iterative

$\vec{F}(\vec{x}) = \vec{0}$ Nonlinear system of EQNS $\vec{F}(\vec{x})$ is the "nonlinear residual"

How to solve?

Possibility ① write as $\vec{x} = \vec{G}(\vec{x})$

(many ways to do this)

given $\vec{x}_0$, $\vec{x}_1 = \vec{G}(\vec{x}_0)$, $\vec{x}_2 = \vec{G}(\vec{x}_1)$

$\vec{x}_{k+1} = \vec{G}(\vec{x}_k)$ Fixed point iteration

If $\vec{r}$ is FP $\vec{r} = \vec{G}(\vec{r})$, need $\|D\vec{G}(\vec{r})\| < 1$ for CVG.

Possibility ② Newton's method

$$\vec{x}_{k+1} = \vec{x}_k - [D\vec{F}(x_k)]^{-1}\vec{F}(\vec{x}_k)$$

$\exists\ M > 0$ s.t.
near root
$\|\vec{e}_{k+1}\| \le M\|\vec{e}_k\|^2$
$\vec{e}_k = \vec{x}_k - \vec{r}$

for "simple root"

If $D\vec{F}(\vec{r})$ is nonsingular at root $\vec{r}$, method is quadratically cvg.

There are other possibilities!

Here $\vec{r}$ is vector root. $\vec{x}^*$ in notes ($\vec{r}$ used for similar residual)

Iterative methods for linear systems

$A\vec{x} = \vec{b}$ $\quad$ B nonsingular and "approximate inverse of A"

system equivalent to $BA\vec{x} = B\vec{b}$

$$\vec{x} = \vec{x} - BA\vec{x} + B\vec{b}$$

$$\vec{x}_{k+1} = (I - BA)\vec{x}_k + B\vec{b}$$ Richardson iteration

$$\boxed{\vec{x}_{k+1} = M\vec{x}_k + \vec{c} \quad \text{"stationary iterative method"}}$$

Newton applied to linear system

let $\vec{F}(\vec{x}) = A\vec{x} - \vec{b}$

$$\vec{F}(\vec{x})(i) = \sum_{j=1}^{n} a_{ij}\,\vec{x}(j) - \vec{b}(i)$$

$$\frac{\partial \vec{F}(\vec{x})(i)}{\partial \vec{x}(\ell)} = \sum_{j=1}^{n} a_{ij}\,\delta_{j\ell} = a_{i\ell}$$

here j^{TH} component of $\vec{x}$ is $\vec{x}(j)$ rather than x_j

shows $D\vec{F}(\vec{x}) = A$

Newton $\vec{x}_{n+1} = \vec{x}_n - A^{-1}(A\vec{x}_n - \vec{b})$

$= A^{-1}\vec{b}$ cvg in one step!

Quasi or inexact Newton method

$$\vec{x}_{k+1} = \vec{x}_k - D^{-1}(A\vec{x}_k - \vec{b})$$ Jacobi method

using inverse of approximate Jacobian

$$= (I - D^{-1}A)\vec{x}_k + D^{-1}\vec{b}$$ Richardson iteration

Will cvg if $\|I - D^{-1}A\| < 1$

(turns out to be true if A "diagonally dominant")

$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for $i = 1, 2, \dots, n$

Jacobi in more detail. Write

$A = L + D + U =$  $+$  $+$ 

(not same L, U in $PA = LU$)

$$\vec{x}_{k+1} = (I - D^{-1}A)\vec{x}_k + D^{-1}\vec{b}$$

$$D^{-1}(L+D+U) = I + D^{-1}(L+U)$$

$$= -D^{-1}(L+U)\vec{x}_k + D^{-1}\vec{b}$$

$$= D^{-1}(\vec{b} - (L+U)\vec{x}_k)$$

solve for diagonal

Jacobi, example

$$\vec{x}_{exact} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} u \\ v \\ w \end{pmatrix}}_{\vec{x}} = \underbrace{-}_{-} \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}}_{(L+U)} \underbrace{\begin{pmatrix} u \\ v \\ w \end{pmatrix}}_{\vec{x}} + \underbrace{\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}}_{\vec{b}}$$

$$\boxed{\begin{aligned} u &= \vec{x}(1) \\ v &= \vec{x}(2) \\ w &= \vec{x}(3) \end{aligned}}$$

$$= \begin{pmatrix} 4 - v + w \\ 1 - 2u - w \\ 1 + u - 2v \end{pmatrix}$$

$$u = \frac{1}{3}(4 - v + w)$$
$$v = \frac{1}{4}(1 - 2u - w)$$
$$w = \frac{1}{5}(1 + u - 2v)$$

Not an iterative method yet

JACOBI

$$\boxed{\begin{aligned} u_{k+1} &= \frac{1}{3}(4 - v_k + w_k) \\ v_{k+1} &= \frac{1}{4}(1 - 2u_k - w_k) \\ w_{k+1} &= \frac{1}{5}(1 + u_k - 2v_k) \end{aligned}}$$

Say w/ guess $\begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Jacobi

$$u_{k+1} = \frac{1}{3}(4 - v_k + w_k)$$
$$v_{k+1} = \frac{1}{4}(1 - 2u_k - w_k)$$
$$w_{k+1} = \frac{1}{5}(1 + u_k - 2v_k)$$

all "old" values on right

Gauss - Seidel

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$$u_{k+1} = \frac{1}{3}(4 - v_k + w_k)$$
$$v_{k+1} = \frac{1}{4}(1 - 2u_{k+1} - w_k)$$
$$w_{k+1} = \frac{1}{5}(1 + u_{k+1} - 2v_{k+1})$$

Use updated values once you have them

From abstract perspective Gauss-Seidel is

$$\vec{x}_{k+1} = \vec{x}_k - (L+D)^{-1}(A\vec{x}_k - \vec{b})$$

$$= \left(I - (L+D)^{-1}A\right)\vec{x}_k + (L+D)^{-1}\vec{b}$$

– Richardson iteration

How to see this?

$$(L+D)\vec{x}_{k+1} = (L+D-A)\vec{x}_k + \vec{b}$$

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$$(L+D)\vec{x}_{k+1} = (L+D-A)\vec{x}_k + \vec{b}$$

$$= -U\vec{x}_k + \vec{b}$$

$A = L+D+U$

$$D\vec{x}_{k+1} = -(L\vec{x}_{k+1} + U\vec{x}_k) + \vec{b}$$

$$\vec{x}_{k+1} = -D^{-1}(L\vec{x}_{k+1} - U\vec{x}_k) + D^{-1}\vec{b}$$

$$= D^{-1}(\vec{b} - L\vec{x}_{k+1} - U\vec{x}_k)$$

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$$\boxed{\vec{x}_{k+1} = D^{-1}(\vec{b} - L\vec{x}_{k+1} - U\vec{x}_k)}$$

take 3x3 example with $\vec{x} = (u, v, w)^T$

$$u_{k+1} = \frac{1}{a_{11}}\left(\vec{b}(1) - u_{12} v_k - u_{13} w_k\right)$$

$$v_{k+1} = \frac{1}{a_{22}}\left(\vec{b}(2) - l_{21} u_{k+1} - u_{23} w_k\right)$$

$$w_{k+1} = \frac{1}{a_{33}}\left(\vec{b}(3) - l_{31} u_{k+1} - l_{32} v_{k+1}\right)$$

Same idea works for larger systems