Solving nonlinear equations: fixed point iteration $\boxed{\text{root2}}$

Problem find a root of $f(x)=0$.

often $x=f(x)$!

Try writing equation as $x=h(x)$. Then given an initial iterate x_0 set

$$x_1 = h(x_0)$$
$$x_2 = h(x_1)$$
$$x_3 = h(x_2)$$

etc

Ex $\underbrace{\sqrt{x+1} - \cos x}_{f(x)} = 0$

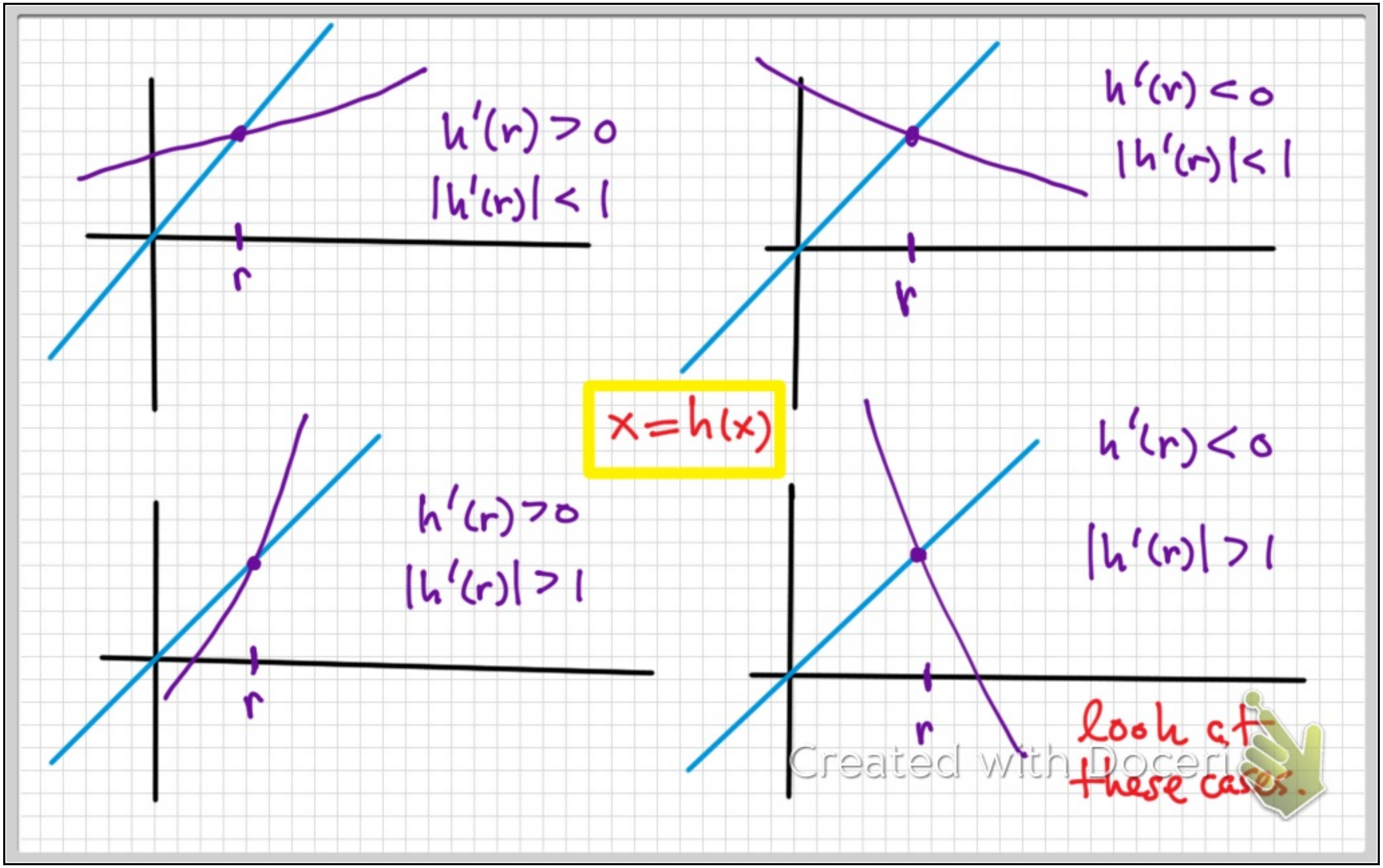
① $x = -1 + \cos^2 x$

② $x = \arccos(\sqrt{x+1})$

When will it work? ($x = f(x)$ in PDF notes)

