Lecture `interp5`: Interpolation: Chebyshev Theorem

October 21, 2022

Summary: Basic facts about Chebyshev points and Chebyshev polynomials.

References: T. Sauer's *Numerical Analysis*, Section 3.3, first edition, pages 160-169.

Basic formulas for Chebyshev polynomials

The nth Chebyshev polynomial is defined by

$$T_n(x) = \cos(n \arccos x), \qquad x \in [-1, 1]. \tag{1}$$

It is not obvious that the expression on the right side is indeed a polynomial in x, although clearly $T_0(x) = 1$ and $T_1(x) = x$. The proof that, so defined, $T_n(x)$ is indeed a degree-n polynomial relies on trigonometric addition-of-angle formulas to establish the following:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{2}$$

To show that this formula holds, first use the $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ to show

$$\begin{aligned}
T_{n+1}(x) &= \cos((n+1)\arccos x) \\
&= \cos(n \arccos x + \arccos x) \\
&= \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x) \\
&= xT_n(x) - \sin(n \arccos x)\sin(\arccos x),
\end{aligned}$$

for $x \in [-1, 1]$, the domain of the arccos. Likewise,

$$T_{n-1}(x) = xT_n(x) + \sin(n \arccos x)\sin(\arccos x),$$

and addition of the last two results gives (2).

Using the recursion (2) and the starting polynomials $T_0(x) = 1$ and $T_1(x) = x$, we see that $T_n(x)$ is indeed a polynomial for all n. Our proof of (1) has taken place for $x \in [-1, 1]$, but once we have the recursion (1), we view it as defining the polynomials $T_n(x)$ for all $x \in \mathbb{R}$. Therefore, while (1) only holds on the standard interval $[-1, 1]$, we view (2) as holding everywhere. With (1) and (2) in hand, we first prove three lemmas which will help establish the main theorem. **For all of these, we assume that the index $n \geq 1$.**

LEMMA 1. *The polynomial $T_n(x)/2^{n-1}$ is monic, that is $T_n(x)/2^{n-1} = x^n + O(x^{n-1})$ has leading coefficient 1.*

The proof simply relies on using (2) to show that

$$T_n(x) = 2^{n-1}x^n + O(x^{n-1}), \qquad n > 0.$$

This follows by inspection, but can also be shown via induction. □

LEMMA 2. *We have*

$$\frac{1}{2^{n-1}}T_n(x) = (x - x_1)(x - x_2)\cdots(x - x_n),$$

where

$$x_k = \cos\frac{(2k-1)\pi}{2n} \quad \text{for } k = 1, \dots, n \tag{3}$$

are the roots of $T_n(x)$.

The proof amounts to showing that the x_k, as defined, are indeed roots of $T_n(x)$. Clearly, each $x_k \in [-1, 1]$, and so each is in the domain of $\arccos x$. Therefore,

$$T_n(x_k) = \cos(n \arccos x_k) = \cos\left(n\frac{(2k-1)\pi}{2n}\right) = \cos\left(\frac{(2k-1)\pi}{2}\right) = 0.$$

The collection $\{x_k\}_{k=1}^n$ gives *all* roots of $T_n(x)$, since $T_n(x)$ is a degree-n polynomial. The expression in the lemma is then the fully factored form of $T_n(x)/2^{n-1}$, where we note that the right hand side is clearly monic. □

LEMMA 3. *On the closed interval* $[-1, 1]$ *the polynomial* $T_n(x)$ *oscillates between the extreme values* -1 *and* 1*, with* $|T_n(x)|$ *reaching its maximum value 1 a total of* $n + 1$ *times. The maximum value* 1 *is attained at both* $x = 1, -1$*, and at* $n - 1$ *interior points in* $(-1, 1)$.

