Strategic Outsourcing Contract Participation and Selection under Cost Uncertainty

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Motivated b y widespread outsourcing practices, this paper studies how a pre-outsourcing client facing uncertain operating costs selects an outsourcing contract between two contract types (the fixed-price contract and the cost-plus contract) proposed by two vendors, who offer contracts only if the expected profits of the contracts satisfy their reservation values. We propose a real options model to determine the optimal policy structure from both the client's and the vendors' perspectives, and we investigate comparative statics of the client's or vendors' optimal policy with respect to key system inputs. Our results help managers gain a better understanding of the closely intertwined relationship between the client's contract selection policy and the vendors' contract participation policies.

Subject classifications: fixed-price contract, cost-plus contract, contract selection, contract participation, outsourcing timing, real options, net present value, elasticity, correlation, correlated relative volatility.
Area of review: Operations and Supply Chain.

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1. Introduction

Among possible outsourcing benefits, such as operational efficiency, capacity pooling, access to external expertise and advanced technologies, cost reduction is a pivotal factor in motivating an outsourcing decision (Li and Kouvelis 1999, Plambeck and Taylor 2006, Aksin et al. 2008). Gauging the possible cost-reduction benefits is complicated by the fact that the operating costs can be highly volatile and stochastically varying over time depending on several factors, including the characteristics of outsourcing countries and industries (Bergin et al. 2007, Jacks et al. 2009), exchange rates (Huchzermeier and Cohen 1996), unpredicted capacity constraints (Seshadri 2005), and prices of raw materials (Wu and Chen 2010). For example, a wild roller-coaster ride of copper prices since 2005 has caused significant stress in industries that use copper as an essential raw material (e.g. the electronics industry). Concurrently, significant price swings, witnessed in many other indispensable

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commodities ranging from oil to fertilizers, further contributes to the complexity of outsourcing cost analyses (Sinha 2015).

In practice, the vast majority payment scheme of outsourcing contracts are variants of simple *fixed-price* and *cost-plus* contracts (Kern et al. 2002, Leimeister 2010, Seshadri 2005), and we shall focus these two types of contracts for their popularity and simplicity. The central focus of this research is to conduct cost analyses on both types of outcourcing contracts and extract managerial insights in response to the first research question: *which contract should the pre-outsourcing firm choose?* In a fixed-price (FP) contract, the client (i.e., the pre-outsourcing firm) pays a constant price rate set forth by the FP agreement regardless of the vendor's (i.e., the contract manufacturer's) operating cost. By sticking to the fixed outsourcing cost, the client essentially shifts its future cost uncertainty to its vendor. Clearly, the client avoids paying a high premium if the vendor's operating cost significantly increases in the future. On the other hand, the client is locked to this relatively high outsourcing price even if the vendor's cost dramatically drops (e.g., the cost of raw materials declines). For instance, METAL contracted with its outsourcing vendor through a 10-year FP agreement and agreed to pay $100 per processed form in 1990 (Lacity and Willcocks (1998)). Yet by 1993, the vendor improved its operating efficiency and was able to offer a much lower price (i.e. $50 per processed form) to other customers. Yet, tied to the FP contract, METAL was unable to achieve this lower cost.

In a cost-plus (CP) contract, the client's outsourcing cost consists of the vendor's operating cost and a fixed proportional markup. Under this cost structure, the client and the vendor share the potential benefits and risk of the fluctuating future cost. Certainly, the CP contract helps the client avoid overcharging from a low-cost vendor. Yet, facing everchanging market conditions, the client's outsourcing cost could spiral out of control if the vendor's cost drastically increases in the future. Therefore, there is no direct dominance relationship between the CP contract and FP contract, and the contract choice between these two contracts hinges on market conditions and its corresponding key system inputs (e.g., volatilities and correlations of cost processes). To better understand the sensitivity of the trade-off between these two contracts, we conduct a systematic study on the second research question: *how is the client's optimal contract selection policy affected by key system inputs?*

Besides the optimal contract selection decision (to whom to outsource), a pre-outsourcing firm also has to determine the optimal execution time (when to receive products/services from its vendor). Because of the continual changes of the market condition, the impact of execution time choice has long been recognized in the industry. For example, Nokia and Motorola realized the benefits of timing flexibility and decided gradually outsourced their partial handset productions to Chinese vendors over time (Alvarez and Stenbacka 2007). To proactively include the execution time flexibility in their outsourcing decisions, the client may choose to negotiate an option contract with the execution time flexibility. Specifically, this option contract grants the client the option, not obligation, to execute/exercise the contract and receive products/services from its vendor anytime within a pre-specified future time period (Li and Kouvelis (1999)). Some examples of the option contract with the execution time flexibility in practice include Hewlett-Packard's procurement risk management and Ben & Jerry's entry into the Japan (Fotopoulos et al. (2008)). In addition, facing purchasing and selling uncertainties, companies including IBM, Hewlett Packard, Sun, Compaq, and Solectron have incorporated various combinations of timing and quantity flexibility into their purchasing contracts (Hu et al. (2012)). Primarily, the client adopts the outsourcing contract with flexible execution time to better leverage the currency/price/cost fluctuations in a certain time period. Therefore to properly evaluate the value of these two contracts with the execution timing flexibility and further select the optimal contract from these two, we further seek the answer to the third research question of *when should the client execute its outsourcing contract?*,

To date, most models are developed from the client's side. Little attention has been paid to the vendor's decision problems. The implicit assumption is that the client's strategic/operational decisions are always acceptable to a passive vendor. However, like the client, the vendor faces the decision of whether to participate in a particular contract via evaluating the net worth of the contract in the presence of both uncertain future costs and uncertain outsourcing timing. The vendor will only accept a contract it deems profitable. In this regard, examining the interaction of both sides, this paper sheds light on the fourth question: *how to measure the value of a time-flexible RO contract from both vendor's and client's perspectives*.

To answer those four research questions, we consider a client deliberating on how and when to outsource its current in-house operation to a vendor: vendor 1 with a FP contract or vendor 2 with a CP contract respectively. The operating costs of those three firms follow correlated time-dependent stochastic processes, which capture the phenomena that future costs are uncertain and potentially correlated (e.g., the price of a raw material may simultaneously affect the operating costs of all firms). The client needs to decide which contract to accept and when to execute the selected contract during the contract effective time window. Furthermore, the outsourcing contract must meet certain participation constraints so that it is profitable for the vendor.

From the client's perspective, we prove that its optimal contract selection decision is a threshold-type policy. In particular, when both contracts are offered, the client's optimal contract selection policy is the *select-up-to* (SUT) policy: select the CP contract from vendor 2 if vendor 2's operating cost is lower than a threshold; otherwise, choose the FP contract from vendor 1.

We further show that the client's optimal execution timing decision also follows threshold-type. In particular, if the client selects the FP contract, then it has the option to execute the contract and pay the vendor a fixed rate agreed ex ante, which is independent of both parties' actual operating costs. We demonstrate that under the FP contract, the client will exercise the option and execute the FP contract when its operating cost reaches or exceeds a certain threshold. On the other hand, if the CP contract is selected, the client should wait and postpone the contract execution if its current *cost efficiency ratio* (i.e., the client's cost over vendor 2's cost) is below a certain threshold. Recall that under CP contract structure, the outsourcing cost depends on vendor 2's cost (i.e., vendor 2's operating cost plus a profit margin). Therefore, the client is better off to keep production in house instead of exercising the CP contract unless its operating cost is much higher than that of vendor 2 (e.g., cost efficiency ratio is high).

Moreover, we examine the value of a CP contract to the client and find out that the client's option value does not necessarily increase in either the vendor's or client's cost volatility, which contradicts the conventional belief in a two stochastic factor RO model (e.g. McDonald and Siegel (1986) and Dixit and Pindyck (1994)). To better explain the comparative statics of the client's option value, we develop a new measure, *correlated relative volatility* (CRV), which gauges the relative cost volatility of the client in relation to the vendor, and show that the CP contract's option value increases as the difference between those two cost volatility becomes more significant (i.e., CRV moves away from 1 in either direction). We argue that this nonmontonicity of CP contract's option value to policy parameters would help align the interests of both firms since a high cost volatility could benefit both the client and the vendor.

Through comparing the FP and CP contract values, we reveal two strategic scenarios, the FP- and CP-oriented mothball regions, in which a pre-outsourcing firm can expect a higher profit by not exercising the immediately exercisable contract but selecting the other unexercisable contract and waiting for its best execution timing. The mothball region scenario shows that the plausible perception that the client should select the contract that is both profitable and immediately exercisable may lead to an incorrect contract commitment as well as suboptimal outsourcing timing.

From the vendors' perspective, we integrate the client's outsourcing timing decision into vendors' contract value functions, and identify their participation policies with respect to two contract types. These two distinct participation policies reflect different tradeoffs faced by two vendors. In the FP contract, the tradeoff is between vendor 1's fixed revenue and its random cost after its execution. Vendor 1 has an incentive to offer FP as long as its fixed revenue stream exceeds its operating cost, and its optimal contract participation policy is characterized by the "*participate-up-to*" (PUT) policy: offer FP if its operating cost is lower than a threshold. In comparison, vendor 2's tradeoff is between the profit margin (i.e., a proportional mark up from its own cost) and the profit discounting due to the client's waiting for a better execution time. Therefore, vendor 2 is willing to offer CP when its operating cost is neither too low (low revenue) nor too high (long wait).

Accordingly, vendor 2's optimal participation policy is prescribed by a lower and upper bound of its operating cost, and we therefore refer to this policy structure as the "*participate-in-between*" (PIB) policy.

Finally, combining previous results, we study the client's contract selection policy under vendors' participation constraints and examine the comparative properties of participating firms' performance with respect to system inputs. We notice that while the client's performance measure has state-independent comparative properties as often observed in the RO literature, the vendor's performance have state-dependent comparative statics with respect to elasticity and other system inputs. This suggests that the system parameters and system states have a *separable* impact on the client's performance, but they have a *joint* impact on the vendor's performance. This observation implies that the outsourcing contract partners do not necessarily have conflicts of interest, and the client's timing flexibility could benefit both the vendor and the client, which favors the outsourcing contract with execution timing flexibility (i.e., the RO contract) and calls both parties to explore the opportunities that are mutually beneficial (win-win).

The rest of this paper is organized as follows. Section 2 reviews the related literature. Section 3 formulates the dynamic programming models for the vendors' and client's decision-marking problems. Sections 4-6 investigate the client's and vendors' outsourcing decisions analytically for a base model. Specifically, Section 4 solves the client's contract selection problem by relaxing the vendor participation constraints; Section 5 study the vendors' optimal participation policies; and Section 6 incorporates the vendor participation constraints into the client's contract selection policy. Section 7 extends the base model and investigates the robustness of our results under a general setting. Finally, Section 8 summarizes the contributions and suggests future research directions. All proofs are presented in Appendices A-J.

2. Literature Review

This paper is related to two research streams: supply chain contracts and real options theory. In this research , we focus on two prevalent contracts, the FP contract and the CP contract. Seshadri (2005) reports that the FP contract prevails in practice for its simplicity, but for more complex projects the CP contract sometimes is preferred over the FP contract according to Bajari and Tadelis (2001). Given the rich literature and studies comparing different contracts, we refer interested readers to Cachon (2003) and Kouvelis et al. (2006) for comprehensive reviews. Several papers study and evaluate the importance of timing issues in supply chains. For example, Ferguson et al. (2005) study the optimal sales timing in a supply chain; and Taylor (2006) shows that a manufacturer with a linear contract prefers earlier selling, while that with a general contract is indifferent to the time of sale. Among those papers, a couple of them have endogenized the execution timing decision in the outsourcing contract and are closely related to this research. In particular, Li and Kouvelis (1999), Milner and Kouvelis (2005), Fotopoulos et al. (2008), and Hu et al. (2012) all consider the time-flexible contract and its variants, which are similar to ours and facilitate the client to choose to execute the outsourcing contract and receive products/services anytime within a pre-specified time period. Yet, different from previous papers, our paper also considers the choice of different contract structures and investigates the vendor's contract selection problem by incorporating the vendor's participation constraints.

Further, the focus of the models in supply chain contracts literature is for pure purchasing firms. In this paper, we focus on outsourcing contracts for a client that is currently producing from an in-house operation. Gilley et al. (2004) point out that outsourcing is not simply a purchasing decision. Outsourcing may arise through the substitution of external purchases for internal activities. In this way, it can be viewed as a discontinuation of internal production and an initiation of procurement from outside suppliers. Thus, analyzing an outsourcing contract simply in terms of purchasing activities does not fully capture the true nature of this issue. Our paper studies the strategic outsourcing decisions of firms and their interactions, specifically the firms' contract selection and contract participation policies in the presence of correlated and volatile operating costs, which differentiates this paper from the purchasing-based supply chain contract literature.

The present paper is also related to the research area in real options. From the real options perspective, the firm with an in-house operation has an option on hand: outsource the in-house operation now or later. Such timing-related managerial flexibility has been intensively discussed in economics (Dixit and Pindyck 1994). Conventionally, an investment decision is seen as acceptable when the net present value (NPV) exceeds zero. However, real options theory views that any action that reduces managerial flexibility is acceptable only if its consequent NPV exceeds the value of those lost options. For this reason, it has become a logical principle in financial economics that no decision should take place unless the net benefit at least compensates for the opportunity loss of "option to wait" (Dixit and Pindyck 1994, McGrath 1997, Huchzermeier and Loch 2001). In the outsourcing literature, the application of real options theory is also growing. Johnstone (2002) formulates a public sector outsourcing problem as an exchange option. Kamrad and Siddique (2004) consider a supplier's reaction option in supply chain contracts.

A stream of recent contract studies explores option contract applications to hedge against the price, inventory, currency exchange rate or demand uncertainty. Boyabatli and Toktay (2004) provided a review of operational hedging. Martinez-de Albeniz and Simchi-Levi (2005) suggest a portfolio of contracts consisting of option contracts to reduce inventory risk. They design an option contract that can procure products from both the supplier and a spot market. Wu and Kleindorfer (2005) derive an optimal portfolio of forwards, options, and spot market procurement under competition. Caldentey and Haugh (2006) consider the general problem of dynamic hedging when the profits of a risk-averse corporation are partially correlated with returns in the financial markets. Ding et al. (2007) study operational and financial hedging decisions when demand and exchange rates are uncertain. Fang and Whinston (2007) examine an option contract as a price discrimination tool under the risk of demand loss. Martnez-de Albniz and Simchi-Levi (2009) analyze the suppliers' pricing strategy in the competition through price and flexibility. Chod et al. (2010) provide an options model to consider capacity investment under demand uncertainty, which leads to two types of risk: mismatch between capacity and demand and profit variability. Kouvelis et al. (2013) study the commodity risks in price and consumption volume via physical inventory and financial hedging.

The multiple stochastic factor model considered in this paper has been studied in the RO literature (see, e.g., McDonald and Siegel (1986) and Dixit and Pindyck (1994)) under other settings. Typically, the literature considers a two-dimensional Brownian motion problem and obtains closed-form solutions for an investor's (like the client in this paper) value function. In contrast, this paper develops closed-form solutions for more general value functions under a correlated three-dimensional Brownian motion model. This allows us to incorporate the client's waiting flexibility into the vendor's value function and obtain the closed-form solution of the vendor's participation policy and comparative statics. These new results fill this RO literature void and advance the existing RO theory. Moreover, previous studies typically determine the impact of correlation, by assuming that the covariance of the two factors and variance of each factor are independent parameters; this assumption leads to the result that an increase of either factor's volatility will increase the option value and the expected option waiting time (see, e.g., McDonald and Siegel (1986) and Dixit and Pindyck (1994)). In contrast, our model emphasizes the role of correlation by treating the correlation of the two factors and the volatility of each factor as independent; this leads to the non-monotone relationship between the option value and the correlated relative volatility. We provide a discussion beyond the technical assumptions and reveal significant economic underpinnings that would justify the two different assumptions. A deep understanding of economic foundation of the operating costs can help the client select the correct model, which may have drastically different strategic implications. We show that this new result has interesting implications in operations management applications, such as the problem studied in this paper.

3. Outsoucing Contract Selection and Participation Model

We consider a pre-outsourcing firm deliberating on outsourcing its current in-house operation to contract manufacturers. Two types of outsourcing contracts, the fixed-price (FP) contract and the cost-plus (CP) contract, are solicited from a competitive global outsourcing market. It is possible that one type of contract might be offered by multiple vendors and that one vendor could offer both contracts. Yet, as a standard practice, many firms adopt the benchmarking clauses in the outsourcing contracts to ensure that the contract terms are competitive and in line with the market price (e.g. see Kane (2013)). Therefore, the terms and cost structure of the outsourcing contract are determined exogenously in the competitive market. For clear exposition and without loss of much generality, we abstract the vendor(s) who offers the FP contract as vendor 1, and the one who offers the CP contract as vendor 2. The operating costs for a single unit product of the client and vendor i, $i=1,2$, are denoted by $W(t)$ and $W_i(t)$, $0\leq t$, which evolve according to the following correlated Geometric Brownian Motions (GBM) processes,

$$dW(t) = \mu W(t)dt + \sigma W(t)dB(t), \tag{1}$$

$$dW_i(t) = \mu_i W_i(t)dt + \sigma_i W_i(t)dB_i(t), \quad i=1,2, \tag{2}$$

where $dB(t)$ and $dB_i(t)$ are the standard Brownian motion (BM) processes with correlation $\rho_i \doteq \mathrm{Cov}[dB(t), dB_i(t)]$, the drift rates of future changes μ and μ_i, and volatilities σ and σ_i, respectively, for i=1,2. For the purpose of this research and to streamline the analyses, we consider both the client and vendors will not financially hedge. The operating costs uncertainty originates from multiple sources, including raw materials, exchange rate, political regulation, industrial related idiosyncratic risk, etc., which could be hardly hedged perfectly due to the lack of trading instruments/markets or simply because certain risks cannot be quantified and hedged financially. Yet, the client could potentially mitigate its operating cost risk by carefully planning its contract selection and execution timing decisions.

