Lecture `interp1`: Introduction to Interpolation

October 10, 2022

Summary: Deals with general motivation for interpolation and basics of well-posedness. Stresses the role of the Vandermonde matrix. Some aspects taken from C. F. Van Loan's *Introduction to Scientific Computing*.

References: See Section 8.1, page 181.

Problem Statement

Suppose we have a data set D_N consisting of ordered pairs $\{(x_i, y_i)\}_{i=1}^N$ for $N > 0$. We assume the data represents samples from a function $y = f(x)$. That is, $y_i = f(x_i)$ for $i = 1, \cdots, N$. *Interpolation*[1] may be used to approximate $f(x)$ at a point x which is *not* one of the x_i. For example, suppose $N = 3$ and $(x_1, y_1) = (3, 2)$, $(x_2, y_2) = (0, 1)$, and $(x_3, y_3) = (2, 4)$. If these data points represent samples from $(x, f(x))$, how might we approximate $f(1)$? The idea is to use the data to construct an *explicit* approximating function $p(x)$ (we use the stem letter p here, since mostly our approximating functions will be polynomials) which likewise satisfies $y_i = p(x_i)$ for $i = 1, \cdots, N$. Then use $p(1)$ in place of $f(1)$. Indeed, we might not know $f(1)$ or it might be expensive to compute.

Example 1 (Linear interpolation)
We are given the two-point data set $(0, 2)$ and $(2, 4)$, and might assume that

$$p(x) = x + 2,$$

which interpolates both $(0, 2)$ and $(2, 4)$. In this case, we would approximate $f(1)$ with $p(1) = 1 + 2 = 3$.

Some theory on Interpolation

Before delving into interpolation, we first recall some linear algebra.

Definition 1 (Linear Independence for vectors)
Let $\{\boldsymbol{v}_i\}_{i=1}^N$ be a collection of vectors. We say that the collection is linearly independent if the only solution $\{c_i\}_{i=1}^N$ to

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_N\boldsymbol{v}_N = \mathbf{0}$$

[1] A denotation of the word is as follows. (Math.) The method or operation of finding from a few given terms of a series, as of numbers or observations, other intermediate terms in conformity with the law of the series. [1913 Webster]

is $c_1 = c_2 = \cdots = c_N = 0$*, that is* $\boldsymbol{c} = \boldsymbol{0}$.

Loosely put, the vectors can not be "canceled against each other".

Definition 2 (Linear independence for functions)
A collection of continuous functions $\{\phi_i\}_{i=1}^N$ *defined on an interval* $I \subset \mathbb{R}$ *is linearly independent if the only solution* $\{c_i\}_{i=1}^N$ *to*

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_N\phi_N(x) = 0 \quad [\textit{equals the zero function on } I]$$

for all x *on* I *is* $c_1 = c_2 = \cdots = c_N = 0$*, that is* $\boldsymbol{c} = \boldsymbol{0}$.

Definition 3 (Interpolant and basis)
Consider the data set $D_N = \{(x_i, y_i)\}_{i=1}^N$*. Choose an interval* $I \subset \mathbb{R}$ *such that* $x_i \in I$ *for all* i*. As a basis for interpolation of the data set* D_N*, choose a collection of* N *linearly independent functions* $\mathcal{B}_N = \{\phi_i\}_{i=1}^N$*. Furthermore, suppose there exist constants* $c_1, c_2, \ldots, c_N$ *such that*

$$c_1\phi_1(x_i) + c_2\phi_2(x_i) + \cdots + c_N\phi_N(x_i) = y_i$$

for all $i = 1, 2, \ldots, N$*. Then we say that* $p(x) = \sum_{i=1}^N c_i\phi_i(x)$ *is an interpolant of the data set* D_N *with respect to the basis* $\mathcal{B}_N$.

We have assumed continuity for the functions ϕ_i in the definitions above (and typically the ϕ_i will be differentiable to some order). This is not necessary, but is simpler. When dealing with a *basis* in linear algebra, one typically introduces the notion of the *span* of the basis, but we will not discuss this notion here. We now have a mathematical definition for the interpolation problem: given D_N and $\mathcal{B}_N$, find the expansion coefficients $\boldsymbol{c}$. But, how do we know if the interpolation problem is "well-posed". That is, what if such interpolants, as defined, do not exist? Or perhaps many such interpolants exist. The following theorem addresses these issues.

