The Characteristic Equation

Geometric Algorithms
Lecture 18

CAS CS 132

Introduction

Recap Problem

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Determine the dimension of the eigenspace of A *for the eigenvalue 4.*

(try not to do any row reductions)

Answer: 2

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ -1 & 2 & -3 & 1 \\ 2 & 4 & 10 & 0 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

Objectives

1. Briefly recap eigenvalues and eigenvectors.
2. Get a primer on determinants.
3. Determine how to find eigenvalues (not just verify them).

Keyword

eigenvectors

eigenvalues

eigenspaces

eigenbases

determinant

characteristic equation

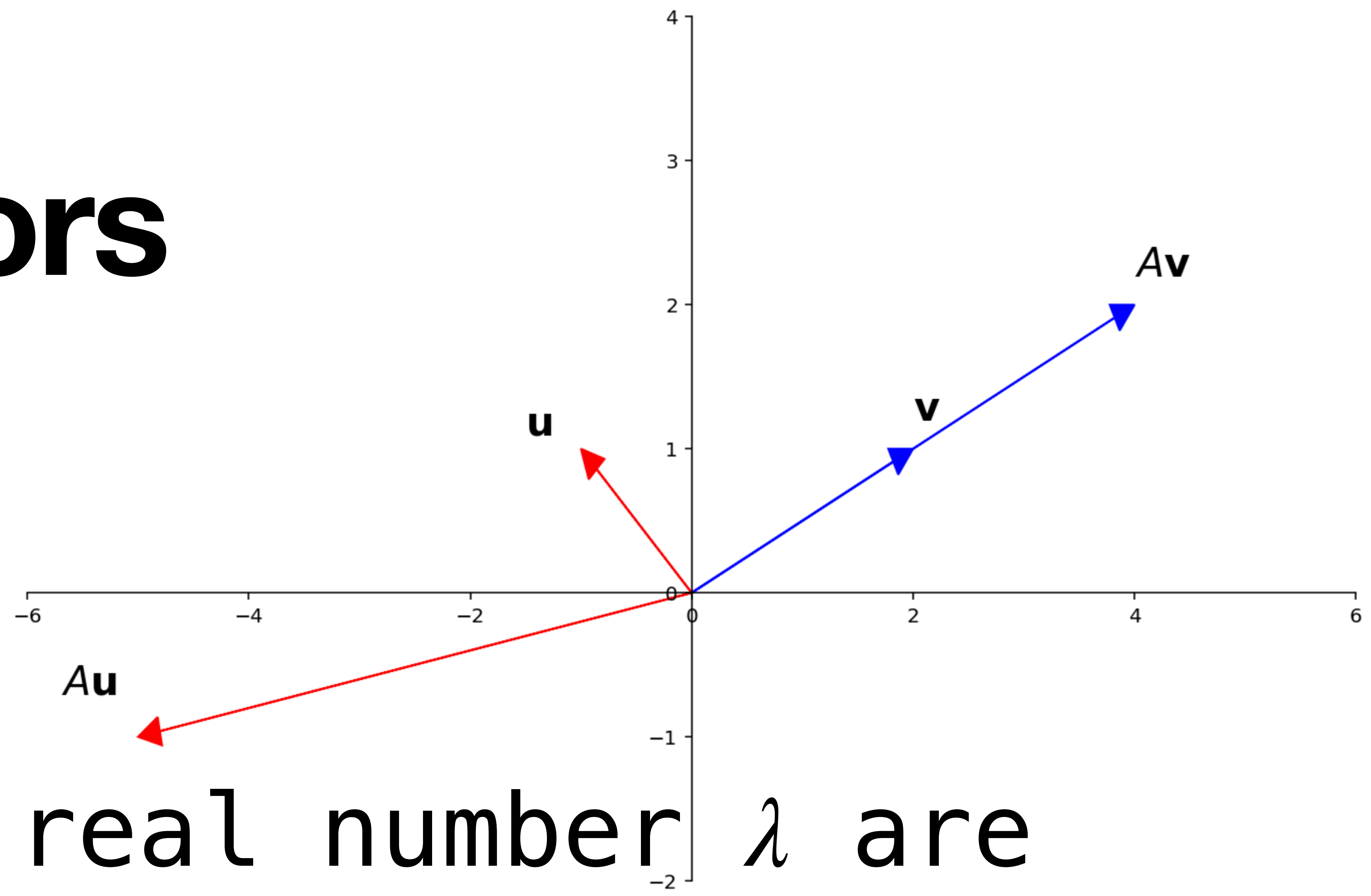
polynomial roots

triangular matrices

multiplicity

Recap

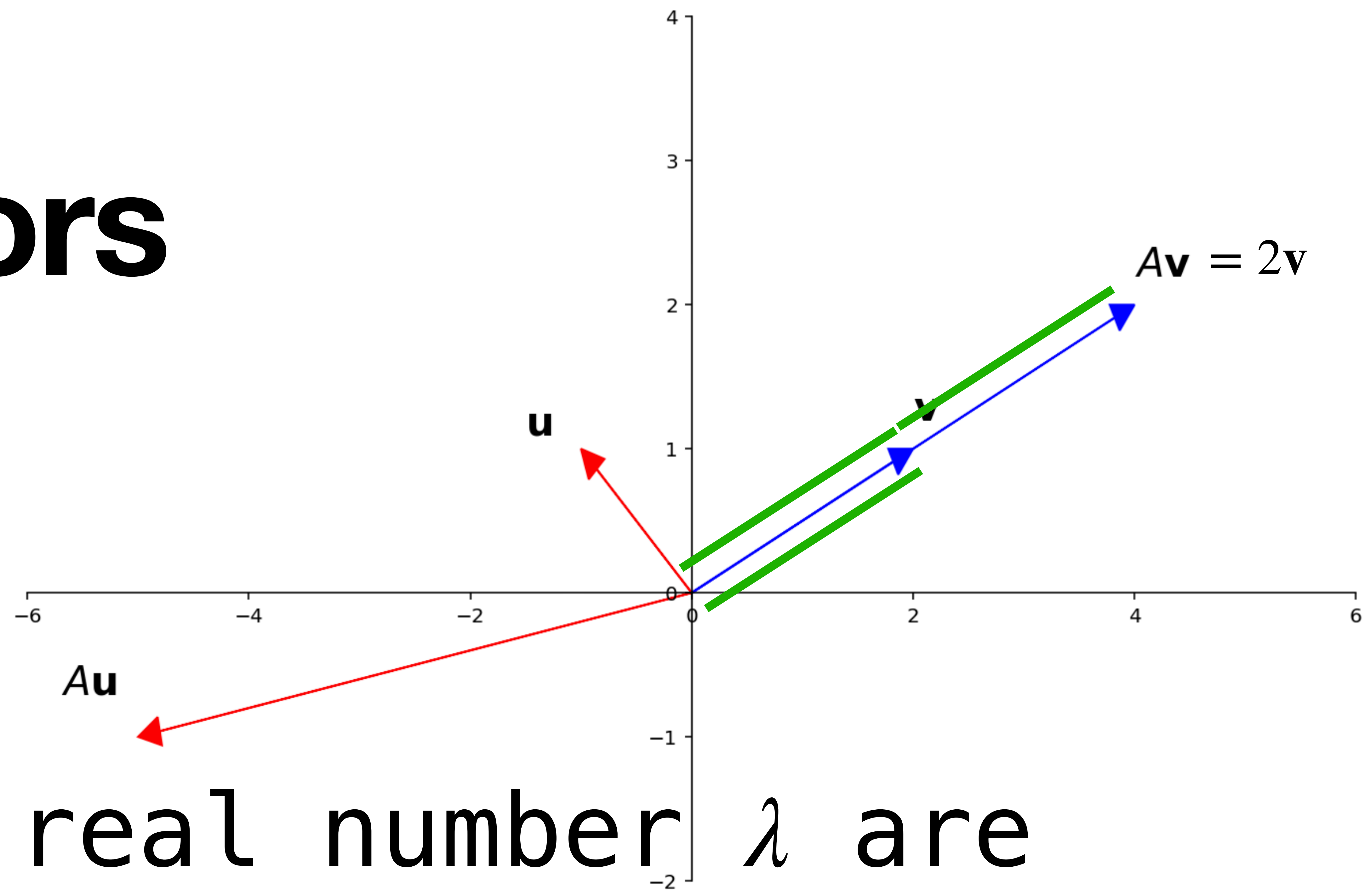
Recall: Eigenvalues/vectors



A *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^n$ and real number λ are an **eigenvector and eigenvalue** for a $n \times n$ matrix A if

$$A\mathbf{v} = \lambda \mathbf{v}$$

Recall: Eigenvalues/vectors



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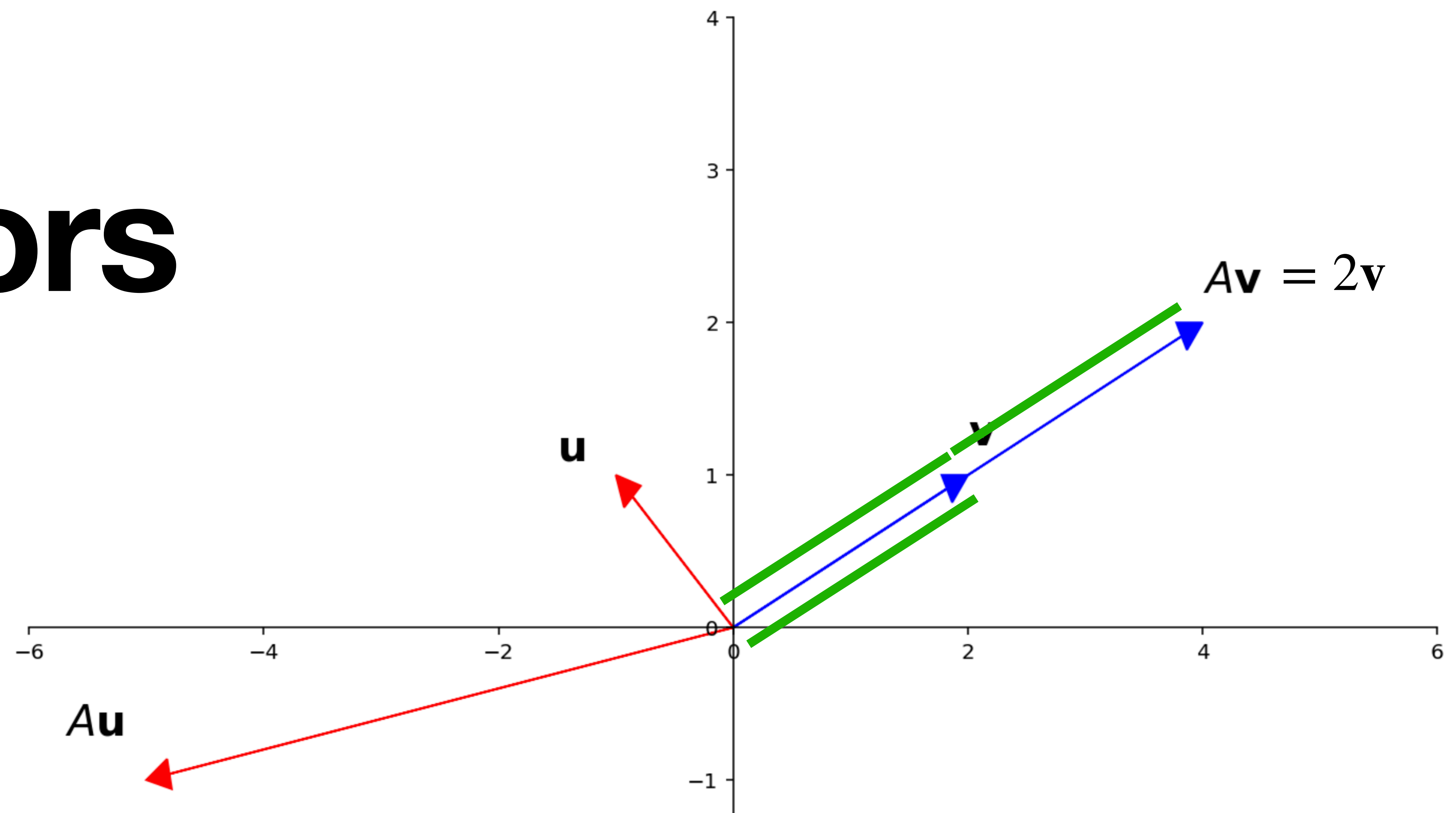