The proof of this lemma is constructive. Indeed, define the $n + 1$ points

$$\xi_k = \cos\frac{k\pi}{n} \quad \text{for } k = 0, \dots, n. \tag{4}$$

Then clearly each ξ_k is in the domain of $\arccos x$ and

$$T_n(\xi_k) = \cos(n \arccos \xi_k) = \cos\left(n\frac{k\pi}{n}\right) = \cos k\pi = (-1)^k. \tag{5}$$

Notice that $T_n(-1) = T_n(\xi_n) = (-1)^n$ and $T_n(1) = T_n(\xi_0) = 1$. Finally, due to (1) we have $|T_n(x)| \le 1$ for $x \in [-1, 1]$. □

Chebyshev Theorem

THEOREM. *The choice of real numbers* $-1 \le x_1, \dots, x_n \le 1$ *that make the value of*

$$\max_{-1\le x\le 1} |(x - x_1)\dots(x - x_n)|$$

as small as possible is (3). In other words, of all monic polynomials with the form

$$P(x) = (x - x_1)\cdots(x - x_n); \qquad x_1, \dots, x_n \in [-1, 1]$$

the polynomial $T_n(x)/2^{n-1}$ *minimizes*

$$\max_{-1\le x\le 1} |P(x)|.$$

To prove the theorem, assume there exists a monic polynomial $P(x)$ with the above form, and such that it has smaller absolute maximum than $T_n(x)/2^{n-1}$. Notice that by LEMMA 3, $T_n(x)/2^{n-1}$ oscillates between $-1/2^{n-1}$ and $1/2^{n-1}$, with $|T_n(x)|/2^{n-1}$ reaching its maximum value $1/2^{n-1}$ a total of $n+1$ times at the points ξ_k given in (4).

By assumption,

$$-1/2^{n-1} < P(x) < 1/2^{n-1} \tag{6}$$

for all $x \in [-1, 1]$. Therefore,

$$\frac{1}{2^{n-1}} T_n(x) - P(x)$$

is alternately positive and negative at the ξ_k. To see why, we use (5) to write

$$\frac{1}{2^{n-1}} T_n(\xi_k) - P(\xi_k) = \frac{(-1)^k}{2^{n-1}} - P(\xi_k).$$

Therefore, for even k

$$\frac{1}{2^{n-1}} T_n(\xi_k) - P(\xi_k) = \frac{1}{2^{n-1}} - P(\xi_k) \geq \frac{1}{2^{n-1}} - |P(\xi_k)| > 0,$$

where the last inequality follows from (6). Similarly, for odd k

$$\frac{1}{2^{n-1}} T_n(\xi_k) - P(\xi_k) = -\frac{1}{2^{n-1}} - P(\xi_k) \leq -\frac{1}{2^{n-1}} + |P(\xi_k)| < 0,$$

where again the last inequality follows from (6).

Therefore, with n invocations of the *Intermediate Value Theorem*, one for each interval $[\xi_k, \xi_{k+1}]$ with $k \in \{0, 1, \ldots, n-1\}$, we establish that

$$\frac{1}{2^{n-1}} T_n(x) - P(x)$$

has n roots. This is a contradiction. Indeed, as the difference of two monic polynomials, the polynomial

$$\frac{1}{2^{n-1}} T_n(x) - P(x) = O(x^{n-1})$$

has degree-$(n-1)$ or less. Therefore, it cannot have n roots. Since the assumption on $P(x)$ leads to a contradiction, we have proved the theorem. □

Chebyshev points for a general interval $[a, b]$

Given a general interval $[a, b]$, not necessarily the standard interval $[-1, 1]$, the Chebyshev points are

$$t_k = \tfrac{1}{2}(b-a)x_k + \tfrac{1}{2}(b+a), \qquad k = 1, \ldots, n, \tag{7}$$

where the x_k are the Chebyshev points (3) relative to the standard interval. Let us now consider the expression

$$\max_{t \in [a,b]} |(t - t_1) \cdots (t - t_n)|.$$

Since we can write $t = \frac{1}{2}(b-a)x + \frac{1}{2}(b+a)$ for $x \in [-1,1]$, with (7) we find

$$t - t_k = \tfrac{1}{2}(b-a)(x - x_k).$$

With this result

$$\max_{t \in [a,b]} |(t-t_1)\cdots(t-t_n)| = \left(\frac{b-a}{2}\right)^n \max_{x \in [-1,1]} |(x-x_1)\cdots(x-x_n)| \leq \frac{\left(\frac{b-a}{2}\right)^n}{2^{n-1}}. \tag{8}$$

This result is of use when analyzing the error formula for an interpolating polynomial defined by the Chebyshev points on $[a,b]$. Notice that $(b-a)/2 = 1$ for the standard interval, that is when $b = 1$ and $a = -1$.