

Matrix Algebra

Geometric Algorithms

Lecture 9

Objectives

1. (From last time) Connect questions about matrix equations and linear transformations
2. Motivate matrix multiplication
3. Define matrix multiplication
4. Look at the algebra of matrix multiplication

Keywords

one-to-one transformation

onto transformation

matrix multiplication

row-column rule

matrix addition and scaling

non-commutativity

Recap

Recall: Matrices as Transformations

Matrices allow us to *transform* vectors.

The transformed vector lies in the span of its columns.

$$\mathbf{x} \mapsto A\mathbf{x}$$

map a vector \mathbf{x} to the vector $A\mathbf{x}$

Recall: Motivating Questions

What kind of functions can we define in this way?

How do we interpret what the transformation does to a set of vectors?

How does this relate back to matrix equations?

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}?$ \equiv is there a vector which A transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A transforms into \mathbf{b}

Recall: A New Interpretation of the Matrix Equation

$A\mathbf{x} = \mathbf{b}$? \equiv is there a vector which A transforms into \mathbf{b} ?

Solve $A\mathbf{x} = \mathbf{b}$ \equiv find a vector which A transforms into \mathbf{b}

What about other questions?

One-to-One and Onto Transformations

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have a solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?

Other Questions Like...

Do the columns of A have full span?

Are the columns of A linearly independent?

Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of \mathbf{b} ?

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Other Questions Like...

Does $A\mathbf{x} = \mathbf{b}$ have at least one solution for any choice of \mathbf{b} ?

Does $A\mathbf{x} = \mathbf{b}$ have at most one solution for any choice of \mathbf{b} ?

Wait, what's going on with this second one?

A New Perspective on Linear Independence

$A\mathbf{x} = \mathbf{0}$ has a unique solution $\equiv A\mathbf{x} = \mathbf{b}$ has at most one solution for any choice of \mathbf{b}

why? :

