A Derivation of the Cosine Rule

Tutoring Centre Ferndale

The cosine rule relates the lengths of the sides of a triangle to the cosine of one of its angles. Given a triangle $\triangle ABC$ with sides labeled as follows:

- a is the length of side BC, opposite angle A.
- b is the length of side AC, opposite angle B.
- c is the length of side AB, opposite angle C.

The cosine rule states:

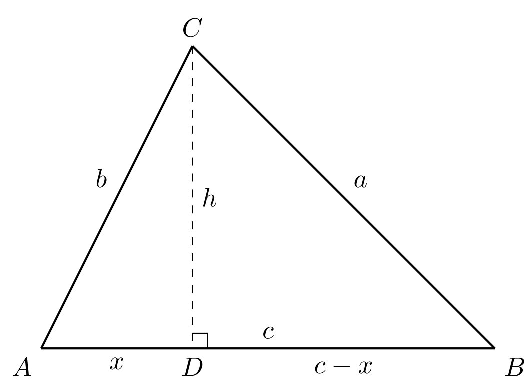
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

We derive this formula using trigonometry and the Pythagorean theorem.

Dropping a perpendicular from vertex C to the base AB, meeting it at point D creates two right-angled triangles $\triangle ADC$ and $\triangle BDC$.

- $AD = x$: the part of AB on the left.
- $DB = c - x$: the remaining part of AB.
- $CD = h$: the height of the triangle.

Thus, $\triangle ADC$ has hypotenuse b, and $\triangle BDC$ has hypotenuse a.



From $\triangle ADC$:

$$\cos A = \frac{x}{b} \Rightarrow x = b\cos A,$$
$$\sin A = \frac{h}{b} \Rightarrow h = b\sin A.$$

From $\triangle BDC$:

$$\cos B = \frac{c-x}{a} \Rightarrow c - x = a\cos B,$$
$$\sin B = \frac{h}{a} \Rightarrow h = a\sin B.$$

Since both expressions for h must be equal:

$$b\sin A = a\sin B.$$

This is the sine rule, but we need to proceed further to derive the cosine rule.

Using Pythagoras' Theorem in $\triangle ADC$:

$$h^2 + x^2 = b^2.$$

Substituting $h = b\sin A$ and $x = b\cos A$:

$$(b\sin A)^2 + (b\cos A)^2 = b^2,$$
$$b^2\sin^2 A + b^2\cos^2 A = b^2.$$

Factoring out b^2:

$$b^2(\sin^2 A + \cos^2 A) = b^2.$$

Using the identity $\sin^2 A + \cos^2 A = 1$:

$$b^2 \cdot 1 = b^2.$$

Thus, this equation holds true.

Now applying Pythagoras' Theorem in $\triangle BDC$:

$$h^2 + (c-x)^2 = a^2.$$

Substituting $h = b\sin A$ and $c - x = a\cos B$:

$$(b\sin A)^2 + (a\cos B)^2 = a^2,$$
$$b^2\sin^2 A + a^2\cos^2 B = a^2.$$

Expanding and rearranging:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

which is the desired cosine rule.