Matrix condition number

$\boxed{\text{cond2}}$

Want to solve $A\vec{x} = \vec{b}$

backward

$\boxed{\text{inputs } A, \vec{b}}$ $\rightarrow$ solution process $\rightarrow$ $\boxed{\text{output } \vec{x}}$

forward

On computer we get approximate solution $\vec{x}_A$

with solves $(A + \delta A)\vec{x}_A = (\vec{b} + \delta\vec{b})$

$\vec{x} = \vec{h}(A, \vec{b})$, $\vec{x}_A = \vec{h}(A + \delta A, \vec{b} + \delta\vec{b})$

How sensitively does $\vec{x}_A$ depend on inputs?

this A not in narration

To address this question we need vector and matrix norms. For $\vec{x} \in \mathbb{R}^n$

vector norms on $\mathbb{R}^n$

$$\|\vec{x}\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$$

$$\|\vec{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

$$\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$$

$$\|\vec{x}\|_\infty = \max_{1 \le k \le n} |x_k|$$

$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ -2 \\ -7.3 \\ 4 \end{pmatrix} \quad \|\vec{x}\|_\infty = 7.3}$$

How do we measure the size of a matrix?

One simple way $\|A\|_F = \left(\sum_{j=1}^{n}\sum_{k=1}^{n} |a_{jk}|^2\right)^{1/2}$

Frobenius norm

Matrix norms induced by vector norms

Given $\|\cdot\|$ on $\mathbb{R}^n$ (say $\|\cdot\|_2$, $\|\cdot\|_\infty$), define

$$\|A\| = \max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \max_{\|\vec{x}\|=1} \|A\vec{x}\|$$

For example

$$\|A\|_\infty = \max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty} = \max_{\|\vec{x}\|_\infty = 1} \|A\vec{x}\|_\infty$$

Key point So defined, an induced matrix norm is consistent with its vector norm.

For any nonzero $\vec{w} \in \mathbb{R}^n$

$$\frac{\|A\vec{w}\|}{\|\vec{w}\|} \leq \max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \|A\| \Rightarrow \boxed{\|A\vec{w}\| \leq \|A\| \, \|\vec{w}\|}$$

Why $\|A\|_\infty$? Turns out $\|A\|_2$, for example, difficult to compute. However,

THM As defined $\|A\|_\infty$ is equivalent to

$$\|A\|_\infty = \max_{1 \le j \le n} \sum_{k=1}^{n} |a_{jk}| \quad \text{max row sum in absolute value}$$

$$A = \begin{pmatrix} 1 & 2.1 & -3 \\ 0.1 & 4 & 0 \\ -7 & 0.2 & 1 \end{pmatrix} \quad \text{row sums} \begin{cases} 6.1 \\ 4.1 \\ 8.2 \end{cases} \qquad \|A\|_\infty = 8.2$$

From now on, assume all $\|\cdot\|$ are $\|\cdot\|_\infty$ (don't want to write subscript ∞ every time).

suppose $\vec{x}_A$ is "approximate" solution to $A\vec{x}=\vec{b}$

residual: $\vec{r} = \vec{b} - A\vec{x}_A$

backward error: $\|\vec{r}\|$, forward error: $\|\vec{x}_A - \vec{x}\|$

$$\text{error magnification} = \frac{\|\vec{x}_A - \vec{x}\| / \|\vec{x}\|}{\|\vec{r}\| / \|\vec{b}\|}$$

$$= \frac{\|\vec{x}_A - \vec{x}\|}{\|\vec{x}\|} \frac{\|\vec{b}\|}{\|\vec{r}\|}$$

EXAMPLE
$$\begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Declare $\vec{x}_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \|\vec{r}\| = 3$$

$$\|\vec{x}_A - \vec{x}\| = \left\| \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\|_\infty = 1$$

$$\text{error mag} = \frac{\|\vec{x}_A - \vec{x}\|}{\|\vec{x}\|} \frac{\|\vec{b}\|}{\|\vec{r}\|} = \frac{1}{2} \frac{3}{3} = \frac{1}{2}$$

