quad2

Error formula for TRAPEZOID RULE $Q_{NC(2)}$

$$\boxed{\left| \int_a^b f(x)\,dx - Q_{NC(2)} \right| \leq \frac{1}{12}(b-a)^3 M_2}$$

where $|f''(x)| \leq M_2$ for all $x \in [a,b]$

Example $\int_0^{1/10} \cos(x^2)\,dx \simeq \frac{1}{20}\left[1 + \cos\left(\frac{1}{100}\right)\right]$

$\simeq 0.0999975000021$

How good is approximation? $\uparrow$ 0.10000

Ex (continued)

$$f(x) = \cos(x^2)$$

$$f'(x) = -2x\,\mathrm{SIN}(x^2)$$

$$f''(x) = -2\,\mathrm{SIN}(x^2) - 4x^2\cos(x^2)$$

So on any $[a,b]$ for $0 = a < b$ we see

$$\begin{aligned} |f''(x)| &= |2\,\mathrm{SIN}(x^2) + 4x^2\cos(x^2)| \\ &\leq 2|\mathrm{SIN}(x^2)| + 4x^2|\cos(x^2)| \\ &\leq 2 + 4b^2 \end{aligned}$$

Therefore,

$$\left| \int_0^{1/10} \cos(x^2)\,dx - \varphi_{NC(2)} \right| \le \frac{1}{12}(b-a)^3 M_2$$

$$= \frac{1}{12}\,\frac{1}{10^3}\left(2+\frac{4}{10^2}\right)$$

$$\simeq 0.00017$$

So we can say value of integral is

$$I = 0.10000 \pm 0.00017$$

Proof of error formula

By definition $Q_{NC(2)} = \int_a^b p_1(x)\,dx$ where

$$p_1(x) = f(a) + f[a,b](x-a)$$

By the error formula for polynomial interpolation

$$f(x) - p_1(x) = \frac{1}{2} f''(\eta_x)(x-a)(x-b)$$

where $\eta_x \in [a,b]$, we see

$$\int_a^b f(x)\,dx - Q_{NC(2)} = \frac{1}{2}\int_a^b f''(\eta_x)(x-a)(x-b)\,dx$$

Then,

$$\left| \int_a^b f(x)\,dx - Q_{NC(2)} \right| = \left| \frac{1}{2} \int_a^b f''(\eta_x)(x-a)(x-b)\,dx \right|$$

$$\leq \frac{1}{2} \int_a^b |f''(\eta_x)(x-a)(x-b)|\,dx$$

$$\leq \frac{1}{2} M_2 \underbrace{\int_a^b (x-a)(b-x)\,dx}_{\frac{1}{6}(b-a)^3} \quad (\blacksquare)$$

Proof in PDF notes quad2 more involved since error formula for poly interpolation is rederived from scratch.

Simpson $\underbrace{\int_a^b f(x)\,dx \simeq \frac{(b-a)}{6}\left[f(a)+4f(c)+f(b)\right]}$

$$\underbrace{\frac{(b-a)}{6}\left[f(a)+4f(c)+f(b)\right]}_{Q_{NC(3)}}$$

$$\left| \int_a^b f(x)\,dx - Q_{NC(3)} \right| \le \frac{(b-a)^5}{2880} M_4$$

very good

general formula

$$\left| \int_a^b f(x)\,dx - Q_{NC}(m) \right| \le |c_m| M_{d+1} \left(\frac{b-a}{m-1} \right)^{d+2}$$

$$\text{for } d = \begin{cases} m-1 & m \text{ even} \\ m & m \text{ odd} \end{cases}$$

$$c_2 = -\frac{1}{12}, \quad c_3 = -\frac{1}{90}, \quad c_4 = -\frac{3}{80}$$

$$c_5 = -\frac{8}{945}, \quad c_6 = -\frac{275}{12096}$$

$m = 3$ (odd)

$$\frac{1}{90} M_4 \left(\frac{b-a}{3-1} \right)^5 = \frac{M_4 (b-a)^5}{\underbrace{2^5 \cdot 90}_{2880}}$$