```
(define talk
  (make-presentation
   (make-title
    "Recursion"
    "Lather. Rinse. Repeat.")
   (make-author
    "Mike Harris" "@MikeMKH")))
```

```
(define talk-goals
  '("intro to solving problems with recursion"
    "help in understanding FP examples"))
```

```
(define topics
 (make-agenda
  '("Recursive Data Type"
    "Structural Recursion"
    "General Recursion")))
```

syntax

5

5

atom

```
plus5 (n : number) : number
  n + 5
```

pseudo

```
(define (plus-5 n)
  (+ n 5))
```

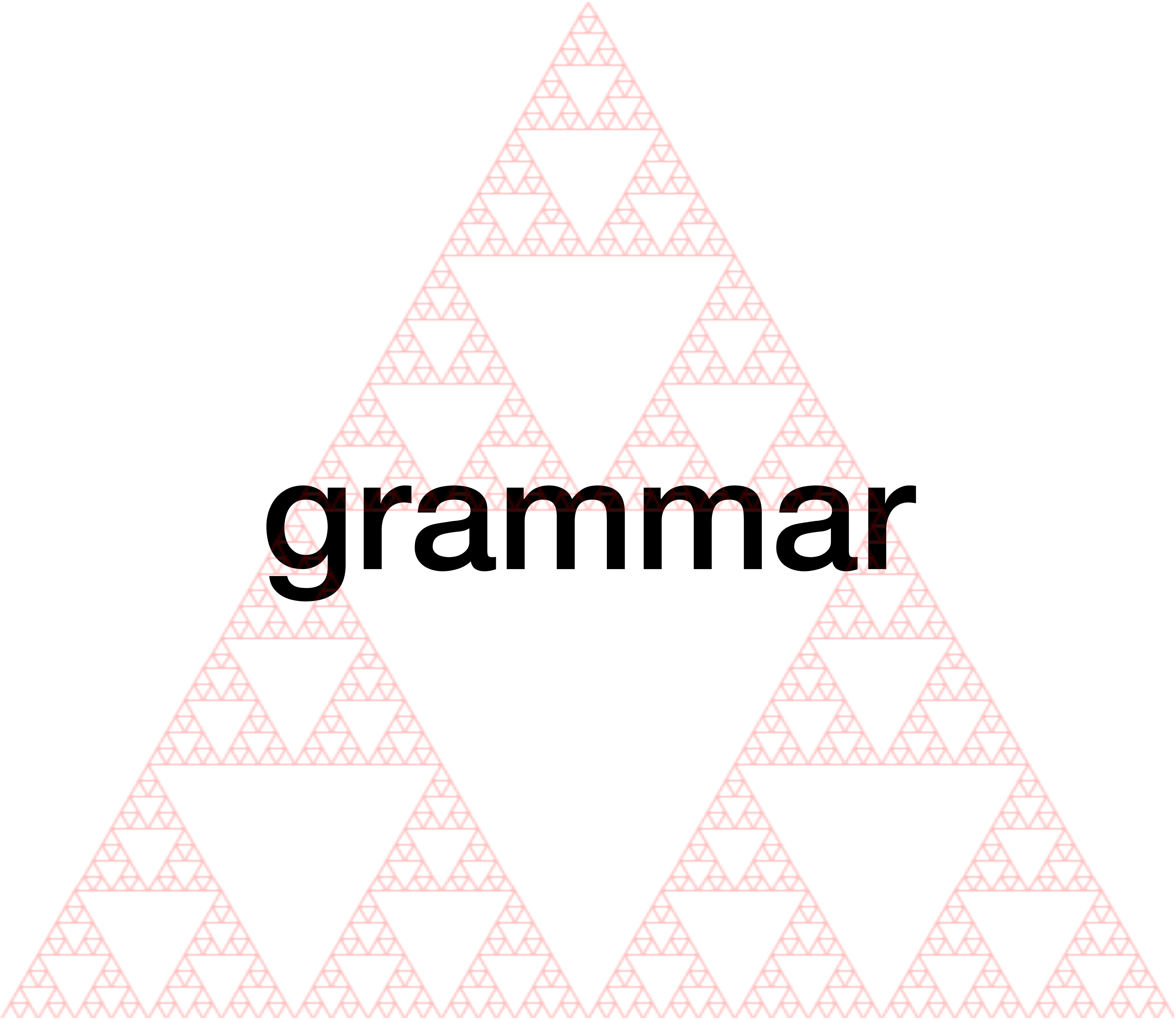
definition*

```
plus5(42)
// 47
```

```
(plus-5 42)
; 47
```

function

```
(define topics
 (make-agenda
  '("Recursive Data Type"
    "Structural Recursion"
    "General Recursion")))
```



grammar

expr = **(name** *expr expr …***)**

| **‘( )**

| *number*

| *boolean*

| *magic*

| **(cond [***expr expr***]** …

[else *expr expr***])**

EBNF

magic = **(define** *name expr***)**
| **(define (***name expr***))**
| **(check-expect** *expr expr***)**
| **(check-satisfied**
*expr expr***)**

cons list

list = **(cons** *value list***)**
| **‘( )**

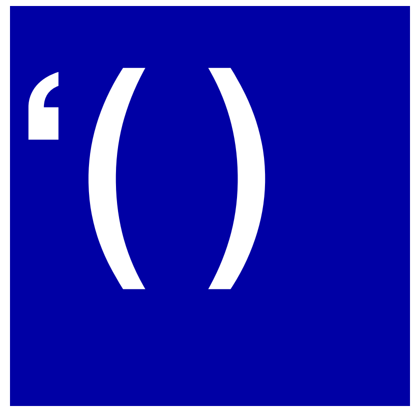
value = *number*
| *string*
| *boolean*

EBNF

constructors

```
; list - constructors
; - (cons value list)
; - '()
```

empty

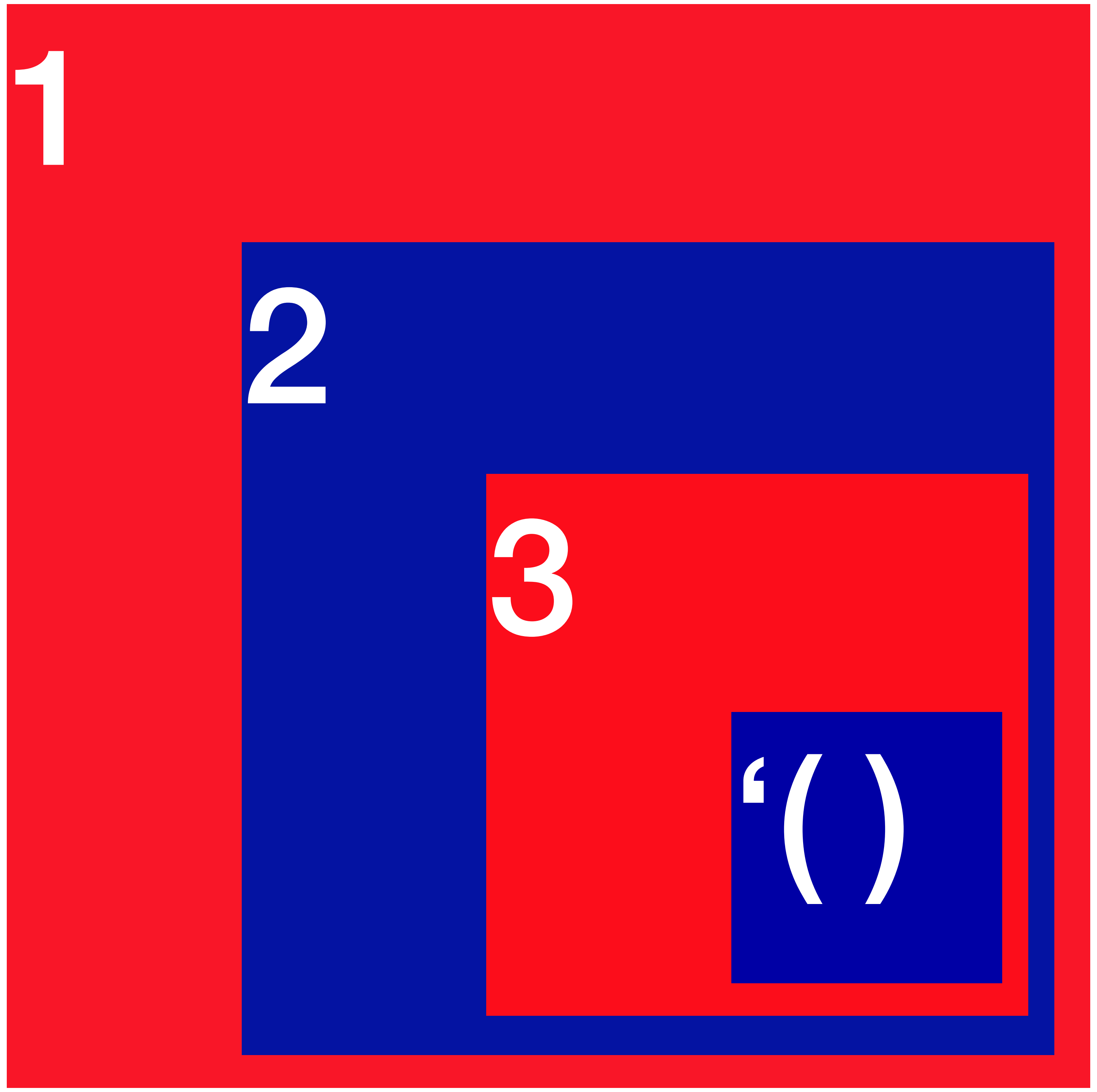


picture

'()

' ( )

cons



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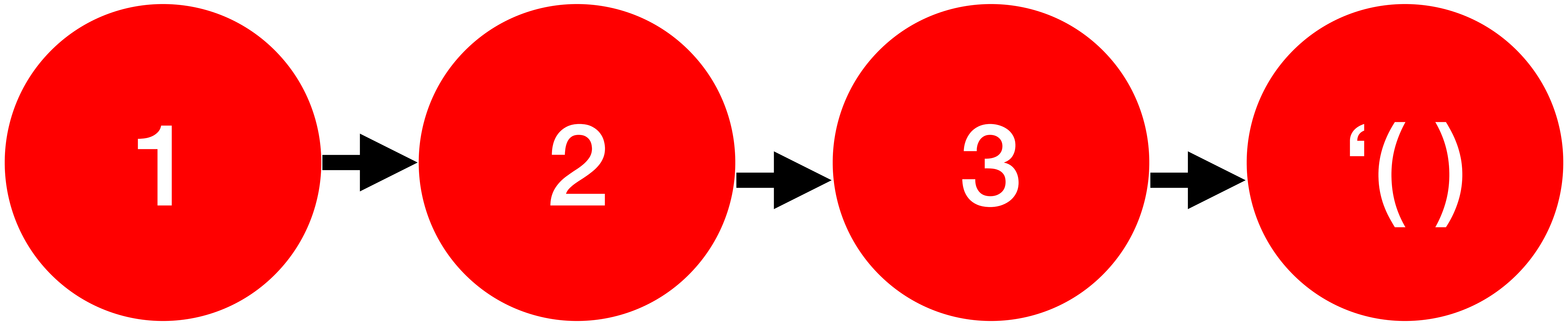
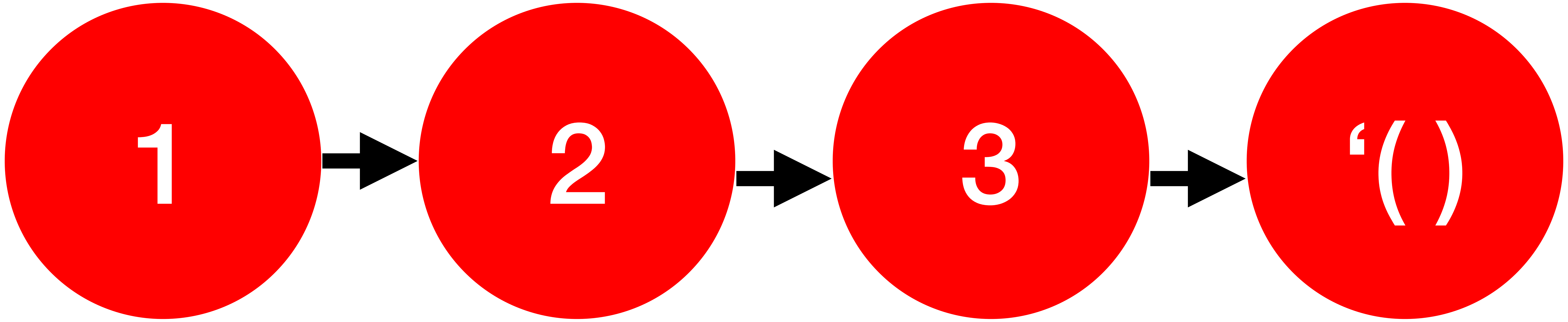
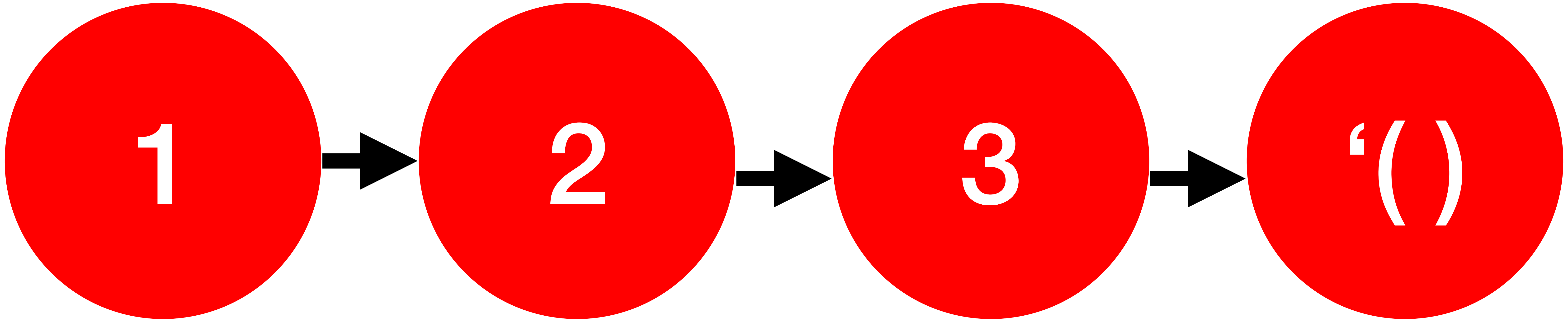



```
(cons 1
      (cons 2
            (cons 3 '())))
```




```
(cons 1 (cons 2 (cons 3 '())))
```



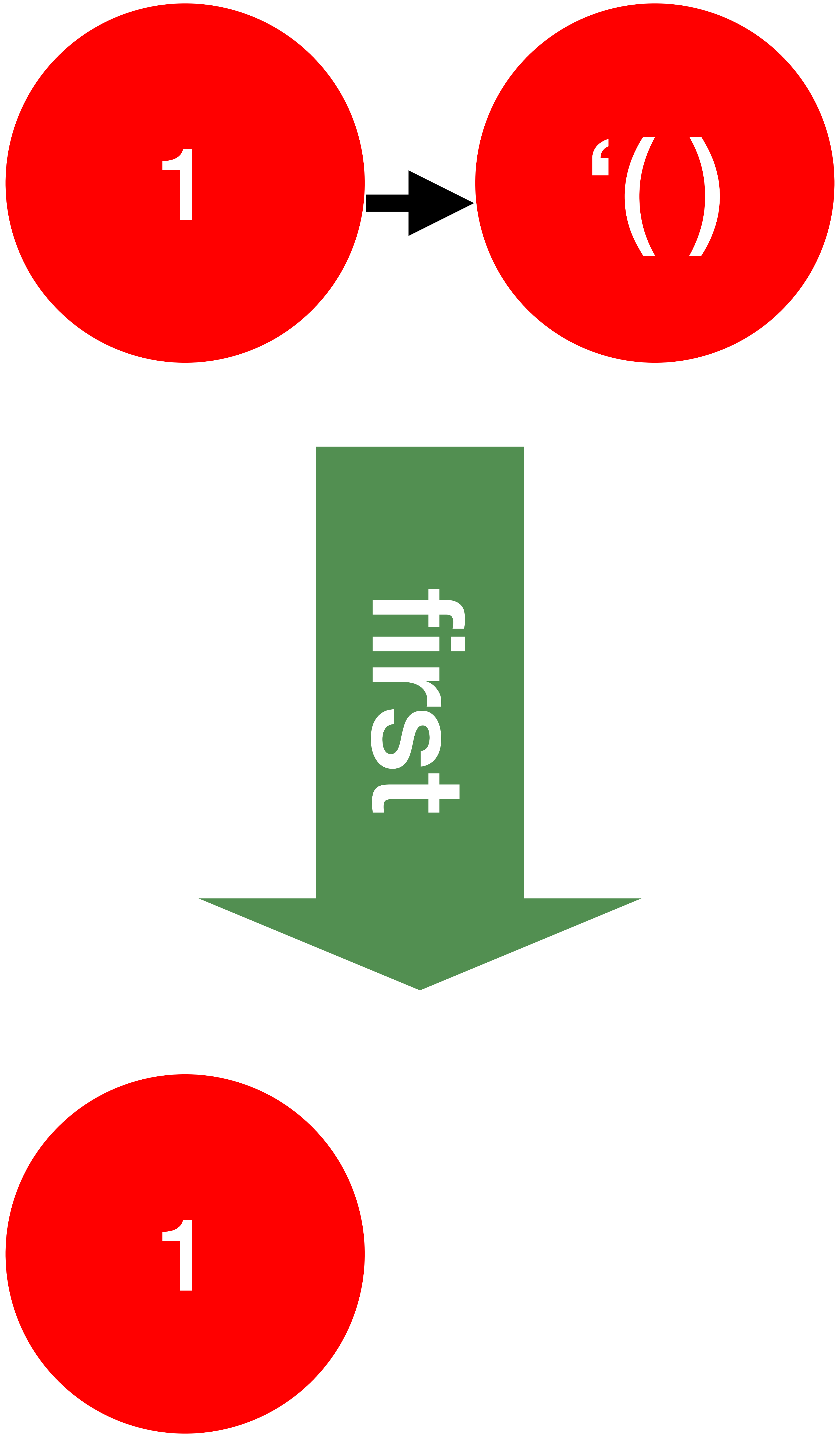

```
'(1 2 3)
```

selectors

```
; list - selectors
; - first : list -> value
; - rest  : list -> list
```

