Vector Equations

Geometric Algorithms
Lecture 4

CAS CS 132

Recap Problem

consider the following system of linear equations

$$\begin{aligned} x + y &= 0 \\ 2x - y &= 0 \\ 3x + 2y &= 0 \end{aligned}$$

write down its augmented matrix and it's reduced echelon form

Recap Problem (Solution)

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we don't even have to do any calculations

I lied a bit on Tuesday

the coefficient matrix is:
identity matrix + zeros on bottom

Objectives

1. motivation
2. define vectors
3. discuss vector operations and vector algebra
4. relate vectors and systems of linear equations

Keywords

vector

vector addition

vector scaling/multiplication

the zero vector

vector equations

linear combinations

span

Sources

Images are from out textbook by Professor Crovella

Demos are from *Interactive Linear Algebra*, a very nice optional text for this course

Motivation

Changing Perspective

$$2^n - 1 = \sum_{i=1}^{n} 2^i$$

Show that this holds for all n

Changing Perspective

$$\boxed{2^n} - \boxed{1} = \boxed{\sum_{i=1}^{n} 2^i}$$

$$\boxed{100...000} - \boxed{000...001} = \boxed{111...111}$$

show that this holds for all n

much easier in binary

Motivation?

vectors will be one of the most important shifts of perspective in this course

the insight is so simple its genius

maybe I'm reaching...

Big Data

a piece of data is a bunch of distinct values (numbers)

How can we tell if two piece of data are similar?

maybe if they are **close together** in a geometric sense

A Note on Algebra

in programming an "interface" is an abstract collection of related functions (e.g., a printing interface, or a comparison interface)

and object then "implements" an interface

doing abstract algebra is like implementing an interface

we're defining an new thing called a "column vector"

we need to define what "equality" and "adding" and "multiplying by a number" means for column vectors

Vectors

What is a vector (in $\mathbb{R}^n$)?

