Lecture `interp4`: Error formula for polynomial interpolation

October 19, 2022

Summary: Derivation of error formula and examples of its use.

References: T. Sauer, *Numerical Analysis*, Section 3.2, first edition, pages 153-159.

Error formula and its derivation

Suppose that we have a data set $D_N = \{(x_i, y_i)\}_{i=1}^N$, where we assume that $y_i = f(x_i)$ for $i = 1, \dots, N$. That is, we assume that the y_i data values stem from evaluation of a known function $f(x)$ on the x_i data points. Further, we assume that f is sufficiently smooth; namely, that its Nth derivative $f^{(N)}(x)$ is itself a continuous function on an open interval which contains $[x_1, x_N]$. For simplification, we also assume $x_1 < x_2 < \cdots < x_{N-1} < x_N$. For many common functions, such as $\sin x$ and e^x, all derivatives are continuous on all of the real line $\mathbb{R}$. The analysis to follow pertains to such functions, among others.

Theorem 1. *Assume the conditions stated in the last paragraph, and let $p(x)$ be the (degree $N-1$ or less) polynomial which interpolates the data set D_N. Then*

$$f(x) - p(x) = (x - x_1)(x - x_2) \cdots (x - x_N) \frac{f^{(N)}(c)}{N!}, \tag{1}$$

where $x \in [x_1, x_N]$ and also $c = c(x) \in [x_1, x_N]$.

Before starting the proof, first notice that the formula "looks right". Indeed, since $p(x)$ interpolates $f(x)$ at the nodal points x_i, we have $f(x_i) - p(x_i) = 0$ for all $i = 1, \dots, N$. The right-hand side of (1) also vanishes at each x_i, since the factor $(x - x_i)$ present in right-hand side will of course vanish when $x = x_i$. The point $c = c(x)$ in general depends on the evaluation point x. Were it a constant (a possibility), then $f(x) = p(x) + (x - x_1)(x - x_2) \cdots (x - x_N) f^{(N)}(c)/N!$ would itself be a polynomial (of degree N or less).

Finally, while (1) is a equality, from it we may derive a more useful inequality. Set

$$M_N = \max_{x_1 \le t \le x_N} |f^{(N)}(t)|.$$

Then, from (1) we obtain

$$|f(x) - p(x)| \le |(x - x_1)(x - x_2) \cdots (x - x_N)| \frac{M_N}{N!}. \tag{2}$$

This error estimate is more useful in practice.

Proof. Now turn to the proof of **Theorem 1**. We start by writing down the Newton formula for the interpolating polynomial $p(x)$,

$$\begin{aligned} p(x) &= \sum_{k=1}^{N} f[x_1,\dots,x_k] \overbrace{\Pi_{i=1}^{k-1}(x-x_i)}^{\text{basis function } \phi_i(x)} \\ &= f(x_1) + f[x_1,x_2](x-x_1) + \cdots + f[x_1,\dots,x_N](x-x_1)\cdots(x-x_{N-1}), \end{aligned} \tag{3}$$

where, as noted before, the last Newton basis function $\phi_N(x)$ does not depend on the last data point x_N (it only depends on x_1 through x_{N-1}). Here is the trick. We now consider the addition of one more data point $(t, f(t))$ to D_N, giving a larger data set $D_{N+1} = \{(x_1, f(x_1)),\dots,(x_N, f(x_N)),(t, f(t))\}$. The trick is that t is essentially arbitrary, except that we assume $t \in [x_1, x_N]$. By the recursive property of the Newton approach, the (degree n or less) polynomial which interpolates this larger data set is

$$q(x) = p(x) + f[x_1,\dots,x_N,t](x-x_1)\cdots(x-x_{N-1})(x-x_N). \tag{4}$$

By construction $q(t) = f(t)$, since $q(t)$ interpolates f at $x = t$. Whence we see that

$$f(t) = p(t) + f[x_1,\dots,x_N,t](t-x_1)\cdots(t-x_{N-1})(t-x_N).$$

In fact, the t here is a "dummy variable"; we could replace all t factors in the last formula with ξ, or s, or, indeed, x. Therefore, replacement of t with x yields

$$f(x) - p(x) = f[x_1,\dots,x_N,x](x-x_1)\cdots(x-x_{N-1})(x-x_N).$$

This is the result (1), provided that we can show $\boxed{f[x_1,\dots,x_N,x] = f^{(N)}(c(x))/N!}$.

To finish the proof, rather than the boxed result we will show $f[x_1,\dots,x_N,t] = f^{(N)}(c(t))/N!$ instead, which is of course the same thing. To this end, refer back to (4) and define

$$h(x) := f(x) - q(x),$$

which, by construction, vanishes at $x_1, x_2, \dots, x_N, t$. Moreover, the Nth derivative $h^{(N)}$ is certainly a continuous function on $[x_1, x_N]$, since we have assumed f is N times continuously differentiable on an open interval *larger* than $[x_1, x_N]$ and, as a polynomial, any derivative of $q(x)$ of any order is a continuous function on all of $\mathbb{R}$ (if q is differentiated more than N times, one gets the zero function which is plenty continuous). Now we repeatedly appeal to Rolle's Theorem (a particular instance of the Mean Value Theorem for Derivatives, a basic result from Calculus): *If a real-valued function g is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $g(a) = g(b)$, then there exists at least one ξ in the open interval (a, b) such that $g'(\xi) = 0$.* Since h has $N + 1$ roots and is continuously differentiable, by Rolle's Theorem, its derivative h' will have N roots. Then h'' will have $N - 1$ roots. By induction, $h^{(N)}$ will have one root, call it $c = c(t)$. That is, $h^{(N)}(c) = 0$, or

$$f^{(N)}(c) = q^{(N)}(c) = p^{(N)}(c) + N!f[x_1,\dots,x_N,t].$$

Since p is a polynomial of degree $\le N - 1$, $p^{(N)}(x) = 0$ and $f^{(N)}(c) = qN!f[x_1,\dots,x_N,t]$. $\square$

