Double and Half Angle Formulae

Tutoring Centre Ferndale

There are special formulae for evaluating double and half of known angles.

Double Angle Formulae

The double angle formulae express trigonometric functions of 2θ in terms of trigonometric functions of θ.

- **Sine:**

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \sin\theta\cos\theta \\ &= 2\sin(\theta)\cos(\theta)\end{aligned}$$

- **Cosine:**

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta) \\ &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2(\theta) - 1\end{aligned}$$

- **Tangent:**

$$\begin{aligned}\tan(2\theta) &= \tan(\theta + \theta) \\ &= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} \\ &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

Half Angle Formulae

The half angle formulae, derived by using the double angle formulae, express trigonometric functions of $\frac{\theta}{2}$ in terms of trigonometric functions of θ.

Sine

We begin with the cosine double-angle formula:

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha)$$

Let $\theta = 2\alpha$. Then $\alpha = \frac{\theta}{2}$. Substituting these into the double-angle formula, we get:

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

Now, solve for $\sin\left(\frac{\theta}{2}\right)$:

$$\begin{aligned} 2\sin^2\left(\frac{\theta}{2}\right) &= 1 - \cos\theta \\ \sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\theta}{2} \\ \sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \end{aligned}$$

The $\pm$ sign indicates that the sign of $\sin\left(\frac{\theta}{2}\right)$ depends on the quadrant of $\frac{\theta}{2}$.

Cosine

We use another form of the cosine double-angle formula:

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1$$

Again, let $\theta = 2\alpha$, so $\alpha = \frac{\theta}{2}$. Substituting:

$$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

Now, solve for $\cos\left(\frac{\theta}{2}\right)$:

$$
\begin{aligned}
2\cos^2\left(\frac{\theta}{2}\right) &= 1 + \cos\theta \\
\cos^2\left(\frac{\theta}{2}\right) &= \frac{1+\cos\theta}{2} \\
\cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1+\cos\theta}{2}}
\end{aligned}
$$

The $\pm$ sign, as before, depends on the quadrant of $\frac{\theta}{2}$.

Tangent

We can derive the tangent half-angle formula using the sine and cosine half-angle formulas:

$$
\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}
$$

Substituting the derived formulas for sine and cosine:

$$
\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}
$$

This is a valid form. We can derive more useful forms by multiplying the numerator and denominator by a strategic term:

Form 1

$$
\begin{aligned}
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \cdot \sqrt{\frac{1-\cos\theta}{1-\cos\theta}} \\
&= \pm\frac{1-\cos(\theta)}{\sqrt{1-\cos^2\theta}} \\
&= \pm\frac{1-\cos(\theta)}{|\sin\theta|}
\end{aligned}
$$

If we restrict θ to an interval where $\sin(\theta)$ is positive:

$$
\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}
$$

Form 2

$$
\begin{aligned}
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \cdot \sqrt{\frac{1+\cos\theta}{1+\cos\theta}} \\
&= \pm\frac{\sqrt{1-\cos^2\theta}}{1+\cos\theta} \\
&= \pm\frac{|\sin\theta|}{1+\cos\theta}
\end{aligned}
$$

Similarly, if we restrict θ to an interval where $\sin\theta$ is positive:

$$
\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos\theta}
$$

These alternative forms are often preferred because they avoid the $\pm$ sign ambiguity (when considering specific intervals for θ) and are sometimes easier to work with algebraically.

Summary

Double Angle Formulas

- **Sine:** $\sin(2\theta) = 2\sin\theta\cos\theta$
- **Cosine:**
  - $\cos 2\theta = \cos^2\theta - \sin^2\theta$
  - $\cos 2\theta = 2\cos^2\theta - 1$
  - $\cos 2\theta = 1 - 2\sin^2\theta$
- **Tangent:** $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Half Angle Formulas

- **Sine:** $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$
- **Cosine:** $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$
- **Tangent:**
  - $\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}$
  - $\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta}$

Note: The $\pm$ sign in the half-angle formulas depends on the quadrant of $\frac{\theta}{2}$.