Parabolas

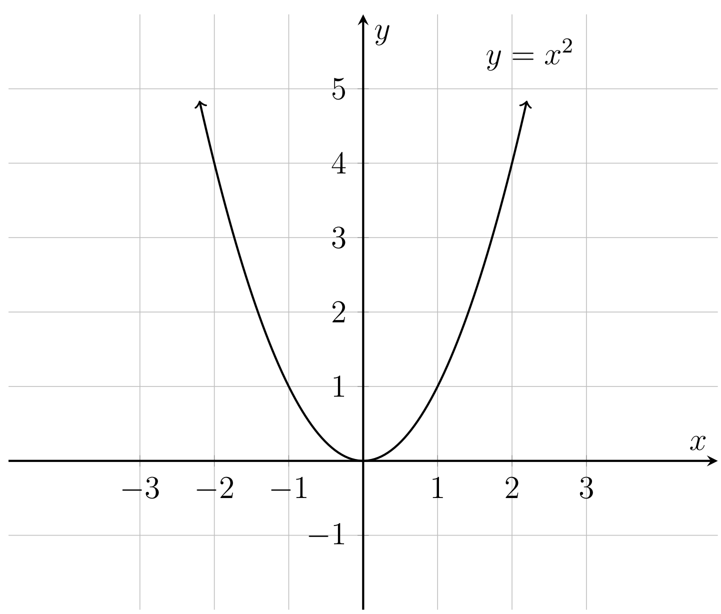
Tutoring Centre Ferndale

Standard form

A parabola is the curve made by plotting the graph of a quadratic equation of the form

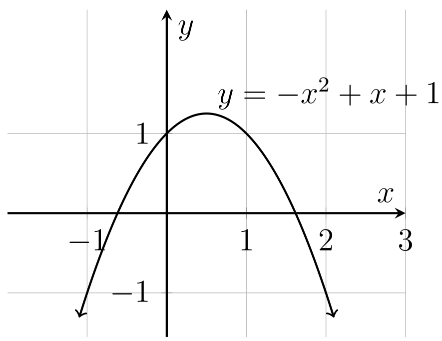
$$y = ax^2 + bx + c$$

where a, b, and c are constants.



Effect of the Leading Coefficient a

- If $|a|$ is large, the parabola is narrow.
- If $|a|$ is small, the parabola is wide.
- If $a > 0$, the parabola opens upwards.
- If $a < 0$, the parabola opens downwards.



Effect of the Linear Coefficient b

- The coefficient b shifts the vertex left or right on the x-axis, but it is also affected by the value of a.

Effect of the Constant Term c

- The coefficient c moves the entire parabola up or down.
- Setting x to 0, the first two terms disappear, leaving only c.
- The parabola passes through $(0, c)$ so c gives the y-intercept.

Finding the Roots

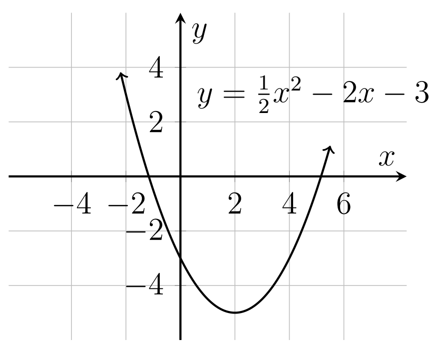
The roots of a quadratic equation are the points where its parabola crosses the x-axis. They are the values of x for which $y = 0$. Roots are found by various methods of factorizing the quadratic equation, by completing the square, or by using the quadratic formula.

The Discriminant

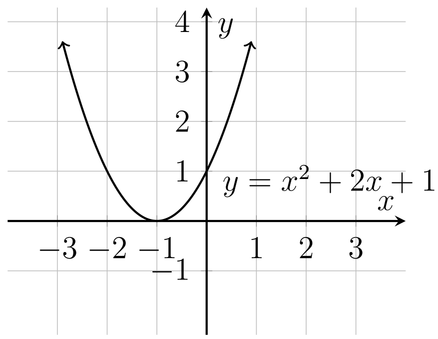
The discriminant $b^2 - 4ac$ shows whether the parabola crosses the x-axis, touches it, or does not cross it.

