Trigonometry

Tutoring Centre Ferndale

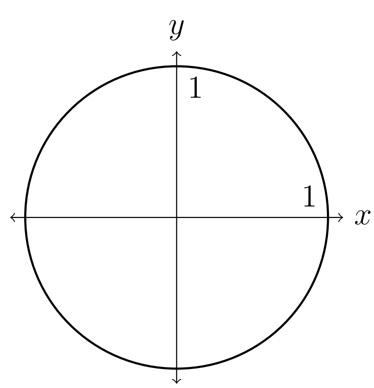
Trigonometry is the study of the relationships between the angles and sides of triangles, particularly right-angled triangles. Trigonometry is a Greek word that means "three side measuring." It has wide applications in various fields such as physics, engineering, and astronomy.

The Unit Circle

The unit circle is a circle with a radius of 1 centered at the origin of the coordinate plane. It is defined by the equation:

$$x^2 + y^2 = 1$$

where x and y are the coordinates of any point on the circle.

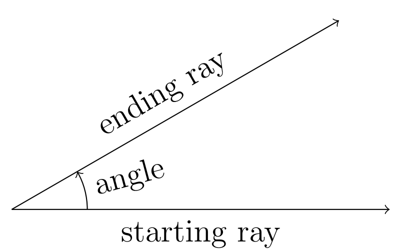


It is a fundamental tool in trigonometry because it provides a reference for the measurement of angles and other geometric shapes.

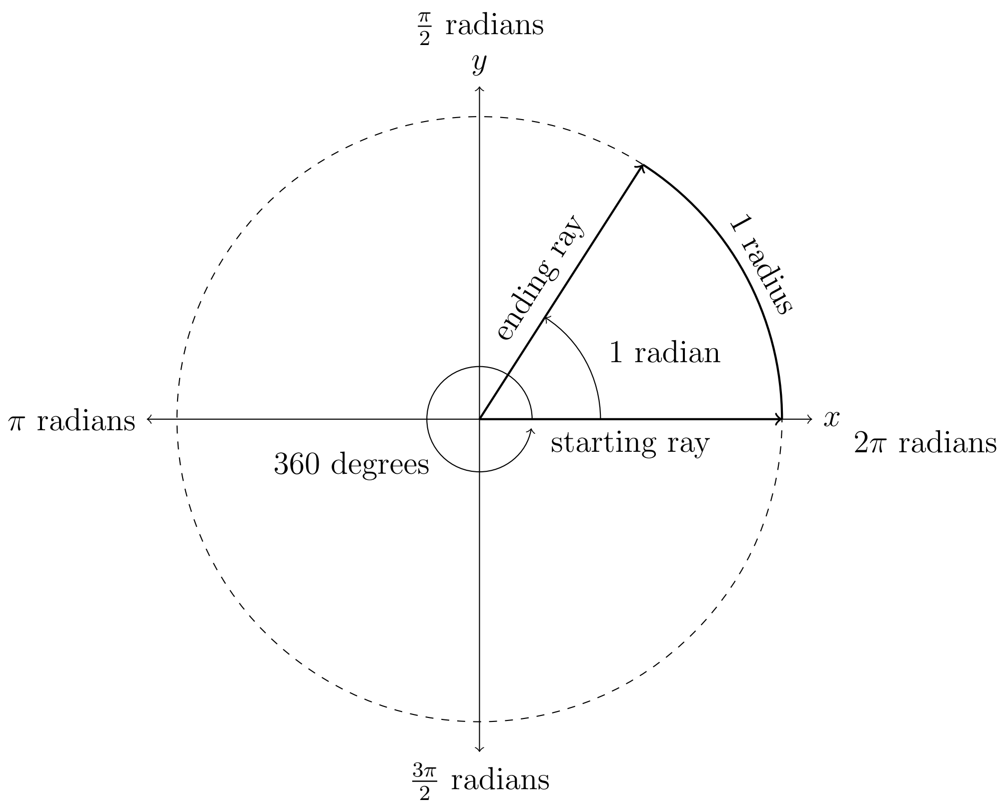
Angles

- **A point** is a specific location in space with no dimensions — no length, width, or height. In the coordinate plane, a point is defined by a pair of coordinates (x,y), which specify its position relative to the origin.
- **A line** is a straight, continuous set of points that extends infinitely in both directions. It is defined by a linear equation in the form y=mx+c, where m is the slope and c is the y-intercept.
- A line segment is a part of a line that is bounded by two distinct endpoints. It contains all the points on the line that lie between those two endpoints. A line segment has a finite length.
- **A ray** is a part of a line that starts at an endpoint and extends infinitely in one direction.
- **An angle** is formed by two rays or line segments that share a common endpoint, known as the vertex.

- An angle is measured by the amount of rotation from the initial side (starting ray) to the terminal side (ending ray).



