Numerics

Geometric Algorithms
Lecture 2

CAS CS 132

Recap (1/2)

Linear equations define <u>hyperplanes</u>

Systems of linear equations define <u>intersections of hyperplanes</u>

We solve systems linear equations by <u>elimination and back substitution</u>

Systems of linear equations can be represented as <u>matrices</u>

Recap (2/2)

elimination and back-substitution can be represented as row operations on matrices

row operations don't change the solution sets

Recap Problem (1/2)

Show that if (s_1, s_2) is a solution to

$$ax + by = c$$
$$dx + ey = f$$

then it is also a solution to

$$ax + by = c$$
$$(a + d)x + (b + e)y = (c + f)$$

Recap Problem (2/2)

Give values of a through f such that

$$(a + d)x + (b + e)y = (c + f)$$

has a solution but

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

does not

don't drop equations when doing replacements

Objectives

1. number representations
2. consequences of floating point representations
3. best practices

Keywords

floating point numbers

IEEE-754

relative error

numpy.isclose

ill-conditioned problems

let's do a quick demo

Significant Figures (Sig Figs)

Have you ever been docked points in a science class for having incorrect sig figs?

when you use a ruler, you can't do better than ± 1mm, so we can't say anything about nanometer differences

we run into a similar problem with decimal numbers in programs

Number Representations

your computer is a collection of fixed size registers

each register holds a sequence of bits

The Goal. represent numbers so they fit in those registers

this is, of course, ~~a lie~~ an abstraction

Number Representations

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Question. How do we slice up our fixed sequence to represent numbers?

<u>things to consider:</u>

- simple idea (easy to understand)
- maximize coverage (not too redundant)
- simple numeric operations (easy to use)

Unsigned Integers

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

`value`

`binary value (we should know this by now)`

`e.g. 10001010 represents`

$$1(2^7) + 0(2^6) + 0(2^5) + 0(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$$

Signed Integers

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
sign	value														

sign bit + binary value

e.g. 10001010 represents

$$-1 \times \left(0(2^6) + 0(2^5) + 0(2^4) + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0)\right)$$

Floating-Point Numbers (Some Figures)

floats in python use 64 bits

That's 1.8×10^{19} possible values

We can't represent everything. We'll have to choose and then round

Question. Which ones should we represent?

Floating-Point Numbers (An Idea)

Integers work because they are **discrete and evenly spaced**

What if we evenly discretize a range of values?

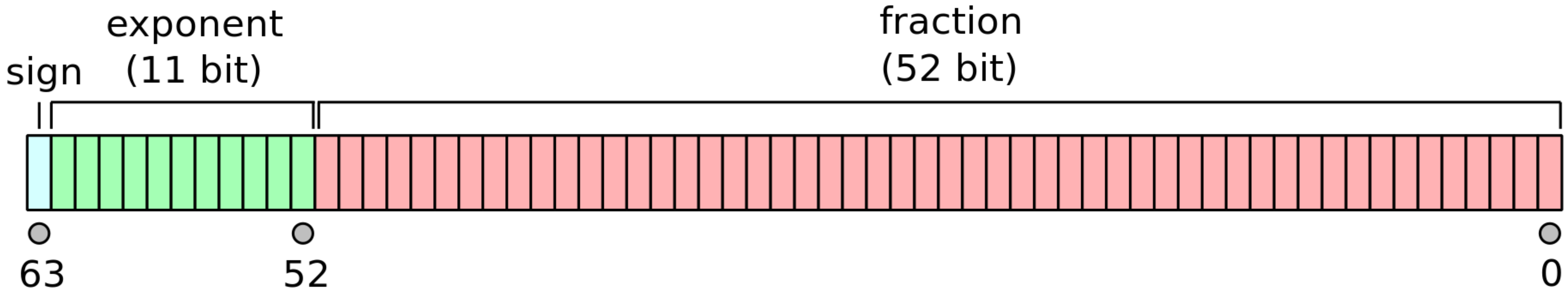
i.e., represent

..., −0.001, 0, 0.0001, 0.002, 0.003, 0.004,...

