Practice problems Name: ______________________

CS/Math 375

Test Two

1. (10 points)

(a) Construct by hand the LU–factorization of the matrix

$$T = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

(b) Assume that your ancient computer can solve 500-by-500 linear system $A\mathbf{x} = \mathbf{b}$ by LU factorization in 10 seconds (with A full and not tridiagonal). How many floating point operations per second (`flops/sec`) does this performance amount to? Based on this `flops/sec` value, estimate how long it would take to solve a 500-by-500 tridiagonal system $T\mathbf{x} = \mathbf{b}$.

2. (10 points) Consider the unit upper bidiagonal matrix

$$U = \begin{pmatrix} 1 & f_1 & & & \\ & 1 & f_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & f_{n-1} \\ & & & & 1 \end{pmatrix}.$$

(a) Write down an efficient algorithm to solve the linear system $U\mathbf{x} = \mathbf{b}$.

(b) Compute the exact number of floating-point operations (include additions, subtractions, multiplications, and divisions in your count) required by the algorithm. What is the cost of the algorithm in terms of the order symbol ("big-$\mathcal{O}$" notation)?

3. (10 points) Consider the linear system

$$\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

with exact solution $\mathbf{x} = (1, -1)^T$. Relative to the approximate solution $\mathbf{x}_a = (0, -2)^T$, compute the relative forward error, relative backward error, and error magnification. Also compute the condition number of the coefficient matrix and compare it to the error magnification. Is the result expected? Use infinity norms throughout.

4. (10 points) Consider the interpolation problem associated with the following data nodes:

$$x_1 = 0, \qquad x_2 = 1, \qquad x_3 = 3.$$

For the following you need not simplify your answers.

(a) Write down the monomial polynomial basis and Vandermonde matrix corresponding to these nodes.

(b) Write down the Lagrange polynomial basis and Vandermonde matrix corresponding to these nodes.

(c) Write down the Newton polynomial basis and Vandermonde matrix corresponding to these nodes.

5. (10 points) **(a)** Find the degree-2 interpolating polynomial $p(x)$ through the points $(0, 0)$, $(\pi/2, 1)$, and $(\pi, 0)$.

(b) Calculate $p(\pi/4)$, an approximation for $\sin(\pi/4)$.

(c) Give an error bound for the approximation in part (b).

(d) Using a calculator or Matlab, compare the actual error to your error bound.

6. (10 points) Consider the data set $\mathcal{D}_3 = \{(0, \frac{1}{4}), (2, \frac{1}{6}), (4, \frac{1}{8})\}$.

(a) Construct the divided–difference table associated with $\mathcal{D}_3$, and then write down the associated degree-2 Newton interpolating polynomial $p_2(x)$.

(b) Suppose $P(x)$ is the (degree $n-1$ or less) polynomial which interpolates the smooth function $f(x)$ at the points $x_1, \ldots, x_n$, that is $P(x_k) = f(x_k)$ for $k = 1, \ldots, n$. Write down an exact formula for the *interpolation error* $f(x) - P(x)$. Define any notation you introduce.

(c) Using the formula from **(b)**, estimate the error $|f(1) - p_2(1)|$ in using $p_2(1)$ to approximate $f(1)$, where $f(x) = 1/(x+4)$ gives the data $\mathcal{D}_3$ from **(a)**. Compare with the exact error.

7. (10 points) Consider the following data set

$$\{(0,0),\ (1,1),\ (2,5)\},$$

and the piecewise cubic function

$$C(x) = \begin{cases} x + (1-s_1)x(x-1) + (s_1+s_2-2)x^2(x-1) & \text{if } x \in [0,1] \\ 1 + 4(x-1) + (4-s_2)(x-1)(x-2) + (s_2+s_3-8)(x-1)^2(x-2) & \text{if } x \in [1,2]. \end{cases}$$

Write down the system of equations defining a clamped spline for the data with endpoint conditions $C'(0) = \alpha$ and $C'(2) = \beta$.

8. (10 points) Using the basis $\mathcal{B}_2 = \{1, x\}$, find the line which best fits the data

$$\mathcal{D}_3 = \{(0,0), (1,1), (2,8)\}$$

in the least squares sense. To do so, use the normal equations approach.

9. (10 points) Use the Givens rotation approach to find a thick QR-factorization for the following matrix.

$$A \equiv \begin{pmatrix} \sqrt{\frac{1}{2}} & 1 \\ \sqrt{\frac{1}{2}} & 1 \\ 0 & 1 \end{pmatrix}$$

The system

$$A\mathbf{x} = \mathbf{b} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is overdetermined. *Without finding* $\mathbf{x}_{LS}$, compute the magnitude $\|A\mathbf{x}_{LS} - \mathbf{b}\|$ of the residual vector corresponding to solution the $\mathbf{x}_{LS}$ of the system in the least squares sense.

10. (10 points) Use the Gram-Schmidt process to find a thin QR-factorization for the following matrix.

$$A \equiv \begin{pmatrix} \sqrt{\frac{1}{2}} & 1 \\ \sqrt{\frac{1}{2}} & 1 \\ 0 & 1 \end{pmatrix}$$

Use your factorization to solve the system

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

in the least squares sense.