Quadratic Equations

Tutoring Centre Ferndale

Quadratic Equations

- **Quadratic** is a Latin word that means to do with squares. This is because the variable in a quadratic equation is squared, meaning raised to the second power. Quadratic equations were originally studied with regard to geometric problems involving squares and rectangles. Finding the areas and the side lengths of squares and rectangles are quadratic problems.
- **Degrees** of equations refer to the highest power of the variables in the equation. This tells you the complexity of the relationship between the variables and helps in identifying the type of equation.
- **Polynomials** are mathematical expressions that consist of variables and coefficients, combined using addition, subtraction, and multiplication, but no division by variables. They are made up of terms, each term being a product of a number (called the coefficient) and a variable raised to a non-negative integer power.
- **A Quadratic Equation** is a second-degree polynomial equation in a single variable x, with the standard form:

  $$ax^2 + bx + c = 0$$

  where a, b, and c are constants, and $a \neq 0$.

Terms of a Quadratic Equation

- The first term is called the quadratic term.
(because x is to the second power, or squared.)
- The second term is called the linear term.
(because x is to the first power, as in linear equations.)
- The third term is called the constant term.

$$\underbrace{ax^2}_{\text{quadratic term}} \quad + \quad \overbrace{bx}^{\text{linear term}} \quad + \quad \underbrace{c}_{\text{constant term}} \quad = 0$$

Roots of a Quadratic Equation

Because the product of two negative numbers is a positive number, the square of a number has both a positive root and a negative root.

$$\text{e.g. } \sqrt{4} = \pm 2 : \text{ both } 2 \times 2 = 4 \text{ and } -2 \times -2 = 4.$$

Because one of its terms is a square, the solutions to a quadratic equation, the value or values of x which make it equal to zero, are also called its roots.

$$\text{e.g. } 6x^2 + 5x - 4 = 0 : \text{ This equation is true only for } x = \frac{1}{2} \text{ and } x = -\frac{3}{4}.$$

A quadratic equation can have either no roots, meaning there is no possible solution for the equation, or only one root, or two roots.

Factorizing

If a quadratic equation has at least one root, then it can be factored by expressing it as the product of two linear binomials.

For example, $x^2 + 5x + 6 = (x + 2)(x + 3)$.

Knowing how to factorize a quadratic equation is important because the factors are used to find its roots.

Null Factor Law

The null factor law is that if the product of a multiplication is zero then one or more factors must also be zero.

The roots of a quadratic equation are found by applying the null factor law to its factors.

For example, $x^2 - 5x + 6 = 0$ can be factored as $(x - 2)(x - 3) = 0$.

Setting each factor to 0, according to the null factor law:

$(x - 2) = 0 \implies x = 2$, or $(x - 3) = 0 \implies x = 3$

So the roots are $x = 2$ and $x = 3$.

Monic and Non-monic Quadratic Equations

- A monic quadratic equation means one where $a = 1$.
- A non-monic quadratic equation means, of course, one where $a \neq 1$.
- Monic quadratic equations are simpler to factorize than non-monic quadratic equations, so the first step of factorizing a quadratic equation, if it is non-monic, is to express it as a monic quadratic equation.

Normalizing

Normalizing means adjusting values so that they are standardised within a given range, usually between 0 and 1. This makes comparisons and visualizations of data much easier. This is used in various fields such as statistics, probability, and data processing.

Normalizing a quadratic equations means to rearrange the equation so that its leading coefficient is equal to 1, making it a monic quadratic equation that is then easier to solve.

- Normalizing is done by dividing the entire equation by the coefficient.

  For example, $2x^2 + 2x - 12 = 0$:

  $$\frac{2x^2 + 2x - 12}{2} = \frac{0}{2} \implies x^2 + x - 6 = 0$$

  which factors as $(x - 2)(x + 3) = 0$.

- It may also be possible to find a common factor for just the quadratic and linear terms resulting in a pair of factors, one of which is a monic quadratic expression which can be factored.