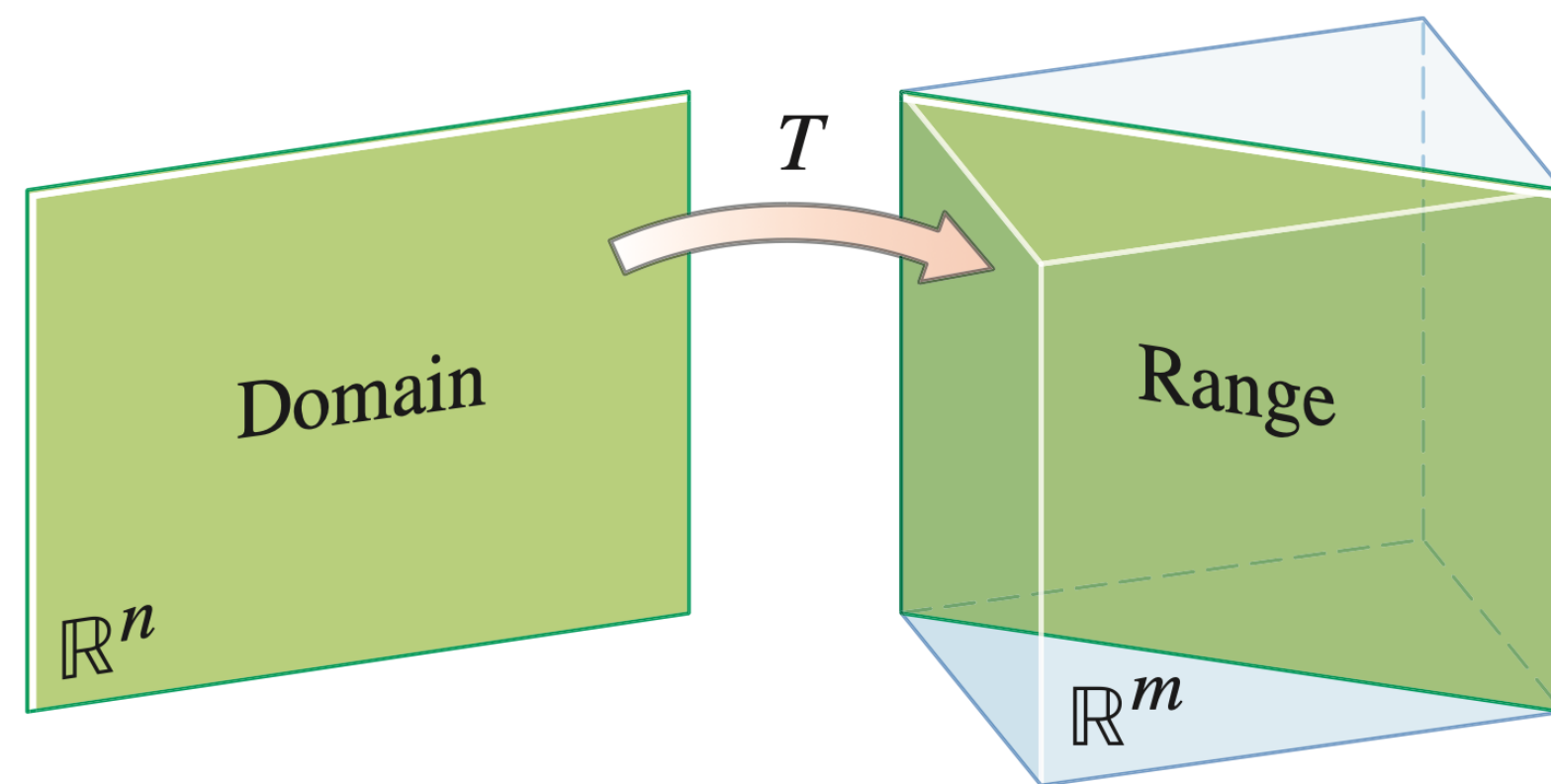
Onto Transformations

Onto Transformations

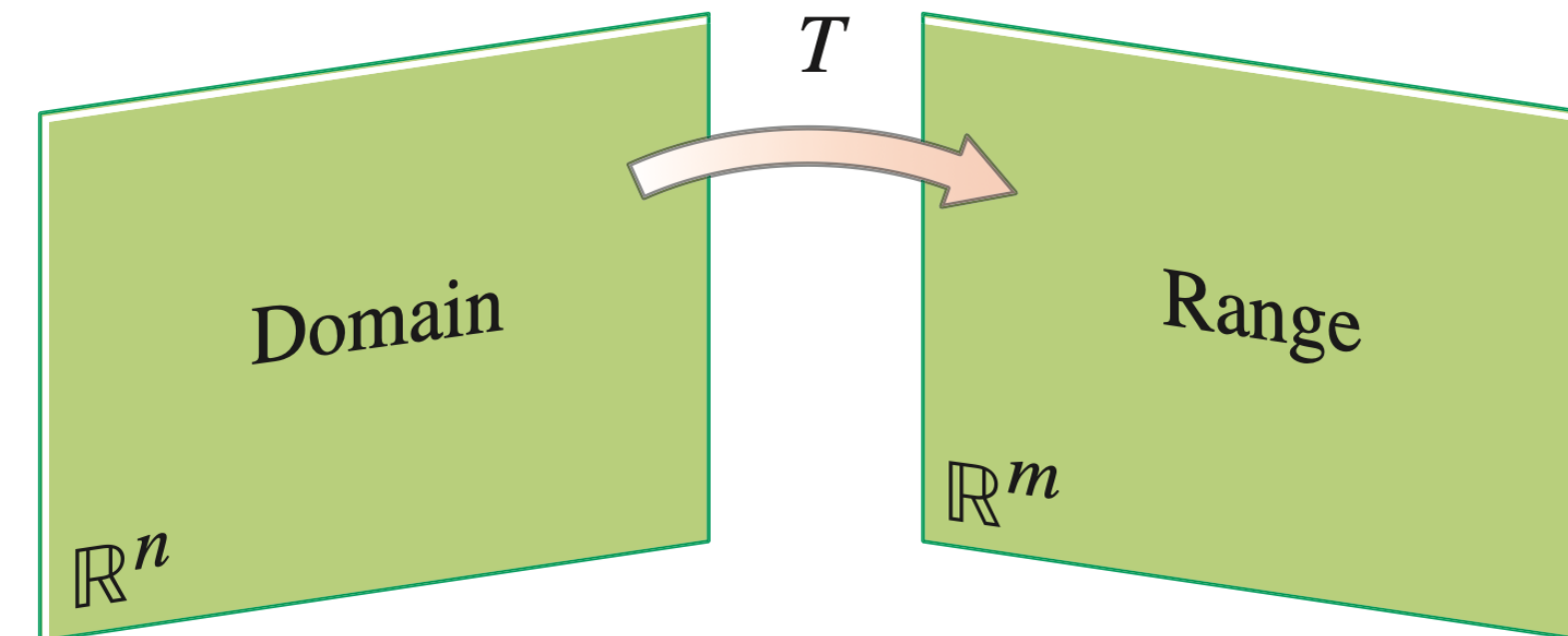
Definition. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is ***onto*** if any vector \mathbf{b} in \mathbb{R}^m is the **image of at least one vector** \mathbf{v} in \mathbb{R}^n (where $T(\mathbf{v}) = \mathbf{b}$).

