Root Finding for systems

$\boxed{\text{root4}}$

Ex $\vec{f}(\vec{u}) = \begin{pmatrix} -u^3 + v \\ u^2 + v^2 - 1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

Solve $\vec{f}(\vec{u}) = \vec{0}$

$$\boxed{\begin{aligned} -u^3 + v &= 0 \\ u^2 + v^2 - 1 &= 0 \end{aligned}}$$

$$-u^3 + v = 0$$
$$u^2 + v^2 - 1 = 0$$

$$u = v^{1/3}$$
$$u = \frac{1}{2}u + \frac{1}{2}v^{1/3}$$

fixed point iteration

Method ①

$$v_{n+1} = u_n^3$$
$$u_{n+1} = \sqrt{1 - v_n^2}$$

$$\boxed{\text{DOES NOT WORK}}$$

method ②

$$u_{n+1} = \frac{1}{2}u_n + \frac{1}{2}v_n^{1/3}$$
$$v_{n+1} = \sqrt{1 - u_n^2}$$

$$\boxed{\text{WORKS}}$$

$$-u^3 + v = 0 \qquad \vec{f}(\vec{u}) = \vec{0}, \quad \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$$
$$u^2 + v^2 - 1 = 0$$

$$D\vec{f}(\vec{u}) = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} -3u^2 & 1 \\ 2u & 2v \end{pmatrix}$$

Scalar Newton: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

System Newton $\vec{u}_{n+1} = \vec{u}_n - \left[D\vec{f}(\vec{u}_n)\right]^{-1} \vec{f}(\vec{u}_n)$

$$\vec{u}_{n+1} = \vec{u}_n - \left[D\vec{f}(\vec{u}_n)\right]^{-1} \vec{f}(\vec{u}_n)$$

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \begin{pmatrix} -3u_n^2 & 1 \\ 2u_n & 2v_n \end{pmatrix}^{-1} \begin{pmatrix} -u_n^3 + v_n \\ u_n^2 + v_n^2 - 1 \end{pmatrix}$$

---

solve $\begin{pmatrix} -3u_n^2 & 1 \\ 2u_n & 2v_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -u_n^3 + v_n \\ u_n^2 + v_n^2 - 1 \end{pmatrix}$

update $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} u_n \\ v_n \end{pmatrix} - \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$