Study Notes for Topology

Mike Mingcheng Wei

May 8, 2020

Contents

Chapter 1

Set Theory and Logic

1.1 1 Fundamental Concepts

Remark 1 *Vacuously True: If hypothesis, $x^2 < 0$, does not hold, then conclusion, $x = 23$, is always true regardless of the conclusions. This vacuously true is used to define a empty set belongs to a non-empty set.*

Definition 2 *Given a statement of the form "If P, then Q", its **contrapositve** is defined to be the statement "If Q is not true, then P is not ture."*

Definition 3 *Given a statement of the form "P", its **negation** is defined to be the statement "not P". (To form the negation of statment, one replaces the quantifer "for every" by the quantifier "for at least one", and one replaces statement "P" by its negation, "not P")*

1.2 2 Functions

Definition 4 *A **Rule of Assignment** is a subset r of the cartesian product $C \times D$ of two sets, having the property that each element of C appears as the first coordinate of at most one ordered pair belonging to r.*

$$[(c, d) \in r \text{ and } (c, d^{'}) \in r] \Rightarrow [d = d^{'}]$$

Definition 5 *Given a rule of assignment r, the **domain** of r is defined to be the subset of C consising of all first coordinates of elements of r, and the **image set** of r is defined as the subset of D consisting of all second coordinates of elements of r.*

Definition 6 *A **function** f is a rule of assignment r, together with a set B that contains the image set of r. The domain A of the rule r is also called the **domain** of the function f; the image set of r is also called the **image set** of f; and the set B is called the **range** of f.*

Definition 7 *Given functions $f : A \to B$ and $g : B \to C$, we define the **composite** $g \circ f$ of f and g as the function $g \circ f : A \to C$ defined by the equation $(g \circ f)(a) = g(f(a))$.*

Definition 8 *A function $f : A \to B$ is said to be **injective** (or one-to-one) if for each pair of distinct points of A, their images under f are distinct. It is said to be **surjective** (or onto) if every element of B is the image of some element of A under the function f. If f is both injective and surjective, it is said to be **bijective** (or one-to-one correspondence)*

Remark 9 *The composite of two injective functions is injective, and the composite of two surjective functions is surjective, and the composite of two bijective functions is bijective.*

Definition 10 *If f is bijective, there exists a function from B to A called the **inverse** of f, denoted by f^{-1}.*

Remark 11 *If f is bijective, then f^{-1} is bijective.*

Lemma 12 *(Lemma 2.1)Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a$ for every a in A and $f(h(b)) = b$ for every b in B, then f is bijective and $g = h = f^{-1}$.*

Definition 13 *Let $f : A \to B$. If A_0 is a subset of A, we denote by $f(A_0)$ the set of all images of points of A_0 under the function f; this set is called the **image** of A_0 under f. On the other hand, if B_0 is a subset of B, we denote by $f^{-1}(B_0)$ the set of all elements of A whose images under f lie in B_0; it is called the **preimage** of B_0 under f (or the "counterimage" or the "inverse image" of B_0).*

1.3 3 Relations

Definition 14 *A **relation** on a set A is a subset C of the cartesian product $A \times A$.*

Definition 15 *An **equivalence relation** on a set A is a relation C on A having the following three properties:*

1. *(Reflexivity) xCx for every x in A;*
2. *(Symmetry) if xCy, then yCx;*
3. *(Transitivity) if xCy and yCz, then xCz.*

Lemma 16 *(3.1) Two equivalence classes E and $E^{'}$ are either disjoint or equal. ($S/\tilde{}$: the set of equivalent classes.)*

Definition 17 *A **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.*

Definition 18 *A relation C on a set A is called on **order relation** (or a simple order, or a linear order) if it has the following properties:*

1. *(Comparability) For every x and y in A for which $x \neq y$, either xCy or yCx;*
2. *(Nonreflexivity) For no x in A does the relation xCx hold;*
3. *(Transitivity) If xCy and yCz, then xCz.*

*(If relations only satisfy 2 and 3, it is called **strict partial order** relations.)*

Definition 19 *If X is a set and $<$ is an order relation on X, and if $a < b$, we use the notation (a, b) to denote the set*

$$\{x|a < x < b\};$$

*it is called an **open interval** in X. If this set is empty, we call a the **immediate predecessor** of b, and we call b the **immediate successor** of a.*

Definition 20 *Suppose that A and B are two sets with order relations $<_A$and $<_B$respectively. We say that A and B have the **same order type** if there is a bijective corespondence between them that preserves order; that is, if there exists a bijective function $f : A \to B$ such that*

$$a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2)$$

Definition 21 *An ordered set A is said to have the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound. Analogously, the set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.*

(A has the least upper bound property if and only if it has the greatest lower bound property.)

1.4 4 The Integers and the Real Numbers

Definition 22 *A subset A of the real numbers is said to be **inductive** if it contains he number 1, and if for every x in A, the number $x+1$ is also in A. Let Δ be the collection of all inductive subsets of $\mathbb{R}$. Then the set $\mathbb{Z}_+$ of **positive integers** is defined by the equation*

$$\mathbb{Z}_+ = \bigcap_{A \in \Delta} A$$

Remark 23 *The basic properties of $\mathbb{Z}_+$, which follow readily from the definition, are the following:*
1). $\mathbb{Z}_+$ is inductive.
2). (Principle of induction) If A is an inductive set of positive integers, then $A = \mathbb{Z}_+$.