first : list -> value

picture



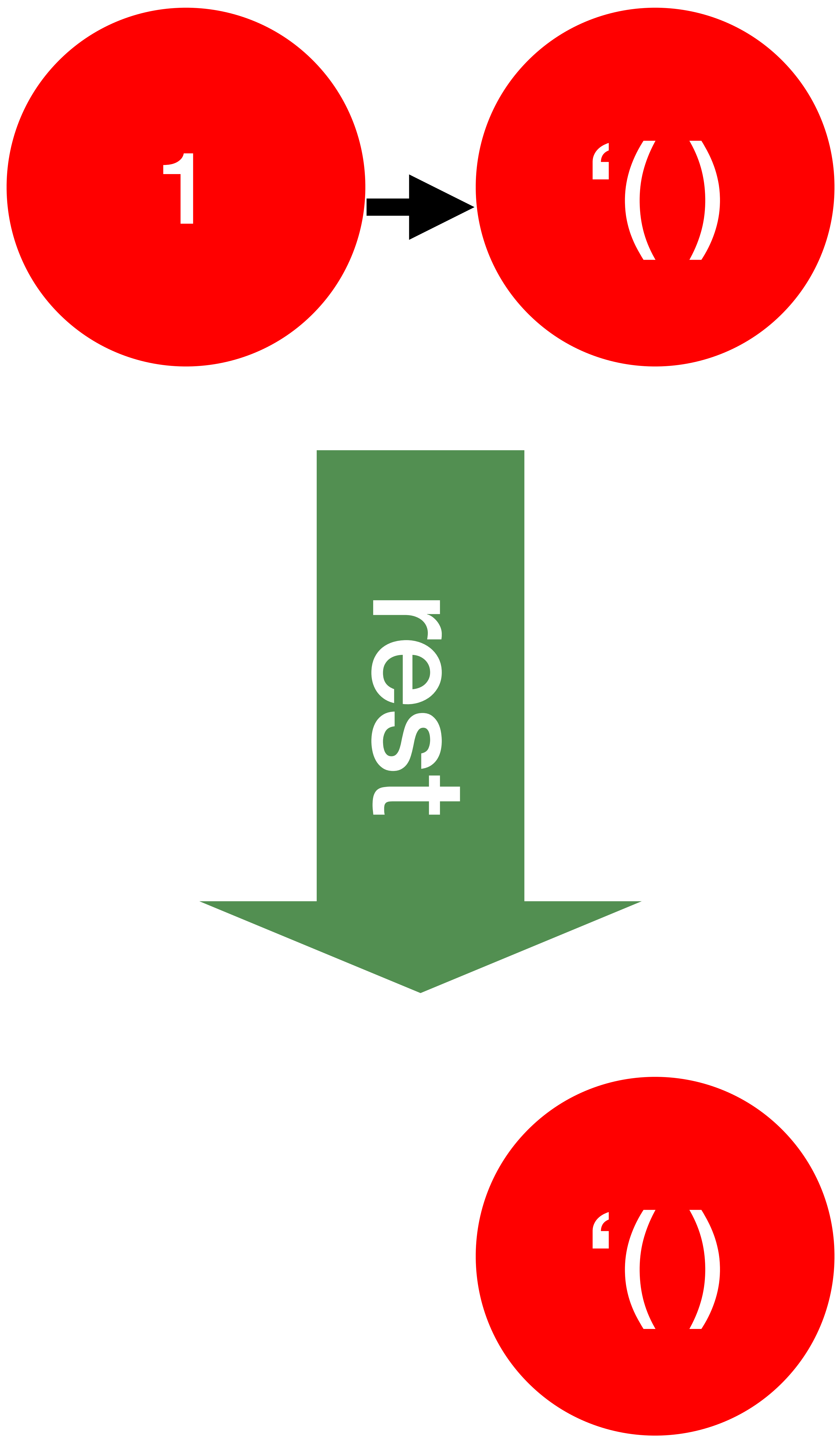
picture

```
(check-expect
 (first (cons 1 '()))
 1)
```

```
(check-expect
 (first '(1))
 1)
```

rest : list -> list

picture



picture

```
(check-expect
 (rest (cons 1 '()))
 '())
```

```
(check-expect
 (rest '(1))
 '())
```

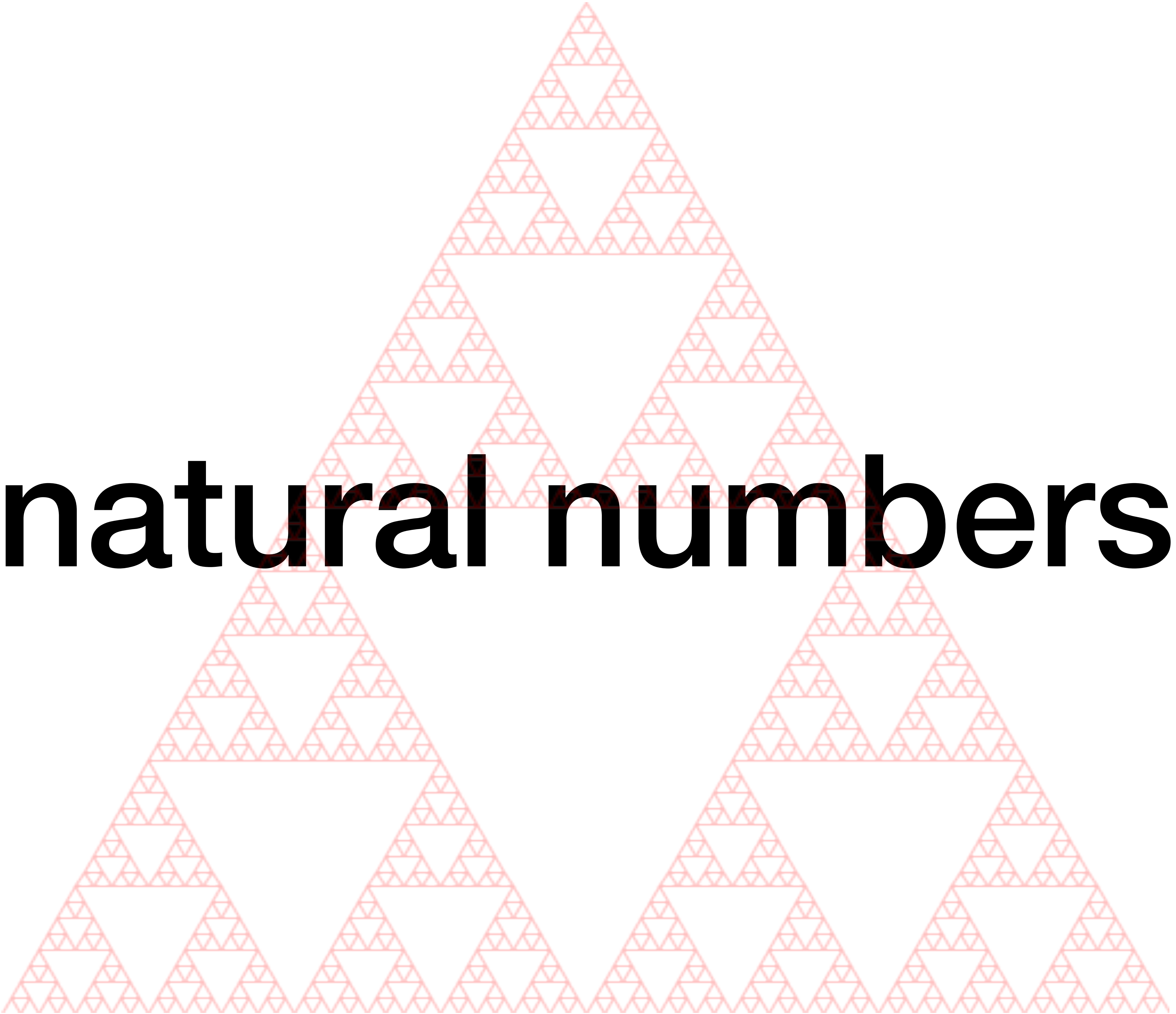
predicates

```
; list - predicates
; - empty? : list -> boolean
```

empty? : list -> boolean

```
(check-satisfied
 '()
 empty?)
```

natural numbers



nat = **0**
    | **(add1** *nat***)**

EBNF

constructors

```
; nat - constructors
; - 0
; - (add1 nat)
```

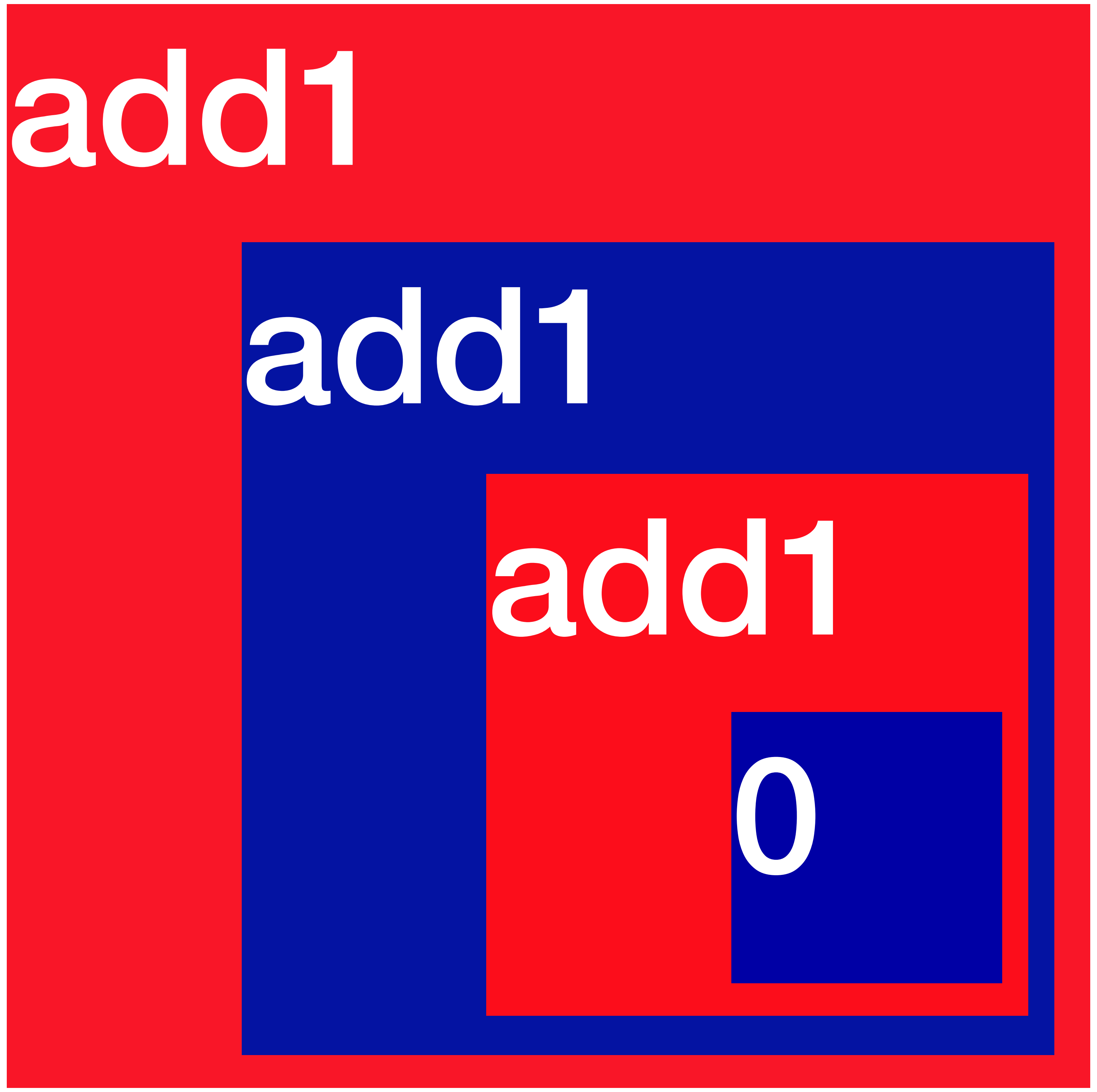
zero

0

picture

0

successor



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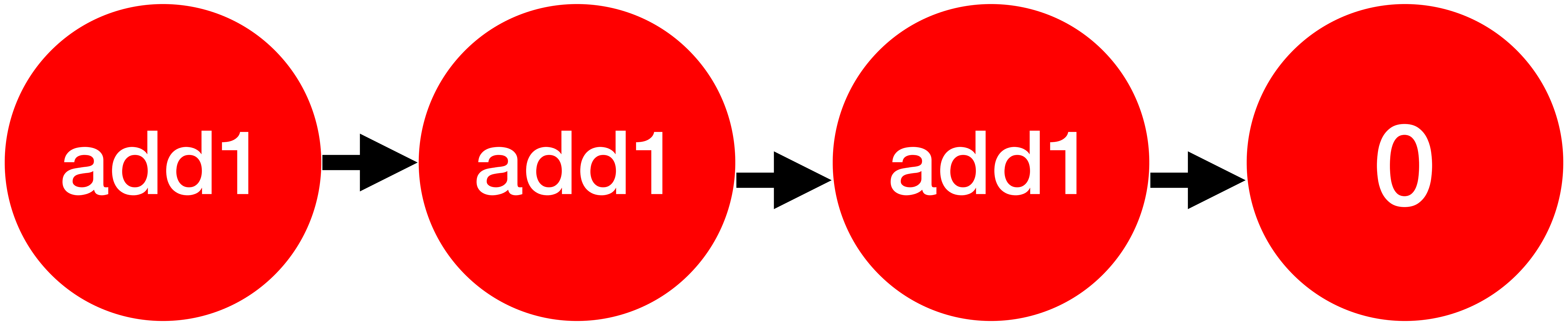
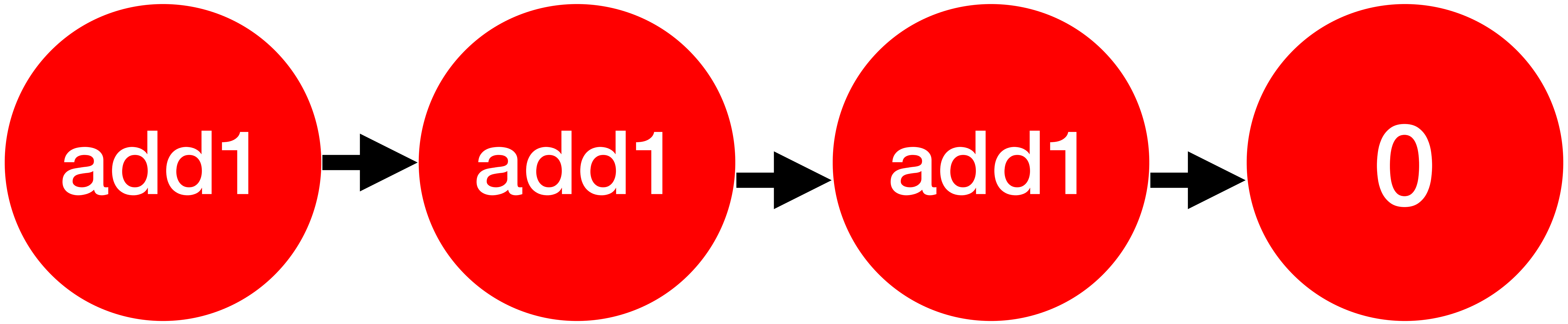
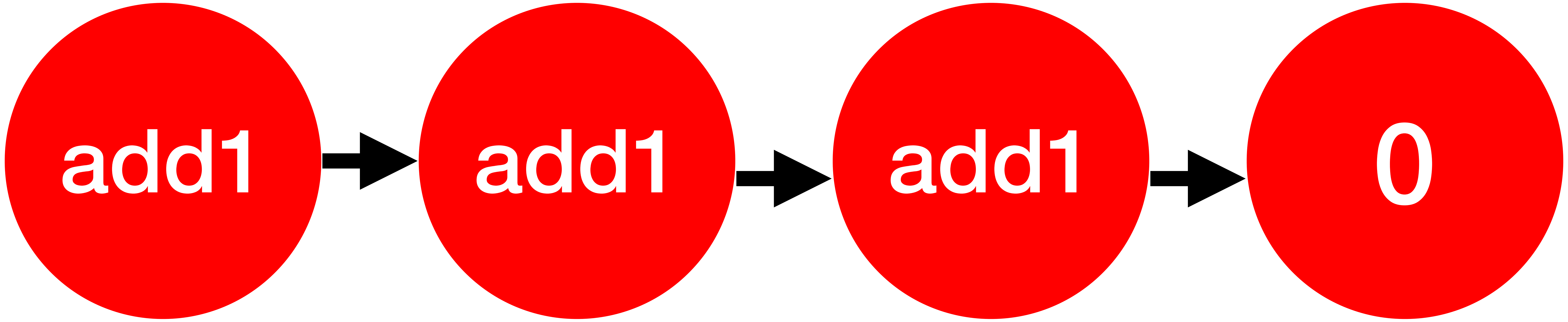



```
(add1
  (add1
    (add1 0)))
```




```
(add1 (add1 (add1 0)))
```

