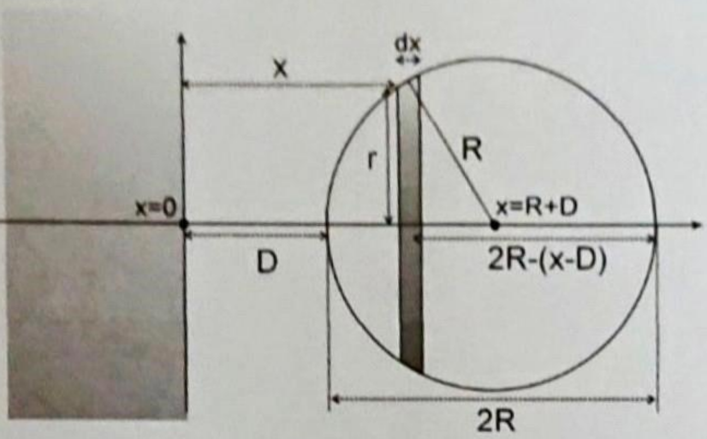
Problem – Dispersion interactions between a slab and a sphere

Derive an expression for the dispersion forces which act between a sphere of radius R and a semi-infinite solid, for separations D >> R, if the number density of atoms in the solid and sphere are n1 and n2 respectively.



To begin we will follow the same steps

1) what are the symmetries?

Be careful here. One might think we need to create hoops but all atoms in each slice of the sphere are x from the slab. This works because the slab is infinite in y + z.

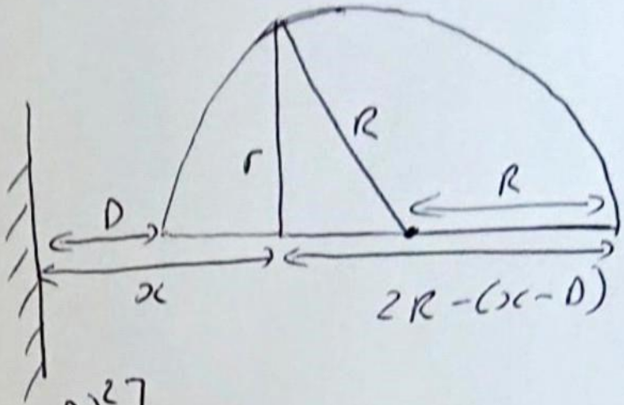
2) work out the interaction from this elemental volume.

As before $U_{atom/slab} = -\frac{n_1 \pi C}{6x^3}$

- The slice has an area πr^2 + thickness dx
- No. atoms in slice, $N = n_2 \pi r^2 dx$ (n_2 = number density)

we must now do some geometry. We cannot use r because it is different for every value of x. We must therefore rewrite r in terms of our variable x + constants R + D (see diagram) using pythagoras:

$$R^2 = r^2 + \left[(2R - (x-D)) - R\right]^2$$



$$R^2 = r^2 + (R - (x-D))^2$$

$$r^2 = R^2 - (R - (x-D))^2$$

$$= R^2 - \left[R^2 - 2R(x-D) + (x-D)^2\right]$$

$$r^2 = 2R(x-D) - (x-D)^2$$

This gives us an expression for r in terms of a variable x which we can integrate & constants.

So no. atoms in disk:

$$N_{disk} = n_2 \pi \overbrace{\left(2R(x-D) - (x-D)^2\right)}^{r^2} dx$$

Total interaction between disk & surface

$$dU_{disk} = N_{disk} U_{atom\text{-}slab}$$

$$= n_2 \pi \left(2R(x-D) - (x-D)^2\right) dx \left(-\frac{n_1 \pi C}{6x^3}\right)$$

$$= \frac{-n_1 n_2 \pi^2 C}{6} \left(\frac{2R(x-D) - (x-D)^2}{x^3}\right) dx$$

$$= -\frac{n_1 n_2 \pi^2 C}{6} \left[\frac{2Rx}{x^3} - \frac{2RD}{x^2} - \frac{x^2}{x^3} + \frac{2xD}{x^3} - \frac{D^2}{x^3}\right] dx$$

$$= \frac{-n_1 n_2 \pi^2 C}{6} \left[\frac{2(R+D)}{x^2} - \frac{D(2R+D)}{x^3} - \frac{1}{x}\right] dx$$

Here we collate terms of same power of x to make subsequent integration easier.