- If $b^2 - 4ac > 0$, there are two real roots (the parabola crosses the x-axis twice).
- If $b^2 - 4ac = 0$, there is one real root (the vertex lies on the x-axis).
- If $b^2 - 4ac < 0$, there are no real roots (the parabola does not cross the x-axis).

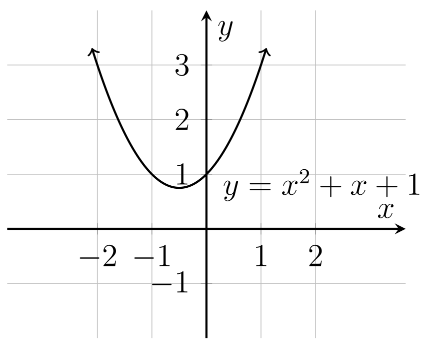
$b^2 - 4ac > 0$**: Parabola with Two Distinct Real Roots (Crosses the x-axis twice)**



$b^2 - 4ac = 0$: Parabola with One Real Root (Vertex on the x-axis) (A perfect square quadratic)



$b^2 - 4ac < 0$: Parabola with No Real Roots (Does not cross the x-axis)



Vertex Form

The vertex of the parabola, also called the point of symmetry, is the point where the curve changes direction.

The quadratic equation can be written in vertex form:

$$y = a(x - h)^2 + k$$

where (h, k) is the vertex of the parabola.

Axis of Symmetry

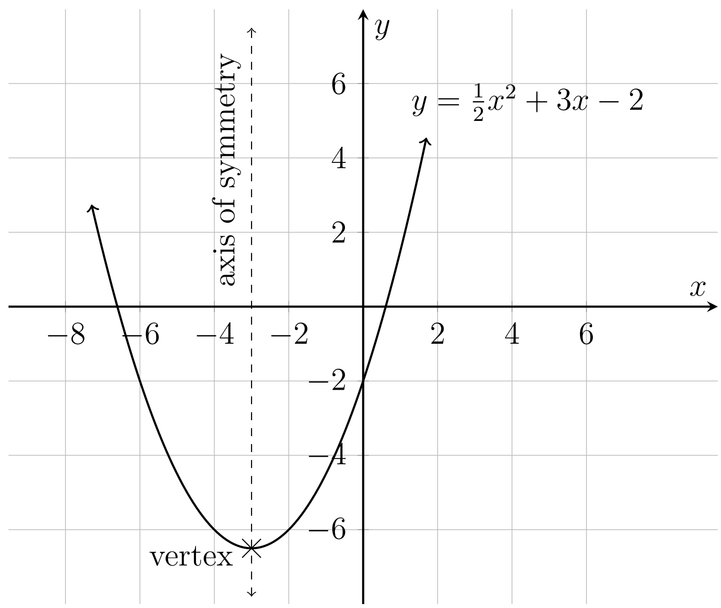
The x-coordinate of the vertex is given by

$$x_v = -\frac{b}{2a}$$

The x-coordinate of the vertex gives the axis of symmetry of the parabola, which is the line $x = -\frac{b}{2z}$.

The y-coordinate of the vertex is given by:

$$y_v = \frac{4ac - b^2}{4a}$$



Standard form to Vertex form

An equation in standard quadratic form is converted to vertex form by completing the square.

For example, $y = 2x^2 + 8x + 5$:

1. Normalize by factoring the first two terms: $y = 2(x^2 + 4x) + 5$
2. Complete the square (add and subtract $\frac{b}{2}^2$) : $y = 2(x^2 + 4x + 4 - 4) + 5$
3. Perfect Square: $y = 2((x + 2)^2 - 4) + 5$
4. Distribute the factor of 2: $y = 2(x + 2)^2 - 8 + 5$
5. Vertex form: $y = 2(x + 2)^2 - 3$

Vertex form to Standard form

A quadratic equation in vertex form is changed to standard form by multiplying it out and rearranging.

For example, $y = 2(x + 2)^2 - 3$:

1. $y = 2(x + 2)^2 - 3 \implies y = 2(x + 2)(x + 2) - 3$
2. FOIL: $y = 2(x^2 + 2x + 2x + 4) - 3$
3. Distribute the factor of 2: $y = (2x^2 + 4x + 4x + 8) - 3$
4. Combine like terms: $y = 2x^2 + 8x + 5$ (Standard form.)