It is worth noting that we do not assume any dependent structure among ρ_i, σ and σ_i, $i=1,2$, and they are treated as *independent* system inputs throughout this paper. In the literature, ρ_i is also called the instantaneous correlation between $W(t)$ and $W_i(t)$, by Shreve (2004). Intuitively, this is because, from Eqs. (1) and (2), $\mathrm{Cov}[dW(t), dW_i(t)] = \rho_i \sigma W(t) \sigma_i \mathrm{W_i(t)}$, implying the correlation of the two cost processes satisfies $\mathrm{Cor}[dW(t), dW_i(t)] = \frac{\mathrm{Cov}(dW(t), dW_i(t))}{\sigma W(t) \sigma_i \mathrm{W_i(t)}} = \rho_i$, a constant when volatilities σ and σ_i change.

In the FP contract, vendor 1 charges a fixed transaction cost, C, for the outsourcing operation. While in the CP contract, vendor 2 collects a variable transaction cost based on a proportional markup of its operating cost at that time: $(1+\alpha)W_2(t)$, where α is vendor 2's markup. Without much loss of generality, both types of contracts facilitate the client to outsource a unit of product to its vendors. Typically in the current competitive global outsourcing environment, the contract terms, such as the fixed-price in the FP contract (i.e. C) and the profit margin in the CP contract (i.e. α), are exogenously determined by the market or via bidding or benchmarking (i.e., the vendors do not have the pricing power). After a contract is selected at time 0, the client can exercise the selected contract at any time before its expiration time T_E, or let the contract expire without exercising it (T_E may also be understood as the planning horizon of the client). The client's execution timing flexibility reflects the fact that the outsourcing decision is in the hands of a pre-outsourcing firm in a competitive environment, whereas a vendor may require a minimum level of utility that must be warranted for a contract to be acceptable. These assumptions together conform with current outsourcing practices in global markets.

Vendors' internal cost structure and operating conditions could influence their incentive to offer outsourcing contracts in the first place. Specifically, for vendor i to offer an outsourcing contract, its expected profit $U_i(W,W_i)$ should be no less than $\xi_i \geq 0$:

$$U_1(W,W_1) \doteq E\left[\int_{T_1^*}^{T_E} (C - W_1(t))e^{-rt}dt\right] \geq \xi_1, \tag{3}$$

$$U_2(W,W_2) \doteq E\left[\int_{T_2^*}^{T_E} \alpha W_2(t)e^{-rt}dt\right] \geq \xi_2, \tag{4}$$

where $T_i^* \leq T_E$, $i=1,2$ is the client's optimal contract execution time and cash flows are discounted with a constant rate r. Note that ξ_i can be interpreted as the vendor's setup cost or opportunity cost minuses the possible transfer payment (to sign the contract or reserve the vendor's capacity) from the client. To ensure the convergency of the profit function, we require $\mu_i < r$ and $\mu < r$, which essentially implies that the net present value of the future profit is decreasing as the time increases.

We denote the client's revenue rate per unit time by R, which is assumed to be invariant over time. This assumption is for expositional simplicity: the results remain valid for a time-dependent revenue stream. Immediately after the client exercises a contract, it incurs a liquidation cost K. Particularly, the client may lay off employees (a positive liquidation cost) and sell its related assets (a negative liquidation cost or a liquidation revenue) after exercising the outsourcing contract. Therefore, the liquidation cost K can be negative or positive.

Let $W \doteq W(0)$ and $W_i \doteq W_i(0)$ be the costs at time 0. We further denote $V_1(W)$ and $V_2(W,W_2)$ as the expected maximum discounted profits of the client during interval $[0,T_E)$ under the FP and CP contracts, respectively. Notably, the client's profit under CP, $V_2(W,W_2)$, depends on costs W and W_2, whereas its profit under FP, $V_1(W)$, depends only on its own cost W, because the client pays a fixed price no matter how W_1 varies. If the client selects the FP contract at time 0, then its optimal FP outsourcing timing problem can be solved by the following stochastic optimal stopping problem:

$$V_1(W) \doteq \sup_{0 \leq T_1 \leq T_E} E\left[\int_0^{T_1} (R - W(t))e^{-rt}dt + \int_{T_1}^{T_E} (R - C)e^{-rt}dt - K\ e^{-rT_1} I_{0 \leq T_1 < T_E}\right], \tag{5}$$

where T_1 is the execution time of FP contract. If $T_1 = T_E$, the client will not outsource during $[0, T_E)$. The first and second terms of Eq. (5) are the client's expected profit during the in-house operation period and outsourcing operation period respectively, and the third term represents the liquidation cost at time $T_1 < T_E$.

Similarly, if the client selects the CP contract at time 0, then we can present the client's expected profit by solving the following stochastic optimal stopping problem. In particular, denote T_2 as the stopping (execution) time of the CP contract. Then

$$V_2(W,W_2) \doteq \sup_{0\le T_2\le T_E} E\left[\int_0^{T_2}(R-W(t))e^{-rt}dt+\int_{T_2}^{T_E}(R-(1+\alpha)W_2(t))e^{-rt}dt-Ke^{-rT_2}I_{0\le T_2<T_E}\right]. \quad (6)$$

It is worth noting that the client may obligate to pay an up-front fee to reserve its client's capacity upon signing the outsourcing contract. Yet, if this fee is not too large so that it is profitable for the client to choose either contract (e.g., the expected option contract values under both FP and CP contracts are positive), then our model facilitates us to incorperate such transfer payment through adjusting ξ_i and R without affecting the analysis (see more discussion at Section 7). When both contracts are offered, the client will select the contract to maximize its expected profit. Accordingly, the value function of the client can be denoted as

$$V(W,W_2) \doteq \max\{V_1(W), V_2(W,W_2)\}. \quad (7)$$

In the next three sections (i.e., Section 4 to 6), we start with a *base model* of this contract decision problem formulated in Eqs. (3)–(7) by assuming the expiration time $T_E = \infty$ (i.e. the contract never expires) and the liquidation cost $K = 0$. These assumptions enable an explicit analytical solution for each firm. It is worth noting that the explicit analytical solution is possible for FP contract even if $K \neq 0$, but for expositional consistency we will focus on the base model where $K = 0$ and $T_E = \infty$ for both FP and CP contracts. We then complete our analyses by establishing the robustness of our results via considering a general case in Section 7 where the selected contract has a finite expiration date, $T_E < \infty$, and a non-zero liquidation cost, $K \neq 0$.

4. Client's Unconstrained Contract Selection Problem

In this section, we study the client's unconstrained contract selection problem formulated in Eqs. (5)–(7) by relaxing constraints (3) and (4), i.e. assuming both contracts are offered at the client's disposal. Specifically, using a dynamic programming approach, we first solve the client's operational decision (i.e., outsourcing timing decision) and calculate the values for FP and CP contracts in Sections 4.1 and 4.2, respectively. Then, we compare the values of those two contracts and determine the client's contract selection policy in Section 4.3.

4.1. Fixed Price Contract

We start with the client's optimal execution timing decision for FP contract (W^*) and gauge the the value of such contract($V_1(W)$). Using the standard American option theory, we obtain the following proposition.

PROPOSITION 1. *Let $T_E=\infty$ and $K=0$. Under FP contract, it is optimal for the client to follow a threshold-type execution policy. Namingly, the client will exercise FP contract if its operating cost exceeds certain threshold W^*; otherwise, the client should wait. The execution cost threshold W^* and the corresponding value function $V_1(W)$ under the FP contract are given as follows:*

$$W^* = \frac{\beta_1}{\beta_1-1}\left(\frac{C}{r}\right)(r-\mu), \tag{8}$$

$$V_1(W) = \begin{cases} (\frac{W}{W^*})^{\beta_1}(\frac{W^*}{r-\mu}-\frac{C}{r})+(\frac{R}{r}-\frac{W}{r-\mu}), & \text{if } W<W^*, \\ \frac{R-C}{r}, & \text{if } W\geq W^*, \end{cases} \tag{9}$$

where β_1 is the elasticity of the waiting option value with respect to its cost in the FP contract and

$$\beta_1 \doteq \frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}\geq 1. \tag{10}$$

Clearly, if the client's initial cost W is higher than the threshold W^*, then the client should exercise FP immediately. As stated in Eq. (9), the value of FP contract is directly given by the difference between NPV of the operating profit, $\frac{R-C}{r}$. On the other hand, when the client's initial cost is below such threshold, it should wait until its cost first reaches W^*. Therefore, under such scenario, the value of FP contract contains two components: the NPV of keeping the operation in house forever, $\frac{R}{r}-\frac{W}{r-\mu}$, and the NPV once FP is exercised at time T_1^*, $\mathcal{O}_1(W)\doteq(\frac{W}{W^*})^{\beta_1}(\frac{W^*}{r-\mu}-\frac{C}{r})$. Here, $(\frac{W}{W^*})^{\beta_1}$ is the discount factor at the optimal stopping time T_1^*, and $\frac{W^*}{r-\mu}-\frac{C}{r}$ is the value after FP is exercised at T_1^*. Because $\frac{\partial\mathcal{O}_1(W)}{\partial W}=\frac{\beta_1\mathcal{O}_1(W)}{W}$, we can present β_1 in terms of $\frac{\partial\mathcal{O}_1(W)/\mathcal{O}_1(W)}{\partial W/W}$, the *elasticity* of the option value with respect to its cost in the FP contract.

Next, we examine comparative statics of policy parameters with respect to system inputs/states in the FP contract. Specifically, we will call $V_1(W),\mathcal{O}_1$ and W^* the policy parameters, σ, σ_1, ρ_1 and β_1 the system inputs, and W and W_1 the system states hereafter. First, it is clear that each policy parameter in FP is independent of σ_1 and ρ_1. Other comparative static results are displayed in Table 1 (proofs are provided in the appendix), where '$-$' and '$+$' represent the negative (i.e. increasing in the system input will decrease the value of its corresponding policy parameter) and positive (increasing in the system input will increase the value of its corresponding policy parameter) relationships, respectively, between a system input and a policy parameter.

The comparative statics results of $\mathcal{O}_1$ and W^* displayed in Table 1 support our intuition that when the client's cost W becomes more volatile, the option value $\mathcal{O}_1$ becomes higher due to the asymmetry between the bounded downward risk and the unbounded upward potential–an

Table 1 Comparative Statics of Client's policy parameters: the FP contract with $T_E = \infty$ and $K = 0$.

		Policy Parameter	
System input	β_1	$\mathcal{O}_1$, V_1	W*
σ	$-$	$+$	$+$
W		$-$	
β_1		$-$	$-$

observation often supported by the real options literature. Table 1 also establishes the relationship between volatility (σ) and elasticity (β_1), and shows that they have opposite effects on policy parameters: a higher volatility leads to a lower elasticity, which improves the client's option value $\mathcal{O}_1$. Note that the dependence of a policy parameter (e.g., $\mathcal{O}_1$, V_1, and W^*) on σ is only through elasticity (proposition 1), which implies that the client can use elasticity as the key input to guide its outsourcing timing decision in FP. Furthermore, we will show in later sections that the client's operational decision in CP contract also depends on its corresponding elasticity (Section 4.2), and the client's strategic decision (i.e., contract selection decision) is solely driven by the relationship between relevant elasticities (Section 4.3).

4.2. Cost-Plus Contract

We now compute the client's execution timing decision under CP contract for the base case where $T_E = \infty$ and $K = 0$. Similar to the FP case, we define the NPV of CP contract that is exercised at the optimal time T_2^* as

$$\mathcal{O}_2(W, W_2) \doteq V_2(W, W_2) - \left(\frac{R}{r} - \frac{W}{r-\mu}\right). \tag{11}$$

Since $\frac{R}{r} - \frac{W}{r-\mu}$ is the client's contract value if CP is never exercised, $\mathcal{O}_2(W, W_2)$ can also be interpreted as the option value of CP. Note that we can directly show that $\mathcal{O}_2(W, W_2)$ is a homogeneous function with degree one, which implies that $\mathcal{O}_2(W, W_2)/W_2$ becomes a one-dimensional problem with respect to the *cost efficiency ratio* process $\{\lambda_2(t) \doteq W(t)/W_2(t),\ t \geq 0\}$. Thus, instead of solving the original two-dimensional problem (i.e., $\mathcal{O}_2(W, W_2)$), it is equivalent to solve the following one-dimensional equation:

$$v_2(\lambda_2) \doteq \mathcal{O}_2(W, W_2)/W_2, \tag{12}$$

where, $\lambda_2 \doteq W/W_2$ is the cost efficiency ratio at time zero. From Ito's quotient rule (Fries 2007), we know that $\lambda_2(t)$ is a GBM with the following drift rate and volatility

$$\mu_{\lambda_2} \doteq \mu - \mu_2 - \rho_2\sigma\sigma_2 + \sigma_2^2 \text{ and } \sigma_{\lambda_2} \doteq \sqrt{\sigma^2 - 2\rho_2\sigma\sigma_2 + \sigma_2^2}. \tag{13}$$

Then following similar solution approach as in Proposition 1 to solve Eq. (12), we present the client's optimal execution timing decision under CP contract in the following proposition.

PROPOSITION 2. *Let $T_E = \infty$ and $K = 0$. The client will execute CP contract if its cost efficiency ratio $\lambda_2(t)$ exceeds certain threshold λ_2^*; otherwise, the client should wait. The cost efficiency ratio threshold λ_2^* and the value function $V_2(W, W_2)$ under the CP contract are given by*

$$V_2(W,W_2) = \begin{cases} (\frac{\lambda_2}{\lambda_2^*})^{\beta_2}(\frac{\lambda_2^*}{r-\mu} - \frac{1+\alpha}{r-\mu_2})W_2 + (\frac{R}{r} - \frac{W}{r-\mu}), & \textit{if } \lambda_2 < \lambda_2^* \ , \\ \frac{R}{r} - \frac{W_2(1+\alpha)}{(r-\mu_2)}, & \textit{if } \lambda_2 \geq \lambda_2^* \ , \end{cases} \tag{14}$$

$$\lambda_2^* = \frac{\beta_2}{\beta_2 - 1}\frac{(r-\mu)}{(r-\mu_2)}(1+\alpha), \tag{15}$$

where β_2 is the elasticity of the client's option value to its cost under the CP contract satisfying

$$\beta_2 \doteq \frac{1}{2} - \frac{\mu - \mu_2}{\sigma_{\lambda_2}^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu-\mu_2}{\sigma_{\lambda_2}^2}\right)^2 + \frac{2(r-\mu_2)}{\sigma_{\lambda_2}^2}} \ > 1. \tag{16}$$

The client's value $V_2(W, W_2)$ under CP bears the similar interpretation as that under FP, where $\left(\frac{\lambda_2}{\lambda_2^*}\right)^{\beta_2}$ is the discount factor at the stopping time T_2^* and $(\frac{\lambda_2^*}{r-\mu} - \frac{1+\alpha}{r-\mu_2})W_2$ is the value after CP is exercised at T_2^*. In addition, it can be directly shown that β_2 is the *elasticity* of the client's option value to its cost under the CP contract (i.e., $\frac{\partial \mathcal{O}_2(W,W_2)/\mathcal{O}_2(W,W_2)}{\partial W/W}$).

To better explain the comparative statics of policy parameters in CP contract, we need to define a new system measure. In particular, we denote $\mathrm{Cov}_2 \doteq \rho_2\sigma\sigma_2$ as the covariance of the client's and vendor 2's costs, and then define $\mathrm{Cov}_2/\sigma_2^2$ as the *correlated relative volatility* (CRV) of the two costs. The CRV combines volatilities and correlation of two stochastic factors into a single measure. In our problem context, CRV represents the client's cost volatility in relation to vendor 2's cost volatility measured on the percentage basis with the latter normalized to 1. In particular, CRV ≤ 1 (≥ 1) indicates the client's cost movement has a smaller (larger) magnitude compared to vendor 2's cost movement, which plays a key role in determining the comparative statics of policy parameters in CP contract.

PROPOSITION 3. *Let β_2, $\mathcal{O}_2(W, W_2)$ and λ_2^* be the client's elasticity, option value, and efficiency ratio threshold under the CP contract. Then:*

(a). σ_{λ_2} is a decreasing function of ρ_2 and β_2 is a decreasing function of σ_{λ_2};

(b). $\mathcal{O}_2(W, W_2)$ and λ_2^* are increasing in σ_{λ_2} and decreasing in β_2.

(c). σ_{λ_2} is decreasing (increasing) in σ_2 if $CRV \geq 1$ ($CRV \leq 1$).

For better visibility, we summarize Proposition 3 and its related results in Table 2. It is worth noting that Part (c) also directly implies that because of symmetry, σ_{λ_2} is also decreasing (increasing) in σ if $CRV \geq \frac{\sigma^2}{\sigma_2^2}$ ($CRV \leq \frac{\sigma^2}{\sigma_2^2}$). Combining these two observations, we demonstrate that volatility σ_{λ_2} of the cost ratio process is *non-monotone* in volatility of each cost process (i.e. σ_2 or σ). In the following, we will illustrate and explain this non-monotone result via the vendor 2's volatility (σ_2), and that of the client's volatility (σ) can be similarly discussed due to symmetry. Specifically, notice that σ_{λ_2} first decreases in σ_2 until the CRV reaches 1 and increases in σ_2 afterward.

Table 2 Comparative Statics of the Client's Policy Parameters: the CP Contract with $T_E = \infty$ and $K = 0$

			Policy Parameter	
System Input	σ_{λ_2}	β_2	$\mathcal{O}_2, V_2$	λ_2^*
ρ_2	$-$	$+$	$-$	$-$
σ_2 (if CRV≥ 1)	$-$	$+$	$-$	$-$
σ_2 (if CRV≤ 1)	$+$	$-$	$+$	$+$
β_2	$-$		$-$	$-$
σ_{λ_2}		$-$	$+$	$+$

Similar non-monotone patterns are also observed in β_2, $\mathcal{O}_2(W, W_2)$, and ${\lambda_2}^*$ with respect to σ_2, which are significantly different from McDonald and Siegel (1986) page 714 and Dixit and Pindyck (1994) page 211, who presented comparative static analyses of the two stochastic factors model and stated that an increase in either factor's volatility will increase the option value. Those two apparently contradictory results can be traced back to the parameter dependency relationship of system inputs. Specifically, we treat ρ_2, σ, and σ_2 as independent inputs, and covariance Cov_2 as a function of ρ_2, σ, and σ_2; in contrast, McDonald and Siegel (1986) and Dixit and Pindyck (1994) take Cov_2, σ, and σ_2 as independent inputs, and ρ_2 as a function of Cov_2, σ and σ_2. This begs the question of how do the two different assumptions affect volatility of the cost ratio process σ_{λ_2}?

