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Kinematic Diagram

The following kinematic diagram was created using the Denavit-Hartenberg convention for frame orientation. There are four mandatory rules:

1. The *z-axis* must be the axis of rotation for a revolute joint or the direction of motion for a prismatic joint.
2. The *x-axis* of frame *n* must be perpendicular to the *z-axis* of frame *n-1*.
3. The *x-axis* of frame *n* must intersect the *z-axis* of frame *n-1*
4. The *y-axis* of each frame must be drawn using the right-hand rule - which states that with the thumb extended in the direction of the *z-axis* and fingers extended in the direction of the *x-axis*, the palm will face the direction of the *y-axis*.

Additional general kinematic diagram rules used:

- One frame is required for each joint and an additional frame is required for the end-effector.
- Axes must be drawn either up, down, left, right or in the first or third quadrant. Axes should never be drawn in the second or fourth quadrant.

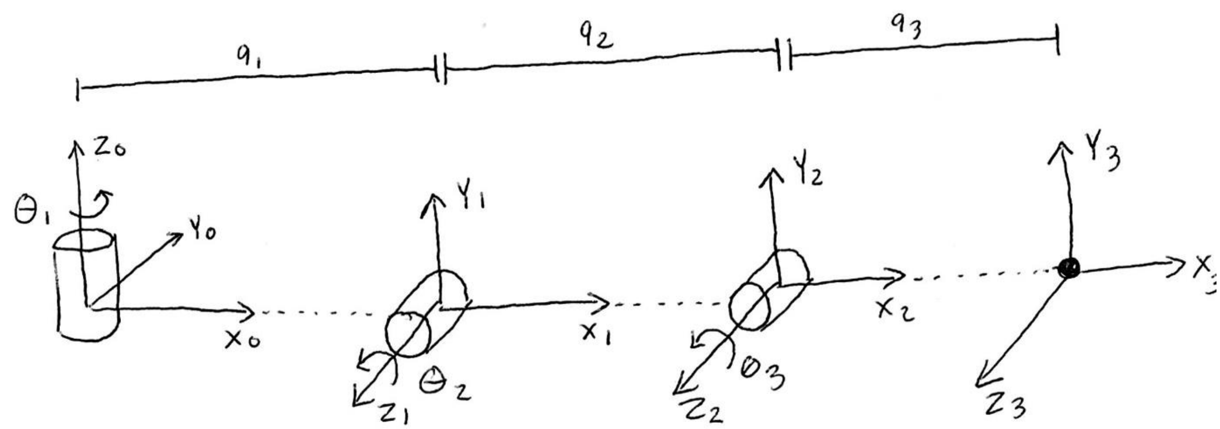

Figure 1-1: Kinematic Diagram

In the above diagram frame 0 represents the shoulder/coxa servo, frame 1 represents the femur servo and frame 2 represents the tibia servo and frame 3 represents the end-effector (foot). a_1 is the coxa length, a_2 is the femur length, a_3 is the tibia length, θ_1 is the coxa angle, θ_2 is the femur angle, and θ_3 is the tibia angle.

To simplify the math, the frame representing the foot is oriented in a straight line along the x_2 axis. The tibia servo will be attached to the servo horn set at 0° but mapped from 45° for maximum PWM value and -135° for the minimum PWM value, allowing increased rotation on the lower arc. The limits of the femur will be -90° to 90° and the limits of the coxa will be 0° to 90°. The centered, standing position will be 45° at the coxa, 0° at the femur and -90° at the tibia.