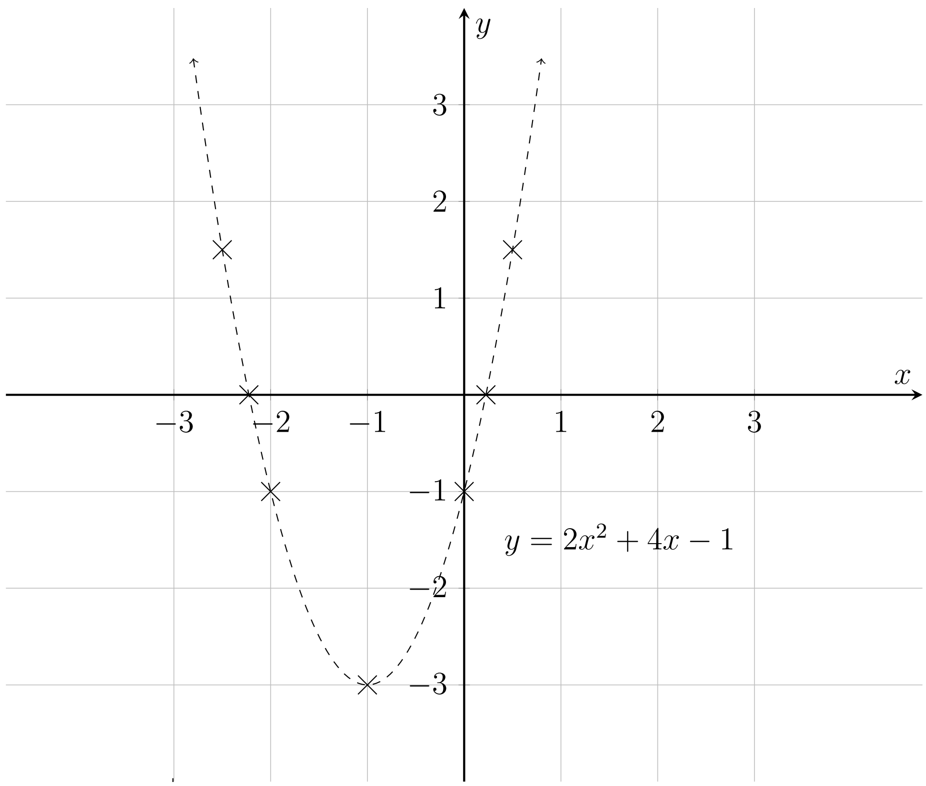
Drawing a Parabola from its equation

Given a quadratic equation, five point of the parabola can be plotted and the curve sketched.

- If a is positive the parabola opens upwards.
- The roots of the parabola are given by the quadratic formula.
- The coordinates of the vertex are $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$.
- The y-intercept, where $x = 0$, is given by c.
- The symmetry point to the y-intercept is $(2h, c)$.
- More points can be plotted by calculating values of y for given values of x. Use the axis of symmetry as the middle x-value and find x-values on either side.

For example, to graph $y = 2x^2 + 4x - 1$:

- a is positive so the parabola opens upwards.
- From the quadratic formula,
  the roots are $\frac{-2+\sqrt{6}}{2}$ and $\frac{-2-\sqrt{6}}{2}$ (≈ 0.225 & $-$ 2.225.)
- The x-coordinate of the vertex, which also gives the axis of symmetry, is $x = -\frac{b}{2a} = -\frac{4}{2\cdot 2} = -1$.
- The y-coordinate of the vertex, by substituting $x = -1$ into the equation, is $y = 2(-1)^2 + 4(-1) - 1 = 2 - 4 - 1 = -3$.
- The y-intercept is given by $c = -1$.
- The symmetry point to the y-intercept is $(2h, c) = (-2, -1)$.
- Plotting a couple of extra points either side of the axis of symmetry:
  For $x = -2.5, y = 2(-2.5)^2 + 4(-2.5) - 1 = 1.5$.
  For $x = 0.5, y = 2(0.5)^2 + 4(0.5) - 1 == 1.5$.
- That's 7 points to plot which makes it easy to draw the curve through them.



Solving quadratic equations graphically

Instead of doing the maths to factorize a quadratic equation or calculate its roots with the quadratic formula, it is sometimes more practical to plot the parabolic curve of the equation and read the solution from the graph.