Under McDonald and Siegel (1986), covariance σ_{λ_2} is not a function of $\rho\sigma\sigma_2$ that is considered as fixed. Under our model, $\rho\sigma\sigma_2$ varies along with σ_2, and σ_{λ_2}, and takes the minimum value when $CRV = 1$ as σ_2 increases. Thus, σ_{λ_2} is a non-monotone function of σ_2. This explains the intuition behind Part (c) of Proposition 3: when CRV is in the neighbourhood of 1, the cost ratio is the most stable and the option value elasticity β_2 takes the largest value. As CRV moves away from 1 in either direction, the cost ratio becomes more volatile, leading to a higher CP option value. In the literature, both assumptions that treat either covariance or correlation as a fixed input have been discussed by researchers (e.g., Lund (2005) page 315). However, whether to treat Cov_2 or ρ_2 as a fixed system input is more than a mere technicality. A deep understanding of economic underpinnings of the operating costs can help the client select the correct model, which may have drastically different strategic implications. We substantiate such difference and related insights by the following example.

Example 1. Consider a case where the operating cost includes uncorrelated (e.g., labor) and correlated (e.g., commodity) costs. In particular, let

$$W(t) = W_L(t) + W_C(t), \;\; W_2(t) = W_{2L}(t) + W_{2C}(t) \tag{17}$$

be the client's and vendor 2's total operating costs, respectively, including correlated commodity costs $(W_C(t), W_{2C}(t))$ and independent labor costs $(W_L(t), W_{2L}(t))$. We show in the following two scenarios that the source of increasing cost volatility has different ramifications on the choice of models.

1. **Volatility of the labor cost increases:** In this case, covariance of the total cost equals covariance of the commodity cost, denoted as Cov_{2C}, which is unaffected by volatilities of labor costs. However, correlation of the total operating costs, $\rho_2 = \mathrm{Cov}_{2C}/(\sigma\sigma_2)$, decreases as volatility of either party's labor cost increases. In fact, this is a special case studied by Lund (2005), who shows that when the increase in volatility of a random factor is due to the addition or multiplication of a random variable that is independent of the two random factors, covariance is unaffected, but correlation $\rho_2 = \mathrm{Cov}_2\sigma\sigma_2$ decreases as σ or σ_2 increases. In such cases, by McDonald and Siegel (1986) and Dixit and Pindyck (1994), the option value is an increasing function of σ or σ_2.
2. **Volatility of the commodity cost increases:** Let the cost models be given in (17), with the same dependence structure. Clearly, covariance of the two GBMs increases as either party's commodity cost volatility increases (as discussed in the Introduction section), with a fixed ρ. As we have shown, whether this increase in volatility of a commodity will benefit the client or not will depend on its relation to CRV, due to the impact of covariance.

In practice, managers should use empirical data to justify which assumption is more appropriate for their application. In the case that changes of the volatility come from correlated sources (e.g., the commodity cost in the previous example), our result reveals a deep insight: in the two random factors analysis, it is the *correlated relative volatility* (i.e. the combined volatilities and correlation of two stochastic factors), rather than only the volatility of each individual factor, that affects the property of the option value. In the context of the CP contract, our result implies that the client's timing flexibility becomes more significant when its cost volatility is either much lower or much higher relative to vendor 2's cost volatility.

In conclusion, because of the structural difference between the CP and FP contracts, the client's profit in FP depends only on the client's cost $W(t)$ (a single factor), whereas in the CP contract profit depends on the cost efficient ratio $\lambda_2(t) = \frac{W(t)}{W_2(t)}$ (two factors). This induces the client to follow different operational policies in FP and CP (W^* vs. λ_2^*). Further, the option value of FP contract ($\mathcal{O}_1(W)$) improves when $W(t)$ has a higher volatility, but the option value of CP contract ($\mathcal{O}_2(W, W_2)$) improves when $\lambda_2(t)$ has a higher volatility, which happens when CRV is moving away from 1 in either direction (i.e., when σ_2 either increases or decreases). The implies that a CP vendor whose cost movement is either significantly larger or smaller relative to its own cost movement (measured by CRV) is more preferable to the client.

4.3. Contract Selection with Both Contracts Available

Comparing the values of FP and CP contracts, we can show that there also exists a unique threshold function $S(W)$ that partitions the initial cost space $\{W, W_2\}$ (i.e. the client's and vendor 2's operating costs at time zero) into the FP and CP selection regions such that for a given W, the client will select CP (FP) if and only if vendor 2's initial cost W_2 is below (above) $S(W)$. To see, let $S(W)$ be vendor 2's cost W_2 for which the client's values under the two contracts are the same:

$$S(W) \doteq \{W_2 : \ W \geq 0, W_2 \geq 0, V_1(W) \equiv V_2(W, W_2)\}. \tag{18}$$

We refer to $S(W)$ as the contract selection/switching curve. To rigorously derive $S(W)$, we need to consider the following two cases.

Case 1 $\beta_1 \leq \beta_2$: First note that two threshold lines for the FP and CP contracts, $W = W^*$ and $W_2 = W/\lambda_2^*$ (Proposition 1 and 2), partition the cost space $\{W \geq 0, W_2 \geq 0\}$ into four regions (see Figure 1), wherein both contracts are not immediately exercisable (UU region), FP is exercisable and CP is not (EU region), CP is exercisable and FP is not (UE region), and both contracts are exercisable (EE region). The value functions $V_1(W)$ in Eq. (9) and $V_2(W, W_2)$ in Eq. (14) have different expressions in each region. If $\beta_1 \leq \beta_2$, then as W increases, $S(W)$ will start from the UU region, then enter the UE region at the point $W = W' \leq W^*$, and finally enter the EE region at the point $W = W^*$. In this case, $S(W)$ will first cross the CP threshold line and then the FP threshold line. To derive $S(W)$, we equalize the corresponding expressions in Eqs. (9) and (14) in the UU, UE and EE regions, and obtain

$$S(W) = \begin{cases} S_{UU}(W) = \frac{1}{\lambda_2^*}\left(\frac{\beta_1}{\beta_2}\frac{W^{\beta_2-\beta_1}}{(W^*)^{1-\beta_1}}\right)^{\frac{1}{\beta_2-1}}, & \text{if } W < W', \\ S_{UE}(W) = \left(\frac{W}{r-\mu} - W^{\beta_1}\frac{(W^*)^{1-\beta_1}}{\beta_1(r-\mu)}\right)\frac{r-\mu_2}{1+\alpha}, & \text{if } W' \leq W < W^*, \\ S_{EE}(W) = \frac{C}{r}\frac{r-\mu_2}{1+\alpha}, & \text{if } W \geq W^*, \end{cases} \tag{19}$$

where W' is the unique solution to $S_{UE}(W) \equiv W/\lambda_2^*$.

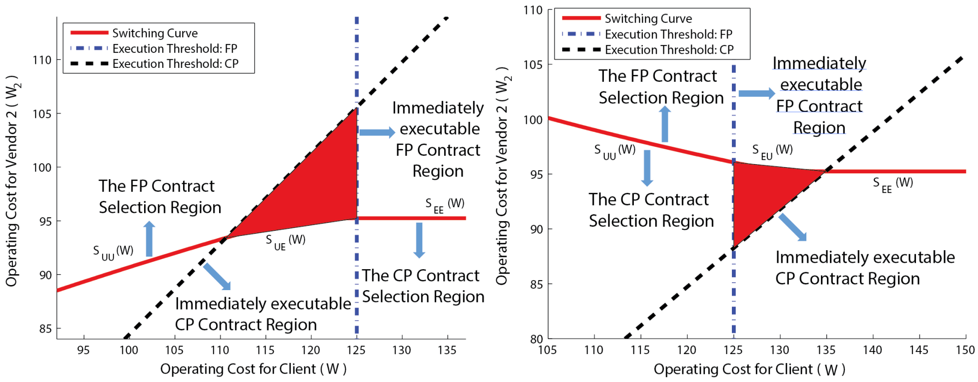
Case 2 $\beta_1 \geq \beta_2$: In this case, $S(W)$ starts from the UU region, enters the EU region at the point $W = W^*$, and finally enter the EE region at the point W''. In other words, $S(W)$ will first intercept the FP threshold line $W = W^*$ and then the CP threshold line $\lambda_2 = \lambda_2^*$. Similar to Case 1, we equalize the corresponding expressions in Eqs. (9) and (14) in the UU, EU and EE regions to obtain $S(W)$:

$$S(W) = \begin{cases} S_{UU}(W) = \frac{1}{\lambda_2^*}\left(\frac{\beta_1}{\beta_2}\frac{W^{\beta_2-\beta_1}}{(W^*)^{1-\beta_1}}\right)^{\frac{1}{\beta_2-1}}, & \text{if } W < W^*, \\ S_{EU}(W) = \left(\frac{\frac{W}{r-\mu} - \frac{C}{r}}{W^{\beta_2}}\frac{\beta_2(r-\mu)}{(\lambda_2^*)^{1-\beta_2}}\right)^{\frac{1}{1-\beta_2}}, & \text{if } W^* \leq W < W'', \\ S_{EE}(W) = \frac{C}{r}\frac{r-\mu_2}{1+\alpha}, & \text{if } W \geq W'', \end{cases} \tag{20}$$

where $W'' \doteq \frac{\beta_2}{\beta_2-1}\frac{\beta_1-1}{\beta_1}W^*$ is the point that S_{EE} intercepts.

Observe from Eqs. (19) and (20) that the effects of system parameters (e.g., σ, σ_1, σ_2, ρ and ρ_2 on $S(W)$) are captured by β_1 and β_2. Therefore, the elasticities of FP and CP are the key drivers of the client's strategic decision. The following theorem summarizes and presents the client's optimal contract selection policy.

Figure 1 **The contract switching curve $S(W)$ for the Base Model: Left figure, $\beta_1 \leq \beta_2$; Right figure, $\beta_1 \geq \beta_2$.**



THEOREM 1. *(The client's unconstrained optimal contract selection policy) Let $T_E = \infty$ and $K = 0$.*

(a). The client with cost W should select CP contract if vendor 2's cost W_2 is below $S(W)$ defined in Eq. (19) and Eq. (20); otherwise the client should select FP contract.

(b). if $\beta_1 \leq \beta_2$, then $S(W)$ is an increasing concave function of W. Specifically, $S_{UU}(W)$ and $S_{UE}(W)$ are increasing concave functions of W, and $S_{EE}(W)$ is an independent function of W.

(c). if $\beta_1 \geq \beta_2$, then $S(W)$ is a decreasing convex function of W. Specifically, $S_{UU}(W)$ and $S_{EU}(W)$ are decreasing convex functions of W, and $S_{EE}(W)$ is an independent function of W.

Part (a) of Theorem 1 states that when both contracts are available, the client's optimal contract selection policy is to select CP if vendor 2's cost is less than $S(W)$, and FP otherwise. Borrowing jargons from the inventory literature, we will refer to the client's optimal contract selection policy as the *select-up-to* (SUT) policy. Part b can be intuitively understood as follows: $\mathrm{S}_{UU}(W)$ is the indifference curve when both contracts are not exercisable. When the option values of the two contracts are the same, $\mathcal{O}_2 \equiv \mathcal{O}_1$, the difference of their marginal option values satisfies $\frac{\partial \mathcal{O}_2}{\partial W} - \frac{\partial \mathcal{O}_1}{\partial W} = (\beta_2 - \beta_1)\frac{\mathcal{O}_2}{W} = (\beta_2 - \beta_1)\frac{\mathcal{O}_1}{W}$, so the contract with the larger option value elasticity will have a higher marginal option value with respect to the cost, and become more attractive to the client as its cost increases. Figure 1 presents $S(W)$ in two cases and shows that the contract with a larger elasticity generates a higher option value with respect to the client's cost and is more beneficial to the client as its cost increases. Note that when $\beta_1 = \beta_2$, $S(W) = \frac{C}{r}\frac{r-\mu_2}{1+\alpha}$ is a constant curve.

There are two strategic scenarios in Figure 1 as a direct consequence of Theorem 1. First look at the "upward triangular" area (i.e., the shaded area on the left-hand-side of Figure 1) constrained by the function $S_{UE}(W)$ in Eq. (19) and the two threshold lines. Recall that when $\beta_1 < \beta_2$, the $S(W)$ first intercepts the CP threshold line and then the FP threshold line. Therefore, in this "upward triangular" area, Theorem 1 suggests a less intuitive contract selecting policy: instead of selecting the immediately exercisable CP contract, the client should choose the currently un-exercisable

FP contract and wait for the best execution time. In other words, even through CP is currently exercisable, the potential benefits from waiting to execute FP in the future (i.e. the option value of FP) is more attractive. We therefore refer to this area as the *FP-oriented mothball region*.

Similarly, when $\beta_1 > \beta_2$, $S(W)$ first intercepts the FP threshold line and then the CP threshold line. We call the "downward triangular" area enclosed by $S_{EU}(W)$ defined in Eq. (20) and the two threshold lines (see the right figure of Figure 1) the *CP-oriented mothball region*, where the client is better off selecting the currently un-exercisable CP contract than choosing the immediately exercisable FP contract.

These two mothball regions are counter-intuitive in the sense that the client in practice tends to select the contract that is immediately exercisable. However, these mothball regions identified in this paper suggest that such a perception may result in not only a sub-optimal outsourcing timing decision, but also an incorrect contract commitment, as partially illustrated by the METAL example in §1. To help the client decide whether an immediately available contract should be passed over in favor of a currently not exercisable contract (i.e. identify the mothball regions), a simple comparison between elasticities can serve as an intuitively appealing criteria.

In the reminder of this section, we further examine the comparative statics of the contract selection curve $S(W)$ with respect to system inputs, which are summarized in the following proposition.

PROPOSITION 4. *Let $T_E = \infty$ and $K = 0$. Then for any client's cost W,*

(a). $S(W)$ is decreasing in ρ_2.

(b). $S(W)$ is decreasing (increasing) in σ_2 if CRV is larger (less) than 1.

(c). $S(W)$ is decreasing in β_1 and increasing in β_2.

We now explain the intuition behind Proposition 4. First, when the cost processes of the client and vendor 2 are more correlated, $V_2(W, W_2)$ decreases but $V_1(W)$ remains the same, which implies that the FP selection threshold $S(W)$ will be reduced. In other words, the client is more likely to select FP when ρ_2 increases. Second, when CRV is greater (less) than 1, the client's cost movement is more (less) volatile relative to vendor 2's cost movement. Directly, increasing σ_2 reduces (increases) volatility σ_{λ_2} and $V_2(W, W_2)$. Since $V_1(W)$ is not affected by σ_2, the FP selection threshold $S(W)$ will be reduced (increased). Therefore, the client is more likely to select FP (CP) when σ_2 increases (decreases) until σ_2 reaches $\rho_2\sigma$ (i.e. CRV reaches 1). Finally, the comparative static of $S(W)$ with respect to β_i, $i = 1, 2$, can be similarly explained, and these results echo our previous discussion: the contract with a larger elasticity generates a higher option value with respect to the client's cost and is more beneficial to the client as its cost increases.

To conclude, we suggest several rules of thumb in the client's strategic decision when both contracts are available: (1) use the simple "select-up-to" (SUT) rule to make the contract selection decision (see Theorem 1 (a)); (2) the contract with the smaller option value elasticity becomes more attractive when the client's initial operating cost increases (see Theorem 1 (b) and (c)); (3) it could be more profitable to select a currently un-exercisable contract with a lower elasticity over an immediately exercisable contract with a higher elasticity (the mothball regions in Figure 1); and (4) FP becomes more favorable when CRV moves toward 1, while CP becomes more attractive when CRV moves away from 1 in either direction (Proposition 4 (b)).

5. Vendors' Contract Participation Problems

In this section, we study the contract selection problem from the vendor's perspective by examining the participation constraints of the FP and CP contracts, formulated in Eqs. (3) and (4) respectively.

5.1. Participation of Fixed Price Contract

When evaluating the value of FP contract, vendor 1 needs to take the client's execution timing decision T_1^* into account (see Eqs. (3)). Incorporating the client's optimal FP contract execution timing decision (Proposition 1), the next proposition explicitly derives the FP contract value from vendor 1's perspective.

PROPOSITION 5. *Let $T_E=\infty$ and $K=0$. Vendor 1's FP contract value is*

$$U_1(W,W_1)=\begin{cases}(\frac{W}{W^*})^{\beta_1}\frac{C}{r}-(\frac{W}{W^*})^{\tilde{\beta}_1}\frac{W_1}{r-\mu_1}, & \text{if } W<W^*,\\ \frac{C}{r}-\frac{W_1}{r-\mu_1}, & \text{if } W\geq W^*,\end{cases} \tag{21}$$

where $\tilde{\beta}_1$ is the elasticity of vendor 1's cost with respect to the client's cost and is given by

$$\tilde{\beta}_1 \doteq \frac{1}{2}-\frac{\mu+\rho_1\sigma\sigma_1}{\sigma^2}+\sqrt{\left(\frac{\mu+\rho_1\sigma\sigma_1}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2(r-\mu_1)}{\sigma^2}}. \tag{22}$$

The two terms in vendor 1's contract value function, Eq. (21), represent its expected revenue (i.e., the client's payment) and operating cost. It is worth noting that $\tilde{\beta}_1$ and β_1 are, respectively, the elasticities of vendor 1's expected cost and revenue with respect to the client's cost W. To see this, first note that T_1^* and $W_1(T_1^*)$ are dependent random variables, and $E[e^{-rT_1^*}W_1(T_1^*)]=(\frac{W}{W^*})^{\tilde{\beta}_1}W_1$, for $W\leq W^*$. As $\frac{\partial(\frac{W}{W^*})^{\tilde{\beta}_1}}{\partial W}=\frac{\tilde{\beta}_1(\frac{W}{W^*})^{\tilde{\beta}_1}}{W}$ and $\frac{\partial(\frac{W}{W^*})^{\beta_1}}{\partial W}=\frac{\beta_1(\frac{W}{W^*})^{\beta_1}}{W}$, we immediately have $\tilde{\beta}_1=\frac{\partial(\frac{W}{W^*})^{\tilde{\beta}_1}\frac{W_1}{r-\mu_1}/(\frac{W}{W^*})^{\tilde{\beta}_1}\frac{W_1}{r-\mu_1}}{\partial W/W}$ and $\beta_1=\frac{\partial(\frac{W}{W^*})^{\beta_1}\frac{C}{r}/(\frac{W}{W^*})^{\beta_1}\frac{C}{r}}{\partial W/W}$. Compared to the client's FP contract value contingent only on W and β_1, vendor 1's FP contract value is more complex and contingent on both parties' costs (W, W_1) and elasticities (β_1, $\tilde{\beta}_1$).