Theorem 24 *(4.1) (Well-ordering property) Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.*

Definition 25 *If n is a positive integer, we use the symbol S_n to denote the set of all positive integers less than n; we call it a **section** of the positive integers.*

$$\{1, 2, ..., n\} = S_{n+1}$$

Theorem 26 *(4.2) (Strong induction principle) Let A be a set of positive integers. Suppose that for each positive integer n, the statement $S_n \subset A$ implies the statement $n \in A$. Then $A = \mathbb{Z}_+$.*

1.5 6 Finite Sets

Definition 27 *A set is said to be **finite** if there is a bijective correspondence of A with some section of the positive integers. That is A is finite if it is empty or if there is a bijection*

$$f : A \to \{1, ..., n\}$$

*for some positive integer n. In the former case, we say that A has **cardinality 0**; in the latter case, we say that A has **cardinality n**.*

Remark 28 *If S is finite set and $f : S \to S$ is a map, then f is injective iff f is surjective.*

Lemma 29 *If S is a finite set, the n defined above is unique.*

Lemma 30 *If S is a finite set and $f : S \to T$ is surjective, then T is a finite set and $|T| \leq |S|$.*

Lemma 31 *(6.1) Let n be a positive integer. Let A be a set; let a_0 be an element of A. Then there exists a bijective correspondence f of the set A with the set $\{1, 2, ..., n+1\}$ if and only if there exists a bijective correspondence g of the set $A - \{a_0\}$ with the set $\{1, 2, ..., n\}$. (Interesting proof by changing the mapping)*

Theorem 32 *(6.2) Let A be a set; suppose that there exists a bijection $f : A \to \{1, ..., n\}$ for some $n \in \mathbb{Z}_+$. Let B be a proper subset of A. Then there exists no bijection $g : B \to \{1, ..., n\}$; but (provided $B \neq \varnothing$) there does exist a bijection $h : B \to \{1, ..., m\}$ for some $m < n$.*

Corollary 33 *(6.3) If A is finite, there is no bijection of A with a proper subset of itself.*

Corollary 34 *(6.4) $\mathbb{Z}_+$ is not finite.*

Corollary 35 *(6.5) The cardinality of a finite set A is uniquely determined by A.*

Corollary 36 *(6.6) If B is a subset of the finite set A, then B is finite. If B is a proper subset of A, then the cardinality of B is less than the cardinality of A.*

Corollary 37 *(6.7) Let B be a nonempty set. Then the following are equivalent:*
1). B is finite;
2). There is a surjective function from a section of the positive integers onto B;
3). There is an injective function from B into section of the positive integers.

Corollary 38 *(6.8) Finite unions and finite cartesian products of finite sets are finite.*

1.6 7 Countable and Uncontable Sets

Definition 39 *A set A is said to be* ***infinite*** *if it is not finite. It is said to be* ***countably infinite*** *if there is a bijective correspondence:*

$$f : A \to \mathbb{Z}_+$$

Definition 40 *A set is said to be countable if it is either finite or* ***countably infinite****. A set that is not countable is said to be* ***uncountable****.*

Theorem 41 *Let B be a nonempty set. Then the following are equivalent:*

1. *B is countable;*
2. *There is a surjective function $f : \mathbb{Z}_+ \to B$.*
3. *There is an injective function $g : B \to \mathbb{Z}_+$.*

Lemma 42 *(7.2) If C is an infinite subset of $\mathbb{Z}_+$, then C is countably infinite.*

Definition 43 ***Principle of recursive definition****: Let A be a set. Given a formula that defines $h(1)$ as a unique element of A, and for $i > 1$ defines $h(i)$ uniquely as an element of A in terms of the values of h for positive integers less than i, this formula determines a unique function $h : \mathbb{Z}_+ \to A$.*

Corollary 44 *(7.3) A subset of a countable set is countable.*

Corollary 45 *(7.4) The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.*

Theorem 46 *(7.5) A countable union of countable sets is countable.*

Theorem 47 *(7.6) A finite product of Countable sets is countable.*

Theorem 48 *(7.7) Let X denote the two element set $\{0, 1\}$. Then the set X^ω is uncountable.*

Theorem 49 *(7.8) Let A be a set. There is no injective map $f : P(A) \to A$, and there is no surjective map $g : A \to P(A)$.*

Remark 50 *If B is a nonempty set, the existance of an injective map $f : A \to B$ implies the existence of a surjective map $g : B \to A$.*

1.7 9 Infinite Sets and the Axiom of Choice

Theorem 51 *(9.1) Let A be a set. The following statements about A are equivalent:*

1. *There exists an injective function $f : \mathbb{Z}_+ \to A$.*
2. *There exists a bijection of A with a proper subset of itself.*
3. *A is infinite.*

Definition 52 ***Axiom of Choice****: Given a collection Δ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of Δ; that is, a set C such that C is contained in the union of the elements of Δ, and for each $A \in \Delta$, the set $C \cap A$ contains a single element.*

Lemma 53 *(9.2) (Existence of a choice function) Given a collection Δ of nonempty sets (not necessarily disjoint), there exists a function*

$$c : \Delta \to \bigcup_{B \in \Delta} B$$

such that $c(B)$ is an element of B, for each $B \in \Delta$. (Proof is based on Axiom of Choice.)

1.8 10 Well-Ordered Sets

Definition 54 *A set A with an order relation $<$ is said to be **well-ordered** if every nonempty subset of A has a smallest element.*

Remark 55 *1. If A is a well-ordered set, then any subset of A is well-ordered in the restricted order relation.*
2. If A and B are well-ordered sets, then $A \times B$ is well-ordered in the dictionary order.

