Matrix Inverses

Geometric Algorithms
Lecture 10

CAS CS 132

Objectives

1. Define a few more important matrix operations
2. Motivate and define matrix inverses
3. Application: Adjacency Matrices

Keywords

Matrix Transpose

Inner Product

Matrix Power

Square Matrix

Matrix Inverse

Invertible Transformation

1–1 Correspondence

numpy.linalg.inv

eterminant

Invertible Matrix Theorem

Recap Problem

Suppose that A, $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ and $C = [\mathbf{c}_1 \ \ \mathbf{c}_2 \ \ \mathbf{c}_3]$ are matrices such that

$$A(B + 5I) = C$$

Find a solution to the equation $A\mathbf{x} = \mathbf{c}_2$.

Answer: $\mathbf{b}_2 + 5\mathbf{e}_2$

More Matrix Operations

Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose (Pictorially)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 \end{bmatrix}$$

Transpose

Definition. For a $m \times n$ matrix A, the **transpose of** A, written A^T, is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}$$

Example.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Algebraic Properties (Transpose)

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ (where c is a scalar)

$(AB)^T = B^T A^T$

Algebraic Properties (Transpose)

$(A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(cA)^T = cA^T$ (where c is a scalar)

$(AB)^T = B^T A^T$ Important: the order reverses!

Challenge Problem (Not In-Class)

Show that $(AB)^T = B^T A^T$.

Example: $\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)^T$

Transposes and Inner Products

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

It's a $1 \times n$ matrix.

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

It's a $1 \times n$ matrix.

For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,
is $\mathbf{u}^T\mathbf{v}$ defined?

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

It's a $1 \times n$ matrix.

For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, is $\mathbf{u}^T\mathbf{v}$ defined?

$$\underset{1 \times n}{\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}} = \underset{1 \times 1}{?}$$

Transposes and Inner Products

For a vector $\mathbf{v} \in \mathbb{R}^n$, what is $\mathbf{v}^T$?

It's a $1 \times n$ matrix.

$\boxed{1} \times \boxed{n}$ $\boxed{n} \times \boxed{1}$ $\boxed{1} \times \boxed{1}$

For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,
is $\mathbf{u}^T\mathbf{v}$ defined?

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \quad ?$$

Transposes and Inner Products

Transposes and Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Transposes and Inner Products

$$\begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Definition. The **inner product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Matrix Powers

Matrix Powers

If A is an $n \times n$ matrix, then the product AA is defined.

Matrix Powers

If A is an $n \times n$ matrix, then the product AA is defined.

Definition. For a $n \times n$ matrix A, we write A^k for the k–fold product of A with itself.

Matrix Powers

If A is an $n \times n$ matrix, then the product AA is defined.

Definition. For a $n \times n$ matrix A, we write A^k for the k–fold product of A with itself.

What should A^0 be?

Matrix Powers

If A is an $n \times n$ matrix, then the product AA is defined.

Definition. For a $n \times n$ matrix A, we write A^k for the k−fold product of A with itself.

What should A^0 be?

$10^0 = 1$, so it stands to reason that $A^0 = I$.

Matrix Powers

If A is an $n \times n$ matrix, then the product AA is defined.

Definition. For a $n \times n$ matrix A, we write A^k for the k−fold product of A with itself.

What should A^0 be?

$10^0 = 1$, so it stands to reason that $A^0 = I$.

(we want $A^0 A^k = A^{0+k} = A^k$)

Final Warnings about Matrix Multiplication

Final Warnings about Matrix Multiplication

1. AB is not necessarily equal to BA, even if both are defined.

Final Warnings about Matrix Multiplication

1. AB is not necessarily equal to BA, even if both are defined.

2. If $AB = AC$ then it is not necessary that $B = C$.

Final Warnings about Matrix Multiplication

1. AB is not necessarily equal to BA, even if both are defined.

2. If $AB = AC$ then it is not necessary that $B = C$.

3. If $AB = 0$ (the zero matrix) it is not necessarily the case that $A = 0$ or $B = 0$.

Question

Find two nonzero 2×2 matrices A and B such that $AB = 0$.

Challenge. *Choose A and B such that they have <u>all</u> nonzero entries.*

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So Far: Matrix Operations

So Far: Matrix Operations

`transpose` A^T

So Far: Matrix Operations

`transpose`	A^T
`scaling`	cA

So Far: Matrix Operations

`transpose`	A^T	
`scaling`	cA	
`addition (subtraction)`	$A + B$	$A + (-1)B = A - B$

So Far: Matrix Operations

`transpose`	A^T	
`scaling`	cA	
`addition (subtraction)`	$A + B$	$A + (-1)B = A - B$
`multiplication (powers)`	AB	A^k

So Far: Matrix Operations

transpose	A^T	
scaling	cA	
addition (subtraction)	$A + B$	$A + (-1)B = A - B$
multiplication (powers)	AB	A^k

What's missing?

Matrix Inverses

Basic Algebra

$$2x = 10$$

Basic Algebra

$$2x = 10$$

`How do we solve this equation?`

Basic Algebra

$$2x = 10$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Basic Algebra

$$2x = 10$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

Basic Algebra

$$2x = 10$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

```
How do we solve this equation?
```

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

Basic Algebra

$$2^{-1}(2x) = 2^{-1}(10)$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$1x = 5$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Basic Algebra

$$x = 5$$

How do we solve this equation?

Divide on both sides by 2 to get $x = 5$.

Multiply each side by $\frac{1}{2}$ *a.k.a.* 2^{-1}.

$\frac{1}{2}$ is the **reciprocal** or **multiplicative inverse** of 2.

Wouldn't it be nice...

$$A\mathbf{x} = \mathbf{b}$$

Wouldn't it be nice...

$$A\mathbf{x} = \mathbf{b}$$

How do we solve this equation?

Wouldn't it be nice...

$$A\mathbf{x} = \mathbf{b}$$

How do we solve this equation?

Multiply each side by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

Wouldn't it be nice...

$$A\mathbf{x} = \mathbf{b}$$

How do we solve this equation?

Multiply each side by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

A^{-1} is the **multiplicative inverse** of A

Wouldn't it be nice...

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

`How do we solve this equation?`

`Multiply each side by` A^{-1} `to get` $\mathbf{x} = A^{-1}\mathbf{b}$`.`

A^{-1} `is the` **`multiplicative inverse`** `of` A

Wouldn't it be nice...

$$I\mathbf{x} = A^{-1}\mathbf{b}$$

How do we solve this equation?

Multiply each side by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

A^{-1} is the **multiplicative inverse** of A

Wouldn't it be nice...

$$\mathbf{x} = A^{-1}\mathbf{b}$$

How do we solve this equation?

Multiply each side by A^{-1} to get $\mathbf{x} = A^{-1}\mathbf{b}$.

A^{-1} is the **multiplicative inverse** of A

Do all matrices have inverses?