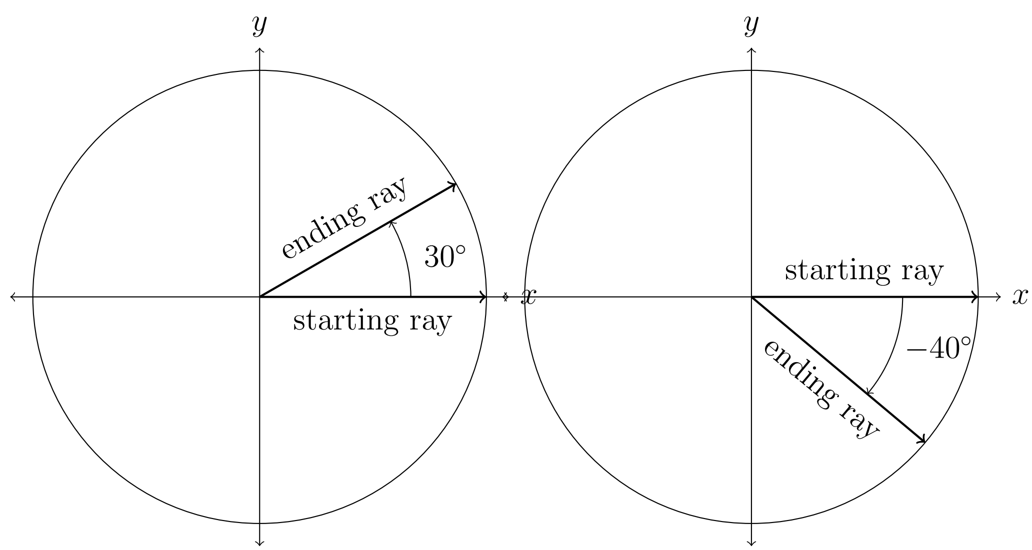
- Angles can be measured in degrees or radians or gradians.
  - **A degree** is $\frac{1}{360}^{\text{th}}$ of a full circle.
  - **A radian** is the angle made by extending the length of one radius along the circumference of a circle between the starting and ending rays.
  - Since π is defined as the ratio between the circumference and the diameter of a circle, $\pi = \frac{C}{D}$, and the diameter of a circle is twice the radius, $\pi = \frac{C}{2r}$, there are 2π radii in a full circle.
  - 1 radian $\approx 57.3°$.
  - **A gradian** (also known as a gon or a grade) is $\frac{1}{400}^{\text{th}}$ of a circle. Gradians are less commonly used than degrees and radians, particularly outside of Europe.



Angles in Standard Position

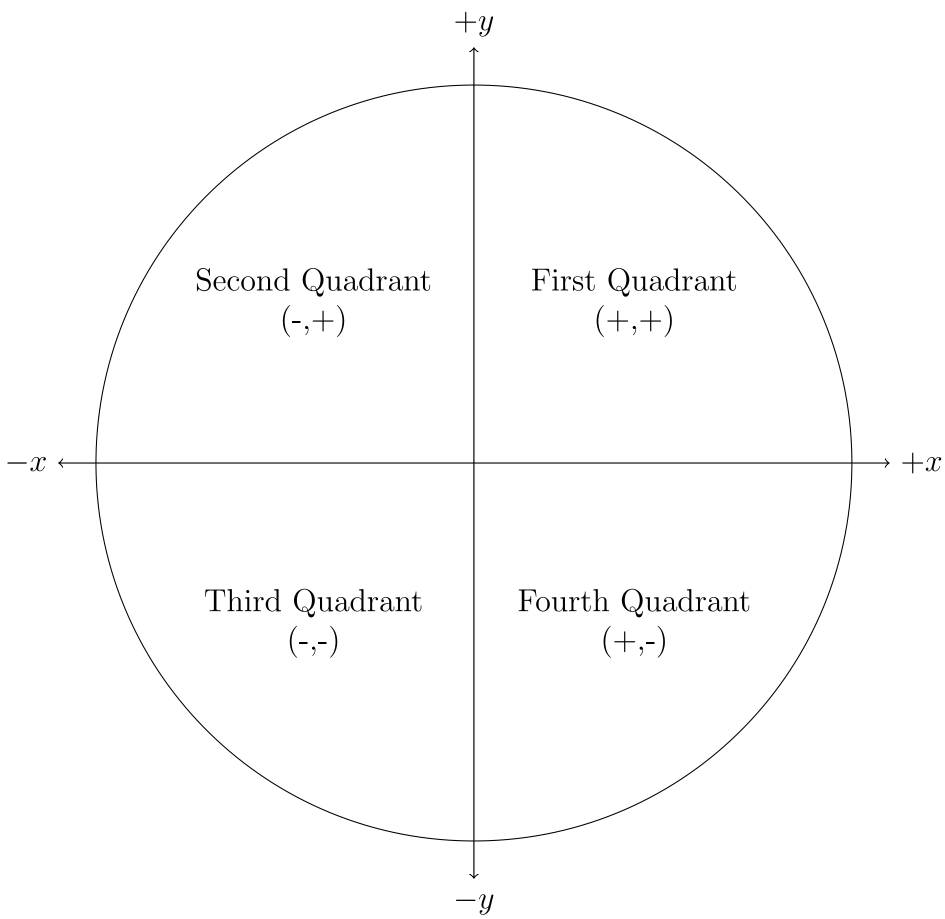
An angle is in standard position when it is drawn in the coordinate plane with its vertex at the origin $(0, 0)$ and its initial side along the positive x-axis.