Theorem 56 *(10.1) Every nonempty finite ordered set has the order type of a section $\{1, ..., n\}$ of $\mathbb{Z}_+$, so it is well-ordered.*

Theorem 57 *(Well-ordering theorem) If A is a set, there exists an order relation on A that is a well-ordering.*

Corollary 58 *There exists an uncountable well-ordered set.*

Definition 59 *Let X be a well-ordered set. Given $\alpha \in X$, le S_α denote the set*

$$S_\alpha = \{x | x \in X \text{ and } x < \alpha\}$$

*It is called the **section of** X **by** α.*

Lemma 60 *(10.2)There exists a well-ordered set A having a largest element Ω, such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.*

Theorem 61 *(10.3) If A is a countable subset of S_Ω, then A has an upper bound in S_Ω.*

Chapter 2

Topological Spaces and Continuous Functions

2.1 12 Topological Spaces

Definition 62 *A* ***Topology*** *on a set X is a collection Υ of subsets of X having the following properties:*

1. *ϕ and X are in Υ;*
2. *The union of the elements of any subcollection of Υ is in Υ;*
3. *The intersection of the elements of any finite subcollection of Υ is in Υ;*

Definition 63 *A set X for which a topology Υ has been specified is called a* ***topological space****.*

Remark 64 *If X is a topological space with topology Υ, we say that a subset U of X is an* ***open set*** *of X if U belongs to the collection Υ. (Hence, open set is defined as a set in topoloby Υ, which is not the "normal" definition in real space. refer to 72) Using this terminology, one can say that a topological space is a set X together with a collection of subsets of X, called open sets, such that ϕ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.*

Definition 65 *Finer, strictly finer, coarse, strictly coarser, comparable.*

2.2 13 Basis for a Topology

Definition 66 *If X is a set, a* ***basis*** *for a topology on X is a collection Ω of subsets of X (called basis elements) such that*

1. For each $x \in X$, there is at least one basis element $B \in \Omega$ containing x.

2. If x belongs to the intersection of two basis elements $B_1 \in \Omega$ and $B_2 \in \Omega$, then there is a basis element $B_3 \in \Omega$ containing x such that $B_3 \subset B_1 \cap B_2$.

Definition 67 *If Ω satisfies these two conditions, then we define the* ***topology*** *Υ* ***generated by*** *Ω as the follows: A subset U of X is said to be open in X (That is, to be an element of Υ) if for each $x \in U$, there is a basis element $B \in \Omega$ such that $x \in B$ and $B \in U$. (Note that each basis element is itself an element of Υ) (Topology Υ generated by Ω is composited by all U which satisfy the above conditions.)*

Remark 68 *(Another way to look at the Topology Υ generated by Ω is that: all open set in Υ can be writen as a union of basis element in Ω)*

Lemma 69 *(13.1 How to generate topology from a basis) Let X be a set; Let Ω be a basis for a topology Υ on X. Then Υ equals the collection of all unions of elements of Ω.*

Lemma 70 *(13.2 How to generate a basis from a topology) Let X be a topological space. Suppose that Ψ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of Ψ such that $x \in C \subset U$. Then Ψ is a basis for the topology of X.*

Lemma 71 *(13.3)(For finer)*

Definition 72 *If Ω is the collection of all open intervals in the real line,*

$$(a, b) = \{x|a < x < b\}$$

*the topology generated by Ω is called the **standard topology** on the real line. Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.*

Definition 73 *If $\Omega^{'}$ is the collection of all half-open intervals of the form,*

$$[a, b) = \{x|a \leq x < b\}$$

*where $a < b$, the topology generated by $\Omega^{'}$ is called the **lower limit topology** on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$.*

Definition 74 *Let K denote the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$, and let $\Omega^{''}$ be the collection of all open intervals (a, b), along with all sets of the form $(a, b) - K$. The topology generated by $\Omega^{''}$ will be called the K**-topology** on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_k$.*

Lemma 75 *(13.4) (For finer)*

Definition 76 *A **subbasis** S for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** S is defined to be the collection Υ of all unions of finite intersections of elements of S.*

Remark 77 *A subbasis S can be a non-basis for a topoloby on X. However, if we denote the set, consisting the subbasis S and collection of all finite intersections of S, as Ω, then Ω is a basis. Hence, by Lemma 13.1, taking all unions of of Ω gives a topology generated by Ω, which is called topology generated by the subbasis S.*

2.3 14 The order topology

Definition 78 *Let X be a set with a simple order relation; assume X has more than one element. Let Ω be the collection of all sets of the following types:*

1. *all open intervals (a, b) in X;*
2. *all intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X;*
3. *all intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X;*

*The collection Ω is a basis for a topology on X, which is called the **order topology***

(If there is no smallest element, there are no sets of type 2; and if has no largest element, there are no sets of type 3.)

Definition 79 *(ray)*

2.4 15 The Product Topology on $X \times Y$

Definition 80 *Let X and Y be topological spaces. **The product topology** on $X \times Y$ is the topology having as basis the collection Ω of all sets of the form $U \times V$, where U is an open subset of X and V is an subset of Y.*

Remark 81 *The basis for $X \times Y$ is Ω, which is composed by all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.*

Remark 82 *Ω itself is not a topology on $X \times Y$. Because the union of two basis elements may not be a basis elements, the topology require the union of two sets to be a set in this topology. (Example in page 87 of Munkres)*

Theorem 83 *If Ω_X is a basis for the topology of X and Ω_Y is a basis for the topology of Y, then the collection*

$$\Omega = \{B \times C | B \in \Omega_X \text{ and } C \in \Omega_Y\}$$

is a basis for the topology of $X \times Y$. (If Ω_X and Ω_Y are basis for X and Y, then the set of $\Omega_X \times \Omega_X$, denoted by Ω, is the basis for $X \times Y$.)