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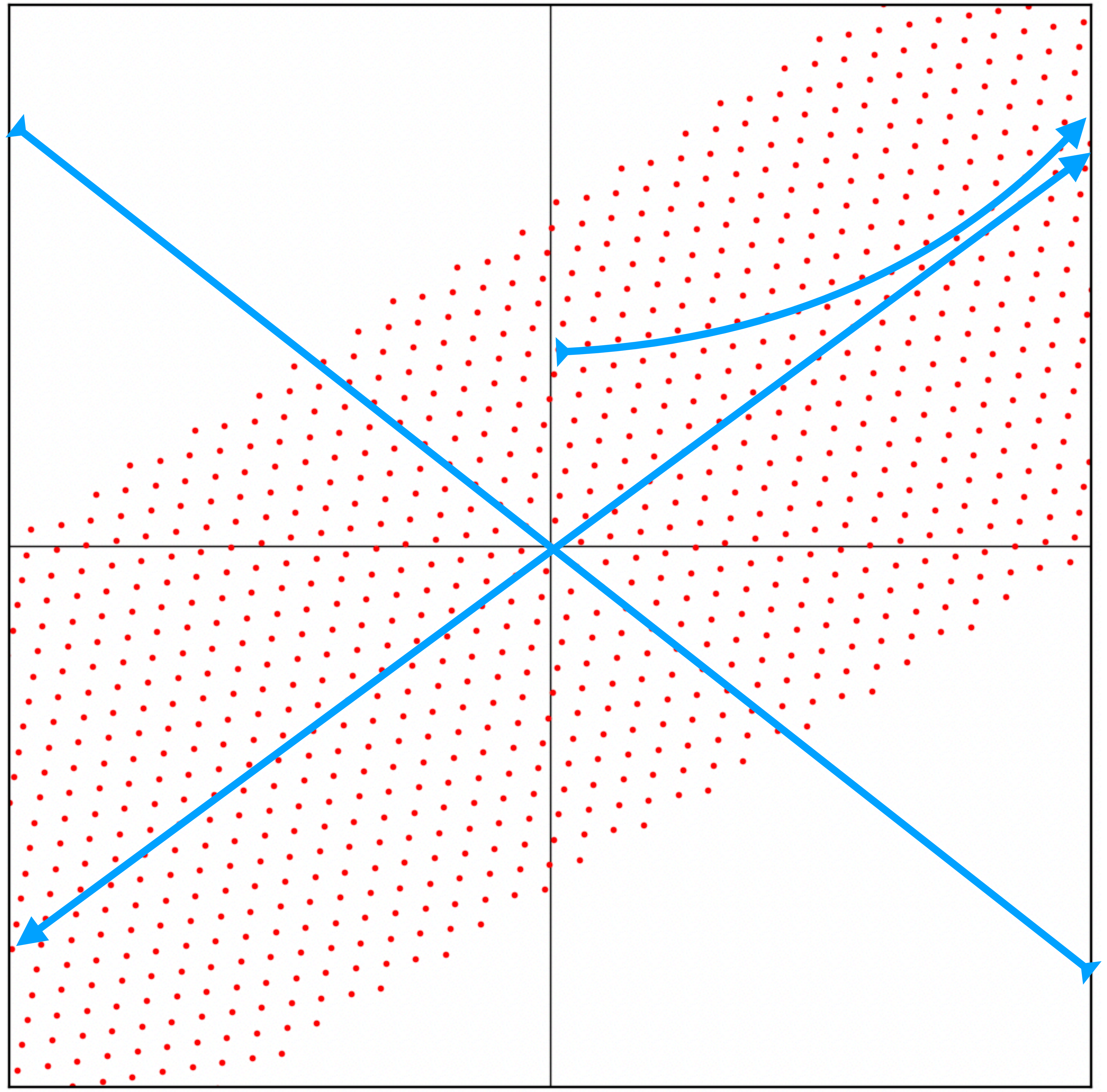
$$A\mathbf{v} = \lambda\mathbf{v}$$

$\mathbf{v}$ is "just scaled" by A, not rotated

Recall: The Picture



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Solution. Easy. Work out the matrix-vector multiplication.

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Example.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix}$$

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$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

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If we don't need the vector we can just show that $A - \lambda I$ is ***not*** *invertible (by IMT).*

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Solution. Find a basis for $\text{Nul}(A - \lambda I)$.

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(we did this for our recap problem)

Finding Eigenvalues

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Question. Determine the eigenvalues of A, along with their associated eigenspaces.

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Solution (Idea). Can we somehow "solve for λ" in the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