Question

Discuss the advantages and disadvantages of this approach

Floating-Point Numbers (IEEE-754)



like scientific notation, but binary

the equation:

$$(-1)^{\text{sign}} \times \left(1 + \frac{1}{\text{fraction}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

it's an accepted standard, not perfect, but it works well

image source

Question

$$(-1)^{\text{sign}} \times \left(1 + \frac{1}{\text{fraction}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

Any ideas why this is better/worse?

Also, why the additive 1?

And why not have a sign bit for the exponent?

$$(-1)^{\text{sign}} \times \left(1 + \frac{1}{\text{fraction}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

Step Size

⟵ step size increases with magnitude ⟶

Definition. step size is the space between two floating-point representations

for fixed exponent n two numbers are at least

$$0.00\ldots001 \times 2^n = 2^{-52} \times 2^n$$

away (why?)

Step size doubles for each exponent

image source

Relative Error

Observation. ± 0.001 is *tiny* error for 10^{20} but *massive* for 10^{-20}

Relative Error.

$$\mathrm{err}_{\mathrm{rel}} = \frac{\mathrm{err}}{\mathrm{val}}$$

IEEE-754 keeps relative error <u>small</u>

Relative Error (Calculation)

$$(-1)^{\mathsf{sign}} \times \left(1 + \frac{1}{\mathsf{fraction}}\right) \times 2^{\mathsf{exponent}-(2^{10}-1)}$$

(fix an exponent n)

error is determined by step-size

$$\text{err} \leq 2^{-52} \times 2^{n}$$

Relative Error (Calculation)

$$(-1)^{\text{sign}} \times \left(1 + \frac{1}{\text{fraction}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

(fix an exponent n)

the smallest number we can represent at least 1.0×2^n

$$\text{val} \geq 1.0 \times 2^n$$

(why do we care about a lower bound on val?)

Relative Error (Calculation)

$$(-1)^{\text{sign}} \times \left(1 + \frac{1}{\text{fraction}}\right) \times 2^{\text{exponent}-(2^{10}-1)}$$

(fix an exponent n)

the relative error is *small*

$$\text{val} \geq 1.0 \times 2^n$$

$$\text{err} \leq 2^{-52} \times 2^n$$

$$\text{err}_{\text{rel}} = \frac{\text{err}}{\text{val}} \leq \frac{2^{-52} \times 2^n}{1.0 \times 2^n} = 2^{-52} \approx 10^{-16}$$

$\approx$16 digits of accuracy

Not bad, but also not great

let's do a quick demo

example from the notes

The Takeaways

operations on floating-point numbers are not exact

properties like $(ab)c = a(bc)$ (commutativity) may not hold

it's a trade-off for large range and low relative error

What do we do about it?

Best Practices

1. don't compare floating points for equality
2. be aware of ill-conditioned problems
3. be aware of small differences

Principle 1: Closeness

When doing floating-point calculations in a program, define an error margin and use that for equality checking

In Practice.

```
Replace    x == y
with       numpy.isclose(x, y)
```

demo

Principle 2: Ill-Conditioned Problems

Make sure your problem is not sensitive to small errors.

In Practice. for example, don't divide by numbers much smaller than your error tolerance

demo

Principle 3: Small Differences

Make sure you understand your error tolerance when looking that the small differences of large numbers.

In Practice. Don't expect $a - b$ to have 16 digits of accuracy even if a and b do

demo

One Last Note: Special Numbers

```
0       (we can't already represent 0?)

nan     stands for not a number, .e.g, sqrt(-2)

inf     symbolic infinity, behaves as expected
```

Summary

floating point numbers are <u>represented</u> in your computer

floating point operations are <u>not</u> exact

this can have unintended consequences

we get <u>16 digits</u> of accuracy