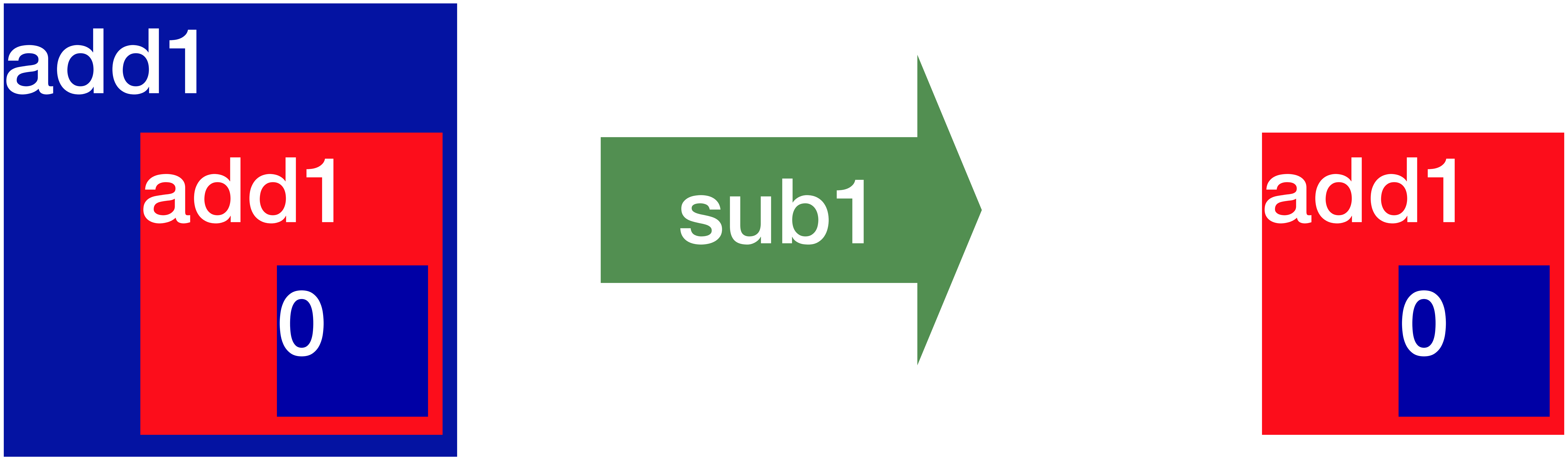


3

selectors

```
; nat - selectors
; - sub1 : nat -> nat
```

sub1 : nat -> nat



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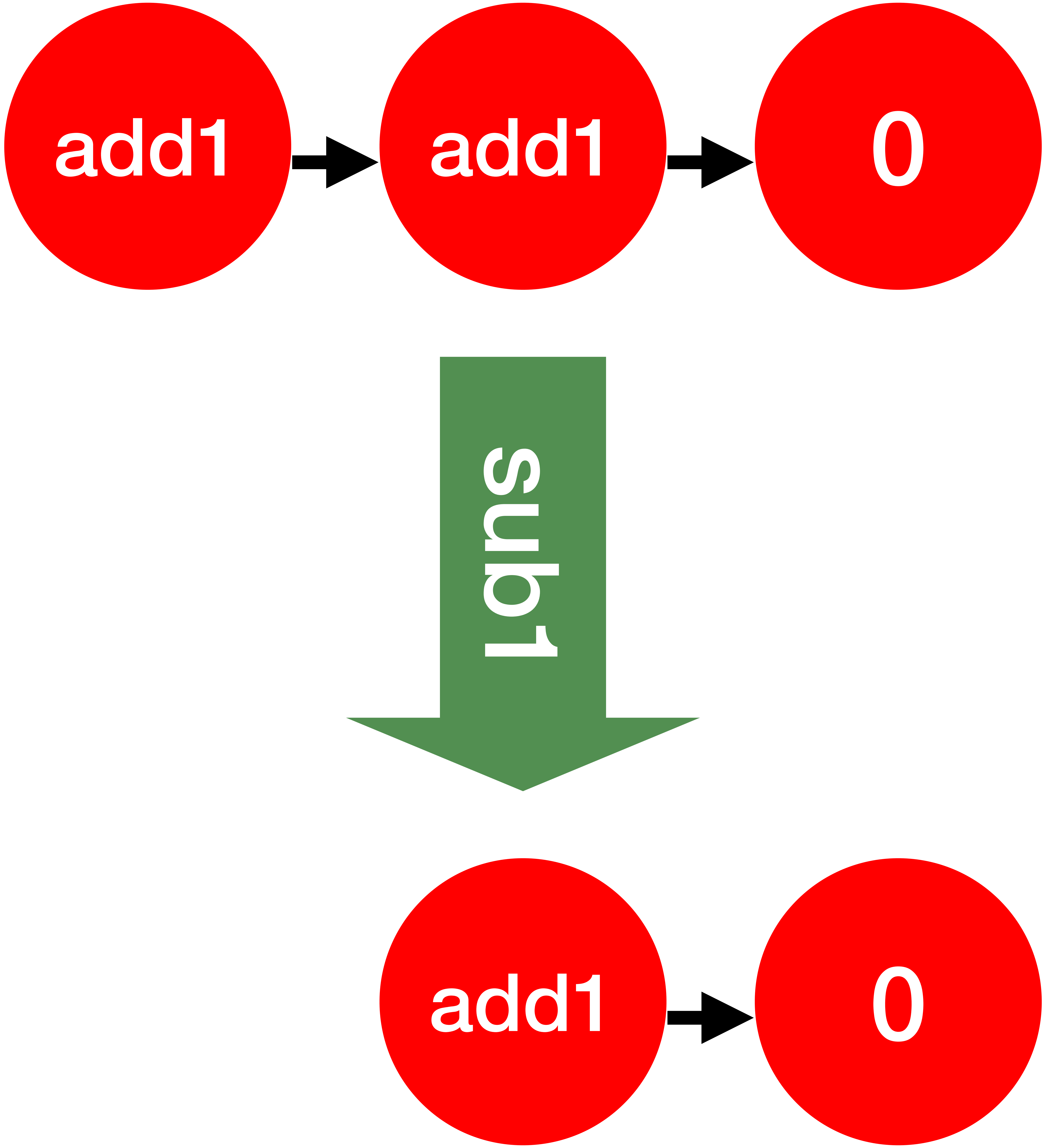


picture

```
(check-expect
 (sub1
  (add1
   (add1 0)))
 (add1 0))
```

```
(check-expect
 (sub1 2)
 1)
```

```
; nat - predicates
; - zero? : nat -> boolean
```

zero? : nat -> boolean

```
(check-satisfied
 0
 zero?)
```

```
(define topics
 (make-agenda
  '("Recursive Data Type"
    "Structural Recursion"
    "General Recursion")))
```

design

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

from 9.0 of How to Design Programs

addition

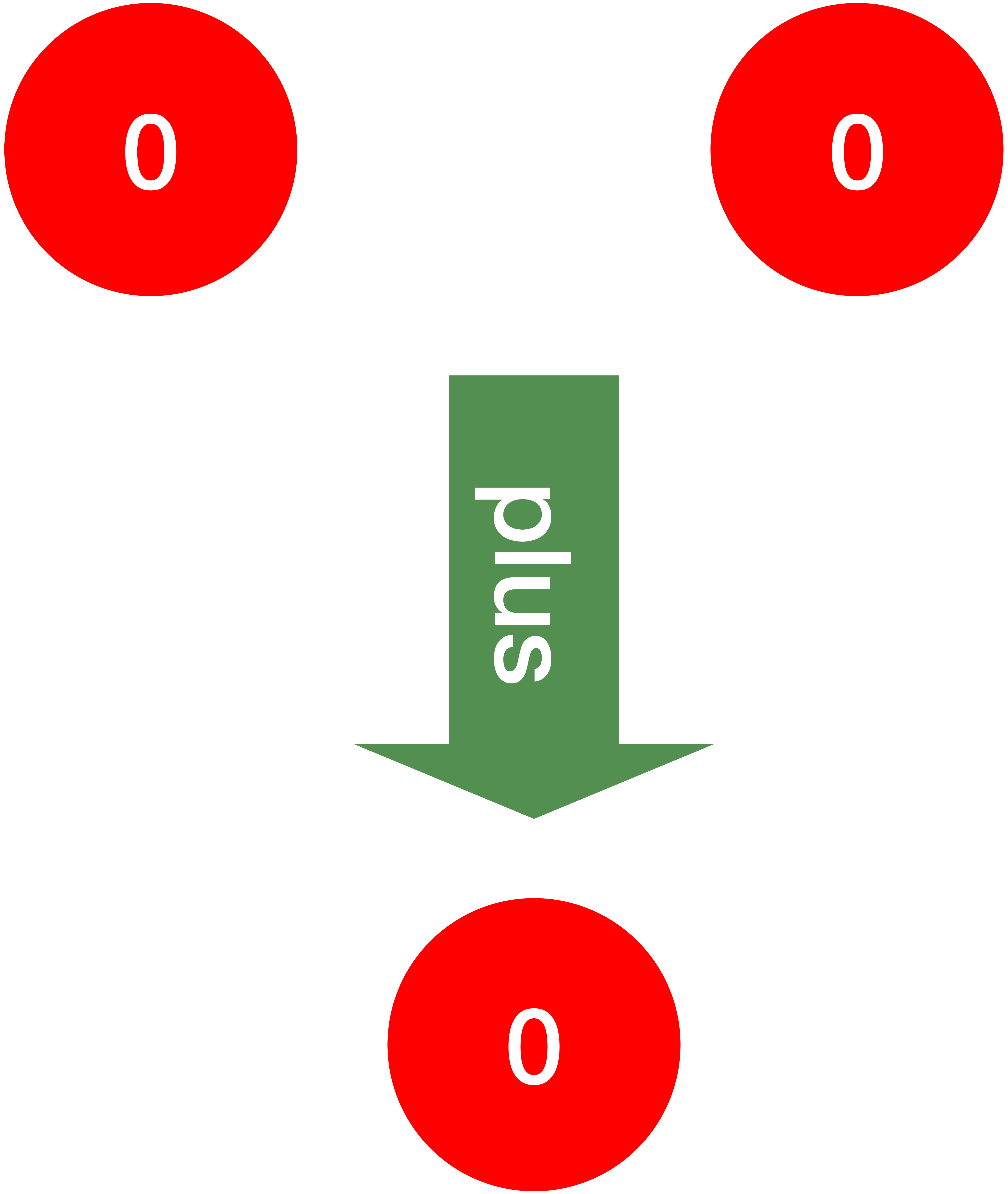
```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

given two natural numbers combine them to yield a natural number which is their total size

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
; plus : nat -> nat -> nat
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```



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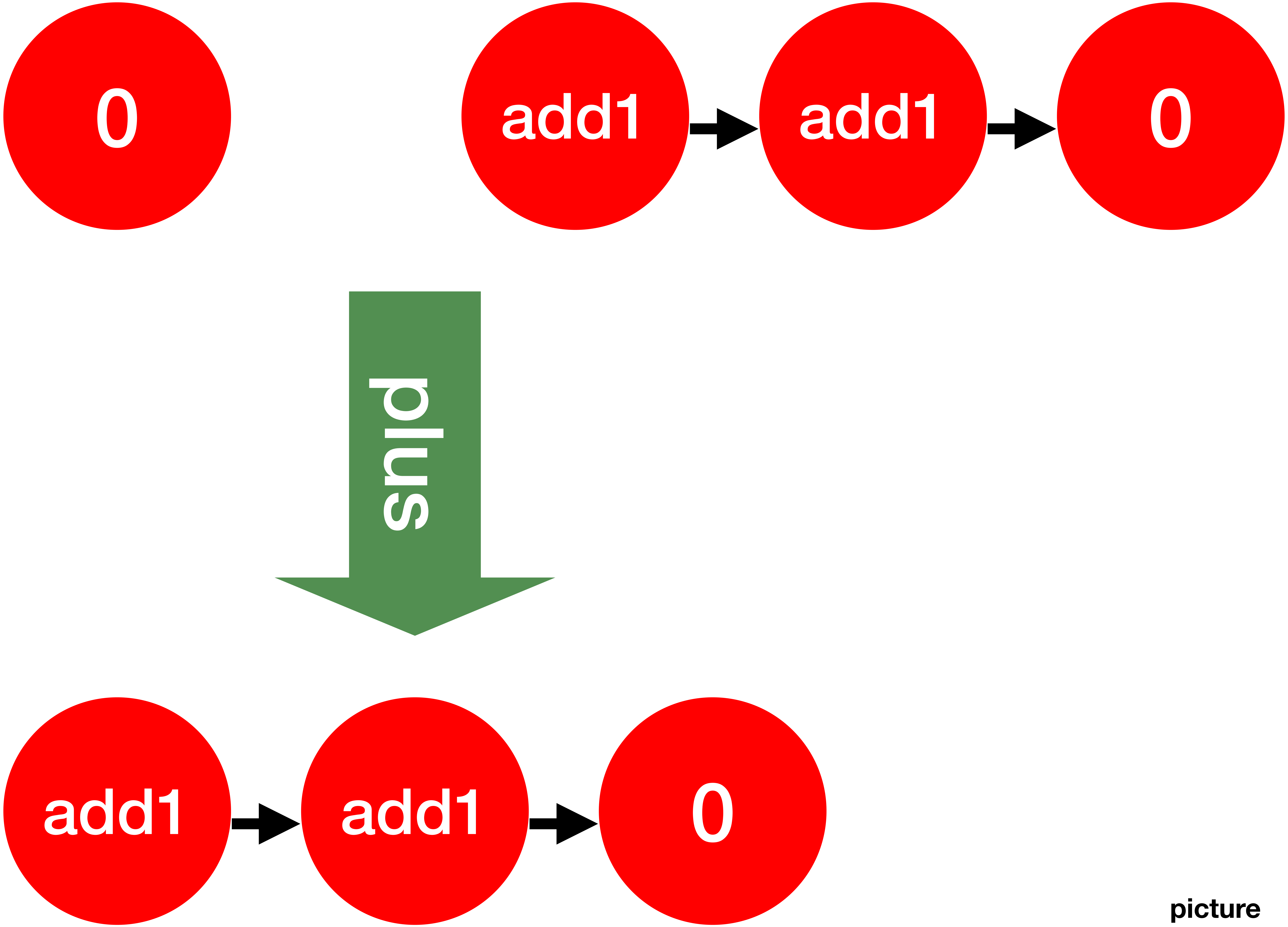

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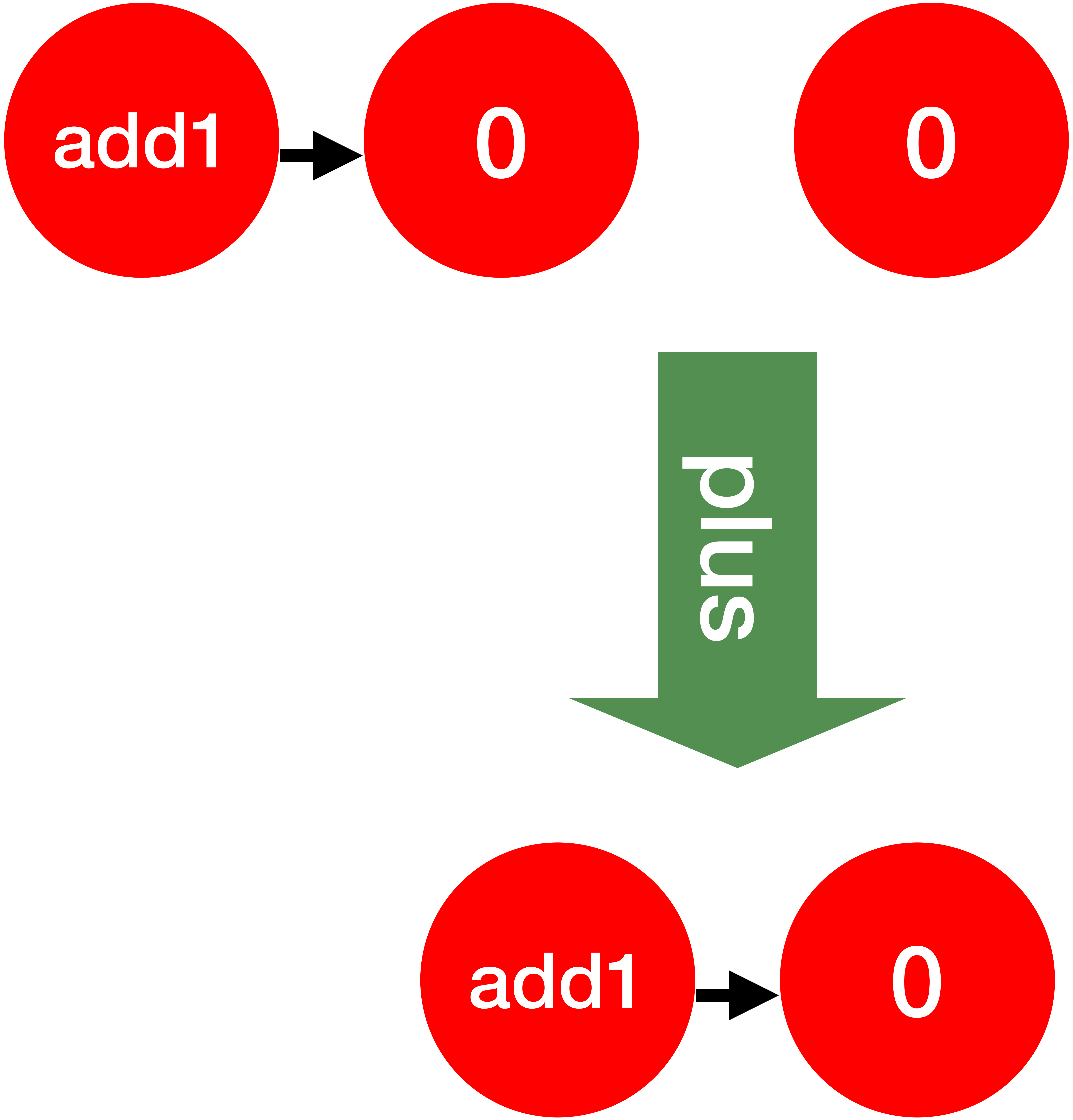

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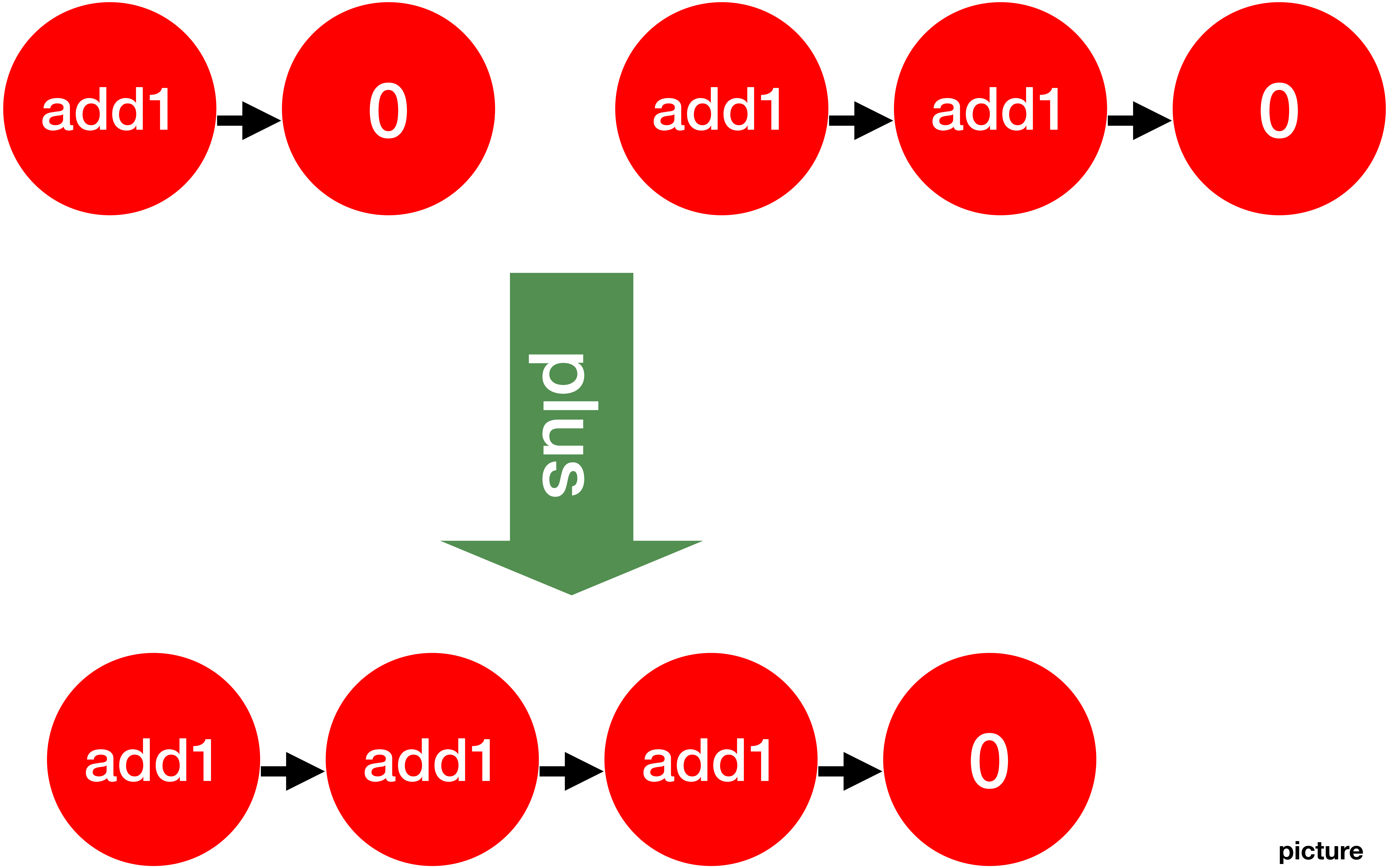


picture