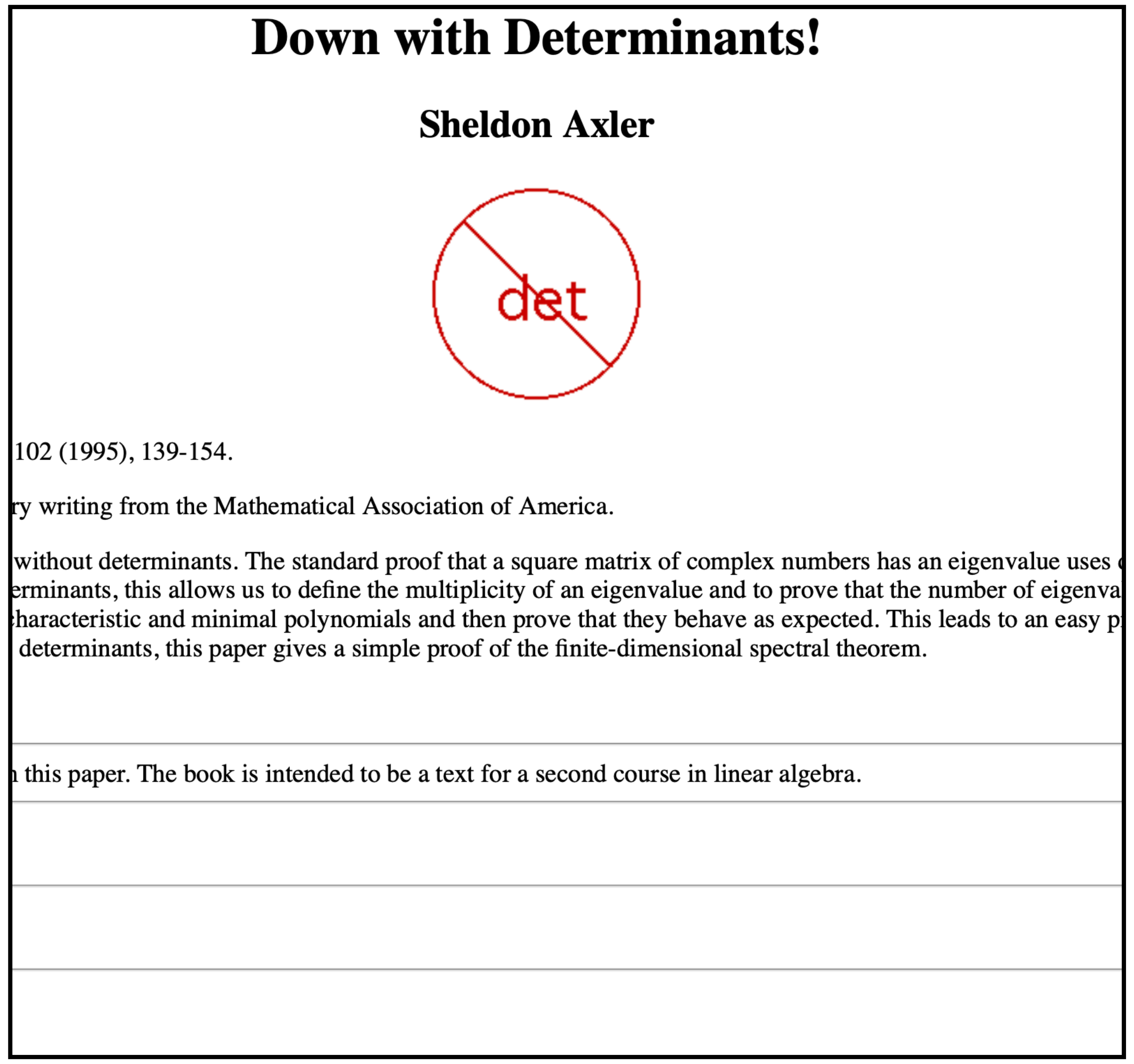
Determinants

An Aside: Determinants are Mysterious

Determinants are strangely polarizing

Some people love them, some people hate them

We'll only scratch the surface...



Down with Determinants!

Sheldon Axler

det

102 (1995), 139-154.

ry writing from the Mathematical Association of America.

without determinants. The standard proof that a square matrix of complex numbers has an eigenvalue uses
erminants, this allows us to define the multiplicity of an eigenvalue and to prove that the number of eigenva
haracteristic and minimal polynomials and then prove that they behave as expected. This leads to an easy p
determinants, this paper gives a simple proof of the finite-dimensional spectral theorem.

this paper. The book is intended to be a text for a second course in linear algebra.

What kind of thing is the determinant?

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Notation. We will write $\det(A)$ for the determinant of A.

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In broad strokes, it's a big sum of products of entries of A.

A Scary-Looking Definition (we won't use)

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\mathsf{sgn}(\sigma)} A_{1\sigma(1)} A_{2\sigma(2)} \ldots A_{n\sigma(n)}$$

`We can think of this function as a procedure:`

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We can think of this function as a procedure:

```
FUNCTION det(A):
  total = 0
  FOR all matrix B we can get by swapping a bunch of rows of A:
    s = 1 IF (# of swaps necessary) is even ELSE -1
    total += s * (product of the diagonal entries of B)
  RETURN total
```

The Determinant of 2×2 Matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

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$$(-1)^{0} ad$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to^1 \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$(-1)^1 cb$$

The Determinant of 3×3 matrices

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

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Another Perspective

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let's row reduce an arbitrary 2×2 matrix:

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So we can yet again extend the IMT:

- » A is invertible
- » $\det(A) \neq 0$
- » 0 is not an eigenvalue

These must be all true or all false.

Determinants (the definition we'll use)

$$\det(A) = \frac{(-1)^s}{c} U_{11} U_{22} \dots U_{nn}$$

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0 if A is not invertible

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Example

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix:

Example (Again)

$$\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Let's find the determinant of this matrix again but with a different sequence of row operations:

The definition holds no matter which sequence of row operations you use.

How To: Determinants

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3. Determine the product of entries along the diagonal of U, call this P.
4. The determinant of A is $\dfrac{sP}{c}$.

The Shorter Version

Beyond small matrices, we'll often just use computers.

With NumPy:

numpy.linalg.det(A)

Properties of Determinants

Properties of Determinants (1)

$$\det(AB) = \det(A)\det(B)$$

It follows that AB is invertible if and only if A and B are invertible

(we won't verify this)

Question

Use the fact that $\det(AB) = \det(A)\det(B)$ *to give an expression for* $\det(A^{-1})$ *in terms of* $\det(A)$*.*

Hint. What is $\det(I)$*?*

Answer: $1/\det(A)$

Properties of Determinants (2)

$$\det(A^T) = \det(A)$$

It follows that A^T is invertible if and only if A is invertible.

(we also won't verify this)

Question

If $A^{-1} = A^T$*, then what are the possible values of* $\det(A)$*?*

Answer: ± 1

Properties of Determinants (3)

Theorem. If A is triangular, then $\det(A)$ is the product of entries along the diagonal.

Verify:

Question

$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -5 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

Find the determinant of the above matrix.