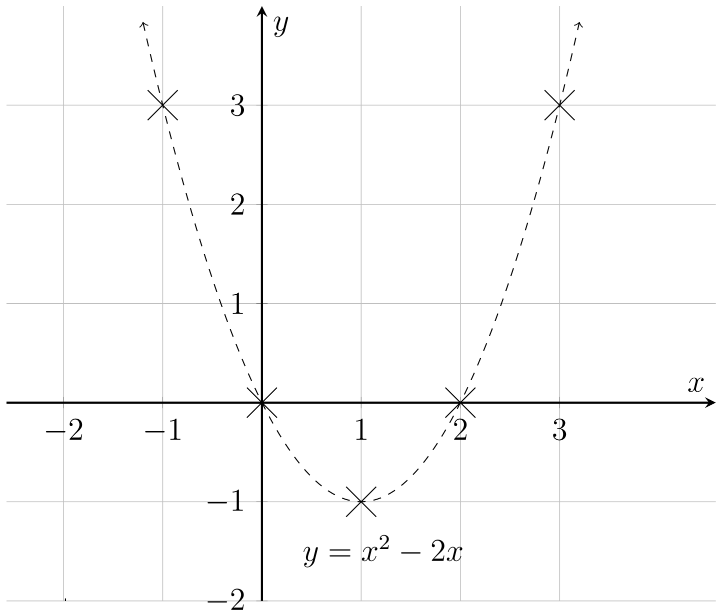
For example, to find the roots and vertex of $y = x^2 - 2x$:

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{-2}{2} = 1$, so calculating some points either side of that:

x	-1	0	1	2	3
y	3	0	-1	0	3

$$y = x^2 - 2x$$

Plotting these coordinates and drawing a curve between them:



The roots can be seen at $(0, 0)$ and $(2, 0)$, and the vertex is at $(1, -1)$.

Finding a quadratic equation given its roots and vertex

Given an observed parabola, such as the path of a thrown object, it is possible to work out its equation.

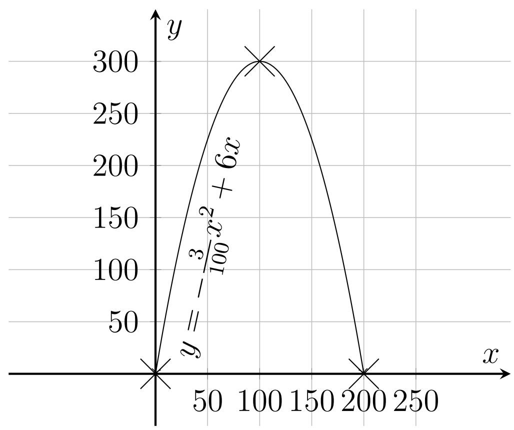
Say an object was fired into the air, reached a height of 300 metres, and landed 200 metres away. That gives us 3 coordinates: The roots are at (0,0) and 200,0). The axis of symmetry is halfway between the starting and ending points, at 100 metres, and the vertex is at (100, 300).

Substituting the vertex (100, 00) into the vertex form of a quadratic equation, we get $y = a(x - 100)^2 + 300$.

To find a, substitute either of the roots into this. Using (0,0): $0 = a(0-100)^2+300 \implies a = -\frac{3}{100}$.

Substituting $a = -\frac{3}{100}$ back into the equation: $y = -\frac{3}{100}(x-100)^2+300$.

Rearranging this to get the equation from vertex form into standard form: $y = -\frac{3}{100}x^2+6x$.



Any point along the curve can now be calculated.

Another method of finding the equation given roots and vertex:

Given the roots (1,0),(-5,0) and the vertex at (-2,-9), form an equation from the roots using the null factor law:

$$\begin{array}{r|r} x=1 & x=-5 \\ \text{x-1=0} & x+5=0 \end{array}$$

$(x-1)(x+5)=0 \implies x^2+4x-5=0$

Other parabolas can share these roots
so verify by substituting in the vertex $(-2,-9)$:

$$\begin{aligned} -2^2+4(-2)-5 &= -9 \\ -9 &= -9 \checkmark \end{aligned}$$

Finding a quadratic equation given the vertex and a point on the parabola

- Vertex: $(2, 3)$ $[h = 2, k = -3]$
- A point on the curve: $(5, 6)$

The quadratic equation in vertex form: $y = a(x - 2)^2 - 3$
Multiply out to get standard form:

$$\begin{aligned}
y &= a(x-2)^2 - 3 \\
y &= a(x-2)(x-2) - 3 \\
y &= ax^2 - 2x - 2x + 4 - 3 \\
y &= ax^2 - 4x + 1
\end{aligned}$$