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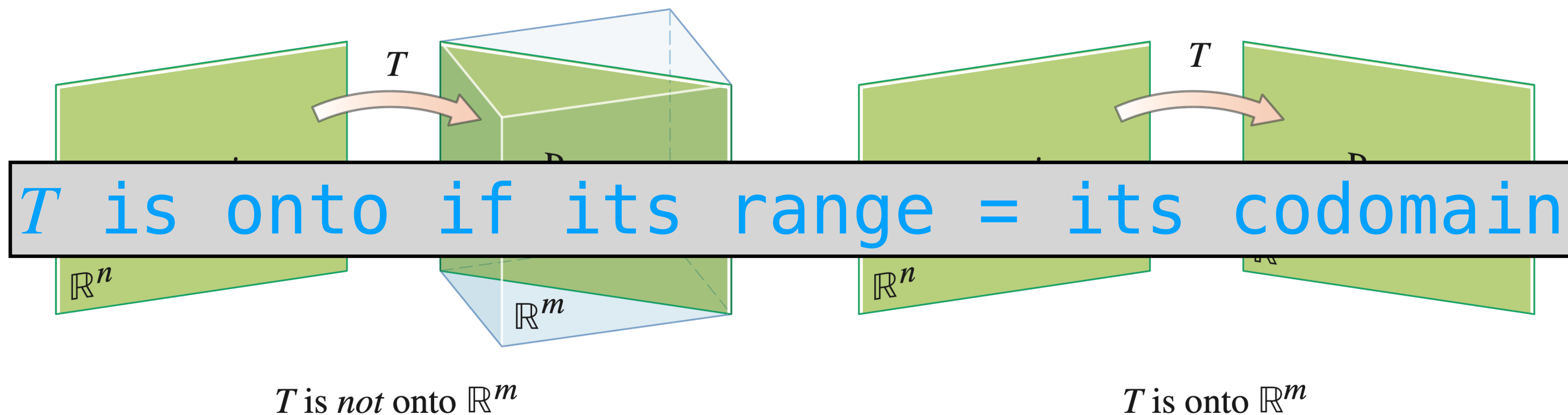
T is not onto \mathbb{R}^m