Recall that vendor 1 will participate if and only if its FP contract value is no less than its reservation/opportunity cost (i.e. , $U_1(W,W_1)\geq\xi_1$). Let $P_1(W)$ be vendor 1's participation indifference curve satisfying $U_1(W, P_1(W)) \equiv \xi_1$. From Eq. (21) in Proposition 5, we can directly show that for a given W, $U_1(W, W_1)$ crosses ξ_1 exactly once. Thus $P_1(W)$ is given by

$$P_1(W) = \begin{cases} (\frac{W}{W^*})^{\beta_1 - \tilde{\beta}_1} \frac{C(r-\mu_1)}{r} - \xi_1 (\frac{W}{W^*})^{-\tilde{\beta}_1}(r-\mu_1), & \text{if } W < W^*, \\ (\frac{C}{r} - \xi_1)(r-\mu_1), & \text{if } W \geq W^*. \end{cases} \tag{23}$$

The following theorem states vendor 1's optimal contract participation policy and related comparative statics of $P_1(W)$.

Theorem 2. *(Vendor 1's optimal contract participation policy)*

(a). Vendor 1 should offer the FP contract if and only if its cost W_1 is no larger than $P_1(W)$ defined in Eq. (23).

(b). $P_1(W)$ is increasing concave function in $\tilde{\beta}_1$, which decreases in ρ_1 and σ_1.

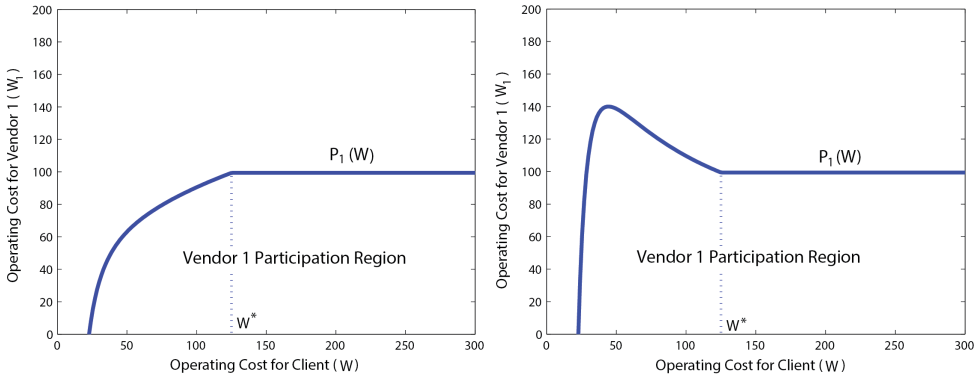
(c). $P_1(W)$ increases in W if $\beta_1 \geq \tilde{\beta}_1(1 - \frac{\xi_1 r}{C})$, or if $\beta_1 \leq \tilde{\beta}_1(1 - \frac{\xi_1 r}{C})$ and $W \leq W^*(\frac{\xi_1 r}{C}/(1 - \frac{\beta_1}{\tilde{\beta}_1}))^{\frac{1}{\beta_1}}$; and decreases in W otherwise.

The first part of Theorem 2 states that given the client's cost W, vendor 1 should offer FP if and only if its own cost W_1 is sufficiently low (i.e., $W_1 \leq P_1(W)$). We therefore refer to vendor 1's optimal contract participation policy as the "*participate-up-to*" (PUT) policy, with the participation threshold determined by the revenue and cost elasticities (see Eq. (23)). The second part shows that as vendor 1's cost becomes less volatile or less correlated with the client's cost, vendor 1's cost elasticity $\tilde{\beta}_1$ increases, which reduces vendor 1's cost and thus the participation threshold $P_1(W)$. This result is not surprising, as each player prefers its own cost to be low.

It is intuitive to believe that $P_1(W)$ increases in W since it expedites the FP execution, which potentially improve the vendor's contract value. However, the last part of Theorem 2 states otherwise and shows that $P_1(W)$ is a non-monotone function of W. For illustration, Figure 2 displays vendor 1's participation regions for two scenarios and shows that a higher W encourages (discourages) vendor 1 to offer FP only when vendor 1's revenue elasticity β_1 is sufficiently large (small) relative to its cost elasticity $\tilde{\beta}_1$. To explain, note that the changes of W lead to two conflicting effects: improve vendor 1's revenue (because the client execute early) and potentially increase vendor 1's production cost (because of the cost correlation between the client and vendor 1). Therefore, only when the percentage change of vendor 1's revenue to the percentage change of W (i.e. β_1) is proportionally higher (lower) than the percentage change of vendor 1's cost to the percentage change of W (i.e., $\tilde{\beta}_1$), vendor 1's participate threshold (i.e., $P_1(W)$) will increase in W.

It is direct to argue that vendor 1's value function, $U_1(W, W_1)$, should share similar comparative static properties of $P_1(W)$, since when vendor 1's value $U_1(W, W_1)$ increases, its participation threshold $P_1(W)$ also increases. Therefore, our discussion of the properties of $P_1(W)$ also applies to $U_1(W, W_1)$. The following proposition summarize the main results of the properties of vendor 1's value function.

Figure 2 **The FP participation curve $P_1(W)$ for the Base Model:**
Left figure, $\beta_1 \geq (1-\frac{\xi_1 r}{C})\tilde{\beta}_1$; Right figure, $\beta_1 \leq (1-\frac{\xi_1 r}{C})\tilde{\beta}_1$.



Proposition 6. *Let $T_E = \infty$, $K = 0$, and $\sigma^2 < 2(r-\mu_1)$.*

(a). $U_1(W, W_1)$ is increasing concave in $\tilde{\beta}_1$.

(b). $(\frac{W}{W^*})^{\beta_1}$ is increasing in β_1 if $\frac{W}{W^*} \geq e^{\frac{-1}{\beta_1-1}}$, and decreasing in β_1 otherwise.

(c). $U_1(W, W_1)$ is decreasing in β_1 if $1 > \frac{W}{W^*} \geq e^{\frac{-1}{\beta_1-1}}$ and $W_1 \leq \tilde{P}_1(W)$, and is increasing in β_1 otherwise, where $\tilde{P}_1(W)$ is vendor 1's cost threshold under which the marginal effect of β_1 to its revenue and cost are the same and is given by

$$\tilde{P}_1(W) \doteq \frac{\beta_1(\beta_1-1)}{\tilde{\beta}_1} P_1(W) \left(ln(\frac{W}{W^*}) + \frac{1}{\beta_1 - 1} \right), \; where \; 1 > \frac{W}{W^*} \geq e^{\frac{-1}{\beta_1-1}}. \tag{24}$$

d. $U_1(W, W_1)$ is increasing in W if $\beta_1 \geq \tilde{\beta}_1 \frac{W_1 r}{C(r-\mu_1)} (\frac{W}{W^*})^{\tilde{\beta}_1 - \beta_1}$, and decreasing in W otherwise.

It is noteworthy that the second part of this proposition presents an unexpected property of the discount factor $(\frac{W}{W^*})^{\beta_1}$ (i.e., $E\left[e^{-rT_1^*} | W < W^*\right]$). Specifically, Harrison (1985) shows that when $\mu > \frac{1}{2}\sigma^2$, the expected stopping time, $E[T_1^* | W < W^*] = \frac{\ln(W^*/W)}{\mu - \frac{1}{2}\sigma^2}$, decrease in β_1. In addition, we have shown that the option value $\mathcal{O}_1(W)$ is decreasing in β_1 (see Table 1). Based on these two results, one could infer that the discount factor should decrease in β_1. This intuition, however, is not true. As seen in part (b) of Proposition 6, the discount factor $(\frac{W}{W^*})^{\beta_1}$ is actually decreasing in β_1 only when W is sufficiently small relative to W^* (i.e., the ratio $\frac{W}{W^*}$ is less than the fraction $e^{\frac{-1}{\beta_1-1}}$), otherwise the opposite is true. We can understand this result as follows: when W is far smaller than W^*, the discount factor indeed becomes bigger with a smaller elasticity value. In this case, vendor 1 benefits from a smaller value of β_1. However, if W is sufficiently close to W^*, the discount factor actually becomes smaller with a smaller value of β_1. In this case, the client tends to expedite the FP execution timing. In other words, the comparative statics of vendor 1's revenue, $(\frac{W}{W^*})^{\beta_1} \frac{C}{r}$, in β_1 depends on system state W. Because of this state-dependent monotonicity of the discount factor explained above, the comparative statics of $U_1(W, W_1)$ in β_1 and W is also state-dependent (i.e., the third and fourth parts of Proposition 6).

To summarize, we obtain several insights for FP contract participation decision: (1) the vendor should use the simple "participate-up-to" (PUT) rule (see Theorem 2 (a)); (2) the vendor is more willing to offer FP contract if cost elasticity ($\tilde{\beta}_1$) is high, cost correlation (ρ_2) is low, cost volatility (σ_2) is low (see Theorem 2 (b)); (3) the vendor's contract value function exhibits a non-monotonic pattern with respect to its revenue elasticity (β_1) and the client's initial cost (W) (see Proposition 6). Those insights help the client identify the FP vendor who is most likely to participate.

5.2. Participation of Cost-Plus Contract

In this subsection, we examine vendor 2's participation condition. Similar to the case of FP contract, we derive and present vendor 2's CP contract value $U_2(W, W_2)$ in the following proposition.

PROPOSITION 7. *Let $T_E = \infty$ and $K = 0$. Then*

$$U_2(W,W_2) = \begin{cases} \frac{\alpha W_2}{r-\mu_2}(\frac{\lambda_2}{\lambda_2^*})^{\beta_2}, & \text{if } \lambda_2 < \lambda_2^*, \\ \frac{\alpha W_2}{r-\mu_2}, & \text{if } \lambda_2 \geq \lambda_2^*. \end{cases} \tag{25}$$

Moreover,

(a). $U_2(W, W_2)$ is decreasing in β_2 and ρ_2 and increasing in σ_{λ_2} if $\lambda_2 \leq e^{\frac{-1}{\beta_2-1}}\lambda_2^*$; and is increasing in β_2 and ρ_2 and decreasing in σ_{λ_2} otherwise.

(b). $U_2(W, W_2)$ increases (decreases) in σ_2 if $\lambda_2/\lambda_2^* \leq e^{\frac{-1}{\beta_2-1}}$ ($\lambda_2/\lambda_2^* > e^{\frac{-1}{\beta_2-1}}$) and $CRV \geq 1$ ($CRV < 1$).

(c). $U_2(W, W_2)$ is a decreasing (an increasing) function of W_2 if $\lambda_2 < \lambda_2^*$ ($\lambda_2 \geq \lambda_2^*$); $U_2(W, W_2)$ is an increasing function of W.

Vendor 2's contract value given in Eq. (25) can be understood intuitively: if $\lambda_2 < \lambda_2^*$, the CP execution will be postponed (Proposition 2) so that vendor 2's revenue is its profit $\frac{\alpha W_2}{r-\mu_2}$ multiplied by the discount factor $(\frac{\lambda_2}{\lambda_2^*})^{\beta_2}$; otherwise, CP will be executed immediately and vendor 2 earns undiscounted profit of $\frac{\alpha W_2}{r-\mu_2}$. The rest of Proposition 7 characterizes the comparative statics of vendor 2's contract value function $U_2(W, W_2)$. Similar to that of Proposition 6, due to the state-dependent nature of the discount factor $(\frac{\lambda_2}{\lambda_2^*})^{\beta_2}$ with respect to β_2 and the non-monotonicity of β_2 with respect to σ_2, the $U_2(W, W_2)$ is also state-dependent and non-monotone in system parameters (e.g., σ_2, W_2, σ_{λ_2}, ρ_2, and β_2).

It is interesting to compare and contrast the comparative statics of the client to those of vendor 2 under the CP contract. Recall that the monotone direction of $V_2(W, W_2)$ with respect to σ_2 depends on whether $CRV \geq 1$ but is *state-independent*. From Table 2 and Proposition 7, we obtain Table 3, in which the thresholds $CRV = 1$ and $\lambda_2 = \lambda_2^* e^{\frac{-1}{\beta_2-1}}$ partition the space $\{\sigma_2 \geq 0, \lambda_2 > 0\}$ into four areas, such that in each area, the two decision makers may have either common interests (the change direction of contract values is consistent for both the client and vendor 2) or conflicts

Table 3 Comparative Statics of V_2 and U_2 with respect to σ_2 : the CP contract with $T_E = \infty$ and $K = 0$

	System state λ_2			
	$\frac{\lambda_2}{\lambda_2^*} \le e^{\frac{-1}{\beta_2 - 1}}$		$\frac{\lambda_2}{\lambda_2^*} \ge e^{\frac{-1}{\beta_2 - 1}}$	
	V_2	U_2	V_2	U_2
σ_2 ($\frac{\text{Cov}_2}{\sigma_2^2} \ge 1$)	$-$	$+$	$-$	$-$
σ_2 ($\frac{\text{Cov}_2}{\sigma_2^2} \le 1$)	$+$	$+$	$+$	$-$

of interest (the contract values have the opposite change directions for the client and vendor 2). In particular, at the lower-left (upper-right) quadrant (i.e., the client's cost movement is smaller (larger) compared to vendor 2's cost movement and the cost ratio is lower (higher) than the threshold $\frac{W}{W_2} = \lambda_2^* e^{\frac{-1}{\beta_2 - 1}}$), then both decision makers benefit from a more volatile (stable) W_2 (i.e., a win-win scenario). The intuition is that at the lower-left (upper-right) quadrant, when σ_2 increases (decreases), CRV will move further to the left (right) of the threshold $CRV = 1$, which improves the client's profit; in the meantime, the two conditions $CRV \ge 1$ and $\frac{W}{W_2} \le \lambda_2^* e^{\frac{-1}{\beta_2 - 1}}$ ($CRV \le 1$ and $\frac{W}{W_2} \ge \lambda_2^* e^{\frac{-1}{\beta_2 - 1}}$) ensure that vendor 2's discount factor increases in λ_2, which subsequently improves its profit. Due to symmetry, all the results in Table 3 remain valid if σ_2 is replaced by σ.

Next, we derive vendor 2's optimal CP contract participation policy. Given (W, W_2), vendor 2 will offer the CP contract if and only if $U_2(W, W_2) \ge \xi_2$. Note that the two expressions of $U_2(W, W_2)$ in Eq. (25) cross each other at $W = \frac{\xi_2 \lambda_2^*(r - \mu_2)}{\alpha}$. This means that when $W < \frac{\xi_2 \lambda_2^*(r - \mu_2)}{\alpha}$, vendor 2 could not meet its participation constraint and thus will not offer CP. When $W \ge \frac{\xi_2 \lambda_2^*(r - \mu_2)}{\alpha}$, we then denote vendor 2's participation indifference curve as $P_2(W)$, i.e., $U_2(W, P_2(W)) \equiv \xi_2$, and present it as follows:

$$P_2(W) \doteq \begin{cases} P_2^H(W) \doteq (\frac{W}{\lambda_2^*})^{\frac{\beta_2}{\beta_2 - 1}} (\frac{\alpha}{(r - \mu_2)\xi_2})^{\frac{1}{\beta_2 - 1}}, & \text{if } \lambda_2 < \lambda_2^*, \\ P_2^L(W) \doteq \frac{(r - \mu_2)\xi_2}{\alpha}, & \text{if } \lambda_2 \ge \lambda_2^*. \end{cases} \tag{26}$$

The following theorem summarizes the above discussion and vendor 2's participation policy.

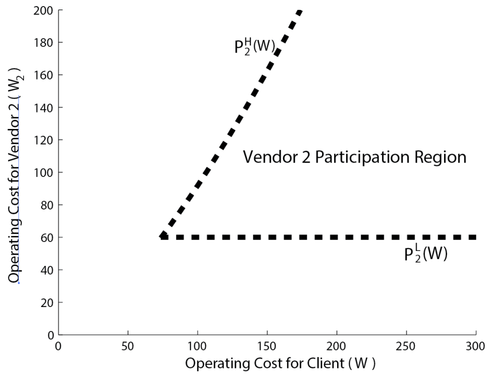
THEOREM 3. *(Vendor 2's optimal CP contract participation policy)*

(a). Vendor 2 will offer the CP contract if and only if $P_2^L(W) \le W_2 \le P_2^H(W)$ *and* $W \ge \frac{\xi_2 \lambda_2^*(r - \mu_2)}{\alpha}$.

(b). When $\lambda_2 \le \lambda_2^*$, $P_2(W)$ *is increasing in* β_2 *if* $\lambda_2^* \frac{(r - \mu_2)\xi_2}{\alpha} e \le W \le \lambda_2^* W_2$, *and decreasing in* β_2 *if* $\lambda_2^* \frac{(r - \mu_2)\xi_2}{\alpha} \le W < \lambda_2^* \frac{(r - \mu_2)\xi_2}{\alpha} e$.

Figure 3 depicts vendor 2's participation region and illustrates that given $W \ge \lambda_2^* \frac{\xi_2(r - \mu_2)}{\alpha}$, vendor 2's participation is determined by two threshold functions $P_2^L(W)$ and $P_2^H(W)$ given in Eq. (26). To understand, note that vendor 2's profit is the highest at the point where $W/W_2 = \lambda_2^*$ (Proposition 7 (c)). When W_2 decreases, the cost efficiency ratio moves to the CP execution region ($W/W_2 \ge \lambda_2^*$) and vendor 2's expected profit $\frac{\alpha W_2}{r - \mu_2}$ is decreasing in W_2. As W_2 reaches the lower bound

Figure 3 Vendor 2's CP participation curve $P_2(W)$ for the Base Model.



