ode1

ODE initial value problems

$$\frac{d\vec{y}}{dt} = \vec{f}(t, \vec{y}), \quad \vec{y}(t_0) = \vec{y}_0$$

Scalar examples

$$y' = 1 - t + 4y, \quad y(0) = 1$$

$$\boxed{\text{SOLUTION} \quad y = \phi(t) = \frac{1}{4}t - \frac{3}{16} + \frac{19}{16}e^{4t}}$$

$$\phi' = 4\left(\frac{1}{16} + \frac{19}{16}e^{4t}\right) = 1 - t + 4\phi \checkmark$$

Another example

$y' = y^2, \; y(0) = 1$

$$\boxed{\text{SOLUTION} \quad y = \phi(t) = \frac{1}{1-t}}$$

$\phi' = \frac{1}{(1-t)^2} = \phi^2$ ✓

How do we theoretical solve ODE? See 316

Focus here will be on Numerical Solution.

System example

$$\vec{y}\,' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{y} + \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{y}\,' = \vec{f}(t, \vec{y}) \qquad \vec{y}(t_0) = \vec{y}_0$$

Really two equations here

$$\left.\begin{aligned} y_1' &= y_1 + y_2 + t \\ y_2' &= 4y_1 - 2y_2 \end{aligned}\right\} \quad \begin{aligned} &\text{coupled system.} \\ &\text{find } y_1 = \phi_1(t) \\ &\phantom{\text{find }} y_2 = \phi_2(t) \end{aligned}$$

Systems example

$$\vec{y}\,' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{y} + \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

SOLUTION $\vec{y} = \vec{\phi}(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}$

$$= \frac{1}{9} \begin{pmatrix} 9e^{2t} + 2e^{-3t} - 3t - 2 \\ 9e^{2t} - 8e^{-3t} - 6t - 1 \end{pmatrix}$$

$$\vec{y}' = \vec{f}(t, \vec{y}), \quad \vec{y}(t_0) = \vec{y}_0$$

and for how long?

① does solution exist?

② is solution unique?

③ does solution $\vec{y} = \vec{\phi}(t)$ depend continuously on $\vec{y}_0$?

well posed

For "reasonable" $\vec{f}(t, \vec{y})$ we get all these.

Difference methods

* partition time $t_k = a + k\Delta t$

$k = 0, 1, \ldots, m$ and $\Delta t = \dfrac{(b-a)}{m}$

$$\underset{\uparrow\, t_0}{a} = t_0 < \underset{\uparrow\, t_0 + \Delta t}{t_1} < \underset{\nwarrow\, t_0 + 2\Delta t}{t_2} < \cdots < t_m = \underset{\uparrow\, T \text{ or } t_F}{b}$$

$\vec{y}_k \simeq \vec{y}(t_k) = \vec{\phi}(t_k)$ Numerical approx

More precisely denote by $\vec{y}_k^{\,\Delta t}$

If $\Delta t = 0.25$, then $\vec{y}_8^{\Delta t}$ approximates $\vec{y}(2)$

If $\Delta t = 0.125$, then $\vec{y}_8^{\Delta t}$ approximates $\vec{y}(1)$

Forward Euler method

$$\frac{y_{k+1} - y_k}{\Delta t} = f(t_k, y_k), \quad y_0 = y(a)$$

$\simeq dy/dt$

$$y_{k+1} = y_k + \Delta t\, f(t_k, y_k) \quad \text{for } k = 1, 2, \ldots, n$$

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IVP : $\dfrac{dy}{dt} = f(t,y)$; $y(0) = y_0$

Forward Euler method for evolving y from $a = t_0$ to $t_F = b$.

* Partition time $t_k = a + k\Delta t$, $\Delta t = \dfrac{b-a}{m}$

$k = 0, 1, 2, \cdots, m$

$$\boxed{y_{k+1} = y_k + \Delta t\, f(t_k, y_k)}$$

<u>Problem</u> $y' = ty \;;\; y(0) = 1$

(a) Write down (Forward) Euler method

(b) Assuming $\Delta t = 0.1$, compute y_1, y_2, y_3

really $y_1^{0.1}, y_2^{0.1}, y_3^{0.1}$
for $y_k^{\Delta t}$

(c) Find exact solution and tabulate the errors $y(t_k) - y_k$

SOLUTION

(a) $y_{k+1} = y_k + \Delta t\, t_k\, y_k$ for $t_0 = 0, y_0 = 1$

(b) from (a) $y_{k+1} = (1 + \Delta t\, t_k)\, y_k$

$y_0 = 1, \quad t_0 = 0, \quad t_1 = \Delta t = 0.1, \quad t_2 = 0.2, \quad t_3 = 0.3$

$$y_1 = (1 + \Delta t\, t_0)\, y_0$$
$$= 1$$

$$y_2 = (1 + \Delta t\, t_1)\, y_1$$
$$= 1 + (0.1)^2 = 1.01$$

$$y_3 = (1 + \Delta t\, t_2)\, y_2$$
$$= (1.02)(1.01) = 1.0302$$

(c) We found in (b) $y_1 = 1$, $y_2 = 1.01$, $y_3 = 1.0302$

$\frac{dy}{dt} = ty$ so $\frac{dy}{y} = t\,dt$ so $\ln|y| = \frac{1}{2}t^2 + C$

$y = Ke^{\frac{1}{2}t^2}$ and since $y(0) = 1$, $K = 1$

$\boxed{y = e^{\frac{1}{2}t^2}}$

$y(t_k) \neq y_k$

$e^{\frac{1}{2}t_1^2} - y_1 = 1.0051 - 1 = 0.00501$

$e^{\frac{1}{2}t_2^2} - y_2 = 1.02020 - 1.01 = 0.01020$

$e^{\frac{1}{2}t_3^2} - y_3 = 1.04603 - 1.0302 = 0.01583$

IVP: $\frac{dy}{dt} = f(t,y)$; $y(t_0) = y_0$

$$\left. y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\, ds \right\} \text{integral form}$$

time-step

$$y(t_{k+1}) = y(t_k) + \underbrace{\int_{t_k}^{t_{k+1}} f(s, y(s))\, ds}$$

left rectangle rule

$$\simeq f(t_k, y(t_k)) \underbrace{(t_{k+1} - t_k)}_{\Delta t}$$

$$y_{k+1} = y_k + \Delta t\, f(t_k, y_k)$$

$$y(t_{k+1}) = y(t_k) + \underbrace{\int_{t_k}^{t_{k+1}} f(s, y(s))\, ds}$$

right rectangle rule $\quad \simeq \Delta t\, f(t_{k+1}, y(t_{k+1}))$

Backward Euler (Implicit)

$$y_{k+1} = y_k + \Delta t\, f(t_{k+1}, y_{k+1})$$

* root equation.

$$y(t_{k+1}) = y(t_k) + \underbrace{\int_{t_k}^{t_{k+1}} f(s, y(s))\, ds}$$

trapezoid rule $\quad \frac{1}{2}\Delta t \left[f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1})) \right]$

$$y_{k+1} = y_k + \frac{\Delta t}{2} \left[f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \right]$$

Explicit trapezoid (an RK2 method)

$$y^* = y_k + \Delta t\, f(t_k, y_k) \quad \text{(Forward Euler)}$$

$$y_{k+1} = y_k + \frac{\Delta t}{2} \left[f(t_k, y_k) + f(t_{k+1}, y^*) \right]$$