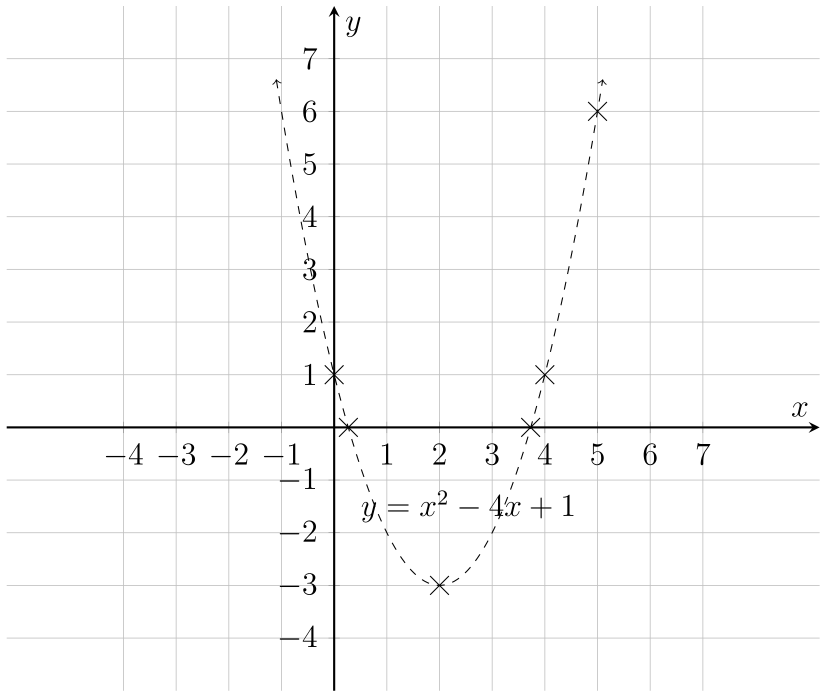
Substitute in the given (x, y) coordinate:

$$\begin{aligned}
6 &= a \cdot 5^2 - 4 \cdot 5 + 1 \\
6 &= a \cdot 25 - 20 + 1 \\
6 &= a \cdot 25 \\
a &= 1
\end{aligned}$$

The equation is $y = x^2 - 4x + 1$

Graphing this:

- Given vertex: $(2, 3)$
- Given point: $5, 6$
- Roots: $x = \frac{-(-4) \pm \sqrt{-4^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{4 \pm \sqrt{12}}{2} \approx 3.73 \ \&0.27$
- y-intercept $= c = 1$
- Symmetrically opposite point to y-intercept: $(2h, c) = (4, 1)$



Finding points of intersection

Intersection of Two Lines

At the point where two lines intersect, that coordinate satisfies both linear equations simultaneously.

Given two lines:

$$y = m_1x + c_1$$

$$y = m_2x + c_2$$

To find their intersection, set the two equations equal to each other:

$$m_1x + c_1 = m_2x + c_2$$

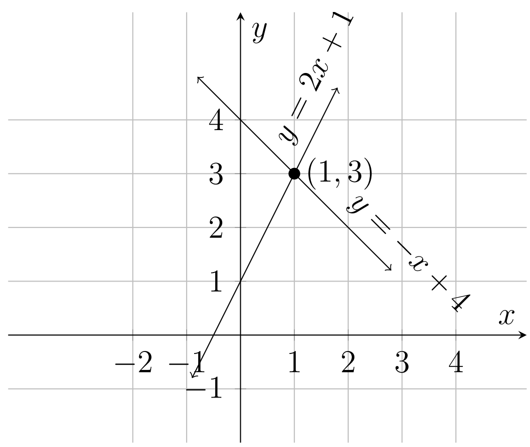
Solve for x, and then substitute x back into either equation to find y.