3) Total interaction - add up contributions by integrating between limits.

There are discs in the sphere which start at $x = D$ and finish at $x = D + 2R$

$$\therefore\ U_{sphere-slab} = \int_D^{D+2R} du_{disk}$$

$$= \frac{-n_1 n_2 \pi^2 C}{6} \int_D^{D+2R} \frac{2(R+D)}{x^2} - \frac{D(2R+D)}{x^3} - \frac{1}{x}\, dx$$

$$= \frac{-n_1 n_2 \pi^2 C}{6} \left[\frac{-2(R+D)}{x} + \frac{D(2R+D)}{2x^2} - \ln x \right]_D^{D+2R}$$

$$= \frac{-n_1 n_2 \pi^2 C}{6} \left[\frac{-2(R+D)}{D+2R} + \frac{D(2R+D)}{2(D+2R)^{\cancel{2}}} - \ln(D+2R) \right.$$

$$\left. + \frac{2(R+D)}{D} - \frac{\cancel{D}(2R+D)}{2D^{\cancel{2}}} + \ln D \right]$$

$$= \frac{-n_1 n_2 \pi^2 C}{6} \left[\frac{-4(R+D) + D}{2(D+2R)} + \frac{4(R+D) - (2R+D)}{2D} \right.$$

$$\left. - \ln\left(\frac{D+2R}{D}\right) \right]$$

$$= \frac{-n_1 n_2 \pi^2 C}{6} \left[\frac{-4R - 3D}{2(D+2R)} + \frac{2R + 3D}{2D} - \ln\left(\frac{D+2R}{D}\right) \right]$$

$$= \frac{-n_1 n_2 \pi^2 C}{6} \left[\frac{-8RD - \cancel{6D^2} + 4R(D+2R) + 6D(\cancel{D}+2R)}{4D(D+2R)} \right.$$

$$\left. - \ln\left(\frac{D+2R}{D}\right) \right]$$

$$= \frac{-n_1 n_2 \pi^2 C}{6}\left[\frac{8RD + 8R^2}{4D(D+2R)} - \ln\left(\frac{D+2R}{D}\right)\right]$$

$$= \frac{-n_1 n_2 \pi^2 C}{6}\left[\frac{2R(R+D)}{D(2R+D)} - \ln\left(\frac{D+2R}{D}\right)\right]$$

In the limit $D >> R$ $\qquad D+2R \rightarrow D$

$\qquad D+R \rightarrow D$

$$U_{tot} \simeq -\frac{n_1 n_2 \pi^2 C}{6}\left[\frac{2R\cancel{D}}{D\cancel{D}} - \cancel{\ln\left(\frac{D}{D}\right)}\right]$$

$$\simeq -\frac{n_1 n_2 \pi^2 C}{6}\frac{2R}{D}$$

$$\boxed{F = -\frac{dU}{dD} \simeq -\frac{n_1 n_2 \pi^2 C R}{3D^2}}$$

Even if $R >> D$ this equation is a reasonable approximation.

$2R + D \rightarrow 2R$

$R + D \rightarrow R$

This limit is common at the nanoscale when we want to know about the forces between 2 objects for example an AFM tip – see later in the course.

$$U = -\frac{n_1 n_2 \pi^2 C}{6}\left[\frac{R}{D} - \ln\frac{2R}{D}\right]$$

if $R = 100\,nm$ + $D = 1\,nm$

$\frac{R}{D} = 100$ whilst $\ln\left(\frac{2R}{D}\right) = 5.3$

so although not tiny

$$U \simeq -\frac{n_1 n_2 \pi^2 C R}{6D} \quad \text{still reasonable.}$$