Do all matrices have inverses?

No.

When does a matrix have an inverse?

Square Matrices

Definition. A $m \times n$ matrix A is **square** if $m = n$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

i.e., it has same number of rows as columns.

Why are square matrices special?

Why are square matrices special?

```
They are the only kind of matrices...
```

Why are square matrices special?

They are the only kind of matrices...

» that can have a pivot in every row <u>and</u> every column.

Why are square matrices special?

They are the only kind of matrices...

» that can have a pivot in every row <u>and</u> every column.

» whose transformations can be both 1–1 and onto.

Why are square matrices special?

They are the only kind of matrices...

» that can have a pivot in every row <u>and</u> every column.

» whose transformations can be both 1–1 and onto.

» whose columns can have full span and be linearly independent.

Why are square matrices special?

They are the only kind of matrices...

» that can have a pivot in every row <u>and</u> every column.

» whose transformations can be both 1-1 and onto.

» whose columns can have full span and be linearly independent.

» that can have inverses.

Dimension Tracking

$$A \quad \mathbf{x} = \mathbf{b}$$

Dimension Tracking

$$A^{-1}A \quad \mathbf{x} = A^{-1}\mathbf{b}$$

Dimension Tracking

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Dimension Tracking

$$\mathbf{x} = A^{-1}\mathbf{b}$$

The only way for the dimensions to make sense is if A is square

Matrix Inverses

Matrix Inverses

Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

Matrix Inverses

Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

A is **invertible** if it has an inverse. Otherwise it is **singular.**

Matrix Inverses

Definition. For a $n \times n$ matrix A, an **inverse** of A is a $n \times n$ matrix B such that

$$AB = I_n \text{ and } BA = I_n$$

A is **invertible** if it has an inverse. Otherwise it is **singular**.

Example. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Example: Geometric

Reflection across the x_1–axis in $\mathbb{R}^2$ is it's own inverse.

Verify:

Example: No inverse

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

`Verify:`

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

Inverses are Unique

Theorem. If B and C are inverses of A, then $B = C$.

Verify:

If A is invertible, then we write A^{-1} for *the* inverse of A.

Solutions for Invertible Matrix Equations

Theorem. For a $n \times n$ matrix A, if A is invertible then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution for any choice of $\mathbf{b}$.

Verify:

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» exactly one solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a <u>unique</u> solution for any choice of $\mathbf{b}$, then it has

- <u>at least one</u> solution for any choice of $\mathbf{b}$
- <u>at most one</u> solution for any choice of $\mathbf{b}$

Unique Solutions

If $A\mathbf{x} = \mathbf{b}$ has a unique solution for any choice of $\mathbf{b}$, then it has

» T is onto

» T is one-to-one

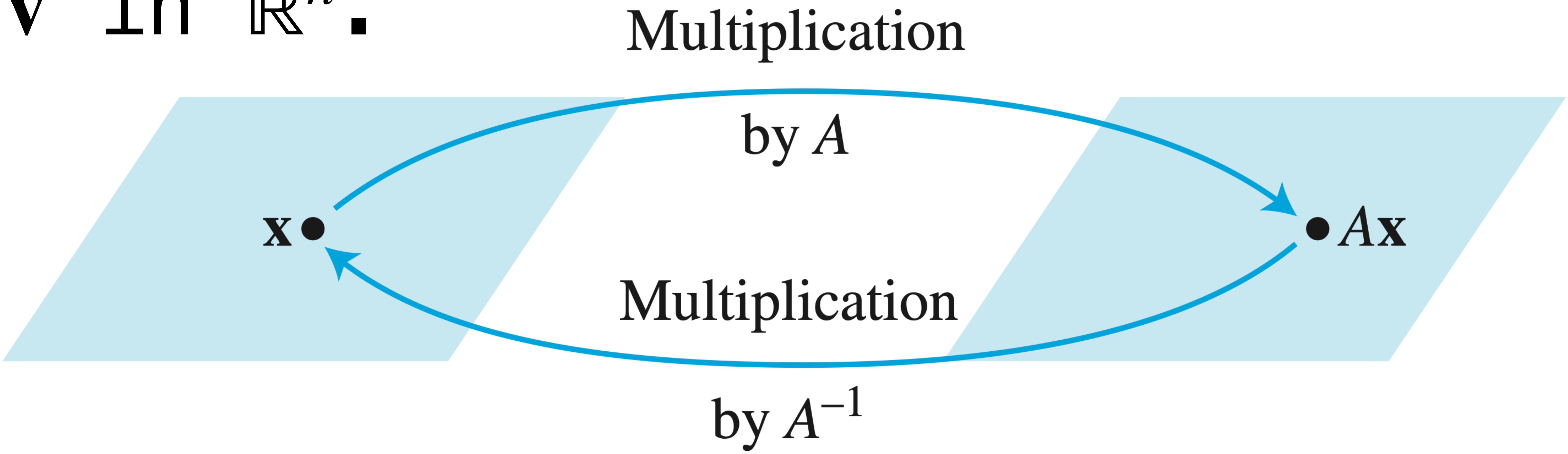
where T is implemented by A

Connection to Transformations

Definition. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a linear transformation S such that

$$S(T(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T(S(\mathbf{v})) = \mathbf{v}$$

for any $\mathbf{v}$ in $\mathbb{R}^n$.



Connection to Transformations

Connection to Transformations

Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

Connection to Transformations

Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

A matrix is invertible if it's possible to "undo" its transformation without "losing information".

Connection to Transformations

Theorem. A $n \times n$ matrix A is invertible if and only if the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible.

A matrix is invertible if it's possible to "undo" its transformation without "losing information".

Non-Example. Projection onto the x_1-axis.

Connection to Transformations

Connection to Transformations

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

Connection to Transformations

Definition. A transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

A transformation is a 1-1 correspondence if it is 1-1 and onto.