  For example, $2x^2 + 8x + 6 = 0$:

  $2(\frac{2x^2+8x}{2}) + 6 = 0 \implies 2(x^2 + 4x) + 6 = 0.$

  Then the quadratic expression $x^2 + 4x$
  can be factorized as $x(x + 4) \implies 2(x(x + 4)) + 6 = 0$

Difference of Squares

- Quadratics in the form $x^2 - n^2$ factor as $(x-n)(x+n)$.
- Quadratics in the form $mx^2 - n^2$
  factor as $(\sqrt{m}x)^2 - n^2 = (\sqrt{m}xn)(\sqrt{m}x+n)$.
- Quadratics in the form $m^2x^2 - n^2 (m > 0)$
  factor as $(mx)^2n^2 = (mx-n)(mx+n)$.

For example, $x^2 - 16 = 0$:

1. Express 16 as a square: $x^2 - 4^2 = 0$
2. Difference of Squares: $(x-4)(x+4) = 0$
3. Null factor law: the roots are $x = 4$ and $x = -4$.

Another example: $2x^2 - 18 = 0$:

1. Take out the common factor: $2(x^2 - 9) = 0$
2. Divide both sides by that factor: $\frac{2(x^2-9)}{2} = \frac{0}{2} \implies x^2 - 9 = 0$
3. Express 9 as a square: $x^2 - 3^2 = 0$
4. Difference of Squares: $(x+3)(x-3) = 0$
5. Null factor law: the roots are $x = -3$ and $x = 3$.

And this example: $8x^2 - 50 = 0$:

1. Apply the difference of squares formula:
   $8x^2 - 50 = (\sqrt{8}x)^2 - (\sqrt{50})^2 = (\sqrt{8}x - \sqrt{50})(\sqrt{8}x + \sqrt{50})$.
2. Surd laws:
   $\sqrt{8}x = \sqrt{4 \cdot 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$
   $\sqrt{50} = \sqrt{25 \cdot 2}\sqrt{25}\sqrt{2} = 5\sqrt{2})$
3. Substitute: $(\sqrt{8}x - \sqrt{50})(\sqrt{8}x + \sqrt{50}) = (2\sqrt{2}x - 5\sqrt{2})(2\sqrt{2}x + 5\sqrt{2})$.
4. Null factor law:
   $2\sqrt{2}x - 5\sqrt{2} = 0 \quad \text{or} \quad 2\sqrt{2}x + 5\sqrt{2} = 0$.
5. Simplify: $x = \frac{5\sqrt{2}}{2\sqrt{2}} = \frac{5}{2} \quad \text{or} \quad x = \frac{-5\sqrt{2}}{2\sqrt{2}} = -\frac{5}{2}$.

Perfect Squares

- Quadratics in the form $m^2 + 2mn + n^2$ factor as $(m+n)^2$.
- Quadratics in the form $m^2 - 2mn + n^2$ factor as $(m-n)^2$.
- Quadratics in the form $m^2x^2 + 2mnx + n^2$ factor as $(mx+n)^2$.
- Quadratics in the form $m^2x^2 - 2mnx + n^2$ factor as $(mx-n)^2$.

  (The first and last terms are perfect squares ($m^2x^2 = m \cdot m \cdot x \cdot x = m \cdot x \cdot m \cdot x = (m \cdot x) \cdot (m \cdot x) = (mx)^2$) and the middle term is twice the product of the square roots of the first and last terms.)

For example, $x^2 + 4x + 4 = 0$:

1. Quadratics in the form $m^2 + 2mn + n^2$ factor as $(m+n)^2$.
2. Express the constant term 4 as a square: $x^2 + 4x + 2^2 = 0$
3. Factorize the middle term: $x^2 + 2 \cdot 2x + 2^2 = 0$.
4. This is now in the form of a perfect square: $(x+2)^2 = 0$.

Another example, $x^2 - 10x + 25 = 0$:

1. Quadratics in the form $m^2 - 2mn + n^2$ factor as $(m-n)^2$.
2. Express 25 as a square: $x^2 - 10x + 5^2 = 0$
3. Factorize the middle term: $x^2 - 2 \cdot 5x + 5^2 = 0$
4. This is now in the form of a perfect square: $(x-5)^2 = 0$.