A. an n–tuple of real numbers

B. a point in $\mathbb{R}^n$

C. a 1–column matrix with real values

D. all of the above

E. none of the above?

it's common to conflate points and vectors

Column Vectors

Definition. a ***column vector*** is a matrix with a single column, e.g.,

$$\begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix}$$

bold letter

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A Note on Matrix Size

an $(m \times n)$ matrix is a matrix with m rows and n columns

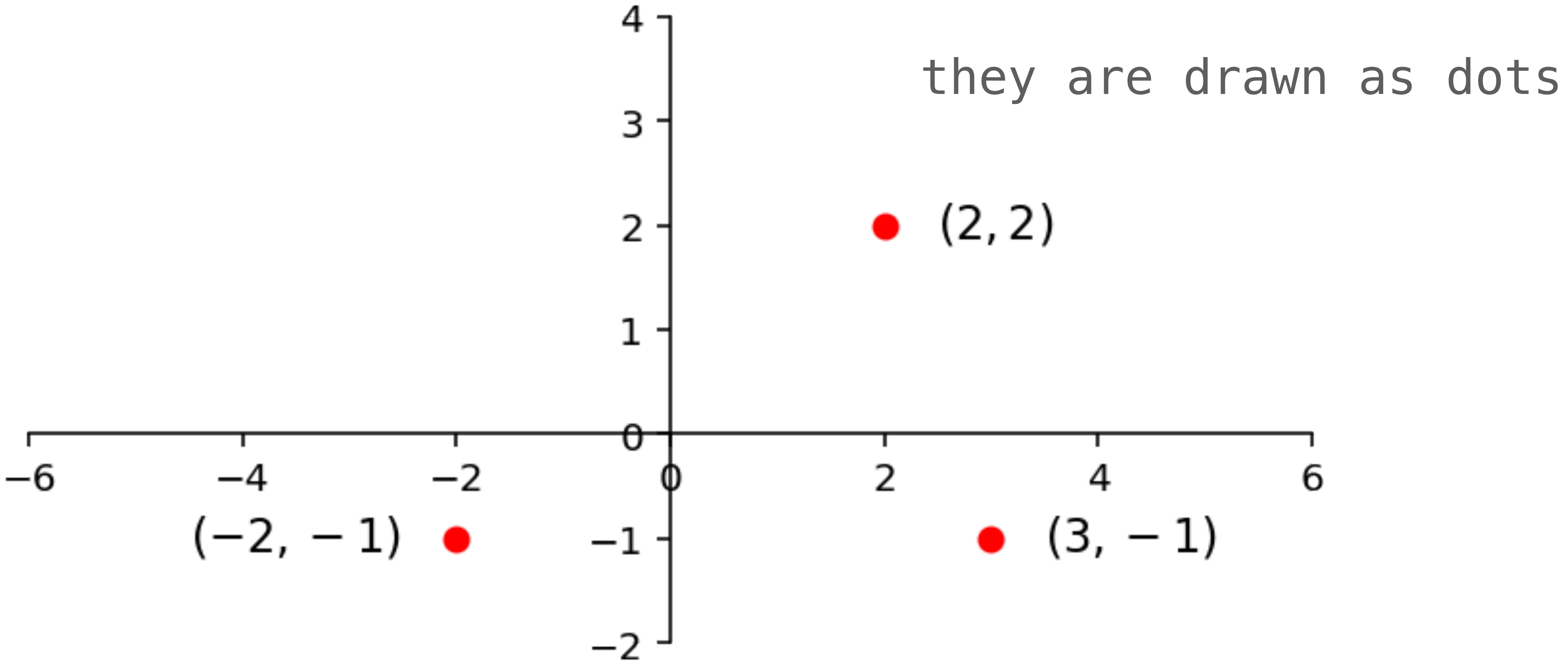
$$\begin{array}{cc} & n \\ m & \begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ * & * & \cdots & * & * \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & 1 \\ 4 & \begin{bmatrix} 2 \\ 3 \\ 0.1 \\ -2 \end{bmatrix} \end{array}$$

the number of rows of a vectors is called its **dimension**

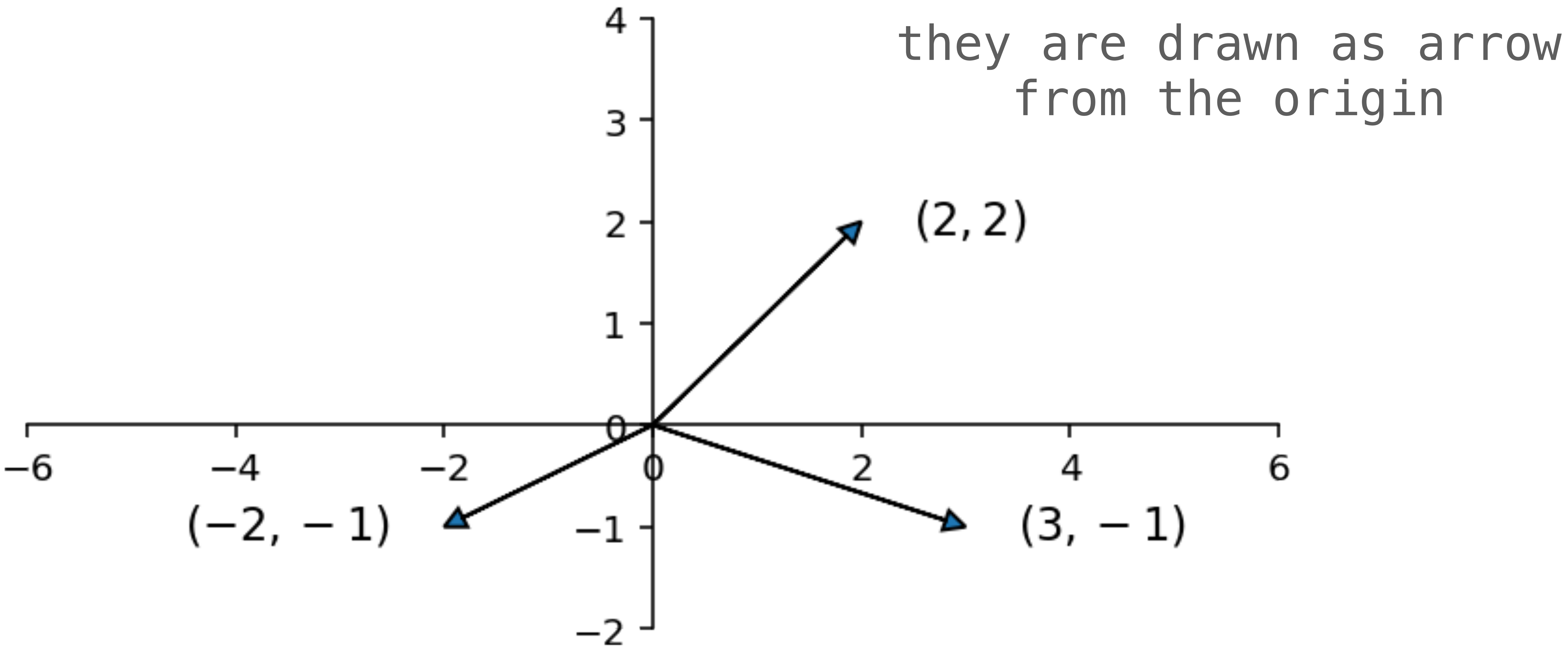
$\mathbb{R}^{m \times n}$ is set of matrices with $\mathbb{R}$ entries