Answer

Characteristic Equation

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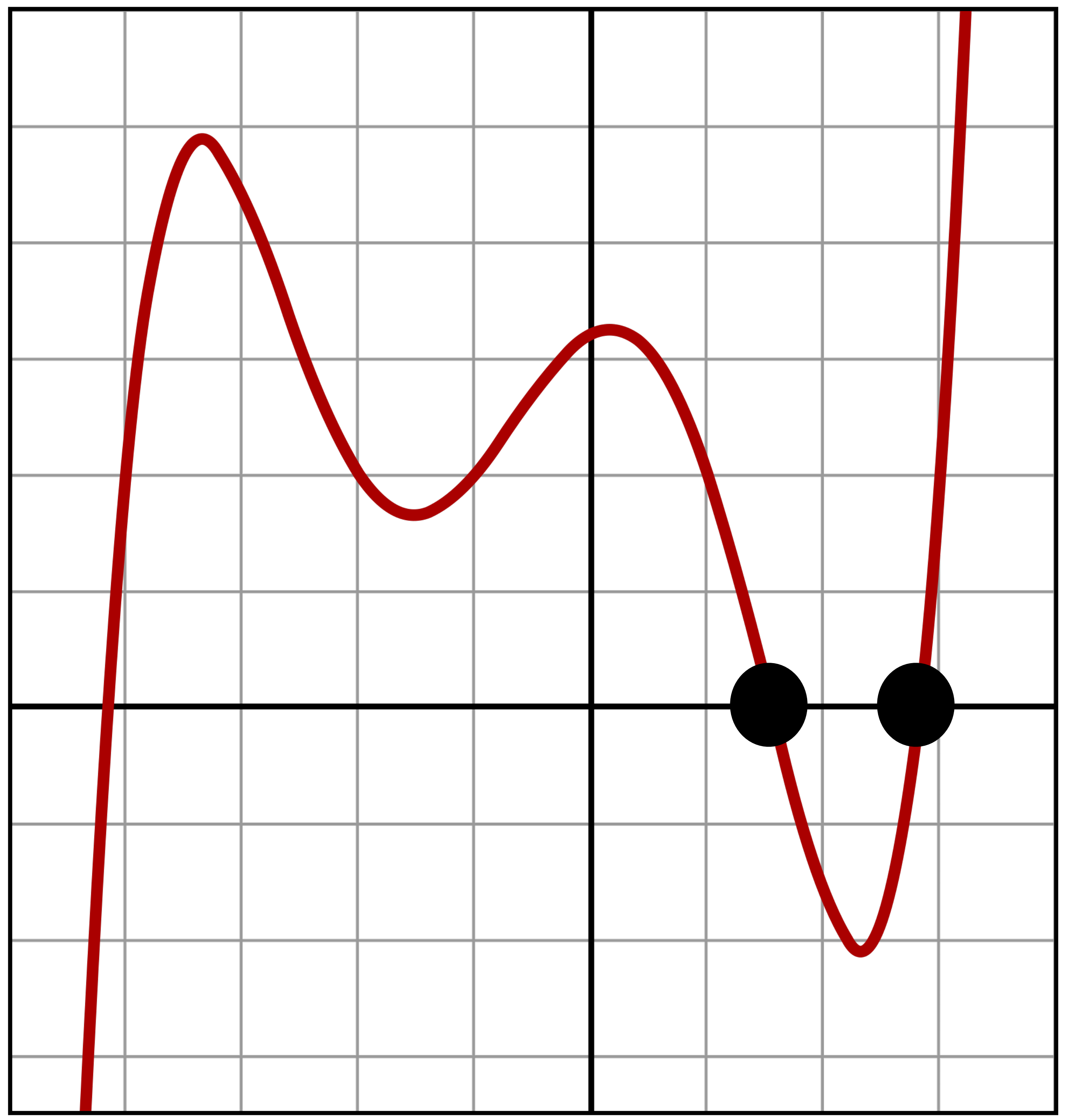
Then $\det(A - \lambda I)$ is a **polynomial.**

Reminder: Polynomial Roots



https://commons.wikimedia.org/wiki/File:Quintic_polynomial.svg

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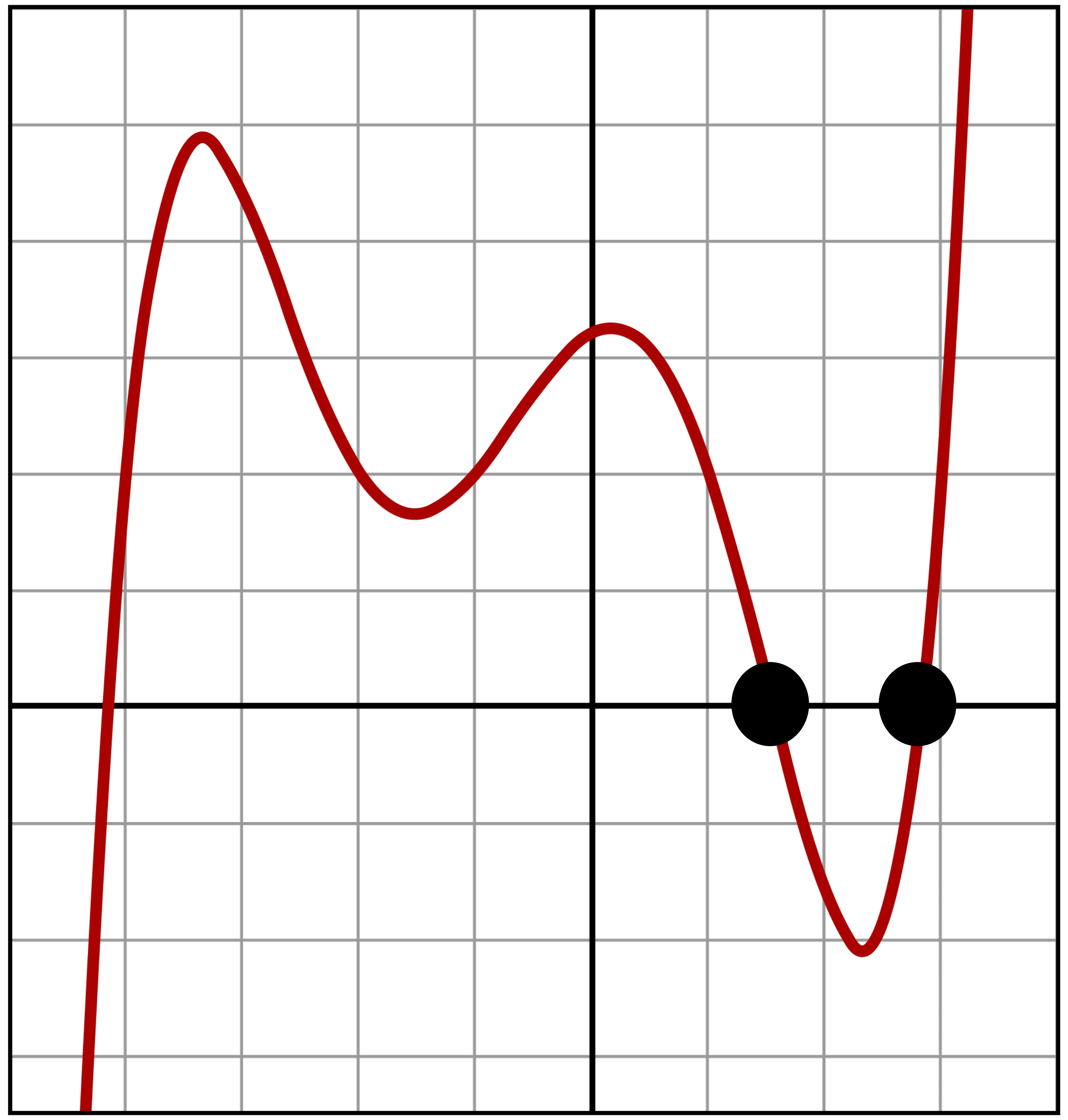
A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.