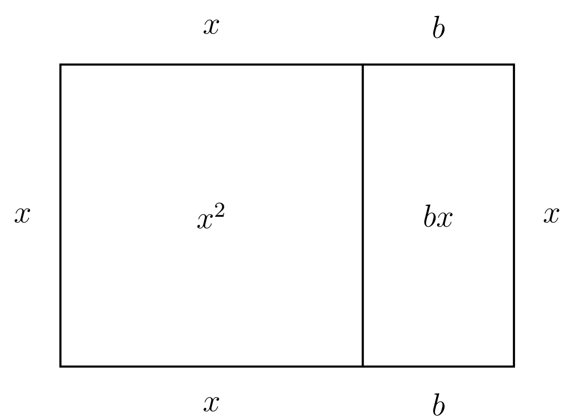
And this example, $9x^2 - 12x + 4$:

1. Quadratics in the form $m^2x^2 - 2mnx + n^2$ factor as $(mx-n)^2$.
2. First check that $9x^2 - 12x + 4$ is in fact a perfect square: The first and last terms should be perfect squares ($9x^2 = (3x)^2$ and $4 = 2^2$) and the middle term should be twice the product of the square roots of the first and last terms ($12x = 2(\sqrt{9x^2} \cdot \sqrt{4}) = 2(3x \cdot 2) = 2(6x) = 12x$.)
3. From the first term $m^2x^2 = 9x^2$ we have $m^2 = 9$ so $m = 3$.
4. From the last term $n^2 = 4$ we have $n = 2$.
5. Substituting these values for m and n: $9x^2 12x + 4 = (3x-2)^2$.

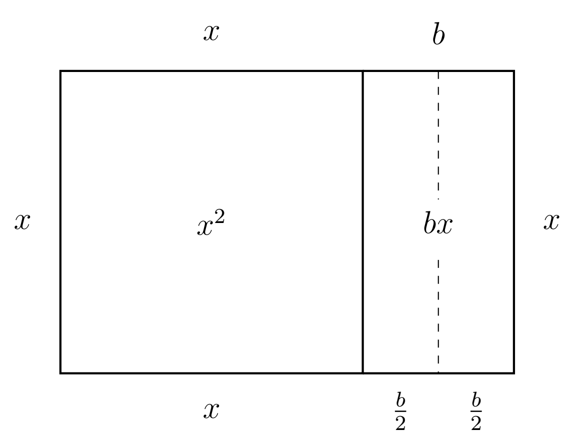
Completing the square

Completing the square is a method of solving a quadratic equation that can't easily be factored by making it into a perfect square that can be factored.

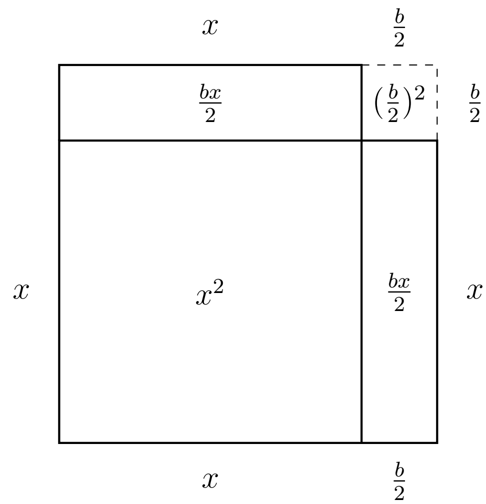
- The term x^2 can be represented as the area of a square with sides of length x.
- The term bx can be represented as the area of a rectangle with sides of length b and x, and area bx.



- The rectangle bx can then be split in two to form two rectangles, each of side lengths $\frac{b}{2}$ and x, and area $\frac{bx}{2}$.



- Positioning these two halves along the sides of the square leaves a small square with sides of length $\frac{b}{2}$ and are $(\frac{b}{2})^2$. This amount completes the square, and it can be used to solve quadratic equations.



Any quadratic expression can be written in the form of a perfect square plus or minus a constant.

Steps to Complete the Square

Given a quadratic equation $ax^2 + bx + c = 0$:

1. If it is a non-monic quadratic
   - check for common factors and factor out the quadratic and linear terms if possible to $a(x^2 + \frac{b}{a}x + ca) = 0$.
   - Otherwise divide the entire equation by a to normalize it to the monic form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.
2. Move the constant term $\frac{c}{a}$ to the other side: $x^2 + \frac{b}{a}x = -\frac{c}{a}$.
3. Add and subtract $(\frac{b}{2})^2$ to the left side: $x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 = -\frac{c}{a}$.

   (The first three terms are now in the form $x^2 + 2ax + a^2$ which is a perfect square.)