THM Assume that $h(x)$ is continuously differentiable in a nbhd of r, that $r = h(r)$, and that $|h'(r)| < 1$. Then FPI converges to r for x_0 chosen sufficiently close to r.

Note: we assume existence of root r.
THM gives criteria that FPI will CVG.
Look at proof later. Pictures for now


h'(r) > 0
|h'(r)| < 1
r
h'(r) < 0
|h'(r)| < 1
r
x = h(x)
h'(r) > 0
|h'(r)| > 1
r
h'(r) < 0
|h'(r)| > 1
r
look at these cases.
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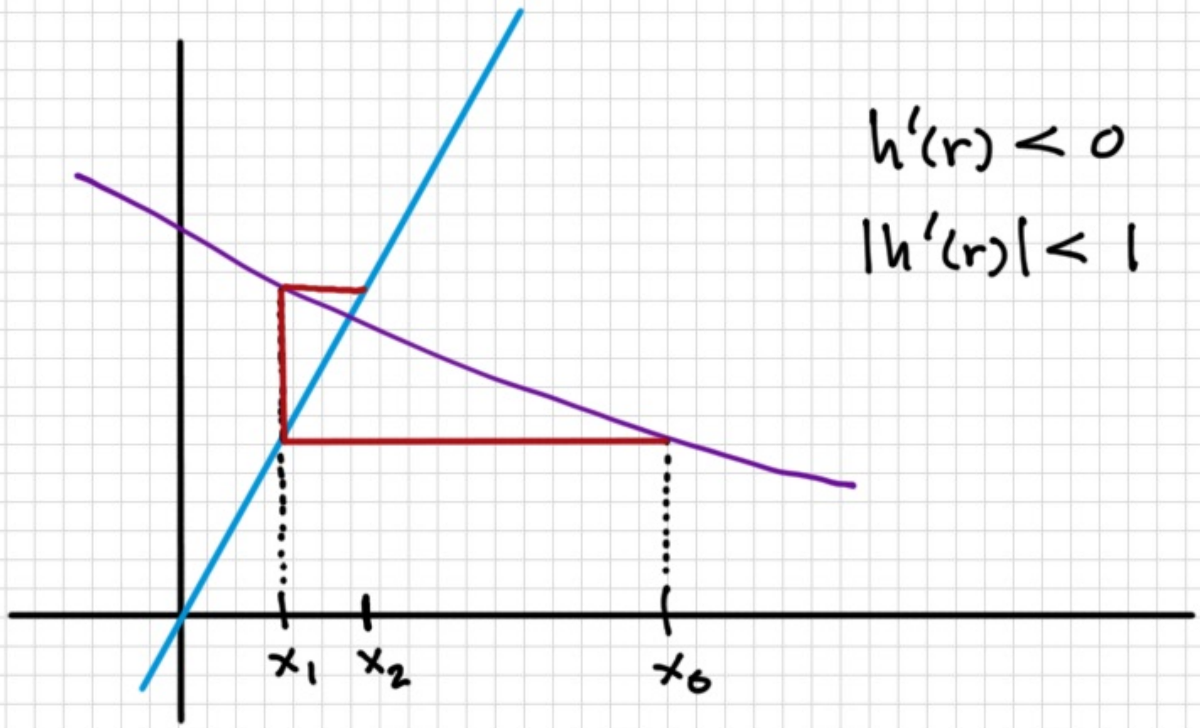