Theorem 1 (Interpolant existence and uniqueness)**.** *For a positive integer* N*, let a data set* $D_N = \{(x_i, y_i)\}_{i=1}^N$ *and a basis* $\mathcal{B}_N = \{\phi_i\}_{i=1}^N$ *be given, and assume* $x_i \neq x_j$ *for all* $i \neq j$ *(the sample points are distinct). In addition, define the vectors* $\boldsymbol{v}_i$ *by*

$$\boldsymbol{v}_i = \begin{pmatrix} \phi_i(x_1) \\ \phi_i(x_2) \\ \vdots \\ \phi_i(x_N) \end{pmatrix}.$$

If the collection $\{\boldsymbol{v}_i\}_{i=1}^N$ *is linearly independent, then the interpolant* f *of the data set* D_N *with respect to the basis* $\mathcal{B}_N$ *exists and is unique.*

Proof. We introduce the *Vandermonde matrix* (which depends only on $\mathcal{B}_N$ and the x_i).

$$V = \begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) & \phi_3(x_1) & \cdots & \phi_N(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \phi_3(x_2) & \cdots & \phi_N(x_2) \\ \vdots & & \vdots & & \vdots \\ \phi_1(x_N) & \phi_2(x_N) & \phi_3(x_N) & \cdots & \phi_N(x_N) \end{pmatrix} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n). \tag{1}$$

The interpolation conditions $y_i = \sum_{k=1}^{N} \phi_k(x_i)$ for $i = 1, \cdots, N$ can then be written as

$$V\boldsymbol{c} = \boldsymbol{y}, \tag{2}$$

where $\boldsymbol{c} = (c_1, \ldots, c_N)^T$ and $\boldsymbol{y} = (y_1, \ldots, y_N)^T$. By assumption, the columns of the Vandermonde matrix are linearly independent, whence V is nonsingular. Therefore, a unique inverse matrix V^{-1} exists, and the unique solution to the interpolation problem is $\boldsymbol{c} = V^{-1}\boldsymbol{y}$. □

The theorem ensures that we can set about finding interpolants with confidence that we are not attempting something unattainable. Note that the theorem makes no mention of the interpolation values y_i. That is, the issues of existence and uniqueness only depend on the nodal values x_i and the basis functions ϕ_i. So long as one picks a collection of functions which is a basis, then interpolation is likely possible. Even if the functions ϕ_i are linearly independent, it could be possible that the choice of nodes x_i gives rise to linearly dependent vectors $\boldsymbol{v}_i$. For example, the collection $\mathcal{B}_2 = \{1, x^2\}$ is linearly independent on $I = (-2, 2)$, since for $c_1 + c_2x^2$ to be the zero function on I we must have $c_1 = 0 = c_2$ (test a few points). For this basis, if the nodes are $x_1 = -1, x_2 = 1$, then the associated Vandermonde matrix (the 2-by-2 matrix of all 1's) is singular. This pathological situation will not arise in practice.

Example 2

Let $\mathcal{B}_4 = \left\{x^{i-1}\right\}_{i=1}^{4}$ and let $D_4 = \{(0,1); (1,4); (2,1); (3,1)\}$ be the basis and data sets. We are using monomials for our basis, and our interpolant has the form

$$p(x) = \sum_{i=1}^{4} c_i x^{i-1} = c_1 + c_2x + c_3x^2 + c_4x^3 \tag{3}$$

Obviously if we can find the constants c_i, then we have determined the interpolant. We enforce the conditions

$$p(0) = 1, \qquad p(1) = 4, \qquad p(2) = 1, \qquad p(3) = 1. \tag{4}$$

Thus, we have four unknowns, the c_i, and four conditions in equation (4). Let us write down the four conditions in (4) by expanding the interpolant using (3).

$$p(0) = c_1 + 0c_2 + 0c_3 + 0c_4 = 1 \tag{5}$$
$$p(1) = c_1 + 1c_2 + 1c_3 + 1c_4 = 4 \tag{6}$$
$$p(2) = c_1 + 2c_2 + 4c_3 + 8c_4 = 1 \tag{7}$$
$$p(3) = c_1 + 3c_2 + 9c_3 + 27c_4 = 1 \tag{8}$$

We now observe that our four conditions form precisely the 4-by-4 linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \boldsymbol{c} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 1 \end{pmatrix} \tag{9}$$

We have thus reduced this interpolation problem to that of solving a linear system. Inverting this system (by hand, or with Matlab), we find that

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ -\frac{15}{2} \\ \frac{3}{2} \end{pmatrix}$$