4. Rewrite these terms as a perfect square: $(x + \frac{b}{2a})^2 - (\frac{b}{2a})^2 = -\frac{c}{a}$.
5. Move the other term to the other side: $(x + \frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$
6. Simplify the right side: $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$.
7. Take the square root of both sides: $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$.
8. Solve for x: $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$.
9. Simplify: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

These steps can be applied to solve most quadratic equations, and they also show the derivation of the general quadratic formula.

Monic example: $x^2 + 6x + 5 = 0$

1. Move the constant term to the other side: $x^2 + 6x = -5$.
2. Add and subtract $(\frac{b}{2})^2$:

   $$x^2 + 6x + (\frac{6}{2})^2 - (\frac{6}{2})^2 = -5 \implies x^2 + 6x + 9 - 9 = -5.$$
3. Rewrite the first three terms as a perfect square: $(x + 3)^2 - 9 = -5$.
4. Simplify: $(x + 3)^2 = 4$.
5. Take the square root of both sides: $x + 3 = \pm 2$.
6. Solve for x: $x = -3 \pm 2$.
7. Thus, $x = -1$ or $x = -5$.

Non-monic Example $2x^2 + 8x + 6 = 0$:

1. Move the constant term to the other side:
   $2x^2 + 8x = -6$

2. Factor out a from the quadratic and linear terms:
   $2(\mathrm{x}^2 + 4x) = -6$

3. Add and subtract $\left(\frac{b}{2}\right)^2 = 4$:
   $2(x^2 + 4x + 4 - 4) = -6$

4. Rewrite the first three terms as a perfect square:
   $2\left((x + 2)^2 - 4\right) = -6$

5. Distribute the factor of 2:
   $2(x + 2)^2 - 8 = -6$

6. Move the constant to the other side:
   $2(x + 2)^2 = 2$

7. Divide by 2:
   $(x + 2)^2 = 1$

8. Take the square root of both sides:
   $x + 2 = \pm 1$

9. Solve for x
   $x = -2 \pm 1$

10. Thus:
    $x = -1 \quad \text{or} \quad x = -3$

General form of completed square

In general, you can use the form:

$$ax^2 + bx + c = 0 \implies a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a} = 0$$

For example, $x^2 + 6x + 7 = 0$:

1. Put values for a, b and c into the formula:

$$(x + \frac{6}{2})^2 + 7 - \frac{6^2}{4} = 0$$

2. Simplify:

$$\begin{aligned}
(x+3)^2 + 7 - \frac{36}{4} &= 0 \\
(x+3)^2 + 7 - 9 &= 0 \\
(x+3)^2 - 2 &= 0 \\
(x+3)^2 &= 2 \\
x + 3 &= \pm 2 \\
x &= \pm\sqrt{2} - 3
\end{aligned}$$

The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the quadratic formula, derived by completing the square for the standard form of a quadratic equation. It can be used to find the roots of any quadratic equation if they exist.

The Discriminant $b^2 - 4ac$

The expression $b^2 - 4ac$ is called the discriminant because it tells how many roots there are for a quadratic equation:

- If $b^2 - 4ac > 0$, there are two roots.
- If $b^2 - 4ac = 0$, there is one root (a perfect square.)
- If $b^2 - 4ac < 0$, there are no real roots.

Checking the discriminant to see if it can be factorized at all in the first place should be the first step of factorizing a quadratic equation.

Factorizing using the Quadratic Formula

If direct factoring is difficult, use the quadratic formula to find the roots, and then express the quadratic in factored form.

If the roots are r_1 and r_2, then the factorized form is:

$$a(x - r_1)(x - r_2) = 0$$

For example, $x^2 - 4x - 5 = 0$:

$$x = \frac{4 \pm \sqrt{-4^2 - 4 \cdot 1 \cdot 5}}{2a}$$

By using the quadratic formula, the roots are $x = 5$ and $x = -1$.
The factorized form is: $(x - 5)(x + 1) = 0$.

Another example, to find the roots of $6x^2 + 11x + 4$:

The discriminant $b^2 - 4ac$ is $11^2 - 4 \cdot 6 \cdot 4 = 25$ so there are two roots.

By the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-11 \pm \sqrt{11^2 - 4 \cdot 6 \cdot 4}}{2 \cdot 6} \\ &= \frac{-11 \pm 5}{12} \\ &= -\frac{1}{2} \text{ or } -\frac{4}{3} \end{aligned}$$

Splitting the middle term bx

If you multiply out the two binomial factors of a quadratic equation, using FOIL, you will find that the quadratic's linear term bx is split and must be simplified to get the quadratic back in standard form.