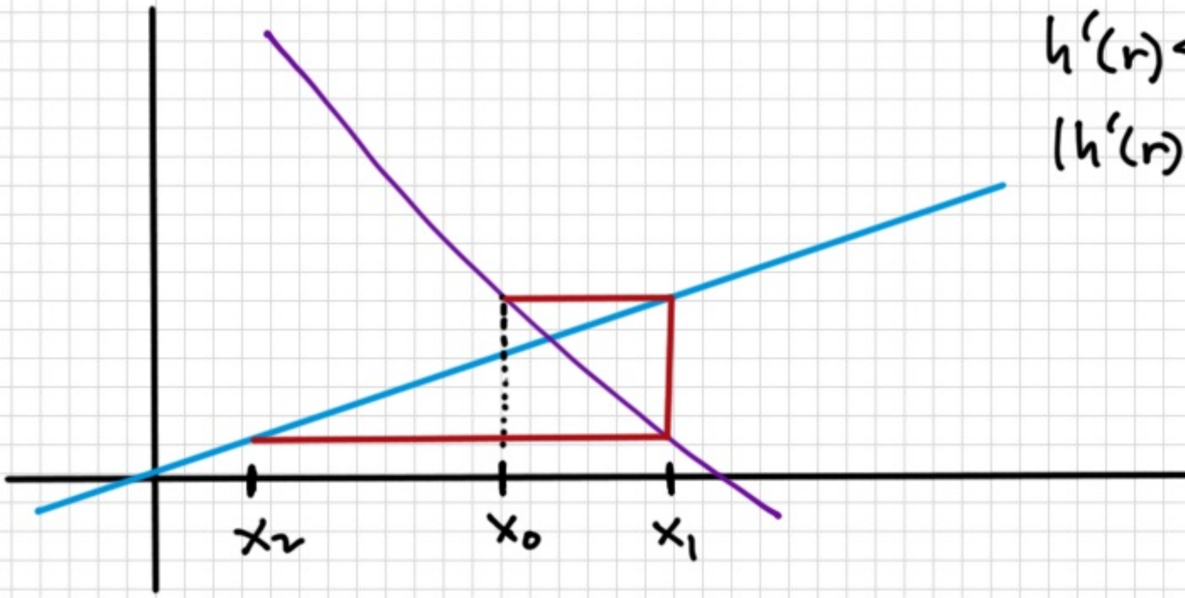
$h'(r) < 0$

$|h'(r)| < 1$

Cobweb diagram, here showing $\lim_{k\to\infty} x_k = r$
(CVG)

