Negation

We often have to negate statements when writing proofs. Negation can be very simple, but some negation forms are not very useful. Furthermore, negating statements with multiple quantifiers can actually be quite complicated.

Goals:

- Negate statements using logically equivalent expressions
- Negate qualified statements
- Negate statements with multiple quantifiers

Consider the following statements.

$$P: \text{ The number 2 is an integer.}$$
$$Q: \text{ The function } g \text{ is continuous.}$$

Recall that negating these statements is essentially putting the phrase "It is not true that" in front of the statement.

$$\sim P: \text{ It is not true that 2 is an integer. } \equiv \text{ The number 2 is not an integer.}$$
$$\sim Q: \text{ It is not true that } g \text{ is continuous. } \equiv \text{ Function } g \text{ is not continuous.}$$

Notice that two forms of the above statements are given. The first is the direct negation, while the second is, in some sense, a more helpful version of the negation.

1. Negate the following simple statements and open sentences. Express the negation in a more natural way if possible. That is, don't simple write "It is not true that" in front of the original statement.

   (a) Matrix A is invertible.

   (b) The function e^x is integrable on $[0, 1]$.

   (c) The sum of 2 and 3 is 7.

It can be more complicated to negate a complicated statement or open sentence. Consider the statement below.

$$\begin{aligned} R:&\ \text{Integer } a \text{ is even and integer } b \text{ is odd.} \\ \sim R:&\ \text{It is not true that } a \text{ is even and } b \text{ is odd.} \end{aligned}$$

Notice that the statement given for $\sim R$ does not tell us anything specific about a or b. Thus, it is the correct negation, but of no practical use. We can see that $R \equiv P \wedge Q$ if P : "Integer a is even" and Q : "Integer b is odd." Recall that by DeMorgan's Laws, $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$. Then we can express $\sim R$ as follows.

$$\sim R:\ \text{Integer } a \text{ is odd } \textit{or} \text{ integer } b \text{ is even.}$$

You have previously shown that $\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$ using truth tables. This logical equivalence offers a much more useful way to express the negation of a conditional statement.

$$\begin{aligned} R:&\ \text{If } f \text{ is differentiable, then } f \text{ is continuous.} \\ \sim R:&\ \text{Function } f \text{ is differentiable and } f \text{ is not continuous.} \end{aligned}$$

2. Negate the following complicated statements using DeMorgan's Laws and the negation of a conditional statement.

   (a) A matrix is singular or it is invertible.

   (b) Integer 2 is a divisor of 10 and 20.

   (c) If $\sqrt{2}$ is rational, then $\sqrt{2} = p/q$ for integers p and q.

   (d) For a sequence a_n to be convergent, it is sufficient that it is absolutely convergent.

(e) For 4 to be a solution of $x^2 - 16 = 0$, it is necessary that $(4)^2 - 16 = 0$.

3. Consider the statements P : "Every real number is an even integer" and Q : "There exists an infinite subset of $\mathbb{N}$."

   (a) Is $\sim P$ true or false? Explain your reasoning.

   (b) What is the most clear way to express $\sim P$ in English?

   (c) Is $\sim Q$ true or false? Explain your reasoning.

   (d) What is the most clear way to express $\sim Q$ in English.

   (e) Suppose S is a set and $P(x)$ is an open sentence concerning x. What is the proper negation of the statement "$\forall\ x \in S,\ P(x)$?"

   (f) What is the proper negation of "$\exists\ x \in S,\ P(x)$?"

Statements or open sentences that contain multiple quantifiers can be tricky to negate. Consider the following statements.

$$\begin{aligned} P &: \ \forall\, m \in \mathbb{Z}, \ \exists\, n \in \mathbb{Z}, m + n = 0 \\ Q &: \ \exists\, k \in \mathbb{Z}, \ \forall\, m \in \mathbb{Z}, km = 0 \\ R &: \ \forall a, b \in \mathbb{N}, \ \exists\, k \in \mathbb{N}, a^n > b \end{aligned}$$

4. Notice that each statement is true. Determine exactly what would have to be true for each statement to be false.

5. Negate the following statements. Determine whether the original statement or the negation is true.

   (a) $\forall\, x \in \mathbb{R}, \ \exists\, y \in \mathbb{R}, \ y^3 = x$

   (b) $\exists\, b \in \mathbb{R}, \ \forall\, a \in \mathbb{Z} - \{0\}\,, \ ab = 1$

(c) For every polynomial $p(x)$, there exists polynomial $q(x)$ such that $p'(x) = q(x)$.

(d) $\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N},\ (n > N \Rightarrow |a_n - L| < \epsilon)$

(e) $\forall\ \epsilon > 0,\ \exists\ \delta > 0, (|x - a| < \delta \Rightarrow |f(x) - L| < \epsilon)$