Notation (Points)



points in $\mathbb{R}^2$ are notated as (a, b)

Notation (Vectors)



vectors in $\mathbb{R}^2$ are notated as $\begin{bmatrix} a \\ b \end{bmatrix}$

Notation

we will often write $(a_1, a_2, \ldots, a_n)$ for the vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

!!IMPORTANT!!

$(a_1, a_2, \ldots, a_n)$ is not the same as $[a_1 \ \ a_2 \ \ \ldots \ \ a_n]$

Vector Equality

two vectors are equal if their entries at each position are equal

(this is also the case for matrices)

!!IMPORTANT!!

ORDER MATTERS

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Vector Equality

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is the same as

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= b_2 \\ &\vdots \\ a_n &= b_n \end{aligned}$$

Vector Operations

Vector "Interface"

addition — what does $\mathbf{u}+\mathbf{v}$ (adding two vectors mean?

scaling — what does $a\mathbf{v}$ (multiplying a vector by a real number) mean?

What properties do they need to satisfy?

Vector Addition

adding two vectors means adding their entries

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

!!IMPORTANT!!

WE CAN ONLY ADD VECTORS OF THE SAME SIZE

Vector Addition (Non-Example)

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 23 \\ 0.5 \\ 3 \\ 0 \end{bmatrix}$$

