Gaussian Quadrature

$\boxed{\text{Quad 4}}$

So far we've focused on (closed) Newton-Cotes rules

$$(b-a)\sum_{k=1}^{m} w_k f(x_k) = Q_{NC(m)} \simeq I = \int_a^b f(x)\,dx$$

Here rule expressed relative to $[0,1]$, but sometimes $[-1,1]$ taken as "standard interval"

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 f(\underbrace{(b-a)s+a}_{x})\,ds$$

$$\int_a^b f(x)\,dx = \frac{(b-a)}{2}\int_{-1}^1 f(\underbrace{\tfrac{1}{2}(b-a)t+\tfrac{1}{2}(b+a)}_{x})\,dt$$

Quadrature rule relative to [0,1]

$$\int_a^b f(x)\,dx \simeq (b-a)\sum_{k=1}^{m} w_k f(x_k)$$

$(b-a)s_k + a$

Quadrature rule relative to [-1,1]

$$\int_a^b f(x)\,dx \simeq \frac{(b-a)}{2}\sum_{k=1}^{m} w_k f(x_k)$$

$\frac{1}{2}(b-a)t_k + \frac{1}{2}(b+a)$

Can base quadrature (Newton-Cotes or Gaussian) on either interval. Here let's use $[-1,1]$. Different from previous notes.

$$x = \frac{1}{2}(b-a)t + \frac{1}{2}(b+a) \quad \text{for } t \in [-1,1]$$

Newton-Cotes:

$$x_k = \frac{1}{2}(b-a)\left(\frac{2k-m-1}{m-1}\right) + \frac{1}{2}(b+a)$$

$$(b-a)\left(\frac{k-1}{m-1}\right) + a$$

goes from -1 to 1 as m goes from m.

3-point NC : $Q_{NC(3)}$

$$\int_{-1}^{1} x^0\,dx = 2 = w_1 + w_2 + w_3$$

$$\int_{-1}^{1} x^1\,dx = 0 = w_1(-1) + w_2(0) + w_3(1)$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = w_1(-1)^2 + w_2(0)^2 + w_3(1)^2$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ \frac{2}{3} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}}$$

integrates x^3 exactly "for free"

3-point Gauss–Legendre : $Q_{GL(3)}$

nonlinear system

$$\int_{-1}^{1} x^0\,dx = 2 = w_1 + w_2 + w_3$$

$$\int_{-1}^{1} x^1\,dx = 0 = w_1 c_1 + w_2 c_2 + w_3 c_3$$

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3} = w_1 c_1^2 + w_2 c_2^2 + w_3 c_3^2$$

$$\int_{-1}^{1} x^3\,dx = 0 = w_1 c_1^3 + w_2 c_2^3 + w_3 c_3^3$$

$$\int_{-1}^{1} x^4\,dx = \frac{2}{5} = w_1 c_1^4 + w_2 c_2^4 + w_3 c_3^4$$

$$\int_{-1}^{1} x^5\,dx = 0 = w_1 c_1^5 + w_2 c_2^4 + w_3 c_3^5$$