Definition 84 *Let $\pi_1 : X \times Y \to X$ be defined by the equation*

$$\pi_1(x, y) = x;$$

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y;$$

*The maps π_1 and π_2 are called the **projections** of $X \times Y$ onto, surjective, its first and second factors, respectively.*

Theorem 85 *The collection*

$$S = \{\pi_1^{-1}(U) | U \text{ open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

2.5 16 The Subspace Topology

Definition 86 *Let X be a topological space with topology Υ. If Y is a subset of X, the collection*

$$\Upsilon_Y = \{Y \cap U | U \in \Upsilon\}$$

*is a topology on Y, called the **subspace topology**. With this topoloby, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.*

(One should be careful when saying "open set": U is open in subspace Y means U belongs to topology of Y, but it may not belongs to the topology of X, which means U is not open in X.)

Lemma 87 *(16.1) If Ω is a basis for the topology of X then the collection*

$$\Omega_Y = \{B \cap Y | B \in \Omega\}$$

is a basis for the subspace topology on Y.

Lemma 88 *(16.2) Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.*

Theorem 89 *(16.3) If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Theorem 90 *(16.4) Let X be an orered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

2.6 17 Closed Sets and Limit Points

Definition 91 *A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.*

Remark 92 *In the discrete topology on the set X, every set is open; it follows that every set is closed as well.*

Theorem 93 *(17.1) Let X be a topological space. Then the following conditions hold:*
1. ϕ and X are closed;
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

Theorem 94 *(17.2) Let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y.*

Theorem 95 *(17.3) Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Definition 96 *Given a subset A of a topological sapce X, the **interior** of A, Int A, is defined as the union of all open sets contained in A, and the **closure** of A, CI A, is defined as the intersection of all closed sets containing A. (A set is closed if and only if this set equals its closure.)*

Theorem 97 *Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.*

Definition 98 *We say that a **set** A **intersects a set** B if the intersection $A \cap B$ is not empty.*

Theorem 99 *(17.5) Let A be a subset of the topological space X:*
a. Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.
b. Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B contiaing x intersects A.

Definition 100 *Shorten the statement "U is an open set containing x" to the phrase"U **is a neighborhood of** x"*

Definition 101 *If A is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself. (Said differently, x is a limit point of A, if it belongs to the closure of $A - \{x\}$, according to Theorem 17.5. The point x may lie in A or not.)*

Theorem 102 *(17.6) Let A be a subset of the topological space X; let $A^{'}$ be the set of all limit points of A. Then*

$$\bar{A} = A \cup A^{'}$$

Corollary 103 *(17.7) A subset of a topological space is closed if and only if it contains all its limit points.*

Definition 104 *A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.*

Definition 105 *The condition that finite point sets be closed has been given a name of tis own: it is called the T_1 **axiom**. (Weaker than Hausdorff space)*

Theorem 106 *(17.8) Every finite point set in a Hausdorff space X is closed.*

Theorem 107 *(17.9) Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Theorem 108 *(17.10) If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Definition 109 *If the sequence x_n of points of the Hausdorff space X converges to the point x of X, we often write $x_n \to x$, and we say that x is the **limit** of the sequence x_n.*

Theorem 110 *(17.11) Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

2.7 18 Continuous Functions

Definition 111 *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

Remark 112 *f^{-1} may be not a function or mapping, but a correspondence. Unless f is indicated as a bijection, we can not assume f^{-1} is a function or mapping.*

Remark 113 *If the topology of the range space Y is given by a basis, then to prove continuity of f it suffices to show that the inverse image of every basis element is open.*

Remark 114 *If the topology on Y is given by a subbasis, to prove continuity of f it will even suffice to show that the inverse image of each subbasis element is open.*

Theorem 115 *(18.1) Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:*
1). f is continuous;
2). For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$;
3). For every closed set B of Y, the set $f^{-1}(B)$ is closed in X;
4). For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Definition 116 *If the condition in 4) holds for the point x of X, we say that f is **continuous at the point** x.*

Definition 117 *Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.*

Remark 118 *Another way to define a homeomorphism is to say that it is a bijective correspondence $f : X \to Y$ such that $f(U)$ is open if and only if U is open.*

Definition 119 ***imbedding***

Theorem 120 *(18.2) (Rules for constructing continuous functions) Let X, Y, and Z be topological spaces.*
a). (Constant function) If $f : X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.
b). (Inclusion) If A is a subspace of X, the inclusion functions $j : A \to X$ is continuous
c). (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.
d). (Restricting the domain) If $f : X \to Y$ is continuous, the if A is a subsapce of X, then the restricted function $f|A : A \to Y$ is continuous.
e). (Restricting or expanding the range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.
f). (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|U_\alpha$ is continuous for each α.

Theorem 121 *(18.3) (The pasting lemma) Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$*

Theorem 122 *(18.4) (Maps into products) Let $f : A \to X \times Y$ be given by the equation*

$$f(a) = (f_1(a), f_2(a))$$

Then f is continuous if and only if the functions

$$f_1 : A \to X \text{ and } f_2 : A \to Y$$

are continuous. The maps f_1 and f_2 are called coordinate functions of f.

Remark 123 *There is no useful criterion for the continuity of a map $f : A \times B \to X$ whose domain is a product space. One might conjecture that f is continuous if it is continuous in each variable separately, but this conjecture is not true. (See exercise 12 at page 112)*

2.8 19 The product Topology

Definition 124 *For cartesian products $X_1 \times X_2 \times ...$, take all sets of the form $U_1 \times U_2 \times ...$ as basis, where U_i is an open set of X_i for each i. This prodcedure define a topology on the cartesian product; we call it the* ***box topology****.*

Definition 125 *We take as a subasis all sets of the form $\pi_i^{-1}(U_i)$, where i is any index and U_i is an open set of X_i. We call this topology the* ***product topology****.*

Remark 126 *These two topologies agree for the finite cartesian product and differ for the infinite product.*

