Linear Equations

Geometric Algorithms

Lecture 1

CAS CS 132

Objectives

1. Motivation
2. Definitions
3. Solve systems of linear equations

Keywords

Systems of linear equations

Solutions

Coefficient matrix

Augmented matrix

Elimination and Back-substitution

Replacement, interchange, scaling

Row Equivalence

(In)consistency

Objectives

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Motivation

1. Lines and line intersections
2. An example from chemistry

Motivation

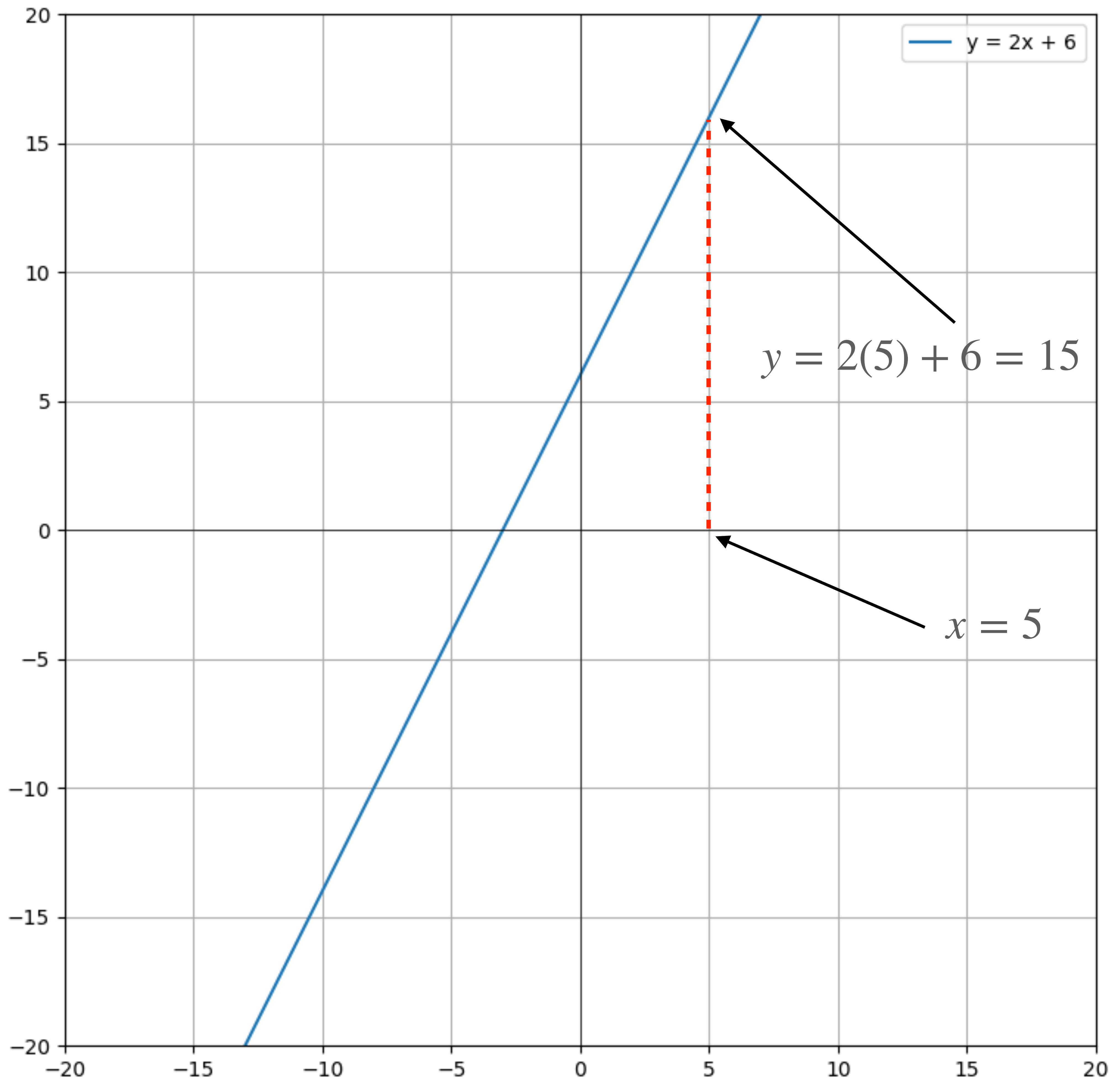
1. **Lines and line intersections**
2. An example from chemistry

Lines (Slope-Intercept Form)

$$y = \underbrace{m}_{\text{slope}}x + \underbrace{b}_{\text{y-intercept}}$$

Given a value of x, I can compute a value of y

Lines (Graph)



Lines (General Form)

