Solving nonlinear equations: Bisection $\boxed{\text{root 1}}$

"root-finding"

$$\boxed{f(x) = 0}$$

consider first scalar case, both $x, f(x) \in \mathbb{R}$.

this is a "provisional equality", a <u>wish</u>

Maybe no solution, e.g. $x^2 + 1 = 0$

Assume f <u>nonlinear</u> in general.

for linear equations, we have complete "existence/uniqueness theory"

$ax = b$

① $a \neq 0$, $\exists!$ solution $x = b/a$

$\underline{f(x) = ax - b}$

② $a = 0$ $\quad$ (i) $b \neq 0$ $\quad$ $0 \cdot x = 7$

no solution

(ii) $b = 0$ $\quad$ $0 \cdot x = 0$

∞-many solutions

only possibilities

$\exists!$ solution, no solution, uncountably ∞-many for general $f(x) = 0$, we don't know what to expect

Ex $\arctan x = 0$ $\exists$? solution

$\sin(x) = 0$ countably ∞-many solutions

$x^2 + 1 = 0$ no solution

$x(x+1)(x-1) = 0$ precisely 3 solutions

Consider equation $f(x) = 0$ (wish). Any $x = r$ s.T. $f(r) = 0$ is called a <u>root</u>.

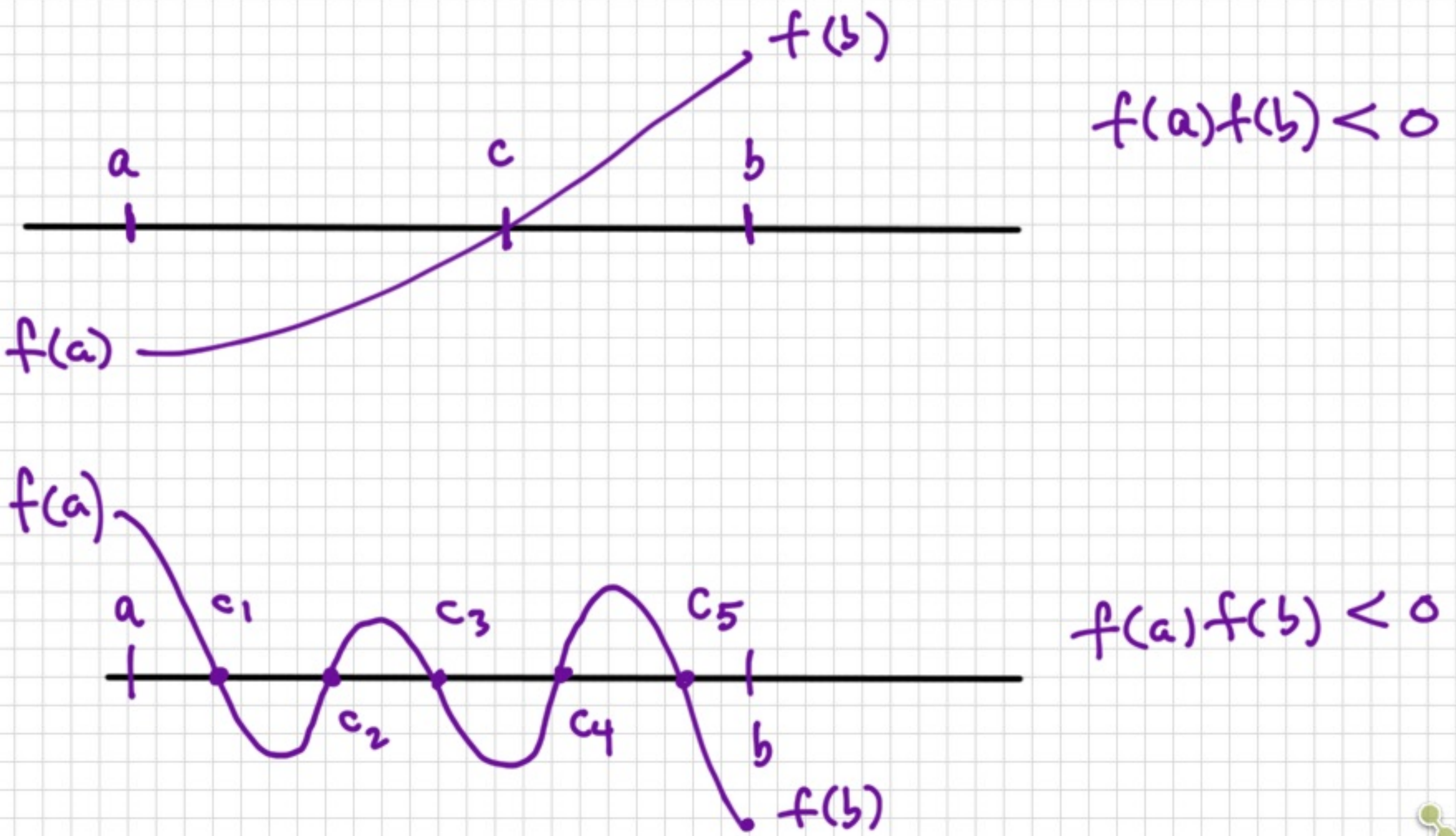
Subject at hand: how to find roots.