Remark 127 *Box topology is in general more finer than the product topology. (Page 115 of Munkres). In finite cartesian prduct, we can generate the basis for box topology, $U_1 \times U_2 \times ...$, by using finite intersection of sets of the form $\pi_i^{-1}(U_i)$, which means the topology generated from subbasis of $\pi_i^{-1}(U_i)$ is equal to the topology generated from basis of $U_1 \times U_2 \times ...$ However, for infinite cartesian product case, $U_1 \times U_2 \times ...$ can not be generated from $\pi_i^{-1}(U_i)$ by finite intersection. However, $\{\underset{finite}{\cap} \pi_i^{-1}(U_i)\} \subset \{U_1 \times U_2 \times ...\}$, so box topology is finer than product topology.*

Definition 128 *Let J be an index set. Given a set X, we define a $J-$tuple of the elements of X to be a function $x : J \to X$. If α is an element of J, we often denote the value of x at α by x_α rather than $x(\alpha)$; we call it the αth coordinate of x. And we oftern denote the function x itself by the symbol*

$$(x_\alpha)_{\alpha \in J}$$

which is as close as we can come to a "tuple notation" for an arbitrary index set J. We denote the set of all $J-$tuples of elemets of X by X^J.

Definition 129 *Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \cup_{\alpha \in J} A_\alpha$. The cartesian product of this indexed family, denoted by*

$$\underset{\alpha \in J}{\Pi} A_\alpha$$

is defnied to be the set of all $J-$tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \to \underset{\alpha \in J}{\cup} A_\alpha$$

such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.

Definition 130 *Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product sapce*

$$\underset{\alpha \in J}{\Pi} X_\alpha$$

the collection of all sets of the form

$$\underset{\alpha \in J}{\Pi} U_\alpha$$

where U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the ***box topology****.*

Definition 131 *Let*

$$\pi_\beta : \underset{\alpha\in J}{\Pi} X_\alpha \to X_\beta$$

be the function assigning to each element of the product space its βth coordinate,

$$\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta$$

*it is called the **projection mapping** associated with the index β.*

Definition 132 *Let S_β denote the collection*

$$S_\beta = \{\pi_\beta^{-1}(U_\beta) | U_\beta \text{ open in } X_\beta\}$$

and let S denote the union of these collections,

$$S = \underset{\beta\in J}{\cup} S_\beta$$

*The topology generated by the subbasis S is called the **product topology**. In the topology $\underset{\alpha\in J}{\Pi} X_\alpha$ is called a **product space**.*

Theorem 133 *(19.1) (Comparison of the box and product topologies) The box topology on ΠX_α has as basis all sets of the form ΠU_α, where U_α is open in X_α for each α. The product topology on ΠX_α has as basis all sets of the form ΠU_α, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.*

Remark 134 *For finite products $\Pi_{\alpha=1}^n X_\alpha$, the two topologies are precisely the same; the box topology is in general finer than the product topology; we can extend the finite products theorem to infinite products cases, but we can not do that for box topology.*

Theorem 135 *(19.2) Suppose the topology on each space X_α is given by a basis Υ_α. The collection of all sets of the form*

$$\underset{\alpha\in J}{\Pi} B_\alpha$$

where $B_\alpha \in \Upsilon_\alpha$ for each α, whill serve as a basis for the box topology on $\Pi_{\alpha\in J} X_\alpha$.

The collection of all sets of the same form, where $B_\alpha \in \Upsilon_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\Pi_{\alpha\in J} X_\alpha$.

Theorem 136 *(19.3) Let A_α be a subspace of X_α, for each $\alpha \in J$. Then ΠA_α is a subspace of ΠX_α if both products are given the box topology, or if both products are given the product topology.*

Theorem 137 *(19.4) If each space X_α is a Hausdorff space, then ΠX_α is a Hausdorff space in both the box and product topologies.*

Theorem 138 *(19.5) Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If ΠX_α is given either the product or the box topology, then*

$$\Pi \overline{A}_\alpha = \overline{\Pi A_\alpha}$$

Theorem 139 *(19.6) (only for product topology) Let $f : A \to \Pi_{\alpha\in J} X_\alpha$ be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha\in J}$$

where $f_\alpha : A \to X_\alpha$ for each α. Let ΠX_α have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

2.9 20 The Metric Topology

Definition 140 *A* ***metric*** *on a set* X *is a function*

$$d : X \times X \to R$$

having the following properties:
1). $d(x, y) \geq 0$ *for all* $x, y \in X$*; equality holds if and only if* $x = y$*.*
2). $d(x, y) = d(y, x)$ *for all* $x, y \in X$*.*
3). (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$*, for all* $x, y, z \in X$*.*

Remark 141 *The number* $d(x, y)$ *is often called the* ***distance*** *between* x *and* y *in the metric* d*. Given* $\epsilon > 0$*, consider the set*

$$B_d(x, \epsilon) = \{y | d(x, y) < \epsilon\}$$

of all points y *whose distance from* x *is less than* ϵ*. It is called the* ***ϵ-ball centered at*** x*.*

Definition 142 *If* d *is a metric on the set* X*, then the collection of all* ϵ*-ball* $B_d(x, \epsilon)$*, for* $x \in X$ *and* $\epsilon > 0$*, is a basis for a topology on* X*, called the* ***metric topology*** *induced by* d*.*

Remark 143 *A set* U *is open in the metric topology induced by* d *if and only if for each* $y \in U$*, there is a* $\delta > 0$ *such that* $B_d(y, \delta) \subset U$*.*

Definition 144 *If* X *is a topological space,* X *is said to be* ***metrizable*** *if there exists a metric* d *on the set* X *that induces the topology of* X*. A* ***metric space*** *is a metrizable space* X *together with a specific metric* d *that gives the topology of* X*.*