Condition # and error magnification

$$\text{error mag} = \frac{\|\vec{x}_A - \vec{x}\|}{\|\vec{x}\|} \frac{\|\vec{b}\|}{\|\vec{r}\|}$$

$$\boxed{\text{condition \#}: \ \|A\| \cdot \|A^{-1}\|}$$

for problem solve $A\vec{x} = \vec{b}$

$\kappa(A)$ here $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$

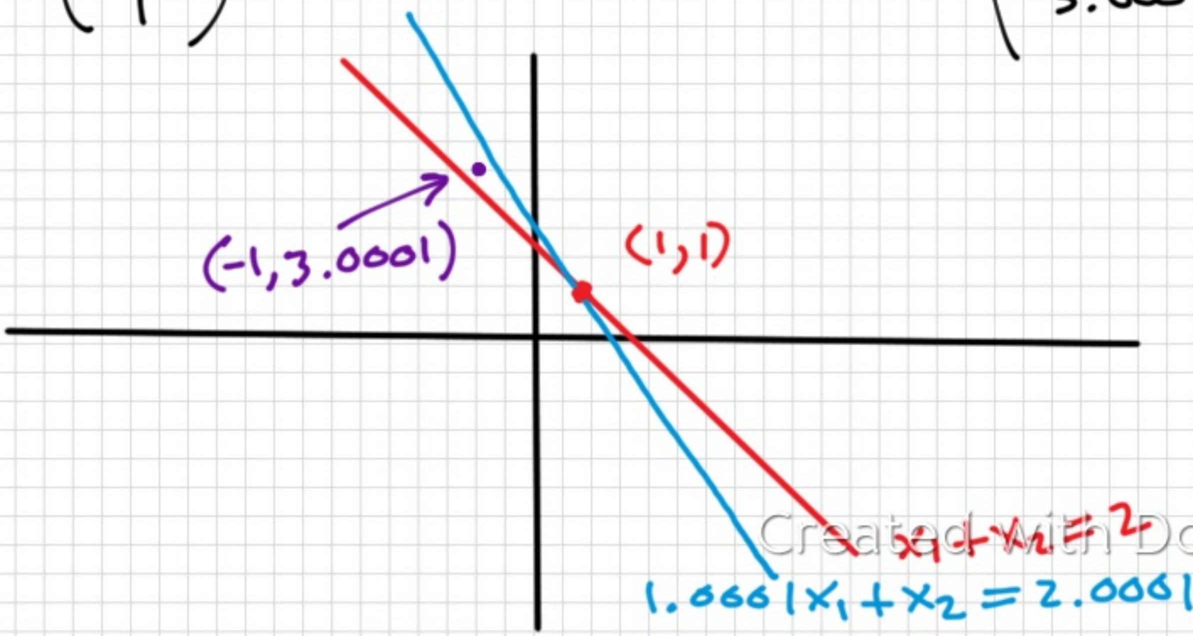
Cond # is largest possible error magnification over all right-hand sides $\vec{b}$.

* goal: show this (or at least motivate)

Example:

$$\begin{pmatrix} 1 & 1 \\ 1.0001 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2.0001 \end{pmatrix}$$