$h'(r) < 0$

$|h'(r)| > 1$



here $\lim_{k \to \infty} x_k \neq r$

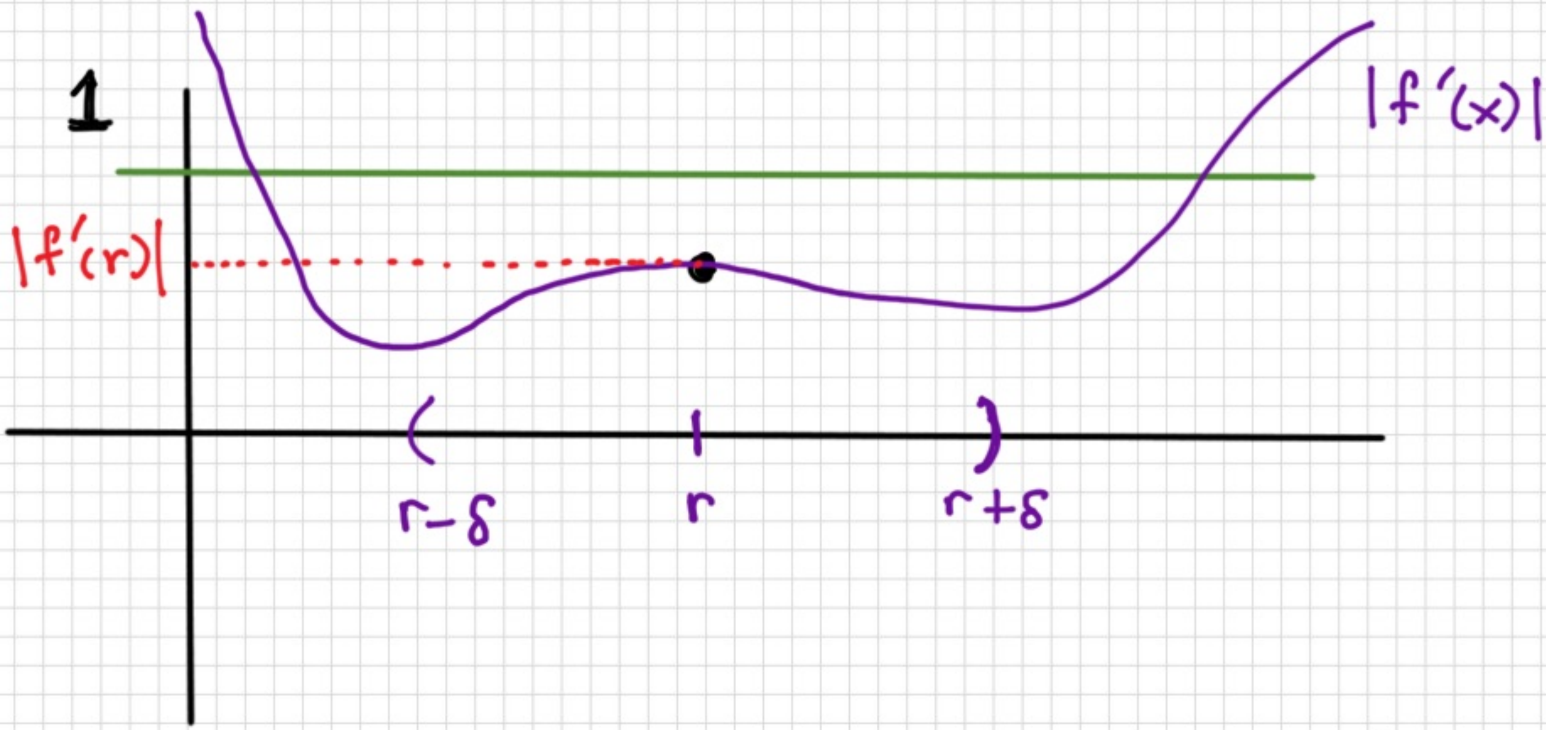
other two cases similar

FPI: $x_{k+1} = f(x_k)$ for $k = 0, 1, 2, \dots$

THM Assume $f(x)$ is continuously differentiable on a nbhd of r, that $r = f(r)$, and that $|f'(r)| < 1$. Then FPI converges to r, provided x_0 is chosen close enough to r.

THM Assume $f(x)$ is continuously differentiable on $I = (r-\delta, r+\delta)$ and $|f'(x)| < Q < 1$ on I, and assume $r = f(r)$. Then FPI cvgs if $x_0 \in I$

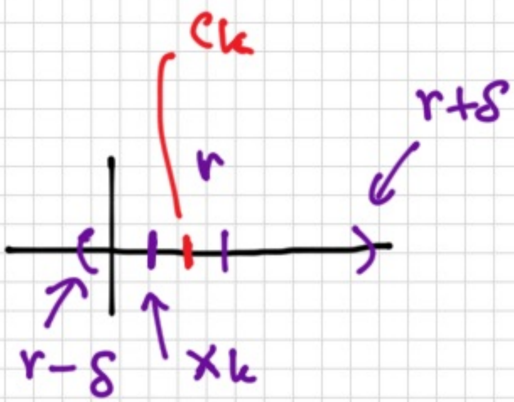
don't lose anything w/ 2^{nd} thm.



