$\boxed{\text{linalg1}}$

Basic Gaussian Elimination

Problem solve square system $A\vec{x} = \vec{b}$

Here $A \in \mathbb{R}^{n\times n}$, $\vec{x} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^n$

Reduce problem to solution of Δ systems.

1. Backward substitution (upper Δ systems)
2. Forward substitution (lower Δ systems)

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Upper triangular system $U\vec{x} = \vec{b}$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Last equation: $u_{33} x_3 = b_3 \Longrightarrow x_3 = b_3 / u_{33}$ known

Middle equation: $u_{22} x_2 + u_{23} x_3 = b_2$ known

$$\Longrightarrow x_2 = \frac{(b_2 - u_{23} x_3)}{u_{22}}$$

Last equation: $u_{11} x_1 + u_{12} x_2 + u_{13} x_3 = b_1$

$$\Longrightarrow x_1 = \frac{(b_1 - u_{12} x_2 - u_{13} x_3)}{u_{11}}$$

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General algorithm

$$U\vec{x} = \vec{b} \iff \sum_{k=1}^{n} u_{jk} x_k = b_j \qquad j = 1, 2, \cdots, n$$

since U upper Δ,

$u_{jk} = 0$ if $k < j$

$$\sum_{k=j}^{n} u_{jk} x_k = b_j \iff u_{jj} x_j + \sum_{k=j+1}^{n} u_{jk} x_k = b_j$$

$$\boxed{x_j = \left(b_j - \sum_{k=j+1}^{n} u_{jk} x_k \right) / u_{jj} \quad \text{for } j = n, n-1, \cdots, 1}$$

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Similar algorithm for $L\vec{x} = \vec{b}$

$\uparrow$ lower Δ

LU factorization

Find P, L, U s.t. $PA = LU$

Then $A\vec{x} = \vec{b} \Rightarrow PA\vec{x} = P\vec{b} \Rightarrow LU\vec{x} = P\vec{b}$

$\Rightarrow \boxed{\vec{x} = U^{-1}L^{-1}P\vec{b}}$ uses forward, then backward substitutions

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EXAMPLE

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & 2 \\ -4 & -4 & 1 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

$$R_1 A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 0 & -4 \\ -4 & -4 & 1 \end{pmatrix}, \quad R_2 R_1 A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 0 & -4 \\ 0 & -8 & 13 \end{pmatrix}$$

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Up to know

$$M = R_2 R_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & 2 \\ -4 & -4 & 1 \end{pmatrix}$$

$$MA = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 0 & -4 \\ 0 & -8 & 13 \end{pmatrix}$$

NOW TAKE $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

permutation

$$PMA = \begin{pmatrix} 1 & -1 & 3 \\ 0 & -8 & 13 \\ 0 & 0 & -4 \end{pmatrix}$$

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$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & 2 \\ -4 & -4 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & -1 & 3 \\ 0 & -8 & 13 \\ 0 & 0 & -4 \end{pmatrix}}_{U}$$

Here $P = P^T = P^{-1}$ so $A = M^{-1}PU$ and $PA = \underbrace{PM^{-1}P}_{L}U$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \leftarrow \text{lower } \Delta \text{ !}$$

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