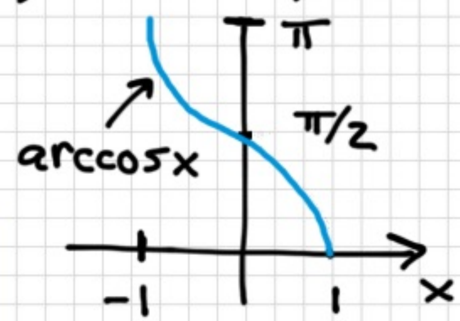
interp 5

Chebyshev (also Tchebyshev) polynomials

$(*)\quad T_n(x) = \cos(n \arccos x)\,, \quad x \in [-1, 1]$

Clearly $T_0(x) = 1$ and $T_1(x) = x$.



$(*)$ Then implies

$$\boxed{T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)}$$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\begin{aligned} T_{n+1}(x) &= \cos((n+1)\arccos x) \\ &= \cos(n\arccos x + \arccos x) \\ &= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x) \end{aligned}$$

$$T_{n-1}(x) = \cos(n\arccos x)\cos(\arccos x) + \sin(n\arccos x)\sin(\arccos x)$$

$T_n(x) = \cos(n \arccos x)$, $T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$

$n \geq 1$ for lemmas

Lemma ① The polynomial $\dfrac{T_n(x)}{2^{n-1}}$ is monic,

That is $T_n(x) = x^n + O(x^{n-1})$ has lead coefficient 1.

$T_0 = 1$
$T_1 = x$
$T_2 = 2x^2 - 1$
$T_3 = 4x^3 - 3x$

Get result by Mathematical Induction

Lemma ② $\dfrac{T_n(x)}{2^{n-1}} = (x-x_1)(x-x_2)\cdots(x-x_n)$

where $\boxed{x_k = \cos\dfrac{(2k-1)\pi}{2n} \text{ for } k=1,2,\cdots,n}$

are the roots of $T_n(x)$

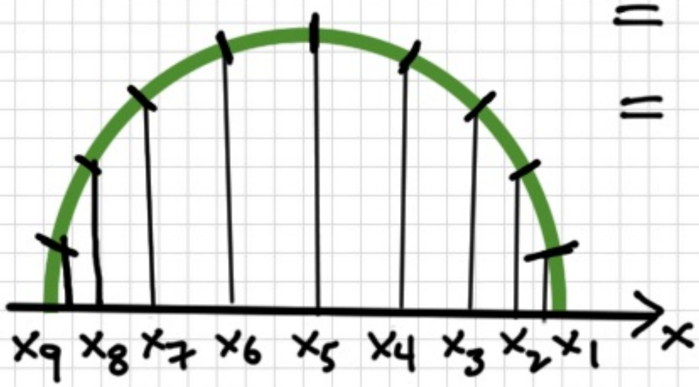
Chebyshev points

Proof

$$T_n(x_k) = \cos(n \arccos x_k)$$

$$= \cos\left(n \arccos\left(\cos\frac{(2k-1)\pi}{2n}\right)\right)$$

$$= \cos\left(\frac{(2k-1)\pi}{2}\right) = 0 \checkmark$$



Lemma ③ on $[-1,1]$ the polynomial $T_n(x)$ oscillates between extreme values -1 and 1, with $|T_n(x)|$ attaining its maximum value 1 a total of $n+1$ times. $|T_n(x)| = 1$ at $x = 1$, $x = -1$ and at $n-1$ interior points in $(-1,1)$.

