Week 11 Discussion Solutions

CAS CS 132: Geometric Algorithms

November 13, 2023

During discussion sections, we will go over three problems.

- The first will be a warm-up question, to help you verify your understanding of the material.
- The second will be a solution to a problem on the assignment of the previous week.
- The third will be a problem similar to one on the assignment of the following week.

The remainder of the time will be dedicated to open Q&A.

1 Eigenvalues, Eigenvectors, Eigenspaces

A. Find an invertible 2×2 matrix with no eigenvalues.

B. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which projects points onto the x_1x_2-plane. Find the eigenvalues and bases for the corresponding eigenspaces of the matrix implementing this transformation *without doing any calculations*. Then write down the matrix implementing this transformation and find its characteristic polynomial. Check that the eigenvalues you get from the characteristic polynomial are the same.

C. Find the eigenvalues and bases for the corresponding eigenspace of

$$\begin{bmatrix} 1 & -4 \\ -3 & 5 \end{bmatrix}$$

Solution.

A. There are two ways to go about this. First, we can consider a linear transformation on $\mathbb{R}^2$ which does not preserve the direction of any vector. The 2D rotation matrix is such an example. Another way to go about this is algebraically. We need to build a 2×2 matrix which has a characteristic polynomial with no roots; for example, the polynomial $\lambda^2 + 1$. One matrix with this as its characteristic polynomial is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

This matrix is invertible since the columns are linearly independent.

B. Let A denote the matrix implementing T. For vectors not on the x_1x_2-plane, T does not preserve their direction, so none of them can be eigenvectors of A. For vectors on the x_1x_2-plane, their direction is preserved and their length is unchanged, so all vectors in the x_1x_2 plane are eigenvectors of A with the eigenvalue 1. The standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ form a basis for this space.

There is one other eigenvalue of this matrix; consider what happens to vectors on the x_3 axis. These vectors are sent to 0, so 0 is an eigenvalue of A and the standard basis vector $\mathbf{e}_3$ by itself form a basis of this eigenspace.

All together A has the eigenvalues $\lambda = 1$ and $\lambda = 0$ with bases

$$\{\mathbf{e}_1, \mathbf{e}_2\} \qquad \text{and} \qquad \{\mathbf{e}_3\}$$

respectively.

Now, to find the matrix implementing T, we need to determine how it behaves on standard basis vectors:

$$T(\mathbf{e}_1) = \mathbf{e}_1 \qquad T(\mathbf{e}_2) = \mathbf{e}_2 \qquad T(\mathbf{e}_3) = \mathbf{0}$$

Therefore, A above is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of this matrix is the determinant of

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

which is $-\lambda(\lambda-1)^2$. The roots of this polynomial are 0 and 1, which is consistent to what we got above.

C. First, we need to find the characteristic polynomial of this matrix, which is the determinant of

$$\begin{bmatrix} 1-\lambda & -4 \\ -3 & 5-\lambda \end{bmatrix}$$

Using our formula for determinants of 2×2 matrices, this is

$$(1-\lambda)(5-\lambda) - (-3)(-4) = 5 - \lambda - 5\lambda + \lambda^2 - 12 = \lambda^2 - 6\lambda - 7$$

Factoring this polynomial gives us $(\lambda-7)(\lambda+1)$, so the eigenvalues of this matrix are 7 and -1. To find a basis for the eigenspace corresponding to 7, we need to find a basis for the null space of $A - 7I$ which is

$$\begin{bmatrix} -6 & -4 \\ -3 & -2 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 2/3 \\ 0 & 0 \end{bmatrix}$$

Therefore, a general-form solution for the equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} &x_1 = (-2/3)x_2 \\ &x_2 \text{ is free} \end{aligned}$$

and the vectors in this solution set can be written as a linear combination of vectors which free variables as weights:

$$x_2 \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$$

Therefore, $\{[(-2/3)\ 1]^T\}$ is basis for the eigenspace of A corresponding to the eigenvalue 7. By a similar calculation, we can find that $\{[2\ 1]^T\}$ is a basis for the eigenspace of A corresponding to the eigenvalue 1.

2 Complement of the Column Space

Let A be a $5 \times n$ matrix such that $\operatorname{rank} A = 4$, which has an LU decomposition where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 & 1 \end{bmatrix}$$

Determine if $\mathbf{v}$ in $\operatorname{Col} A$, where

$$\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ -11 \\ 5 \\ -12 \end{bmatrix}$$

Solution. Since $\operatorname{rank} A = 4$, every echelon form of A has exactly one row of all zeros. We have to verify that the last row of an echelon form of the matrix $[A\ \mathbf{v}]$ does represent an inconsistent equation. We can read from L a sequence of row operations which takes A to an echelon form.

$$\begin{aligned} R_2 &\leftarrow R_2 + R_1 \\ R_4 &\leftarrow R_4 - 2R_1 \\ R_3 &\leftarrow R_3 - 4R_2 \\ R_5 &\leftarrow R_5 - 3R_2 \\ R_5 &\leftarrow R_5 + 3R_3 \end{aligned}$$

If we apply these operations to $\mathbf{v}$, we get the vector $[2\ -3\ 1\ 1\ 0]^T$, which means the equation $A\mathbf{x} = \mathbf{v}$ has a solution (since the last row doesn't represent an inconsistent equation).

3 Characteristic Polynomials

Find the characteristic polynomial for the matrix

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 2 & -4 \\ 0 & 1 & 5 \end{bmatrix}$$

Use this to determine the eigenvalues of A.

Solution. To find the characteristic polynomial of A, we need to find the determinant of

$$\begin{bmatrix} 1-\lambda & -1 & 5 \\ 0 & 2-\lambda & -4 \\ 0 & 1 & 5-\lambda \end{bmatrix}$$

Since this matrix is not triangular, we cannot immediately apply our formula for the determinant of a triangular matrix. We have to do at least one row operation in order to make it triangular. We will do it in two row operations to simplify the calculation. After $R_3 \leftarrow (2-\lambda)R_3$ and $R_3 \leftarrow R_3 - R_2$, we get the matrix

$$\begin{bmatrix} 1-\lambda & -1 & 5 \\ 0 & 2-\lambda & -4 \\ 0 & 0 & (5-\lambda)(2-\lambda)-4 \end{bmatrix}$$

This is now in echelon form, so we can look at the product of the entries along the diagonal:

$$(1-\lambda)(2-\lambda)((5-\lambda)(2-\lambda)-4)$$

but since we did one scaling by $(2-\lambda)$ (and no swaps) in this process, the determinant of the matrix is

$$\frac{(1-\lambda)(2-\lambda)((5-\lambda)(2-\lambda)-4)}{2-\lambda} = (1-\lambda)((5-\lambda)(2-\lambda)-4)$$

If we multiply out the second term, we get $\lambda^2 - 7\lambda + 10 - 4 = \lambda^2 - 7\lambda + 6$. So the final factorization of the characteristic polynomial is

$$(1-\lambda)^2(6-\lambda)$$

This is the characteristic polynomial of A and it tells us that 1 and 6 are the eigenvalues of A.