First method: <u>Bisection</u>

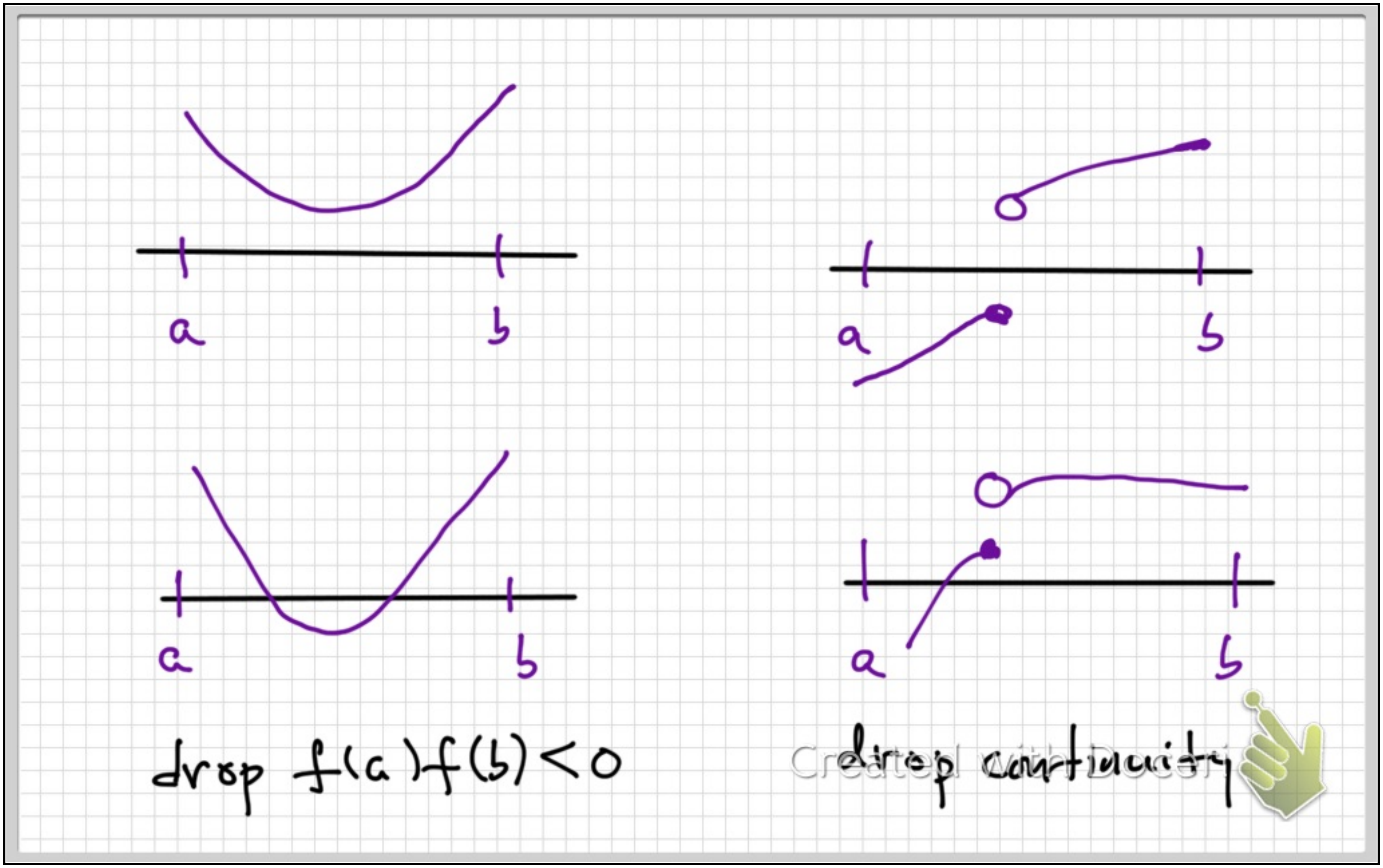
More precise setting

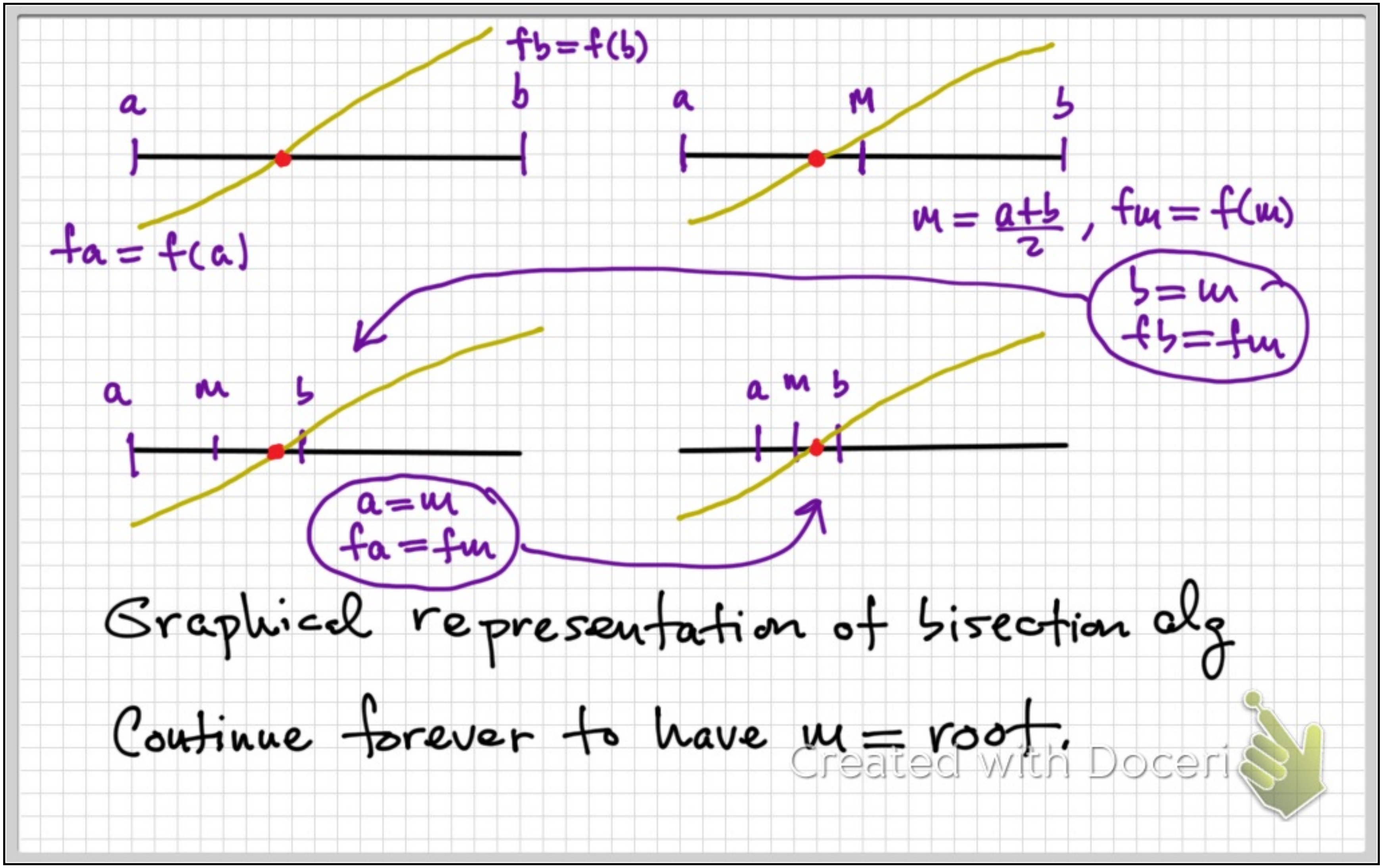
assume $f:[a,b] \to \mathbb{R}$ continuous and $f(a)f(b) < 0$.

<u>Intermediate Value THM</u>: If $f(x)$ is continuous on $[a,b]$ and $f(a)f(b) < 0$, then $\exists$ at least one $c \in (a,b)$ s.T. $f(c) = 0$.