`This is nonsensical`

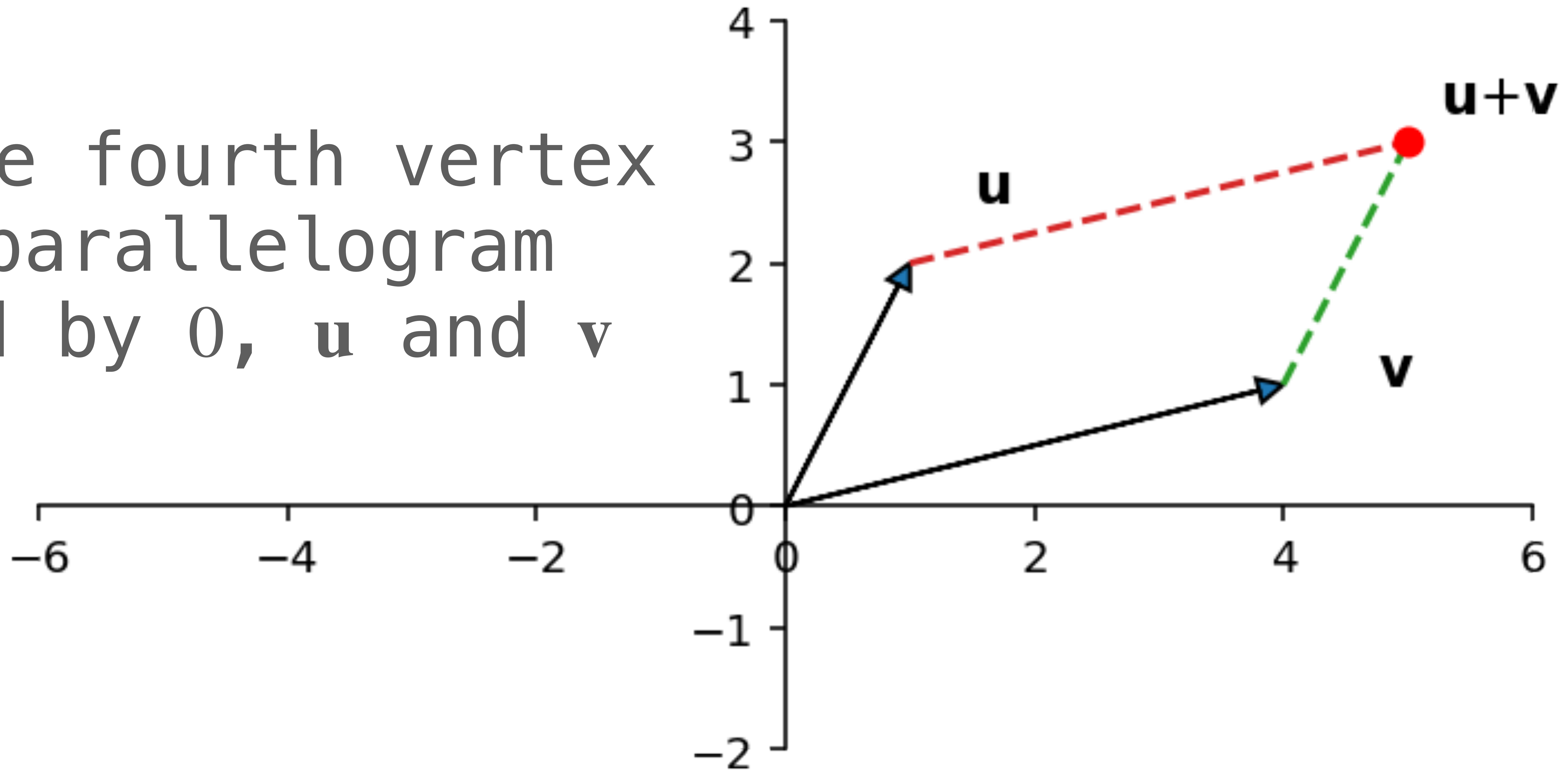
Vector Addition (Example)

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 23 \\ 0.5 \\ 3 \end{bmatrix} = \begin{bmatrix} 23 + 2 \\ 3 + 0.5 \\ 4 + 3 \end{bmatrix} = \begin{bmatrix} 25 \\ 3.5 \\ 7 \end{bmatrix}$$

Vector Addition (Geometrically)