- **Initial Side:** The initial side of the angle is always placed along the positive x-axis.
- **Terminal Side:** The terminal side is the ray that rotates about the vertex (the origin) to form the angle. The amount of rotation determines the measure of the angle.
- **Sign of the Angle:**
  - Counterclockwise Rotation: If the terminal side rotates counterclockwise from the initial side, the angle is considered positive.
  - Clockwise Rotation: If the terminal side rotates clockwise, the angle is considered negative.



- **The Greek letter theta** θ is commonly used in trigonometry to label angles. It was originally chosen, perhaps, because it is composed of a circle and a line which suggests the idea of an angle or part of a circle.
- **Degrees** are abbreviated as a small raised circle, originally chosen, perhaps, because a degree is a measure of a circle.

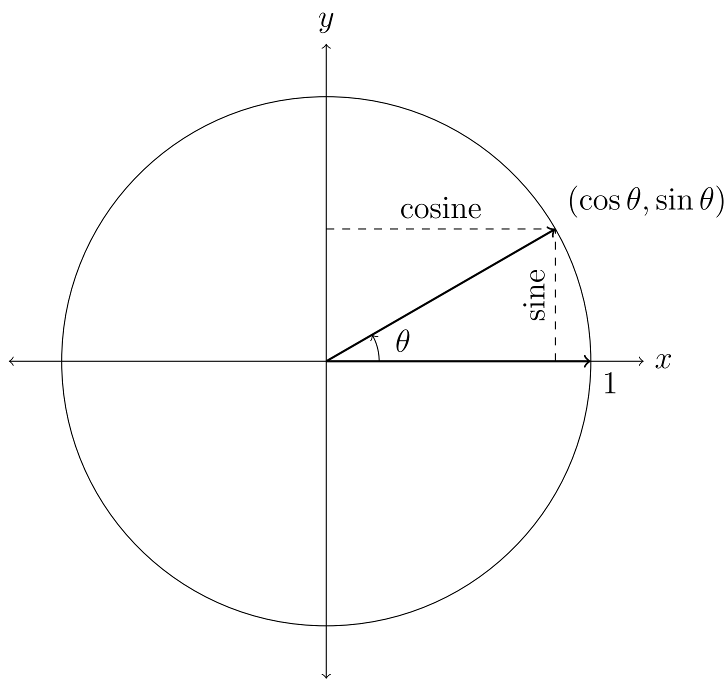
- **Quadrants:** Depending on the rotation, the terminal side of the angle may lie in one of the four quadrants of the coordinate plane:
  - First Quadrant: If the terminal side is between the positive x-axis and the positive y-axis.
  - Second Quadrant: If the terminal side is between the positive y-axis and the negative x-axis.
  - Third Quadrant: If the terminal side is between the negative x-axis and the negative y-axis.
  - Fourth Quadrant: If the terminal side is between the negative y-axis and the positive x-axis.



Sine and Cosine

For any angle θ measured from the positive x-axis, the coordinates of the corresponding point on the unit circle are the sine and cosine of θ, abbreviated $(\cos\theta, \sin\theta)$.

Sine is a Latin word for curve that is a mistranslation of an Arabic word for bowstring.



Inverse Functions

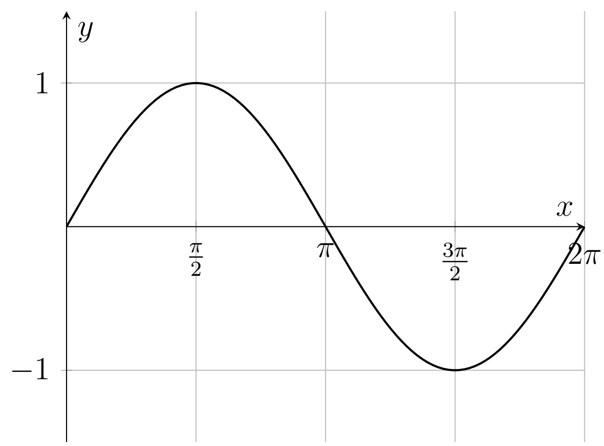
- The **arcsine** (arcsin) or $\sin^{-1}$ is the inverse of the sine, giving the angle whose sine is a given value.
- The **arccosine** (arccos) or $\cos^{-1}$ is the inverse of the cosine, giving the angle whose cosine is a given value.
- The *arc-* prefix refers to the arc of the unit circle measured between the rays of an angle.

The Sine Function

The sine function is defined as:

$$y = \sin(x)$$

The graph of the sine function is a smooth, periodic curve that oscillates between $y = -1$ and $y = 1$. The period of the sine function is 2π, meaning that the function repeats itself every 2π units along the x-axis.