```
; (plus 0 0) ; 0
; (plus 0 b) ; b
; (plus a 0) ; a
; (plus a b) ; a + b
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
(define (plus a b)
  (cond
    [(zero? a) b]
    [else
     (add1
      (plus (sub1 a)
                b))]))
```

```
(define recusive-process
  '("identify principal"
    "test basis"
    "reduced recursion"
    "combine results"))
```

plus : (a : nat) -> nat -> nat

principal	a : nat
basis	zero?
reducer	sub1
combine	add1

```
(define (plus a b)
  (cond
    [(zero? a) b]
    [else
     (add1
      (plus (sub1 a)
            b))]))
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
; (plus 0 0) ; 0

(check-satisfied
 (plus 0 0)
 zero?)
```

```
; (plus 0 b) ; b

(check-expect
 (plus 0 2)
 2)
```

```
; (plus a 0) ; a

(check-expect
 (plus 1 0)
 1)
```

```
; (plus a b) ; a + b

(check-expect
 (plus 1 2)
 3)
```

```
; (plus a b) ; a + b

(check-expect
 (plus
  (add1 0)
  (add1 (add1 0)))
 (plus
  0
  (add1 (add1 (add1 0)))))
```


take

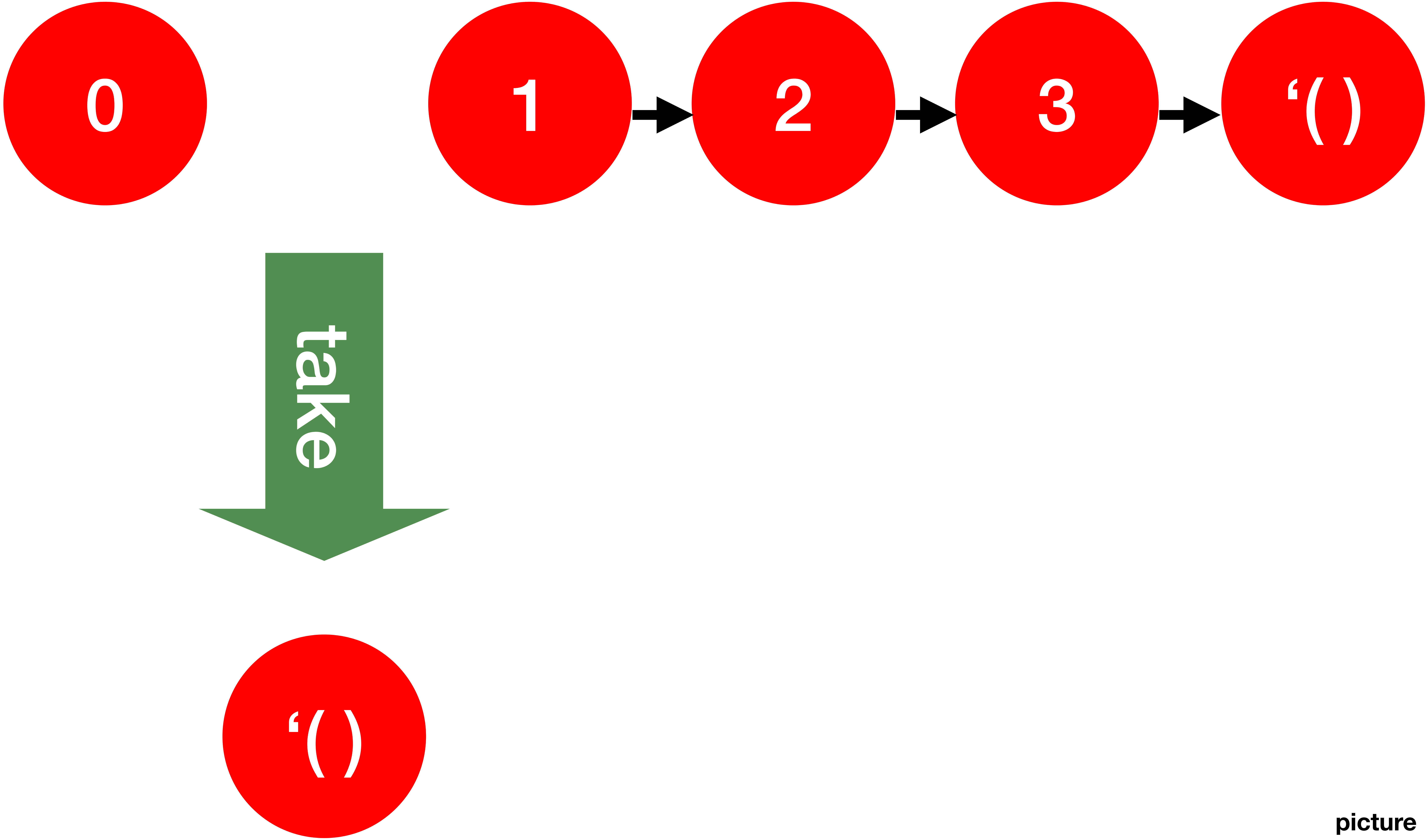
```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

given a natural number n and a list
yield a list with the first n elements

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
; take : nat -> list -> list
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```



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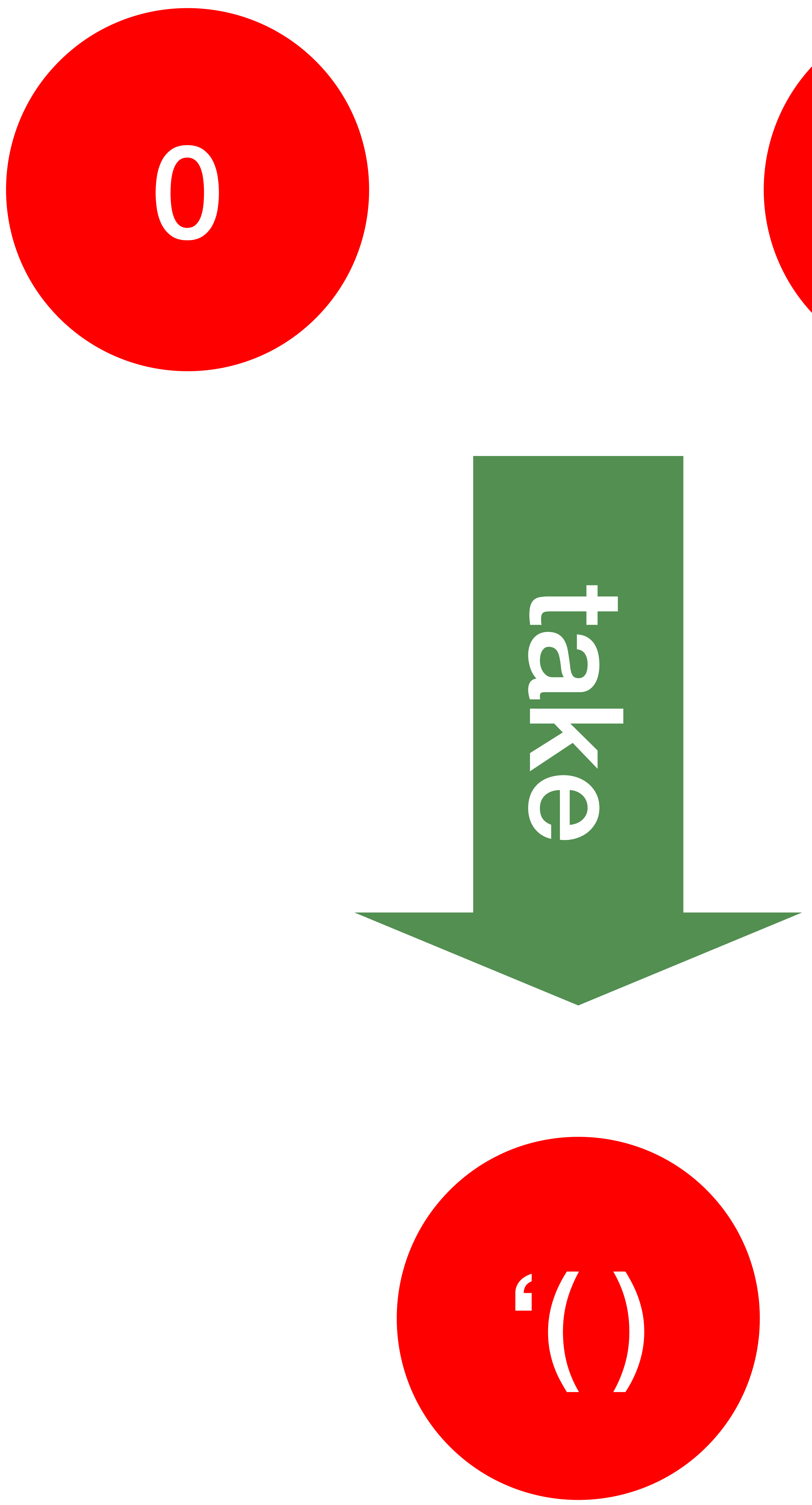

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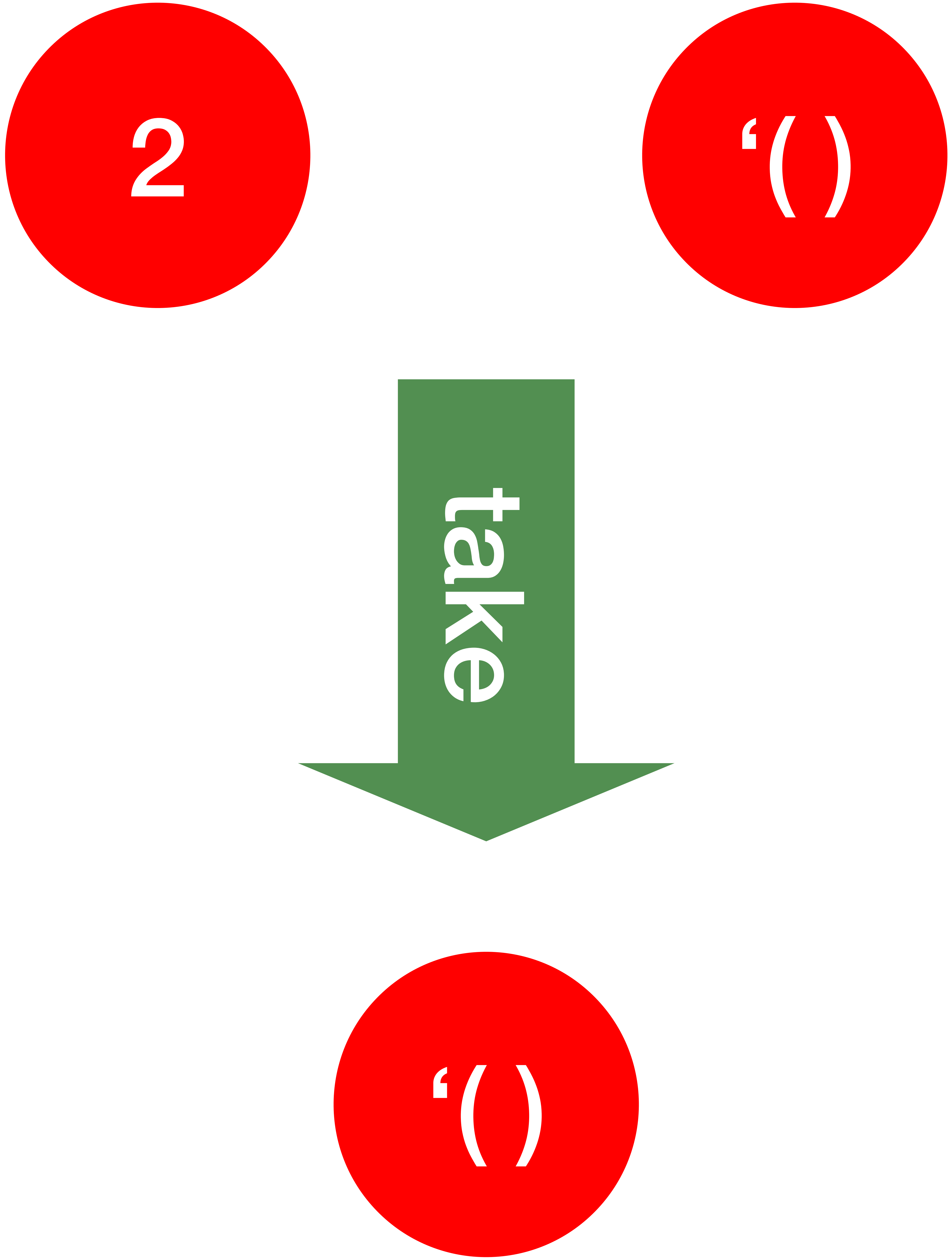

