QR factorization: Givens Rotations

linalg 5

$$Q(\alpha,\beta) = \frac{1}{\sqrt{\alpha^2+\beta^2}} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

$$Q^T(\alpha,\beta)\, Q(\alpha,\beta) = \frac{1}{\alpha^2+\beta^2} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

$$= \frac{1}{\alpha^2+\beta^2} \begin{pmatrix} \alpha^2+\beta^2 & 0 \\ 0 & \alpha^2+\beta^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

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$$Q(\alpha,\beta)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{\alpha^2+\beta^2}}\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{\alpha^2+\beta^2}}\begin{pmatrix} \alpha x_1 + \beta x_2 \\ -\beta x_1 + \alpha x_2 \end{pmatrix}$$

$\therefore$

$$Q(x_1,x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{x_1^2+x_2^2}}\begin{pmatrix} x_1^2 + x_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{x_1^2+x_2^2} \\ 0 \end{pmatrix}$$

<u>NOTE</u>: $\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \sqrt{x_1^2+x_2^2} = \left\| \begin{pmatrix} \sqrt{x_1^2+x_2^2} \\ 0 \end{pmatrix} \right\|$

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$$V = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \underset{\uparrow \text{ find}}{=:} QR$$

Technique: Apply orthogonal matrices $G_1, G_2, G_3, \dots$ to V until it becomes upper Δ

$$\underbrace{G_6 G_5 G_4 G_3 G_2 G_1}_{Q^T} V = R$$

Then $V = QR$.

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$$V = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ \boxed{1} & 2 & 4 \end{pmatrix}$$

want to zero out entry in red box. Note:

$$\Phi(1,1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{\Phi(1,1)}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

$$G_1 V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ \sqrt{2} & \frac{3}{\sqrt{2}} & \frac{5}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$$

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$$G_1 V = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ \boxed{\sqrt{2}} & \frac{3}{\sqrt{2}} & \frac{5}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$$

Want to zero out entry in red box. Note:

$$Q(1,\sqrt{2})\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}$$

$$G_2 G_1 V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ \sqrt{2} & \frac{3}{\sqrt{2}} & \frac{5}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ \sqrt{3} & \sqrt{3} & \frac{5}{\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{5}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$$

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$$G_2G_1V = \begin{pmatrix} 1 & -1 & 1 \\ \boxed{\sqrt{3}} & \sqrt{3} & \frac{5}{\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{5}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{6}} \end{pmatrix}$$

Want to zero out entry in red box. Notice

$$\Phi(1,\sqrt{3})\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$G_3G_2G_1V = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ \sqrt{3} & \sqrt{3} & \frac{5}{\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{5}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{5}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$$

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$$G_3 G_2 G_1 V = \begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{5}{\sqrt{6}} \\ 0 & \boxed{\frac{1}{\sqrt{2}}} & \frac{3}{\sqrt{2}} \end{pmatrix}$$

Want to zero out entry in red box. Notice:

$$\Phi\left(\sqrt{\tfrac{3}{2}}, \sqrt{\tfrac{1}{2}}\right)\begin{pmatrix} \sqrt{\frac{3}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{3}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{3}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$$

$$G_4 G_3 G_2 G_1 V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{4}} & \sqrt{\frac{1}{4}} \\ 0 & 0 & -\sqrt{\frac{1}{4}} & \sqrt{\frac{3}{4}} \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{5}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{2} & \sqrt{8} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$$

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$$G_4G_3G_2G_1V = \begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \boxed{\sqrt{2}} & 2\sqrt{2} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Want to zero out entry in red box. Notice:

$$\varphi(\sqrt{3},\sqrt{2})\begin{pmatrix} \sqrt{3} \\ \sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} \sqrt{3} & \sqrt{2} \\ -\sqrt{2} & \sqrt{3} \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix}$$

$$G_5G_4G_3G_2G_1V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} & 0 \\ 0 & -\sqrt{\frac{2}{5}} & \sqrt{\frac{3}{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{2} & 2\sqrt{2} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & \sqrt{\frac{10}{3}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$$

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$$G_5G_4G_3G_2G_1V = \begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & \sqrt{\frac{10}{3}} \\ 0 & 0 & \boxed{\sqrt{\frac{2}{3}}} \end{pmatrix}$$

Want to zero out entry in red box. Notice

$$\Phi\left(\sqrt{\tfrac{10}{3}}, \sqrt{\tfrac{2}{3}}\right)\begin{pmatrix} \sqrt{\frac{10}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{10}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{10}{3}} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{10}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$G_6G_5G_4G_3G_2G_1V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{5}{6}} & \sqrt{\frac{1}{6}} \\ 0 & 0 & -\sqrt{\frac{1}{6}} & \sqrt{\frac{5}{6}} \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & \sqrt{\frac{10}{3}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}$$

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Now form

$$G_6 G_5 G_4 G_3 G_2 G_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} & -\frac{3}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} \end{pmatrix}$$

this is Q^T

$$Q^T V = R = \begin{pmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow V = QR$$

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