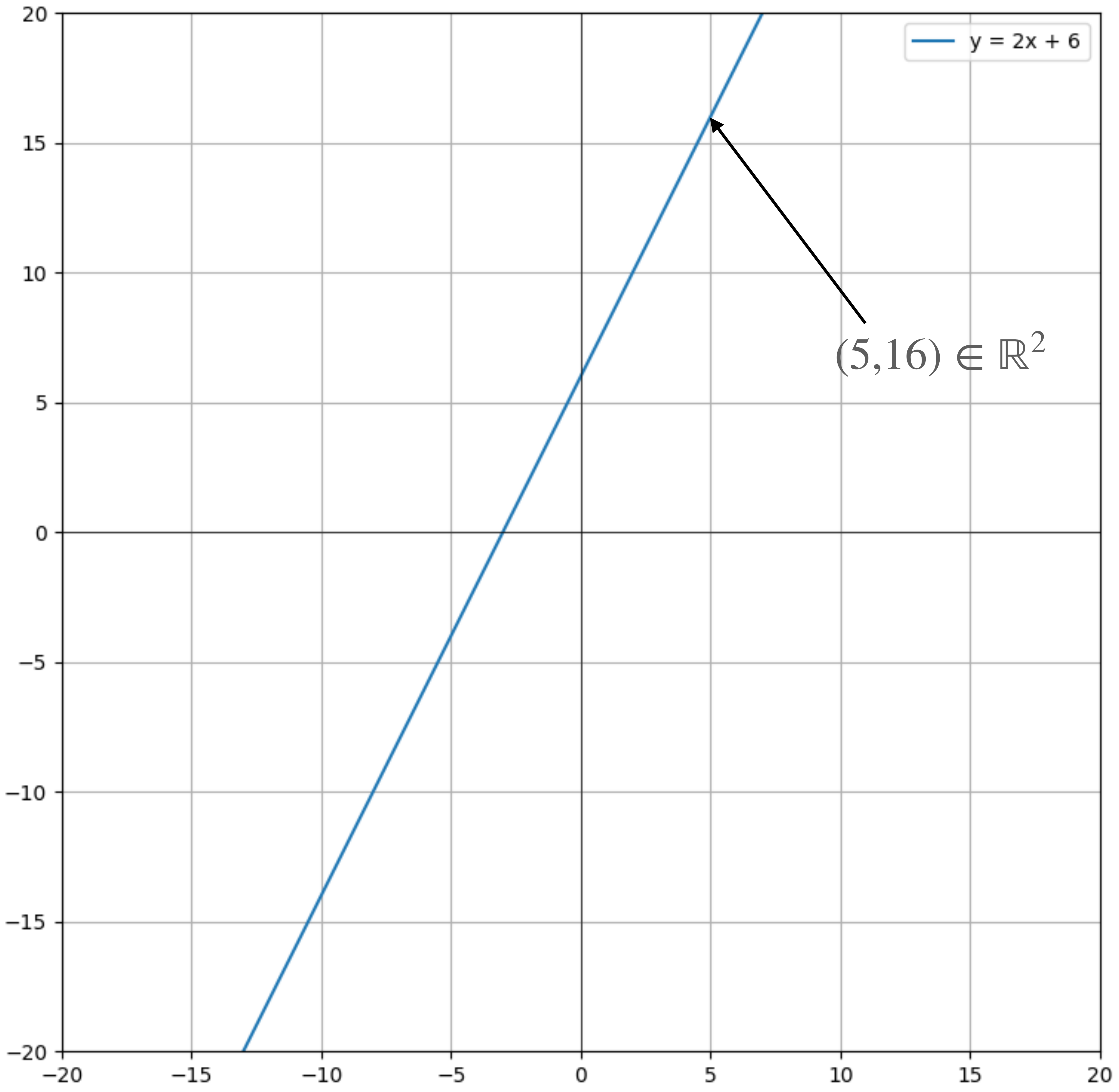
$$\boxed{a}x + \boxed{b}y = \boxed{c}$$

x-intercept: $\boxed{\frac{c}{a}}$

y-intercept: $\boxed{\frac{c}{b}}$

What values of x and y make the equality hold?

Lines (Graph)



$$\{(x, y) : (-2)x + y = 6\}$$

Lines

slope-int → general

$$(-m)x + y = b$$

general → slope-int

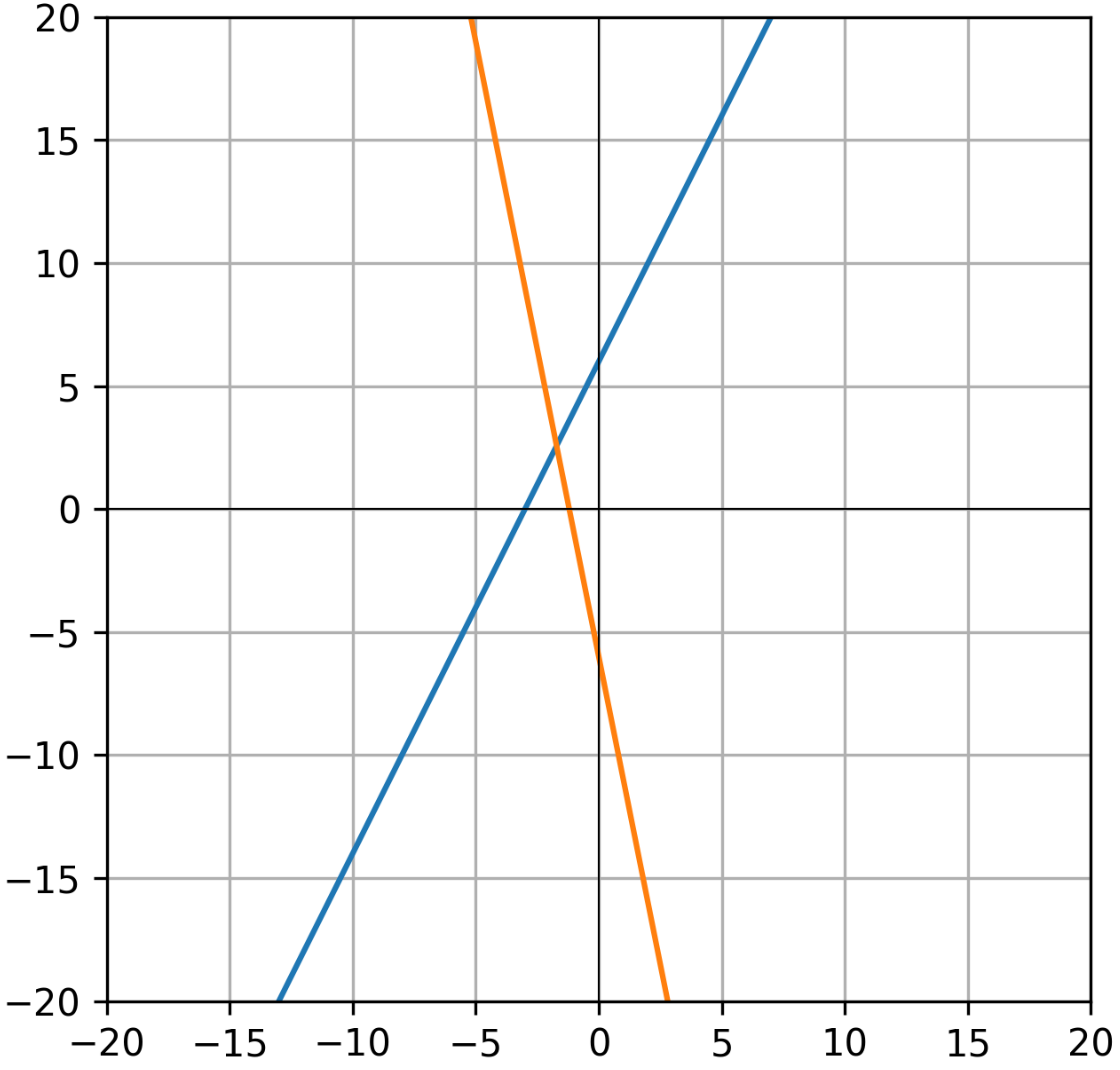
$$y = \left(\frac{-a}{b}\right)x + \frac{c}{b}$$