Forward Kinematics

Based upon the kinematic diagram, rotation and matrices were created. The homogenous transformation matrices on the right (H) were manually calculated by combining the appropriate rotation matrices (R) and displacement vectors (d). Reference the ASU courses Robotics 1 2017 and Robotics 2 2018 for additional information regarding transformation matrices.

$$R_1^0 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \quad d_1^0 = \begin{bmatrix} a_1\cos\theta_1 \\ a_1\sin\theta_1 \\ 0 \end{bmatrix} \quad H_1^0 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 & a_1\sin\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad d_2^1 = \begin{bmatrix} a_2\cos\theta_2 \\ a_2\sin\theta_2 \\ 0 \end{bmatrix} \quad H_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3^2 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad d_3^2 = \begin{bmatrix} a_3\cos\theta_3 \\ a_3\sin\theta_3 \\ 0 \end{bmatrix} \quad H_3^2 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & a_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & a_3\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 1-2: Rotation matrices, displacement vectors and transformation matrices

The homogeneous transformation matrices can also be created using the Denavit-Hartenberg parameters table, which uses a standard form (included after the table) to create each transformation matrix. The table below was created to cross check the manually calculated transformation matrices above.

- n represents the frame
- θ is the angle required to rotate frame $n\text{-}1$ around z_{n-1} such that x_{n-1} is aligned with x_n
- α is the angle required to rotate frame $n\text{-}1$ around axis x_n such that z_{n-1} is aligned with z_n
- r is the displacement in the x_n direction from the center of frame $n\text{-}1$ to the center of frame n
- d is the displacement in the z_{n-1} direction from the center of frame $n\text{-}1$ to the center of frame n

n	θ	α	r	d
1	θ_1	90	a_1	0
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0

Figure 1-3: DH Parameters Table

$$H_n^{n-1} = \begin{bmatrix} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & r_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & r_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 1-4: DH Standard Form

$$H_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1\cos(90) & \sin\theta_1\cos(90) & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1\cos(90) & -\cos\theta_1\sin(90) & a_1\sin\theta_1 \\ 0 & \sin(90) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 & a_1\sin\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2\cos(0) & \sin\theta_2\sin(0) & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2\cos(0) & -\cos\theta_2\sin(0) & a_2\sin\theta_2 \\ 0 & \sin(0) & \cos(0) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3\cos(0) & \sin\theta_3\sin(0) & a_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3\cos(0) & -\cos\theta_3\sin(0) & a_3\sin\theta_3 \\ 0 & \sin(0) & \cos(0) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^2 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & a_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & a_3\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 1-5: Transformation matrices generated using DH standard form

Note that the DH generated HTMs match the manually calculated HTMs. Next, the appropriate matrices must be multiplied to get the transformation matrices from frame 0 to frame 2 and from frame 0 to frame 3. These matrices will give us the cumulative rotation and displacement components from frame 0 to 2 and 0 to 3 and will be needed for FK testing as well as constructing the Jacobian matrix.:

$$H_2^0 = H_1^0 H_2^1$$

$$H_2^0 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 & a_1\sin\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2 + 0 + 0 + 0 & \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2 + 0 + 0 + 0 & 0 + 0 + \mathrm{s}\,\theta_1 + 0 & \mathrm{c}\,\theta_1\, a_2 \mathrm{c}\,\theta_2 + 0 + 0 + a_1 \mathrm{c}\,\theta_1 \\ \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2 + 0 + 0 + 0 & -\mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2 + 0 + 0 + 0 & 0 + 0 - c\theta_1 + 0 & s\theta_1\, a_2 \mathrm{c}\,\theta_2 + 0 + 0 + a_1 \mathrm{s}\,\theta_1 \\ 0 + s\theta_2 + 0 + 0 & 0 + c\theta_2 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + a_2 s\theta_2 + 0 + 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2 & \sin\theta_1 & a_2\cos\theta_1\cos\theta_2 + a_1\cos\theta_1 \\ sin\theta_1\cos\theta_2 & -sin\theta_1\sin\theta_2 & -\cos\theta_1 & a_2\sin\theta_1\cos\theta_2 + a_1 sin\,\theta_1 \\ sin\theta_2 & cos\theta_2 & 0 & a_2 sin\,\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = H_2^0 H_3^2$$

