P. 15 Basic Induction

Proof by induction is a technique that allows us to prove a variety of statements that have one thing in common: the "objects" the statement refers to are in correspondance with the natural numbers. Often times we use induction to prove statements about the natural numbers, but the method applies in a surprising variety of situations.

Unlike many of our proof techniques, a proof by induction does follow a certain form to which we must adhere. This form is necessary for both the validity of the method, and for others to be able to understand our proofs.

> Goals:
>
> - Follow along with the steps of an induction proof
> - Write a basic induction proof

1. Follow along with the induction proof below and answer the questions as you go.

   Claim 1. *For all* $n \in \mathbb{N}$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

   (a) This claim establishes a lot. If we let $P(n)$ be the statement "$1+2+3\cdots+n = \frac{n(n+1)}{2}$," then our claim becomes $\forall\ n \in \mathbb{N}, P(n)$. What is the statement $P(1)$? What about $P(2)$? Are these statements true?

   (b) By checking that $P(1)$ is true above, you actually did the first step of the induction proof. This step is known as the "Basis" step. Checking that $P(2)$ is true is not part of the basis step, but it never hurts to check a few values.

   (c) The next step of induction is the "induction step." The goal here is to show that $P(n) \Rightarrow P(n+1)$. That is, we will show that "If $P(n)$ is true, then $P(n+1)$ is true." What do you think is the purpose of the induction step, keeping in mind that we've already established that $P(1)$ is true?

(d) We almost always use a direct proof to show that $P(n) \Rightarrow P(n+1)$. That is, we will suppose that $P(n)$ is true and use it to show that $P(n+1)$ is true. What is the statement $P(n)$? What is the statement $P(n+1)$?

(e) The proof below shows that $P(n) \Rightarrow P(n+1)$. Follow along with the proof and justify each step.

Proof. Suppose $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$. Then

$$\begin{aligned} 1+2+3+\cdots+n+(n+1) &= (1+2+3+\cdots+n)+(n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2}. \end{aligned}$$

Therefore, if $1+2+\cdots+n = \frac{n(n+1)}{2}$ then $1+2+\cdots+n+(n+1) = \frac{(n+1)((n+1)+1)}{2}$. □

(f) Together, the basis step and the induction step make up a proof by induction.

2. The following is a proof of the same claim. The proof correctly follows the format of a proof by induction, but contains a severe flaw. See if you can find the mistake.

Claim 2. *For all* $n \in \mathbb{N}$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Proof. We proceed by mathematical induction.
Basis Step: Notice that when $n = 1$ the statement becomes $1 = \frac{1(1+1)}{2}$, which is obviously true.

Induction Step: Suppose $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. Then

$$
\begin{aligned}
1 + 2 + 3 + \cdots + n + (n+1) &= \frac{(n+1)((n+1)+1)}{2} \\
(1 + 2 + 3 + \cdots + n) + (n+1) &= \frac{(n+1)(n+2)}{2} \\
\frac{n(n+1)}{2} + (n+1) &= \frac{n^2+3n+2}{2} \\
n+1 &= \frac{n^2+3n+2}{2} - \frac{n^2+n)}{2} \\
n+1 &= \frac{n^2+3n+2-(n^2+n)}{2} \\
n+1 &= \frac{2n+2}{2} \\
n+1 &= n+1.
\end{aligned}
$$

Since the last statement if obviously true, if $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ then $1 + 2 + \cdots + n + 1 = \frac{(n+1)(n+2)}{2}$. $\square$

3. Prove the following claim using mathematical induction.

Claim 3. *If* $n \in \mathbb{N}$, *then* $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Proof. We proceed by mathematical induction.
Basis Step:

Induction Step:

$\square$