The general form of the sine function is:

$$y = A\sin(Bx + C) + D$$

where:

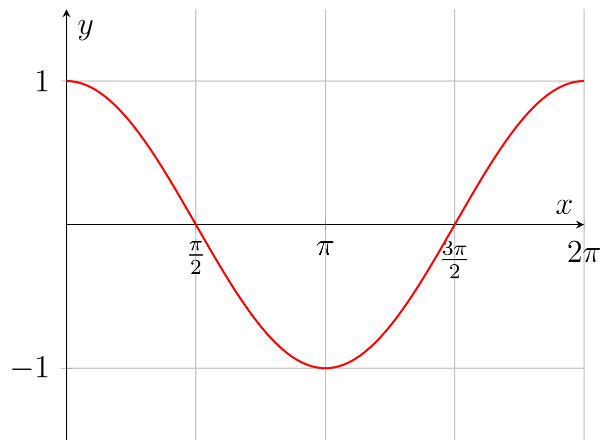
- A is the amplitude (the maximum height of the wave).
- B affects the period (the wave frequency, which is $\frac{2\pi}{|B|}$).
- C causes a horizontal shift.
- D causes a vertical shift.

The Cosine Function

The cosine function is defined as:

$$y = \cos(x)$$

Like the sine function, the cosine function oscillates between $y = -1$ and $y = 1$ with a period of 2π.



The general form of the cosine function is similar to the sine function:

$$y = A\cos(Bx + C) + D$$

The parameters A, B, C, and D have the same effects on the cosine function as they do on the sine function.

Tangent and Cotangent

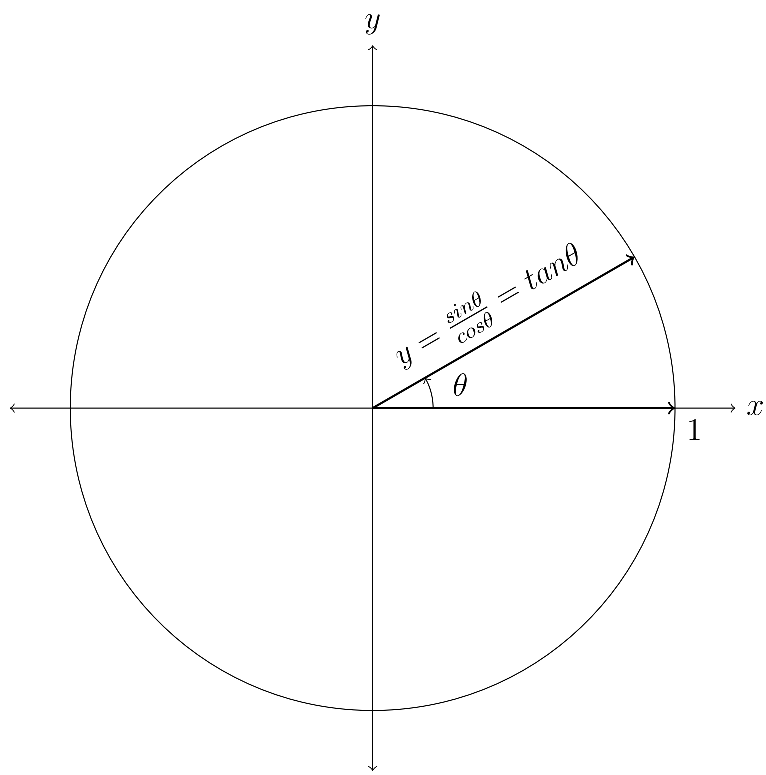
- The tangent of the angle θ is the ratio of the sine to the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

- The cotangent of the angle θ is the ratio of the cosine to the sine:

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Given the equation of a line $y = mx + b$ where m is the slope $m = \frac{rise}{run}$, the tangent can be interpreted as the slope of the line connecting the origin to the point $(\cos\theta, \sin\theta)$ on the unit circle.