Line Intersection

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

Question. Given two lines, where do they intersect?

Line Intersection (Graph)



Line Intersection (Alternative)

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

Question. Given two (general form) lines, what values of x and y satisfy ***both*** equations?

This is the same question

Motivation

1. `Lines and line intersections`
2. **`An example from chemistry`**

Example: Balancing Chemical Equations

$$C_6H_{12}O_6 \rightarrow C_2H_5OH + CO_2$$

Glucose Ethanol

We want to know how much ethanol is produced by fermentation (for science)

The number of atoms has to be preserved on each side of the equation

Balancing Chemical Equations

$$\alpha C_6H_{12}O_6 \rightarrow \beta C_2H_5OH + \gamma CO_2$$

Glucose Ethanol

$$6\alpha = 2\beta + \gamma \qquad (C)$$

$$12\alpha = 6\beta \qquad (H)$$

$$6\alpha = \beta + 2\gamma \qquad (O)$$

Balancing Chemical Equations

$$\alpha C_6H_{12}O_6 \rightarrow \beta C_2H_5OH + \gamma CO_2$$

Glucose Ethanol

$$6\alpha - 2\beta - \gamma = 0 \qquad \text{(C)}$$

$$12\alpha - 6\beta = 0 \qquad \text{(H)}$$

$$6\alpha - \beta - 2\gamma = 0 \qquad \text{(O)}$$

Objectives

1. Motivation
2. **Definitions**
3. Solve systems of linear equations

Defining Systems of Linear Equations

1. Linear equations
2. Systems of linear equations
3. Consistency
4. Matrix representations

Defining Systems of Linear Equations

1. **Linear equations**
2. Systems of linear equations
3. Consistency
4. Matrix representations

Linear Equations

Definition. A ***linear equation*** in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$\overbrace{a_1}^{\text{coefficients}} \overbrace{x_1}^{\text{unknowns}} + a_2 x_2 + \ldots + a_n x_n = b$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers ($\mathbb{R}$)

Linear Equations (Point sets)

Linear equations describe ***point sets***:

$$\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : a_1 s_1 + a_2 s_2 + \ldots + a_n s_n = b\}$$

The collections of numbers such that the equation holds.

These points are also called ***vectors***, and $\mathbb{R}^3$ is an example of a ***vector space***

Linear Equations (Geometrically)

If a 2D linear equation is a ***line*** then a 3D linear equation is...

Not a line...

Example 1

$$0x + 0y + z = 15$$

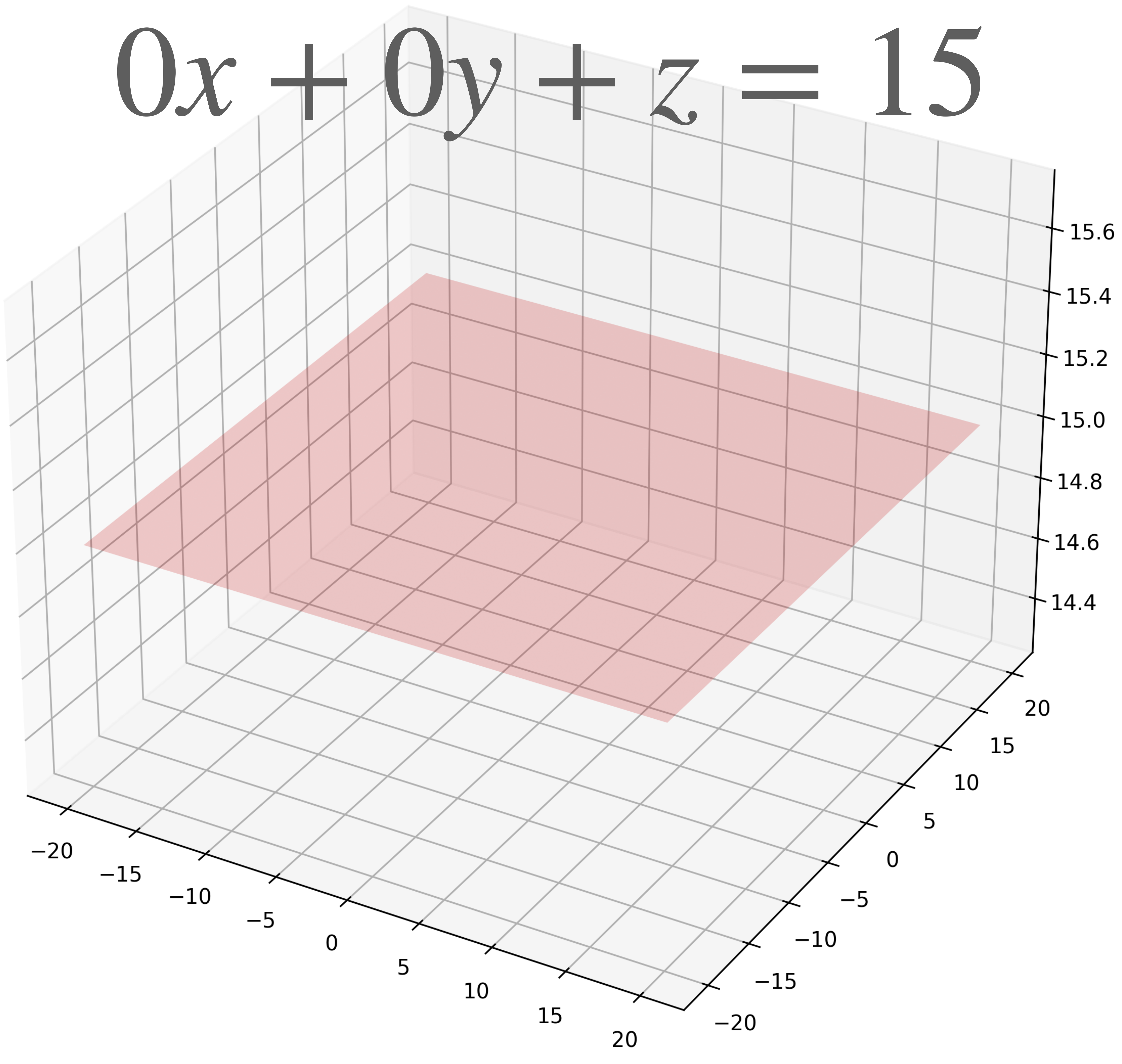
This equation describes the solution set