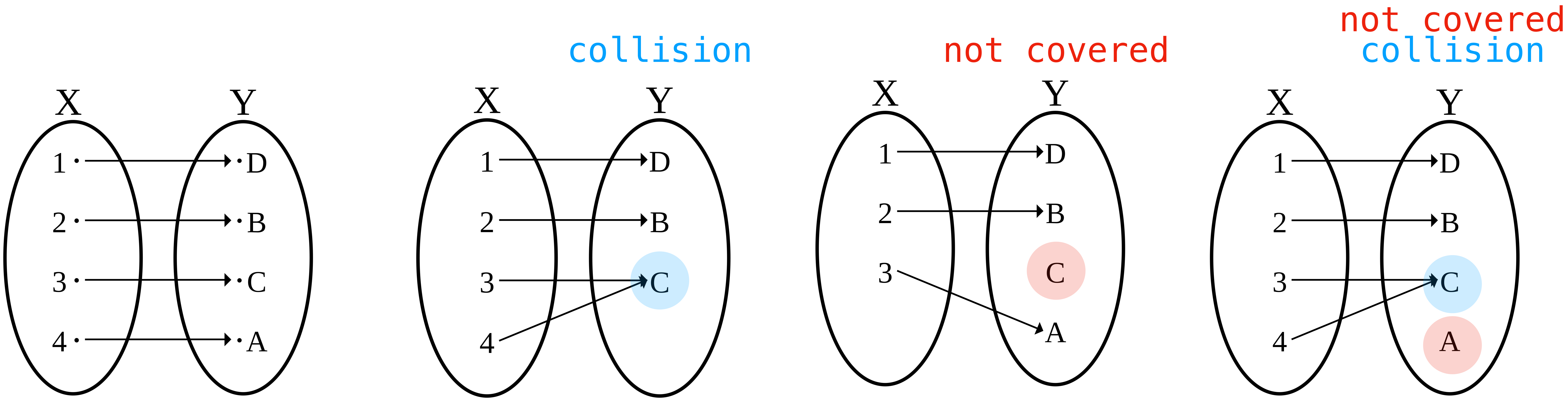
Connection to Transformations

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a **one-to-one correspondence** (bijection) if any vector $\mathbf{b}$ in $\mathbb{R}^n$ is the image of **exactly** one vector $\mathbf{v}$ in $\mathbb{R}^n$ (where $T(\mathbf{v}) = \mathbf{b}$).

A transformation is a 1-1 correspondence if it is 1-1 and onto.

Invertible transformations are 1-1 correspondences.

Kinds of Transformations (Pictorially)



1–1 correspondence

onto, not 1–1

1–1 not onto

not 1–1, not onto

Computing Matrix Inverses

In General

$$A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = I$$

Can we solve for each $\mathbf{b}_i$?:

How To: Matrix Inverses

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

How To: Matrix Inverses

Question. Find the inverse of an invertible $n \times n$ matrix A.

Solution. Solve the equation $A\mathbf{x} = \mathbf{e}_i$ for every standard basis vector. Put those solutions $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ into a single matrix

$$\begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_n \end{bmatrix}$$

How To: Matrix Inverses

How To: Matrix Inverses

Question. Find the inverse of the $n \times n$ matrix A.

How To: Matrix Inverses

Question. Find the inverse of the $n \times n$ matrix A.

Solution. Row reduce the matrix $[A \quad I]$ to a matrix $[I \quad B]$. Then B is the inverse of A.

How To: Matrix Inverses

Question. Find the inverse of the $n \times n$ matrix A.

Solution. Row reduce the matrix $[A \quad I]$ to a matrix $[I \quad B]$. Then B is the inverse of A.

This is really the same thing. It's a simultaneous reduction.

How To: Matrix Inverse Computationally

How To: Matrix Inverse Computationally

Question. Find the inverse of the $n \times n$ matrix A.

How To: Matrix Inverse Computationally

Question. Find the inverse of the $n \times n$ matrix A.

Solution. Use `numpy.linalg.inv()`

How To: Matrix Inverse Computationally

Question. Find the inverse of the $n \times n$ matrix A.

Solution. Use `numpy.linalg.inv()`

Warning: this only works if the matrix is invertible.

demo

Special Case: 2×2 Matrice Inverses

Special Case: 2×2 Matrice Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ac - bd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Special Case: 2×2 Matrice Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ac - bd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The **determinant** of a 2×2 matrix is the value $ad - bc$.

Special Case: 2×2 Matrice Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ac - bd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The **determinant** of a 2×2 matrix is the value $ad - bc$.

The inverse is defined only if the determinant is nonzero.

Special Case: 2×2 Matrice Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ac - bd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The **determinant** of a 2×2 matrix is the value $ad - bc$.

The inverse is defined only if the determinant is nonzero.

(see the notes on linear transformations for more information about determinants)

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

Is the above matrix invertible?

Example

$$\begin{bmatrix} -6 & 14 \\ 3 & -7 \end{bmatrix}$$

Is the above matrix invertible?

No. The determinant is $(-6)(-7) - 14(3) = 42 - 42 = 0$

Algebra of Matrix Inverses

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A

$$(A^{-1})^{-1} = A$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrix A, the matrix A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Verify:

Algebraic Properties (Matrix Inverses)

Theorem. For a $n \times n$ invertible matrices A and B, the matrix AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Verify:

Question

Suppose that A is a $n \times n$ invertible matrix such that $A = A^T$ and B is a $m \times n$ matrix.

Simplify the expression $A(BA^{-1})^T$ using the algebraic properties we've seen.

$$A(BA^{-1})^T$$

$$A = A^T$$

Answer: B^T

Invertible Matrix Theorem

High Level

How do we know if a matrix is invertible?

By connecting everything we've said so far.

Invertible Matrix Theorem (IMT)

Invertible Matrix Theorem (IMT)

1. A is invertible

Invertible Matrix Theorem (IMT)

1. A is invertible
2. A^T is invertible

Invertible Matrix Theorem (IMT)

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$

Invertible Matrix Theorem (IMT)

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$

Invertible Matrix Theorem (IMT)

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$

Invertible Matrix Theorem (IMT)

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
6. A has n pivots (per row and per column)

Invertible Matrix Theorem (IMT)

1. A is invertible
2. A^T is invertible
3. $A\mathbf{x} = \mathbf{b}$ has at least one solution for any $\mathbf{b}$
4. $A\mathbf{x} = \mathbf{b}$ has at most one solution for any $\mathbf{b}$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$
6. A has n pivots (per row and per column)
7. A is row equivalent to I

Invertible Matrix Theorem (IMT)

Invertible Matrix Theorem (IMT)

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

Invertible Matrix Theorem (IMT)

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent

Invertible Matrix Theorem (IMT)

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$

Invertible Matrix Theorem (IMT)

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto

Invertible Matrix Theorem (IMT)

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one

Invertible Matrix Theorem (IMT)

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence

Invertible Matrix Theorem (IMT)

8. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
9. The columns of A are linearly independent
10. The columns of A span $\mathbb{R}^n$
11. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto
12. $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one
13. $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one correspondence
14. $\mathbf{x} \mapsto A\mathbf{x}$ is invertible

We get a lot of information for free

We get a lot of information for free

Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

We get a lot of information for free

Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

We only need to check one of these.

We get a lot of information for free

Theorem. If A is square, then

A **is 1–1** if and only if A **is onto**

We only need to check one of these.

Warning. Remember this only applies square matrices.

