interp1

Interpolation

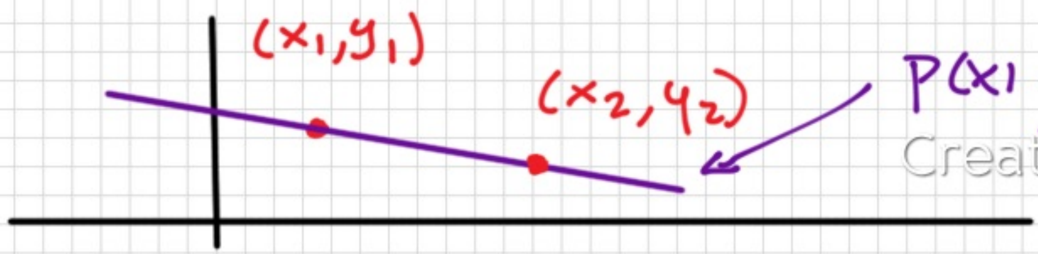
Problem: given data $\mathcal{D}_N = \{(x_i, y_i)\}_{i=1}^{N}$

basis $\mathcal{B}_N = \{\phi_k(x)\}_{k=1}^{N}$

find $p(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_N\phi_N(x)$

s.t. $y_i = p(x_i)$ for all i.

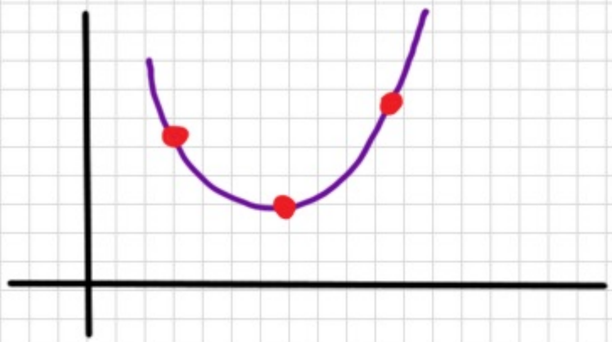
Ex $\mathcal{D}_2 = \{(x_1, y_1), (x_2, y_2)\}$, $\mathcal{B}_2 = \{1, x\}$

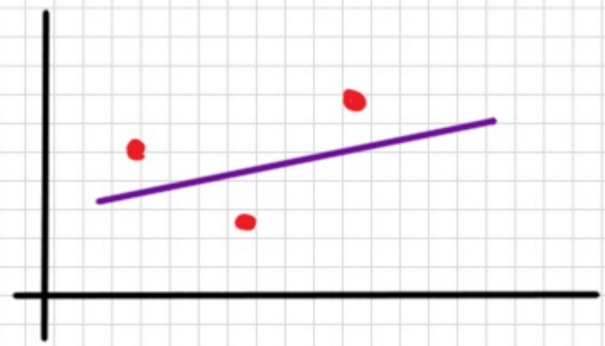


$p(x) = c_1\phi_1(x) + c_2\phi_2(x)$

$= c_1 + c_2 x$

Ex $\mathcal{D}_3 = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$





$$p(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) = c_1 + c_2 x + c_3 x^2$$