$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and declare $\vec{x}_A = \begin{pmatrix} -1 \\ 3.0001 \end{pmatrix}$



$$\vec{r} = \begin{pmatrix} 2 \\ 2.0001 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1.0001 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3.0001 \end{pmatrix}$$

$$= \begin{pmatrix} -0.0001 \\ 0.0001 \end{pmatrix}$$

$$\boxed{\frac{\|\vec{b}\|}{\|\vec{r}\|} = \frac{2.0001}{10^{-4}} = 20001}$$

$$\|\vec{x}_A - \vec{x}\| = \left\| \begin{pmatrix} -2 \\ 2.0001 \end{pmatrix} \right\| = 2.0001$$

$$\text{error mag} = \frac{\|\vec{x}_A - \vec{x}\|}{\|\vec{x}\|} \, \frac{\|\vec{b}\|}{\|\vec{r}\|} = 40004.0001$$

$$\vec{A} = \begin{pmatrix} 1 & 1 \\ 1.0001 & 1 \end{pmatrix} \qquad \|A\| = 2.0001$$

$$\vec{A}^{-1} = -10^4 \begin{pmatrix} 1 & -1 \\ -1.0001 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -10000 & 10000 \\ 10001 & -10000 \end{pmatrix} \qquad \|A^{-1}\| = 20001$$

$$\|A\|\,\|A^{-1}\| = 40004.0001 \qquad \underline{\text{cond \#}}$$

$$\boxed{\text{if instead } \vec{x}_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ error Mag} = 2.0001}$$

Analysis try to view $\vec{x}_A$ as exact solution to perturbed system $(A+\delta A)\vec{x}_A = \vec{b} + \delta\vec{b}$

Always possible. For example $\delta A = 0$

$$\delta\vec{b} = \underbrace{\vec{b}_A}_{A\vec{x}_A} - \vec{b}$$

let $\vec{x}_A = \vec{x} + \delta\vec{x}$

$$A(\vec{x} + \delta\vec{x}) = \vec{b} + \delta\vec{b}$$

$$A\,\delta\vec{x} = \delta\vec{b} \quad \text{so} \quad \delta\vec{x} = A^{-1}\,\delta\vec{b}$$

(note $\delta\vec{x} = \vec{x}_A - \vec{x}$ and $\delta\vec{b} = \vec{b}_A - \vec{b}$)

$$\delta\vec{x} = A^{-1}\,\delta\vec{b}$$

$$\|\delta\vec{x}\| \le \|A^{-1}\|\,\|\delta\vec{b}\|$$

$$\boxed{\begin{aligned}\delta\vec{x} &= \vec{x}_A - \vec{x}\\ \delta\vec{b} &= -\vec{r}\end{aligned}}$$

$$\frac{\|\delta\vec{x}\|}{\|\vec{x}\|} \le \frac{\|A^{-1}\|}{\|\vec{x}\|}\,\|\vec{b}\|\,\frac{\|\delta\vec{b}\|}{\|\vec{b}\|}$$

but $\|\vec{b}\| = \|A\vec{x}\| \le \|A\|\,\|\vec{x}\|$

$$\le \|A\|\,\|A^{-1}\|\,\frac{\|\delta\vec{b}\|}{\|\vec{b}\|}$$

$$\text{error mag} = \frac{\|\delta\vec{x}\|}{\|\vec{x}\|}\Big/\frac{\|\delta\vec{b}\|}{\|\vec{b}\|} \le \|A\|\,\|A^{-1}\| = \kappa(A)$$

Practical issues

solve $A\vec{x} = \vec{b}$ numerically to get $\vec{x}_A$

Typically $\dfrac{\|\vec{r}\|}{\|\vec{b}\|} \simeq \varepsilon_{mach}$

so $\dfrac{\|\vec{x}_A - \vec{x}\|}{\|\vec{x}\|} \leq \kappa(A) \dfrac{\|\vec{r}\|}{\|\vec{b}\|} \lesssim \kappa(A)\, \varepsilon_{mach}$

What if $\kappa(A) = 10^p$?

$$\kappa(A)\,\varepsilon_{mach} \simeq 10^{p-16}$$

ex $p = 16$, $\kappa(A) = 10^{10}$

$$\frac{\|\vec{x}_A - \vec{x}\|}{\|\vec{x}\|} \lesssim 10^{-6}$$

only about 6 digits of accuracy in relative sense.