in $\mathbb{R}^2$ it's called the *parallelogram rule*

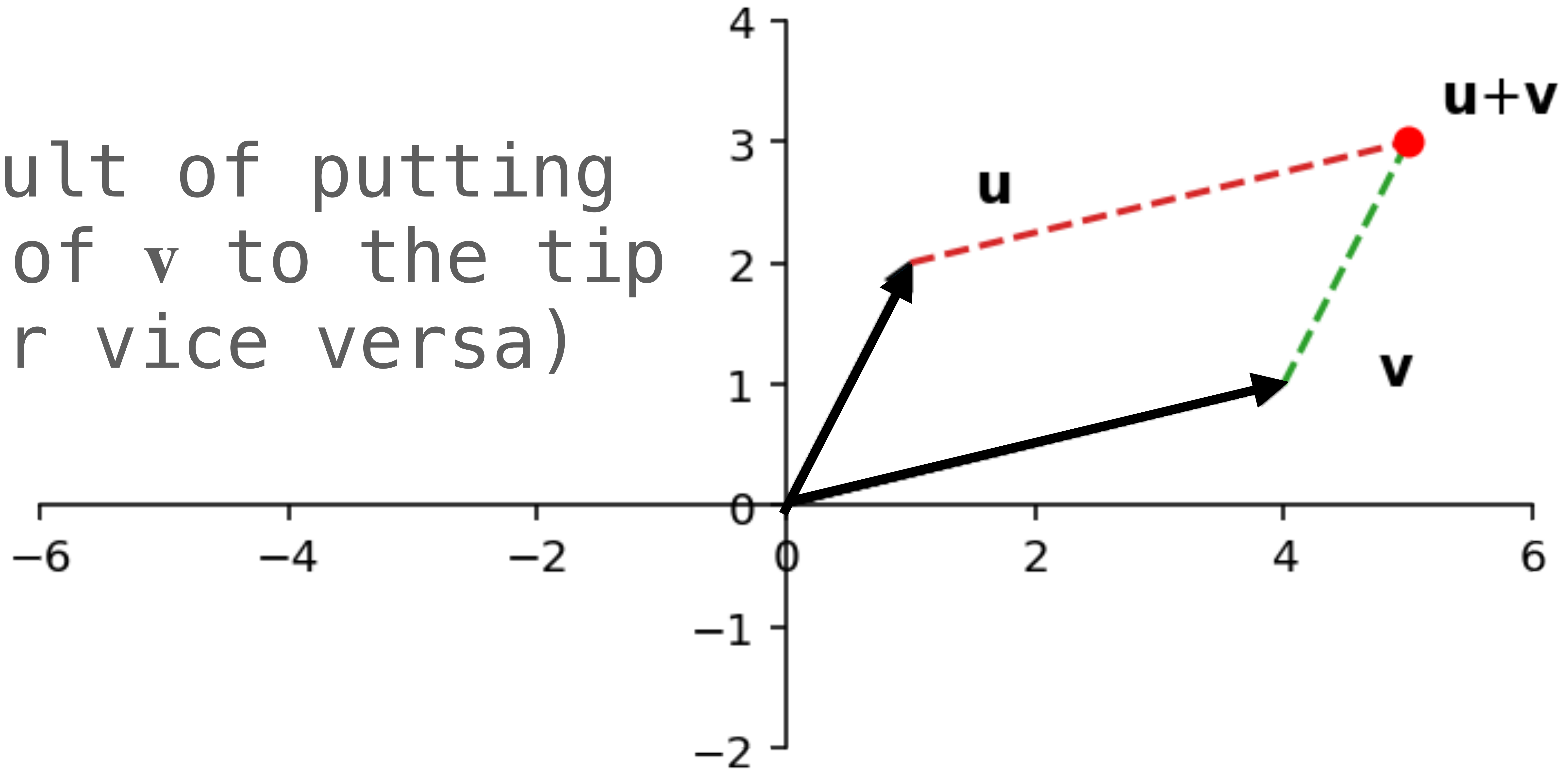
$\mathbf{u}+\mathbf{v}$ is the fourth vertex of the parallelogram generated by 0, $\mathbf{u}$ and $\mathbf{v}$



Vector Addition (Geometrically)

or the *tip–to–tail rule*

$\mathbf{u} + \mathbf{v}$ result of putting the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$ (or vice versa)



demo
(from ILL)