$P_2^L(W) = \frac{(r-\mu_2)\xi_2}{\alpha}$, vendor 2's profit reaches the participation indifference point $U_2(W, \frac{(r-\mu_2)\xi_2}{\alpha}) = \xi_2$. Therefore, when the cost efficiency ratio is in the CP execution region, vendor 2 prefers its cost to be as high as possible (so that the revenue from the client will be high as well), and will not offer CP if its cost is too low. Conversely, when W_2 increases and the cost efficiency ratio moves from λ_2^* to the CP non-execution region ($\frac{W}{W_2} \le \lambda_2^*$), then because the profit increases at a linear rate but the discounting occurs at a faster rate $\beta_2 > 1$, vendor 2's expected profit $\frac{\alpha W_2}{r-\mu_2}(\frac{\lambda_2}{\lambda_2^*})^{\beta_2}$ is decreasing in W_2. When W_2 reaches the upper bound $P_2^H(W)$, vendor 2's profit reaches the break-even point $U_2 = \xi_2$. Therefore, when the cost ratio is in the CP non-execution region, vendor 2 prefers its cost to be as low as possible to induce a faster CP execution.

Hereafter, we will refer to vendor 2's optimal CP participation policy as the "*participate-in-between*" (PIB) policy. Note that for the special case where $\xi_2 = 0$, we have $P_2^L(W) = 0$ and $P_2^H(W) = \infty$, which is as expected: since when $U_2 \ge 0$, vendor 2 will always offer the CP contract.

In summary, in this section we derive vendor 2's contract value and its optimal participation policy. We show that vendor 2 should offer CP only when W is not too low and should use the "participate-in-between" rule (see Theorem 3 (a)). Via the analyses of comparative statics, we identify the key characteristics that induce a higher likelihood of vendor 2's participation: (1) a low elasticity β_2 if λ_2 is sufficiently low, or a high elasticity β_2 if λ_2 is sufficiently high (Proposition 7 (a)); (2) a cost efficiency ratio that is sufficiently close to λ_2^* (Proposition 7 (c)); and (3) in certain cost regions both players prefer a lower elasticity value β_2 (Table 3). Accordingly, these characteristics offer the guidelines for the client to select a CP vendor who is most likely to participate.

6. Client's Constrained Contract Selection

Now, we return to the original client's contract selection problem with the vendor's participation consideration. Through combining the client's contract selection policy (Theorem 1) with the vendor participation constraints (Theorem 2 and 3), the following theorem presents the client's optimal contract selection policy.

THEOREM 4. *The client's optimal constrained contract selection policy is determined by the following selection function:*

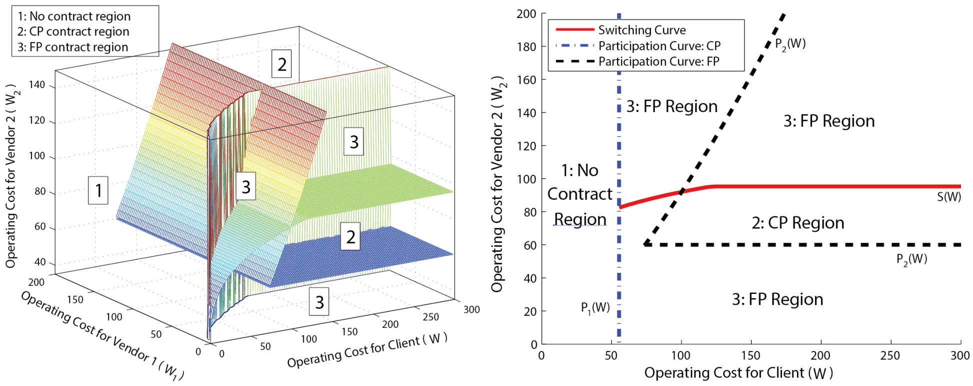
$$S_C(W,W_1,W_2) \doteq \begin{cases} \text{No}, & \text{if } W_1 > P_1(W) \text{ and } W < \frac{\xi_2\lambda_2^*(r-\mu_2)}{\alpha}, \\ & \text{or if } W_1 > P_1(W),\ W \geq \frac{\xi_2\lambda_2^*(r-\mu_2)}{\alpha},\ \text{and } P_2^H(W) < W_2 \text{ or } P_2^L(W) > W_2; \\ \text{CP}, & \text{if } W_1 > P_1(W),\ W \geq \frac{\xi_2\lambda_2^*(r-\mu_2)}{\alpha},\ \text{and } P_2^L(W) \leq W_2 \leq P_2^H(W), \\ & \text{or if } W_1 \leq P_1(W),\ \frac{\xi_2\lambda_2^*(r-\mu_2)}{\alpha} < W,\ \text{and } P_2^L(W) \leq W_2 \leq \min\{S(W), P_2^H(W)\}; \\ \text{FP}, & \text{otherwise}. \end{cases} \tag{27}$$

Theorem 4 can be visualized by coupling the vendor's participation functions $P_1(W)$ and $P_2(W)$ (Figures 2 and 3) with the client's contract selection function $S(W)$ (Figure 1). In particular, the intercepts of the three curves, $S(W)$, $P_1(W)$ and $P_2(W)$, partition the three-dimensional cost space $\{W \geq 0, W_1 \geq 0, W_2 \geq 0\}$ into CP, FP, and No contract (both vendors decline to participate) selection regions. The No contract regions follow from the condition that FP will not be offered (see Theorem 2 (a)) and the condition that CP will not be offered (Theorem 3 (a)). Further, we can identify two regions in which CP will be selected: in the first region only CP is offered (Theorems 2 and 3); in the second region both CP and FP are offered, but by Theorem 1, the client selects CP because $W_2 \leq S(W)$. In all other areas, the FP contract will be selected: either because only FP is offered or because both FP and CP are offered but FP is in favor.

There are 4 scenarios of the partition $S_C(W,W_1,W_2)$ depending on $\beta_1/(1-\frac{\xi_1 r}{C}) \geq (\leq)\tilde{\beta}_1$ and $\beta_1 \geq (\leq)\beta_2$, which determine the functional properties of $P_1(W)$ and $S(W)$ (see Figure 1 and 2). Through a systematic and extensive numerical study of those 4 scenarios, we observe that all the scenarios share similar structure. Therefore, we only present and illustrate, in Figure 4, an example of the partition $S_C(W,W_1,W_2)$ under the scenario $\beta_1/(1-\frac{\xi_1 r}{C}) \geq \tilde{\beta}_1$ and $\beta_1 \leq \beta_2$ in the three-dimensional space (the left figure). In this figure, the vertical surface is the participation curve for FP contract (i.e., $P_1(W)$), and vendor 1 will offer this contract as long as its own operation cost (i.e., W_1) is lower than its participation curve; the participation curve of CP contract (i.e., $P_2(W)$) consists of two pieces–the horizontal $P_2^L(W)$ surface and the increasing $P_2^H(W)$ surface; the client's contract selection/switching curve (i.e., $S_C(W,W_1,W_2)$) intercepts and separates vendors' participation curves (i.e., $P_1(W)$ and $P_2^H(W)$).

For a better visualization, we also provide a two-dimensional figure (the right figure of Figure 4), in which we fix vendor 1's initial cost at 45, i.e., $W_1 = 45$. Clearly, as vendor 1's initial operating cost is fixed, it will offer FP contract when the client's initial cost is not too low (e.g., the right region to the dash-dot $P_1(W)$ line according to Theorem 2,); vendor 2 offers CP contract when its operating cost is neither too low nor too high comparing to the client's cost (e.g., enclosed by the dash $P_2(W)$ lines according to Theorem 3). Therefore, when both vendors prefer not to offer contracts (e.g., $W \leq 50$), we observe the *No Contract Region.* As the client's operating cost increases (e.g.,$W > 50$), vendor 1 starts offering FP contract. Obviously, it is always beneficial for

Figure 4 **The client's contract selection regions in the Base Model (i.e., $T_E = \infty$ and $K = 0$): Left figure, 3 dimensions; Right figure, 2 dimensions where W_1 is fixed at 45.**



Notes: The specific parameter values used to generate this figure are $r = 0.06$, $R = 300$, $C = 100$, $\alpha = 0.05$, $\mu = \mu_1 = \mu_2 = 0.01$, $\sigma = \sigma_1 = \sigma_2 = 0.15$, $\rho_1 = \rho_2 = 0.5$, and $\xi_1 = \xi_2 = 50$.

the client to have a contract with the execution flexibility than without, so the client chooses the only available contract–the FP contract (e.g., *FP Region*). As the client's operating cost further increases, vendor 2 begins offering CP contract. Facing the choice of these two contracts, the client will select CP contract when vendor 2's operating cost is not too high (so that the client' outsourcing cost, proportionally to vendor 2's operating cost, is not too high). In other words, the *CP Region* is below the switching curve $S(W)$. Finally, it is worth noting that if we use a higher vendor 1's initial operating cost (W_1) to generate this two-dimensional figure, then we should expect that the vendor 1 is less willing to offer FP contract (i.e., $P_1(W)$ curve move to the right) and CP contract is more likely to be chosen (i.e., CP Region becomes larger comparing to FP Region).

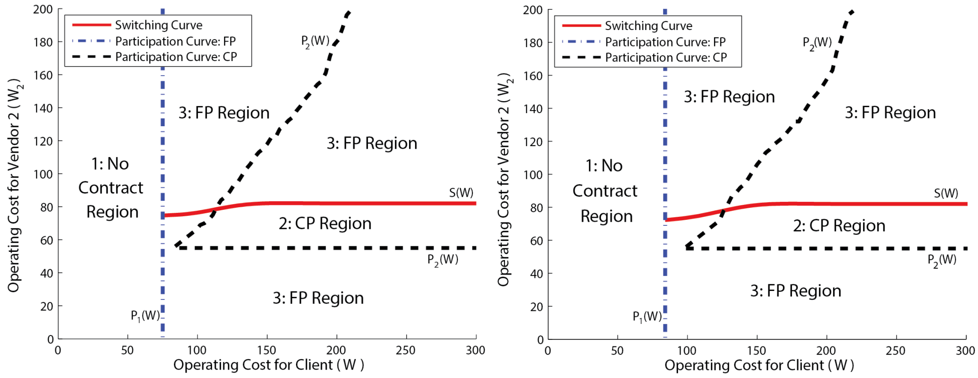
Evidently, with vendor's participation constraints imposed, the client's contract selection space is much limited. Nevertheless, by soliciting two contracts of different types rather than one contract of a particular type, the client can significantly increase its selection space and enlarge the likelihood of reaching an agreement with a willing partner.

7. Analyses of the General Case

In previous sections, for analytical tractability, we focus on the base model with an infinite contract expiration time and a negligible liquidation cost (i.e., $T_E = \infty$ and $K = 0$). In this section, we will relax these assumptions and examine the robustness of our results and insights gained from the previous sections.

Similar to the analyses of the base model, we first consider the value function under FP, defined in Eq. (5), with $T_E < \infty$ and $K \neq 0$. This problem is equivalent to a finite-horizon American option and we refer to Shreve (2000, Section 8.4) for analytical details, which are based on optimal stopping theory developed by Van Moerbeke (1979). The FP contract execution problem consists of

Figure 5 **The client's contract selection regions in the general case (W_1 is fixed at 45): Left figure, $T_E = 50$ and $K = 0$; Right figure, $T_E = 50$ and $K = 200$.**



Notes: All parameter values, except T_E and K, are consistent with those of Figure 4.

identifying a time-dependent threshold function $W^*(t)$, for $0 \le t \le T_E$, such that FP is exercised if and only if $W(t) \ge W^*(t)$ to maximize the client's value starting from time 0. On the other hand, when CP contract is selected, the optimal execution timing problem consists of identifying a threshold function $\lambda_2^*(t)$, for $0 \le t \le T_E$, such that CP is exercised if and only if $\lambda_2(t) \ge \lambda_2^*(t)$. Then, the optimal outsourcing time functions $W^*(t)$ and $\lambda_2^*(t)$, are embedded in vendors' value functions $U_1(W, W_1)$ and $U_2(W, W_2)$, respectively, to determine vendors' contract participation functions $P_1(W)$ and $P_2(W)$ and client's contract selection function $S(W)$. At last, we calculate those functions through implementing the least-squares method proposed by Longstaff and Schwartz (2001). The procedure is based on Monte Carlo simulation with 10,000 paths and 50 time steps.

Using the same parameter combinations as in the base model, we conduct the robustness test for all four scenarios of the general case, depending on $\beta_1/(1 - \frac{\xi_1 r}{C}) \ge (\le)\tilde{\beta}_1$ and $\beta_1 \ge (\le)\beta_2$, through varying the value of the contract expiration time and the liquidation cost (i.e., T_E and K). We find that the FP and CP selection regions and the switching curves in the general case ($T_E < \infty$ and $K \neq 0$) resemble their counterparts in the base model and therefore conform with the explanations derived before. Although our results and insights from the base model are qualitatively resilient to the general case, we observe that the contract expiration time T_E and the liquidation cost K could quantitatively influence FP and CP selection regions and the switching curves. For illustration (other cases are omitted for similarity), we extend the two-dimentional figure of Figure 4 (i.e., the base model) to the general case, and present the FP and CP selection regions in Figure 5. Particularly, in the left figure of Figure 5, we hold the liquidation cost at zero (i.e., $K = 0$) and allow both FP and CP contracts to expire at time $T_E = 50$. In the the right figure of Figure 5, not only both contracts expire at time $T_E = 50$, the client will incur a positive liquidation cost to execute any outsourcing contract (e.g., $K = 200$).

Similar to the base case, the movement of the client's switching curve, $S(W)$, depends on other system parameters, but the magnitude of the influences of T_E and K on $S(W)$ is relatively small (e.g., comparing the right figure of Figure 4 to Figure 5). This observation is not surprising: changing terms of contracts have a marginal influence on the client's choice over these two contracts, as long as the same changes are applied to both contracts. On the other hand, from vendors' perspectives, the changes of T_E and K have significant impacts in valuating these two contracts, and the movement of the vendors' participation curves actually exhibit monotonic properties with respect to T_E and K. Specifically, decreasing contract expiration time (i.e. outsourcing the client's operations for a shorter period of time) reduces the value of the contract to its vendor. Accordingly, vendors will only offer contracts to the client with a high operating cost, under which the client will execute its contract earlier and therefore increases the value of contract to its vendor. In other words, the FP and CP participation curves move to the right (i.e.,outsourcing contracts will be offered to client with a high initial operation cost). Similarly, the client favors an early execution when facing a low liquidation cost (i.e., K is small or even negative), which increases the contract value to its vendor. In turn, vendors are more willing to offer outsourcing contracts if the client has a low liquidation cost–FP and CP participation curves move to the left as liquidation cost decreases.

In previous sections, following the literature, we assume both FP and CP option contracts are "free" to the client (i.e., the client does not need to pay at time zero when signing the contract with its vendor). Yet, committing to the contract, the vendor needs to build up its capacity or reserve its partial capacity, which can be quite costly. Therefore, it seems to be fair and reasonable to compensate the vendor with a monetary payment, P, up-front when the contract is signed. As discussed in Section 3, we can incorporate this contract price (as long as the price is not too large so that it is not profitable for the client to consider the outsourcing contract) by setting the revised opportunity cost for vendor i, $\tilde{\xi}_i$, and the revised client's revenue rate, $\tilde{R}$, as follows: $\tilde{\xi}_i = \xi_i - P$ and $\tilde{R} = R - ((P \cdot r)/(1 - e^{-r \cdot T_E}))$. We observe that as the contract price increases, vendor 1's participation curve, $P_1(W)$, increases (i.e., move to the left in the 2 dimensions figure where W_1 is fixed); vendor 2's upper participation curve $P_1^H(W)$ increases and lower participation curve $P_1^L(W)$ decreases (i.e., $P_1^H(W)$ moves up and $P_1^L(W)$ moves down in the 2 dimensions figure); and the client's contract selection curve $S(W)$ stays unchanged. All those observations can be directly proved for the base model. Intuitively, we should not be surprised by those results. With the additional payment from the client, the vendor will be more willing to accept the contract from the client who has a low initial production cost (therefore more likely to postpone contract execution) and is less likely to attract the participation of its vendors otherwise. On the other hand, as the client will pay the same price for either contract, its contract selection decision/curve is hardly influenced by the contract price (as long as both contracts are still attractive).

8. Conclusion

This paper uses the real options framework to investigate a pre-outsourcing firm's optimal outsourcing decisions at both the strategic (contract selection) and operational (outsourcing timing) levels, subject to vendors' participation constraints. From the modeling perspective, this paper consists of several unique constructs that have not been studied in the literature: the contract selection problem faced by the client; the contract participation problem faced by a vendor; and the interactions between the client and vendors (i.e., the impact of the client's execution timing flexibility to vendors, and the impact of vendors' participation constraints to the client).

Through modeling from both the client's and the vendors' perspectives, we advance the theoretical RO theory and derive appealing strategic and operational policies. In particular, from the client's side, we enhance the RO theory by developing strong analytical results for the client's optimal value function and contract selection policy structure (i.e. the select-up-to policy), and investigate their comparative statics. From the vendors' sides, we incorporate the client's waiting flexibility into vendors' value functions to obtain the vendors' optimal participation policies (i.e. the participate-up-to policy for FP contract and participate-in-between policy for CP contract). These new results propel the existing RO theory and make the RO approach more complete and appealing to practitioners.

We also offer several new managerial insights that either have not been discussed in the literature or are contrary to conventional beliefs. By comparing the FP and CP contract values, we reveal two strategic scenarios, the FP- and CP-oriented mothball regions, in which the client is better off not exercising the immediately exercisable contract but selecting the other currently unexercisable contract and waiting for its best execution timing. The mothball region scenarios caution managers against early commitment to an incorrect contract form, even if this contract is profitable and immediately exercisable.

In addition, we reveal the significant role played by elasticities in each firm's strategic contract selection and participation decisions, and obtain simple rules of thumb to determine the impact of elasticities on the firm's strategic policy, which appeal to managerial intuition and ease implementation. Contrary to the conventional belief (e.g. McDonald and Siegel (1986) and Dixit and Pindyck (1994)), we also show that the client's option value of CP contract does not necessarily increase in either vendor's or client's cost processes. To explain, we develop a new measure, Correlated Relative Volatility (CRV), which gauges the relative cost volatility of the client in relation to the vendor, and show that CRV is the key driver to determine the comparative statics of the optimal contract value and participation functions. In particular, we demonstrate that the CP contract's option value increases as the difference between those two cost volatility becomes more significant (i.e., CRV moves away from 1 in either direction). This nonmontonicity of policy parameters in CP contract value also offers opportunities to both firms where a high cost volatility can benefit both the client and the vendor. Further, we capture the fundamental difference between the vendor and clients comparative properties: while the former is state-dependent (either U or reverse U shaped) in a system input (such as volatility), the latter is typically monotone and state-independent. An implication of this result is that the outsourcing contract partners do not necessarily have conflicts of interest so both parties should explore the opportunities that are mutually beneficial. These managerial insights provide practical guidance on how outsourcing contracts should be effectively managed.