We get a lot of information for free

We get a lot of information for free

Theorem. If A is square, then

A **is invertible** $\quad \equiv \quad$ $A\mathbf{x} = \mathbf{0}$ **implies** $\mathbf{x} = \mathbf{0}$

We get a lot of information for free

Theorem. If A is square, then

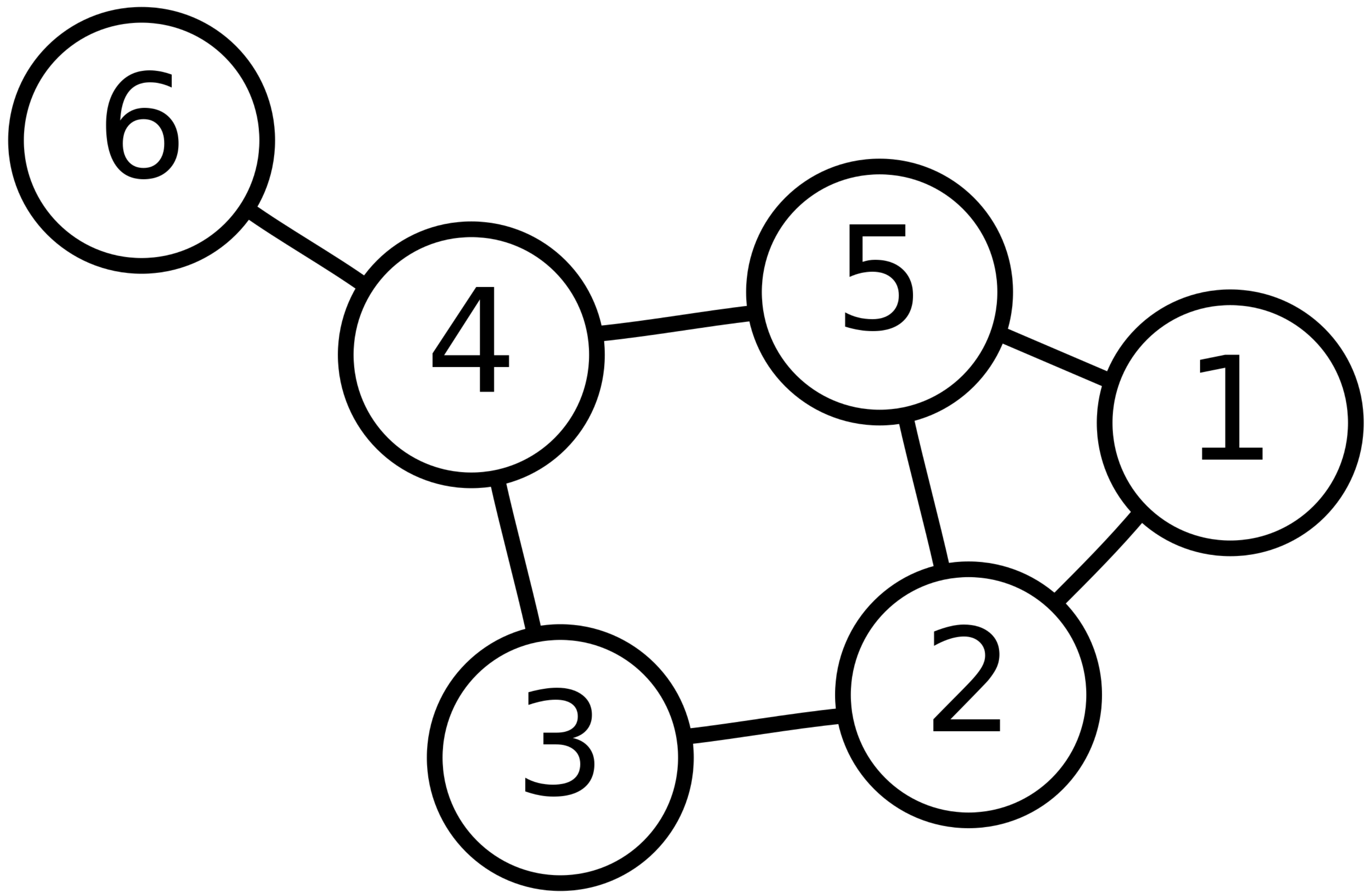
$$A \textbf{ is invertible} \quad \equiv \quad A\mathbf{x} = \mathbf{0} \textbf{ implies } \mathbf{x} = \mathbf{0}$$

Invertibility is completely determined by how A behaves on $\mathbf{0}$.

Application: Adjacency Matrices

Graphs

Definition (Informal). An **undirected graph** is a collection of nodes with edges between them.

