Practice Test 3 Name: ________________________________

CS/Math 375

09 December 2022

Please show all work in the spaces provided, using the backs of pages if necessary. Please give exact answers where possible. For example, write $\frac{1}{3}$ *not* 0.333 *and* π *not* 3.14.

1. (10 points) The equation for a sphere of radius R centered at the point (c_1, c_2, c_3) in 3-dimensional space is

$$(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = R^2.$$

Consider the problem of finding two points in common of three given spheres. Respectively, the three spheres have centers $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$, and each as radius 1.

(a) Consider the use of multivariate Newton's method to solve the above problem. Write down pseudocode for Matlab (or Python) functions which return the relevant nonlinear residual and Jacobian.

(b) In terms of the functions you defined in **(a)** write down pseudocode for solving the problem via Newton's method. Discss your choice for initial guess.

2. (10 points) Consider the 3-point left Gauss-Radau quadrature rule

$$Q_{GR(3)} = h\big[w_1 f(a + c_1 h) + w_2 f(a + c_2 h) + w_3 f(a + c_3 h)\big], \qquad h = (b - a),$$

for approximating integrals of the form $\int_a^b f(x)dx$. For the left rule we demand that the left endpoint $a + c_1 h = a$ is a quadrature point, that is $c_1 = 0$ is fixed.

(a) Write down a nonlinear system which defines the remaining initially *unknown* quantities c_2, c_3, w_1, w_2, w_3.

(b) $Q_{GR(3)}$ will *exactly* integrate polynomials up to and including what degree? Justify your answer.

3. (10 points) Relative to the interval $[-2, 2]$, consider the fixed (open) quadrature points $x_k = -2 + k\Delta x$, where $\Delta x = \frac{4}{5}$ and k runs from 1 to 4.

(a) Why are these points referred to as "open"?

(b) Write down a linear system of equations defining weights in the quadrature rule

$$Q^{\text{open}}_{NC(4)} = 4\sum_{k=1}^{4} w_k f(x_k)$$

for approximating $\int_{-2}^{2} f(x)dx$ if the rule is to exactly integrate polynomials of the highest possible degree.

(c) if the rule is applied to $f(x) = x^3 - x + 4$ then what is the result. Justify your answer.

4. (10 points) Suppose f is a function which is everywhere twice continuously differentiable. The exact error formula for the trapezoid rule is expressible as

$$\int_a^b f(x)dx - Q_{NC(2)} = c_2(b-a)^3 f''(\eta),$$

where c_2 is a constant, and η is a point on $[a,b]$.

(a) Compute $\int_a^b (x-a)^2 dx$ exactly.

(b) Compute the quadrature rule $Q_{NC(2)} = \frac{1}{2}h[f(a)+f(b)]$ for $f(x) = (x-a)^2$.

(c) Find the constant c_2.

5. (10 points) The error formula for the trapezoid rule is

$$\left| \int_a^b f(x)dx - Q_{NC(2)} \right| \leq \frac{1}{12}(b-a)^3 M_2,$$

where $|f''(x)| \leq M_2$ for $x \in [a, b]$.

(a) Using this formula, estimate the error in approximating the integral $I = \int_0^{\frac{1}{5}} \cos(x^2)dx$ with the trapezoid rule $Q_{NC(2)}$.

(b) Compute the error in the trapezoid approximation $Q_{NC(2)}$ of I relative to the Simpson approximation $Q_{NC(3)}$ (that is, pretend that the Simpson approximation is the exact value of the definite integral). Compare with the result from **(a)** and discuss.

6. (10 points) Approximate $\int_0^2 \arctan(x)dx$ using $Q^{(3)}_{NC(3)}$, that is the composite Simpson Rule based on three subintervals. Compute the error in your approximation against the exact answer obtained by the Fundamental Theorem of Calculus.

7. (10 points)

(a) Write down the forward Euler and explicit trapezoid methods for numerical integration of an initial value problem (assume the scalar case for simplicity).