For example,

Find the intersection of the lines $y = 2x + 1$ and $y = -x + 4$.

$$\begin{aligned} 2x + 1 &= -x + 4 \\ 3x &= 3 \\ x &= 1 \\ y &= 2(1) + 1 = 3 \end{aligned}$$

The intersection point is $(1, 3)$:



Intersection of a Line and a Parabola

The intersection points of a line and a parabola occur where the coordinates satisfy both equations. That point is found by substituting the linear equation into the quadratic equation and solving the resulting quadratic equation.

Given a line $y = mx + c$ and a parabola $y = ax^2 + bx + d$, substitute the linear equation into the quadratic:

$$mx + c = ax^2 + bx + d$$

Rearrange to form a quadratic equation:

$$ax^2 + (b - m)x + (d - c) = 0$$

Solve this quadratic equation for x. Then, substitute the solutions back into the linear equation to find the corresponding y values.

For example,

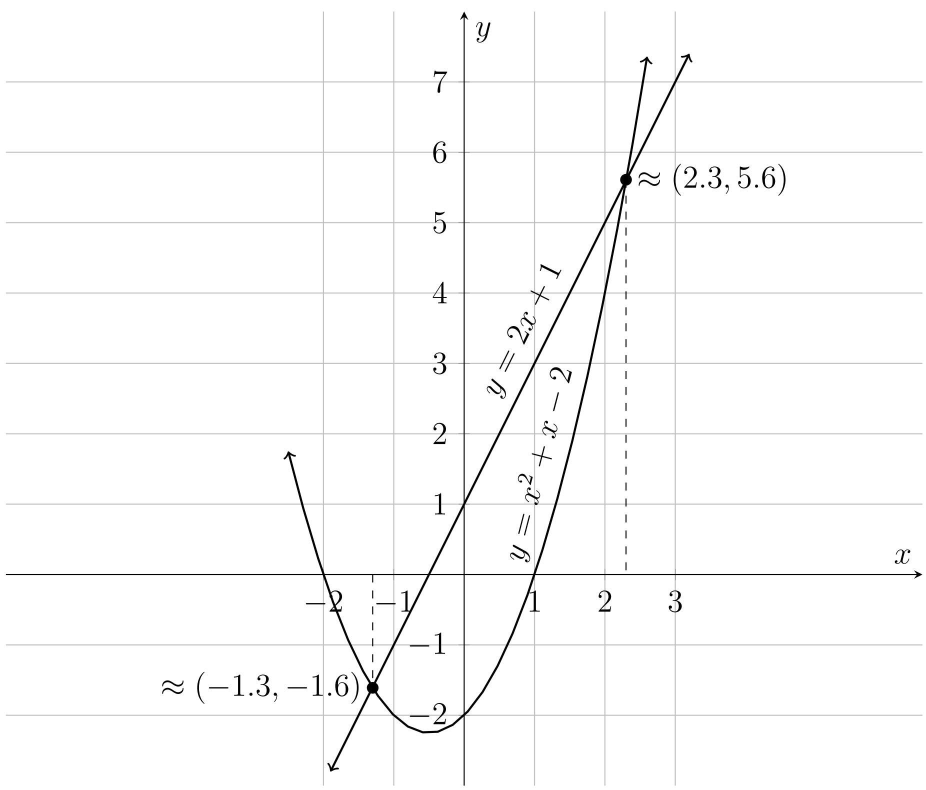
Find the intersection of the line $y = 2x+1$ and the parabola $y = x^2+x-2$:

$$\begin{aligned} 2x + 1 &= x^2 + x - 2 \\ 0 &= x^2 - x - 3 \\ x &= \frac{1 \pm \sqrt{13}}{2} \approx 2.30 \ \& -1.30 \end{aligned}$$

Substitute these into the line equation to find the corresponding y values:

$$y = 2\left(\frac{1 \pm \sqrt{13}}{2}\right) + 1 \approx 5.61 \ \& -1.61$$

So the points of intersection are $\approx (2.30, 5.61)$ and $\approx (-1.30, -1.61)$.



Solving a quadratic equation by adding a line

Given any parabola formed by the equation $y = ax^2 + bx + c$,
any quadratic equation $Ax^2+Bx+C = 0$ can be solved by drawing a straight line.

