P. 19 Injections and Surjections and Bijections...Oh My!

While functions can have any number of various properties, none are more important than injectivity, surjectivity, and bijectivity. More commonly you know of injective as "one-to-one" and surjective as "onto."

> Goals:
>
> - Define injection, surjection, and bijection
> - Prove that functions have these properties

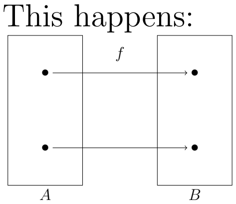
You've undoubtedly heard the terms "one-to-one" and "onto" at some point in your mathematics education, but to be fair you probably don't have a strong grasp on these concepts. Part of this is because the functions that you normally see that are defined on the real numbers make these concepts more complicated than they need to be. Let's start with injectivity.

1 Injections

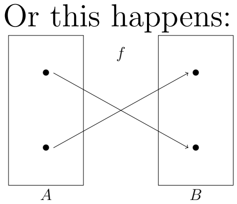
Definition. Let $f : A \to B$. For $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$, then we say that f is *injective* or *one-to-one*.

What this definition says is that a function is injective if any output that gets hit only gets hit by one input. So

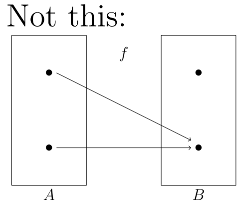
This happens:



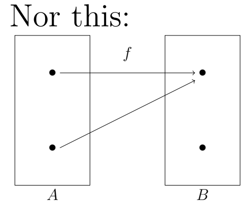
Or this happens:



Not this:



Nor this:



A function might be one-to-one as opposed to two-to-one. The common function $f(x) = x^2$ is two-to-one because each output of the function gets hit by two inputs. For instance 4 gets hit by 2 and -2, 9 gets hit by 3 and -3, etc. By definition, if something is one-to-(a number bigger than one) then it's not a function. We call such things *maps*.

1. Is the Birthmonth function injective for your group? Why or why not?

2. Is the archery function injective? Explain your answer.

The definition given above for injection is the most common because it's the simplest to understand (that's debatable). The next definition requires that you recall what the range of f, denoted R_f, is.

3. For $f : A \to B$, what is R_f? Give the name of R_f and its description in set-builder notation.

Definition. The function $f : A \to B$ is *injective* if for all $b \in R_f$ there exists a unique $a \in A$ such that $f(a) = b$.

The benefit of this new definition is that it tells us that to prove a function is injective, we need to write a uniqueness proof.

4. How do you prove something is unique?

5. Read the following injection claim and proof and justify the steps.

Claim 1. *The function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = -2x + 1$ *is injective.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -2x + 1$ and for sake of contradictions suppose there exists distinct real numbers a and b such that $f(a) = f(b)$. Then

$$\begin{aligned} -2a + 1 &= -2b + 1 \\ -2a &= -2b \\ a &= b. \end{aligned}$$

This is a contradiction, so $f(a) \neq f(b)$ when $a \neq b$. Therefore f is injective. $\square$

6. Prove that $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3$ is injective.

7. Prove that $h : \mathbb{R}^+ \to \mathbb{R}$ given by $h(t) = \sqrt{t}$ is injective.

8. Prove that $s : \mathbb{R} \to \mathbb{R}$ given by $s(t) = \sin(t)$ is not injective.

2 Surjections

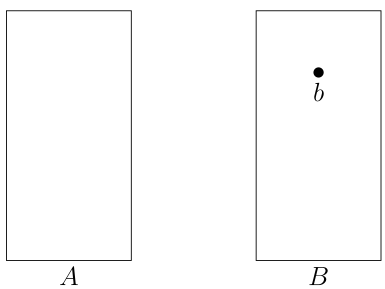
The idea of a surjection (or onto function) is actually really simple. Dr. Wright is convinced that the only reason anyone ever struggles with it is because they weren't taught the difference between a range and a codomain. Two equivalent definitions for surjective follow. The first one tells us what surjective really means, but the second tells us how to prove a function is surjective.

Definition. Let $f : A \to B$. Then f is *surjective* or *onto* if $R_f = B$.

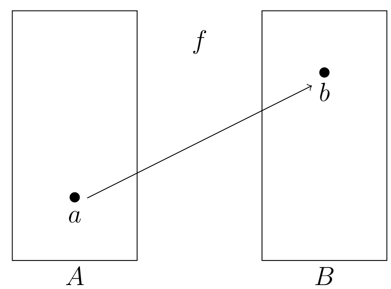
Definition. Let $f : A \to B$. Then f is *surjective* or *onto* if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

In pictures,

If we have this



Then we have this



From these definition, we can tell that proving that a function is surjective means writing an existence proof.

1. Read the following proof that $f(x) = \sqrt[3]{x}$ is surjective and answer the questions that follow.

   Proof. Suppose $f(x) = \sqrt[3]{x}$ and suppose $b \in \mathbb{R}$. Let $a = b^3$. Then $f(a) = f(b^3) = \sqrt[3]{b^3} = b$. Thus, for all $b \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = b$. $\square$

   (a) Does the proof tell us where letting $a = b^3$ came from? Should the proof tell us how to find the appropriate a?

(b) Remember that one way to think about a surjection is $R_f = B$. The proof has shown that $B \subseteq R_f$. Why doesn't it need to show that $R_f \subseteq B$?

2. Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5x + 7$ is surjective.

3. Prove that $g : \mathbb{Z} - \{0\} \to \mathbb{N}$ given by $g(x) = x^2$ is surjective.

4. Another way to say that function is surjective is to say is "maps A onto B." For a function f with domain D, is it true that f maps D onto R_f? Why or why not?

3 Bijectivity

The best thing about bijections is that they're really nothing new.

Definition. For $f : A \to B$, f is a *bijection* if it is both injective and surjective.

1. For each of the following, determine if the function is an injection, surjection, or bijection. Prove your conclusion either with an appropriate proof or counterexample.

   (a) When $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = e^x$.

   (b) When $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \tan(x)$.

(c) When $f : \mathbb{Z} \to \{0, 1, \ldots, 9\}$ is given by $f(x) = 2x \pmod{10}$.

(d) When $f : \mathbb{Z} \to \{0, 1, 2\}$ is given by $f(x) = 2x \pmod{3}$.