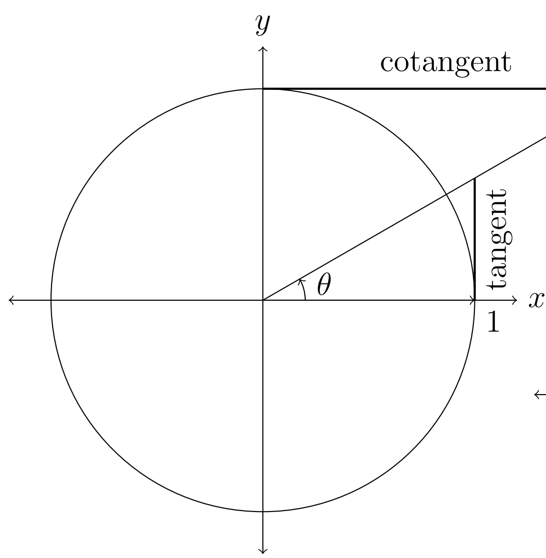


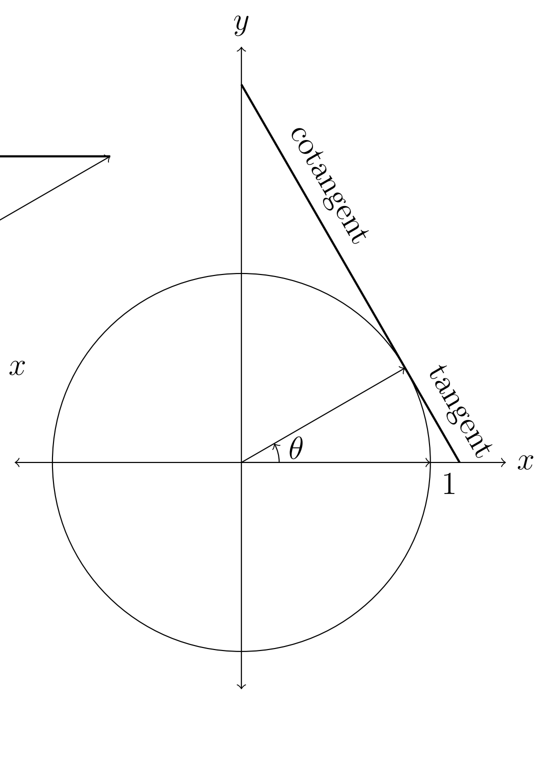
- Tangent is from a Latin word that means to touch. A tangent is any line that touches a circle at a single point without crossing it.

- In trigonometry, the tangent and cotangent of an angle can be visualized in two ways:

  - The tangent of an angle is the length of a vertical line tangent to the unit circle from the x-axis to the terminating ray.
    The cotangent of an angle is the length of a horizontal line tangent to the unit circle from the y-axis to the terminating ray.

  - The tangent of an angle is the length, from the x-axis to the terminating ray, of a line tangent to the unit circle at the point of intersection of the terminating ray.
    The cotangent of an angle is the length, from the y-axis to the terminating ray, of a line tangent to the unit circle at the point of intersection of the terminating ray.





Inverse Functions

- The **arctangent** (arctan) or $\tan^{-1}$ is the inverse of the tangent, giving the angle whose tangent is a given value.
- The **arccotangent** (arccot) or $\cot^{-1}$ is the inverse of the cosine, giving the angle whose cotangent is a given value.

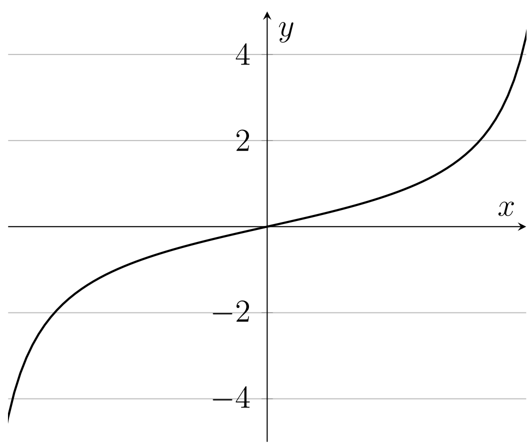
The Tangent Function

The tangent function is defined as:

$$y = \tan(x)$$

Unlike sine and cosine, the tangent function is not bounded. It has vertical asymptotes at $x = \frac{\pi}{2} + n\pi$, where n is an integer. The period of the tangent function is π, and it repeats every π units.

Below is the graph of the tangent function:



The general form of the tangent function is:

$$y = A\tan(Bx + C) + D$$

The parameters A, B, C, and D influence the tangent function similarly to how they influence the sine and cosine functions, with B affecting the period, C shifting the graph horizontally, and D shifting it vertically. However, the concept of amplitude does not apply to the tangent function as it is unbounded.

Secant and Cosecant

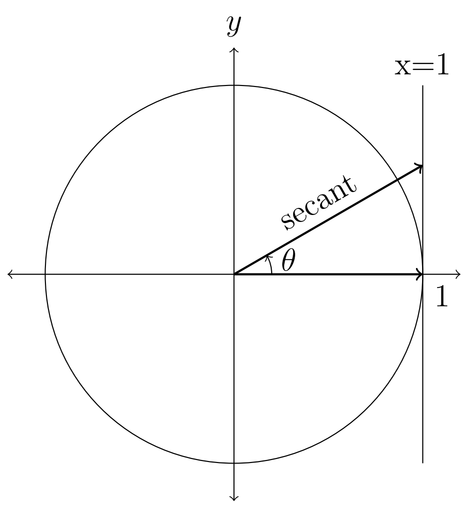
- The secant is the reciprocal of the cosine: $\sec\theta = \frac{1}{\cos\theta}$.

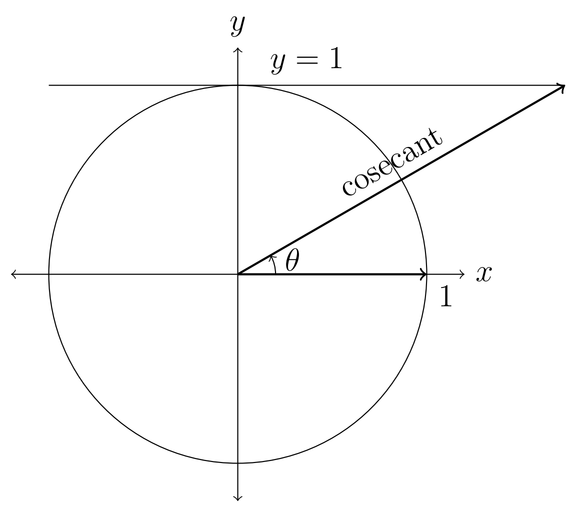
  The secant of an angle is the length from the origin to the point on the terminating ray that intersects a vertical line at $x = 1$.