(b) The initial value problem (IVP)

$$y' = 1 - t - 4y, \qquad y(0) = 1$$

has exact solution $y = \phi(t) = (1/16)[4t - 3 + 19\exp(4t)]$. Suppose that you numerically integrate this IVP to final time $T = 0.5$ with time step size h, thereby producing a numerical approximation $K \simeq \phi(0.5)$. You do so for various h and also using the forward Euler, explicit trapezoid, and Runge Kutta 4 methods. Results are tabulated below. Which column corresponds to which numerical ODE method and why?

h	K from ODE solver 1	K from ODE solver 2	K from ODE solver 3
1/16	7.015551567077637e+00	8.711540012367770e+00	8.561418538052010e+00
1/32	7.755109579282565e+00	8.711971940934486e+00	8.670473447444508e+00
1/64	8.200916774197337e+00	8.712001999173960e+00	8.701107935042835e+00
1/128	8.447452931436011e+00	8.712003981596910e+00	8.709214384796084e+00
1/256	8.577357879339992e+00	8.712004108876201e+00	8.711298390612702e+00

8. (10 points) Bogacki and Shampine have devised two low-order Runge Kutta schemes which are used together for error control. To advance a numerical ODE solution y_n at time t_n to the solution y_{n+1} at time $t_{n+1} = t_n + h$, they first define the following.

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1)$$

$$k_3 = f(t_n + \tfrac{3}{4}h, y_n + \tfrac{3}{4}hk_2)$$

$$k_4 = f(t_n + h, y_n + h(\tfrac{2}{9}k_1 + \tfrac{1}{3}k_2 + \tfrac{4}{9}k_3))$$

Next, the solution is advanced by one of the following rules:

$$\text{rule A: } y_{n+1} = y_n + h(\tfrac{2}{9}k_1 + \tfrac{1}{3}k_2 + \tfrac{4}{9}k_3)$$
$$\text{rule B: } y_{n+1} = y_h + h(\tfrac{7}{24}k_1 + \tfrac{1}{4}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{8}k_4)$$

(a) These methods are applied to the initial value problem $y' = t^3/y^2, y(0) = 1$, yielding the following approximations to $y(1)$.

h	rule A	rule B
1/4	1.204752189e+00	1.206066959e+00
1/8	1.205031371e+00	1.205341541e+00
1/16	1.205066228e+00	1.205142083e+00
1/32	1.205070525e+00	1.205089311e+00
1/64	1.205071057e+00	1.205075733e+00
1/128	1.205071123e+00	1.205072289e+00
1/256	1.205071131e+00	1.205071422e+00

What are the order of these schemes? It may help to know that the exact solution to the stated IVP is $y(t) = \left(\frac{3}{4}t^4 + 1\right)^{1/3}$.

(b) Write down the Butcher table for the Bogacki-Shampine embedded method.

9. (10 points) Consider the initial value problem

$$y' = -4t^3y^2, \qquad y(1) = \tfrac{1}{2},$$

with exact solution $y = \phi(t) = (t^4 + 1)^{-1}$.

(a) Verify by hand that the stated solution indeed solves the IVP.

(b) Write down the forward Euler method for this problem with $h = \frac{1}{4}$, and by hand compute the forward Euler approximation $y_1 \simeq \phi(t_1)$.

(c) Write down the explicit trapezoid method for this problem with $h = \frac{1}{4}$, and by hand compute the explicit trapezoid approximation $y_1 \simeq \phi(t_1)$.

10. (10 points) **(a)** The Butcher table

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

specifies the classical Runge-Kutta 4 method. For a scalar problem $y' = f(t, y)$ write down this method for advancing a numerical solution with state (t_n, y_n) to $(t_{n+1} = t_n + h, y_{n+1})$.

(b) Consider the initial value problem

$$y' = y, \qquad y(0) = 1,$$

with solution $y = \phi(t) = e^t$. Consider the Classical Runge-Kutta 4 method for generating numerical approximations $y_n \simeq \phi(t_n)$. Show for one time-step y_1 is an $O(h^5)$ approximation to $\phi(t_1)$. **Hint:** recall the Maclaurin series $e^t = \sum_{k=0}^{\infty} t^k/k!$.