If $|f'(x)|$ is continuous, can always find small window $(r-\delta, r+\delta)$ on which $|f'(x)| < Q < 1$ if $|f'(r)| < 1$.

Proof:

$$x_{k+1} - r = f(x_k) - f(r)$$



$$= \frac{f(x_k) - f(r)}{x_k - r}\,(x_k - r)$$

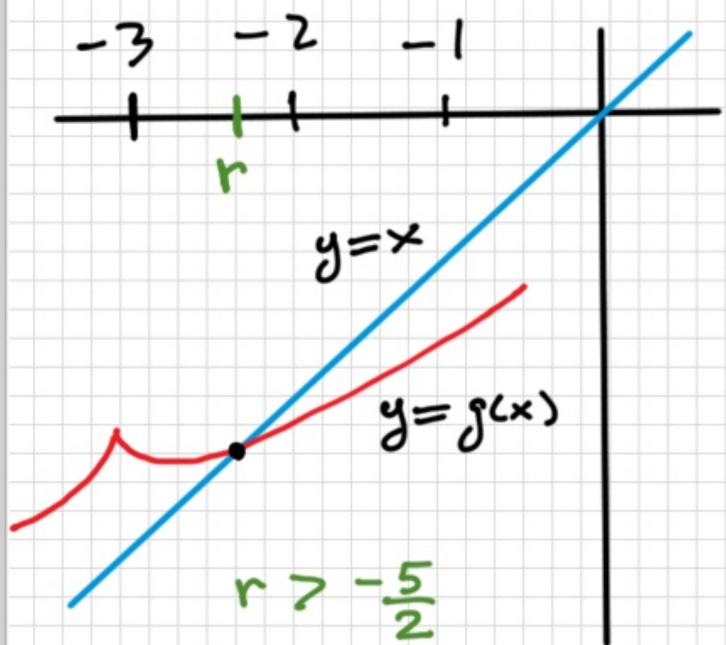
$$= f'(c_k)\,(x_k - r)$$

for some c_k between x_k and r

so $|x_{k+1} - r| = |f'(c_k)|\,|x_k - r| < Q\,|x_k - r|$

$\Rightarrow |x_k - r| < Q^k\,|x_0 - r| \longrightarrow 0$ as $k \to \infty$.

EX $\quad x = \underbrace{\frac{1}{8}\sin x - \frac{1}{2}\sqrt{|x+3|} + \frac{3}{4}x}_{g(x)}$



From graph we can see there is a root.

Will FPI converge?

① can just try it.