Definition 145 *Let* X *be a metric space with metric* d*. A subset* A *of* X *is said to be* ***bounded*** *if there is some number* M *such that*

$$d(a_1, a_2) \leq M$$

for every pair a_1, a_2 *of points of* A*. If* A *is bounded and nonempty. the* ***diameter*** *of* A *is defined to be the number*

$$diam\ A = \sup\{d(a_1, a_2) | a_1, a_2 \in A\}$$

Theorem 146 *(20.1)Let* X *be a metric space with metric* d*. Define* $\overline{d} : X \times X \to R$ *by the equation*

$$\overline{d}(x, y) = \min\{d(x, y), 1\}$$

Then $\overline{d}$ *is a metric that induces the same topology as* d*. Then metric* $\overline{d}$ *is called the* ***standard bounded metric*** *corresponding to* d*.*

Definition 147 *Given* $x = (x_1, ..., x_n)$ *in* R^n*, we define the* ***norm*** *of* x *by the equation*

$$\|x\| = (x_1^2 + ... + x_n^2)^{1/2}$$

and we define the ***euclidean metric*** d *on* R^n *by the equation*

$$d(x, y) = \|x - y\| = [(x_1 - y_1)^2 + ... + (x_n - y_n)^2]^{1/2}$$

We define the ***square metric*** ρ *by the equation*

$$\rho(x, y) = \max\{|x_1 - y_1|, ..., |x_n - y_n|\}$$

Lemma 148 *(20.2) Let* d *and* $d^{'}$ *be two metrics on the set* X*: let* Υ *and* $\Upsilon^{'}$ *be the topologies they induce, respectively. Then* $\Upsilon^{'}$ *is finer than* Υ *if and only if for each* x *in* X *and each* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon)$$

Remark 149 *If $d(x,y) < d^{'}(x,y)$ for every x, y in X, then $\Upsilon^{'}$ is finer than Υ.*

Theorem 150 *(20.3) The topologies on R^n induced by the euclidean metric d and the square metric ρ are the same as the product topology on R^n. (Use 20.2 to prove they are finer to each other, which indicate they are the same.)*

Definition 151 *Given an index set J, and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of R^J, let us define a metric $\overline{\rho}$ on R^J by the equation*

$$\overline{\rho}(x,y) = \sup\{\overline{d}(x_\alpha, y_\alpha) | \alpha \in J\}$$

where $\overline{d}$ is the standard bounded metric on R. It is easy to check that $\overline{\rho}$ is indeed a metric; it is called the ***uniform metric*** *on R^J, and the topology it induces is called the* ***uniform topology****.*

Theorem 152 *(20.4) The uniform topology on R^J is finer than the product topology and coarser than the box topology; these three topologies are all different if J is infinite.*

Theorem 153 *(20.5) Let $\overline{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on R. If x and y are two points of R^w, define*

$$D(x,y) = \sup\{\frac{\overline{d}(x_i, y_i)}{i}\}$$

Then D is a metric that induces the product topology on R^w.

2.10 21 The Metric Topology (continued)

Theorem 154 *(21.1) Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

Theorem 155 *(21.2) (The sequence lemma) Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.*

Theorem 156 *(21.3) Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.*

Lemma 157 *(21.4) The addition, subtraction, and multiplication operations are continuous functions from $R \times R$ into R; and the quotient operation is a continuous function from $R \times (R - \{0\})$ into R.*

Theorem 158 *(21.5) If X is a topological space, and if $f, g : X \to R$ are continuous functions, then $f + g$, $f - g$, $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.*

Definition 159 *A space X is said to have* ***a countable basis at the point*** *x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n. A space X that has a countable basis at each of its points is said to satisfy* ***the first countability axiom****.*

Remark 160 *Metric space automatically is the first countable.*

Definition 161 *Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n)* ***converges uniformly*** *to the functin $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that*

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all x in X.

Theorem 162 *(21.6) (Uniform limit theorem) Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. if (f_n) converges uniformly to f, then f is continuous.*

2.11 22 The Quotient Topology

Definition 163 *Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X. (Strong continuity)*

Remark 164 *Two kinds of quotient maps are the open maps and the closed maps.*

Definition 165 *A map $f : X \to Y$ is said to be an **open map** if for each open set U of X, the set $f(U)$ is open in Y.*

Definition 166 *A map $f : X \to Y$ is said to be an **closed map** if for each closed set U of X, the set $f(U)$ is closed in Y.*

Remark 167 *It follows immediately from the definition that if $p : X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map. However, there are quotient maps that are neither open nor closed.*

Definition 168 *If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology Υ on A relative to which p is a quotient map; it is called the **quotient topology** induced by p.*

Definition 169 *Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that arries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.*

Chapter 3

Connectedness and Compactness

Intermediate Value Theorem, Maximum Value Theorem, and Uniform Continuity Theorem are three basic fundemental theorems in Calculus about continuous functions. Intermediate Value Theorem need connectedness, and the other two need compactness.

3.1 23 Connected Spaces

Definition 170 *Let X be a topological space. A* ***separation*** *of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be* ***connected*** *if there does not exist a separation of X.*

Remark 171 *If X is connected, so is any space homeomorphic to X;*

A space X is connected iff the only subsets of X that are both open and closed in X are empty set and X itself.

Lemma 172 *(23.1) If Y is a subspace of X, a separation of Y is equavelent to a pair of disjoint nonempty sets A and B, whose union is Y, then neither of A and B contains a limit point of the other. If there exists no separation of Y, then the space Y is connected.*

Remark 173 *This Lemma is another way to define separation and connectness for subspace. Hence, the original definition still holds for subspace. For example, a separation of Y is a pair sets A and B is disjoint nonempty open subsets of Y, whose union is Y.*

Remark 174 *X is connected iff there does not exist nonempty open sets A and B such that 1. $\overline{A} \cap B = \phi$ 2. $A \cap \overline{B} = \phi$ 3. $A \cup B = X$;*

Lemma 175 *(23.2) If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.*

Theorem 176 *(23.3) The union of a collection of connected subspaces of X that have a point in common is connected.*

Theorem 177 *(23.4) Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.*

(If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.)