For example,
$(x+1)(x+4) = 0 \implies x^2 + 4x + x + 4 = 0 \implies x^2 + 5x + 4.$

Reversing that process, splitting bx, results in terms that can be factorized. Splitting bx in this way is a key method of factorizing quadratic equations.

Splitting bx for Monic Quadratics

Look for two numbers that multiply to c and add to b.

For example, $x^2 + 5x + 6 = 0$:

- Find two numbers that multiply to 6 and add to 5.
- These numbers are 2 and 3, so the factorization is: $(x+2)(x+3) = 0$.

Splitting bx for Non-monic Quadratics

- Find two numbers that multiply to ac and add to b.

- Rewrite the middle term (bx) using these two numbers.
- Factor by grouping.

For example, $2x^2 + 7x + 3 = 0$

- Multiply a and c: $2 \times 3 = 6$.
- Find two numbers that multiply to 6 and add to 7: 6 and 1.
- Rewrite the middle term: $2x^2 + 6x + x + 3 = 0$.
- Factor by grouping: $2x(x + 3) + 1(x + 3) = 0$
  $\implies (2x + 1)(x + 3) = 0$

Another example, $6x^2 + 19x + 10$

- Look for factors of ac (60) that add to b (19):
  Factors of $60 : 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$.
- Split bx with these values: $6x^2 + \underbrace{15x + 4x} + 10$
- Factorize by grouping:
  $(6x^2 + 15x) + (4x + 10) = 3x(2x + 5) + 2(2x + 5)$
- Factorize by the distributive law ($ab + ac = a(b + c)$):
  $= (3x + 2)(2x + 5)$
- Check: $(3x \cdot 2x) + (3x \cdot 5) + (2 \cdot 2x) + (2 \cdot 5)$
  $= 6x^2 + 15x + 4x + 10 = 6x^2 + 19x + 10$ ✓

Another example, $2x^2 + 11x + 12$

- Look for factors of ac (24) that add to b (11):
  Factors of 24 : $(1, 24)(2, 12)(3,8)(4, 6)$

- Split bx with these values: $2x^2 + \underbrace{8x + 3x} + 12$

- Factorize by grouping:
  $(2x^2 + 8x) + (3x + 12) = 2x(x + 4) + 2(x + 4)$

- Factorize by the distributive law $(ab + ac = a(b + c))$:
  $= (2x + 3)(x + 4)$

- Check: $(2x \cdot x) + (2x \cdot 4) + (3 \cdot x) + (3 \cdot 4)$
  $= 2x^2 + 8x + 3x + 12 = 2x^2 + 11x + 12$ ✓

Another example, $-50x^2 - 115x - 60$

- Take out the common factor: $= -5(10x^2 + 23x + 12)$

- Look for factors of ac $(10 \times 12 = 120)$ that add to b (23):

factors	sum	
(1,120)	121	
(2,60)	62	
(3,40)	43	
(4,30)	34	
(5,24)	29	
(6,20)	26	
(8,15)	23	$\longleftarrow$
(10,12)	22	

- Split bx with these values: $= -5(10x^2 + \underbrace{15x + 8x} + 12)$

- Factorize by grouping: $= -5(5x(2x + 3) + 4(2x + 3))$

- Factorize by the distributive law: $= -5(5x + 4)(2x + 3)$

- Check: $-5(5x \cdot 2x) + (5x \cdot 3) + (4 \cdot 2x) + (4 \cdot 3)$
  $=$ -5(10x$^2 + 15x + 8x + 12)$
  $= -5(10x^2 + 23x + 12)$
  $= -50x^2 - 115x - 60$ ✓

Cross-Product Method

This is a systematic way to find factors of ac that equal b.

1. List out the factors of a and the factors of c.
2. Write a factor of a with its matching factor under it.
3. To the right of that, write a factor of c with its matching factor under it.
4. Cross-multiply the two pairs.
5. Add their products.
6. Keep testing products of factor pairs
   until their cross-products add to bx.
7. Circle these two rows.

- That pair of products is the the split for bx.
- The linear factors of the quadratic are the top and bottom pairs.