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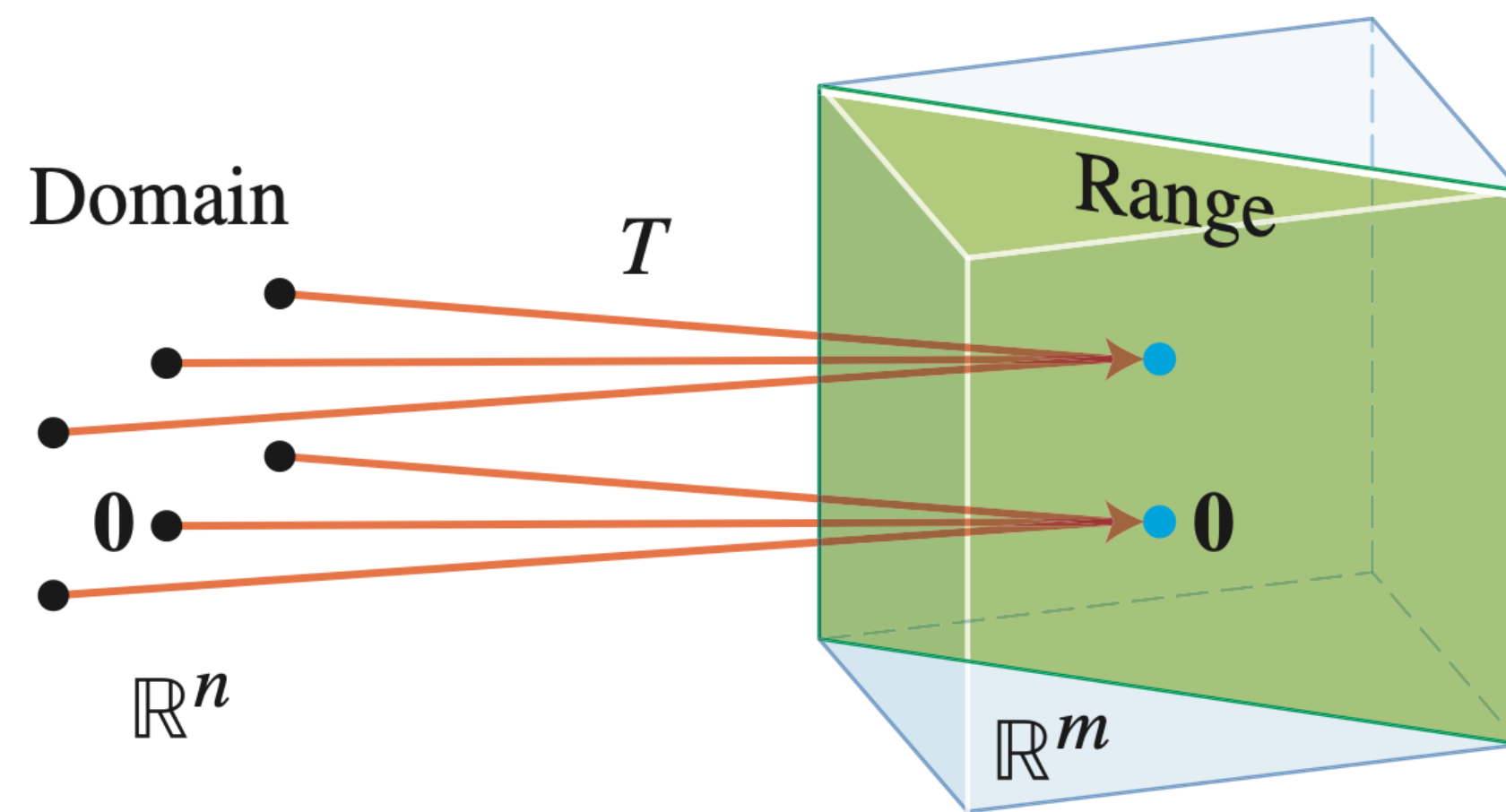
One-to-one Transformations