Proof Set $\xi_k = \cos\frac{k\pi}{n}$ for $k = 0, 1, \cdots, n$

$$\begin{aligned} T_n(\xi_k) &= \cos(n \arccos \xi_k) \\ &= \cos\left(n \arccos\left(\cos\frac{k\pi}{n}\right)\right) \\ &= \cos(k\pi) = (-1)^k \end{aligned}$$

<u>Chebyshev THM</u> The choice of real numbers $-1 \le x_1, x_2, \cdots, x_n \le 1$ that minimizes the expression

$$\boxed{\max_{-1 \le x \le 1} \left| (x-x_1)(x-x_2)\cdots(x-x_n) \right|}$$

are the Chebyshev points. $x_k = \cos \frac{(2k-1)\pi}{2n}$

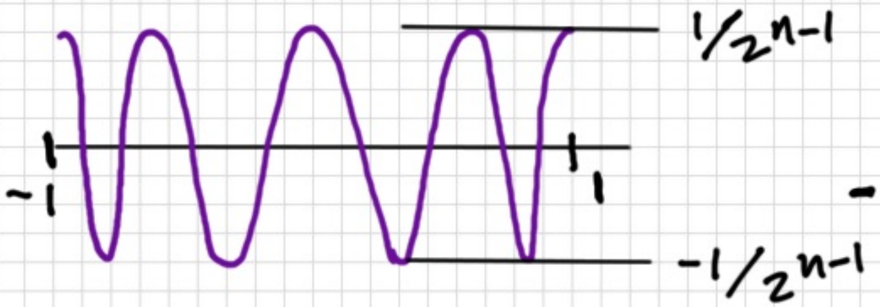
Before proof recall error formula for polynomial interpolation

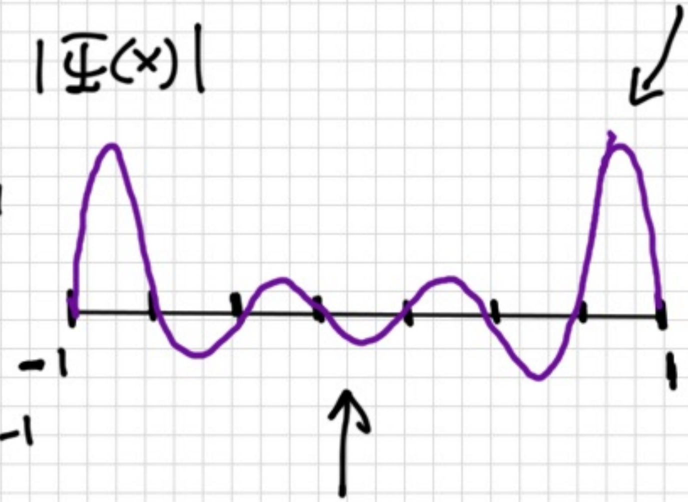
$$f(x) - P(x) = \underbrace{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}_{\Psi(x)} \frac{f^{(n)}(c(x))}{n!}$$

the Chebyshev points minimize $\max_{-1 \le x \le 1} |\Psi(x)|$

Put differently, of all monic polynomials $\Phi(x)$

$$\frac{T_n(x)}{2^{n-1}} \text{ minimizes } \max_{-1\le x\le 1} |\Phi(x)|$$





Conclusion: Chebyshev points are good for interpolation.

Proof of Chebyshev Theorem

$\xi_k = \cos\frac{k\pi}{n}$

$\frac{T_n(x)}{2^{n-1}}$ oscillates between $-\frac{1}{2^{n-1}}$ and $\frac{1}{2^{n-1}}$ $n+1$ times on $[-1,1]$, with $\left|\frac{T_n(x)}{2^{n-1}}\right| = \frac{1}{2^{n-1}}$ at the ξ_k points.

Assume, to the contrary, that there exists a "better" monic polynomial $\Phi(x)$ which obeys $-\frac{1}{2^{n-1}} < \Phi(x) < \frac{1}{2^{n-1}}$. Then $\boxed{\frac{1}{2^{n-1}}T_n(x) - \Phi(x)}$ alternates sign at the $n+1$ points ξ_k. Then boxed poly has n roots on $(-1,1)$. But $\frac{1}{2^{n-1}}T_n(x) - \Phi(x) = O(x^{n-1})$

$\Rightarrow\Leftarrow$