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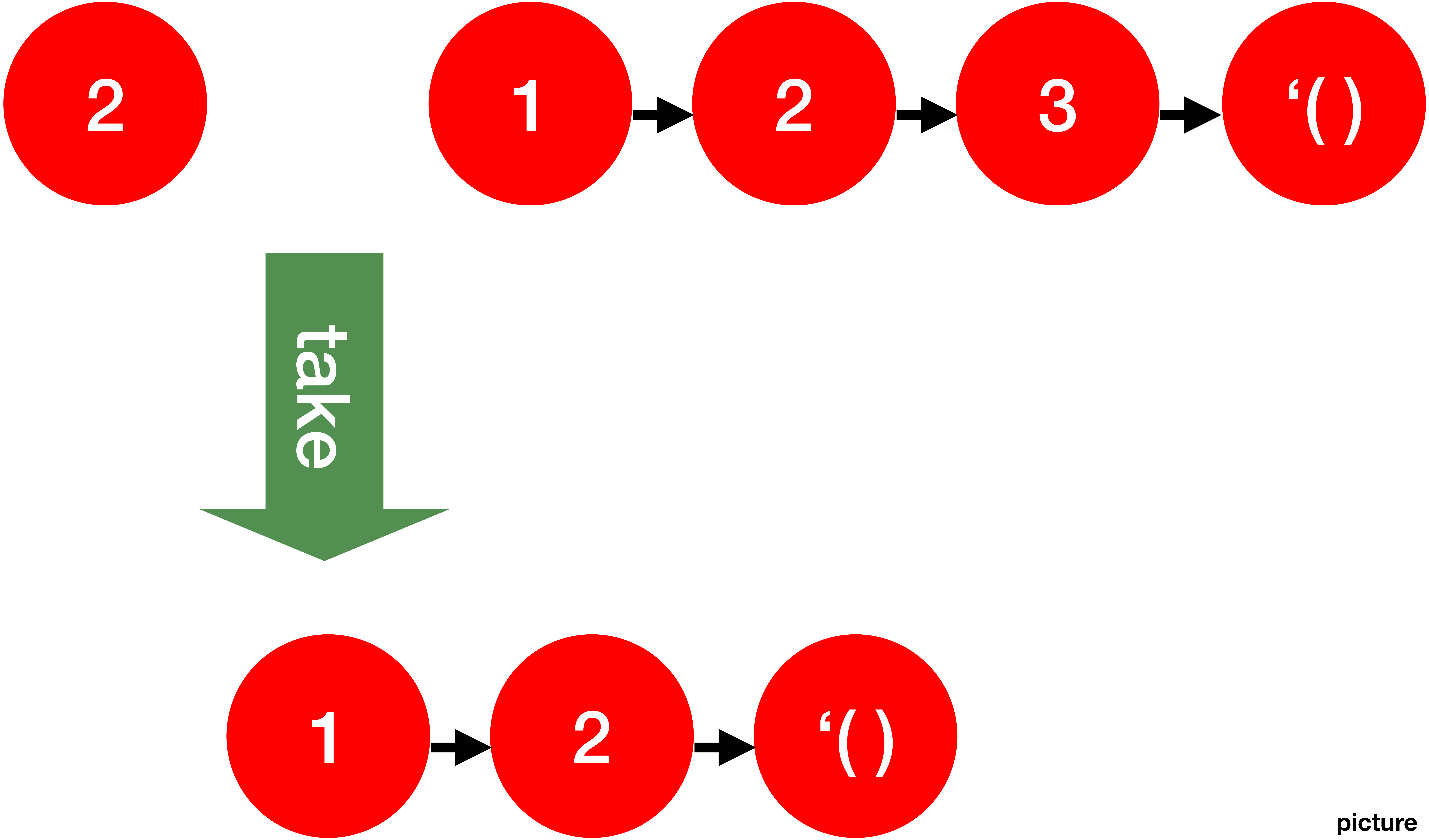

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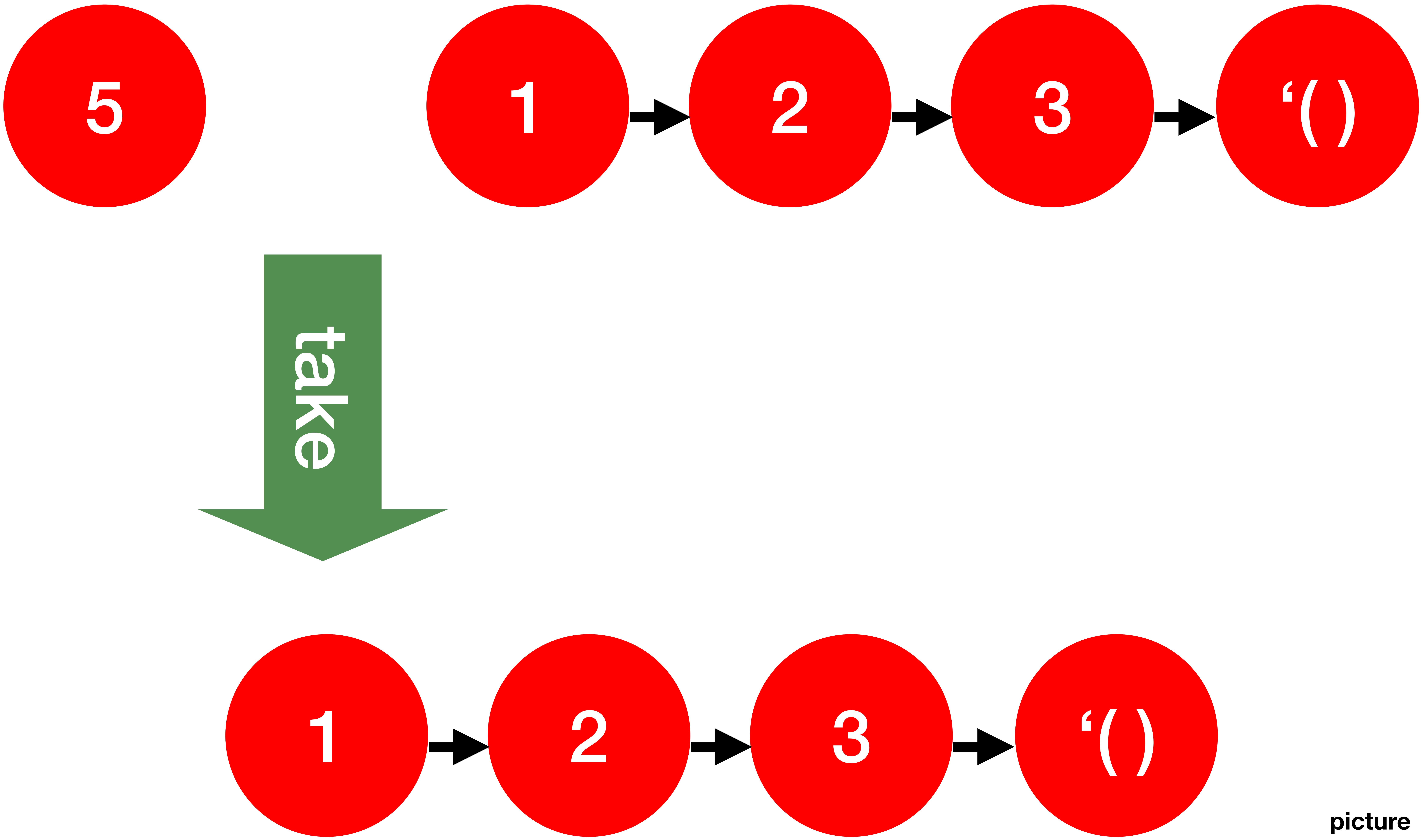

5
1
2
3
‘()
take
1
2
3
‘()
picture

```
; (take 0 l) ; '()
; (take 0 '()) ; '()
; (take n '()) ; '()
; (take n l of size > n) ; l of size n
; (take n l of size < n) ; l
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
(define (take n l)
  (cond
    [(zero? n) '()]
    [(empty? l) ‘()]
    [else
     (cons (first l)
           (take (sub1 n)
                 (rest l)))]))
```

```
(define recusive-process
  '("identify principal"
    "test basis"
    "reduced recursion"
    "combine results"))
```

take : (n : nat) -> (l : list) -> list

principal	n : nat	l : list
basis	zero?	empty?
reducer	sub1	rest
combine		cons

```
(define (take n l)
  (cond
    [(zero? n) '()]
    [(empty? l) ‘()]
    [else
     (cons (first l)
           (take (sub1 n)
                 (rest l)))]))
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```racket
; (take 0 l) ; '()

(check-satisfied
 (take 0 '(1 2 3))
 empty?)
```

```
; (take 0 '()) ; '()

(check-satisfied
 (take 0 '())
 empty?)
```

```
; (take n '()) ; '()

(check-satisfied
 (take 2 '())
 empty?)
```

```
; (take n l of size > n) ; l of size n

(check-expect
 (take 2 '(1 2 3))
 '(1 2))
```

```
; (take n l of size > n) ; l of size n

(check-expect
 (take
  (add1 (add1 0))
  (cons 1 (cons 2 (cons 3 '()))))
 (cons 1 (cons 2 '())))
```

```
; (take n l of size > n) ; l of size n
(check-expect
 (take
  (add1 (add1 0))
  '(1 2 3))
 (cons (first '(1 2 3))
       (cons (first '(2 3))
             '())))
```

```
; (take n l of size < n) ; l

(check-expect
 (take 5 '(1 2 3))
 '(1 2 3))
```


take

```
(define (take n l)
  (cond
    [(zero? n) '()]
    [(empty? l) ‘()]
    [else
     (cons (first l)
           (take (sub1 n)
                 (rest l)))]))
```


drop

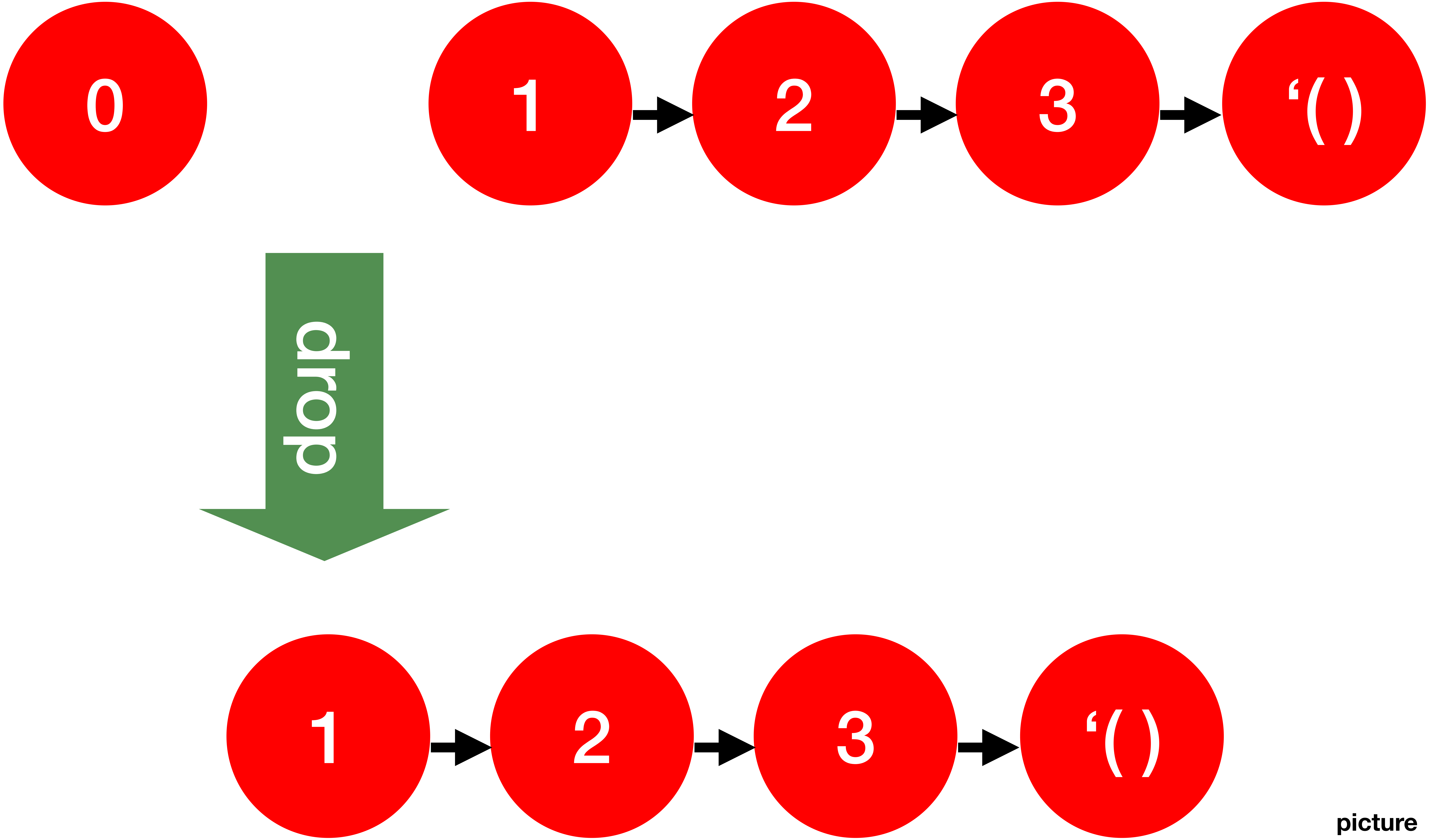
```racket
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

given a natural number n and a list yield a list without the first n elements

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
; drop : nat -> list -> list
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```



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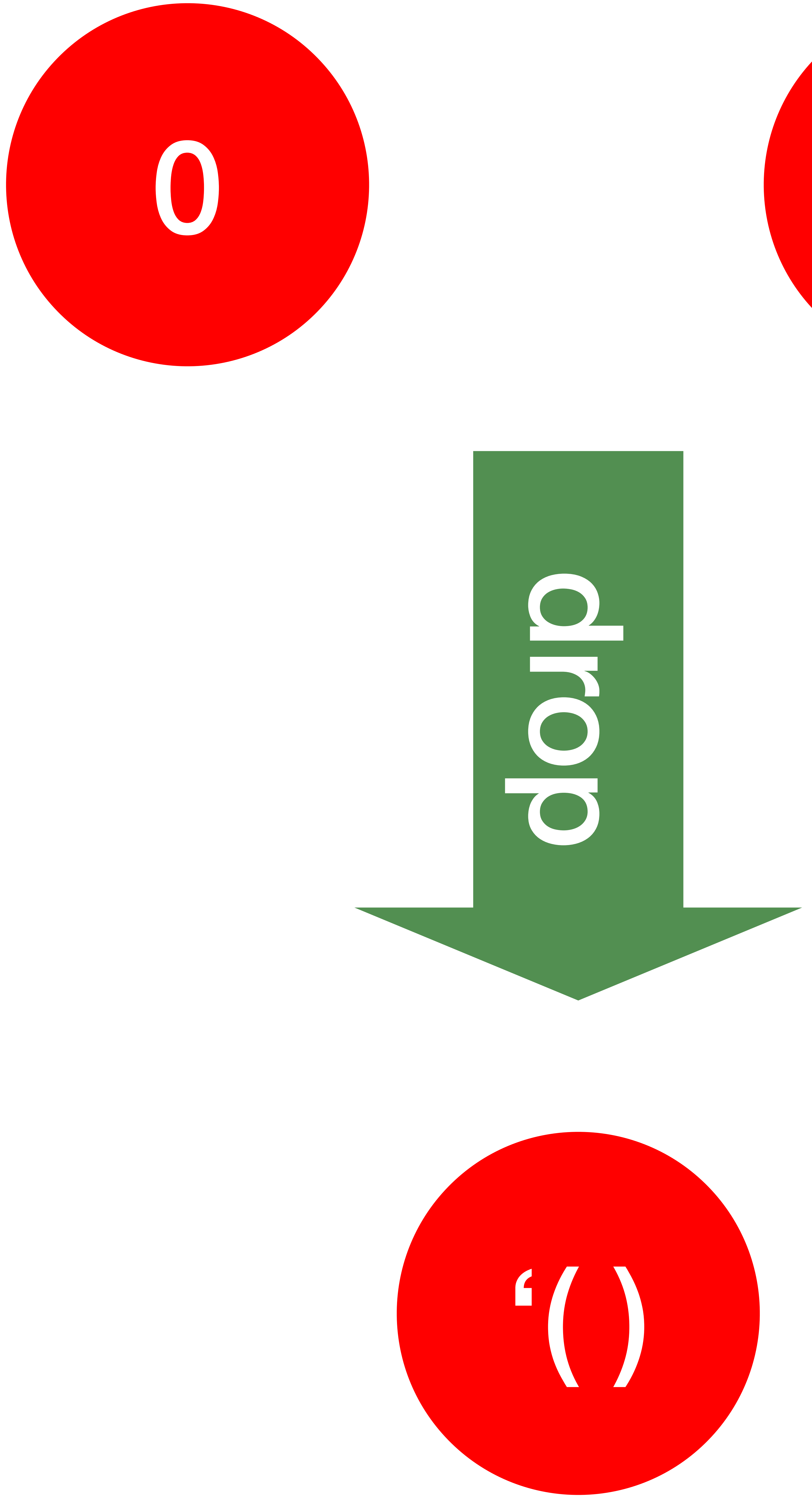

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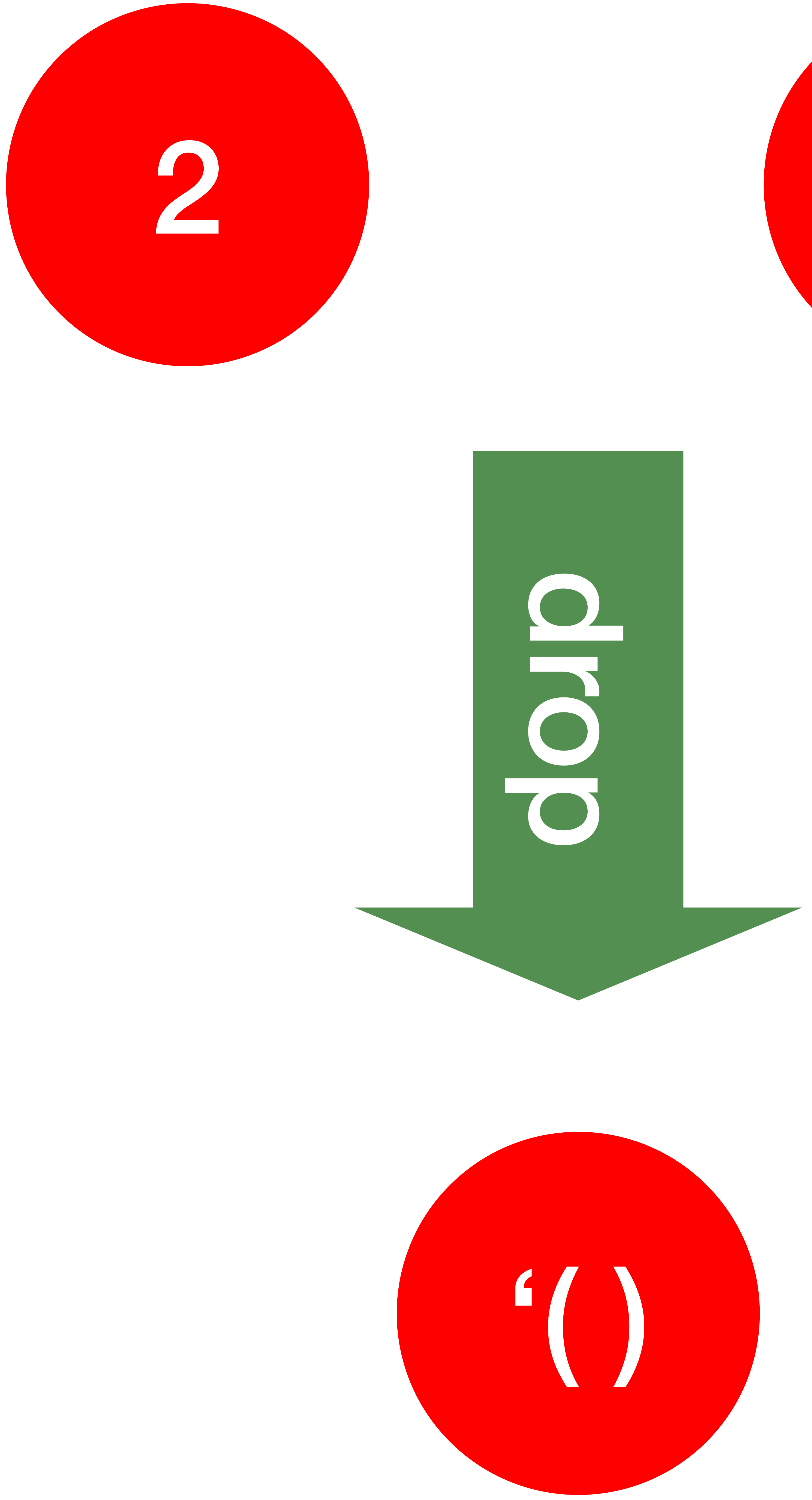

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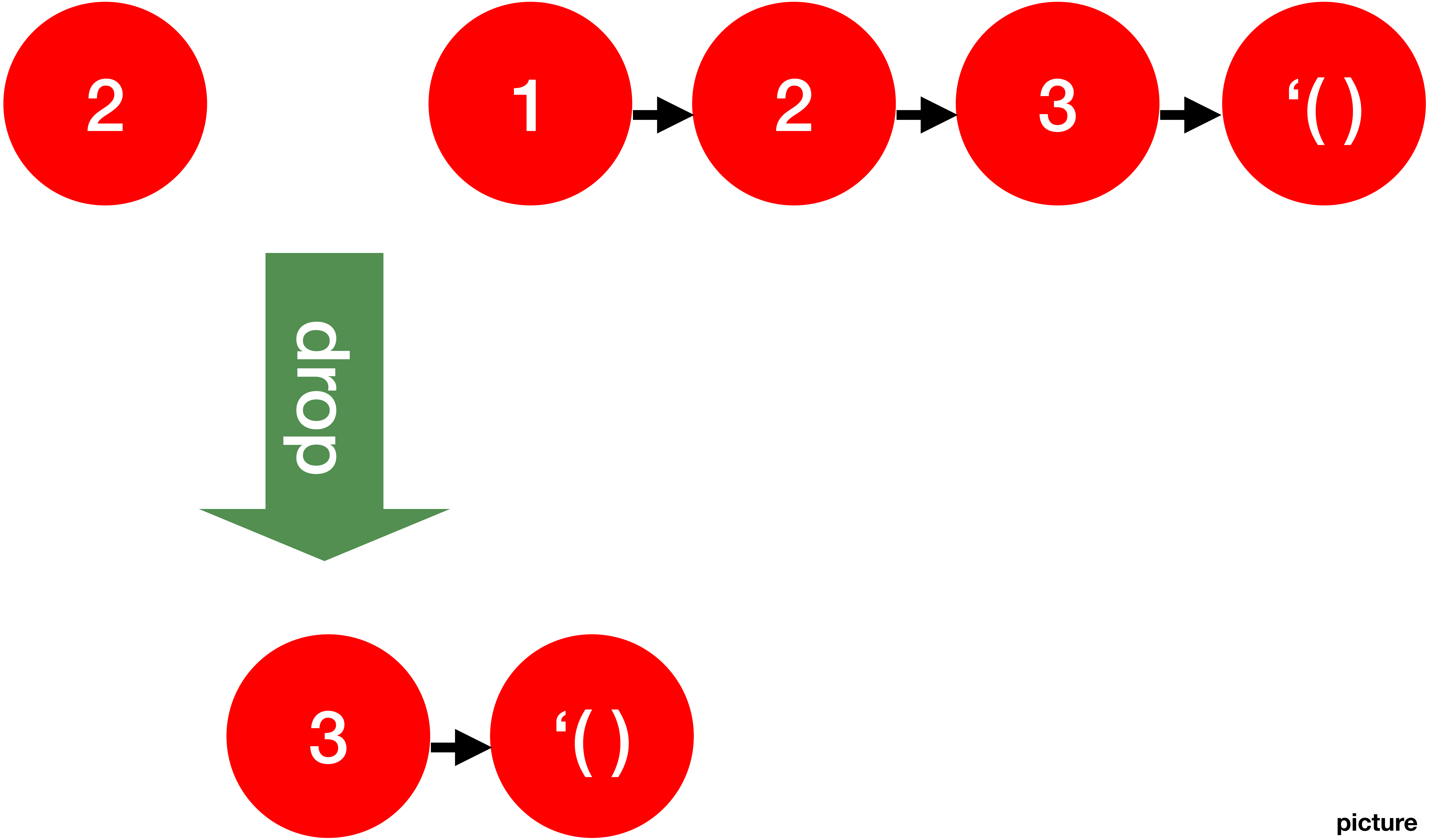

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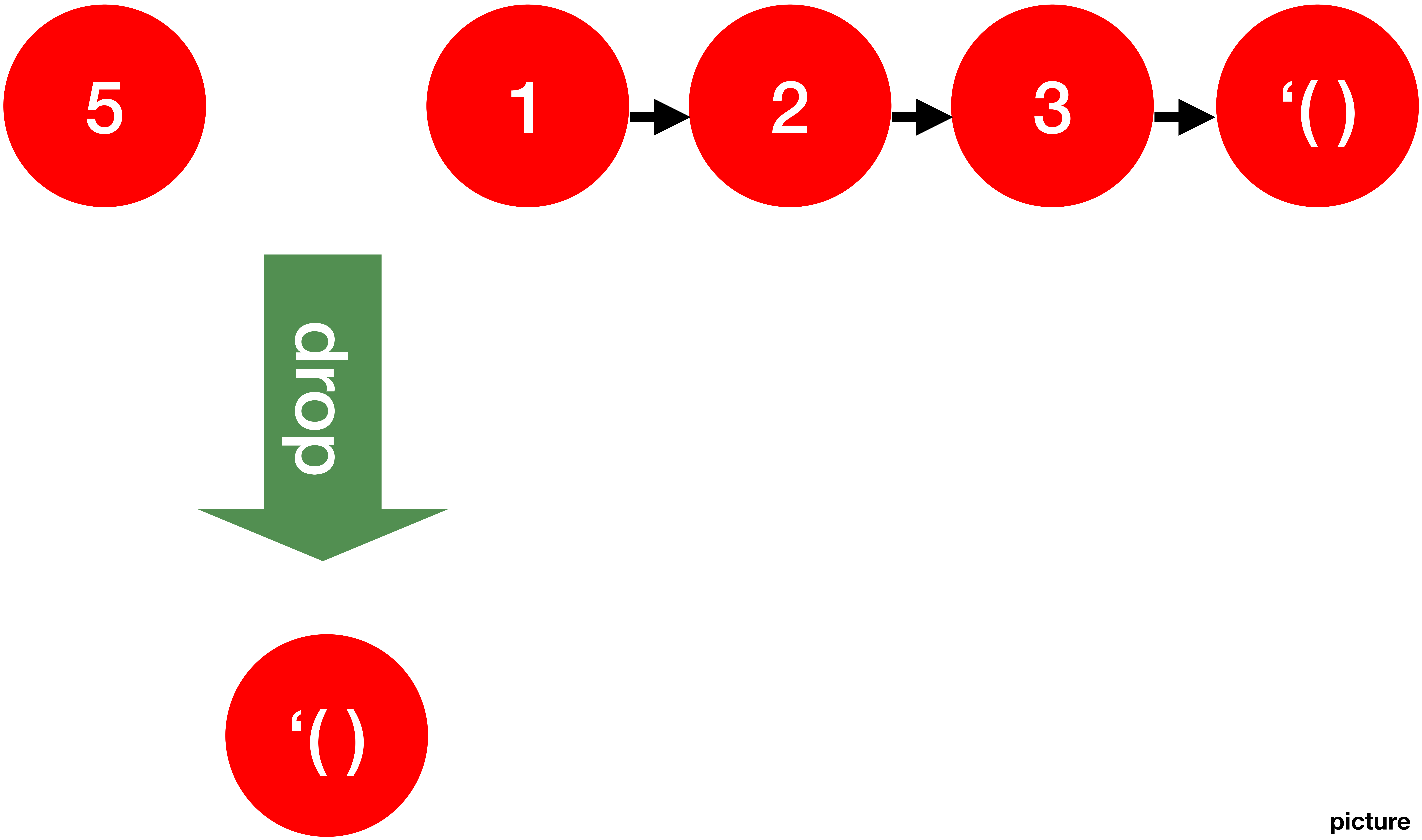


picture

