QR - factorization : Gram – Schmidt

$\boxed{\text{linalg } 6}$

Ways to compute a QR

1. Gram-Schmidt
   - (a) classical 1883
   - (b) modified

2. Givens rotations

3. Householder reflections 1950s

Good for least squares and square systems!

Example

$$\begin{pmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix} \qquad \hat{Q}\hat{R}$$

Thin QR:

$$\begin{pmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{pmatrix}\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

silent column

Thick QR:

$$\begin{pmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \\ 0 & 0 \end{pmatrix}$$

QR

$$\begin{pmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{pmatrix} = (\vec{a}_1, \vec{a}_2) = (\vec{q}_1, \vec{q}_2)\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

$$\therefore \underbrace{\vec{a}_1 = r_{11}\vec{q}_1}_{1} \quad \text{and} \quad \underbrace{\vec{a}_2 = r_{12}\vec{q}_1 + r_{22}\vec{q}_2}_{2}$$

from 1: $r_{11} = \left\| \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\| = 3 \Rightarrow \vec{q}_1 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$

from 2: $r_{12} = \vec{q}_1^{\,T}\vec{a}_2 = 2 + 4 + 0 = 6$, so

$$r_{22}\vec{q}_2 = \vec{a}_2 - r_{12}\vec{q}_1 = \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix} - 6\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$r_{22}\vec{q}_2 = \begin{pmatrix}-1\\-2\\-2\end{pmatrix} \text{ so } r_{22} = 3 \text{ and } \vec{q}_2 = \begin{pmatrix}-1/3\\-2/3\\-2/3\end{pmatrix}$$

We have
$$\begin{pmatrix}2 & 3\\-2 & -6\\1 & 0\end{pmatrix} = \begin{pmatrix}2/3 & -1/3\\-2/3 & -2/3\\1/3 & -2/3\end{pmatrix}\begin{pmatrix}3 & 6\\0 & 3\end{pmatrix}$$

THICK

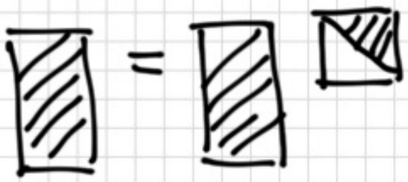
$$\begin{pmatrix}2 & 3\\-2 & -6\\1 & 0\end{pmatrix} = \begin{pmatrix}2/3 & -1/3 & 2/3\\-2/3 & -2/3 & 1/3\\1/3 & -2/3 & -2/3\end{pmatrix}\begin{pmatrix}3 & 6\\0 & 3\\0 & 0\end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2/3 & -1/3 & 2/3 \\ -2/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & -2/3 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix}$$

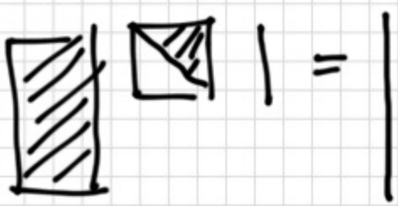
$$\begin{pmatrix} 3 & 6 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/3 & -2/3 & 1/3 \\ -1/3 & -2/3 & -2/3 \\ 2/3 & 1/3 & -2/3 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix}$$

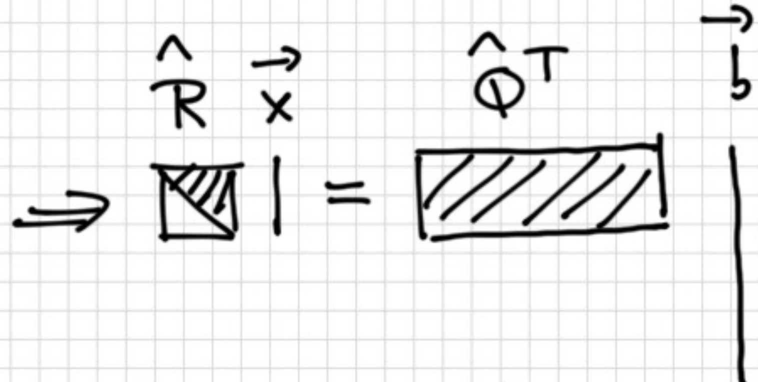
$$\begin{pmatrix} 3 & 6 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -3 \end{pmatrix} \Rightarrow \vec{x}_{LS} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \text{ and } \|A\vec{x}_{LS} - \vec{b}\| = \|R\vec{x}_{LS} - \vec{w}\| = 3$$

$$A = \hat{Q}\hat{R}$$



$$\hat{Q}\hat{R}\vec{x} = \vec{b}$$





$$\Rightarrow \quad \hat{R}\vec{x} = \hat{Q}^T\vec{b}$$