References

Aksin, O. Zeynep, Francis de Vericourt, Fikri Karaesmen. 2008. Call center outsourcing contract analysis and choice. *Management Science* **54**(2) 354–368.

Alvarez, L. H. R., R. Stenbacka. 2007. Partial outsourcing: A real options perspective. *International Journal of Industrial Organization* **25**(1) 91–102.

Bajari, P., S. Tadelis. 2001. Incentives versus transaction costs: A theory of procurement contracts. *The RAND Journal of Economics* **32**(3) 387–407.

Bergin, P. R., R. C. Feenstra, G. H. Hanson. 2007. Outsourcing and volatility. *NBER Working Paper No. W13144* .

Boyabatli, O., L.B. Toktay. 2004. Operational hedging: A review with discussion. *Working paper, INSEAD, Fontainebleau, France* .

Cachon, G. P. 2003. Supply chain coordination with contracts. *Handbooks in Operations Research and Management Science: Supply Chain Management* .

Caldentey, R., M.B. Haugh. 2006. Supply contracts with financial hedging. *Operations Research* **57**(1) 47–65.

Chod, J., N. Rudi, J.A. Van Mieghem. 2010. Operational flexibility and financial hedging: Complements or substitutes? *Management Science* **56**(6) 1030–1045.

Ding, Q., L. Dong, P. Kouvelis. 2007. On the integration of production and financial hedging decisions in global markets. *Operations Research* **55**(3) 470.

Dixit, Avinash K., Robert S. Pindyck. 1994. *Investment under Uncertainty*. Princeton University Press.

Fang, F., A. Whinston. 2007. Option contracts and capacity management-enabling price discrimination under demand uncertainty. *Production and Operations Management* **16**(1) 125–137.

Ferguson, M. E., Gregory A. DeCroix, Paul H. Zipkin. 2005. Commitment decisions with partial information updating. *Naval Research Logistics* **52**(8) 780–795.

Fotopoulos, S.B., X. Hu, C.L. Munson. 2008. Flexible supply contracts under price uncertainty. *European Journal of Operational Research* **191**(1) 253–263.

Fries, C. P. 2007. *Mathematical Finance: Theory, Modeling, Implementation*. Wiley.

Gilley, K.M., C.R. Greer, A.A. Rasheed. 2004. Human resource outsourcing and organizational performance in manufacturing firms. *Journal of Business Research* **57**(3) 232–240.

Harrison, JM. 1985. Brownian motion and stochastic flow systems. *New York* .

Hu, X., C.L. Munson, S.B. Fotopoulos. 2012. Purchasing decisions under stochastic prices: Approximate solutions for order time, order quantity and supplier selection. *Annals of Operations Research* **201**(1) 287–305.

Huchzermeier, A., M. A. Cohen. 1996. Valuing operational flexibility under exchange rate risk. *Operations Research* **44**(1) 100–113.

Huchzermeier, A., C. H. Loch. 2001. Project management under risk: Using the real options approach to evaluate flexibility in r&d. *Management Science* **47**(1) 85–101.

Jacks, D. S., K. H. O'Rourke, J. G. Williamson. 2009. Commodity price volatility and world market integration since 1700. *NBER Working Paper No. W14748* .

Johnstone, D. 2002. Public sector outsourcing as an exchange option. *Abacus* **38**(2) 153–176.

Kamrad, B., A. Siddique. 2004. Supply contracts, profit sharing, switching, and reaction options. *Management Science* **50**(1) 64–82.

Kane, L. 2013. Benchmarks still count: Keys to effective use of contractual measurement clauses. *Boeing* .

Kern, T., L. P. Willcocks, E. van Heck. 2002. The winner's curse in it outsourcing: strategies for avoiding relational trauma. *California Management Review* **44**(2) 47–69.

Kouvelis, P., C. Chambers, H. Wang. 2006. Supply chain management research and production and operations management: Review, trends, and opportunities. *Production and Operations Management* **15**(3) 449–469.

Kouvelis, P., R. Li, Q. Ding. 2013. Managing storable commodity risks: The role of inventory and financial hedge. *Manufacturing & Service Operations Management* **15**(3) 507–521.

Lacity, M. C., L. P. Willcocks. 1998. An empirical investigation of information technology sourcing practices: Lessons from experience. *MIS Quarterly* **22**(3) 363–408.

Leimeister, S. 2010. *IT Outsourcing Governance: Client Types and Their Management Strategies*. Springer.

Li, C. L., P. Kouvelis. 1999. Flexible and risk-sharing supply contracts under price uncertainty. *Management Science* **45**(10) 1378–1398.

Longstaff, F. A., E. S. Schwartz. 2001. Valuing american options by simulation: A simple least-squares approach. *Review of Financial Studies* **14**(1) 113–147.

Lund, D. 2005. How to analyze the investmentuncertainty relationship in real option models? *Review of Financial Economics* **14** 311–322.

Martinez-de Albeniz, V., D. Simchi-Levi. 2005. A portfolio approach to procurement contracts. *Production and Operations Management* **14**(1) 90–114.

Martnez-de Albniz, V., D. Simchi-Levi. 2009. Competition in the supply option market. *Operations Research* **57**(5) 1082–1097.

McDonald, Robert, Daniel Siegel. 1986. The value of waiting to invest. *The Quarterly Journal of Economics* **101**(4) 707–728.

McGrath, R. G. 1997. A real options logic for initiating technology positioning investments. *The Academy of Management Review* **22**(4) 974–996.

Milner, J.M., P. Kouvelis. 2005. Order quantity and timing flexibility in supply chains: The role of demand characteristics. *Management Science* **51**(6) 970–985.

Plambeck, E. L., T. A. Taylor. 2006. Partnership in a dynamic production system with unobservable actions and noncontractible output. *Management Science* **52**(10) 1509.

Seshadri, S. 2005. *Sourcing strategy*. Springer.

Shreve, S. E. 2004. *Stochastic Calculus for Finance: Continuous-time models*. Springer.

Sinha, A. A. 2015. Uncertain times: How can analytics help you combat purchase price variance. URL http://www.thesmartcube.com/insights/blog/blog-details/insights/2015/03/18/uncertain-times-how-can-analytics-help-you-combat-purchase-price-variance.

Taylor, T. A. 2006. Sale timing in a supply chain: When to sell to the retailer. *Manufacturing Service Operations Management* **8**(1) 23.

Van Moerbeke, P. 1979. On optimal stopping and free boundary problems. *Arch. Rational Mech. Anal.* **60** 101–148.

Wu, D. J., P. R. Kleindorfer. 2005. Competitive options, supply contracting, and electronic markets. *Management Science* **51**(3) 452–466.

Wu, O., H. Chen. 2010. Optimal control and equilibrium behavior of production-inventory systems. *Management Science* **56**(8) 1362–1379.

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Appendix A: Proof of Proposition 1

It is optimal for the client to exercise the FP contract as soon as its operating cost $W(t)$ exceeds a threshold value W^* (Shreve 2004, Section 8). Eq. (5) needs to satisfy (i) the boundary condition $V_1(0)=\frac{R}{r}$, because cost process $W(t)$, modeled as a GBM, has an absorbing barrier 0; (ii) the "value matching" condition $V_1(W^*)=\frac{R-C}{r}-K$, since the client receives a net payoff $\frac{R-C}{r}$ at the execution time of FP; and (iii) the "smooth pasting" (or "high contact") condition $\frac{\partial V_1(W^*)}{\partial W}=0$ coming from optimality (Dixit and Pindyck 1994).

While the operation is still in-house at time t , the client is in the continuation region and produces a net cash flow $R-W(t)$ over an infinitesimal time interval dt. As shown in Dixit and Pindyck (1994, Chapter 4), the Bellman equation in the continuation region of $V_1(W(t))$ is given by $[rV_1(W(t))-(R-W(t))]dt=E[dV_1(W(t))]$. Using Ito's Lemma, the Bellman equation becomes

$$\frac{1}{2}\sigma^2W^2(t)V_1^{''}(W(t))+\mu W(t)V_1^{'}(W(t))-rV_1(W(t))+R-W(t)=0. \tag{28}$$

The Bellman equation (28) is a second-order nonhomogeneous differential equation. Hence the solution to Eq. (28) takes the form $V_1(W)=V_1^h(W)+V_1^s(W)$, where $V_1^h(W)$ is the general solution of the corresponding homogeneous equation to Eq. (28), given below,

$$\frac{1}{2}\sigma^2W^2V_1^{''}(W)+\mu WV_1^{'}(W)-rV_1(W)=0, \tag{29}$$

and $V_1^s(W)$ is a special solution of Eq. (28). Now, the general solution to the homogeneous equation (29) takes the form $V_1^h(W)=A_1(W)^{\beta_1}+B_1(W)^{\beta_1'}$, where A_1 and B_1 are the constants to be determined by the boundary conditions, and β_1 and β_1' are the two roots of the characteristic function of Eq. (29), $\frac{1}{2}\sigma^2\beta^2+(\mu-\frac{1}{2}\sigma^2)\beta-r=0$, which can be worked out as

$$\begin{aligned}\beta_1 &= \frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}>\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2}>1,\\ \beta_1' &= \frac{1}{2}-\frac{\mu}{\sigma^2}-\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}<0,\end{aligned} \tag{30}$$

where the first root $\beta_1>1$ is due to our assumption $r>\mu$. To satisfy boundary condition of $V_1(0)=R/r$, we must have $B_1=0$, so the general solution to homogeneous equation (29) takes the form of $V_1^h(W_c)=A_1(W)^{\beta_1}$. It is easily verified that $V_1^s(W)=\frac{C}{r}-\frac{W}{r-\mu}$, the value of keeping the operation in house forever, is a special solution of the Bellman equation (28). Further, from boundary conditions of $V_1(W^*)$ and $\partial V_1(W^*)/\partial W$, we can obtain Eq. (8). Therefore, the solution to the Bellman equation (28) is given by Eq. (9). Finally, from Eqs. (8)–(9), we can verify that $V_1(W)$ is continuous at $W=W^*$, a piecewise twice continuously differentiable function, and a decreasing function of W.

Appendix B: Proof of Table 1

The proof for the results with respect to β_1, $\mathcal{O}_1$ and W^* can be done by checking the first order conditions and is omitted.

(a) We need to show that the first derivative of β_1 with respect to σ is non-positive.

$$\frac{\partial \beta_1}{\partial \sigma} = \frac{2\mu\left[\sqrt{(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}-(\frac{\mu}{\sigma^2}-\frac{1}{2}+\frac{r}{\mu})\right]}{\sigma^3\sqrt{(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}} \tag{31}$$

$$= \frac{2(\mu\beta_1 - r)}{\sigma^3\sqrt{(\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}} < 0. \tag{32}$$

The last inequality holds because, if $\mu = 0$, then the numerator reduces to $-2r < 0$, hence $\frac{\partial \beta_1}{\partial \sigma} < 0$; if $\mu < 0$, since $\beta_1 > 1$, again $\frac{\partial \beta_1}{\partial \sigma} < 0$; if $0 < \mu < r$, hence from Eq. (31), $\frac{\mu}{\sigma^2} - \frac{1}{2} + \frac{r}{\mu} \geq 0$, we obtain $(\frac{\mu}{\sigma^2}-\frac{1}{2}+\frac{r}{\mu})^2 = (\frac{\mu}{\sigma^2}-\frac{1}{2})^2 + \frac{2r}{\sigma^2} + \frac{r}{\mu}(\frac{r}{\mu}-1) > (\frac{\mu}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}$, therefore $\frac{\partial \beta_1}{\partial \sigma} < 0$.

(b) We first exam the derivative of $\mathcal{O}_1 = (\frac{W}{W^*})^{\beta_1}(\frac{W^*}{r-\mu} - \frac{C}{r})$ with respect to β_1.

$$\begin{aligned}
\frac{\partial \mathcal{O}_1}{\partial \beta_1} &= \frac{\partial}{\partial \beta_1}\left\{\frac{W}{r-\mu}\frac{1}{\beta_1}\left(\frac{\beta_1}{\beta_1-1}\frac{C(r-\mu)}{Wr}\right)^{1-\beta_1}\right\} \\
&= \frac{W}{r-\mu}\frac{1}{\beta_1}\left(\frac{\beta_1}{\beta_1-1}\frac{C(r-\mu)}{Wr}\right)^{1-\beta_1}\left(\frac{-1}{\beta_1} - \ln\left(\frac{\beta_1}{\beta_1-1}\frac{C(r-\mu)}{Wr}\right) + \frac{1}{\beta_1}\right) \\
&= \frac{W}{r-\mu}\frac{1}{\beta_1}\left(\frac{W^*}{W}\right)^{1-\beta_1}\left(-\ln\left(\frac{W^*}{W}\right)\right) \leq 0,
\end{aligned}$$

where the last inequality holds since $W < W^* = \frac{\beta_1}{\beta_1 - 1}\left(\frac{P}{r}\right)(r-\mu)$. This further implies that $\frac{\partial \mathcal{O}_1}{\partial \sigma} > 0$. In addition, it is easily seen from Eq. (8) that $\frac{\partial W^*}{\partial \beta_1} < 0$. Together with $\frac{\partial \beta_1}{\partial \sigma} < 0$, we conclude that $\frac{\partial W^*}{\partial \sigma} > 0$. From $E(T_1^*|W \leq W^*) = \frac{\ln(W^*/W)}{\mu - \frac{1}{2}\sigma^2}$, we also see that $\frac{\partial E[T_1|W\leq W^*]}{\partial \sigma} > 0$.

Appendix C: Proof of Proposition 2

The Bellman equation for $v_2(\lambda_2)$ satisfies the second-order ordinary differential equation in the continuation region (Dixit and Pindyck (1994, p210), McDonald and Siegel (1986, A.4)):

$$\frac{1}{2}(\sigma^2 - 2\rho_2\sigma\sigma_2 + \sigma_2^2)\lambda_2^2 v_2''(\lambda_2) + (\mu_2 - \mu)\lambda_2 v_2'(\lambda_2) - (r - \mu_2)v_2(\lambda_2) = 0.$$

Similar to the FP contract, it can be shown that there exists a threshold λ_2^*, such that it is optimal to exercise CP the first time $\lambda_2(t) \geq \lambda_2^*$. Eq. (33) needs to satisfy the "smooth pasting" condition $\frac{\partial v_2(\lambda_2^*)}{\partial \lambda_2} = \frac{1}{r-\mu}$, the boundary condition $v_2(0) = 0$, and the "value matching" condition $v_2(\lambda_2^*) = \frac{\lambda_2^*}{r-\mu} - \frac{1+\alpha}{r-\mu_2}$. Following a similar solution procedure that solves the value function for FP, we can obtain the closed-form solution for $v_2(\lambda_2)$ and the optimal threshold λ_2^*. Then using Eqs. (11) and (12), we convert $v_2(\lambda_2)$ back to $V_2(W, W_2)$, and present it as in the proposition:

Appendix D: Proof of Proposition 3

(a) Fixing σ and σ_2, we have $\frac{\partial\sigma_{\lambda_2}}{\partial\rho_2} = \frac{-2\sigma\sigma_2}{\sigma_{\lambda_2}} < 0$. Furthermore, we need to show $\frac{\partial\beta_2}{\partial\sigma_{\lambda_2}} < 0$. From Eq. (16),

$$\frac{\partial\beta_2}{\partial\sigma_{\lambda_2}} = \frac{2(\mu-\mu_2)\left[\sqrt{(\frac{(\mu-\mu_2)}{\sigma^2_{\lambda_2}}-\frac{1}{2})^2+\frac{2(r-\mu_2)}{\sigma^2_{\lambda_2}}}-(\frac{(\mu-\mu_2)}{\sigma^2_{\lambda_2}}-\frac{1}{2}+\frac{(r-\mu_2)}{(\mu-\mu_2)})\right]}{\sigma^3_{\lambda_2}\sqrt{(\frac{(\mu-\mu_2)}{\sigma^2_{\lambda_2}}-\frac{1}{2})^2+\frac{2(r-\mu_2)}{\sigma^2_{\lambda_2}}}} \tag{33}$$

$$= \frac{2(\mu-\mu_2)\beta_1 - 2(r-\mu_2)}{\sigma^3_{\lambda_2}(\beta_1-\frac{1}{2}+\frac{\mu-\mu_2}{\sigma^2_{\lambda_2}})}. \tag{34}$$

We use Eq. (33) to prove $\frac{\partial\beta_2}{\partial\sigma_{\lambda_2}} < 0$. Because the denominator of Eq. (33) is positive, we need to show that its numerator is non-positive, using the facts that $r > \mu_2$ and $r > \mu$. If $\mu = \mu_2$, then the numerator reduces to $-2(r-\mu_2) < 0$; if $\mu < \mu_2$, then the numerator is negative because $\frac{\mu-\mu_2}{\sigma^2_{\lambda_2}} - \frac{1}{2} + \frac{r-\mu_2}{\mu_{\lambda_2}} < 0$; if $\mu > \mu_2$, we have $\frac{\mu-\mu_2}{\sigma^2_{\lambda_2}} - \frac{1}{2} + \frac{r-\mu_2}{\mu_{\lambda_2}} > 0$ and $(\frac{\mu-\mu_2}{\sigma^2_{\lambda_2}} - \frac{1}{2} + \frac{r-\mu_2}{\mu_{\lambda_2}})^2 = (\frac{\mu-\mu_2}{\sigma^2_{\lambda_2}} - \frac{1}{2})^2 + \frac{2(r-\mu_2)}{\sigma^2_{\lambda_2}} + \frac{r-\mu_2}{\mu_{\lambda_2}}(\frac{r-\mu_2}{\mu_{\lambda_2}} - 1) > (\frac{\mu-\mu_2}{\sigma^2_{\lambda_2}} - \frac{1}{2})^2 + \frac{2(r-\mu_2)}{\sigma^2_{\lambda_2}}$, implying that $\frac{\partial\beta_2}{\partial\sigma_{\lambda_2}} < 0$.