② analyze as follows

$$g(x) = \frac{1}{8}\sin x - \frac{1}{2}\sqrt{|x+3|} + \frac{3}{4}x$$

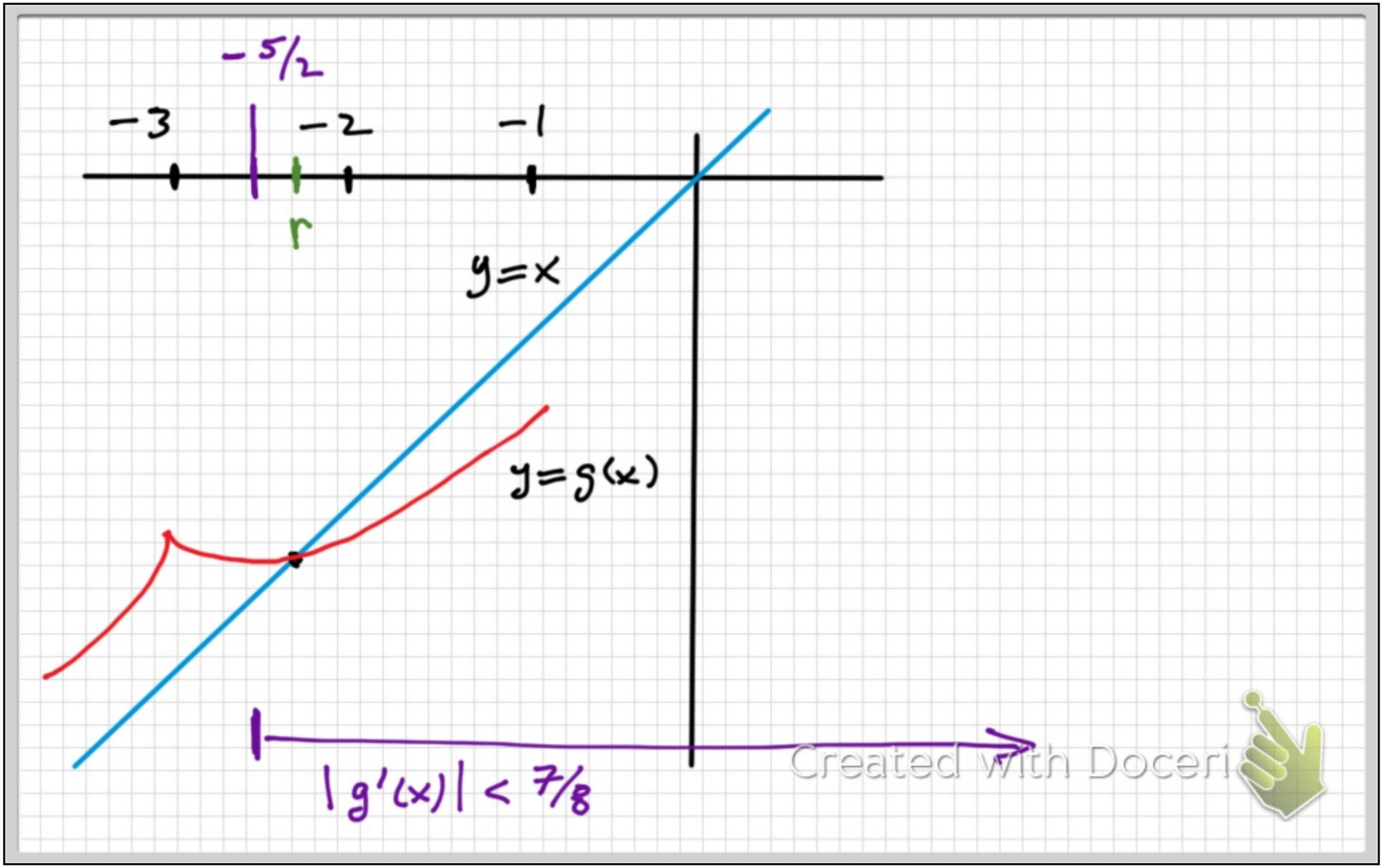
$$g(x) = \frac{1}{8}\sin x - \frac{1}{2}\sqrt{x+3} + \frac{3}{4}x \quad \text{for } x > -\frac{5}{2}$$

$$g'(x) = \frac{1}{8}\cos x - \frac{1}{4\sqrt{x+3}} + \frac{3}{4}$$

$$|g'(x)| \le \frac{1}{8} + \frac{1}{4}\left|3 - \frac{1}{\sqrt{x+3}}\right| < \frac{1}{8} + \frac{3}{4} = \frac{7}{8} < 1$$

$$x > -\frac{5}{2} \Rightarrow x+3 > \frac{1}{2} \Rightarrow 0 < \frac{1}{x+3} < 2$$

$$\Rightarrow -\sqrt{2} < -\frac{1}{\sqrt{x+3}} < 0 \Rightarrow 3-\sqrt{2} < 3 - \frac{1}{\sqrt{x+3}} < 3$$


$-5/2$
-3
-2
-1
r
$y=x$
$y=g(x)$
$|g'(x)| < 7/8$
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