Theorem 178 *(23.5) The image of a connected space under a continuous map is connected.*

Theorem 179 **(Notes)* Let $f : X \to Y$ be a surjective continnous open map; assume $f^{-1}(y)$ is connected for $\forall y \in Y$. Then X is connected. (The Theorem 23.6 can be proved by this theorem by using projection mapping from $X \times Y$ to Y)*

Theorem 180 *(23.6) A finite cartesian product of connected spaces is connected.*
(In example 7, author proved that R^ω in product topology is connected given R is connected.)
(from theorem 23.3 to 23.6, we should know how to construct new connected spaces out of given ones.)

3.2 24 Connected subspaces of the real line

Definition 181 *A simply ordered set L having more than one element is called a* ***linear continuum*** *if the following hold:*
1). L has the least upper bound property;
2). If $x < y$, there exists z such that $x < z < y$.

Theorem 182 *(24.1) If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.*

Corollary 183 *(24.2) The real line R is connected and so are intervals and rays in R.*

Theorem 184 *(24.3) (Intermediate value theorem) Let $f : X \to Y$ be a continuous map, where X is a connected sapce and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.*

Definition 185 *Given points x and y of the space X, a* ***path*** *in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be* ***path connected*** *if every pair of points of X can be joined by a path in X.*

Remark 186 *Path-connected space X is connected. However, a connected space need not be path connected.*

3.3 25 Components and Local Connectedness

3.4 26 Compact Spaces

Definition 187 *A collection Υ of subsets of a space X is said to* ***cover*** *X, or to be a* ***covering*** *of X, if the union of the elements of Υ is equal to X. It is called* ***open covering*** *of X if its elements are open subsets of X.*

Definition 188 *A space X is said to be* ***compact*** *if every open covering Υ of X contains a finite subcollection that also covers X.*

Definition 189 *If Y is a subspace of X, a collection Υ of subsets of X is said to* ***cover*** *Y if the union of its elements contains Y.*

Lemma 190 *(26.1) Let Y be a subspace of X, then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.*

Theorem 191 *(26.2) Every closed subspace of a compact space is compact.*

Theorem 192 *(26.3) Every compact subspace of a Hausdorff space is closed.*

Lemma 193 *(26.4) If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively. (Standard proof techniques)*

Remark 194 *In R, once we prove that the interval $[a, b]$ is compact, it follows from Theorem 26.2 that any closed subspace of $[a, b]$ is compact. From theorem 26.3, the intervals $(a, b]$ and $[a, b)$ cannot be compact because they are not closed in Hausdorff space R.*

Theorem 195 *(26.5) The image of a compact space under a continuous map is compact.*

Theorem 196 *(26.7) The product of finitely many compact spaces is compact. (For infinite case, we should refer to Tychonoff theorem)*

Lemma 197 *(26.8) (The tube Lemma) Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X. (Standard proof techniques)*

Definition 198 *A collection Υ of subsets of X is said to have the **finite intersection property** if for every finite subcollection*

$$\{C_1, ..., C_n\}$$

of Υ, the intersection $C_1 \cap ... \cap C_n$ is nonempty.

Theorem 199 *(26.9) Let X be a topological space. Then X is compact iff for every collection Υ of closed sets in X having the finite intersection property , the intersection $\cap_{C\in\Upsilon} C$ of all the elements of Υ is nonempty.*

Remark 200 *Given a collection Ψ of subsets of X, let*

$$\Upsilon = \{X - A | A \in \Psi\}$$

be the collection of their complements. Then the following statements hold:
1). Ψ is a collection of open sets iff Υ is a collection of closed sets;
2). The collection Ψ covers X iff the intersection $\cap_{C\in\Upsilon} C$ of all the elements of Υ is empty.
3). The finite subcollection $\{A_1, ..., A_n\}$ of Ψ covers X iff the intersection of the corresponding elements $C_i = X - A_i$ of Υ is empty.

Definition 201 *A continuous map $f : X \to Y$ is called a proper map if the inverse image of a compact set is compact.*

3.5 27 Compact subspaces of the Real Line

Theorem 202 *(27.1) Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.*

Corollary 203 *(27.2) Every closed interval in R is compact.*

Theorem 204 *(27.3) A subspace A of R^n is compact iff it is closed and is bounded in the euclidean metric d or the square metric ρ.*

Theorem 205 *(27.4) (Extreme value theorem) Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

Definition 206 *Let (X, d) be a metric space; let A be a nonempty subset of X. For each $x \in X$, we define the **distance from** x **to** A by the equation*

$$d(x, A) = \inf\{d(x, a) | a \in A\}$$

Remark 207 *$d(x, A)$ is a continuous function of x.*

Lemma 208 *(27.5) (The Lebesgue number lemma) Let Υ be an open convering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of Υ containing it. (The number δ is called a **Lebesgue number** for the covering Υ.)*

Definition 209 *A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be uniformly continuous if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,*

$$d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \epsilon$$

Theorem 210 *(27.6) (Uniform continuity theorem) Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.*

3.6 28 Limit Point Compactness

Definition 211 *A space X is said to be* ***limit point compact*** *if every infinite subset of X has a limit point.*

Theorem 212 *(28.1) Compactness implies limit point compactness, but not conversely.*

Definition 213 *Let X be a topological space. If (x_n) is a sequence of points of X, and if*

$$n_1 < n_2 < ... < n_i < ...$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$, is called a ***subsequence*** *of the sequence (x_n). The space X is said to be* ***sequentially compact*** *if every sequence of points of X has a convergent subsequence.*

Lemma 214 *Let X be a first countable topological space and assume that every infinite subset has a limit point. Then any sequence $\{x_n\}$ has a convergent subsequence.*

Theorem 215 *(28.2) Let X be a metrizable space. Then the following are equivalent:*
1). X is compact;
2). X is limit point compact;
3). X is sequentially compact.