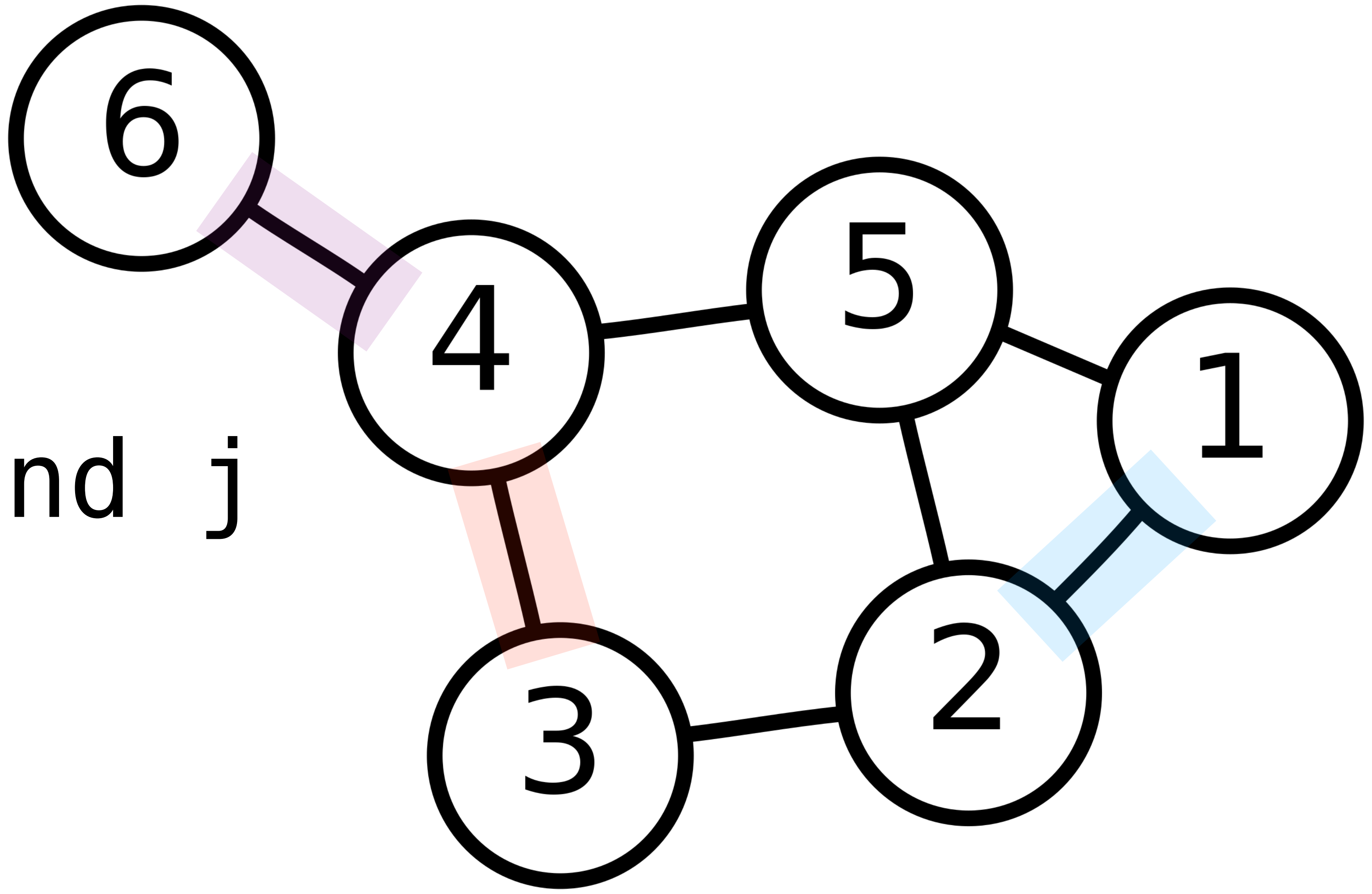


How do we represent these in computers?

Adjacency Matrices

For an undirected graph G we can create the **adjacency matrix** A for G where:

$$
\begin{array}{c c}
 & \begin{array}{cccccc} & A_{12} & & A_{34} & & A_{46} \end{array} \\
\begin{array}{c} \\ A_{21} \\ \\ A_{43} \\ \\ A_{64} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
$$



$$A_{ij} = \begin{cases} 1 & \texttt{there is an edge between i and j} \\ 0 & \texttt{otherwise} \end{cases}$$

Spectral Graph Theory

Once we have an adjacency matrix, we can do linear algebra on graphs.

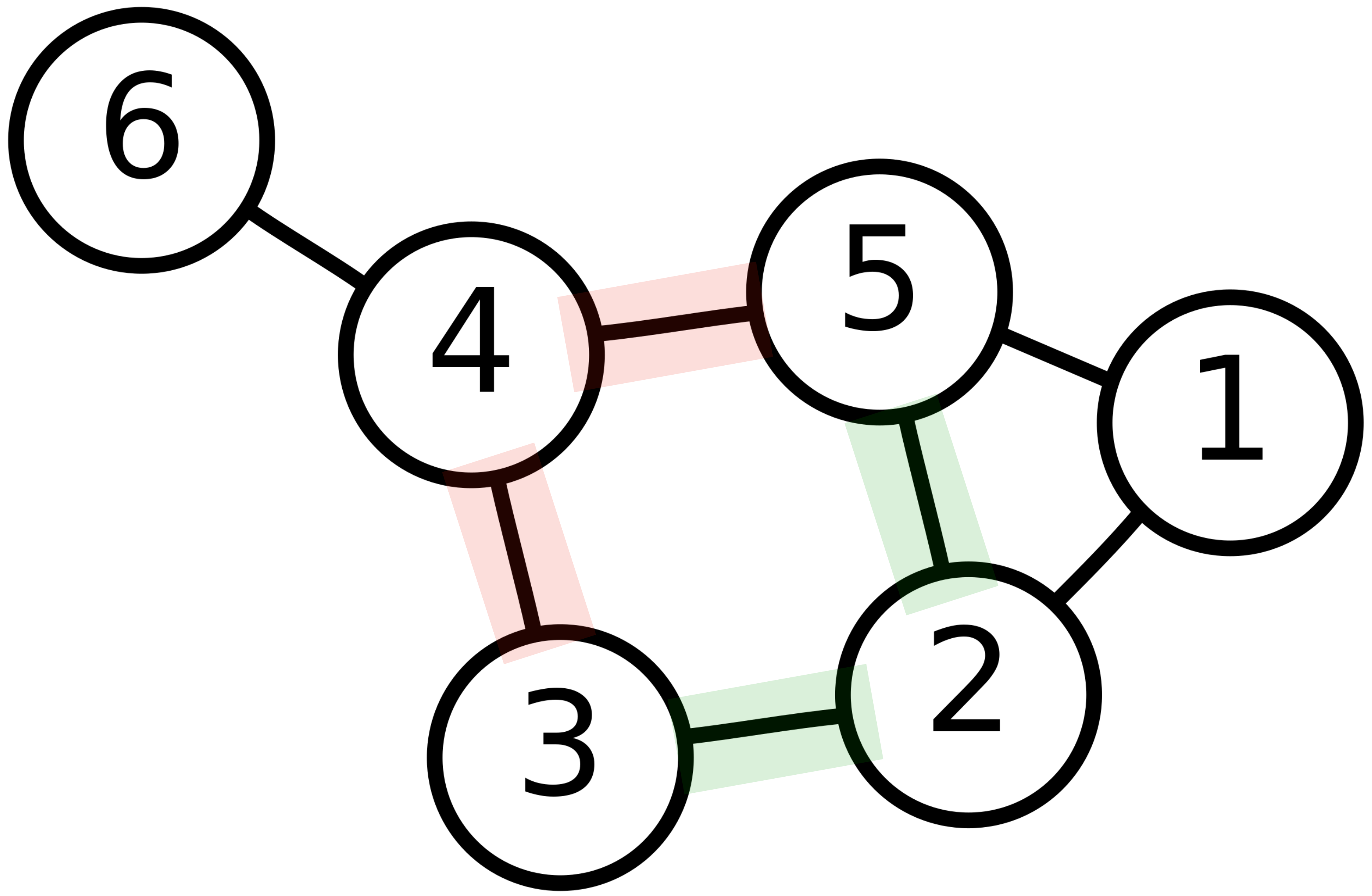
Example: Squared Adjacency Matrices

Given an adjacency matrix A

Can we interpret anything meaningful from A^2?

