interp 4

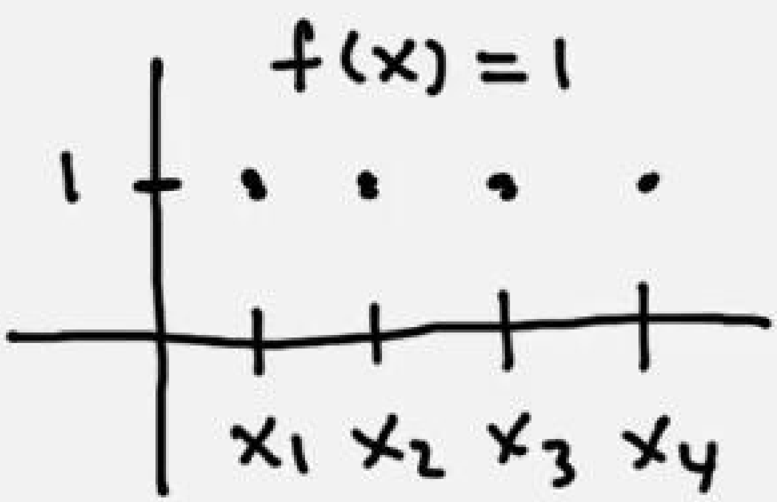
Error formula

$y = f(x)$

data $D_N = \{ \underbrace{(x_1, y_1)}_{f(x_1)}, \underbrace{(x_2, y_2)}_{f(x_2)}, \dots, \underbrace{(x_N, y_N)}_{f(x_N)} \}$

Polynomial which interpolates data: $p(x)$

Has degree $N-1$ or less



$$\begin{array}{c|ccccccc}
x_1 & 1 & & & & & & \\
 & & > & 0 & & & & \\
x_2 & 1 & & & > & 0 & & \\
 & & > & 0 & & & > & 0 \\
x_3 & 1 & & & > & 0 & & \\
 & & > & 0 & & & & \\
x_4 & 1 & & & & & &
\end{array}$$

$p(x) = 1$

Notation

$P_{N-1}(x)$

vs

$P_N(x)$

Error formula

$$f(x) - p(x) = \underbrace{(x-x_1)(x-x_2)\cdots(x-x_N)}_{\Psi(x)}\,\frac{f^{(N)}(c(x))}{N!}$$

Here assume $x_1 < x_2 < \cdots < x_N$

Evaluation point $x \in [x_1, x_N]$

also $c(x) \in [x_1, x_N]$

$$\frac{f^{(N)}(c)}{N!}$$

* Formula looks right!

$$f(x_1) - p(x_1) = 0$$

* $c(x)$ cannot generally be a constant

$$f(x) = p(x) + K\,\Psi(x)$$

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Error formula

$$\circledast \quad f(x)-p(x) = (x-x_1)(x-x_2)\cdots(x-x_N)\frac{f^{(N)}(c(x))}{N!}$$

More useful result

$$|f(x)-p(x)| \le |(x-x_1)(x-x_2)\cdots(x-x_N)|\frac{M_N}{N!}$$

$$\boxed{M_N = \max_{x_1 \le \xi \le x_N} |f^{(N)}(\xi)|}$$

Proof of $\circledast$ in PDF notes. Relies on

$f[x_1] = f(x_1)$ $\quad$ $f[x_1,x_2,x_3] = f''(\eta)/2!$

$f[x_1,x_2] = f'(\xi)$ $\quad$ $f[x_1,x_2,\ldots,x_N,t] = f^{(N)}(c)/N!$

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Problem Construct poly which interpolates $f(x) = \frac{1}{x+4}$ at $x_1 = 0$, $x_2 = 2$, $x_3 = 4$. Then estimate error in using $p(3)$ as an approx for $f(3) = 1/7$.

$$
\begin{array}{c|cccc}
0 & \frac{1}{4} & & & \\
 & & -\frac{1}{24} & & \\
2 & \frac{1}{6} & & \frac{1}{192} & \\
 & & -\frac{1}{48} & & \\
4 & \frac{1}{8} & & &
\end{array}
$$

$$p(x) = \frac{1}{4} - \frac{1}{24}x + \frac{1}{192}x(x-2)$$

$$p(3) = \frac{9}{64}$$

$$|f(3) - p(3)| \le |(3-0)(3-2)(3-4)| \frac{M_3}{3!} = \frac{M_3}{2}$$

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$$f(x) = \frac{1}{x+4}$$

$$f'(x) = -\frac{1}{(x+4)^2}$$

$$f''(x) = \frac{2}{(x+4)^3}$$

$$f'''(x) = -\frac{6}{(x+4)^4}$$

$$|f'''(x)| = \frac{6}{(x+4)^4}$$

Observe on $[x_1, x_3] = [0,4]$

$$|f'''(x)| \le |f'''(0)|$$

$$= \frac{6}{4^4}$$

$$= \frac{3}{128} = M_3$$

so

$$|f(3) - p(3)| \le \frac{3}{3!}\,\frac{3}{128}$$

$$= \frac{3}{256} \approx 0.0117$$

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