$$H_3^0 = \begin{bmatrix} \cos\theta_1 \cos\theta_2 & \cos\theta_1 \sin\theta_2 & \sin\theta_1 & a_2\cos\theta_1\cos\theta_2 + a_1\cos\theta_1 \\ sin\theta_1\cos\theta_2 & -sin\theta_1\sin\theta_2 & -\cos\theta_1 & a_2\sin\theta_1\cos\theta_2 + a_1 sin\,\theta_1 \\ sin\theta_2 & cos\theta_2 & 0 & a_2 sin\,\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & a_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & a_3\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 + 0 + 0 & -\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{s}\,\theta_3 - \mathrm{c}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + 0 + 0 & 0 + 0 + \mathrm{s}\,\theta_1 + 0 & a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3 c\theta_1 s\theta_2 s\theta_3 + 0 + (a_2 c\theta_1 c\theta_2 + a_1 c\theta_1) \\ \mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - \mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 + 0 + 0 & -\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{s}\,\theta_3 - \mathrm{s}\,\theta_1\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + 0 + 0 & 0 + 0 - \mathrm{c}\,\theta_1 + 0 & a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3 s\theta_1 s\theta_2 s\theta_3 + 0 + (a_2 s\theta_1 c\theta_2 + a_1 s\theta_1) \\ \mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + \mathrm{c}\,\theta_2\,\mathrm{s}\,\theta_3 + 0 + 0 & -\mathrm{s}\,\theta_2\,\mathrm{s}\,\theta_3 + \mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 + 0 + 0 & 0 + 0 + 0 + 0 & a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3 c\theta_2 s\theta_3 + 0 + (a_2 s\theta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_1 \mathrm{c}\theta_2 \mathrm{c}\theta_3 - c\theta_1 s\theta_2 s\theta_3 & -c\theta_1 \mathrm{c}\theta_2 s\theta_3 - c\theta_1 s\theta_2 \mathrm{c}\theta_3 & \mathrm{s}\theta_1 & a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3 c\theta_1 s\theta_2 s\theta_3 + a_2 c\theta_1 c\theta_2 + a_1 c\theta_1 \\ s\theta_1 \mathrm{c}\theta_2 \mathrm{c}\theta_3 - s\theta_1 s\theta_2 s\theta_3 & -s\theta_1 \mathrm{c}\theta_2 s\theta_3 - s\theta_1 s\theta_2 \mathrm{c}\theta_3 & -\mathrm{c}\theta_2 & a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3 s\theta_1 s\theta_2 s\theta_3 + a_2 s\theta_1 c\theta_2 + a_1 s\theta_1 \\ s\theta_2 \mathrm{c}\theta_3 + c\theta_2 s\theta_3 & -s\theta_2 s\theta_3 + c\theta_2 \mathrm{c}\theta_3 & 0 & a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3 c\theta_2 s\theta_3 + a_2 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 1-6: Transformation matrices from frame 0 to 2 and frame 0 to 3

Forward Kinematics Testing

Using these transformation matrices, we can calculate the x, y and z coordinates of the foot for a given θ_1 , θ_2 and θ_3. Reference the python code in Appendix A which compares the manually calculated matrices against the DH table derived matrices. To see the results of a calculation print the desired matrix by adding "*print(np.matrix(MATRIX_VARIABLE))"* to the code. By changing the values of θ_1 , θ_2 and θ_3 and referencing the kinematics diagram, the transformation matrix rotation and displacement values can be confirmed.

Jacobian Matrix

The end-effector rate of movement and rotation can be controlled using the Jacobian matrix and to calculate inverse kinematics we can use the inverse of the Jacobian matrix.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = J \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \\ \vdots \\ \dot{q_n} \end{bmatrix}$$

Figure 2-1: Jacobian Matrix

First, we need the standard Jacobian matrix for our diagram. Elements $\dot{x}$, $\dot{y}$ and $\dot{z}$ above represent how fast the end-effector moves in the x, y and z directions, ω_x, ω_y and ω_z represents how fast the end-effector rotates around the x, y and z axes. J represents the Jacobian matrix. $q_1, q_2 \dots q_n$ represent a series of unknown joint types with n equal to the total number of joints. If a joint is revolute, q_i is replaced by θ_i and represents rotation is radians/sec. If the joint type is prismatic, q_i is replaced by d_i and represents movement in meters/sec.

