An Early Proof of Normalization by A.M. Turing

R.O. Gandy

Dedicated to H.B. Curry on the occasion of his 80th birthday

In the extract printed below, Turing shows that every formula of Church's simple type theory has a normal form. The extract is the first page of an unpublished (and incomplete) typescript entitled 'Some theorems about Church's system'. (Turing left his manuscripts to me; they are deposited in the library of King's College, Cambridge). An account of this system was published by Church in 'A formulation of the simple theory of types' (J. Symbolic Logic 5 (1940), pp. 56-68). Church had previously described the system in lectures given at Princeton (1937-38) which Turing attended; he was a graduate student at Princeton 1936-1938. He is mentioned as having contributed to results about the system in footnote 12 of Church's paper. In an undated letter to M.H.A. Newman (which must have been written early in 1942) Turing outlines the contents of his proposed paper (including the normal form theorem); he refers to it as 'forthcoming' in his paper 'The use of dots as brackets in Church's system' (J. Symbolic Logic 7 (1942), pp. 146-156, received 17 June 1942). For some further details about Turing's work on type theory, see my paper 'The simple theory of types' in Logic Colloqium '76, Ed. R.O. Gandy & J.M.E. Hyland, North Holland Pub. Co. Amsterdan 1977, pp. 173-181. Thus Turing's proof antedates by many years any published proof of the theorem.

Turing's proof depends on the rather obvious remark that if one reduces the rightmost (or an innermost) redex $(\lambda x_\beta A_\alpha)B_\beta$ whose head $\lambda x_\beta A_\alpha$ is of highest type in a formula F, then the resulting formula has fewer redexes with head of type $(\alpha\beta)$. The theorem follows by use of simple induction with a π^0_1 predicate, or by transfinite induction up to ω^2 with a primitive recursive predicate. (Turing's use of an ordering of formulae with order-type ω^ω is not necessary). A very meticulous account of this method of proof is given by P.B. Andrews in 'Resolution in type theory' (J. Symbolic Logic 36 (1971), pp. 414-432). Andrews writes in a footnote: 'This proposition is part of the folklore of type-theoretic λ-conversion. The author first heard the idea of the proof given here from Dr. James R. Guard'. The same method of proof, applied to the contractions of proofs in systems of natural reduction, is used by Pravitz in his *Natural Deduction: a proof-theoretical study* (Stockholm 1965).

The earliest published proof known to me is in Curry and Feys' book *Combinatory Logic* (North Holland Pub. Co. Amsterdam 1958; 2nd printing 1968). The normal form theorem is included in their Theorem 9 on page 340. The proof depends on the 'elimination theorem' (Theorem 5, p. 326), which may be viewed as sort of cut-elimination theorem for the theory of functionality. In a very loose sense, one may say that Genzen's Hauptsatz was the first normal form theorem.

I should like to thank J.P. Seldin for his help in providing references.

The Extract

Proof that every type formula has a normal form

We will well-order the formulae of Church's system as follows. We say that the formula has an unreduced part of order n if it has a part of form $(\lambda x_\alpha A_\beta)B_\alpha$ where $\beta\alpha$ is of length n. If we wish to decide which of two formulae precedes we find out what is the highest order of any unreduced part in each. The formula which has an unreduced part of higher order than any part of the other comes later. Suppose however that the maximum orders are the same, n say, then the one which has the more unreduced parts of order n comes later. But these numbers may also be equal, and in this case we compare the number of unreduced parts of order $n-1$, and if we fail with these we go to those of order $n-2$. If eventually there is a difference the formula with the greater number of unreduced parts comes later, if however the numbers remain the same to the end, i.e. as far as those of order 1 then the longer formula comes later. It is not difficult to see that this is a well ordering of formulae, of type ω^ω.

Now when we perform a reduction on a formula, in which we reduce one of the unreduced parts of highest order, we necessarily decrease the number of unreduced parts of the highest order, for we destroy one and we do not create any more: this at any will be the case if we choose the unreduced part of highest order whose λ lies farthest to the right. We therefore reduce the formula to one which is earlier in the sequence, and as the sequence is well-ordered the sequence of reductions must come to an end.

This has been copied verbatim: 'rate' should be inserted after 'any' in the last line but four.