For example, $2x^2 + 5x - 12$:

Factors of $a : (2x, x)$
Factors of $c : (1, -12), (2, -6), (3, -4), (4, -3), (6, -2), (12, -1)$

Cross multiplying and testing factor pairs $(2x, x)$ and $(1, -12)$:

$$\begin{array}{cccc} 2x & 1 & = & x \\ x & -12 & = & -24x \\ \hline & & & -23x \end{array}$$

factors of c

factors of a

$$\begin{array}{cccc} 2x & 1 & = & x \\ x & -12 & = & -24x \\ \hline & & & -23x \end{array}$$

linear factor $2x + 1$

linear factor $x - 12$

$$\begin{array}{ccccc} 2x & 1 & = & x & \\ & & & & \longleftarrow \text{possible } bx \text{ split} \\ x & -12 & = & -24x & \\ \hline & & & -23x & (\neq bx) ✗ \end{array}$$

Keep cross-multiplying and checking each pair of factors:

$$\begin{array}{ccccc} 2x & 1 & = & x & \\ x & -12 & = & -24x & \\ \hline & & & -23x & (\neq bx) ✗ \end{array}$$

$$\begin{array}{ccccc} 2x & -1 & = & -x & \\ x & 12 & = & 24x & \\ \hline & & & 23x & (\neq bx) ✗ \end{array}$$

$$\begin{array}{ccccc} 2x & 2 & = & 2x & \\ x & -6 & = & -12x & \\ \hline & & & -10x & (\neq bx) ✗ \end{array}$$

$$\begin{array}{ccccc} 2x & -2 & = & -2x & \\ x & 6 & = & 12x & \\ \hline & & & 10x & (\neq bx) ✗ \end{array}$$

$$\begin{array}{cccll} 2x & 3 & = & 3x & \\ x & -42 & = & -8x & \\ \cline{4-4} & & & -5x & (\neq bx)\ \text{✗} \end{array}$$

$$\begin{array}{cccll} 2x & -3 & = & -3x & \\ x & 4 & = & 8x & \leftarrow bx \text{ split} \\ \cline{4-4} & & & 5x & (= bx)\ \text{✓} \end{array}$$

linear factor $2x - 3$

linear factor $x + 4$

By testing different combinations of factors we find that bx can be split as $5x = 8x - 3x$ and that $2x^2 + 5x - 12 = 0$ factorizes as $(2x - 3)(x + 4)$.

Here is another example:

$$3x^2 + 4x - 4 = 0$$

$$\begin{array}{cccll} 3x & -1 & = & -x & \\ x & 42 & = & 12x & \\ \cline{4-4} & & & 11x & (\neq bx)\ \text{✗} \end{array}$$

$$\begin{array}{cccll} 3x & 4 & = & 4x & \\ x & -1 & = & -3x & \\ \cline{4-4} & & & -x & (\neq bx)\ \text{✗} \end{array}$$

$$\begin{array}{cccll} 2x & 3 & = & 3x & \\ x & -42 & = & -8x & \\ \cline{4-4} & & & -5x & (\neq bx)\ \text{✗} \end{array}$$

$$\begin{array}{cccll} 3x & -2 & = & -2x & \\ x & 2 & = & 6x & \\ \cline{4-4} & & & 4x & (= bx)\ \text{✓} \end{array}$$

$$3x^2 + 4x - 4 = 0 \implies (3x - 2)(x + 2) = 0$$

Verify: Although we already have our factors, splitting bx gives us $3x^2 + 4x - 4 = 3x^2 - 2x + 6x - 4$ which can be grouped as $(3x^2 - 2x) + (6x - 4)$ and factored as $x(3x - 2) + 2(3x - 2)$, which distributes as $(3x - 2)(x + 2)$.

Reverse FOIL method

Find factors of a and c that add up to b:

$$ax^2 + bx + c = (\overbrace{\_\_\_\ x + \underbrace{\_\_\_\ ) \cdot (\ \_\_\_}_{}}^{\text{factors of a}}\ x + \_\_\_\ )$$

factors of c

For example, $6x^2 + 5x - 4$

Factors of a (6): (1,6)(2,3) Factors of c (-4): (1,-4)(2,-2)(-1,4)

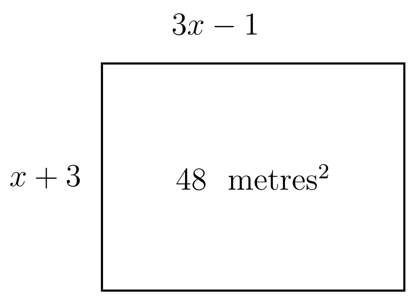
By trying combinations of factors we find: $(2x - 1)(3x + 4)$

Factoring Examples with Rectangles

How long is a piece of string?