```
; (drop 0 l) ; l
; (drop 0 '()) ; '()
; (drop n '()) ; '()
; (drop n l of size n + m) ; l of size m
; (drop n l of size < n) ; '()
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```racket
(define (drop n l)
  (cond
    [(zero? n) l]
    [(empty? l) '()]
    [else
     (drop (sub1 n)
           (rest l))]))
```

```racket
(define recusive-process
  '("identify principal"
    "test basis"
    "reduced recursion"
    "combine results"))
```

drop : (n : nat) -> (l : list) -> list

principal	n : nat	l : list
basis	zero?	empty?
reducer	sub1	rest
combine		

```racket
(define (drop n l)
  (cond
    [(zero? n) l]
    [(empty? l) '()]
    [else
     (drop (sub1 n)
           (rest l))]))
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
; (drop 0 l) ; l

(check-expect
 (drop 0 '(1 2 3))
 '(1 2 3))
```

```
; (drop 0 '()) ; '()

(check-satisfied
 (drop 0 '())
 empty?)
```

```racket
; (drop n '()) ; '()

(check-satisfied
 (drop 2 '())
 empty?)
```

```
; (drop n l of size n + m) ; l of size m

(check-expect
 (drop 2 '(1 2 3))
 '(3))
```

```
; (drop n l of size < n) ; '()

(check-satisfied
 (drop 5 '(1 2 3))
 empty?)
```

drop

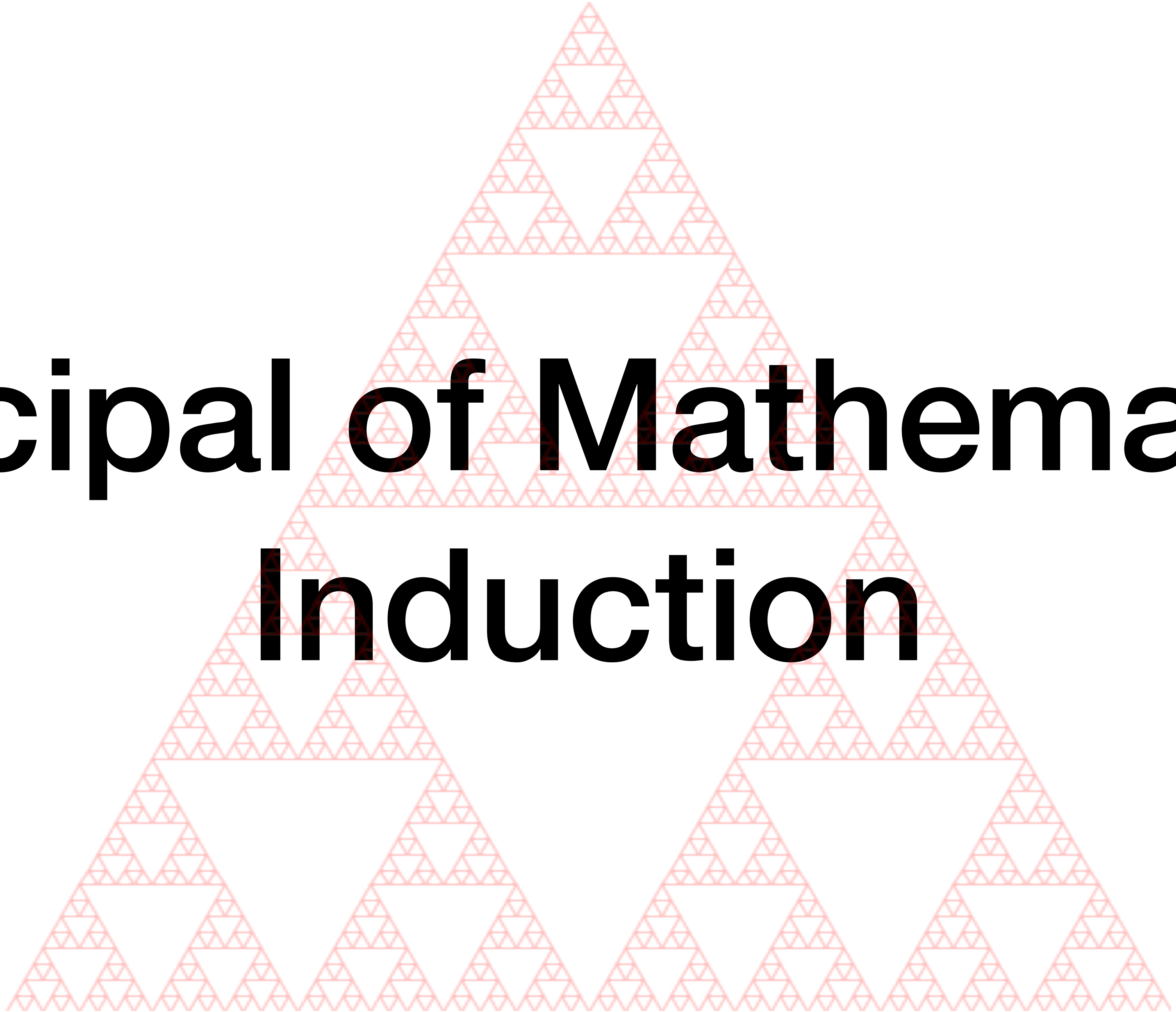
```
(define (drop n l)
  (cond
    [(zero? n) l]
    [(empty? l) '()]
    [else
     (drop (sub1 n)
           (rest l))]))
```

take vs. drop

```
(define (take n l)
  (cond
    [(zero? n) '()]
    [(empty? l) ‘()]
    [else
     (cons (first l)
           (take (sub1 n)
                 (rest l)))]))
```

