Lecture (supplemental): Introduction to Linear Systems

August 22, 2022

Summary: Introduction to systems of linear equations, focusing on the 2–by–2 case. Assumes familiarity with vectors, matrices, and matrix multiplication.

Matrix form of 2–by–2 linear systems

Two algebraic equations linear in two unknowns $\{x_1, x_2\}$ have the form:

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad a_{21}x_1 + a_{22}x_2 = b_2, \tag{1}$$

here in terms of real numbers (although we could also consider systems involving complex numbers). For example,

$$0.3372x_1 + 900.732x_2 = 32.09 \qquad 88.0x_1 + 2.1x_2 = 21.45. \tag{2}$$

The numbers a_{11}, a_{12}, a_{21}, a_{22}, b_1, and b_2 need not be simple; however, to make the algebra easy in order to streamline the lectures, we mostly choose integers (the easiest numbers) to work with. Another example,

$$3x_1 + 2x_2 = 1 \qquad 5x_1 + 4x_2 = 10. \tag{3}$$

This is an example of a *square system*, that is, it has the same number of equations as unknowns. For a square system, the counting is right to hope for a unique solution.

Expressed as a matrix system, (1) has the form

$$Ax = b \qquad \Longleftrightarrow \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{4}$$

Note that Van Loan does not use boldface to denote matrices and/or vectors. It's also common to write, say, $A\mathbf{x} = \mathbf{b}$ or $\mathbf{Ax} = \mathbf{b}$ for such a system of equations. The concrete system (3) has matrix form

$$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}. \tag{5}$$

Formal solution

For a 2–by–2 matrix A, we define $\det A$, the **determinant** of A, as follows:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Longrightarrow \det A = a_{11}a_{22} - a_{12}a_{21}. \tag{6}$$

For example, the matrix in (5) has determinant $\det A = 3 \cdot 4 - 2 \cdot 5 = 2$. Provided $\det A \neq 0$, we may construct the inverse A^{-1} of a 2–by–2 matrix A via the explicit formula

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Longrightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{7}$$

Check that $AA^{-1} = I = A^{-1}A$, where I is the 2–by–2 identity matrix.

A^{-1} is analogous to the reciprocal $1/a$ of a non–zero scalar number a. A scalar number a has a reciprocal (multiplicative inverse) $1/a$, provided $a \neq 0$. Likewise, a square matrix A has an inverse A^{-1}, provided $\det A \neq 0$ (as we will see, this statement is true beyond the 2–by–2 case). If $\det A \neq 0$, we may characterize the solution to a square system of linear equations as follows:

$$Ax = b,\ \det A \neq 0 \qquad \Longleftrightarrow \qquad x = A^{-1}b \tag{8}$$

For example, the inverse of the matrix appearing in (5) is

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}, \qquad \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{9}$$

Therefore, we find

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 10 \end{bmatrix} = \begin{bmatrix} -8 \\ \frac{25}{2} \end{bmatrix}. \tag{10}$$

as the unique solution to the system (5). **Lesson:** if $\det A \neq 0$, then *there exists a unique solution* $x = A^{-1}b$ to $Ax = b$.

What can go wrong?

If $\det A = 0$, the following can go wrong.

- Loss of existence, there is no solution. Consider, for example,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{11}$$

  Can't simultaneously have $x_1 + x_2 = 1$ and $x_1 + x_2 = 2$.

- Loss of uniqueness, there is more than one solution (in fact, there are infinitely many solutions). Consider, for example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{12}$$

  Only one independent equation here. Namely, $x_1 + x_2 = 1$. Therefore, can't uniquely determine two unknowns, and the solution

$$x = \begin{bmatrix} 1 - \alpha \\ \alpha \end{bmatrix} \tag{13}$$

  has a free parameter α (infinitely many solutions).

We have the following summary for a square linear system (today for 2–by–2, but summary also relevant for n–by–n case).

Following are equivalent	**Following are equivalent**
1. A is invertible (nonsingular).	1.′ A is non–invertible (singular).
2. $\det A \neq 0$	2.′ $\det A = 0$.
3. $Ax = b$ has a unique solution for all b.	3.′ Either $Ax = b$ has no solution (loss of existence, occurs when b not in range of A), or has ∞–many solutions (loss of uniqueness).
4. $x = 0$ (zero vector) is the only solution to $Ax = 0$.	4.′ There exists at least one nontrivial solution $x \neq 0$ to $Ax = 0$.
5. Columns (rows) of A are linearly independent.	5.′ Columns (rows) of A are linearly dependent.

The last statements follow since

$$Ac = c_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + c_2 \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}, \tag{14}$$

that is Ac can be viewed as an expansion in the columns of A. By definition the columns are *linearly independent* if

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + c_2 \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} = 0 \tag{15}$$

can only hold provided $c_1 = 0 = c_2$. Since $\det A^T = \det A$, linear dependence of the rows of A is related to $A^T c$.

Matrix form of general n–by–n linear systems

Suppose we have a system of n equations linear in n unknowns $\{x_1, x_2, \cdots, x_n\}$,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ \vdots \qquad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_3. \end{aligned} \tag{16}$$

As with the 2–by–2 case, may again express the system in a matrix form,

$$Ax = b \qquad \Longleftrightarrow \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}. \tag{17}$$

Using permutation tensors, we may extend the concept of determinant and matrix inverse to n–by–n matrices,

$$\begin{aligned} \det A &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \epsilon_{i_1 i_2 \cdots i_n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}, \\ (A^{-1})_{ji} &= \frac{\sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \cdots \sum_{i_n=1}^{n} \sum_{j_2=1}^{n} \sum_{j_3=1}^{n} \cdots \sum_{j_n=1}^{n} \epsilon_{i i_2 i_3 \cdots i_n} \epsilon_{j j_2 j_3 \cdots j_n} a_{i_2 j_2} a_{i_3 j_3} \cdots a_{i_n j_n}}{(n-1)! \det A}, \end{aligned} \tag{18}$$

where $\epsilon_{i_1 i_2 \cdots i_n}$ is the totally antisymmetric tensor of Levi-Civita. Our purpose here is NOT to explain or understand these formulas. The point is just to know that there are such formulas for $\det A$ and A^{-1}. Here's how they look for the 3–by–3 case,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \implies A^{-1} = \frac{1}{\det A} \begin{bmatrix} ei - fh & -(bi - ch) & bf - ce \\ -(di - fg) & ai - cg & -(af - cd) \\ dh - eg & -(ah - bg) & ae - bd \end{bmatrix}, \tag{19}$$

where $\det A = a(ei - fh) - b(di - fg) + c(dh - eg)$. While of theoretical importance, such formulas are often not useful practically (their use requires order $n!$ operations). There is a more efficient way (order n^3 operations) to find determinants, construct inverses, and solve linear systems: Gaussian elimination.

Problems

- Find the inverse A^{-1} of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$

- Verify that $AA^{-1} = I = A^{-1}A$.