  Secant is a Latin word that means to cut because the secant cuts through the circle.

- The cosecant is the reciprocal of the sine: $\csc\theta = \frac{1}{\sin\theta}$.

  The cosecant of an angle is the length from the origin to the point on the terminating ray that intersects a horizontal line at $y = 1$.





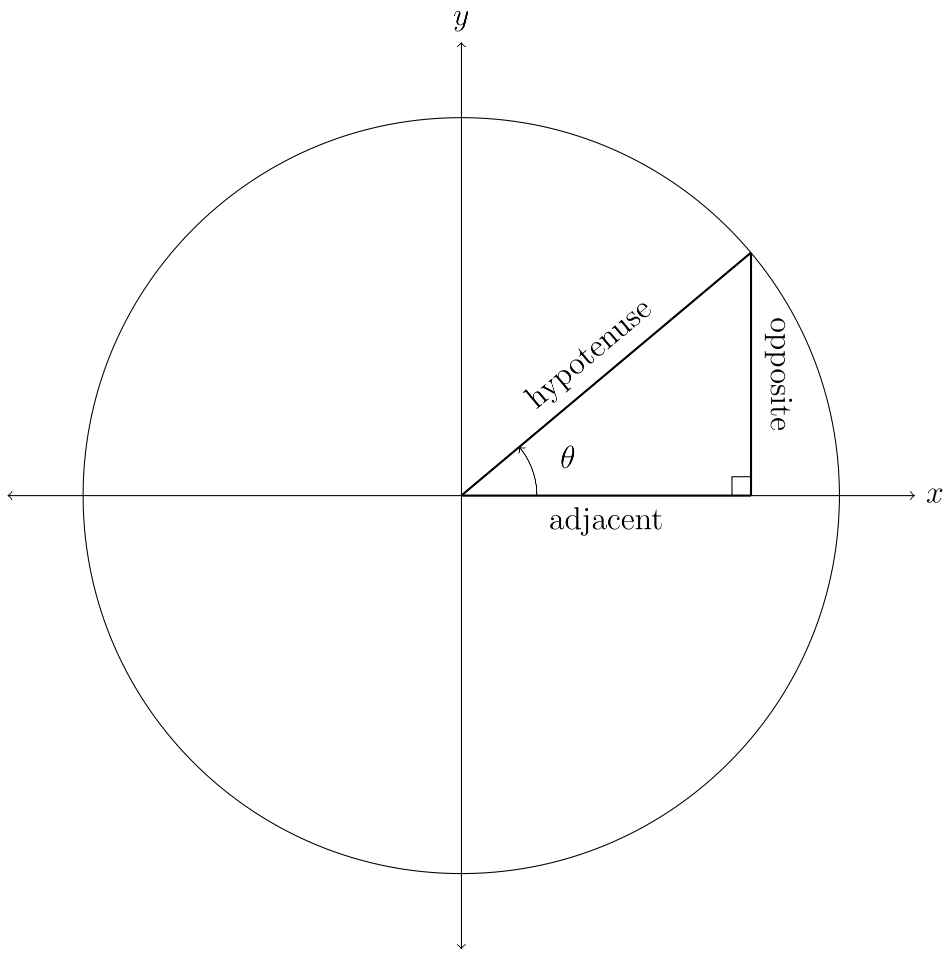
Inverse Functions

- The **arcsecant** (arcsec) or $\sec^{-1}$ is the inverse of the secant, giving the angle whose secant is a given value.

- The **arccosecant** (arccsc) or $\csc^{-1}$ is the inverse of the cosine, giving the angle whose cotangent is a given value.

SOHCAHTOA

The sides of a right triangle are named relative to the angle θ in standard position:



SOHCAHTOA stands for:

$$\textbf{SOH: } \text{Sine} = \tfrac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\textbf{CAH: } \text{Cosine} = \tfrac{\text{Adjacent}}{\text{Hypotenuse}}$$

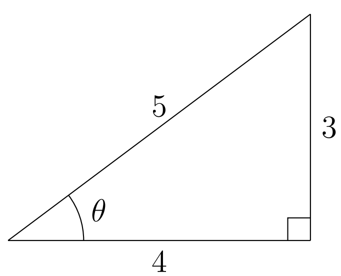
$$\textbf{TOA: } \text{Tangent} = \tfrac{\text{Opposite}}{\text{Adjacent}}$$

Using SOHCAHTOA

Given a right triangle where:

- The length of the opposite side is 3
- The adjacent side is 4
- The hypotenuse is 5

We can calculate the sine, cosine, and tangent of the angle θ:



$$\sin\theta = \tfrac{3}{5}, \quad \cos\theta = \tfrac{4}{5}, \quad \tan\theta = \tfrac{3}{4};$$

The unknown angles can be found using inverse functions:

$$\sin\theta = \frac{3}{5}$$
$$\sin^{-1}\frac{3}{5} \implies \theta \approx 36.9^\circ$$

Then, use the fact that angles of a triangle add up to 180°, so $180 - (90 + 36.9) = 53.1^\circ$, or apply SOHCAHTOA to the other angle:

$$\cos(\text{other angle}) = \frac{3}{5}$$
$$\cos^{-1}\frac{3}{5} \implies \text{other angle} \approx 53.1^\circ$$

Area of a Triangle

The area of a triangle is half the length of its base times its perpendicular height.