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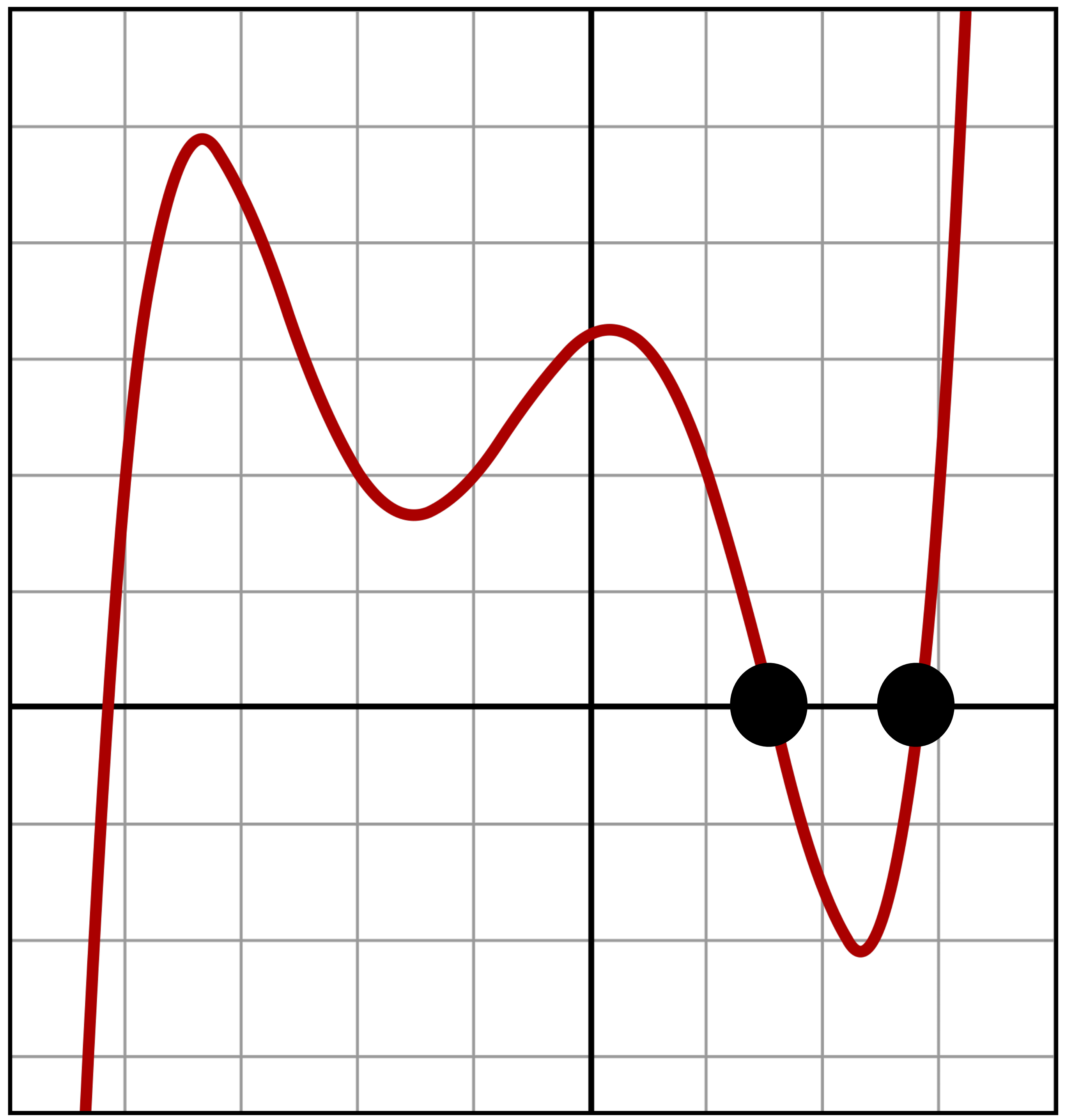

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A **root** of a polynomial $p(x)$ is a value r such that $p(r) = 0$.