(b)

$$\frac{\partial\mathcal{O}_2}{\partial\beta_2} = \frac{\partial(\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2}W_2^{1-\beta_2})}{\partial\beta_2}$$

$$= \frac{\partial}{\partial\beta_2}\left\{\frac{W}{r-\mu}\frac{1}{\beta_2}\left[\frac{\beta_2}{\beta_2-1}\frac{W_2(r-\mu)(1+\alpha)}{W(r-\mu_2)}\right]^{1-\beta_2}\right\}$$

$$= \frac{W}{r-\mu}\frac{1}{\beta_2}\left(\frac{\lambda_2{}^*}{\lambda_2}\right)^{1-\beta_2}\left(-\ln\frac{\lambda_2{}^*}{\lambda_2}\right) \le 0,$$

where the last inequality holds since $\frac{W}{W_2} = \lambda_2 \le \lambda_2^* = \frac{\beta_2}{\beta_2-1}\frac{r-\mu}{r-\mu_2}(1+\alpha)$. Since, by part (a), $\frac{\partial\beta_2}{\partial\sigma_{\lambda_2}} \le 0$, we have $\frac{\partial\mathcal{O}_2}{\partial\lambda_2} = \frac{\partial\mathcal{O}_2}{\partial\beta_2}\frac{\partial\beta_2}{\partial\sigma_{\lambda_2}} \ge 0$. Next, It can be easily seen from Eq. (15) that $\frac{\partial\lambda_2^*}{\partial\beta_2} < 0$, implying $\frac{\partial\lambda_2^*}{\partial\sigma_{\lambda_2}} = \frac{\partial\lambda_2^*}{\partial\beta_2}\frac{\partial\beta_2}{\partial\lambda_2} \ge 0$.

(c) $\frac{\partial\sigma_{\lambda_2}}{\partial\sigma_2} = \frac{\sigma_2-\rho_2\sigma}{\sigma_{\lambda_2}} \ge 0$ if and only if $\rho_2\sigma \le \sigma_2$, or, equivalently, $\frac{\text{Cov}_2}{\sigma_2^2} \le 1$.

Appendix E: Proof of Theorem 1

(a) We derive the switching curve in the following two cases. We shall use the results that both $V_1(W)$ and $V_2(W, W_2)$ are continuous functions. Consequently, the switching curve $S(W)$, i.e., the indifference curve of $V_1(W)$ and $V_2(W, W_2)$, must also be continuous.

Denote the intersection point of the two threshold lines under the two contracts is denoted by W_2^*, i.e., $W_2^* = W^*/\lambda_2^*$. At this intersection point, the corresponding vendor's cost satisfies

$$W_2^* \equiv W^*/\lambda_2^* = \frac{\beta_1}{\beta_1-1}\frac{\beta_2-1}{\beta_2}\frac{P}{1+\alpha}\frac{r-\mu_2}{r}. \tag{35}$$

Case 1 $\beta_2 \ge \beta_1$: We use the left figure of Figure 1 to facilitate our discussion. From Eqs. (19) and (35), we have $W_2^* > \frac{C}{1+\alpha}\frac{r-\mu_2}{r} = S_{EE}(W^*)$, that is, the indifference curve S_{EE}, which is a constant in its defined region, will not intercept the threshold line for the CP contract (the black dash line in Figure 1 (left) , and will intercept the threshold line for the FP contract (the blue dash dotted line in Figure 1 (left) at a value smaller than W_2^*, the interception value of the two threshold lines, as seen in Figure 1 (left). Due to continuity of the switching curve $S(W)$, S_{EE} must be connected with S_{UE} at $W = W^*$. Because S_{UE} is an increasing concave function, as W decreases, S_{UE} and the linear threshold line $W_2 = \frac{1}{\lambda_2^*}W$ will cross only once when $W = W' = (\frac{\beta_1}{\beta_2})^{1/(\beta_1-1)}W^*$, where W' is the unique solution of $S_{UE}(W) = \frac{1}{\lambda_2^*}W$, as illustrated by Figure 1 (upper left). Further, S_{UE} will meet S_{UU} at $W = W'$. Because S_{UU} is increasing and concave, and $S_{UU}(0) = \left(\frac{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}}{\frac{(W*)^{1-\beta_1}}{\beta_1(r-\mu)}}\right)^{1/(\beta_2-1)} > 0$, we know that $W = W'$ is the unique point at which S_{UU} intercepts the threshold line $\frac{1}{\lambda_2^*}W$, as illustrated in Figure 1 (upper left). In summary, the switching curve $S(W)$ is a piecewise increasing concave function constituting indifference curves S_{UU}, S_{UE} and S_{EE} (the red solid line in Figure 1 (left), given in Eq. (19)). The switching curve confirms the intuition that the client with cost W_C should choose the CP contract if the vendor 2's cost W_2 is lower than a threshold, and this threshold is increasing as W increases.

Case 2 $\beta_1 \le \beta_2$: We use Figure 1 (right) to aid discussions in this case. In this case, from Eqs. (20) and (35), we have $W_V^* < \frac{C}{1+\alpha}\frac{r-\mu_2}{r} = S_{EE}(W^*)$, that is, S_{EE}, which is independent of W, will not intercept the threshold line for the FP contract, but will intercept the threshold line of the CP contract, $\frac{1}{\lambda_2^*}W$, at a value greater than W_2^*. Let $W = W'' = \frac{\beta_2}{\beta_2-1}\frac{\beta_1-1}{\beta_1}W^*$ (from Eq. (15) and (20)) be the point that S_{EE} intercepts $\frac{1}{\lambda_2^*}W$. Then, S_{EE} will meet with S_{EU} at $W = W''$, as seen in Figure 1 (right). Because S_{EU} is decreasing and convex, as W decreases, it will intercept the threshold line of the FP contract at W^*, as illustrated by function S_{EU} in Figure 1 (middle left). At point $W = W^*$, S_{EU} is connected with S_{UU}. Because S_{UU} is decreasing and convex and $S_{UU}(0) = \left(\frac{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}}{\frac{(W*)^{1-\beta_1}}{\beta_1(r-\mu)}}\right)^{1/(\beta_2-1)} > 0$, we know that S_{UU} will not intercept the threshold line $\frac{1}{\lambda_2^*}W$ in its region, again as seen in Figure 1 (right). To summarize, the switching curve $S(W)$ with $\beta_2 < \beta_1$ is a continuous, piecewise decreasing convex function consisting of indifference curves S_{UU}, S_{EU} and S_{EE} given in Eq. (20).

(b) (1) From Eq. (19), we take the first derivative of S_{UU} and obtain

$$\frac{\partial S_{UU}(W)}{\partial W} = \left(\frac{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}}{\frac{(W*)^{1-\beta_1}}{\beta_1(r-\mu)}}\right)^{\frac{1}{\beta_2-1}} W^{\frac{1-\beta_1}{\beta_2-1}} \left(\frac{\beta_2-\beta_1}{\beta_2-1}\right),$$

which is positive if $\beta_2 > \beta_1$, negative if $\beta_2 < \beta_1$, and equal to zero if $\beta_2 = \beta_1$. The second derivative of S_{UU} yields

$$\frac{\partial^2 S_{UU}(W)}{\partial W^2} = -\left(\frac{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}}{\frac{(W*)^{1-\beta_1}}{\beta_1(r-\mu)}}\right)^{\frac{1}{\beta_2-1}} W^{\frac{2-\beta_1-\beta_2}{\beta_2-1}} \left(\frac{\beta_2-\beta_1}{\beta_2-1}\right)\left(\frac{\beta_1-1}{\beta_2-1}\right),$$

which is negative if $\beta_2 > \beta_1$, positive if $\beta_2 < \beta_1$, and equal to zero if $\beta_2 = \beta_1$.

(2) From Eq.(20), we take the first derivative of S_{EU} with respect to W and obtain

$$\frac{\partial S_{EU}(W)}{\partial W} = \frac{1}{1-\beta_2}\left(\frac{\frac{W}{r-\mu}-\frac{C}{r}}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2}}\right)^{\frac{\beta_2}{1-\beta_2}} \frac{W^{\beta_2}-\left(\frac{W}{r-\mu}-\frac{C}{r}\right)\beta_2 W^{\beta_2-1}}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{2\beta_2}}$$
$$= \frac{1}{1-\beta_2}\left(\frac{\frac{W}{r-\mu}-\frac{C}{r}}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2}}\right)^{\frac{\beta_2}{1-\beta_2}} \frac{\left(\frac{(1-\beta_2)W}{r-\mu}+\frac{C\beta_2}{r}\right)}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2+1}}. \tag{36}$$

We know $\frac{1}{1-\beta_2}<0$ since $\beta_2>1$. Since $W\geq W^*$, from Eq. (8), $W\geq \frac{\beta_1}{\beta_1-1}\frac{C}{r}(r-\mu)>\frac{C}{r}(r-\mu)$, thus $\frac{W}{r-\mu}-\frac{C}{r}>0$. Because $\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}>0$, we have $\left(\frac{\frac{W}{r-\mu}-\frac{C}{r}}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2}}\right)>0$. Finally, since $W<\lambda_2^*W$ and $\beta_2>1$, we have

$$\begin{aligned} 0 &\leq W^{1-\beta_2}-(\lambda_2^*W_2)^{1-\beta_2} \\ &= W^{1-\beta_2}-(\lambda_2^*)^{1-\beta_2}\left(\frac{W}{r-\mu}-\frac{C}{r}\right)\left(\frac{\beta_2(r-\mu)}{(\lambda_2^*)^{1-\beta_2}W^{\beta_2}}\right) \\ &= W^{-\beta_2}(r-\mu)\left(\frac{(1-\beta_2)W}{r-\mu}+\frac{C\beta_2}{r}\right), \end{aligned} \tag{37}$$

where the first equality uses the identify $W_2=S_{EU}(W)=\left(\frac{\frac{W}{r-\mu}-\frac{C}{r}}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2}}\right)^{1/(1-\beta_2)}$. Therefore, the last term in Eq.(36) is positive. Thus we conclude $\frac{dS_{EU}(W)}{dW}<0$.

Now, we take the second derivative of $S_{EU}(W)$ and after some simplifications,

$$\frac{\partial^2 S_{EU}(W)}{\partial W^2} = \frac{\beta_2}{(1-\beta_2)^2}\left(\frac{\frac{W}{r-\mu}-\frac{C}{r}}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2}}\right)^{\frac{2\beta_2-1}{1-\beta_2}} \frac{\left(\frac{(1-\beta_2)W}{r-\mu}+\frac{C\beta_2}{r}\right)^2}{\left(\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2+1}\right)^2}$$
$$-\frac{1}{1-\beta_2}\left(\frac{\frac{W}{r-\mu}-\frac{C}{r}}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2}}\right)^{\frac{\beta_2}{1-\beta_2}} \frac{\beta_2}{\frac{(\lambda_2^*)^{1-\beta_2}}{\beta_2(r-\mu)}W^{\beta_2+2}}\left[\frac{(1-\beta_2)W}{r-\mu}+\frac{(\beta_2+1)C}{r}\right].$$

Similar to our argument in the proof of $\frac{dS_{EU}(W)}{dW}>0$, the first expression is positive. The second expression is also positive due to Eq.(37).

(3) First, $\frac{\partial S_{UE}(W)}{\partial W}=\left(\frac{1}{r-\mu}-\beta_1 W^{\beta_1-1}\frac{(W*)^{1-\beta_1}}{\beta_1(r-\mu)}\right)\frac{r-\mu_2}{1+\alpha}>0$, where the inequality comes from Eqs. (8) and (19), and $W<W^*$. Moreover, we can show $\frac{\partial^2 S_{UE}(W)}{\partial W^2}=-\beta_1(\beta_1-1)W^{\beta_1-2}\frac{(W*)^{1-\beta_1}}{\beta_1(r-\mu)}\frac{r-\mu_2}{1+\alpha}<0$.

(4) From Eq. (19), we can shown $\frac{\partial S_{EE}(W_C)}{\partial W_C}=0$.

Appendix F: Proof of Proposition 4

(a) First note that $V_1(W)$, defined in Eq. (9), is independent of W and ρ_2. We write $V_2(W,W_2,\rho_2)$ and $S(W,\rho_2)$ to indicate, explicitly, that each function depends on ρ. By Eq. (18), $S(W,\rho_2)$ is vendor 2's cost W_2 that equalizes the values of the two contracts, i.e., $V_1(W)=V_2(W,S(W,\rho_2),\rho_2)$. By Eq. (15), $V_2(W,W_2,\rho_2)$ is decreasing in W_2. In addition, part (a) of Proposition 3, in conjunction with Eq. (15), implies that $V_2(W,W_2,\rho_2)$ is decreasing in ρ_2. This implies, for $\rho_2^-<\rho_2^+$,

$$V_1(W)=V_2(W,S(W,\rho_2^-),\rho_2^-)\geq V_2(W,S(W,\rho_2^-),\rho_2^+).$$

This further implies $S(W,\rho_2^-) \geq S(W,\rho_2^+)$ since $V_2(W,W_2,\rho_2^+)$ is decreasing in W_2.

(b) We write $V_2(W,W_2,\sigma_2)$ and $S(W,\sigma_2)$ to explicitly indicate that each function depends on σ_2. Again $V_1(W)$ is independent of W_2 and σ_2 and $V_2(W,W_2,\sigma_2)$ is decreasing in W_2. By Eq. (18), $S(W,\sigma_2)$ is the value of W_2 that equalizes the values of the two contracts, i.e., $V_2(W) = V_2(W,S(W,\sigma_2),\sigma_2)$. In addition, parts (a) and (c) of Proposition 3, in conjunction with Eq. (15), imply that $V_2(W,W_2,\sigma_2)$ is decreasing in σ_2 for $\sigma_2 \leq \rho_2\sigma$. Then, for $\sigma_2^- < \sigma_2^+ \leq \rho_2\sigma$,

$$V_1(W) = V_2(W,S(W,\sigma_2^-),\sigma_2^-) \geq V_2(W,S(W,\sigma_2^-),\sigma_2^+).$$

This further implies $S(W,\sigma_2^-) \geq S(W,\sigma_2^+)$, for $\sigma_2^- < \sigma_2^+ \leq \rho_2\sigma$, since $V_2(W,W_2,\sigma_2^-)$ is decreasing in W_2. On the other hand, parts (a) and (c) of Proposition 3 and Eq.(15) imply that $V_2(W,W_2,\sigma_2)$ is increasing in σ_2 for $\sigma_2 \geq \rho_2\sigma$. Therefore, for $\rho_2\sigma \leq \sigma_2^- < \sigma_2^+$,

$$V_1(W) = V_2(W,S(W,\sigma_2^-),\sigma_2^-) \leq V_2(W,S(W,\sigma_2^-),\sigma_2^+).$$

Thus, $S(W,\sigma_2^-) \leq S(W,\sigma_2^+)$, $\rho_2\sigma \leq \sigma_2^- < \sigma_2^+$, since $V_{2(W,W_2,\sigma_2^-)}$ is decreasing in W_2.

(c) This follows from the facts that $\partial V_1(W)/\partial\beta_1 \leq 0$, $\partial V_1(W)/\partial\beta_2 = 0$, $\partial V_2(W,W_2)/\partial\beta_1 = 0$, $\partial V_2(W,W_2)/\partial\beta_2 \leq 0$. Remaining proof is similar as that in the above part (a) and (b).

Appendix G: Proof of Proposition 5

When $T_E = \infty$, we can write $U_1(W,W_1) = E\left[\int_{T_1^*}^{\infty}(C - W_1(t))e^{-rt}dt\right] = E\left[\int_{T_1^*}^{\infty}Ce^{-rt}dt\right] - E\left[\int_{T_1^*}^{\infty}W_1(t))e^{-rt}dt\right] = \frac{C}{r}E\left[e^{-rT_1^*}\right] - W_1 E\left[\int_{T_1^*}^{\infty}e^{(\mu_1-\sigma_1^2/2-r)t+\sigma_1 B_1(t)}dt\right]$.

Now write $B_1(t) = \rho_1 B(t) + \sqrt{1-\rho_1^2}\tilde{B}(t)$, where $\tilde{B}(t)$ is a standard Brownian motion such that $Corr[dB(t),d\tilde{B}(t)] = 0$. Note that $E[e^{\sigma_1 B_1(t)}|T_1^*] = e^{\sigma_1 B_1(T_1^*)+\frac{1}{2}\sigma_1^2(t-T_1^*)}$, where the conditional expectation $E[\cdot|T_1^*]$ is taken with respect conditioning on information available at time T_1^*. Moreover, note that

$$B(T_1^*) = \frac{1}{\sigma}(ln(\frac{W^*}{W}) - (\mu - \frac{1}{2}\sigma^2)T_1^*) \text{ and } E[e^{\sigma_1\sqrt{1-\rho_1^2}\tilde{B}(T_1^*)-\frac{1}{2}\sigma_1^2(1-\rho_1^2)T_1^*}] = 1.$$

Let $\gamma_1 \equiv r - \mu_1 + \frac{1}{2}\rho_1^2\sigma_1^2 + \frac{\sigma_1\rho_1}{\sigma}(\mu - \frac{\sigma^2}{2})$. Then

$$\begin{aligned}
& E\left[\int_{T_1^*}^{\infty} e^{(\mu_1-\sigma_1^2/2-r)t+\sigma_1 B_1(t)}dt\right] = E\left[E\left[\int_{T_1^*}^{\infty} e^{(\mu_1-\frac{1}{2}\sigma_1^2-r)t+\sigma_1 B_1(t)}dt|T_1^*\right]\right] \\
&= E\left[\int_{T_1^*}^{\infty} e^{(\mu_1-\frac{1}{2}\sigma_1^2-r)t}E\left[e^{\sigma_1 B_1(t)}|T_1^*\right]dt\right] = E\left[\int_{T_1^*}^{\infty} e^{(\mu_1-r)t-\frac{1}{2}\sigma_1^2T_1^*+\sigma_1 B_1(T_1^*)}dt\right] \\
&= E\left[\int_{T_1^*}^{\infty} e^{(\mu_1-r)t-\frac{1}{2}\sigma_1^2T_1^*+\sigma_1(\rho_1 B(T_1^*)+\sqrt{1-\rho_1^2}\tilde{B}(T_1^*))}dt\right] \\
&= \frac{1}{r-\mu_1}E\left[e^{-(\mu_1-r)T_1^*-\frac{1}{2}\sigma_1^2T_1^*+\frac{\sigma_1\rho_1}{\sigma}(ln(\frac{W^*}{W})-(\mu-\frac{1}{2}\sigma^2)T_1^*)+\frac{1}{2}\sigma_1^2(1-\rho_1^2)T_1^*}\right] \\
&= \frac{1}{r-\mu_1}E\left[e^{-\gamma_1 T_1^*}\right](\frac{W^*}{W})^{\frac{\sigma_1\rho_1}{\sigma}} \qquad (38)
\end{aligned}$$

For any $y>0$, let $\beta_{1,y} \equiv \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2y}{\sigma^2}}$. Using standard real options theory (see Dixit and Pindyck (1994) and Shreve (2004)), it can be shown

$$E\left[e^{-yT_1^*}\right] = (\frac{W}{W^*})^{\beta_{1,y}}. \tag{39}$$

From Eq. (22), $\tilde{\beta}_1 = \beta_{1,\gamma_1} - \frac{\sigma_1\rho_1}{\sigma}$. From Eqs. (38) and (39), we have

$$U_1(W,W_1) = (\frac{W}{W^*})^{\beta_1}\frac{C}{r} - (\frac{W}{W^*})^{\tilde{\beta}_1}\frac{W_1}{r-\mu_1}.$$

Appendix H: Proof of Proposition 6 and Theorem 2

The proof of part (a) of Theorem 2 is straightforward and hence omitted. The proof of part (a) of Proposition 6 and parts (b) of Theorem 2 are combined since one can be easily derived from another. The proof of part (c) of Proposition 6 and parts (c) of Theorem 2 are also combined. In the following proof, (a)-(d) correspond to proof of parts (a)-(d) of Proposition 6.