Vector Scaling/Multiplication

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scaling/multiplying a vector by a number means
multiplying each of it's elements
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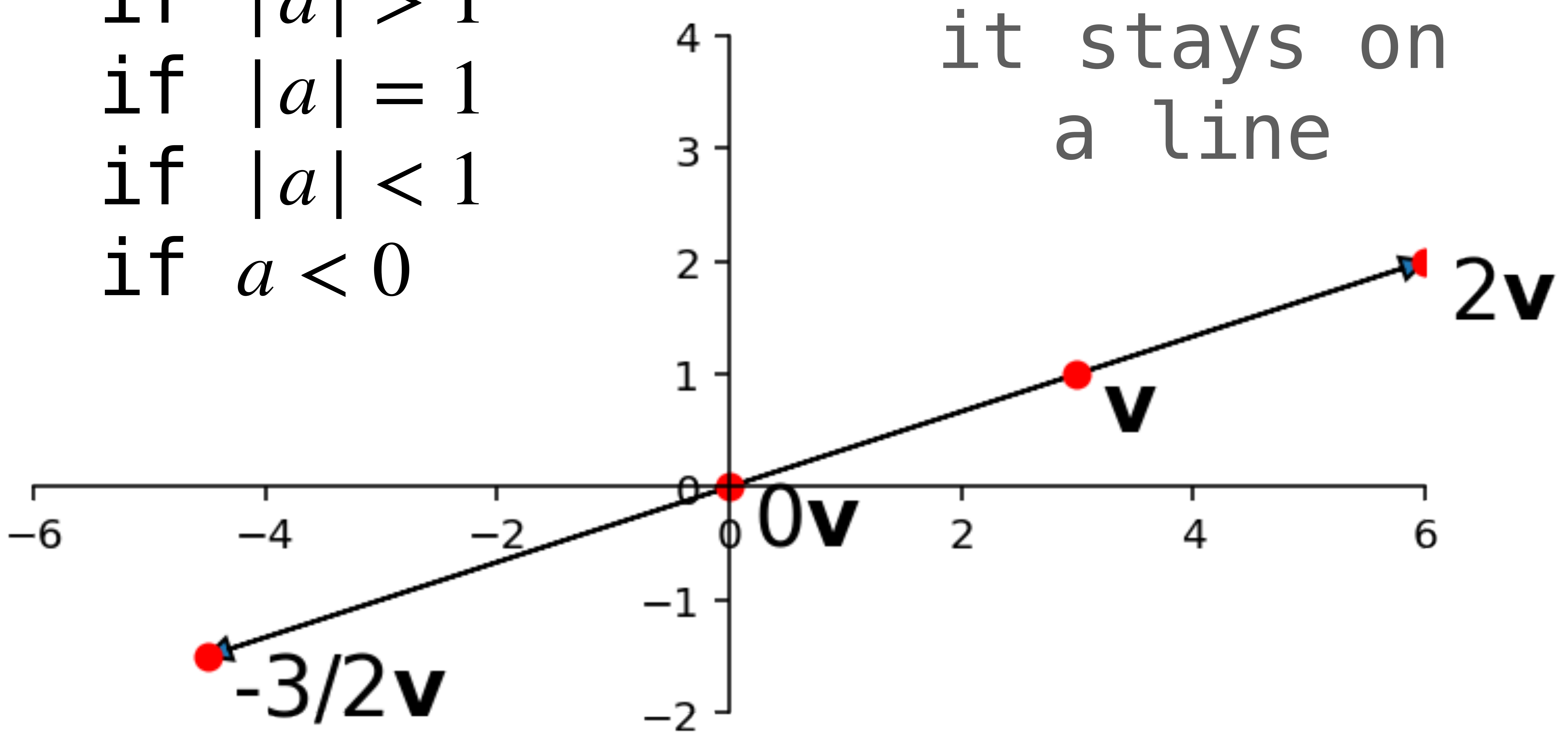
$$a \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} ab_1 \\ ab_2 \\ \vdots \\ ab_n \end{bmatrix}$$

Vector Scaling/Multiplication (Example)

$$3 \begin{bmatrix} 2 \\ 1 \\ 3.5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 1 \\ 3 \cdot 3.5 \\ 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 10.5 \\ 12 \end{bmatrix}$$

Vector Scaling/Multiplication (Geometrically)

longer	if $\|a\| > 1$
the same length	if $\|a\| = 1$
shorter	if $\|a\| < 1$
reversed	if $a < 0$



demo

(from ILL)

Algebraic Properties

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and any real numbers c, d:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

$$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$$

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

these are requirements for any **vector space**
they matter more for *bizarre* vector spaces

Question (Practice)

compute the value of this vector

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Answer

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix}-\begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Answer

$$\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 6 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \\ 2 \\ 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 3 + 4 - (-3) \\ 3 + 0 - 4 \\ 3 + 6 - 2 \\ 3 + (-2) + 0 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 10 \\ -1 \\ 7 \\ 1 \end{bmatrix}$$

In Sum

we can add vectors

we can scale vectors

this gives us a way of generating new vectors from old ones

What vectors can we make in this way?