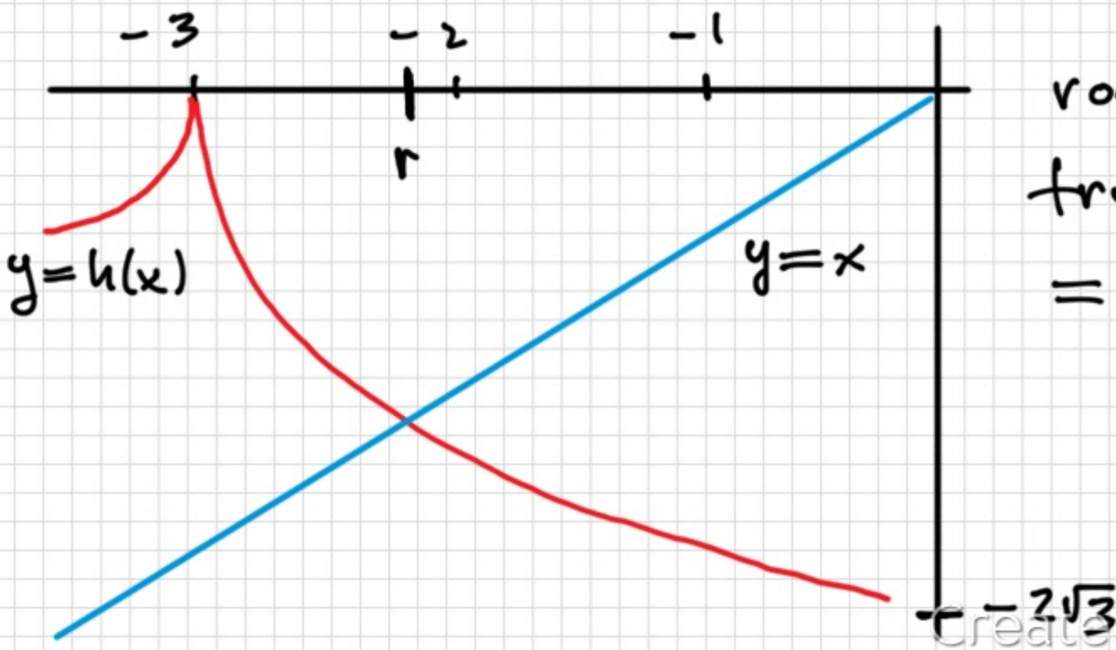
IVT guarantees one root; there may be more


a
b
a
b
a
b
a
b
drop f(a)f(b)<0
drop continuity


fb = f(b)
a
b
fa = f(a)
a
M
b
m = (a+b)/2 , fm = f(m)
b = m
fb = fm
a
m
b
a
m
b
a = m
fa = fm
Graphical representation of bisection alg
Continue forever to have m = root.
Created with Doceri

```
[m, k] = bisection(f, a, b, tol, kmax)        stopping criteria
k = 0, fa = f(a), fb = f(b)
m = 1/2 (a+b), err = 1/2 (b-a)
while (err >= tol)
    fm = f(m), k = k+1
    if fm = 0 , return % got lucky
    if fa * fm < 0
        b = m, fb = fm
    else
        a = m, fa = fm
    end
    m = 1/2 (a+b), err = 1/2 (b-a)
    if k >= kmax , return % for safety
end
```

<u>Ex</u> $f(x) = x - \underbrace{h(x)}_{\frac{1}{2}\sin x - 2\sqrt{|x+3|}}$



root near -2.2

try $[a,b]$ $= [-5, 5]$ $k = 29$

$[-50, 50]$ $k = 33$

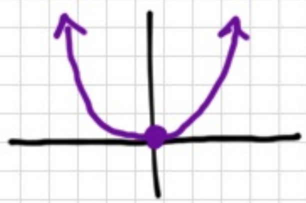
$[-291.02, 632.333]$ $k = 36$

all w/ tol $= 10^{-8}$

Ex $f(x) = x^2$, root $r = 0$

but $f(a)f(b) < 0$ can't be satisfied.

Bisection not applicable



Ex $f_N(x) = \cos(Nx + 0.1)$

look for roots in $x \in [0, \pi]$: $r_i = \frac{(2i-1)\pi}{2N} - \frac{1}{10N}$

$i = 1, 2, \dots, N$

↑ N roots

$f(0) = \cos(0.1)$

$f(\pi) = \cos(N\pi + 0.1) = \cos(N\pi)\cos(0.1)$

$f(0)f(\pi) = \cos(N\pi) = (-1)^N$