$$\{(x, y, z) : z = 15\}$$

so x and y can be whatever we want

Example 1

$$0x + 0y + z = 15$$



Linear Equations (Geometrically)

If a 2D linear equation is a ***line*** then a 3D linear equation is...

A plane(!)

Example 2

$$-x + 0y + z = 15$$

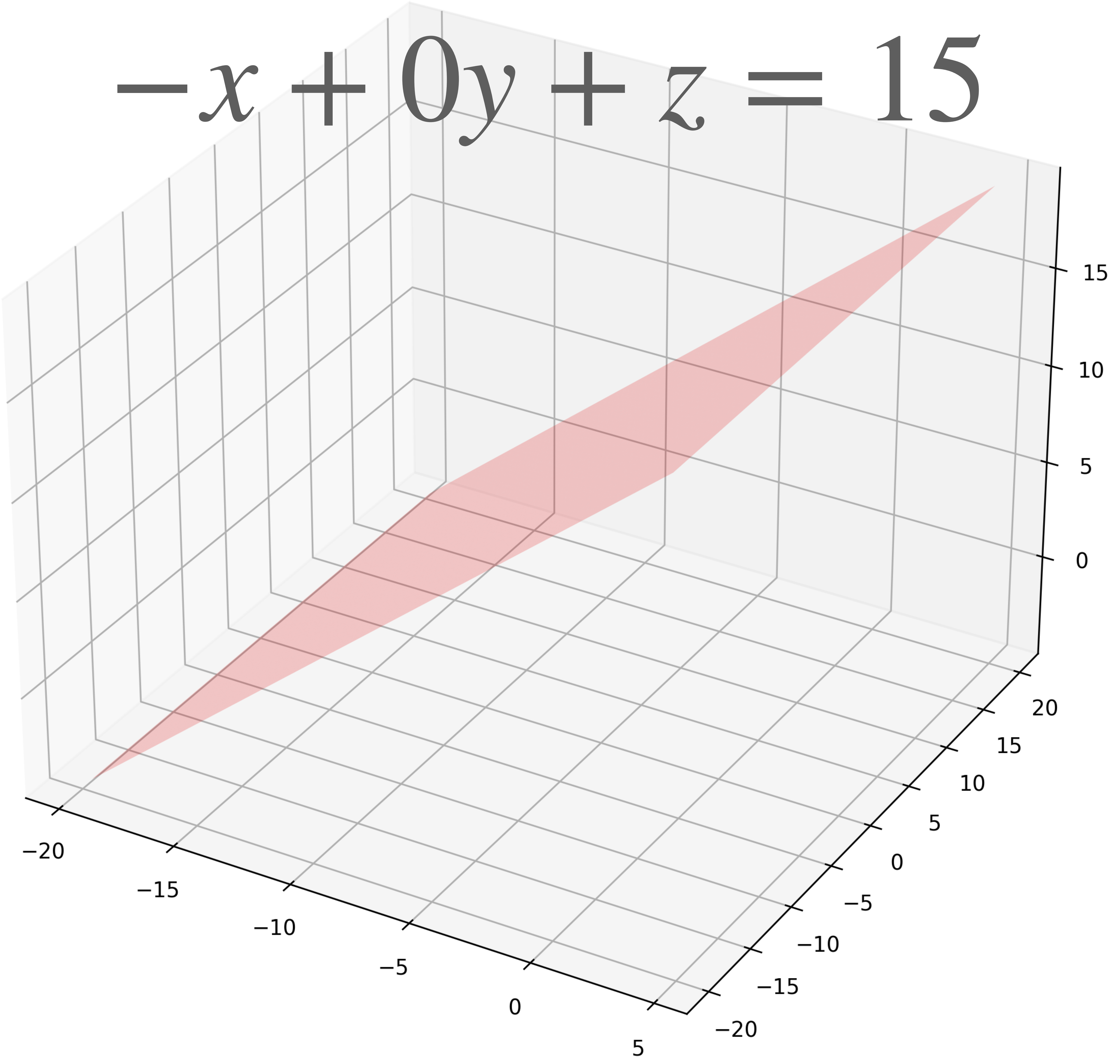