The Jacobian matrix consists of a six-row matrix, with n columns, where n is the number of joints in the system. The following table is used to calculate each column:

Component	Prismatic Joint	Revolute Joint
Linear	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(d_n^0 - d_{i-1}^0\right)$
Rotational	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Figure 2-2 Jacobian Matrix Table

Our kinematic diagram has three revolute joints, so the Jacobian matrix (J) will have six rows and three columns. The components of the matrix are filled in using the table above.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_3^0 - d_0^0) & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_3^0 - d_1^0) & R_2^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_3^0 - d_2^0) \\ R_0^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_1^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & R_2^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Figure 2-3: Jacobian matrix for three revolute joints created using the table above

By using the previously calculated transformation matrices, we can extract the appropriate rotation matrices and displacement vectors to fill in the values. R_2^0 and d_2^0 can be extracted from H_2^0 , while d_3^0 can be extracted from H_3^0. For the first row of matrices we get the following values. Note that R_0^0 is the identity matrix, since there is no change in rotation from frame 0 to frame 0.

For reference, the cross product of two vectors is taken as follows:

$$\begin{bmatrix} a1 \\ a2 \\ a3 \end{bmatrix} \times \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix} = \begin{bmatrix} a2b3 - a3b2 \\ a3b1 - a1b3 \\ a1b2 - a2b1 \end{bmatrix}$$

The manual calculations of the cross products below highlighted in yellow got complicated and need to be double checked. In the python code, I used *numpy.cross()* along axis 0 to calculate the cross product of the two appropriate vectors, which seems to be correct given 0° and 90° angle tests.

Row 1, column 1:

$$R_0^0\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$(d_3^0 - d_0^0) = \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2 + a_1c\theta_1\\a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2 + a_1s\theta_1\\a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$(d_3^0 - d_0^0) = \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2 + a_1c\theta_1\\a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2 + a_1s\theta_1\\a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2\end{bmatrix}$$

$$R_0^0\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_3^0 - d_0^0) = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2 + a_1c\theta_1\\a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2 + a_1s\theta_1\\a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2\end{bmatrix}$$

$$R_0^0\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_3^0 - d_0^0) = \begin{bmatrix}0 - (a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2 + a_1s\theta_1)\\a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2 + a_1c\theta_1 - 0\\0 - 0\end{bmatrix}$$

$$R_0^0\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_3^0 - d_0^0) = \begin{bmatrix}-a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3s\theta_1s\theta_2s\theta_3 - a_2s\theta_1c\theta_2 - a_1s\theta_1)\\a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2 + a_1c\theta_1\\0\end{bmatrix}$$

Row 1, column 2:

$$R_1^0\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\cos\theta_1 & 0 & \sin\theta_1\\\sin\theta_1 & 0 & -\cos\theta_1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\sin\theta_1\\-\cos\theta_1\\0\end{bmatrix}$$

$$(d_3^0 - d_1^0) = \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2 + a_1c\theta_1\\a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2 + a_1s\theta_1\\a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2\end{bmatrix} - \begin{bmatrix}a_1\cos\theta_1\\a_1\sin\theta_1\\0\end{bmatrix}$$

$$(d_3^0 - d_1^0) = \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2\\a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2\\a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2\end{bmatrix}$$

$$R_1^0\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_3^0 - d_1^0) = \begin{bmatrix}\sin\theta_1\\-\cos\theta_1\\0\end{bmatrix} \times \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2\\a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2\\a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2\end{bmatrix}$$

$$R_1^0\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_3^0 - d_1^0) = \begin{bmatrix}((-\cos\theta_1)\,(a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2)) - 0\\0 - ((\sin\theta_1)(a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2))\\((\sin\theta_1)(a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2)) - ((-\cos\theta_1)(a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2)\end{bmatrix}$$

