Practice Final

CAS CS 132: Geometric Algorithms

December 14, 2023

Name:

BUID:

Location:

- You will have approximately 120 minutes to complete this exam.
- Make sure to read every question, some are easier than others.
- Please write your name and BUID **on every page**.

(Extra page)

1 Orthogonal Projections and Linear Equations

Consider the linear equation

$$x_1 - x_2 + x_3 = 0$$

and the vector

$$\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

A. (3 points) Write down the unique vector $\mathbf{z}$ which is orthogonal to the plane given by the above linear equation (that is, the vector which is orthogonal to every solution in its solution set.)

B. (5 points) Find a basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ for the plane given by the above linear equation.

C. (5 points) Find a solution to the vector equation $y_1\mathbf{z} + y_2\mathbf{b}_1 + y_3\mathbf{b}_2 = \mathbf{v}$.

D. (5 points) Find the orthogonal projection of $\mathbf{v}$ onto the plane given by the above linear equation. (*Hint.* Use the previous parts.)

Solution.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

2 True/False Questions

A. (2 points) For any matrix A, if A is square and $\det(A) = 0$, then the columns of A are linearly dependent.

B. (2 points) For any stochastic matrix A, if A has a unique stationary state, then it must be regular.

C. (2 points) For any matrix A, the dimension of the null space of A is at most the rank of A.

D. (2 points) For any matrix A, if A has n distinct eigenvalues, then it is invertible.

E. (2 points) Every orthogonal set is linearly independent.

F. (2 points) For any two matrices A and B, if A is invertible and A is row equivalent to B then B is invertible.

G. (2 points) For any two matrices A and B, if AB is defined then $AB \neq BA$.

H. (2 points) For any matrix A and quadratic form $Q(\mathbf{x})$, if $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, then A is symmetric.

Solution.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

3 Elementary Matrices

A. (5 points) Find the 3×3 matrix E which implements the following row operations:

$$\begin{aligned} &\mathsf{swap}(R_1, R_2) \\ &R_1 \leftarrow 3R_1 \\ &R_3 \leftarrow R_3 + 2R_2 \end{aligned}$$

B. (6 points) Find values for i through m such that E^T implements the following row operations:

$$\begin{aligned} &\mathsf{swap}(R_i, R_j) \\ &R_k \leftarrow 3R_k \\ &R_l \leftarrow R_l + 2R_m \end{aligned}$$

C. (6 points) Compute AE where

$$A = \begin{bmatrix} 11 & 22 & 33 \\ 11 & 22 & 33 \\ 11 & 22 & 33 \end{bmatrix}$$

(*Hint.* Use the previous part and the fact that $(B^T)^T = B$.)

Solution.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

4 Diagonalizability

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

A. (7 points) Find the characteristic polynomial of A.

B. (8 points) Find bases for every eigenspace of A. That is for each eigenvalue λ of A, find a basis for $\text{Nul}(A - \lambda I)$.

C. (3 points) Determine if A is diagonalizable. If it is, provide a diagonalization. Otherwise, justify your answer.

Solution.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

5 Interpreting Matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 7 \\ 0 & 0 & -4 & 1 \\ 3 & -3 & 2 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

A. (2 points) Is A in echelon form?

B. (5 points) Find a basis of $\operatorname{Col} A$ with vectors that are columns of A.

C. (5 points) Find a basis of $\operatorname{Nul} A$.

D. (5 points) Compute $\det B$.

E. (2 points) Is B invertible?

Solution.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)

6 Linear Models

Suppose we are given the data

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$$

A. (5 points) Construct the design matrix for the given data which can be used to find the best-fit curve of the form

$$f_{\beta_1,\beta_2}(\theta) = \beta_1 \cos\theta + \beta_2 \sin\theta$$

where β_1 and β_2 are parameters.

B. (7 points) Consider trying to fit the data with a curve of the form

$$g_\alpha(\theta) = \cos(\theta + \alpha)$$

where α is a parameter. Note that g_α is not linear in its parameters. Given $\hat{\alpha}$ and $\hat{\beta}_1$ and $\hat{\beta}_2$, the parameters for the best-fit curves, show that

$$\sum_{i=1}^{4} \|\hat{\beta}_1 \cos(x_i) + \hat{\beta}_2 \sin(x_i) - y_i\|^2 \leq \sum_{i=1}^{4} \| \cos(x_i + \hat{\alpha}) - y_i\|^2$$

using the trigonometric identity

$$\cos(a + b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

In other words, show that the best-fit curve from part A has error at least as small as the error of the best-fit curve from part B.

C. (4 points, **Extra Credit**) Compute $\hat{\alpha}$ from $\hat{\beta}_1$ and $\hat{\beta}_2$. This implies that, in fact, the errors are equal.

Solution.

Solution. (Continued)

Solution. (Continued)

Solution. (Continued)