Say you have a rectangular yard that you know is 48 square metres in area. You want to measure the lengths of the two sides but all you have is a piece of string of unknown length, which you call x. Measuring the yard with your string you find that one side has a length of $x+3$ and the other has a length of $3x-1$. How long is your piece of string?

You make a drawing:



You write this up as a quadratic equation:

$$(x+3)(3x-1) = 48$$

You expand this out into standard form:

$$3x^2 - x + 9x - 3 = 48 \implies 3x^2 + 8x - 51 = 0$$

You know that to factorize this you need to find the factors of ac ($3 \times -51 = -153$) that add to equal b (8).

You do a prime factor tree to get factors of 153, and you make a table:

```
   153
   / \
  3   51
      / \
     3   17
```

factors	sum		factors	sum	
(-153,1)	-152		(153,1)	152	
(-51,3)	-48		(51,-3)	48	
(-17,9)	-8		(17,-9)	8	
(-9,17)	8	$\leftarrow$	(9,-17)	-8	$\leftarrow$
(-3,51)	48		(3,-51)	-48	
(-1,153)	152		(1,-153)	-153	

You see there are two ways that you could split bx:

$$(3x^2 + 17x) - (9x - 51) = 0$$

$$(3x^2 - 9x) + (17x - 51) = 0$$

You use the second one because it can be easily factored:

$$3x(x - 3) + 17(x - 3) = 0$$

You use the distributive law to get your factors:

$$(3x + 17)(x - 3) = 0$$

You use the null factor law to find the value of x:

$$\begin{array}{rr} 3x + 17 = 0 & x - 3 = 0 \\ x = \frac{-17}{3} & x = 3 \end{array}$$

You see that $\frac{-17}{3}$ isn't a valid length, so your string must be 3 metres long.

Knowing that, you apply it to the original measurements of $(x + 3)(3x - 1) = 48$:

$(3x - 1)$:
$3 \times 3 - 1 = 8$ m

$(x + 3)$:
$3 + 3 = 6$ m

$$\boxed{\begin{array}{c} 6 \text{ m} \times 8 \text{ m} \\ = 48 \text{ metres}^2 \ \checkmark \end{array}}$$

You see that your maths is correct and that the lengths of the sides of your yard are 8 metres by 6 metres.

Another Rectangle:

A rectangle has an area of 45 square centimetres, and is 4 centimetres longer on one side than the other. What are the lengths of its sides?

As an equation that is: $x(x + 4) = 45$

Multiplying it out to standard form: $x^2 + 4x - 45 = 0$

Factor by grouping:

Splitting bx with factors of ac that add to b:

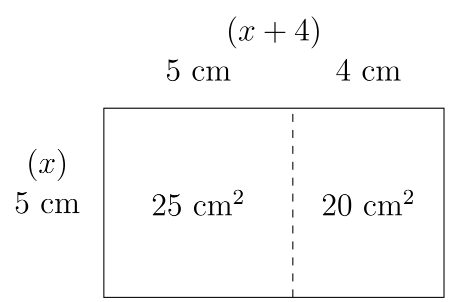
factors	sum		factors	sum	
(-45,1)	-44		(45,-1)	44	
(-9,5)	-4		(15,-3)	12	
(-5,9)	4	←	(9,-5)	4	←
(-15,3)	-12		(5,-9)	-4	
(-3,15)	12		(3,-15)	-12	
(-1,45)	44		(1,-45)	-44	

$$(x^2 - 5x) + (9x - 45) = 0$$

$$(x^2 + 9x) - (5x - 45) = 0$$

The first one looks simpler.
Factor each binomial: $x(x - 5)9(x - 5) = 0$
Distribute: $(x - 9)(x - 5) = 0$
The rectangle is 5×9 cm:

$$x - 5 = 0 \qquad x + 9 = 0$$
$$x = 5 \qquad \underbrace{\text{x=-9}}_{\text{not a valid length}}$$



25 + 20 = 45 cm^2✓