Row 1, column 3:

$$R_2^0\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2 & \sin\theta_1\\ sin\theta_1\cos\theta_2 & -sin\theta_1\sin\theta_2 & -\cos\theta_1\\ sin\theta_2 & cos\theta_2 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\sin\theta_1\\ -\cos\theta_1\\ 0\end{bmatrix}$$

$$(d_3^0 - d_2^0) = \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3 + a_2c\theta_1c\theta_2 + a_1c\theta_1\\ a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3 + a_2s\theta_1c\theta_2 + a_1s\theta_1\\ a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3 + a_2s\theta_2\end{bmatrix} - \begin{bmatrix}a_2\cos\theta_1\cos\theta_2 + a_1\cos\theta_1\\ a_2\sin\theta_1\cos\theta_2 + a_1sin\,\theta_1\\ a_2sin\,\theta_2\end{bmatrix}$$

$$(d_3^0 - d_2^0) = \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3\\ a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3\\ a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3\end{bmatrix}$$

$$R_2^0\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_3^0 - d_2^0) = \begin{bmatrix}\sin\theta_1\\ -\cos\theta_1\\ 0\end{bmatrix} \times \begin{bmatrix}a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3\\ a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3\\ a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3\end{bmatrix}$$

$$R_2^0\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_3^0 - d_2^0) = \begin{bmatrix}((-c\theta_1)(a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3)) - 0\\ 0 - ((s\theta_1)(a_3\,\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_2s\theta_3))\\ \big((s\theta_1)(a_3\,\mathrm{s}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_2s\theta_3)\big) - ((-c\theta_1)(a_3\,\mathrm{c}\,\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1s\theta_2s\theta_3))\end{bmatrix}$$

$$R_2^0\begin{bmatrix}0\\0\\1\end{bmatrix} \times (d_3^0 - d_2^0) = \begin{bmatrix}-a_3\,c\theta_1\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3c\theta_1c\theta_2s\theta_3\\ -a_3\,s\theta_1\mathrm{s}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1c\theta_2s\theta_3\\ a_3\,\mathrm{s}\,\theta_1\,s\theta_1\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 - a_3s\theta_1s\theta_1s\theta_2s\theta_3 + a_3\,\mathrm{c}\,\theta_1c\theta_1\,\mathrm{c}\,\theta_2\,\mathrm{c}\,\theta_3 + a_3c\theta_1c\theta_1s\theta_2s\theta_3\end{bmatrix}$$

Row 2, column 1:

$$R_0^0\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Row 2, column 2:

$$R_1^0\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\cos\theta_1 & 0 & \sin\theta_1\\ \sin\theta_1 & 0 & -\cos\theta_1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\sin\theta_1\\ -\cos\theta_1\\ 0\end{bmatrix}$$

Row 2, column 3:

$$R_2^0\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2 & \sin\theta_1\\ sin\theta_1\cos\theta_2 & -sin\theta_1\sin\theta_2 & -\cos\theta_1\\ sin\theta_2 & cos\theta_2 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\sin\theta_1\\ -\cos\theta_1\\ 0\end{bmatrix}$$

To simplify writing the above equations, let the matrix row and column values be equal to the variables J11, J12, J13, where J11 represents the first row, first column, J12 represents the first row, second column, etc. This naming convention will be used in code:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} J11 & J12 & J13 \\ J21 & J22 & J23 \\ J31 & J32 & J33 \\ J41 & J42 & J43 \\ J51 & J52 & J53 \\ J61 & J62 & J63 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$\dot{x} = J11\dot{\theta}_1 + J12\dot{\theta}_2 + J13\dot{\theta}_3$$

$$\dot{y} = J21\dot{\theta}_1 + J22\dot{\theta}_2 + J23\dot{\theta}_3$$

$$\dot{z} = J31\dot{\theta}_1 + J32\dot{\theta}_2 + J33\dot{\theta}_3$$

$$\omega_x = J41\dot{\theta}_1 + J42\dot{\theta}_2 + J43\dot{\theta}_3$$

$$\omega_y = J51\dot{\theta}_1 + J52\dot{\theta}_2 + J53\dot{\theta}_3$$

$$\omega_z = J61\dot{\theta}_1 + J62\dot{\theta}_2 + J63\dot{\theta}_3$$

Figure 2-4: Jacobian variable naming convention

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} J11 & J12 & J13 \\ J21 & J22 & J23 \\ J31 & J32 & J33 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Figure 2-5: Simplified Jacobian for the quadruped leg