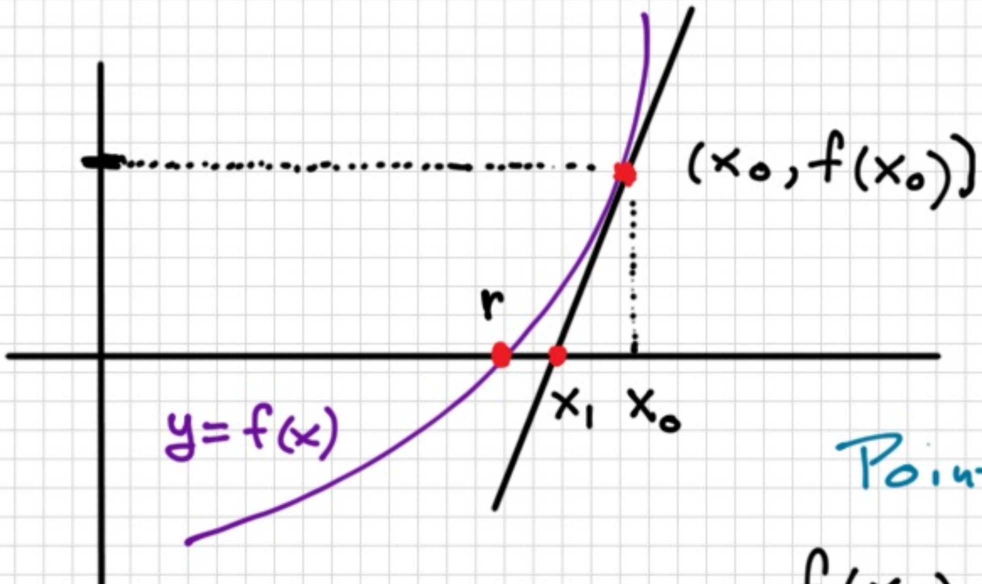
Newton's method for root-finding

root3

Problem
Solve $f(x)=0$



Point-slope form

$$\frac{f(x_0) - \overset{0}{y_1}}{x_0 - x_1} = f'(x_0)$$

EQN of tangent line

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$ ← general formula

Algorithm

handles for $f(x)$ and $f'(x)$

```
[x, k] = newton (f, df, x0, tol, kmax)

    x = x0 ; err = 100 ; k = 0
    while err >= tol
        y = f(x0)
        x = x0 - y/f'(x0)
        err = max (abs(y), abs (x-x0))
        x0 = x
        k = k+1 ; if k >= kmax, return
    end
```

In while loop

$$y = f(x_0)$$

$$x = x_0 - y/f'(x_0)$$

$$err = max(abs(y), abs(x-x_0))$$

backward error

proxy for forward error

berr ≤ ferr   ferr ≤ berr

Could have used similar in fixed-point iteration

$$err = max(abs(x_0 - y), abs(x - x_0))$$

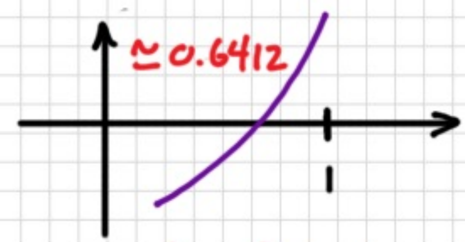
not in Narration really same thing for FPI

$x = f(x)$ not $f(x) = 0$

Examples

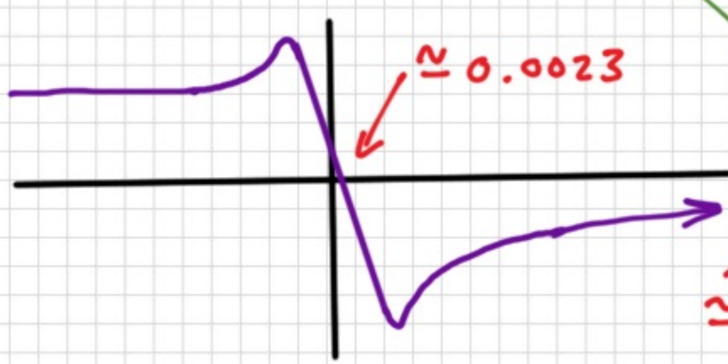
$$f(x) = x\,2^x - 1$$

$$f'(x) = 2^x + x\,2^x \ln 2$$



$$f(x) = \frac{1}{2} - \frac{x}{|x|^{1.1} + \frac{1}{300}}$$

at first sight may not look differentiable at $x=0$. But ok!



≃ 1023.98

$$f'(x) = \frac{0.1\,|x|^{1.1} - \frac{1}{300}}{\left(|x|^{1.1} + \frac{1}{300}\right)^2}$$

Convergence (CVG)

Compare fixed-point to Newton. Consider

$$f(x) = 0 \quad \text{or} \quad \underbrace{x - g(x)}_{f(x)} = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = g(x_n)$$

How fast do these methods CVG?

Fixed point iteration

$$x_{n+1} = g(x_n)$$

$$r = g(r)$$

$$x_{n+1} - r = \frac{g(x_n) - g(r)}{x_n - r}(x_n - r)$$

$$\simeq g'(r)(x_n - r)$$

$$e_{n+1} \simeq \overbrace{|g'(r)|}^{S} e_n, \text{ where } e_n = |x_n - r|$$

$$\boxed{e_{n+1} \simeq S e_n}$$

Newton's method

$$x_{n+1} = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{H(x_n)}$$

Where $H(x) = x - \dfrac{f(x)}{f'(x)}$. Notice

$$H'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{[f'(x)]^2}$$

$$\boxed{H'(r) = 0}$$

$\leftarrow$ assume smooth f and $f'(r) \neq 0$.

Closer look: write $r = x_n + (r - x_n)$

$$0 = f(x_n + (r - x_n))$$

$$= f(x_n) + f'(x_n)(r - x_n) + \frac{1}{2} f''(\xi)(r - x_n)^2$$

where ξ lies between x_n and $\underbrace{x_n + (r - x_n)}_{r}$,

Taylor series w/ remainder

$\Longrightarrow$ NOW REARRANGE

$$x_n - \frac{f(x_n)}{f'(x_n)} - r = \frac{1}{2} \frac{f''(\xi)}{f'(x_n)} (r - x_n)^2$$

From last slide

$$x_n - \frac{f(x_n)}{f'(x_n)} - r = \frac{1}{2} \frac{f''(\xi)}{f'(x_n)} (r - x_n)^2$$

$$x_{n+1} - r = \frac{1}{2} \frac{f''(\xi)}{f'(x_n)} (r - x_n)^2$$

$$\boxed{e_{n+1} \simeq M e_n^2, \text{ where } M = \frac{1}{2} \frac{f''(r)}{f'(r)}}$$