$\boxed{\text{interpolation}}$

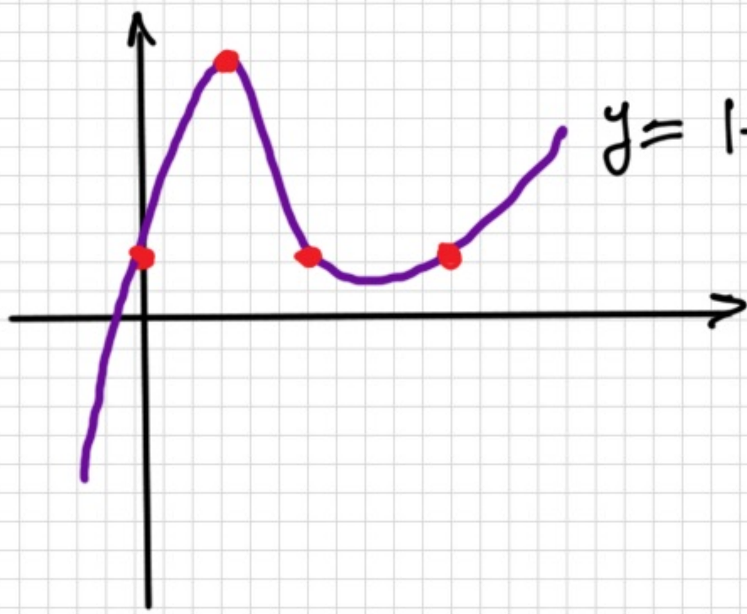
focus for now

$$p(x) = \alpha_1\phi_1(x) + \alpha_2\phi_2(x) = \alpha_1 + \alpha_2 x$$

$\boxed{\text{least squares}}$

Ex $\mathcal{D}_4 = \{(x_i, y_i)\}_{i=1}^{4}$

$= \{(0,1), (1,4), (2,1), (3,1)\}$



$y = 1 + 9x - \frac{15}{2}x^2 + \frac{3}{2}x^3$

How do we find this polynomial?

Monomial basis $\mathcal{B}_4 = \{\phi_1 = 1, \phi_2 = x, \phi_3 = x^2, \phi_4 = x^3\}$

$$\boxed{\mathcal{B}_N = \{\phi_k(x) = x^{k-1}\}_{k=1}^{N}}$$

$$c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + c_4\phi_4(x) = p(x) = y$$

$$c_1\phi_1(0) + c_2\phi_2(0) + c_3\phi_3(0) + c_4\phi_4(0) = 1$$

$$c_1\phi_1(1) + c_2\phi_2(1) + c_3\phi_3(1) + c_4\phi_4(1) = 4$$

$$c_1\phi_1(2) + c_2\phi_2(2) + c_3\phi_3(2) + c_4\phi_4(2) = 1$$

$$c_1\phi_1(3) + c_2\phi_2(3) + c_3\phi_3(3) + c_4\phi_4(3) = 1$$

First eqn $p(x_1) = y_1$ for $x_1 = 0,\ y_1 = 1$

$$c_1\phi_1(0) + c_2\phi_2(0) + c_3\phi_3(0) + c_4\phi_4(0) = 1$$

$$\begin{pmatrix} \phi_1(0) & \phi_2(0) & \phi_3(0) & \phi_4(0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 1$$

Vandermonde system

$$\begin{pmatrix} \phi_1(0) & \phi_2(0) & \phi_3(0) & \phi_4(0) \\ \phi_1(1) & \phi_2(1) & \phi_3(1) & \phi_4(1) \\ \phi_1(2) & \phi_2(2) & \phi_3(2) & \phi_4(2) \\ \phi_1(3) & \phi_2(3) & \phi_3(3) & \phi_4(3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 1 \end{pmatrix}$$

Solution $\vec{c} = \begin{pmatrix} 1 \\ 9 \\ -15/2 \\ 3/2 \end{pmatrix}$

$$p(x) = 1 + 9x - \frac{15}{2}x^2 + \frac{3}{2}x^3$$

General problem

$$p(x) = \sum_{k=1}^{N} c_k \phi_k(x)$$

Here $B_N = \{\phi_k(x) = x^{k-1}\}_{k=1}^{N}$ but other possibilities. For example

$$B_5 = \{1, \cos x, \sin x, \cos 2x, \sin 2x\}$$

"trigonometric interpolation"

$$\mathcal{D}_N = \{(x_i, y_i)\}_{i=1}^{N}, \quad \mathcal{B}_N = \{\phi_k(x)\}_{k=1}^{N}$$

demand

$$c_1\phi_1(x_i) + c_2\phi_2(x_i) + \cdots + c_N\phi_N(x_i) = p(x_i) = y_i$$

$$\begin{pmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_N(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_N(x_2) \\ \vdots & \vdots & & \vdots \\ \phi_1(x_N) & \phi_2(x_N) & \cdots & \phi_N(x_N) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

Vandermonde system $V\vec{c} = \vec{y}$