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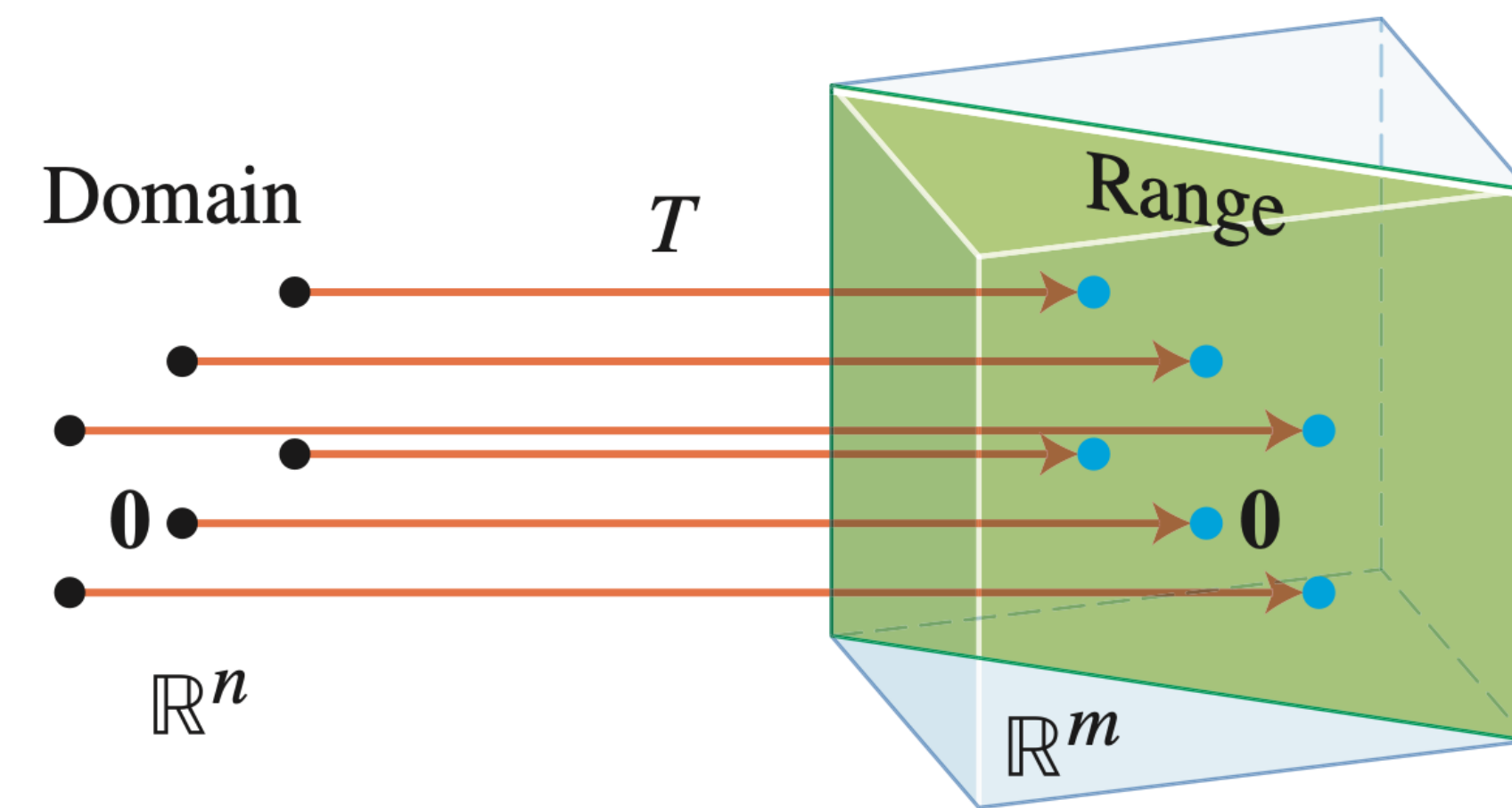