3.7 29 Local Compactness

Definition 216 *A space X is said to be* ***locally compact at*** *x if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be* ***locally compact****.*

Remark 217 *A compact space is automatically locally compact.*

Remark 218 *R is locally compact; R^n is locally compact; R^ω is not locally compact.*

Remark 219 *Every simly ordereed set X having the least upper bound property is locally compact. (Use theorem 27.1)*

Theorem 220 *(29.1) Let X be a space. Then X is locally compact Hausdorff iff there exists a space Y satisfying the following conditions:*
1). X is a subspace of Y;
2). The set $Y - X$ consist of a single point;
3). Y is a compact Hausdorff space.

Definition 221 *If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be* ***a compactifcation of*** *X. If $Y - X$ equals a single point, then Y is called the* ***one-point compactification of*** *X.*

Remark 222 *In Theorem 29.1, we have show that X has a one-point compactification Y iff X is a locally compact Hausdorff space that is not itself compact. We speak of Y as "the" one-point compactification because Y is uniquely determined up to homeomorphism.*

Theorem 223 *(29.2) Let X be a Hausdorff space. Then X is locally compact iff given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.*

Corollary 224 *(29.3) Let X be locally compact Hausdorff; let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.*

Chapter 4

Countability and Separation Axioms

4.1 30 The Countability Axioms

Definition 225 *A space X is said to have a* ***countable basis at*** *x if there is a countable collection Υ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of Υ. A space that has a countable basis at each of its points is said to satisfy the* ***first countability axiom****, or to be* ***first-countable****.*

Remark 226 *Every metrizable space satisfies this axiom.*

Theorem 227 *(30.1) Let X be a topological space.*

a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is first-countable;

b) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.

Definition 228 *If a space X has a countable basis for its topology, then X is said to satisfy the* ***second countability axiom****, or to be* ***second-countable****.*

Remark 229 *The second countability axiom implies the first countability axiom.*

Remark 230 *The real line R, with all open interval (a, b) with rational end points, has countable basis; similarly, R^n and R^ω has countable basis; however, R^ω in uniform topology does not have countable basis.*

Theorem 231 *(30.2) A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable.*

A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

Definition 232 *A subset A of a space X is said to be dense in X if $\overline{A} = X$.*

Theorem 233 *(30.3) Suppose that X has a countable basis. Then:*

a). Every open covering of X contains a countable subcollection covering X.

b). There exists a countable subset of X that is dense in X.

Definition 234 *A space for which every open covering contains a countable subcovering is called a* ***Lindelof Space****.*

Definition 235 *A space having a countable dense subset is often said to be* ***separable****.*

4.2 31 The Separation Axioms

Definition 236 *Suppose that one-point sets are closed in X. Then X is said to be* ***regular*** *if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be* ***normal*** *if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.*

Remark 237 *A normal space is regular; (Sorgenfrey plane is regular but not normal)*
A regular space is Hausdorff. (R_K is Hausdorff but not regular)

Lemma 238 *(31.1) Let X be a topological space. Let one-point sets in X be closed.*
a) X is regular iff given a point x of X and a neighborhood U of x, there is a neighborhodd V of x such that $\overline{V} \subset U$.
b). X is normal iff given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$.

Theorem 239 *(31.2) a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff;*
b) A subspace of a regular space is regular; a product of regular space is regular.
(There is no analogous theorem for normal spaces.)

4.3 32 Normal Spaces

Theorem 240 *(32.1) Every regular space with a countable basis is normal.*

Theorem 241 *(32.2) Every metrizable space is normal.*

Theorem 242 *(32.3) Every compact Hausdorff space is normal.*

Theorem 243 *(32.4) Every well-ordered set X is normal in the order topology. (In fact, every order topology is normal.)*

4.4 33 The Urysohn Lemma

Theorem 244 *(33.1) (Urysohn Lemma) Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map*

$$f : X \to [a, b]$$

such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.

Remark 245 *The Urysohn Lemma says that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function. The converse is trivial, for if $f : X \to [0, 1]$ is the function, then $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint open sets containing A and B, respectively.*

Definition 246 *If A and B are two subsets of the topological space X, and if there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B* ***can be separated by a continuous function****.*

Definition 247 *A space X is* ***completely regular*** *if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.*

Remark 248 *A normal space is completely regular, by the Urysohn lemma;*

A completely regular space is regular.

The separation axiom, listed in order of increasing strength, were Hausdorff, Regular, completely regular, normal, and completely normal. (The spaces R_l^2 are completely regular but not normal)

Theorem 249 *(33.2) A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.*

4.5 34 The Urysohn Metrization Theorem

Theorem 250 *(34.1) (Urysohn metrization theorem) Every regular space X with a countable basis is metrizable.*

Theorem 251 *(34.2) Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha\in J}$ is an indexed family of continuous functions $f_\alpha : X \to R$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is positive at x_0 and vanishes outside U. Then the function $F : X \to R^J$ defined by*

$$F(x) = (f_\alpha(x))_{\alpha\in J}$$

is an imbedding of X in R^J. If f_α maps X into $[0,1]$ for each α, then F imbedes X in $[0,1]^J$.

Theorem 252 *(34.3) A space X is completely regular iff it is homeomorphic to a subspace of $[0,1]^J$ for some J.*

Chapter 5

The Tychonoff Theorem

5.1 37 The Tychonoff Theorem

Theorem 253 *(37.3) (Tychonoff Theorem) An arbitrary product of compact spaces is compact in the product topology.*