(a) From Eq. (22),

$$\frac{\partial\tilde{\beta}_1}{\partial\rho_1} = \frac{1-\sqrt{(\frac{\mu+\rho_1\sigma_1}{\sigma^2}-\frac{1}{2})^2+\frac{2(r-\mu_1)}{\sigma^2}}}{\sigma\sqrt{(\frac{\mu+\rho_1\sigma_1}{\sigma^2}-\frac{1}{2})^2+\frac{2(r-\mu_1)}{\sigma^2}}} < 0, \tag{40}$$

the last inequality satisfies when $\frac{2(r-\mu_1)}{\sigma^2} > 1$. Similarly we can prove that $\frac{\partial\tilde{\beta}_1}{\partial\sigma_1} < 0$.

From Eq. (21), $\frac{\partial U_1(W,W_1)}{\partial\tilde{\beta}_1} \geq 0$, then $\frac{\partial U_1(W,W_1)}{\partial\rho_1} \leq 0$. Similarly we have $\frac{\partial U_1(W,W_1)}{\partial\sigma_1} \leq 0$. Furthermore, $\frac{\partial^2 U_1(W,W_1)}{\partial(\tilde{\beta}_1)^2} \geq 0$.

From Eq. (23), $\frac{\partial P_1(W)}{\partial\tilde{\beta}_1} \geq 0$, then $\frac{\partial P_1(W)}{\partial\rho_1} \leq 0$. Similarly we have $\frac{\partial P_1(W)}{\partial\sigma_1} \leq 0$. Furthermore, $\frac{\partial^2 P_1(W)}{\partial(\tilde{\beta}_1)^2} \geq 0$.

(b) We show the property of $(\frac{W}{W^*})^{\beta_1}$ with respect to β_1, which directly translates to the property of $(\frac{W}{W^*})^{\beta_1}$ with respect to σ, since β_1 is decreasing in σ. The derivative of $(\frac{W}{W^*})^{\beta_1}$ work out as:

$$\frac{\partial(\frac{W}{W^*})^{\beta_1}}{\partial\beta_1} = (\frac{W}{W^*})^{\beta_1}\left(ln(\frac{W}{W^*}) + \frac{1}{\beta_1-1}\right).$$

This expression is positive if and only if $\frac{W}{W^*} \geq e^{\frac{-1}{\beta_1-1}}$.

(c) From Eq. (21), when $W<W^*$, $\frac{\partial U_1(W,W_1)}{\partial\beta_1} = \frac{\partial(\frac{W}{W^*})^{\beta_1}}{\partial\beta_1}\frac{C}{r} - \frac{\partial(\frac{1}{W^*})^{\tilde{\beta}_1}}{\partial\beta_1}\frac{W^{\tilde{\beta}_1}W_1}{r-\mu_1}$. The derivatives of $\frac{\partial(\frac{1}{W^*})^{\tilde{\beta}_1}}{\partial\beta_1}$ and $\frac{\partial(\frac{W}{W^*})^{\beta_1}}{\partial\beta_1}$ work out as:

$$\begin{aligned}
\frac{\partial(\frac{1}{W^*})^{\tilde{\beta}_1}}{\partial\beta_1} &= \tilde{\beta}_1(\frac{1}{W^*})^{\tilde{\beta}_1-1}(\frac{-1}{(w^*)^2})\frac{(-1)}{(\beta_1-1)^2}\left(\frac{C}{r}\right)(r-\mu_1) \\
&= (\frac{1}{W^*})^{\tilde{\beta}_1}\frac{\tilde{\beta}_1}{w^*}\frac{\beta_1}{\beta_1(\beta_1-1)^2}\left(\frac{C}{r}\right)(r-\mu_1) \\
&= (\frac{1}{W^*})^{\tilde{\beta}_1}\frac{\tilde{\beta}_1}{\beta_1(\beta_1-1)} && (41) \\
\frac{\partial(\frac{W}{W^*})^{\beta_1}}{\partial\beta_1} &= (\frac{W}{W^*})^{\beta_1}\left(ln(\frac{W}{W^*}) + \frac{1}{\beta_1-1}\right) && (42)
\end{aligned}$$

After substituting, we obtain

$$\begin{aligned}\frac{\partial U_1(W,W_1)}{\partial \beta_1} &= (\frac{W}{W^*})^{\beta_1}\left(ln(\frac{W}{W^*})+\frac{1}{\beta_1-1}\right)\frac{C}{r}-(\frac{W}{W^*})^{\tilde{\beta}_1}(\frac{\tilde{\beta}_1}{\beta_1(\beta_1-1)})\frac{W_1}{r-\mu_1}\\ &= \frac{(\frac{W}{W^*})^{\tilde{\beta}_1}}{(r-\mu_1)}\left[(\frac{W}{W^*})^{\beta_1-\tilde{\beta}_1}\left(ln(\frac{W}{W^*})+\frac{1}{\beta_1-1}\right)\frac{C(r-\mu_1)}{r}-\frac{\tilde{\beta}_1 W_1}{\beta_1(\beta_1-1)}\right]\\ &= \frac{(\frac{W}{W^*})^{\tilde{\beta}_1}}{(r-\mu_1)}\left[P(W)\left(ln(\frac{W}{W^*})+\frac{1}{\beta_1-1}\right)-\frac{\tilde{\beta}_1 W_1}{\beta_1(\beta_1-1)}\right]\end{aligned} \tag{43}$$

where the last equality uses the relationship $P_1(W)=(\frac{w}{W^*})^{\beta_1-\tilde{\beta}_1}\frac{C(r-\mu_1)}{r}$. If $W\le W^* e^{\frac{-1}{\beta_1-1}}$, both terms in Eq (43) are negative, implying $U_1(W,W_1)$ is decreasing in β_1. If $W\ge W^* e^{\frac{-1}{\beta_1-1}}$, then the first term is positive and the second term is negative. Define $\tilde{P}_1(W)$

$$\tilde{P}_1(W)=\begin{cases}0 & W\le W^* e^{\frac{-1}{\beta_1-1}}\\ \frac{\beta_1(\beta_1-1)}{\tilde{\beta}_1}P_1(W)\left(ln(\frac{W}{W^*})+\frac{1}{\beta_1-1}\right), & W^* e^{\frac{-1}{\beta_1-1}}\le W\le W^*\end{cases} \tag{44}$$

In words, $P_1(W)$ is the value of W_1 so that Eq. (43) equals zero if $W\ge e^{\frac{-1}{\beta_1-1}}$. Note that $\tilde{P}_1(W)\le P_1(W)$. Clearly, U_1(W,W$_1$) is decreasing in β_1 if $W_1\le\tilde{P}_1(W)$ and increasing in β_1 if $W_1\ge\tilde{P}_1(W)$:

$$\frac{\partial U_1(W,W_1)}{\partial\beta_1}=\begin{cases}\le 0 \text{ if } W_1\le\tilde{P}_1(W)\\ \ge 0 \text{ if } W_1\ge\tilde{P}_1(W)\end{cases}$$

In words, function $\tilde{P}_1(W)$ partitions vendor 1's participation region, $W_1\le P_1(W)$, into the β_1-decreasing region, $W_1\le\tilde{P}_1(W)$, and the β_1-increasing region, $\tilde{P}_1(W)\le W_1\le P_1(W)$.

(d) From Eq. (23), when $W<W^*$,

$$\frac{\partial P_1(W)}{\partial\beta_1}=P_1(W)\left(ln(\frac{W}{W^*})+\frac{\beta_1-\tilde{\beta}_1}{\beta_1(\beta_1-1)}\right) \tag{45}$$

If $\beta_1\le\tilde{\beta}$, both items of Eq. (45) are negative, implying that $P_1(W)$ is a decreasing function of β_1. If $\beta_1\ge\tilde{\beta}$, then the first term is positive and the second term is negative. Therefore, $P_1(W)$ is an increasing function of β_1 if $W\le W^* e^{\frac{-(\beta_1-\tilde{\beta}_1)}{\beta_1(\beta_1-1)}}$ and a decreasing function of β_1 if $W\ge W^* e^{\frac{-(\beta_1-\tilde{\beta}_1)}{\beta_1(\beta_1-1)}}$.

(5) From Eq. (23), $\frac{\partial P_1(W)}{\partial W}=\frac{(r-\mu_1)C}{Wr}(\frac{W}{W^*})^{\beta_1-\tilde{\beta}_1}[\beta_1-\tilde{\beta}_1(1-\frac{\xi_1 r}{C}(\frac{W}{W^*})^{-\beta_1})]$, then $P_1(W)$ is increasing in W if $\beta_1\ge\tilde{\beta}_1(1-\frac{\xi_1 r}{C})$, or if $\beta_1\le\tilde{\beta}_1(1-\frac{\xi_1 r}{C})$ and $W\le W^*(\frac{\frac{\xi_1 r}{C}}{1-\frac{\beta_1}{\tilde{\beta}_1}})^{\frac{1}{\beta_1}}$; and decreasing in W otherwise.

Appendix I: Proof of Proposition 7

Now write $B_2(t)=\rho_{\lambda_2,2}B_{\lambda_2}(t)+\sqrt{1-\rho^2_{\lambda_2,2}}\tilde{B}_{\lambda_2}(t)$, where $\rho_{\lambda_2,2}=\frac{\rho_2\sigma-\sigma_2}{\sigma_{\lambda_2}}$, $\tilde{B}_{\lambda_2}(t)$ is a standard Brownian motion such that $Corr[dB_{\lambda_2}(t),d\tilde{B}_{\lambda_2}(t)]=0$. Note that $E[e^{\sigma_2 B_2(t)}|T_2^*]=e^{\sigma_2 B_2(T_2^*)+\frac{1}{2}\sigma_2^2(t-T_2^*)}$, $B_{\lambda_2}(T_2^*)=\frac{1}{\sigma_{\lambda_2}}(ln(\frac{\lambda_2^*}{\lambda_2})-(\mu_{\lambda_2}-\frac{1}{2}\sigma^2_{\lambda_2})T_2^*)$, $E[e^{\sigma_{\lambda_2}\sqrt{1-\rho^2_{\lambda_2,2}}\tilde{B}(T_2^*)-\frac{1}{2}\sigma^2_{\lambda_2}(1-\rho^2_{\lambda_2,2})T_2^*}]=1$, then

$$U_2(W,W_2)=E\left[\int_{T_2^*}^{\infty}\alpha W_2(t)e^{-rt}dt\right]=\alpha W_2 E\left[\int_{T_2^*}^{\infty}e^{(\mu_2-\sigma_2^2/2-r)t+\sigma_2 B_2(t)}dt\right]$$

$$=\alpha W_2 E\left[E\left[\int_{T_2^*}^{\infty}e^{(\mu_2-\frac{1}{2}\sigma_2^2-r)t+\sigma_2 B_2(t)}dt|T_2^*\right]\right]=\alpha W_2 E\left[\int_{T_2^*}^{\infty}e^{(\mu_2-\frac{1}{2}\sigma_2^2-r)t}E\left[e^{\sigma_2 B_2(t)}dt|T_2^*\right]\right]$$

$$= \alpha W_2 E\left[\int_{T_2^*}^{\infty} e^{(\mu_2-r)t-\frac{1}{2}\sigma_2^2 T_2^*+\sigma_2 B_2(T_2^*)}dt\right] = \frac{\alpha W_2}{r-\mu_2}E\left[e^{(\mu_2-r)T_2^*-\frac{1}{2}\sigma_2^2 T_2^*+\sigma_2 B_2(T_2^*)}\right]$$
$$= \frac{\alpha W_2}{r-\mu_2}E\left[e^{(\mu_2-r)T_2^*-\frac{1}{2}\sigma_2^2 T_2^*+\sigma_2(\rho_{\lambda_2,2}B_{\lambda_2}(T_2^*)+\sqrt{1-\rho_{\lambda_2,2}^2}\tilde{B}_{\lambda_2}(T_2^*))}\right]$$
$$= \alpha W_2 E\left[e^{(\mu_2-r)T_2^*-(\frac{1}{2}\rho_{\lambda_2,2}^2\sigma_2^2+\frac{\sigma_2\rho_{\lambda_2,2}}{\sigma_{\lambda_2}}(\mu_\lambda-\frac{1}{2}\sigma_{\lambda_2}^2))T_2^*+\frac{\sigma_2\rho_{\lambda_2,2}}{\sigma_{\lambda_2}}ln(\frac{\lambda_2^*}{\lambda_2})}\right] = \frac{\alpha W_2}{r-\mu_2}(\frac{\lambda_2^*}{\lambda_2})^{\frac{\sigma_2\rho_{\lambda_2,2}}{\sigma_{\lambda_2}}}E\left[e^{-\gamma_2 T_2^*}\right] \quad (46)$$

where $\gamma_2 \equiv r-\mu_2+\frac{1}{2}\rho_{\lambda_2,2}^2\sigma_2^2+\frac{\sigma_2\rho_{\lambda_2,2}}{\sigma_{\lambda_2}}(\mu_\lambda-\frac{1}{2}\sigma_{\lambda_2}^2)$.

For any $y>0$, and $\beta_{2,\gamma_2} \equiv \frac{1}{2}-\frac{\mu_\lambda}{\sigma_\lambda^2}+\sqrt{\left(\frac{\mu_\lambda}{\sigma_\lambda^2}-\frac{1}{2}\right)^2+\frac{2y}{\sigma^2}}$, it can be shown (see Dixit and Pindyck (1994) and Shreve (2004))

$$E\left[e^{-yT_2^*}\right] = (\frac{\lambda_2}{\lambda_2^*})^{\beta_{2,y}}. \quad (47)$$

From Eqs. (46) and (47), and note that $\beta_{2,\gamma_2}-\frac{\sigma_2\rho_{\lambda_2,2}}{\sigma_{\lambda_2}}=\beta_2$, we have

$$U_2(W,W_2) = \frac{\alpha W_2}{r-\mu_2}(\frac{W}{W_2\lambda_2^*})^{\beta_{2,\gamma_2}-\frac{\sigma_2\rho_{\lambda_2,2}}{\sigma_{\lambda_2}}} = \frac{\alpha W_2}{r-\mu_2}(\frac{W}{W_2\lambda_2^*})^{\beta_2}.$$

As a remark, note that an alternative derivation of U_2 can follow from $E\left[W_2(T_2^*)e^{-rT_2^*}\right] = W_2(\frac{W}{W_2\lambda_2^*})^{\beta_2}$, which can be derived similarly as $\tilde{V}_2$ in Section 4.2.

From Eq. (25), when $\lambda_2<\lambda_2^*$, we take the derivative of U_2 with respect to β_2 and obtain

$$\frac{\partial U_2(W,W_2)}{\partial\beta_2} = \frac{\alpha W_2}{r-\mu_2}(\frac{W}{W_2\lambda_2^*})^{\beta_2}(ln(\frac{W}{W_2\lambda_2^*})+\frac{1}{\beta_2-1})\begin{cases}>0 & \text{if } \lambda_2>\lambda_2^* e^{\frac{-1}{\beta_2-1}}\\ \leq 0 & \text{if } \lambda_2\leq\lambda_2^* e^{\frac{-1}{\beta_2-1}}\end{cases}$$

From the other first derivatives of U_2, the remaining proof is straightforward.

Appendix J: Proof of Part (c) of Theorem 3

When $\lambda_2\geq\lambda_2^*$, $\frac{\partial P_2(W)}{\partial\beta_2}=0$. When $\lambda_2<\lambda_2^*$,

$$\begin{aligned}\frac{\partial P_2(W)}{\partial\beta_2} &= P_2(W)[\frac{-1}{(\beta_2-1)^2}ln(\frac{W}{\lambda_2^*})+\frac{-1}{(\beta_2-1)^2}+\frac{-1}{(\beta_2-1)^2}ln(\frac{\alpha}{(r-\mu_2)\xi_2}))]\\ &= P_2(W)\frac{-1}{(\beta_2-1)^2}[(ln(\frac{W}{\lambda_2^*}\frac{\alpha}{(r-\mu_2)\xi_2})-1)]\\ &\begin{cases}>0 & \text{if } \lambda_2^*\frac{(r-\mu_2)\xi_2}{\alpha}e\leq W\leq\lambda_2^* W_2\\ \leq 0 & \text{if } \lambda_2^*\frac{(r-\mu_2)\xi_2}{\alpha}\leq W\leq\lambda_2^*\frac{(r-\mu_2)\xi_2}{\alpha}e \text{ and } W\leq\lambda_2^* W_2\end{cases}\end{aligned}$$