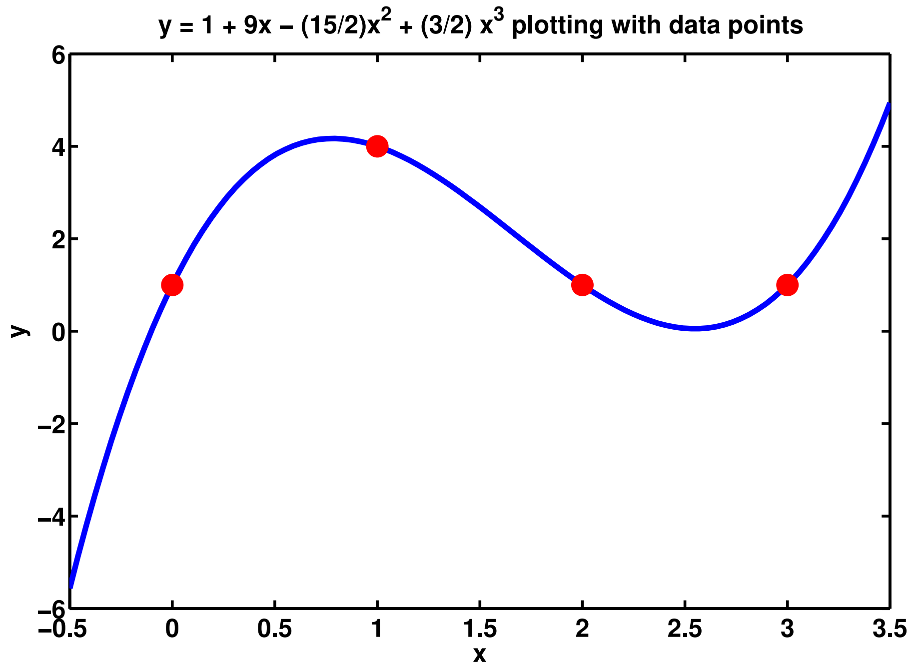

Figure 1: Plot of the interpolant in example 2

To see the results, we can use the following MATLAB code which produced Figure 1.

```
>> a = [1 0 0 0; 1 1 1 1; 1 2 4 8; 1 3 9 27];
>> data = [0 1; 1 4; 2 1; 3 1];
>> b = data(:,2);
>> c = inv(a)*b;
>> x = linspace(-0.5, 3.5, 100);
>> p = c(1) + c(2)*x + c(3)*x.^2 + c(4)*x.^3;
>> plot(x,p,'b', data(:,1), data(:,2), 'r.')
```

Example 3 (Polynomial interpolation)
The above example is a special case of polynomial interpolation. In general, let $\phi_i = x^{i-1}$ for every $i = 1, 2, \ldots, N$ on any finite interval of $\mathbb{R}$. Then $\mathcal{B}_N = \{\phi_i\}_{i=1}^N$ is a basis. The polynomials (monomials here) are a very important basis set. In future lectures, we shall survey a few methods for polynomial interpolation.

Example 4 (Trigonometric interpolation)
Let $x \in [0, 2\pi]$, and define the ϕ_i in the following way.

$$\begin{aligned}
\phi_1(x) &= 1 \\
\phi_2(x) &= \sin(x) \\
\phi_3(x) &= \cos(x) \\
\phi_4(x) &= \sin(2x) \\
\phi_5(x) &= \cos(2x) \\
\phi_6(x) &= \sin(3x) \\
&\vdots
\end{aligned}$$

These are periodic functions, and they are linearly independent. The interpolation using this basis and a particular set of nodal values x_i (not specified here) constitutes the Discrete Fourier Transform, a ubiquitously employed method in science, engineering, and mathematics.

Example 5 (Crazy basis)
Let $x \in [1, 5]$, and define the ϕ_i in the following way.

$$\begin{aligned}
\phi_1(x) &= 1 \\
\phi_2(x) &= \ln(x) \\
\phi_3(x) &= x\ln(x) \\
\phi_4(x) &= x\ln^2(x) \\
\phi_5(x) &= x^2\ln^2(x) \\
\phi_6(x) &= x\ln^3(x) \\
&\vdots
\end{aligned}$$

This is not a particularly important collection of functions. However, they are linearly independent and thus do form a basis with which we can interpolate data points. The point of this example is not that this is a good basis to use; rather that one can dream up all sorts of function collections to use for interpolation.