition of a recursively enumerable (r.e.) set.

c. Show that given $A, B \subseteq \mathbb{N}$, if A is recursive and $B = A \cap \mathbb{P}$ then B is recursive (here $\mathbb{P}$ denotes the set of even numbers). Does the converse hold? I.e., is it the case that if $B = A \cap \mathbb{P}$ is recursive then A is recursive?

Solution:

1. A set $A \subseteq \mathbb{N}$ is recursive if the characteristic function $\chi_A : \mathbb{N} \to \mathbb{N}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

   is computable.

2. A set $A \subseteq \mathbb{N}$ is r.e. if the semi-characteristic function $sc_A : \mathbb{N} \to \mathbb{N}$ defined by

$$sc_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

   is computable.

3. Let $A, B \subseteq \mathbb{N}$ with A recursive and let $B = A \cap \mathbb{P}$. Just observe that $\mathbb{P}$ is recursive (in fact $\chi_{\mathbb{P}}(y) = \overline{sg}(rm(2, y))$). Hence B is the intersection of recursive sets which is known to be recursive.

   The converse is false. In fact, consider the set $A = \{2x + 1 \mid x \in K\}$. We have that $B = A \cap \mathbb{P} = \varnothing$ is recursive. However, A is not recursive since $K \leqslant_m A$. The reduction function can simply be $f(x) = 2x + 1$. Clearly it is total and computable and $x \in K$ iff $f(x) \in A$.

Exercise 2

State the s-m-n theorem and use it to prove that there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $W_{s(x)} = \mathbb{P}$ and $E_{s(x)} = \{z \in \mathbb{N} \mid z \geqslant x\}$ (where again $\mathbb{P}$ is the set of even numbers).

Solution: We can define, for instance,

$$f(x,y) = \begin{cases} x + y/2 & \text{if } y \in \mathbb{P} \\ \uparrow & \text{otherwise} \end{cases}$$

which is clearly computable. In fact

$$f(x,y) = x + qt(2,y) + \mu w.rm(2,y)$$

Seen as a function of y, it has as domain $\mathbb{P}$ and as codomain $\{z \mid z \geqslant x\}$. Then one can use the smn theorem to get a function $s : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = f(x,y) = \begin{cases} x + y/2 & \text{if } y \in \mathbb{P} \\ \uparrow & \text{otherwise} \end{cases}$$

Then s is the desired function, in fact:

- $W_{s(x)} = \mathbb{P}$, by construction;
- $E_{s(x)} = \{\varphi_{s(x)}(y) \mid y \in W_{s(x)}\} = \{\varphi_{s(x)}(y) \mid y \in \mathbb{P}\} = \{x + y/2 \mid y \in \mathbb{P}\} = \{x + y' \mid y' \in \mathbb{N}\} = \{z \mid z \geqslant x\}$, as desired.

Exercise 3

Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x = E_x\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated since it can be expressed as $A = \{x \mid \varphi_x \in \mathcal{A}\}$ with $\mathcal{A} = \{f \mid dom(f) = cod(f)\}$.

By Rice-Shapiro's theorem:

- A is not r.e.
  In fact the constant 1 function $\mathbf{1} \notin \mathcal{A}$, since $dom(\mathbf{1}) = \mathbb{N} \neq \{1\} = cod(\mathbf{1})$.

  However, consider $\theta = \emptyset$, the function which is always undefined. Then $\theta \subseteq \mathbf{1}$ and $\theta \in \mathcal{A}$ since $dom(\theta) = \emptyset \neq \{0\} = cod(\theta)$.

- $\bar{A}$ is not r.e.
  In fact if we consider the predecessor $pred(x) = x \dot{-} 1$
  $$pred \notin \overline{\mathcal{A}} = \{f \mid dom(f) \neq cod(f)\},$$
  since $dom(pred) = \mathbb{N} = cod(pred)$. However, if we consider the finite subfunction $\theta \subseteq pred$,
  $$\theta(x) = \begin{cases} 0 & \text{if } x \leqslant 1 \\ \uparrow & \text{otherwise} \end{cases}$$
  we have $\theta \in \overline{\mathcal{A}}$. In fact $dom(\theta) = \{0, 1\} \neq cod(\theta) = \{0\}$.

Exercise 4
Classify the following set from the point of view of recursiveness
$$B = \{x \in \mathbb{N} \mid \exists z.\ \varphi_x(z) > x\},$$
i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable. Also establish if B is saturated.

Solution: The set B is not recursive since $K \leqslant_m B$. To show this one can consider the function
$$g(x, y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$
This function is computable, given that $g(x, y) = y \cdot sc_k(x)$. So by the smn theorem, there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x, y)$ for each $x, y \in \mathbb{N}$. It is easy to show that s is a reduction function of K to B. Indeed

- If $x \in K$ then $\varphi_{s(x)}(y) = g(x, y) = y$ for each $y \in \mathbb{N}$. Hence $E_{s(x)} = \mathbb{N}$ and thus $z = s(x) + 1 \in E_{s(x)}$, with $z > s(x)$. Thus $s(x) \in B$.
- If $x \notin K$ then $\varphi_{s(x)}(y) = g(x, y) \uparrow$ for every $y \in \mathbb{N}$. Thus $E_{s(x)} = \varnothing$ and therefore certainly there is no $z > s(x)$ such that $z \in E_{s(x)}$. Thus $s(x) \notin B$.

Moreover B is r.e., since its characteristic function is computable. In fact it can be expressed as:
$$\begin{aligned} sc_B(x) &= \mathbf{1}(\mu(y, z, t).S(x, y, z, t) \wedge z > x) \\ &= \mathbf{1}(\mu(y, z', t).S(x, y, x + 1 + z', t)) \\ &= \mathbf{1}(\mu w.S(x, (w)_1, x + 1 + (w)_2, (w)_3)) \end{aligned}$$
Since B is r.e. and not recursive, necessarily $\bar{B}$ is not r.e. (and hence not recursive).

Concerning the second point, B is not saturated. we first observe that there is $e \in \mathbb{N}$ such that for all $y \in \mathbb{N}$.

$$\varphi_e(y) = e + 1$$

To this aim define $g : \mathbb{N}^2 \to \mathbb{N}$ as

$$g(x, y) = x + 1$$

This is clearly computable and thus, by smn theorem, there exists $s : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$ we have

$$\varphi_{s(x)}(y) = g(x, y) = x + 1$$

By the second recursion theorem there is $e \in \mathbb{N}$ such $\varphi_{e)}(y) = \varphi_{s(e)}(y)$ and thus for all $y \in \mathbb{N}$

$$\varphi_e(y) = \varphi_{s(e)}(y) = e + 1$$

Thus $e \in B$.

Now, there are infinitely many indexes for the function φ_e, hence there is $e' \in \mathbb{N}$, $e' > e$ such that $\varphi_e = \varphi_{e'}$ and thus for all $y \in \mathbb{N}$ $\varphi_{e'}(y) = \varphi_e(y) = e + 1 \leqslant e'$. Hence $e' \notin B$.

Summing up, $e \in B$, $e' \notin B$ and $\varphi_e = \varphi_{e'}$. Hence B is not saturated.

Note: Each exercise contributes with the same number of points (8) to the final grade.