Examples

Example 1
Consider the data set $\mathcal{D}_3 = \{(0, \frac{1}{4}), (2, \frac{1}{6}), (4, \frac{1}{8})\}$. **(a)** Construct the divided–difference table associated with $\mathcal{D}_3$, and then write down the associated degree-2 Newton interpolating polynomial $p(x)$. **(b)** Using the formula from **(a)**, estimate the error $|f(3) - p(3)|$ in using $p(3)$ to approximate $f(3)$, where $f(x) = 1/(x+4)$ gives the data $\mathcal{D}_3$ from **(a)**. Compare with the exact error.
Solution. The divided difference table is as follows.

$$
\begin{array}{c|cccc}
0 & \frac{1}{4} & & \\
 & & -\frac{1}{24} & \\
2 & \frac{1}{6} & & \frac{1}{192} \\
 & & -\frac{1}{48} & \\
4 & \frac{1}{8} & &
\end{array}
$$

From the table the interpolating polynomial is

$$p(x) = \tfrac{1}{4} - \tfrac{1}{24}x + \tfrac{1}{192}x(x-2).$$

Therefore, we see that

$$p(3) = \tfrac{1}{4} - \tfrac{3}{24} + \tfrac{3}{192} = \tfrac{9}{64} \simeq \mathtt{0.14062}$$

The exact error between $p(3)$ and $f(3) = 1/(3+4) = \frac{1}{7}$ is $f(3) - p(3) = \frac{1}{7} - \frac{9}{64} = \frac{1}{448}$, and so

$$|f(3) - p(3)| = \tfrac{1}{448} \simeq \mathtt{0.0022321}.$$

The third derivative of f is $f'''(x) = -6/(x+4)^4$, and $M_3 = |f'''(0)| = \frac{3}{128}$ is a bound on $|f'''(x)|$ over $[0,4]$, indeed over $[0,\infty)$. So the estimate (2) becomes

$$|f(3) - p(3)| \le \frac{|(3-0)(3-2)(3-4)|}{3!} \cdot \frac{3}{128} \simeq \mathtt{0.011719}.$$

Note that the estimate for the error is a bit worse than the exact error: $\mathtt{0.011719} > \mathtt{0.0022321}$.

Example 2
(*Model for the sine function key on a digital calculator*). Construct the polynomial p which interpolates $f(x) = \sin x$ at 4 equally spaced points on $[0, \frac{\pi}{2}]$, and then estimate the error in using $p(1)$ as an approximation to $\sin(1)$.
Solution. The data set is $\mathcal{D}_4 = \{(0,0), (\frac{\pi}{6}, \frac{1}{2}), (\frac{\pi}{3}, \frac{\sqrt{3}}{2}), (\frac{\pi}{2}, 1)\}$, with the associated divided difference table as follows.

$$
\begin{array}{c|ccccccc}
0 & 0 & & & & & & \\
 & & \frac{3}{\pi} & & & & & \\
\frac{\pi}{6} & \frac{1}{2} & & \frac{9(\sqrt{3}-2)}{\pi^2} & & & & \\
 & & \frac{3(\sqrt{3}-1)}{\pi} & & \frac{18(5-3\sqrt{3})}{\pi^3} & & & \\
\frac{\pi}{3} & \frac{\sqrt{3}}{2} & & \frac{9(3-2\sqrt{3})}{\pi^2} & & & & \\
 & & \frac{3(2-\sqrt{3})}{\pi} & & & & & \\
\frac{\pi}{2} & 1 & & & & & &
\end{array}
$$

We then read-off the interpolating polynomial from the top of the divided difference table,

$$
\begin{aligned}
p(x) &= 0 \cdot 1 + \frac{3}{\pi}(x-0) + \frac{9(\sqrt{3}-2)}{\pi^2}(x-0)(x-\tfrac{\pi}{6}) + \frac{18(5-3\sqrt{3})}{\pi^3}(x-0)(x-\tfrac{\pi}{6})(x-\tfrac{\pi}{3}) \\
&= \frac{3}{\pi}x + \frac{9(\sqrt{3}-2)}{\pi^2}x(x-\tfrac{\pi}{6}) + \frac{18(5-3\sqrt{3})}{\pi^3}x(x-\tfrac{\pi}{6})(x-\tfrac{\pi}{3}).
\end{aligned}
$$

Next, from this formula we calculate

$$
p(1) = \frac{3}{\pi} + \frac{9(\sqrt{3}-2)}{\pi^2}(1-\tfrac{\pi}{6}) + \frac{18(5-3\sqrt{3})}{\pi^3}(1-\tfrac{\pi}{6})(1-\tfrac{\pi}{3}) \simeq \texttt{0.841086}.
$$

Since the 4th derivative of the sine function is again the sine function, the M_n factor in (2) is $M_4 = 1$. Therefore, the error estimate becomes

$$
|\sin(1) - p(1)| \leq \frac{|(1-0)(1-\frac{\pi}{6})(1-\frac{\pi}{3})(1-\frac{\pi}{2})|}{4!} \simeq \texttt{5.3476e-04}.
$$

So the approximation $p(1)$ is good to better than 3-digits in a relative sense: $|\sin(1) - p(1)|/|\sin(1)| \simeq \texttt{6.3551e-04}$.