Example: Squared Adjacency Matrices

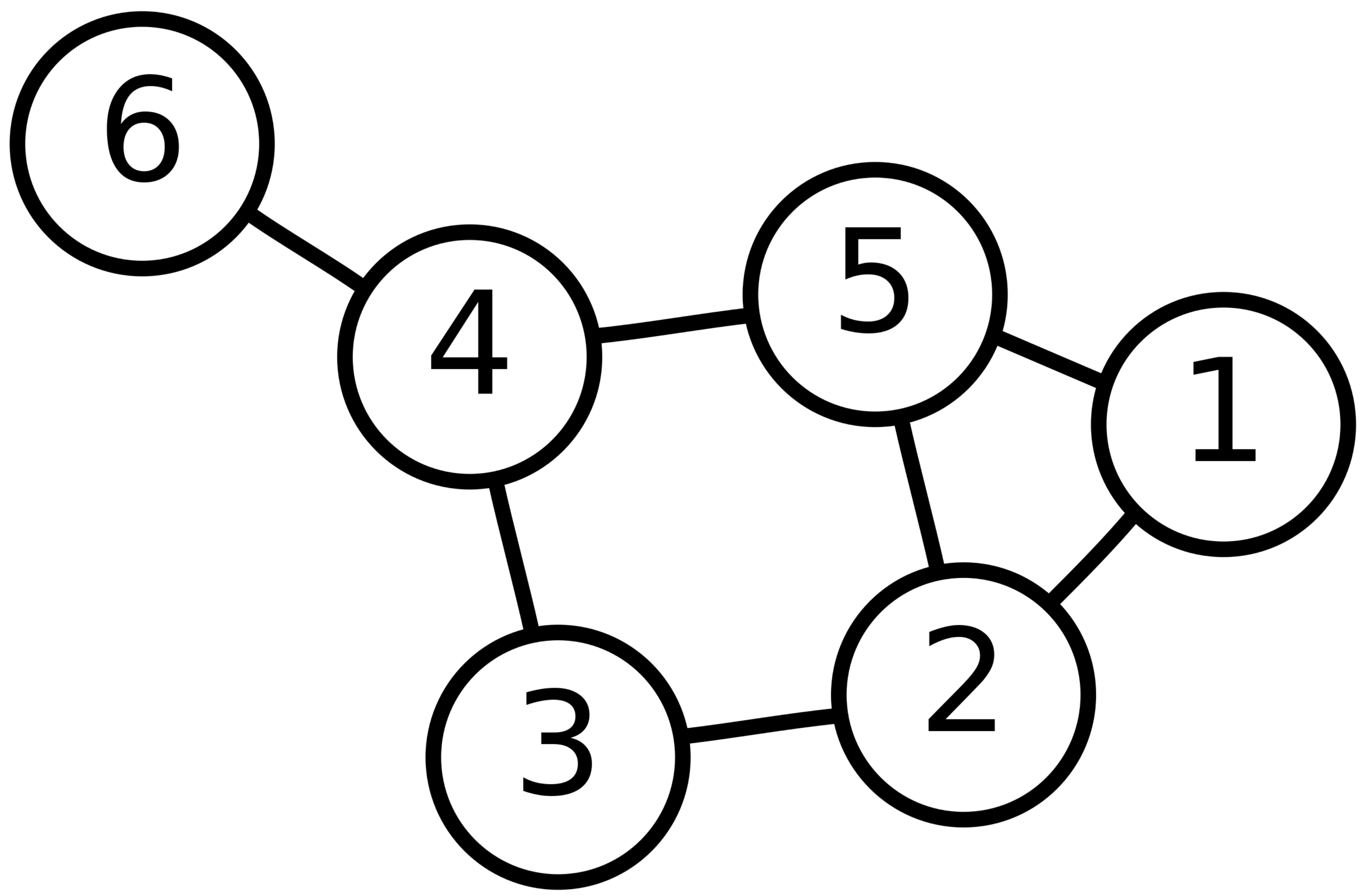
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$



$$(A^2)_{53} = 1(0) + 1(1) + 0(0) + 1(1) + 0(0) + 0(0) = 2$$

Example: Squared Adjacency Matrices

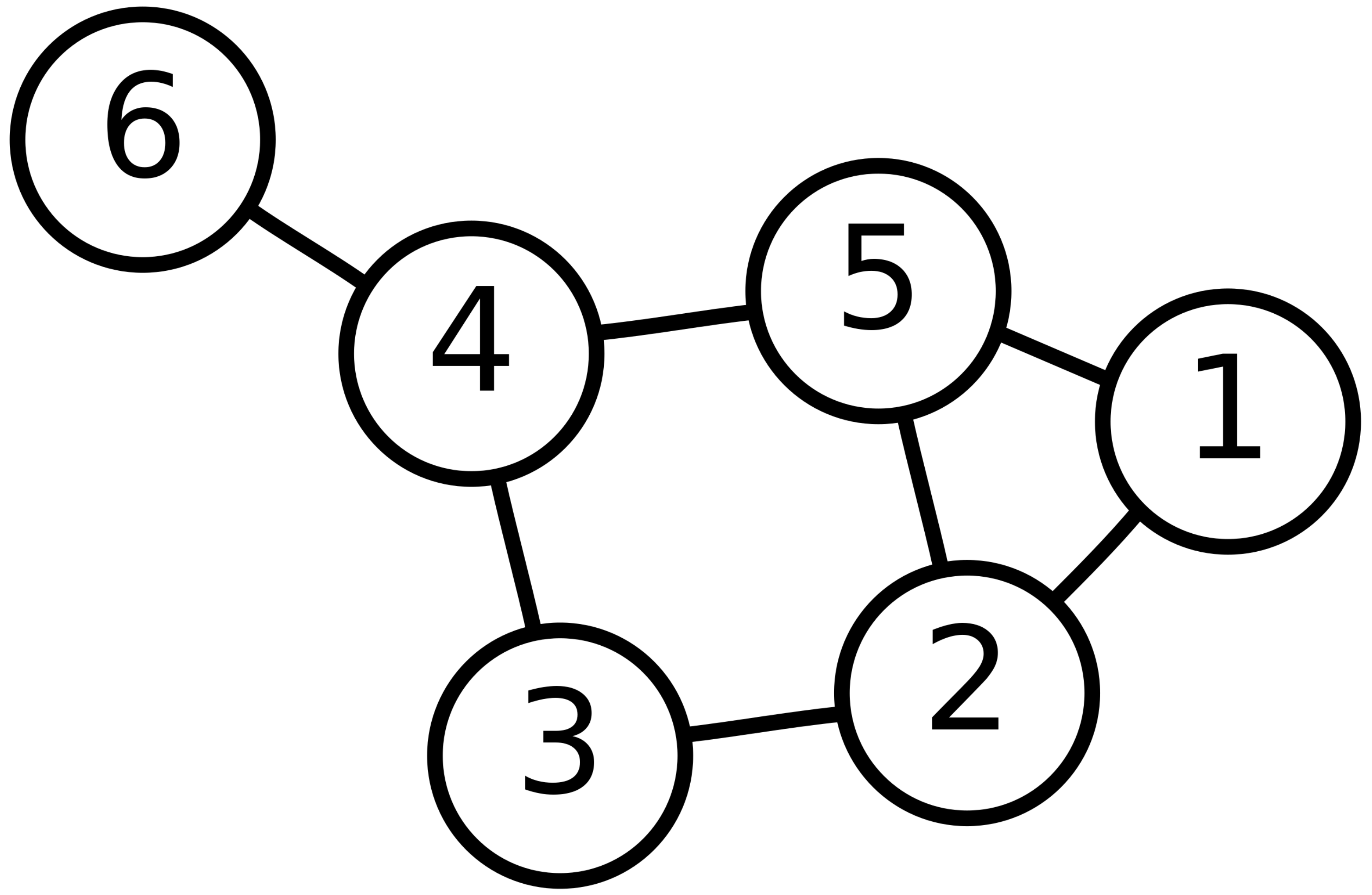
$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$



Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$

$$A_{ik}A_{kj} = \begin{cases} 1 & \texttt{there are edges from i to k and k to j} \\ 0 & \texttt{otherwise} \end{cases}$$



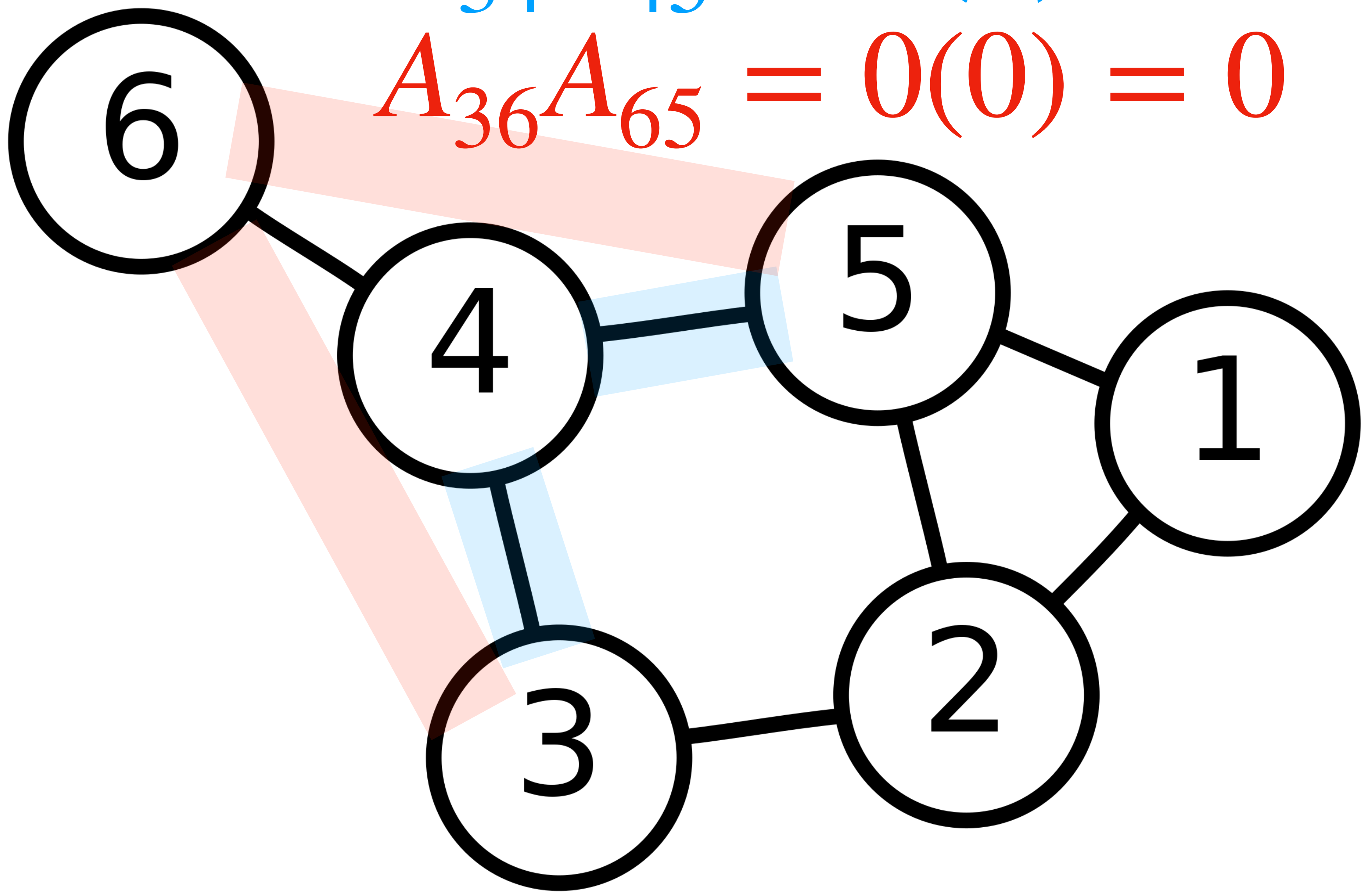
Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$

$$A_{ik}A_{kj} = \begin{cases} 1 & \texttt{there are edges from i to k and k to j} \\ 0 & \texttt{otherwise} \end{cases}$$