This equation describes the point set

$$\{(x, y, z) : z = x + 15\}$$

so y can be whatever we want

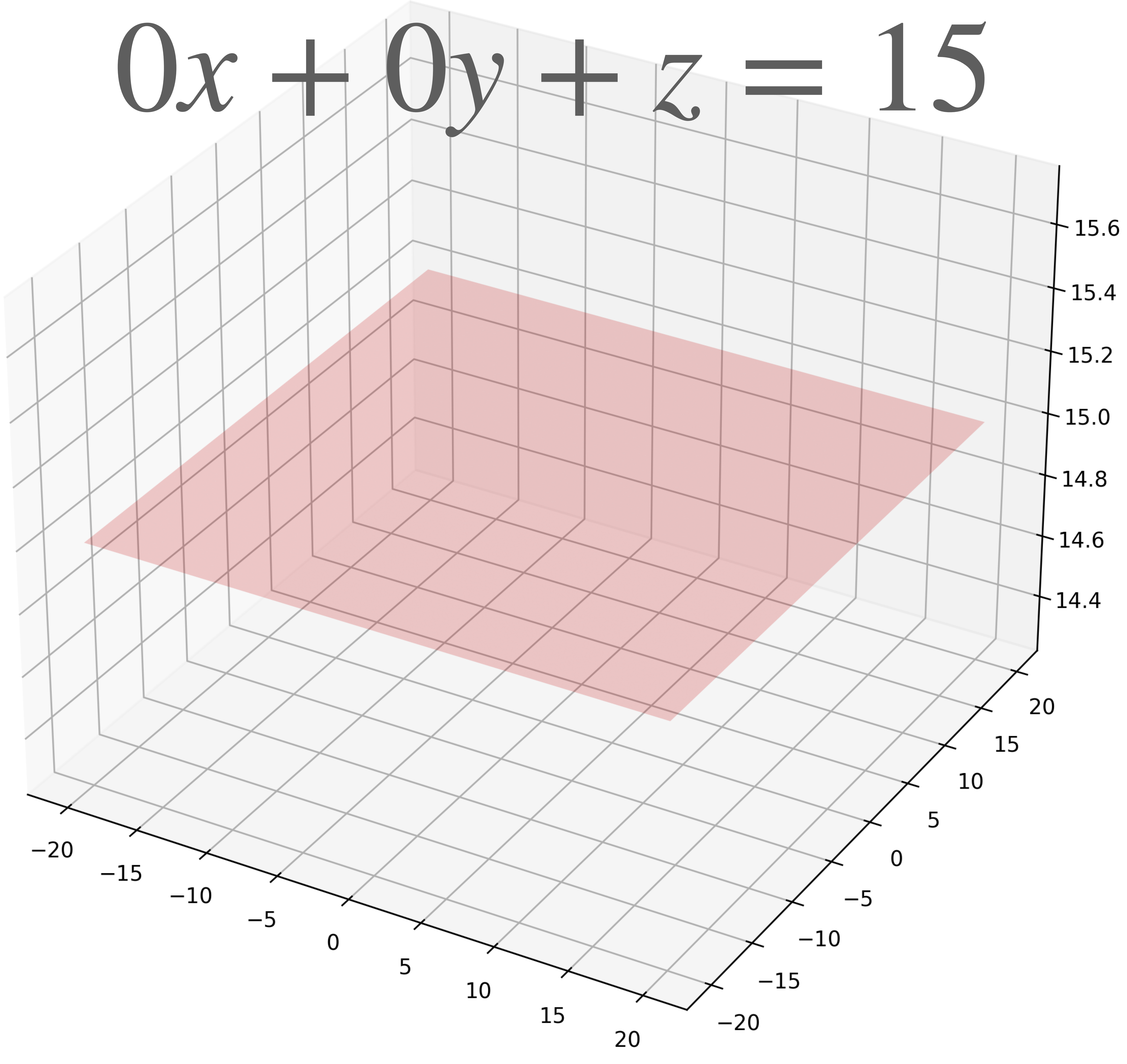
Example 2

$$-x + 0y + z = 15$$



Example 1

$$0x + 0y + z = 15$$



Example 3

$$-x + -y + z = 15$$

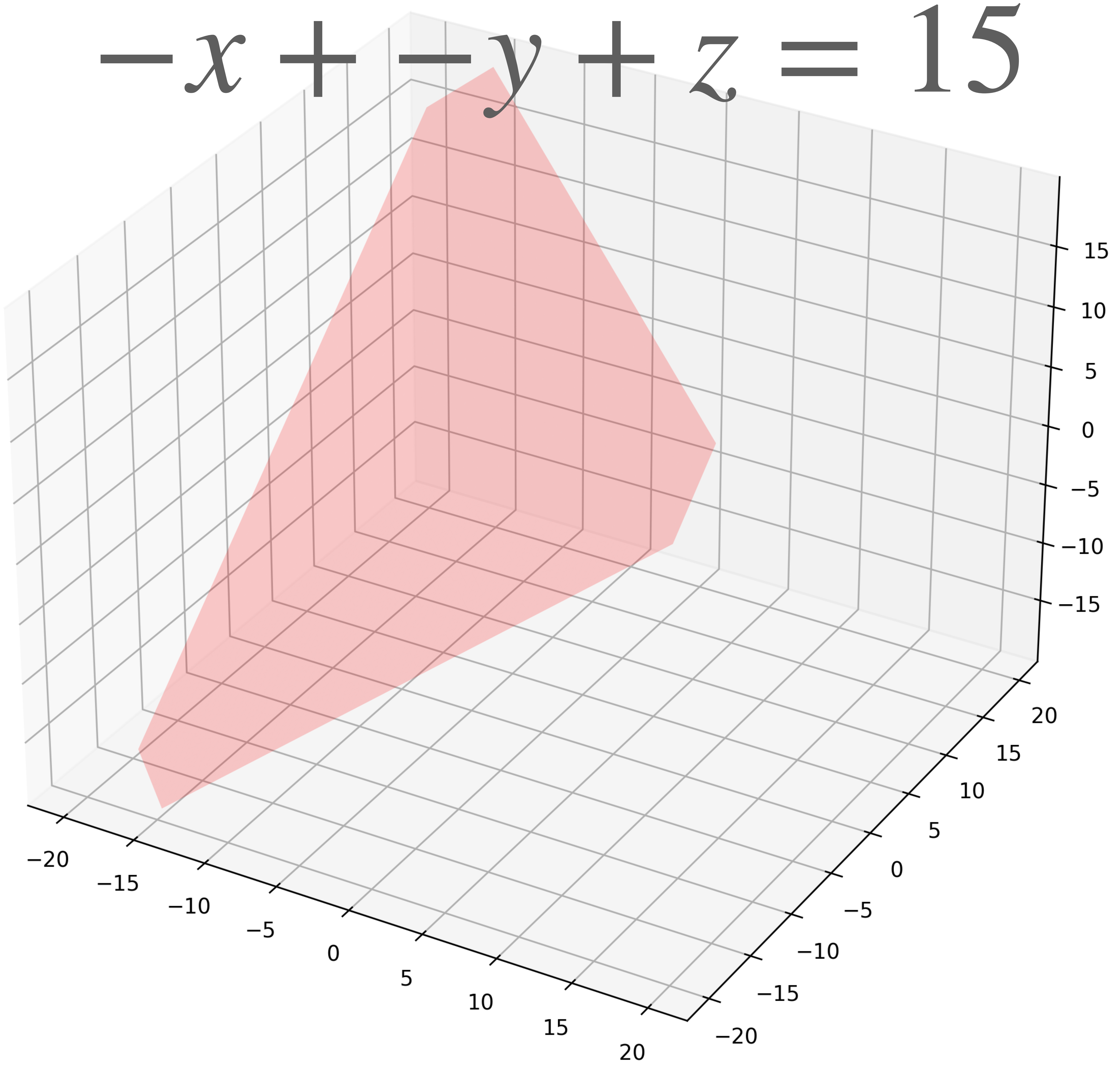
This equation describes the solution set