```
(define (drop n l)
  (cond
    [(zero? n) l]
    [(empty? l) '()]
    [else
     ; combine-solutions
      (drop (sub1 n)
            (rest l))]))
```

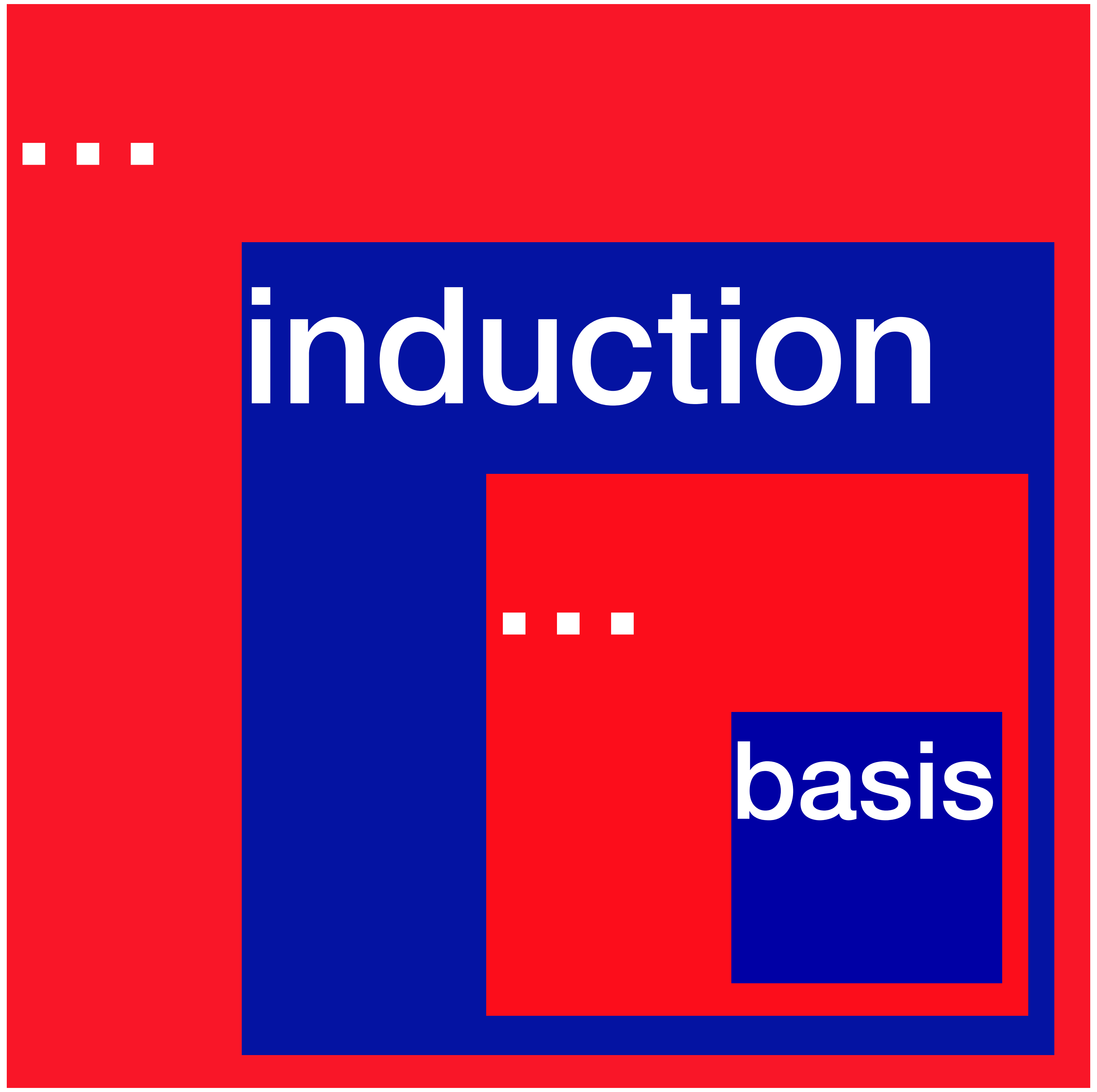
Principal of Mathematical Induction



P holds for all values 0, 1, 2, … if

basis step: **P**(0) is true

induction step: **P**(k) is true => **P**(k + 1) is also true



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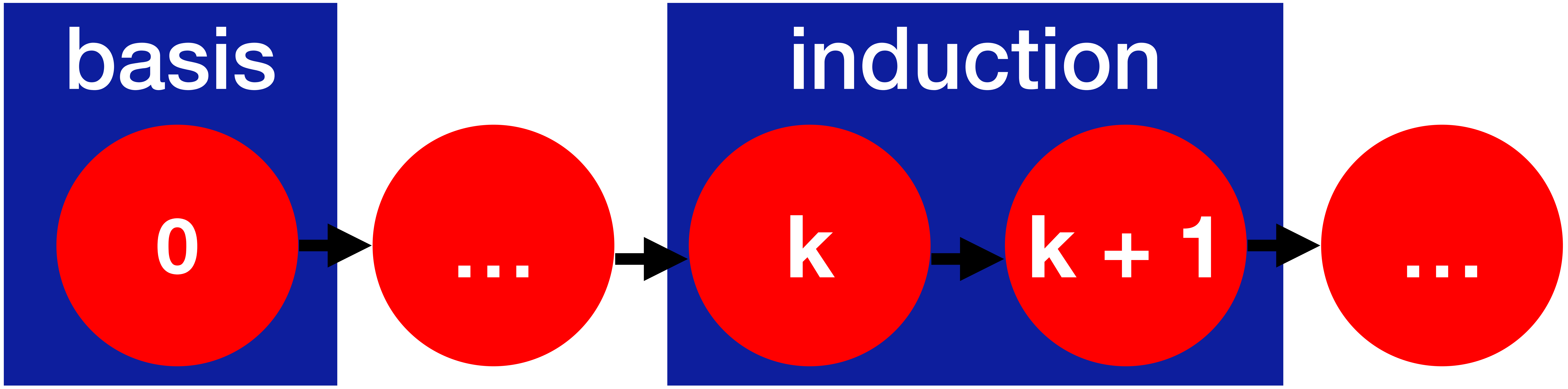
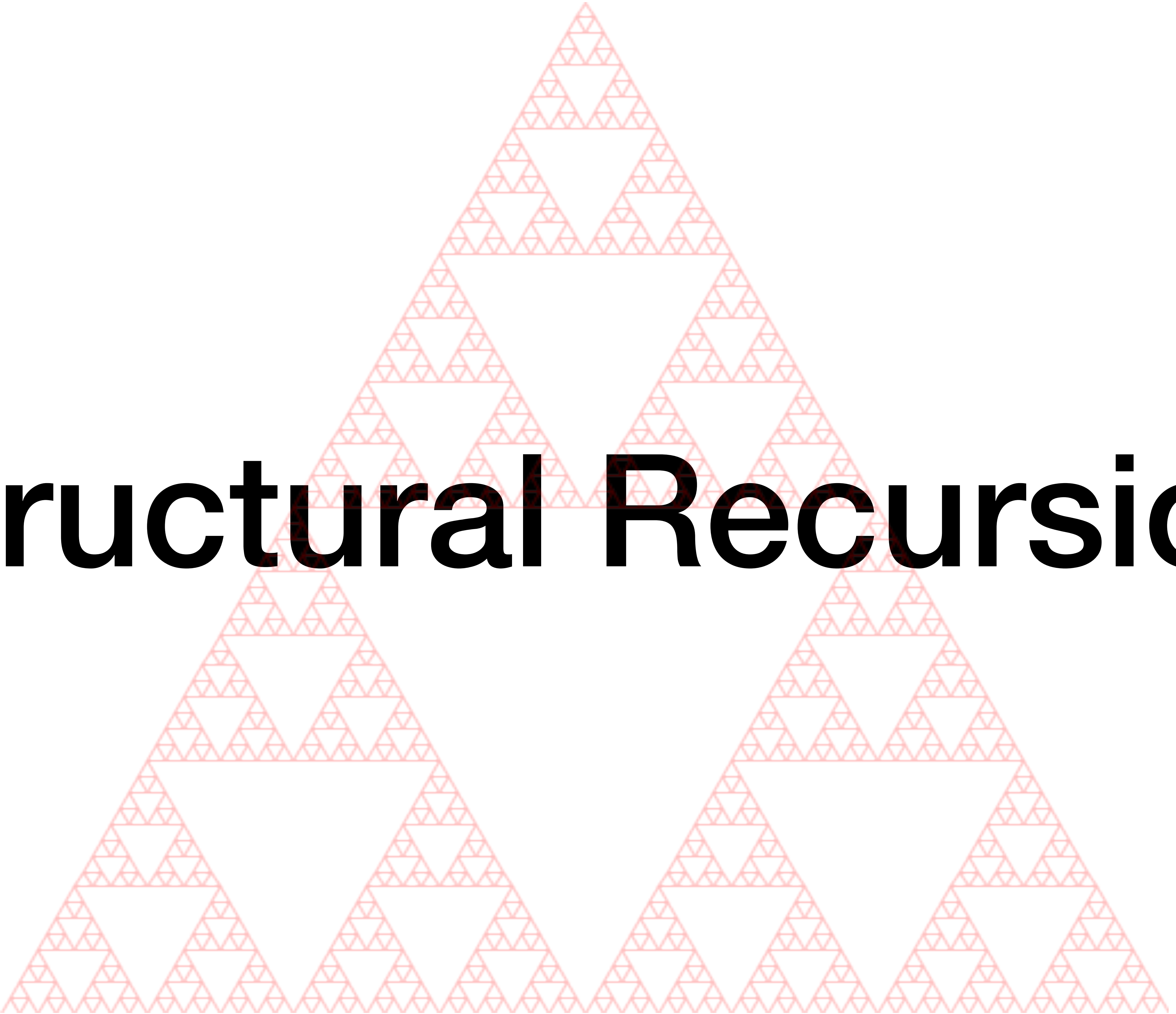


picture

Structural Recursion



```
(define (structural P)
  (cond
    [(basis? P) (solve P)]
    [else
     (combine-solutions
       P
       (structural
         (induction P)))]))
```

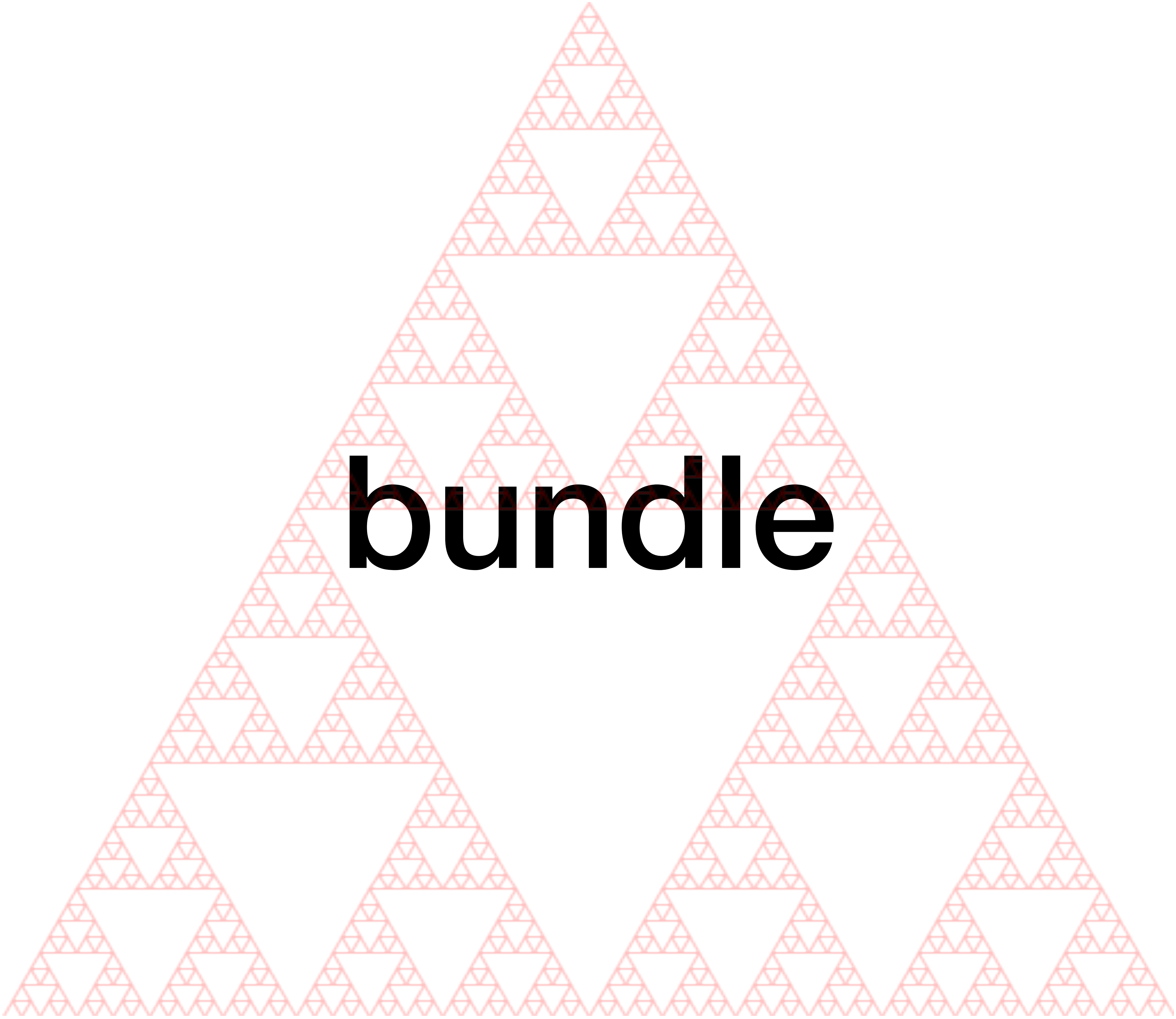
template

take vs. drop

```racket
(define (take n l)
  (cond
    [(zero? n) '()]
    [(empty? l) '()]
    [else
     (cons (first l)
           (take (sub1 n)
                 (rest l)))]))
```

```racket
(define (drop n l)
  (cond
    [(zero? n) l]
    [(empty? l) '()]
    [else
    ; combine-solutions
     (drop (sub1 n)
           (rest l))]))
```

```
(define topics
 (make-agenda
  '("Recursive Data Type"
    "Structural Recursion"
    "General Recursion")))
```



bundle

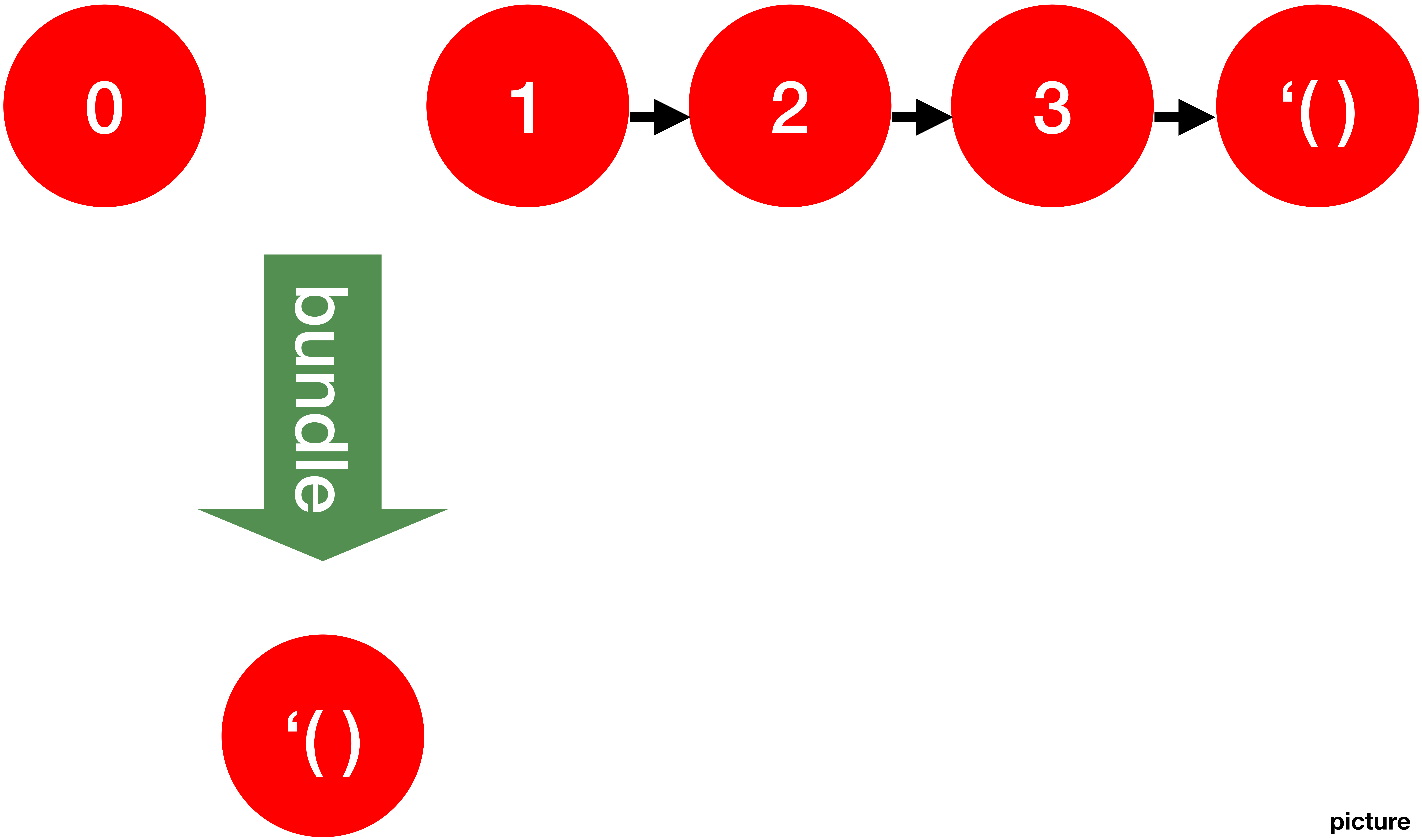
```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

given a natural number n and a list yield a list of lists each with at least n elements

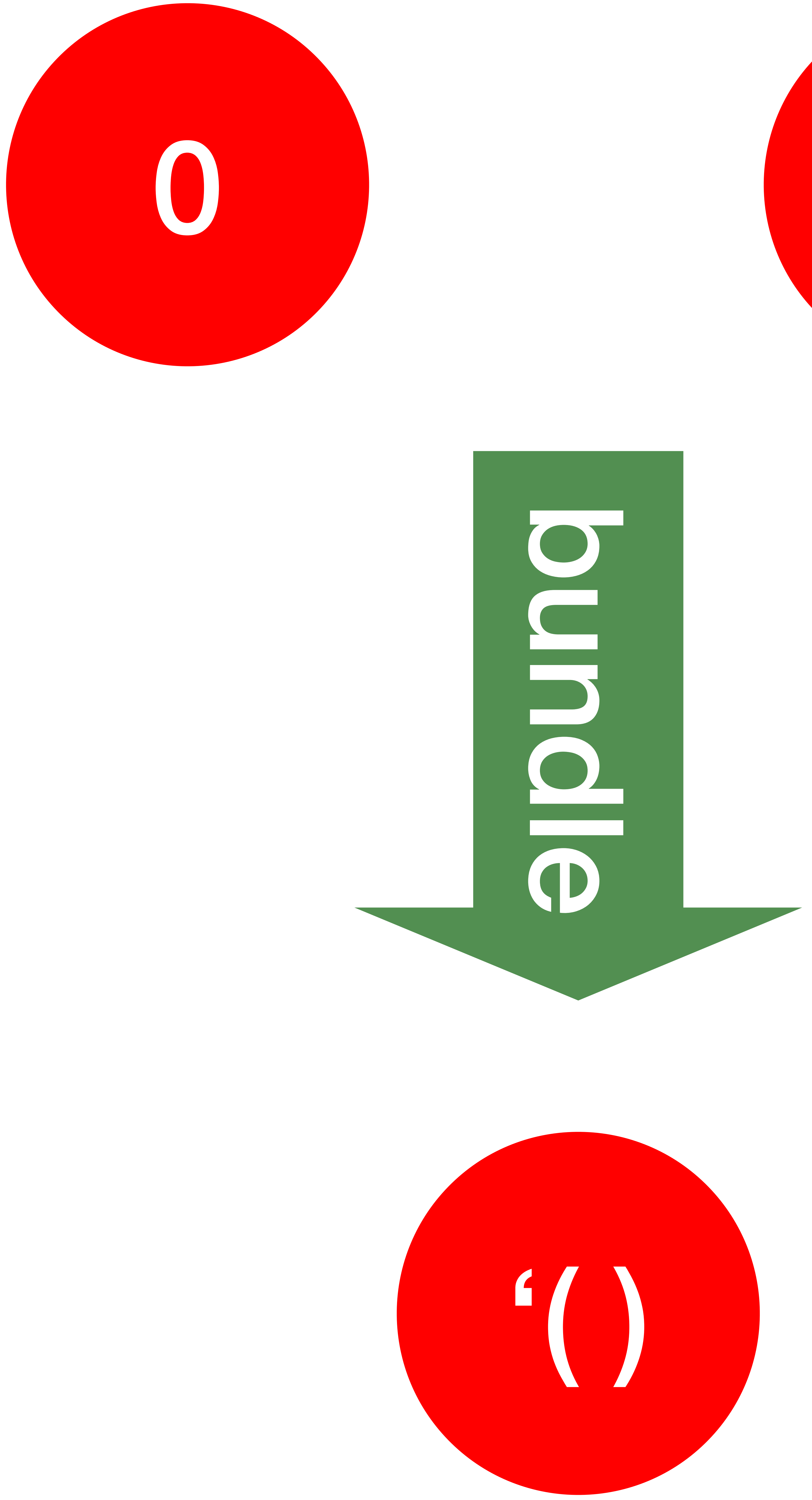
```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
; bundle : nat -> list -> list
```

