Hyperbolic Equations

Tutoring Centre Ferndale

A hyperbolic equation is similar to a quadratic equation but involves two variables instead of one. The standard form of a hyperbolic equation is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are constants ≥ 0.

Hyperbolic equations are important in understanding shapes and curves in geometry. The hyperbola, which is the graph of a hyperbolic equation, was first studied by ancient Greek mathematicians.

Hyperbolas

The graph of a hyperbolic equation is called a hyperbola.

- The standard hyperbolic equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

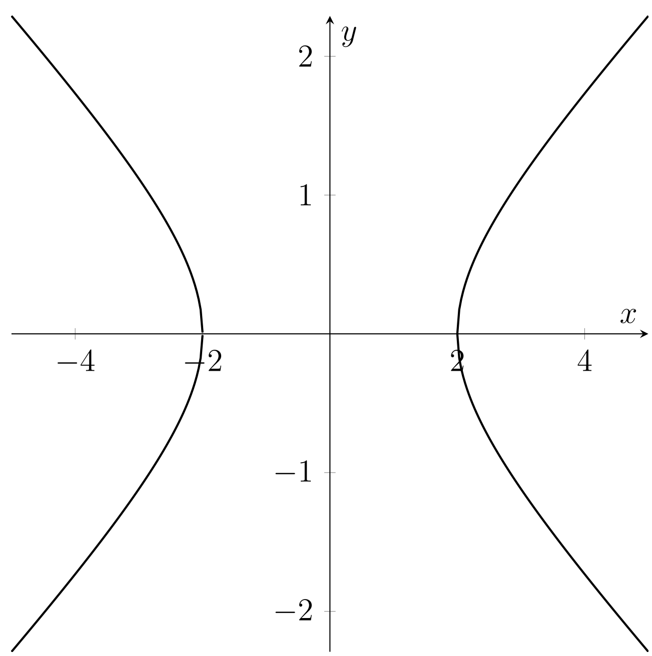
  makes a hyperbola that opens horizontally.

- The equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

  makes a hyperbola that opens vertically.

- Here is what a hyperbola looks like for the equation $\frac{x^2}{4} - \frac{y^2}{1} = 1$:



Vertices

The **vertices** of a hyperbola are the innermost points where the curve turns.

- For standard form, the vertices are located at $(\pm a, 0)$.
- For a hyperbola that opens vertically, the vertices are located at $(0, \pm a)$.

Transverse Axis and Conjugate Axis

- The **transverse axis** is the line segment passing through the center and connecting the two vertices of the hyperbola.

- The **conjugate axis** is the line segment perpendicular to the transverse axis and passes through the center of the hyperbola.

Asymptotes

The **asymptotes** of a hyperbola are the lines that the hyperbola approaches but never intersects as it extends to infinity. These lines provide a visual boundary that guides the shape of the hyperbola.

- For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the asymptotes are given by $y = \pm \frac{b}{a} x$.
- For $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, the asymptotes are given by $y = \pm \frac{a}{b} x$.

Foci

In a hyperbola, the **foci** (plural of focus) are two points located along the transverse axis.

- The defining property of a hyperbola is that the difference in distances from any point on the hyperbola to the two foci is a constant, equal to $2a$.

- For a hyperbola given by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the foci are positioned at $(\pm c, 0)$, where $c = \sqrt{a^2 + b^2}$.

These foci lie outside the vertices of the hyperbola, at a distance greater than a from the center.
The foci of a hyperbola are used in various applications, such as satellite communications or the design of optical systems, to focus light or other forms of radiation.

Effect of Constants a and b

The constants a and b control the overall dimensions and steepness of the hyperbola:

- **The constant** a determines the distance from the center of the hyperbola to the vertices.
- **The constant** b affects the distance from the center to the foci along the conjugate axis, thereby influencing the steepness of the hyperbola's branches. A larger value of b leads to steeper branches, while a smaller b results in shallower curves.

Hyperbola not Centred at the Origin

If the hyperbola is centered at (h, k) rather than the origin, the equations are adjusted accordingly.

Horizontal Transverse Axis

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

- Vertices: $(h \pm a, k)$
- Asymptotes: $y - k = \pm\frac{b}{a}(x - h)$

Vertical Transverse Axis

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

- Vertices: $(h, k \pm a)$
- Asymptotes: $y - k = \pm\frac{a}{b}(x - h)$

Applications

Hyperbolic equations and hyperbolas appear in many real-world situations:

- **Navigation**: Hyperbolas are used in GPS technology to pinpoint locations.
- **Astronomy**: The paths of some celestial objects follow hyperbolic trajectories.
- **Engineering**: Hyperbolic shapes are used in structures like cooling towers.

Exercise

1. Graph the hyperbolic equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Answer

1. To graph $\frac{x^2}{16} - \frac{y^2}{9} = 1$, plot the vertices at $(4, 0)$ and $(-4, 0)$, and draw the asymptotes $y = \pm\frac{3}{4}x$. Then sketch the hyperbola opening left and right.