$$\{(x, y, z) : z = x + y + 15\}$$

so all variables depend on each other

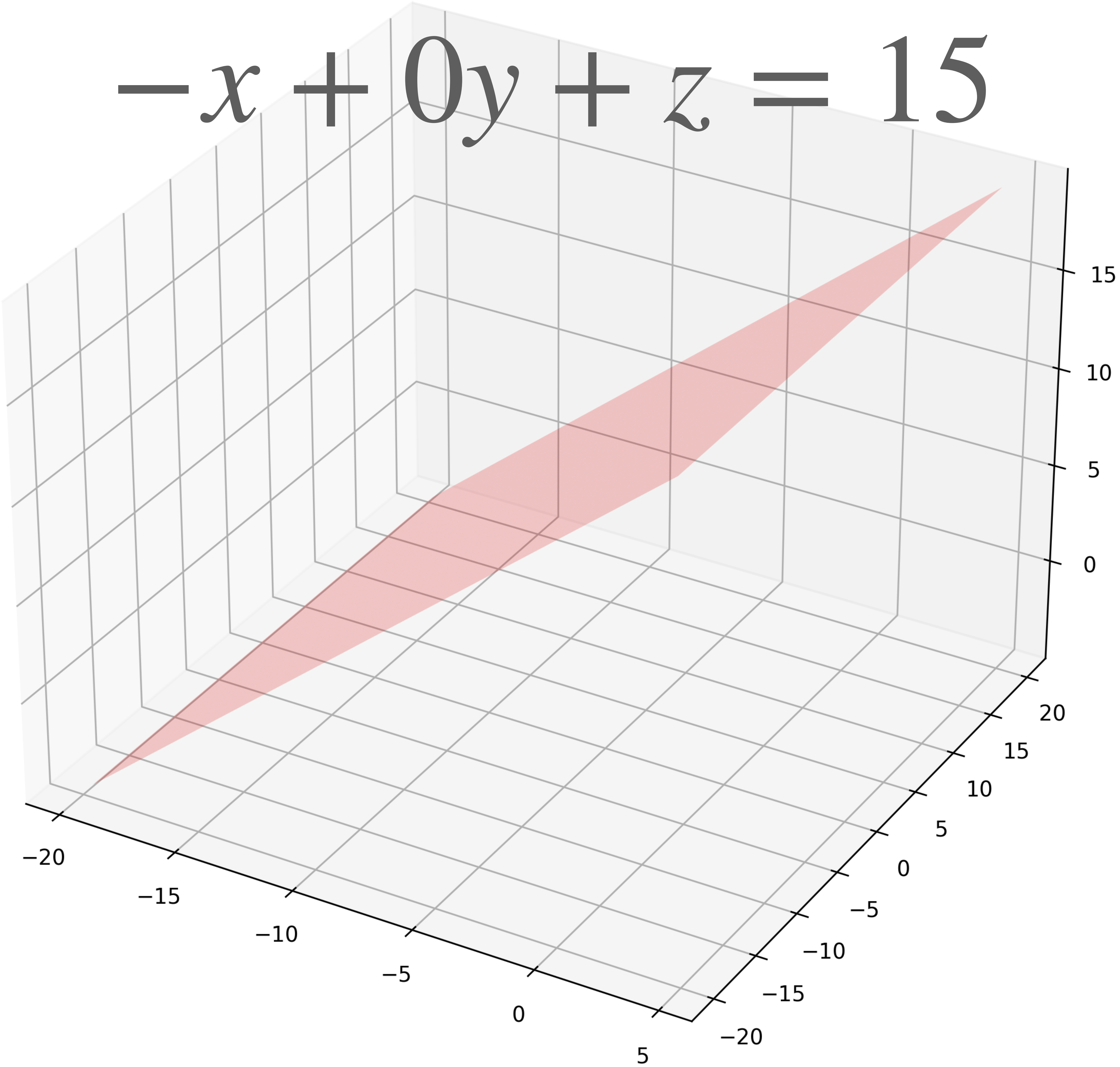
Example 3

$$-x + -y + z = 15$$

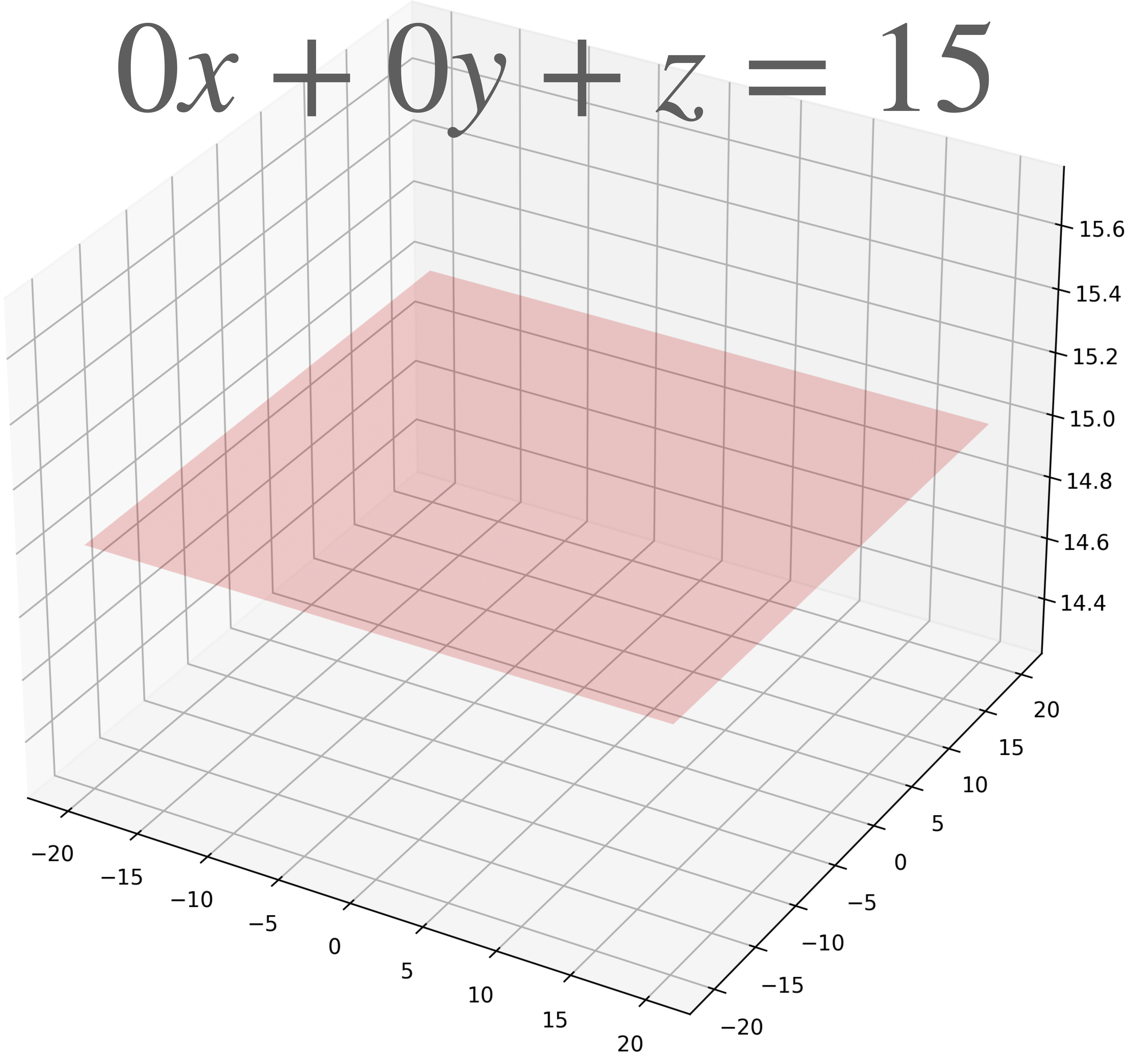


Example 2

$$-x + 0y + z = 15$$



Example 1


$0x + 0y + z = 15$
15.6
15.4
15.2
15.0
14.8
14.6
14.4
−20
−15
−10
−5
0
5
10
15
20
20
15
10
5
0
−5
−10
−15
−20

XYZ-intercepts

$$ax + by + cz = d$$

Just like with lines, we can define

x-intercept: $\frac{d}{a}$ y-intercept: $\frac{d}{b}$ z-intercept: $\frac{d}{c}$

These three points define the plane

Question

I just lied

Give an example of a linear equation that defines a plane with an x–intercept and y–intercept but no z–intercept

Hyperplanes

after three dimensions, we can't visualize planes

the point set of a linear equation is called a ***hyperplane***

Defining Systems of Linear Equations

1. `Linear equations`
2. **`Systems of linear equations`**
3. `Consistency`
4. `Matrix representations`

Systems of Linear Equations

Definition. A ***system of linear equations*** is just a collection of linear equations

Definition. A ***solution*** to a system is a point (vector) that satisfies all its equations <u>simultaneously</u>

System of Linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Does a system have a solution?

How many solutions are there?

What are its solutions?

Defining Systems of Linear Equations

1. Linear equations
2. Systems of linear equations
3. **Consistency**
4. Matrix representations

Consistency

Definition. A system of linear equations is ***consistent*** if it has a solution

It is ***inconsistent*** if it has <u>no</u> solutions

Question

give an example of a 2D system of linear equations with no solutions

Can two lines intersect at more than one point?