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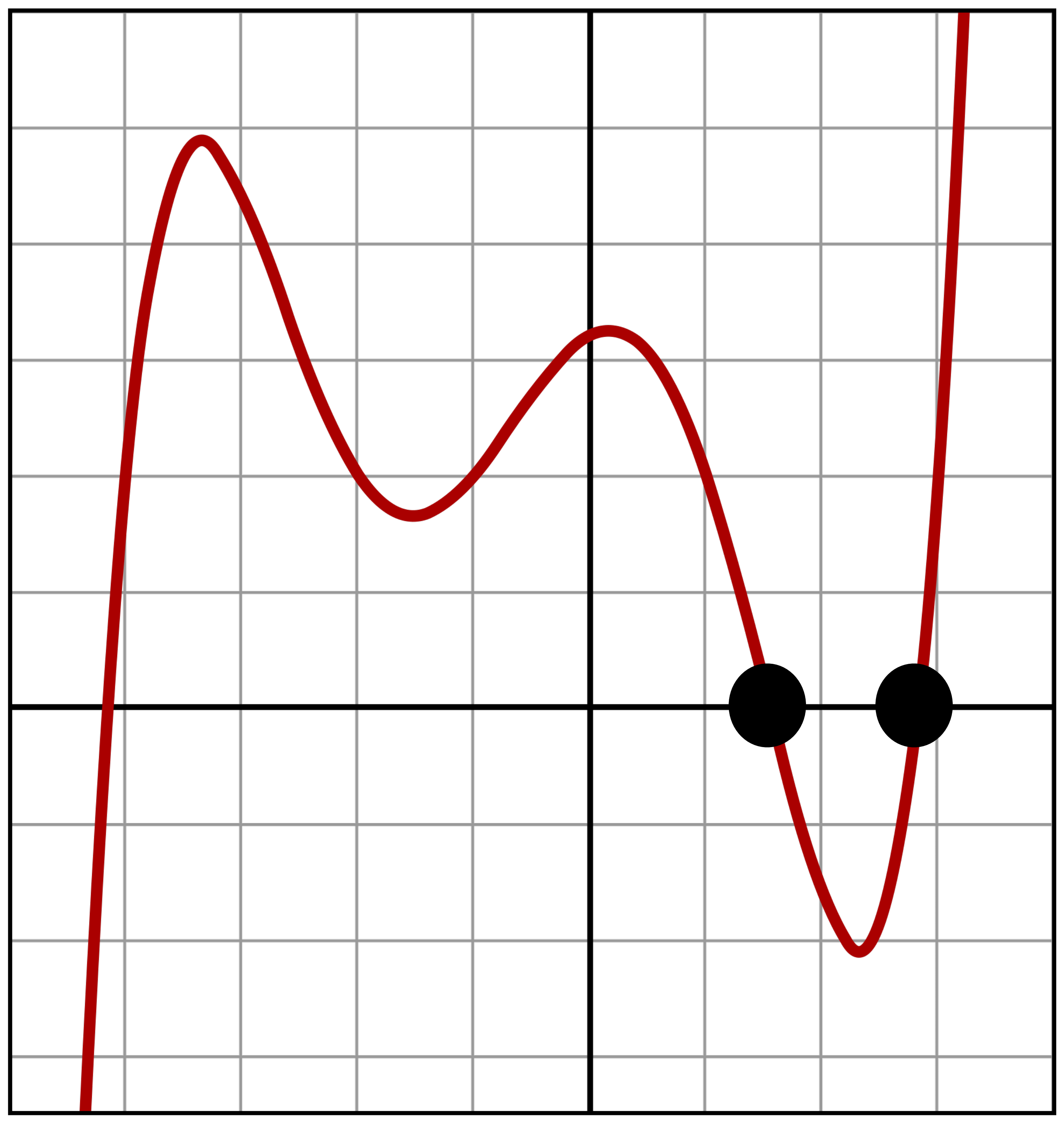
If r is a root of $p(x)$, then it is possible to find a polynomial $q(x)$ such that

$$p(x) = (x - r)q(x)$$



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This is a polynomial with the eigenvalues of A as roots.

So we can "solve" for the eigenvalues in the equation

$$\det(A - \lambda I) = 0$$

Example: 2×2 Matrix*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Let's find the characteristic polynomial of this matrix:

*we won't deal explicitly with matrices beyond 2×2, though there may be conceptual questions about larger matrices

Example: 2×2 Matrix*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Let's find the characteristic polynomial of this matrix:

An Aside: What is this matrix?

*we won't deal explicitly with matrices beyond 2×2, though there may be conceptual questions about larger matrices

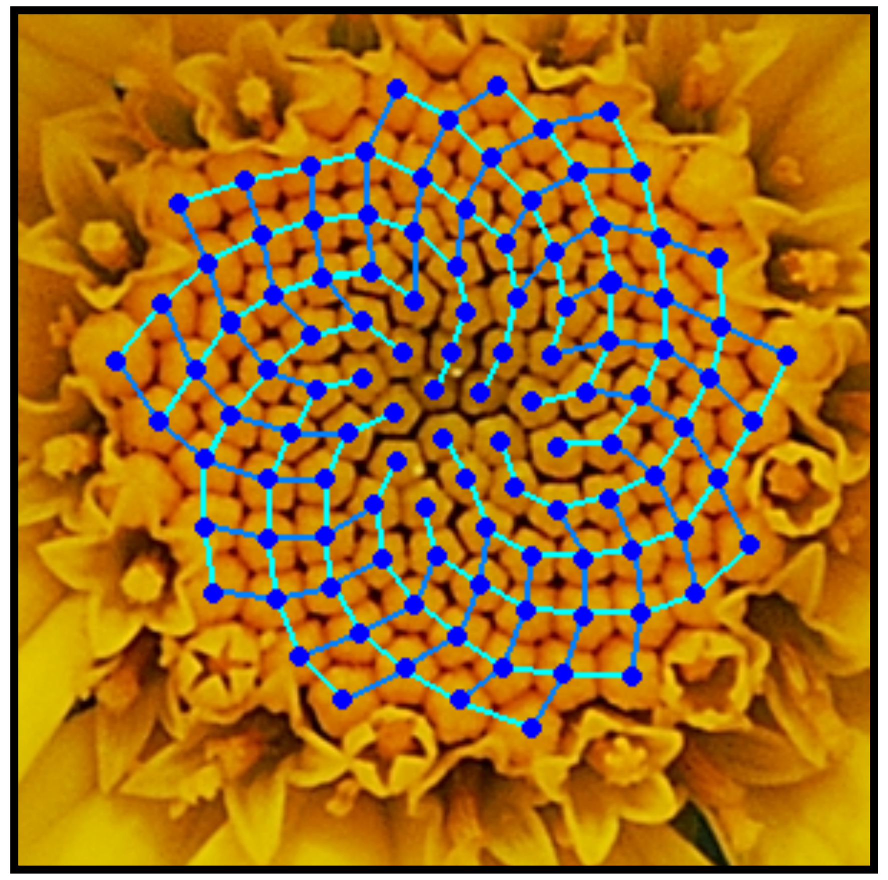
A Special Linear Dynamical System

$$\mathbf{v}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}_k \qquad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Consider the system given by the above matrix.

