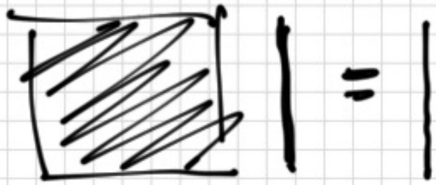
Triangular and tridiagonal systems

$\boxed{\text{linalg2}}$

1. $A\vec{x} = \vec{b}$

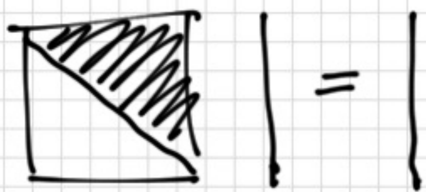


general full system

$O(n^3)$ solve

2. upper Δ

$U\vec{x} = \vec{b}$

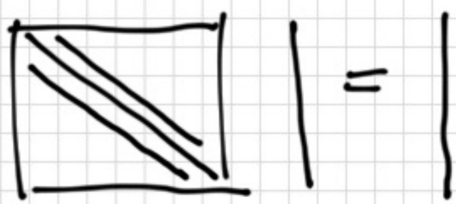


$O(n^2)$ solve

same for lower Δ

3. tridiagonal

$T\vec{x} = \vec{b}$



$O(n)$ solve

② first $\sum_{k=1}^{j} l_{jk} x_k = b_j \Longleftrightarrow x_j = \dfrac{\left(b_j - \sum_{k=1}^{j-1} l_{jk} x_k\right)}{l_{jj}}$

$x_1 = b_1$ 0 flops

$x_2 = (b_2 - l_{21} x_1)$ 2 flop

$x_3 = (b_3 - l_{31} x_1 - l_{32} x_2)$ 4 flops

work row $(k) = 2(k-1)$ flops

total work $= 2 \sum_{k=1}^{n} (k-1) = 2 \sum_{q=1}^{n-1} q$

$= (n-1)n = n^2 - n = O(n^2)$

Solve $A\vec{x}=\vec{b}$

later

① factor $PA=LU$ so $LU\vec{x}=P\vec{b}$ $O(n^3)$

② Solve $L\vec{y}=P\vec{b}$ for $\vec{y}$ $O(n^2)$

③ Solve $U\vec{x}=\vec{y}$ for $\vec{x}$ $O(n^2)$

+ follow same process for $T\vec{x}=\vec{b}$, but now each step is $O(n)$.

Consider

$$\begin{pmatrix} d_1 & f_1 & 0 & 0 \\ e_2 & d_2 & f_2 & 0 \\ 0 & e_3 & d_3 & f_3 \\ 0 & 0 & e_4 & d_4 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ l_2 & 1 & & \\ & l_3 & 1 & \\ & & l_4 & 1 \end{pmatrix} \begin{pmatrix} u_1 & f_1 & & \\ & u_2 & f_2 & \\ & & u_3 & f_3 \\ & & & u_4 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 & f_1 & 0 & 0 \\ l_2 u_1 & u_2 + l_2 f_1 & f_2 & 0 \\ 0 & l_3 u_2 & u_3 + l_3 f_2 & f_3 \\ 0 & 0 & l_4 u_3 & u_4 + l_4 f_3 \end{pmatrix}$$

$$\boxed{\begin{aligned} u_1 &= d_1 \\ l_2 &= e_2 / u_1 \\ u_2 &= d_2 - l_2 f_1 \\ l_3 &= e_3 / u_2 \\ u_3 &= d_3 - l_3 f_2 \\ l_4 &= e_4 / u_3 \\ u_4 &= d_4 - l_4 f_3 \end{aligned}}$$

Tridiagonal LU factorization (no permutations

$u_1 = d_1$

for $k = 2$ to n

  $\ell_k = e_k / u_{k-1}$

  $u_k = d_k - \ell_k f_{k-1}$

end

MISTAKE IN NARRATION
THIS SHOULD BE u_{k-1}

3 flops per instance of k.

Total cost $3(n-1)$

$O(n)$ work

Bidiagonal solves $\quad L = \begin{pmatrix} 1 & & & & \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \ddots & \ddots & \\ & & & l_n & 1 \end{pmatrix}$

$L\vec{y} = \vec{b}$

$y_1 = b_1$

$y_2 = b_2 - l_2 y_1$

$y_3 = b_3 - l_3 y_2$

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y1 = b1
for k=2 to n
    yk = bk - lk yk-1
end
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$2(n-1)$ cost

$O(n)$

Bidiagonal solves

$$U = \begin{pmatrix} u_1 & f_1 & & & \\ & u_2 & f_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & f_{n-1} \\ & & & & u_n \end{pmatrix}$$

$$U\vec{x} = \vec{y}$$

$$x_n = y_n / u_n$$

$$x_{n-1} = (y_{n-1} - f_{n-1} x_n) / u_{n-1}$$

```
x_n = y_n/u_n
for k = n-1 down to 1
    x_k = (y_k - f_k x_{k+1})/u_k
end
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$O(n)$