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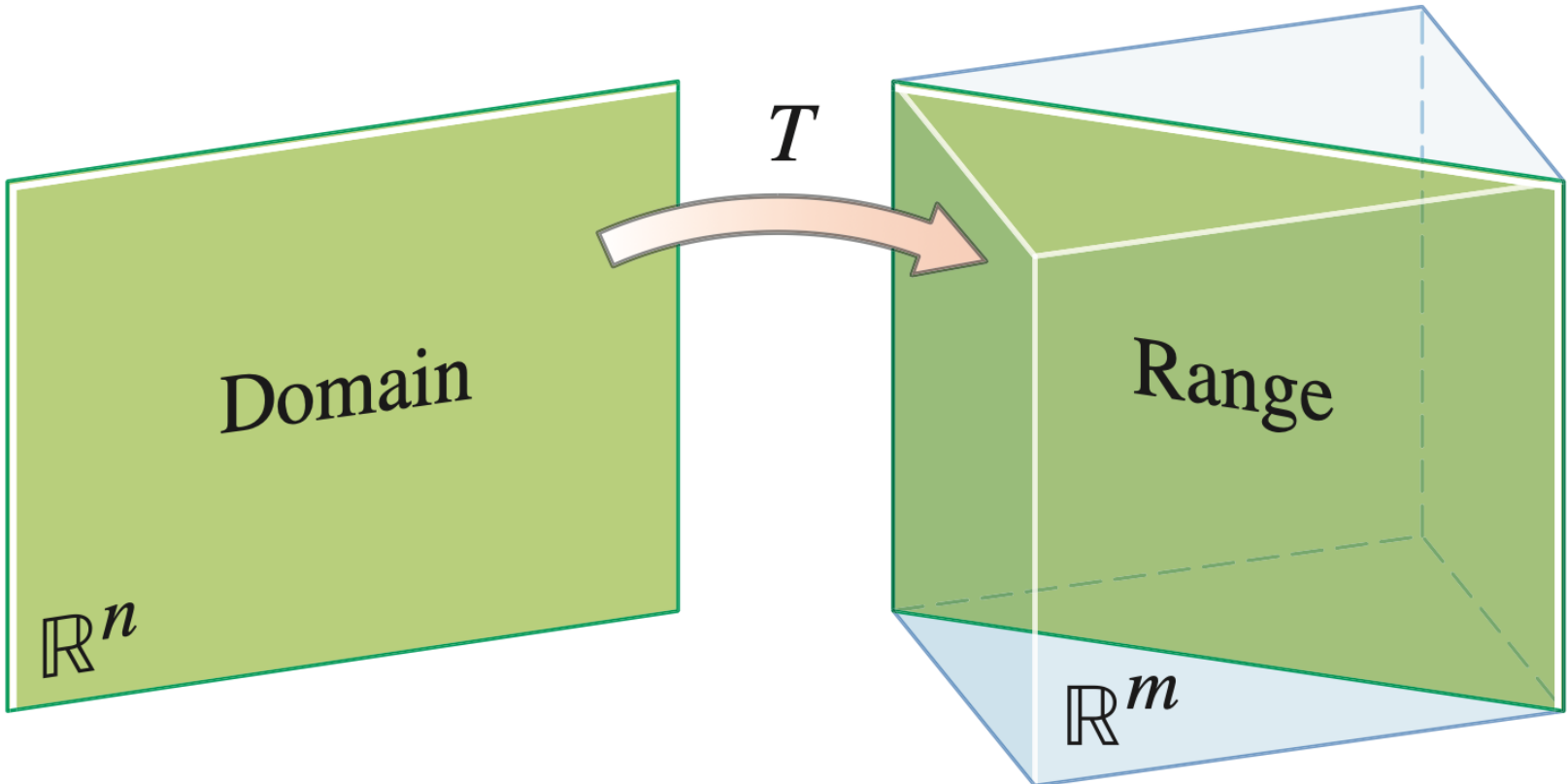


