Newton's approach (polynomial interp)

$\boxed{\text{interp 2}}$

$$\mathcal{D}_N = \{(x_k, y_k)\}_{k=1}^{N}$$

$$\mathcal{B}_N = \{\phi_k\}_{k=1}^{N}$$

	Monomial	Newton
ϕ_1	1	1
ϕ_2	x	$(x-x_1)$
ϕ_3	x^2	$(x-x_1)(x-x_2)$
$\vdots$	$\vdots$	$\vdots$
ϕ_N	x^{N-1}	$(x-x_1)(x-x_2)\cdots(x-x_{N-1})$

Interpolant: $y = p(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_N\phi_N(x)$

$$\begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_N(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_N(x_2) \\ \vdots & \vdots & & \vdots \\ \phi_1(x_N) & \phi_2(x_N) & \cdots & \phi_N(x_N) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

Vandermonde system

$V\vec{c} = \vec{y}$

N=3 case — Newton basis

$\phi_1(x) = 1$

$\phi_2(x) = (x - x_1)$

$\phi_3(x) = (x - x_1)(x - x_2)$

$$\begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) & \phi_3(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \phi_3(x_2) \\ \phi_1(x_3) & \phi_2(x_3) & \phi_3(x_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & 0 \\ 1 & x_3 - x_1 & (x_3 - x_1)(x_3 - x_2) \end{pmatrix}$$

$\Rightarrow$ lower-Δ Vandermonde system $V\vec{c} = \vec{y}$

Solve for $\vec{c}$ by forward substitution

$$p(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2)$$

Newton divided differences

(think of $y = f(x)$ so $y_k = f(x_k)$)

$$\boxed{c_k = f[x_1, x_2, \cdots, x_k]}$$

$\underline{f[x_1]} = f(x_1)$, $f[x_2] = f(x_2)$, $f[x_3] = f(x_3)$ } zeroth order D.D.

$$f[\underline{x_1, x_2}] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$ } 1^{ST} order D.D.

$$f[\underline{x_1, x_2, x_3}] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$ } 2^{ND} order D.D.

$$p(x) = f(x_1) + f[x_1, x_2](x - x_1) + f[x_1, x_2, x_3](x - x_1)(x - x_2)$$

EX $\mathcal{D}_3 = \{(0,1), (2,2), (3,4)\}$

$$
\begin{array}{c|c}
0 & 1 \\
 & & \dfrac{2-1}{2-0} = \dfrac{1}{2} \\
2 & 2 & & \dfrac{2-\frac{1}{2}}{3-0} = \dfrac{1}{2} \\
 & & \dfrac{4-2}{3-2} = 2 \\
3 & 4
\end{array}
$$

$$p(x) = f(x_1) + f[x_1,x_2](x-x_1) + f[x_1,x_2,x_3](x-x_1)(x-x_2)$$

$$
\boxed{
\begin{aligned}
p(x) &= 1 + \tfrac{1}{2}x + \tfrac{1}{2}x(x-2) \\
&= 1 - \tfrac{1}{2}x + \tfrac{1}{2}x^2
\end{aligned}
}
$$

what we would get w/ monomials

$\underline{\text{Ex}}$ $\mathcal{D}_4 = \mathcal{D}_3 \cup \{(1,0)\}$ (add a point)

$$= \{\underbrace{(0,1),\ (2,2),\ (3,4)}_{\mathcal{D}_3},\ (1,0)\}$$

$$
\begin{array}{c|ccccccc}
0 & \underline{\underline{1}} & & & & & & \\
 & & > \underline{\underline{\tfrac{1}{2}}} & & & & & \\
2 & 2 & & > \underline{\underline{\tfrac{1}{2}}} & & & & \\
 & & > 2 & & > \dfrac{0-\frac{1}{2}}{1-0} = \underline{\underline{-\tfrac{1}{2}}} & & & \\
3 & 4 & & > 0 & & & & \\
 & & > 2 & & & & & \\
1 & 0 & & & & & &
\end{array}
$$

$$p(x) = 1 + \tfrac{1}{2}x + \tfrac{1}{2}x(x-2)$$

$$q(x) = 1 + \tfrac{1}{2}x + \tfrac{1}{2}x(x-2) - \tfrac{1}{2}x(x-2)(x-3)$$