Consider a triangle ABC with sides a, b, and c opposite vertices A, B, and C, draw a line h perpendicular to the base c of the triangle at H to the vertex of angle C. This creates two right triangles. The area of these triangles is given by:

$$Area = \frac{1}{2} \times c \times h.$$

In the right triangle HBC we have:

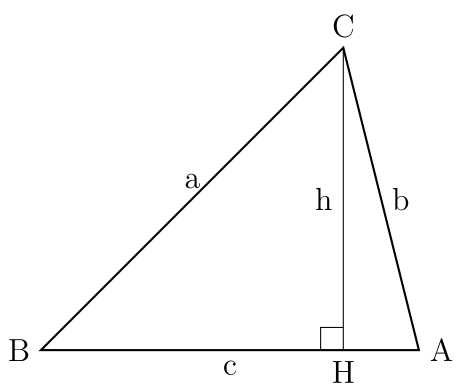
$$\sin B = \frac{h}{a} \implies h = a \sin B.$$

Substituting this into the area formula, we get:

$$Area = \frac{1}{2} \times c \times h = \frac{1}{2} \times c \times a \sin B.$$

Thus, the area of the triangle can be expressed as:

$$Area = \frac{1}{2} ac \sin B.$$



$$\begin{aligned} \text{Area} &= \frac{1}{2} ac \sin B \\ &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} bc \sin A \end{aligned}$$

Special Angles

Angle	sin	cos	tan $(\frac{\sin}{\cos})$
$0°$	0	1	0
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	1	0	undefined
$180°$	0	-1	0
$270°$	-1	0	undefined

Related Angles

$-\theta$:	$\sin -\theta = -\sin\theta$	$\cos -\theta = \cos\theta$
$90° - \theta$:	$\sin 90° - \theta = \cos\theta$	$\cos 90° - \theta = \sin\theta$
$90° + \theta$:	$\sin 90° + \theta = \cos\theta$	$\cos 90° + \theta = -\sin\theta$
$180° - \theta$:	$\sin 180° - \theta = \sin\theta$	$\cos 180° - \theta = -\cos\theta$
$180° + \theta$:	$\sin 180° + \theta = -\sin\theta$	$\cos 180° + \theta = -\cos\theta$

The Sine Rule

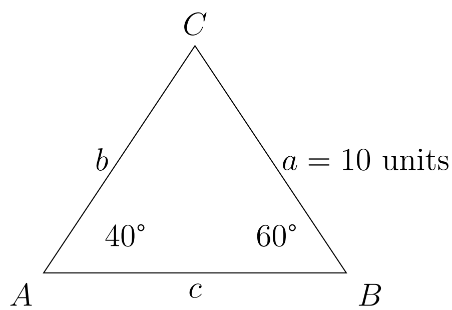
The Sine Rule relates the sides of a triangle to the sines of its angles. It is useful for finding unknown sides or angles in any triangle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Using the Sine Rule

Consider a triangle ABC where:

- Angle A = 40 degrees
- Angle B = 60 degrees
- Side a = 10 units



We want to find the length of side b.

1. Substitute the known values:

$$\frac{10}{\sin 40^\circ} = \frac{b}{\sin 60^\circ}$$

2. Solve for b:

$$b = 10 \cdot \frac{\sin 60^\circ}{\sin 40^\circ} \approx 12.85 \text{ units}$$

The Cosine Rule

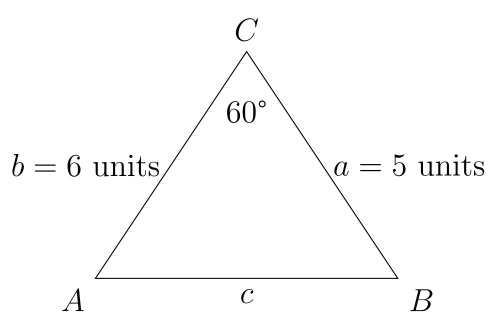
The Cosine Rule relates the sides of a triangle to the cosine of one of its angles. It is useful for finding the length of a side when two sides and the included angle are known:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Using The Cosine Rule

Given a triangle with:

- Angle C = 60°
- Side a = 5 units
- Side b = 6 units



We can find the side c using the Cosine Rule:

$$c^2 = 5^2 + 6^2 - 2 \times 5 \times 6 \times \cos 60^\circ$$

$$c = \sqrt{25 + 36 - 60 \times \frac{1}{2}} = \sqrt{31} \approx 5.57$$

Trigonometric Identities

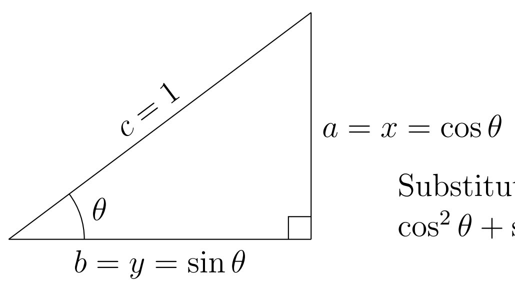
Trigonometric identities are equations involving trigonometric functions that are true for every value of the variable for which both sides are defined. They are useful for simplifying trigonometric expressions and solving trigonometric equations.