What does this system represent?:

Fibonacci Numbers



$$F_0 = 0$$

$$F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2}$$

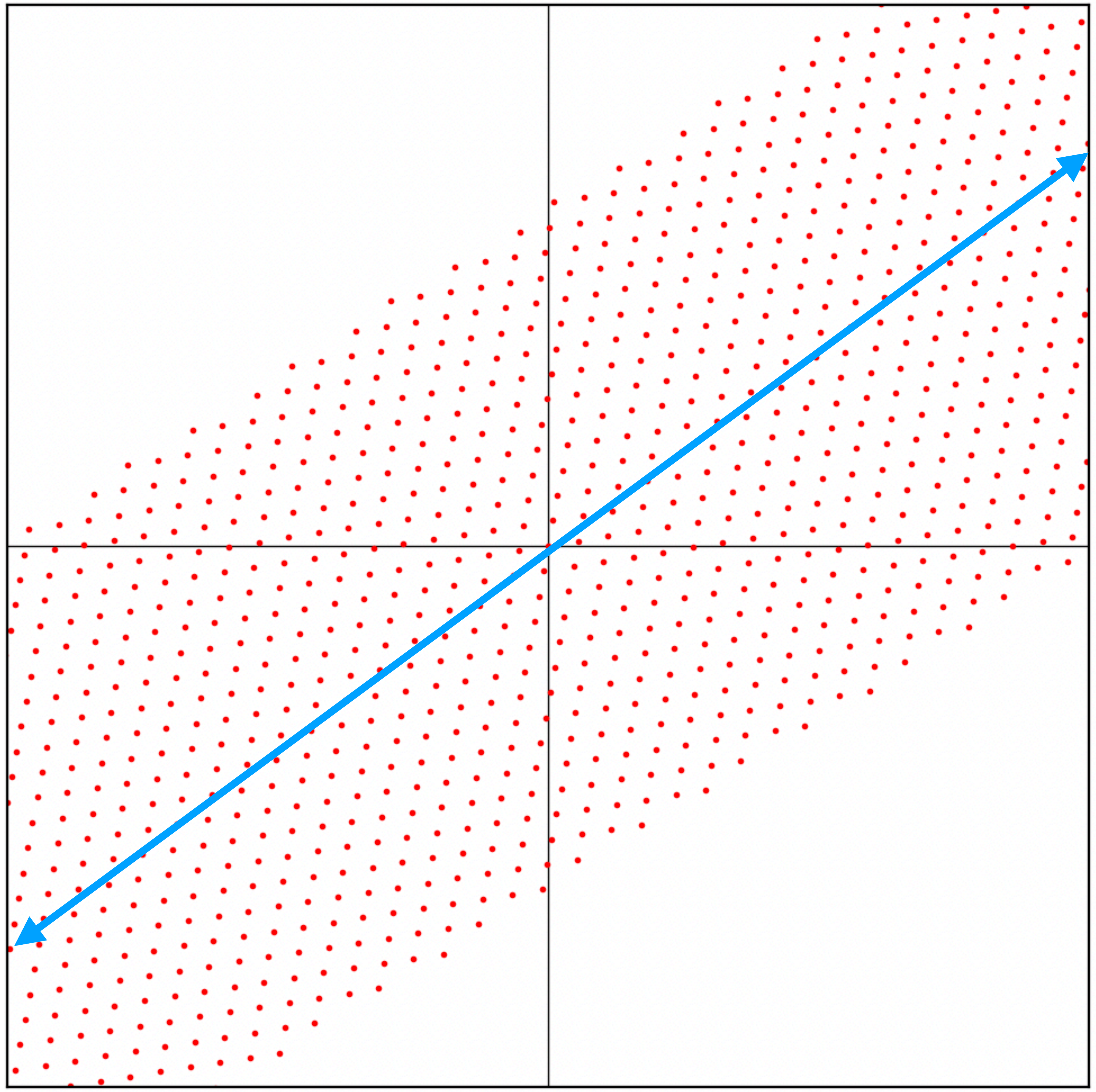
```
define fib(n):
  curr, next ← 0, 1
  repeat n times:
    curr, next ← next, curr + next
  return curr
```

The Fibonacci numbers are defined in terms of a recurrence relation.

They seem to crop-up in nature, engineering, etc.

https://commons.wikimedia.org/wiki/File:FibonacciChamomile.PNG

Recall: The Fibonacci Matrix



The largest eigenvalue is the slope of this line

Golden Ratio

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \frac{F_{k+1}}{F_k} \to \varphi \text{ as } k \to \infty$$

This is the largest eigenvalue of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$**.**

To Come. The "long term behavior" is the Fibonacci sequence is defined by the golden ratio.

Example: Triangular matrix

$$\begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The characteristic polynomial of a triangular matrix comes pre-factored:

How To: Finding Eigenvalues

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Solution. Find the roots of the characteristic polynomial of A.

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In Reality. We'll mostly just use

numpy.linalg.eig(A)

An Observation: Multiplicity

$$\boxed{\lambda^{1}}(\lambda - 1)^{\boxed{2}}(\lambda - 4)^{\boxed{1}}$$ multiplicities

In the examples so far, we've seen a number appear as a root multiple times.

This is called the **multiplicity** of the root.

Is the multiplicity meaningful in this context?

Multiplicity and Dimension

Theorem. The dimension of the eigenspace of A for the eigenvalue λ is <u>at most</u> the multiplicity of λ in $\det(A - \lambda I)$.

The multiplicity is an upper bound on "how large" the eigenspace is.

Example

Let A be a 5×5 matrix with characteristic polynomial $(x-1)^3(x-3)(x+5)$.

» What is $\mathrm{rank}(A)$?

» What is the minimum possible rank of $A - I$?

Application: Similar Matrices

Definition. Two square matrices A and B are **similar** if there is an invertible matrix P such that

$$A = P^{-1}BP$$

Application: Similar Matrices

Theorem. Similar matrices have the same eigenvalues.

Verify:

Summary

The determinant of a matrix is an arithmetic expression of its entries.

The characteristic polynomial is the determinant of $A - \lambda I$ viewed as a polynomial of λ, and it tells us what the eigenvalues of a matrix are.