Midterm Review Solutions

CAS CS 132: Geometric Algorithms

1 Solving Systems of Linear Equations

Find a solution the following system of linear equations.

$$\begin{aligned} x_1 + x_2 - x_3 &= -9 \\ x_2 - 2x_3 &= -1 \\ x_1 + x_2 &= -10 \end{aligned}$$

Solution. This system has the augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 & -9 \\ 0 & 1 & -2 & -1 \\ 1 & 1 & 0 & -10 \end{bmatrix}$$

which has the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

2 LAA 1.2.18

Determine all values of h such that the following matrix is the augmented matrix of a consistent linear system.

$$\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix}$$

Solution. By the reduction $R_2 \leftarrow R_2 - 5R_1$, we get the matrix

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{bmatrix}$$

This matrix is in echelon form, so h can be any value other than 15. If $h = 15$, then there a row which represents an inconsistent equation.

3 General Form Solutions

Consider the following system of linear equations.

$$\begin{aligned} x_1 - 5x_2 - x_3 - 2x_4 &= 3 \\ x_3 - 2x_4 &= 11 \\ (-2)x_3 + 5x_4 &= -24 \end{aligned}$$

A. Write down a general form solution which describes the solution set of the following system of linear equations.

B. Write down a different general form solution which describes the same solution set (i.e., one in which a different variable is free).

Solution.

A. The reduced echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & -5 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

We can write down a solution in general form from this matrix:

$$\begin{aligned} &x_1 = 6 + 5x_2 \\ &x_2 \text{ is free} \\ &x_3 = 7 \\ &x_4 = -2 \end{aligned}$$

B. Since x_1 is written in terms of the free variable x_2, we can instead write x_2 in terms of x_1.

$$\begin{aligned} &x_1 \text{is free} \\ &x_2 = (1/5)x_1 - (6/5) \\ &x_3 = 7 \\ &x_4 = -2 \end{aligned}$$

4 LAA 1.3.13

Consider the following matrix A and vector $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ h \end{bmatrix}$$

A. Determine if $\mathbf{b}$ can be written as a linear combination of the columns of A if $h = -3$.

B. For what values of h can $\mathbf{b}$ be written as a linear combination of the columns of A.

Solution.

A. The augmented matrix $[A\ \mathbf{b}]$ is row equivalent to the echelon form

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

since this represents an inconsistent system, $\mathbf{b}$ cannot be written as a linear combination of the columns of A.

B. If $h = -6$, then the last row does not represent an inconsistent equation. In this case, $\mathbf{b}$ can be written as a linear combination of the columns of A.

5 Intersections of Spans

Consider the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Find a nonzero vector which lies in both $\mathsf{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathsf{span}\{\mathbf{v}_3, \mathbf{v}_4\}$.

Solution. The span of the first pair of vectors is exactly the x_1x_2-plane, which is described by the equation $x_3 = 0$. Therefore, it suffices to find a vector in $\mathsf{span}\{\mathbf{v}_3, \mathbf{v}_4\}$ such whose third component is 0. We can take $2\mathbf{v}_3 + \mathbf{v}_4$, which is

$$\begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}$$

6 LAA 1.7.31

Find a nontrivial solution to matrix equation $A\mathbf{x} = \mathbf{b}$ without performing any row reductions. (Hint. What is the relationship between the first two columns and the last column of A?)

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution. Write A as $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ then $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$. Therefore, a solution to this equation is the vector

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

7 Linearly Independent Vectors

Consider three arbitrary vectors

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

and suppose that they are linearly independent.

A. What is the maximum number of entries of these vectors which can be 0? Note that the solution will be a number between 0 and 9.

B. What is the minimum number?

In each case provide an example.

Solution.

A. There can be at most 6 zero entries. If there were 7, then some vector would have to be all zeros, which would automatically make the set linearly dependent. An example is the standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

B. It is possible for there to be no zero entries. We can take as an example

$$\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

8 Drawing Linear Transformations

Draw the unit square after being transformed by the matrix transformation implemented by

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution.

9 Matrices of Linear Transformations

Find the matrix which implemented the following transformation.

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \mapsto \begin{bmatrix} v_2 \\ v_1 \\ v_3 + v_4 \end{bmatrix}$$

Solution. In order to determine the matrix implementing a linear transformation, we have to determine how the transformation affects the standard basis.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then we put these together into a single matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

10 3D Linear Transformations

Considering the transformation T implemented by the following matrix.

$$\begin{bmatrix} \cos 2 & 0 & -\sin 2 \\ 0 & 1 & 0 \\ \sin 2 & 0 & \cos 2 \end{bmatrix}$$

Describe geometrically what T does. Then find a vector $\mathbf{v}$ whose span is not changed by this transformation (i.e., $\mathsf{span}\{\mathbf{v}\} = \mathbf{span}\{T(\mathbf{v})\}$).

Solution. This transformation rotates vectors around the x_2 axis. The the span of the vector $\mathbf{e}_2$ is not changed by this transformation since $T(\mathbf{e}_2) = \mathbf{e}_2$.