The Pythagorean Identity

The most fundamental trigonometric identity is the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1$$

This follows from the Pythagorean Theorem $a^2 + b^2 = c^2$ and the definition of sine and cosine on the unit circle:



Substituting these into $a^2 + b^2 = c^2$:
$\cos^2\theta + \sin^2\theta = 1$

Other Important Identities

$$\begin{aligned}
1 + \tan^2\theta &= \sec^2\theta \\
\sin(2\theta) &= 2\sin\theta\cos\theta \\
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
&= 2\cos^2\theta - 1 \\
&= 1 - 2\sin^2\theta \\
\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}$$

Simplifying Expressions

For example, consider the expression:

$$\frac{\sin(2\theta)}{\cos\theta}$$

We can use the identity for $\sin(2\theta)$:

$$\frac{2\sin\theta\cos\theta}{\cos\theta} = 2\sin\theta$$

Thus, the expression simplifies to $2\sin\theta$.

Solving Equations

For example, solve the equation:

$$\sin^2\theta - \sin\theta = 0$$

Factor the equation:

$$\sin\theta(\sin\theta - 1) = 0$$

This gives two solutions:

$$\sin\theta = 0 \quad \text{or} \quad \sin\theta = 1$$

So, $\theta = n\pi$ or $\theta = \frac{\pi}{2} + 2n\pi$ for integer n.

Graphical Solutions

Graphical solutions are a powerful tool for solving trigonometric equations, especially when algebraic solutions are complex or unknown. By plotting the functions involved and identifying their intersections, we can visually determine approximate solutions.

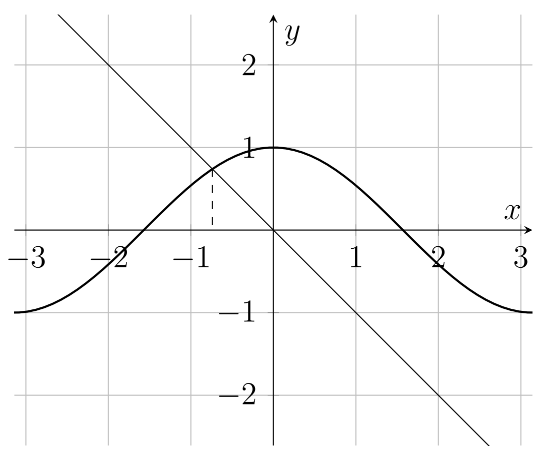
Example 1: Solving $x + \cos(x) = 0$

To solve this equation graphically, rewrite it as:

$$\cos(x) = -x$$

Then graph the functions $y = \cos(x)$ and $y = -x$ and find their point of intersection.

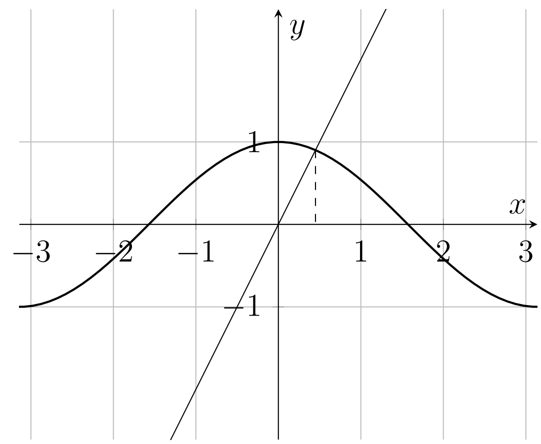
Graph of $y = \cos(x)$ and $y = -x$:



The solution to $x + \cos(x) = 0$ is where these two graphs intersect. The intersection occurs near $x \approx 0.74$.

Example 2: Solving $\cos(x) = 2x$

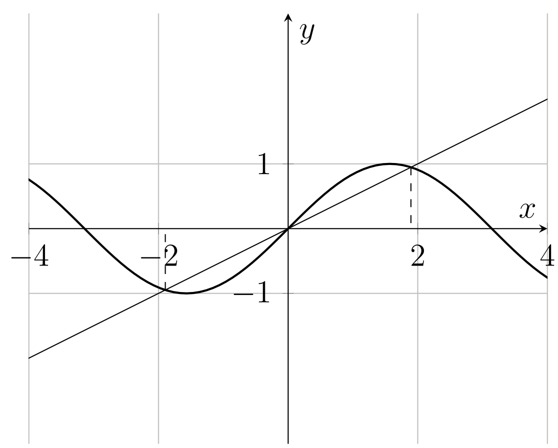
Plot $y = \cos(x)$ and $y = 2x$ and find the intersection.



The graphs intersect at $x \approx 0.45$.

Example 3: Solving $\sin(x) = \frac{x}{2}$

Plot $y = \sin(x)$ and $y = \frac{x}{2}$ to find their intersection:



The intersections occur near $x \approx 0.93, x = 0$, and $x \approx 0.93$.