Quadrature

$\boxed{\text{quad1}}$

Problem: compute $\int_0^1 \cos(x^2)\,dx$

Calculus answer: find $F(x)$ s.t. $\frac{dF}{dx} = \cos(x^2)$

then $\int_0^1 \cos(x^2)\,dx = F(1) - F(0)$

However, $\cos(x^2)$ has no elementary antiderivative.

NOT PRACTICAL.

Approximate

$$\int_a^b f(x)\,dx \doteq I \simeq Q = (b-a)\sum_{k=1}^{m} w_k f(x_k)$$

points : x_k; $k=1,\dots,m$

weights : w_k; $k=1,\dots,m$

We'll consider two approaches

today

1. Newton – Cotes
2. Gauss – Legendre

already in last HW

Newton-Cotes

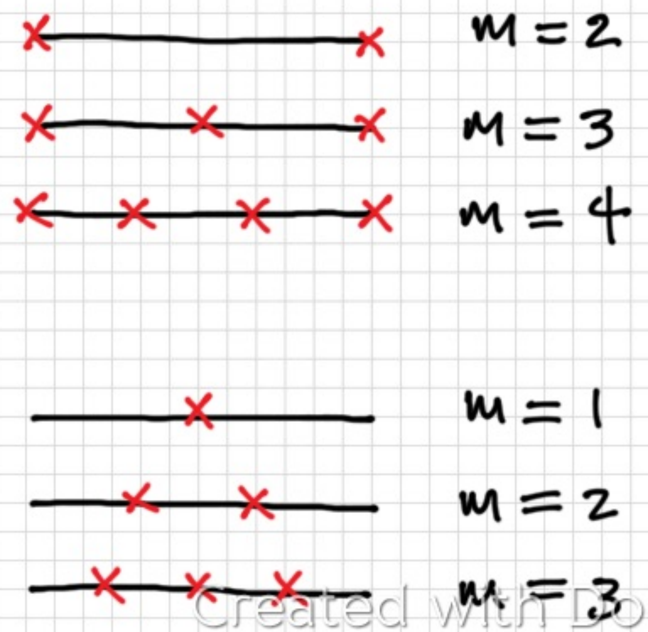
Points are picked in advance.

Closed rules:

$$\boxed{\begin{aligned} x_k &= a + \frac{(k-1)}{(m-1)}(b-a) \\ &= a + c_k(b-a) \end{aligned}}$$

Open rules:

$$\boxed{x_k = a + \frac{k}{m+1}(b-a)}$$



Newton-Cotes (closed)

write down interpolating polynomial

$$P_{m-1}(x) = \sum_{k=1}^{m} f[x_1, \ldots, x_k]\, \phi_k(x)$$

Newton

$$= \sum_{k=1}^{m} f(x_k)\, \ell_k(x)$$

Lagrange

DEFINE : $\boxed{Q_{NC(m)} = \int_a^b P_{m-1}(x)\, dx}$

$$Q_{NC(m)} = \int_a^b p_{m-1}(x)\,dx$$

$$p_{m-1}(x) = \sum_{k=1}^{m} f(x_k)\,\ell_k(x)$$

$$= \sum_{k=1}^{m} f(x_k) \underbrace{\int_a^b \ell_k(x)\,dx}_{(b-a)\int_0^1 \ell_k((b-a)t+a)\,dt \;=\; (b-a)w_k}$$

let $x = (b-a)t + a$

$$= (b-a) \sum_{k=1}^{m} w_k f(x_k)$$

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$\underline{n=2}$ (Trapezoid rule)

$$\boxed{\begin{gathered} x_k = a + \frac{(k-1)}{2-1}(b-a) \\ x_1 = a\ ,\ x_2 = b \end{gathered}}$$

$$P_1(x) = f(a)\,\ell_1(x) + f(b)\,\ell_2(x)$$ Lagrange

$$= f(a) + f[a,b](x-a)$$ Newton

$$Q_{NC(2)} = \int_a^b \left\{ f(a) + f[a,b](x-a) \right\} dx$$

$$= f(a)(b-a) + \frac{f(b)-f(a)}{b-a}\,\frac{(b-a)^2}{2}$$

$$= (b-a)\left[\tfrac{1}{2} f(a) + \tfrac{1}{2} f(b)\right]$$

$w_1 = \frac{1}{2} = w_2$

Closed rules

Trapezoid $w = [\frac{1}{2}\ \frac{1}{2}]$

$$Q_{NC(2)} = \frac{1}{2}(x_2 - x_1)\left[f(x_1) + f(x_2)\right]$$

Simpson $w = [\frac{1}{6}\ \frac{2}{3}\ \frac{1}{6}]$

$$Q_{NC(3)} = \frac{1}{6}(x_3 - x_1)\left[f(x_1) + 4f(x_2) + f(x_3)\right]$$

Simpson 3/8 $w = [\frac{1}{8}\ \frac{3}{8}\ \frac{3}{8}\ \frac{1}{8}]$

$$Q_{NC(4)} = \frac{1}{8}(x_4 - x_1)\left[f(x_1) + 3f(x_2) + 3f(x_3) + f(x_4)\right]$$

In notes quad1, weights listed through $n = 11$