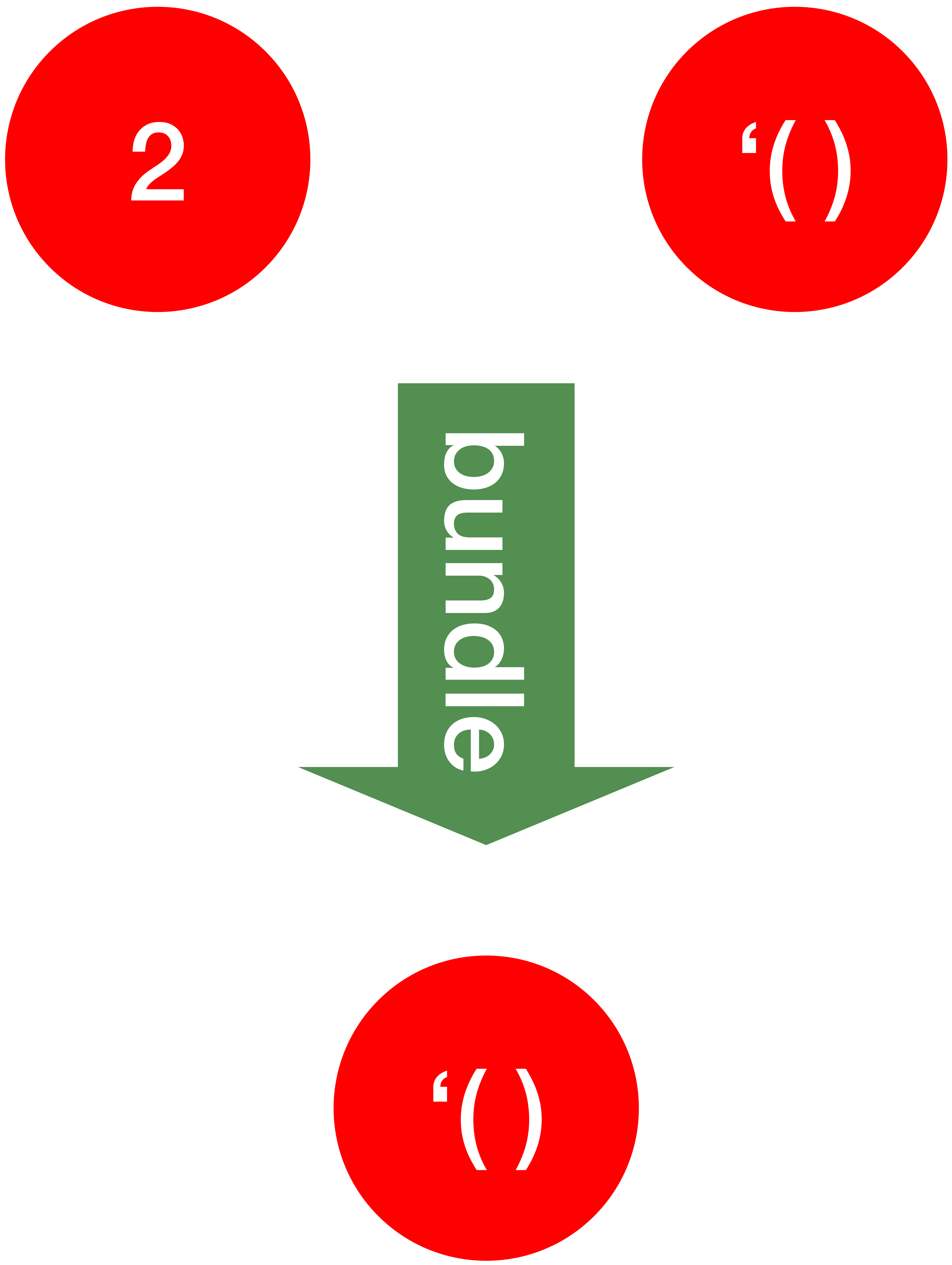
```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```



picture



picture



picture

picture


3
1
...
7
‘()
bundle
1
2
3
‘()
4
5
6
‘()
7
‘()
list
picture

```
; (bundle 0 l) ; '()
; (bundle 0 '()) ; '()
; (bundle n '()) ; '()
; (bundle n l of size n * m) ; '(l of size m, ...)
; (bundle n l of size n * m + p)
    ; '(l of size m, ..., l of size p)
; (bundle n l of size < n) ; '(l)
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
(define (bundle n l)
  (cond
    [(zero? n) '()]
    [(empty? l) '()]
    [else
     (cons (take n l)
       (bundle n (drop n l)))]))
```

```
(define recusive-process
  '("identify principal"
    "test basis"
    "reduced recursion"
    "combine results"))
```

bundle : (n : nat) -> (l : list) -> list list

principal		l : list
basis	zero?	empty?
reducer		take n
combine		cons

```
(define (bundle n l)
  (cond
    [(zero? n) '()]
    [(empty? l) '()]
    [else
     (cons (take n l)
       (bundle n (drop n l)))]))
```

```
(define design-steps
  '("problem analysis"
    "function signature"
    "examples"
    "function definition"
    "tests"))
```

```
; (bundle 0 l) ; '()

(check-satisfied
 (bundle 0 '(1 2 3))
 empty?)
```

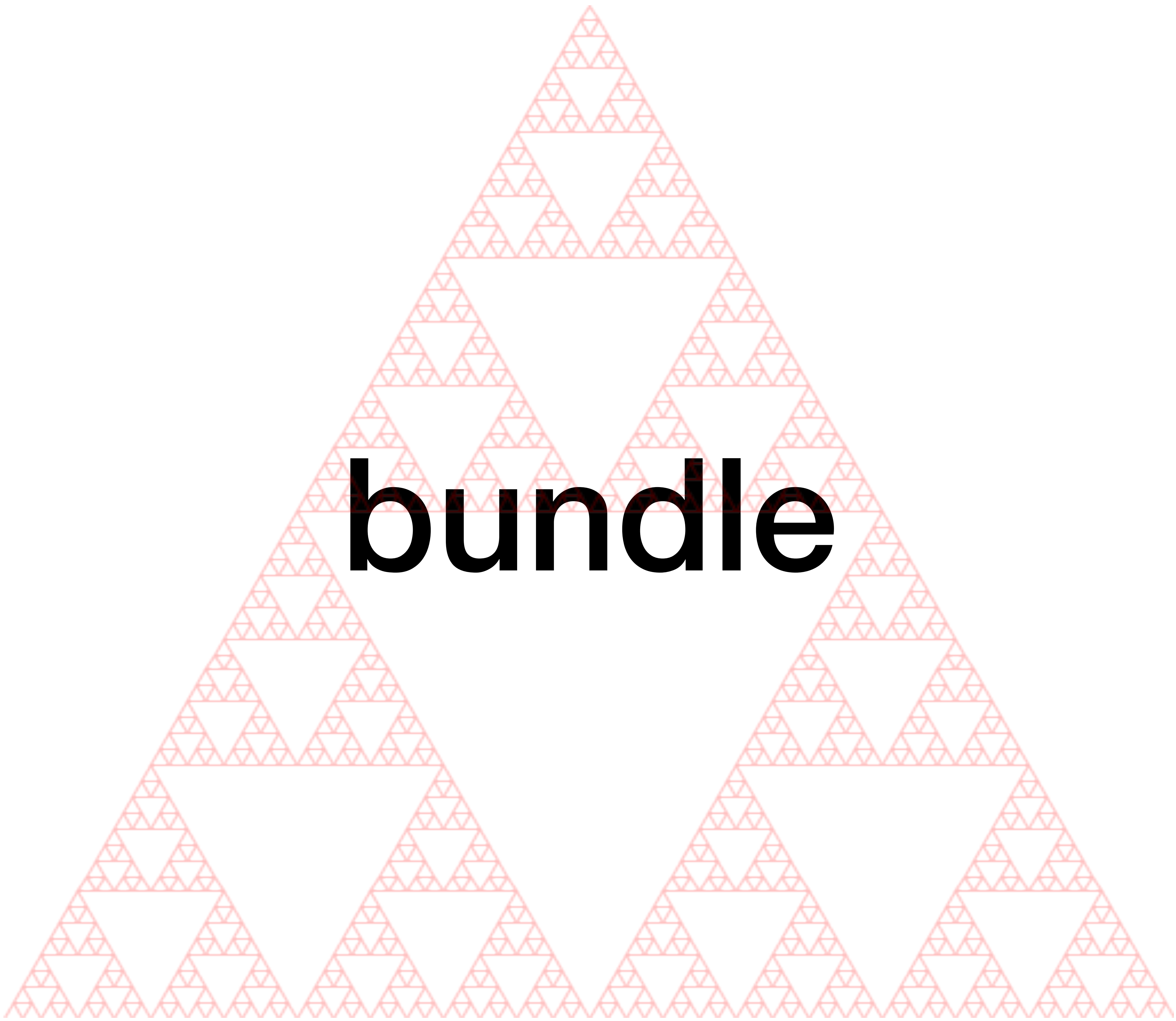
```
; (bundle 0 '()) ; '()

(check-satisfied
 (bundle 0 '())
 empty?)
```

```racket
; (bundle n '()) ; '()

(check-satisfied
 (bundle 2 '())
 empty?)
```

```
; (bundle n l of size n * m) ; '(l of size m, ...)

(check-expect
 (bundle 2 '(1 2 3 4))
 '((1 2) (3 4)))
```

```
; (bundle n l of size n * m + p)
;    '(l of size m, ..., l of size p)

(check-expect
 (bundle 3 '(1 2 3 4 5 6 7))
 '((1 2 3) (4 5 6) (7)))
```

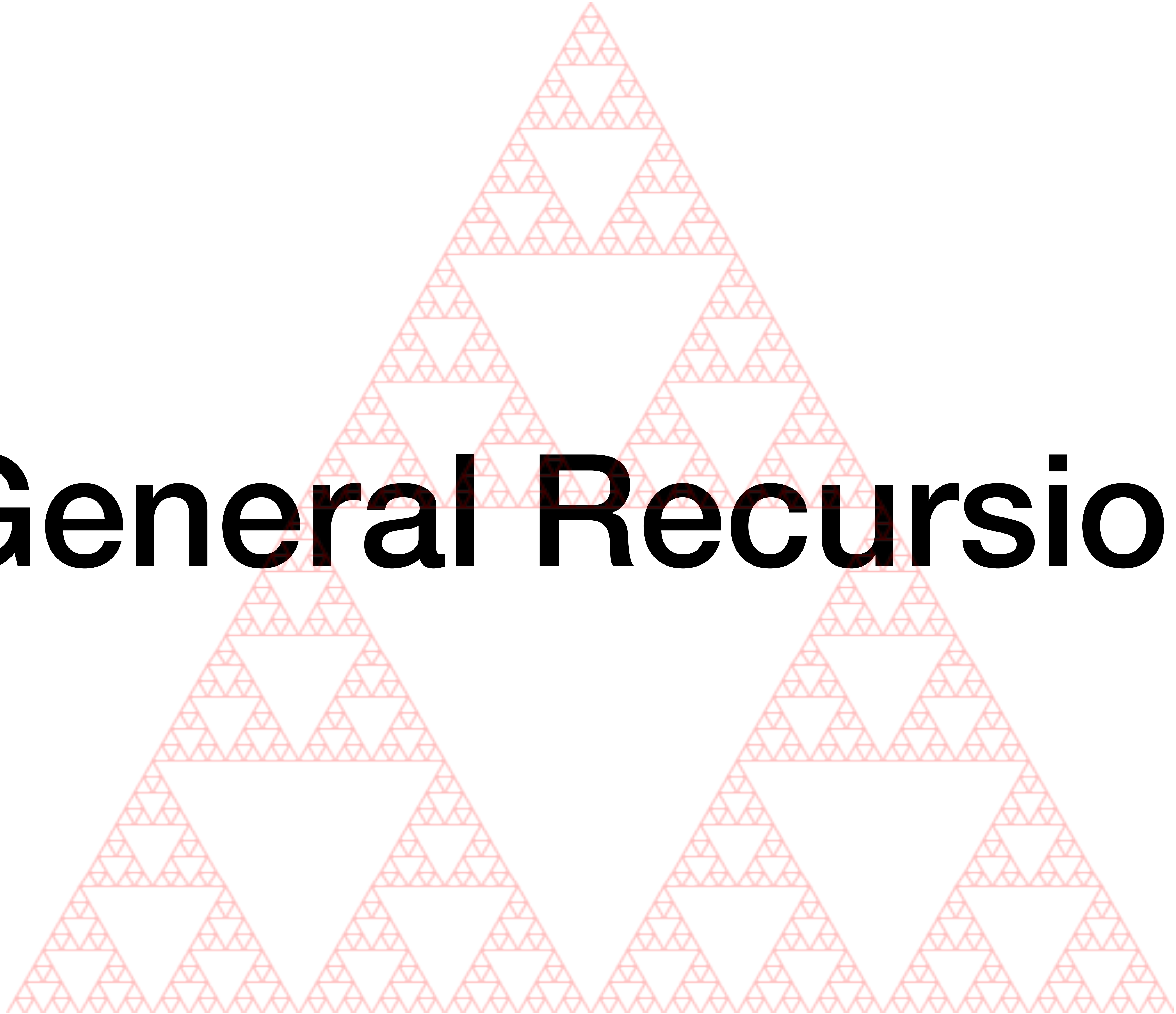
```
; (bundle n l of size < n) ; '(l)

(check-expect
 (bundle 3 '(1 2))
 '((1 2)))
```


bundle

```
(define (bundle n l)
  (cond
    [(zero? n) '()]
    [(empty? l) '()]
    [else
     (cons (take n l)
       (bundle n (drop n l)))]))
```

```
(define recusive-process
  '("identify principal"
    "test basis"
    "reduced recursion"
    "combine results"))
```

General Recursion



```
(define (general P)
  (cond
    [(trivial? P) (solve P)]
    [else
     (combine-solutions
       P
       (general
         (generate P)))]))
```

template

general vs. structural

```
(define (general P)
  (cond
    [(trivial? P) (solve P)]
    [else
     (combine-solutions
       P
       (general
         (generate P)))]))
```

```
(define (structural P)
  (cond
    [(basis? P) (solve P)]
    [else
     (combine-solutions
       P
       (structural
         (induction P)))]))
```

template

structural is a special case of general recursion

```
(define topics
 (make-agenda
  '("Recursive Data Type"
    "Structural Recursion"
    "General Recursion")))
```

Things to Consider

- How would we write a tail call version of plus, take, and bundle?
- What would these examples look like in a strongly typed language?
- How did we get around not having to use the Y combinator or a fix point?

Further Readings

- **How to Design Programs** 2nd edition by Matthias Felleisen et. al.
  https://htdp.org/2018-01-06/Book/
- **An Introduction to Functional Programming Through Lambda Calculus** by Greg Michaelson
  http://store.doverpublications.com/0486478831.html
- **Type-Driven Development with Idris** by Edwin Brady
  https://www.manning.com/books/type-driven-development-with-idris
- **Coq'Art: The Calculus of Inductive Constructions** by Yves Bertot and Pierre Castéran
  https://www.springer.com/us/book/9783540208549

```
(define programmer-recursive-joke
  (make-joke
   "Why did the programmer run out of shampoo?"
   "The instructions said: lather, rinse, repeat."))
```

Thank you

Mike Harris

Recursion -
Lather. Rinse. Repeat.
https://bit.ly/2qjVHU0

@MikeMKH

Source

- examples and slides: https://github.com/MikeMKH/talks/tree/master/recursion-lather-rinse-repeat

Images

- Racket logo: by Matthias.f at en.wikipedia, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=14868979
- Self: taken by Kelsey Harris at StrangeLoop

appendix
(plus 1 2) expanded using DrRacket

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(plus 1 2)
```

➡

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(cond
 ((zero? 1) 2)
 (else
  (add1 (plus (sub1 1) 2))))
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(cond
 ((zero? 1) 2)
 (else
  (add1 (plus (sub1 1) 2))))
```

→

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(cond
 (#false 2)
 (else
  (add1 (plus (sub1 1) 2))))
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(cond
 (#false 2)
 (else
  (add1 (plus (sub1 1) 2))))
```

→

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(cond
 (else
  (add1 (plus (sub1 1) 2))))
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(cond
 (else
  (add1 (plus (sub1 1) 2))))
```

→

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1 (plus (sub1 1) 2))
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1 (plus (sub1 1) 2))
```

→

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1 (plus 0 2))
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1 (plus 0 2))
```

→

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1
 (cond
  ((zero? 0) 2)
  (else
   (add1 (plus (sub1 0) 2)))))
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1
 (cond
  ((zero? 0) 2)
  (else
   (add1 (plus (sub1 0) 2)))))
```

→

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1
 (cond
  (#true 2)
  (else
   (add1 (plus (sub1 0) 2)))))
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1
 (cond
  (#true 2)
  (else
   (add1 (plus (sub1 0) 2)))))
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1 2)
```

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
(add1 2)
```

→

```
(define (plus a b)
  (cond
   ((zero? a) b)
   (else
    (add1 (plus (sub1 a) b)))))
3
```