Number of Solutions

```
zero the system is inconsistent

one  the system has a unique solution

many the system has infinity solutions
```

Defining Systems of Linear Equations

1. Linear equations
2. Systems of linear equations
3. Consistency
4. **Matrix representations**

Matrix Representations

always writing down the unknowns is <u>exhausting</u>

we will write down linear systems as ***matrices***, which are just 2D grids of numbers with <u>fixed</u> width and height

a matrix is just a representation

Matrix Representations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

`augmented matrix`

Matrix Representations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

`coefficient matrix`

Matrix Representations

$$6\alpha - 2\beta - \gamma = 0 \qquad \text{(C)}$$

$$12\alpha - 6\beta = 0 \qquad \text{(H)}$$

$$6\alpha - \beta - 2\gamma = 0 \qquad \text{(O)}$$

Matrix Representations

$$\begin{bmatrix} 6 & -2 & -1 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & -1 & -2 & 0 \end{bmatrix}$$

Objectives

1. Motivation
2. Definitions
3. **Solve systems of linear equations**

Solving Systems of Linear Equations

1. Some simple examples
2. Elimination and Back-Substitution
3. Row Equivalence

Solving Systems of Linear Equations

1. Some simple examples
2. Elimination and Back-Substitution
3. Row Equivalence

Solving Systems with Two Variables

$$2x + 3y = -6$$

$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$2x = (-3)y - 6$$
$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$
$$4x - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$4((-3/2)y - 3) - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-6y - 12 - 5y = 10$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$-11y = 22$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)y - 3$$

$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = (-3/2)(-2) - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 3 - 3$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$x = 0$$
$$y = -2$$

The Approach

Solve for x in terms of y in EQ1

Substitute result for x in EQ2 and solve for y

Substitute result for y in EQ1 and solve for x

Solving Systems with Two Variables

$$2x + 3y = -6$$
$$4x - 5y = 10$$

The Approach

Eliminate x from the EQ2 and solve for y

Eliminate y from EQ1 and solve for x

Solving Systems of Linear Equations

1. Some simple examples
2. **Elimination and Back-Substitution**
3. Row Equivalence

Solving Systems with Three Variables

$$\begin{aligned} x - 2y + z &= 5 \\ 2y - 8z &= -4 \\ 6x + 5y + 9z &= -4 \end{aligned}$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$6(5 + 2y - z) + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$30 + 12y - 6z + 5y + 9z = -4$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$17y + 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(8z - 4)/2 + 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$17(4z - 2) - 3z = -34$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$

$$2y - 8z = -4$$

$$68z - 34 - 3z = 26$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$71z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2y + \boxed{0} = 5$$

$$2y - 8\boxed{(0)} = -4$$

$$\boxed{z = 0}$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$
\begin{aligned}
x - 2y &= 5 \\
2y &= -4 \\
z &= 0
\end{aligned}
$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x - 2(-2) = 5$$

$$y = -2$$

$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$
\begin{aligned}
x &= 1 \\
y &= -2 \\
z &= 0
\end{aligned}
$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Solving Systems with Three Variables

$$x = 1$$
$$y = -2$$
$$z = 0$$

The Approach

Eliminate x from the EQ2 and EQ3

Eliminate y from EQ3

Elimination

Eliminate z from EQ2 and EQ1

Eliminate y from EQ1

Back-Substitution

Verifying the Solution

$$x - 2y + z = 5$$
$$2y - 8z = -4$$
$$6x + 5y + 9z = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Verifying the Solution

$$(1) - 2(-2) + (0) = 5$$
$$2(-2) - 8(0) = -4$$
$$6(1) + 5(-2) + 9(0) = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Verifying the Solution

$$1 + 4 + 0 = 5$$
$$-4 + 0 = -4$$
$$6 - 10 + 0 = -4$$

$$x = 1$$
$$y = -2$$
$$z = 0$$

Verifying the Solution

$$5 = 5$$
$$-4 = -4$$
$$-4 = -4$$

The solution simultaneously satisfies the equations

$$x = 1$$
$$y = -2$$
$$z = 0$$

Solving Systems of Linear Equations

1. Some simple examples
2. Elimination and Back-Substitution
3. **Row Equivalence**

Solving Systems as Matrices

How does this look with matrices?

Observation. Each intermediate step of elimination and back-substitution gives us a new linear system with the <u>same solutions</u>

Can we represent these intermediate steps as operations on matrices?

Elementary Row Operations

scaling	multiply a row by a number
interchange	switch two rows
replacement	add two rows (and replace one with the sum)

These operations don't change the solutions

Scaling Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 4x + 6y &= -12 \\ 4x - 5y &= 10 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 4 & 6 & -12 \\ 4 & -5 & 10 \end{bmatrix}$$

Interchange Example

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 4x - 5y &= 10 \\ 2x + 3y &= -6 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 4 & -5 & 10 \\ 2 & 3 & -6 \end{bmatrix}$$

Replacement

$$\begin{aligned} 2x + 3y &= -6 \\ 4x - 5y &= 10 \end{aligned} \longrightarrow \begin{aligned} 2x - 3y &= -6 \\ 6x - 2y &= 4 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

Question

Describe how to perform substitution (substituting a variable in one equation with the its value in another equation) via row operations

Elementary Row Operations

scaling	multiply a row by a number
interchange	switch two rows
replacement	add two rows (and replace one with the sum)
rep. + scl.	add a scaled equation to another

Example: Row Reductions

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & 3 & -6 \\ 0 & -11 & 22 \end{bmatrix}$$

$$\xrightarrow{R_2/(-11)} \begin{bmatrix} 2 & 3 & -6 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_1/2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Example: Row Reductions

$$R_2 \to R_2 - 2R_1$$
$$R_2 \to R_2/(-11)$$

elimination

$$R_1 \to R_1 - 3R_2$$
$$R_1 \to R_1/2$$

substitution

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Equivalence

Definition. Two matrices are ***row equivalent*** if one can be transformed into the other by a sequence of row operations

$$\begin{bmatrix} 2 & 3 & -6 \\ 4 & -5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

We can compute solutions by sequence of row operations

Row Equivalence and Inconsistency

If a system is inconsistent, it is row equivalent to a system with a row of the form

$$0\ 0\ \ldots\ 0\ k$$

for $k \neq 0$

(what happens if $k = 0$?)

Summary

Linear equations define hyperplanes

Systems of linear equations may or may not have solutions

Linear systems can be represented as matrices, which makes them more convenient to solve