Linear Combinations

Linear Combinations

Definition. a ***linear combination*** of vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

Looks suspiciously like a linear equation...

where $\underbrace{\alpha_1, \alpha_2, \ldots, \alpha_n}_{\text{weights}}$ are in $\mathbb{R}$

Linear Combinations (Example)

$$3\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 2\begin{bmatrix}2\\0\\3\\-1\end{bmatrix} - \begin{bmatrix}-3\\4\\2\\0\end{bmatrix}$$

Linear Combinations (Geometrically)

`demo`

`(from ILL)`

The Fundamental Concern

Can $\mathbf{u}$ be written as a linear combination of

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n?$$

That is, are there weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots \alpha_n\mathbf{v}_n = u?$$

Why is this fundamental?

I'm going to ask that you suspend your disbelief...

For now, how do we solve this problem?

Vector Equations and Systems of Linear Equations

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$\begin{bmatrix} x_1 \\ (-2)x_1 \\ (-5)x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$\begin{bmatrix} x_1 + 2x_2 \\ (-2)x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The Fundamental Connection

we don't know the weights, that's want we want to find

what if we write them as *unknowns*?

$$x_1 + 2x_2 = 7$$
$$(-2)x_1 + 5x_2 = 4$$
$$-5x_1 + 6x_2 = -3$$

we get a system of linear equations we know how to solve

The Fundamental Connection

`More generally:`

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The Fundamental Connection

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{1m}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{21}x_2 \\ \vdots \\ a_{2m}x_1 \end{bmatrix} + \ldots + \begin{bmatrix} a_{n1}x_n \\ a_{n2}x_n \\ \vdots \\ a_{nm}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector scaling`

The Fundamental Connection

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

`by vector addition`

The Fundamental Connection

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

`by vector equality`

The Fundamental Connection

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

augmented matrix

this is our big shift in perspective

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

system of linear equations

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{1m} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{21} \\ \vdots \\ a_{2m} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

vector equation

HOW TO: Linear Combination Problems

Question. Can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$?

Solution. Solve the system of linear equations with the augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

this is notation for building a matrix out of column vectors

A solution to this system is a set of weights to define $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$

Question (Practice)

Can $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ *be written as a linear combination of* $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$*?*

Answer: Yes

$$3\begin{bmatrix}1\\-2\\-5\end{bmatrix}+2\begin{bmatrix}2\\5\\6\end{bmatrix}=\begin{bmatrix}7\\4\\-3\end{bmatrix}$$

Spans

Definition. the ***span*** of a set of vectors is the set of all possible linear combinations of them

$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots \alpha_n\mathbf{v}_n : \alpha_1, \alpha_2, \ldots, \alpha_n$ are in $\mathbb{R}\}$

read: $\mathbf{u}$ is an element of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

$\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ exactly when $\mathbf{u}$ can be expressed as a linear combination of those vectors

Spans (Geometrically)

for one vector

$$\text{span}\{\mathbf{v}\} = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$$

this is **all scalar multiple of** $\mathbf{v}$

the span of one vector is a **line**

Spans (Geometrically)

the span of **two** vectors is a **plane**

the span of **three** vectors is a **hyperplane**

!!IMPORTANT!!

In all cases they pass through the origin

Spans (Geometrically)

demo

(from ILL)

HOW TO: Span Problems

Question. Is $\mathbf{b} \in \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$?

Solution. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$

you know how to do this now

Question (Conceptual)

What does it mean geometrically if $\mathbf{b} \notin \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$*?*

demo
(from ILL)

HOW TO: Inconsistency and Spans

Question. find a vector $\mathbf{b}$ which *does not* appear in $\text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$

Solution. Choose $\mathbf{b}$ so that

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

is the augmented matrix of an *inconsistent* system

There is **no way** to write $\mathbf{b}$ as a linear combination

Summary

vectors are fundamental objects

we can think of them as the columns of a linear system

we can scale them and add them together

they can span spaces which represent hyperplanes