Given:

$$Ax^2 + Bx + C = 0$$

Divide by A and multiply by a:

$$ax^2 + \frac{aB}{A}x + \frac{aC}{A} = 0$$

Isolate ax^2:

$$ax^2 = -\frac{aB}{A}x - \frac{aC}{A}$$

Substitute:

$$ax^2 + bx + c = -\frac{aB}{A}x - \frac{aC}{A} + bx + c$$

Factorize into standard linear form:

$$ax^2 + bx + c = (b - \frac{aB}{A})x + (c - \frac{aC}{A})$$

The intersection of $y = ax^2 + bx + c$ and $y = (b - \frac{aB}{A})x + (c - \frac{aC}{A})$ then solves $y = Ax^2 + Bx + C = 0$.

A graphical solution such as this can be a simpler method when an exact solution is not required.

For example, Using $y = x^2 - 2x - 4$ to solve $0 = 2x^2 - 3x - 6$:

You could do:

Set the equations equal:

$$x^2 - 2x - 4 = -\frac{1}{2}x - 1$$

Move all terms to one side:

$$2(x^2 - 2x - 4) = 2(-\frac{1}{2}x - 1)$$
$$2x^2 - 4x - 8 = -x - 2$$
$$2x^2 - 4x - 8 + x + 2 = 0$$
$$2x^2 - 3x - 6 = 0$$

Use the quadratic formula to find the roots:

$$x = \frac{3 \pm \sqrt{57}}{4} \approx 2.64 \ \& - 1.14$$

But an easier solution could be to draw the parabola and to find the equation of the line to draw across it:

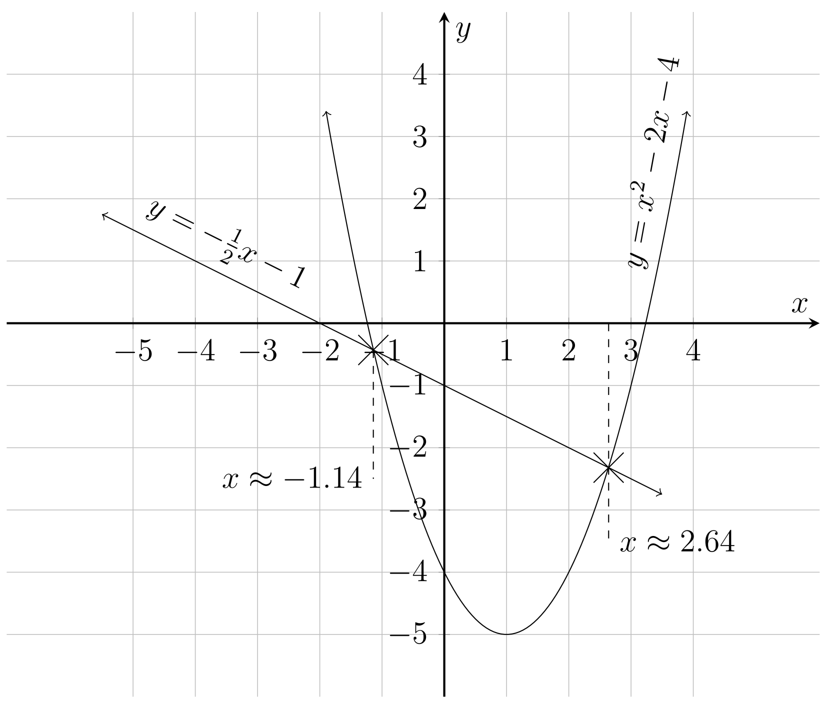
$$y = (b - \frac{aB}{A})x + (c - \frac{aC}{A})$$

$$y = (-2 - \frac{1 \cdot -3}{2})x + (-4 - \frac{1 \cdot -6}{2})$$

$$y = (-2 + \frac{3}{2})x + (-4 + 3)$$

$$y = -\frac{1}{2}x - 1$$

Plotting these curves and reading off the roots:



Using $y = x^2$ to solve a quadratic equation

In the same way that adding a line can be a simple way to solve a quadratic equation, the parabola $y = x^2$ can be used to solve quadratic equations.

For example, Using $y = x^2$ to solve $3x^2 - 6x - 4 = 0$:

$$\begin{aligned} 3x^2 - 6x - 4 &= 0 \\ x^2 - 2x - \frac{4}{3} &= 0 \\ x^2 &= 2x + \frac{4}{3} \end{aligned}$$

So plot $y = 2x + \frac{4}{3}$ and $y = x^2$: