

Midterm Review Solutions

CAS CS 132: Geometric Algorithms

1 Solving Systems of Linear Equations

Find a solution the following system of linear equations.

$$\begin{aligned}x_1 + x_2 - x_3 &= -9 \\x_2 - 2x_3 &= -1 \\x_1 + x_2 &= -10\end{aligned}$$

Solution. This system has the augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 & -9 \\ 0 & 1 & -2 & -1 \\ 1 & 1 & 0 & -10 \end{bmatrix}$$

which has the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

2 LAA 1.2.18

Determine all values of h such that the following matrix is the augmented matrix of a consistent linear system.

$$\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix}$$

Solution. By the reduction $R_2 \leftarrow R_2 - 5R_1$, we get the matrix

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & h + 15 & 3 \end{bmatrix}$$

This matrix is in echelon form, so h can be any value other than -15 . If $h = -15$, then there is a row which represents an inconsistent equation.

3 General Form Solutions

Consider the following system of linear equations.

$$\begin{aligned}x_1 - 5x_2 - x_3 - 2x_4 &= 3 \\x_3 - 2x_4 &= 11 \\(-2)x_3 + 5x_4 &= -24\end{aligned}$$

- A. Write down a general form solution which describes the solution set of the following system of linear equations.
- B. Write down a different general form solution which describes the same solution set (i.e., one in which a different variable is free).

Solution.

- A. The reduced echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & -5 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

We can write down a solution in general form from this matrix:

$$\begin{aligned}x_1 &= 6 - 5x_2 \\x_2 &\text{ is free} \\x_3 &= 7 \\x_4 &= -2\end{aligned}$$

- B. Since x_1 is written in terms of the free variable x_2 , we can instead write x_2 in terms of x_1 .

$$\begin{aligned}x_1 &\text{ is free} \\x_2 &= (6/5) - (1/5)x_1 \\x_3 &= 7 \\x_4 &= -2\end{aligned}$$

4 LAA 1.3.13

Consider the following matrix A and vector \mathbf{b} .

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ h \end{bmatrix}$$

- A. Determine if \mathbf{b} can be written as a linear combination of the columns of A if $h = -3$.
- B. For what values of h can \mathbf{b} be written as a linear combination of the columns of A .

Solution.

- A. The augmented matrix $[A \ \mathbf{b}]$ is row equivalent to the echelon form

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

since this represents an inconsistent system, \mathbf{b} cannot be written as a linear combination of the columns of A .

- B. If $h = -6$, then the last row does not represent an inconsistent equation. In this case, \mathbf{b} can be written as a linear combination of the columns of A .

5 Intersections of Spans

Consider the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Find a nonzero vector which lies in both $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\text{span}\{\mathbf{v}_3, \mathbf{v}_4\}$.

Solution. The span of the first pair of vectors is exactly the x_1x_2 -plane, which is described by the equation $x_3 = 0$. Therefore, it suffices to find a vector in $\text{span}\{\mathbf{v}_3, \mathbf{v}_4\}$ such whose third component is 0. We can take $2\mathbf{v}_3 + \mathbf{v}_4$, which is

$$\begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$$

6 LAA 1.7.31

Find a nontrivial solution to matrix equation $A\mathbf{x} = \mathbf{b}$ without performing any row reductions. (Hint. What is the relationship between the first two columns and the last column of A ?)

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution. Write A as $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ then $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$. Therefore, a solution to this equation is the vector

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

7 Linearly Independent Vectors

Consider three arbitrary vectors

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

and suppose that they are linearly independent.

- A. What is the maximum number of entries of these vectors which can be 0?
Note that the solution will be a number between 0 and 9.
- B. What is the minimum number?

In each case provide an example.

Solution.

- A. There can be at most 6 zero entries. If there were 7, then some vector would have to be all zeros, which would automatically make the set linearly dependent. An example is the standard basis vectors \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 .
- B. It is possible for there to be no zero entries. We can take as an example

$$\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

8 Drawing Linear Transformations

Draw the unit square after being transformed by the matrix transformation implemented by

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution.

9 Matrices of Linear Transformations

Find the matrix which implemented the following transformation.

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \mapsto \begin{bmatrix} v_2 \\ v_1 \\ v_3 + v_4 \end{bmatrix}$$

Solution. In order to determine the matrix implementing a linear transformation, we have to determine how the transformation affects the standard basis.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Then we put these together into a single matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

10 3D Linear Transformations

Considering the transformation T implemented by the following matrix.

$$\begin{bmatrix} \cos 2 & 0 & -\sin 2 \\ 0 & 1 & 0 \\ \sin 2 & 0 & \cos 2 \end{bmatrix}$$

Describe geometrically what T does. Then find a vector \mathbf{v} whose span is not changed by this transformation (i.e., $\mathbf{span}\{\mathbf{v}\} = \mathbf{span}\{T(\mathbf{v})\}$).

Solution. This transformation rotates vectors around the x_2 axis. The the span of the vector \mathbf{e}_2 is not changed by this transformation since $T(\mathbf{e}_2) = \mathbf{e}_2$.