$A_{34}A_{45} = 1(1) = 1$

$A_{36}A_{65} = 0(0) = 0$



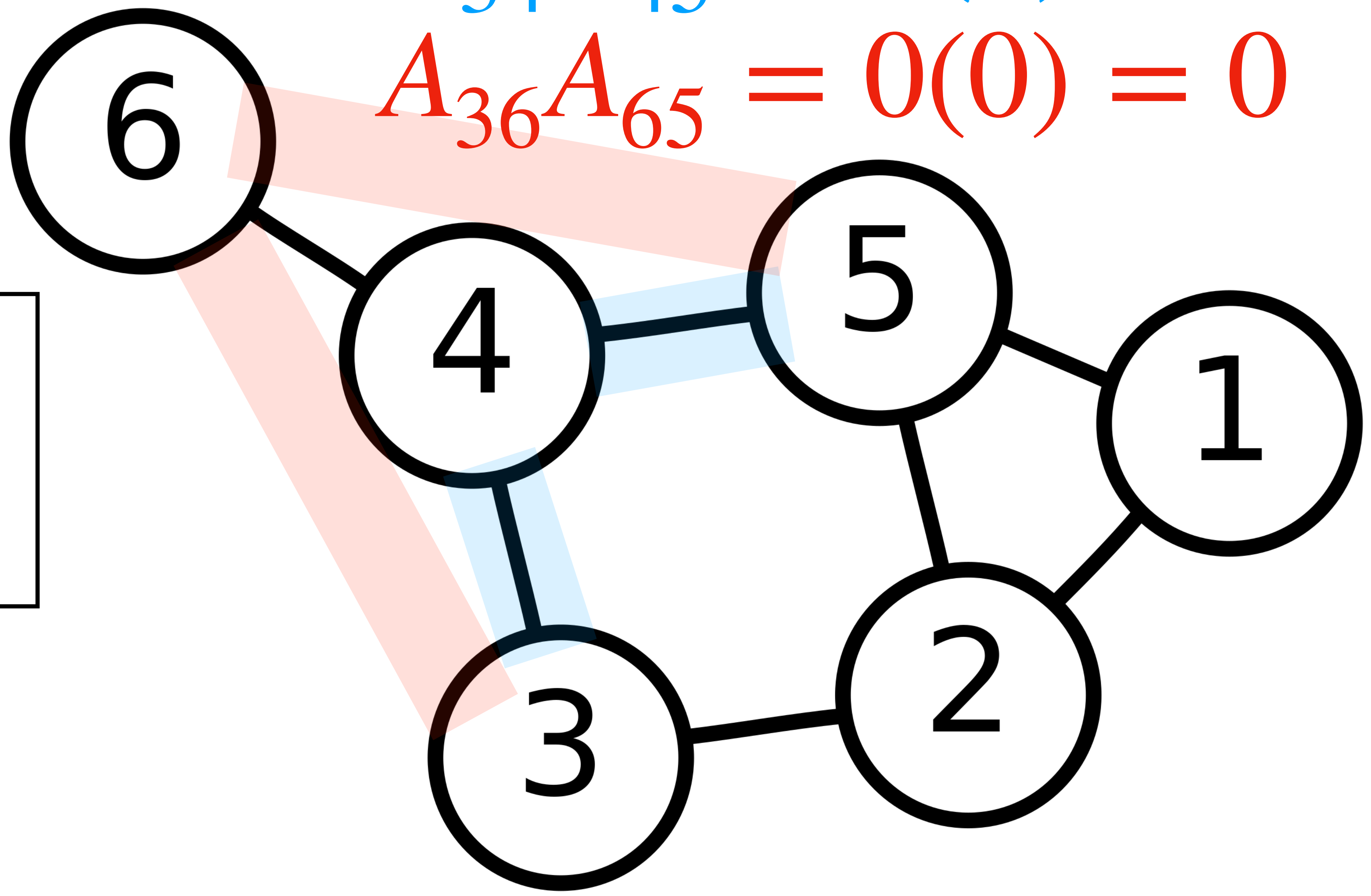
Example: Squared Adjacency Matrices

$$(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \ldots + A_{in}A_{nj}$$

$$A_{ik}A_{kj} = \begin{cases} 1 & \texttt{there are edges from i to k and k to j} \\ 0 & \texttt{otherwise} \end{cases}$$

$$A_{34}A_{45} = 1(1) = 1$$

$$A_{36}A_{65} = 0(0) = 0$$

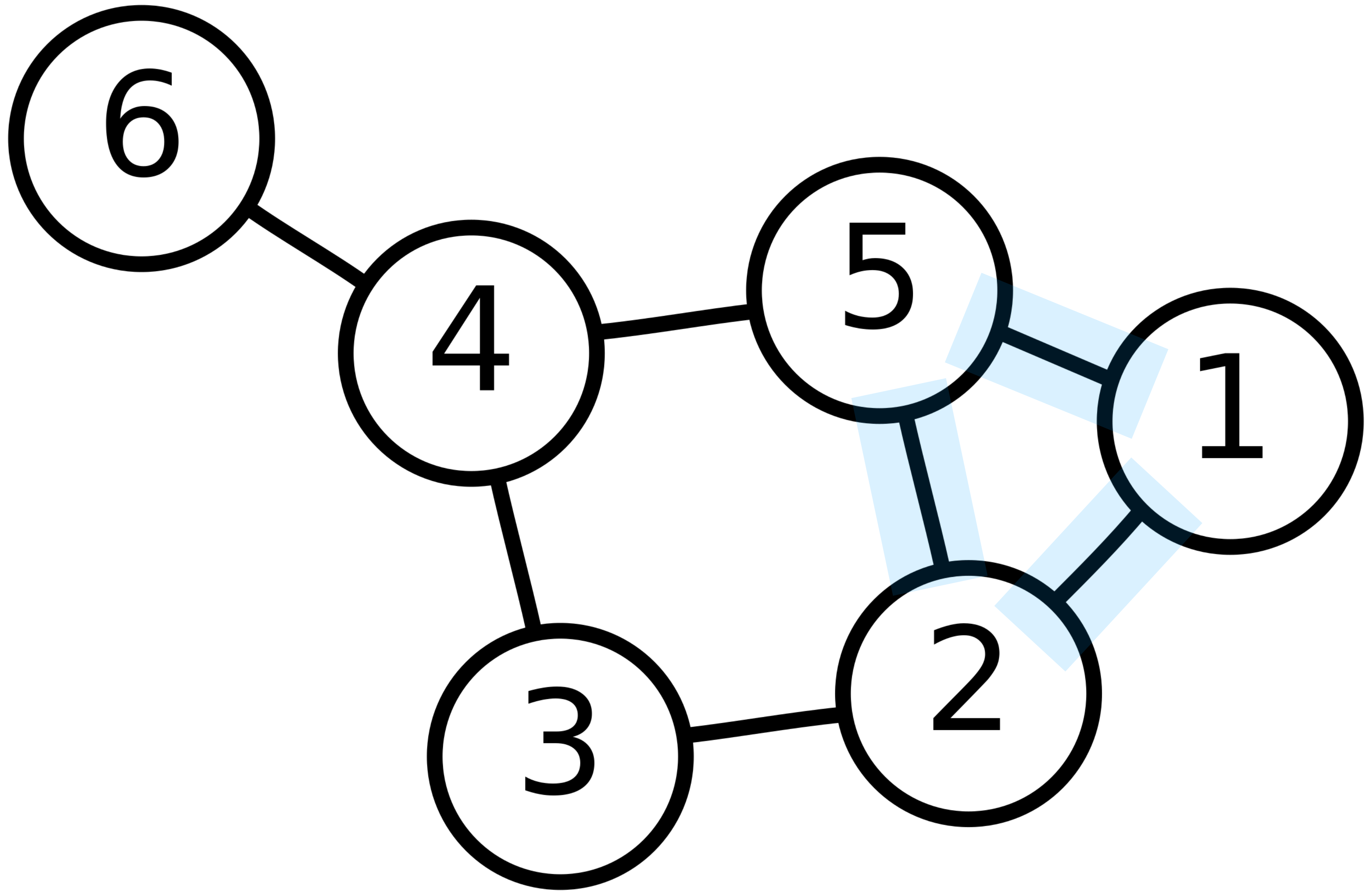


$$(A^2)_{ij} = \boxed{\texttt{number of 2-step paths from i to j}}$$

Application: Triangle Counting

A **triangle** in an undirected graph is a set of three distinct nodes with edges between every pair of nodes.

Triangles in a social network represent mutual friends and tight cohesion (among other things)



Application: Triangle Counting

Theorem. For an adjacency matrix A, the number of triangle containing the edge (i, j) is

$$(A^2)_{ij} A_{ij}$$

Application: Triangle Counting

```
FUNCTION tri_count(A):
    compute A^2
    count ← sum of (A^2)_ij A_ij for all distinct i and j
    RETURN count / 6        # why divided by 6?
```

FUNCTION tri_count(A):

compute A^2

count $\leftarrow$ sum of $(A^2)_{ij}A_{ij}$ for all distinct i and j

RETURN count / 6 # why divided by 6?

Summary

We can solve matrix equations by inverting the matrix, though not all matrices have inverses.

We can compute matrix inverses a simultaneous row reduction.

We can connect all the concepts we've defined so far by thinking about them in terms of invertibility (for square matrices).