T is *not* one-to-one

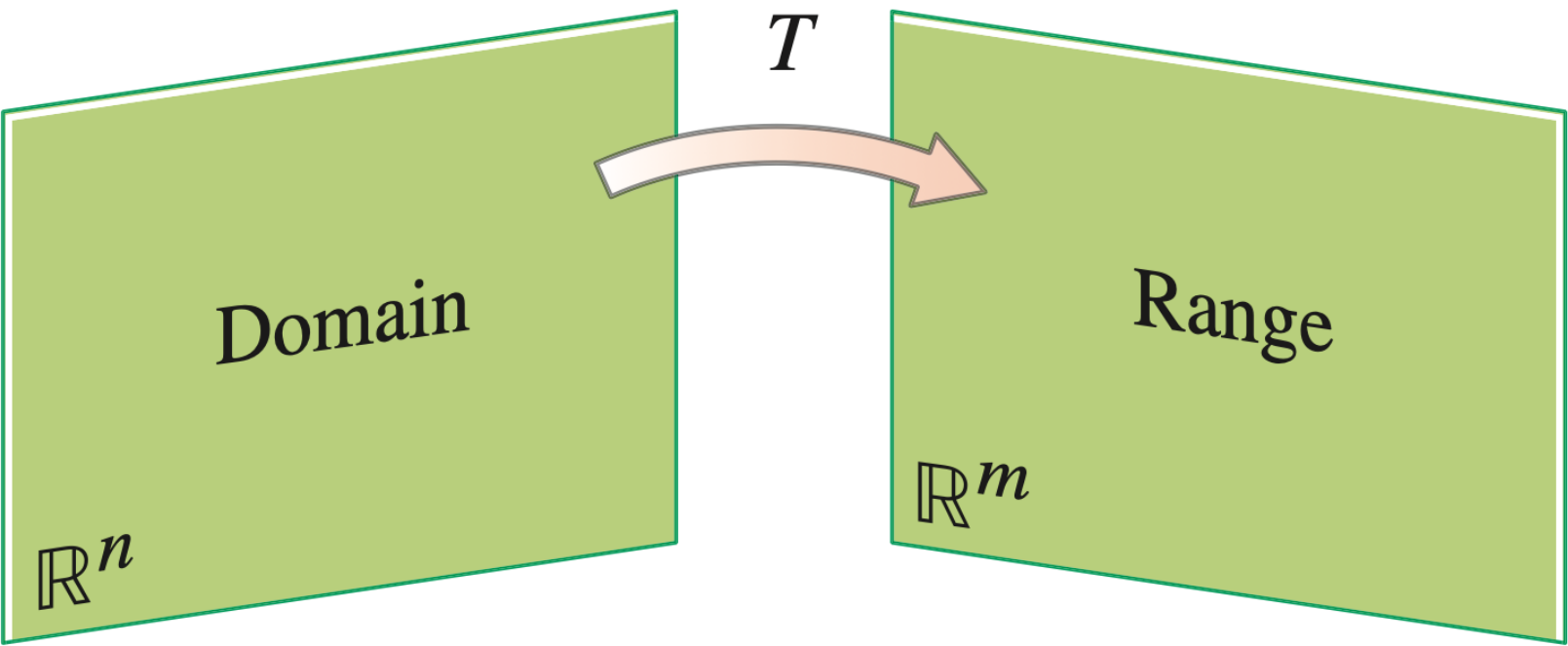


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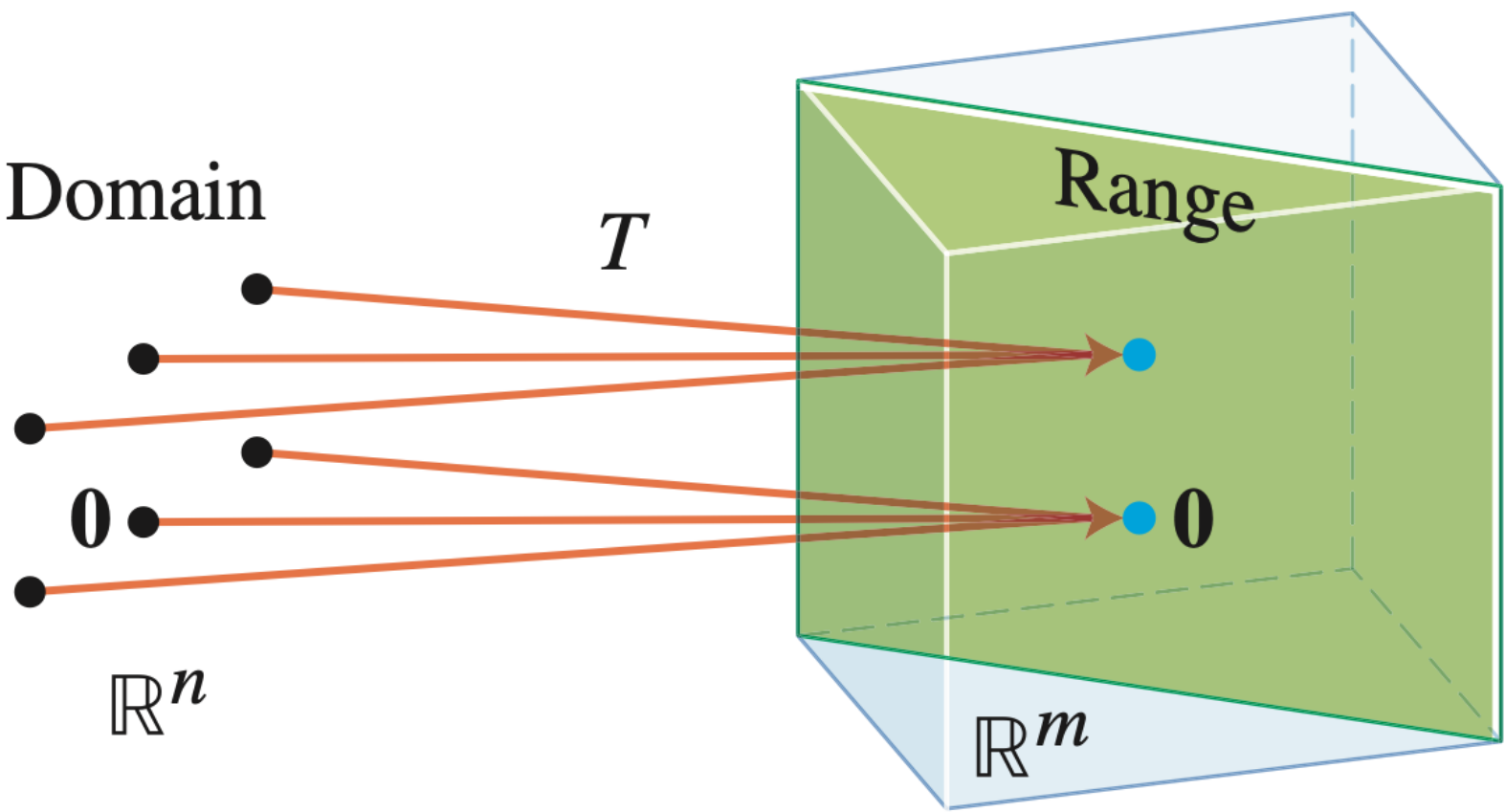
Comparing Pictures



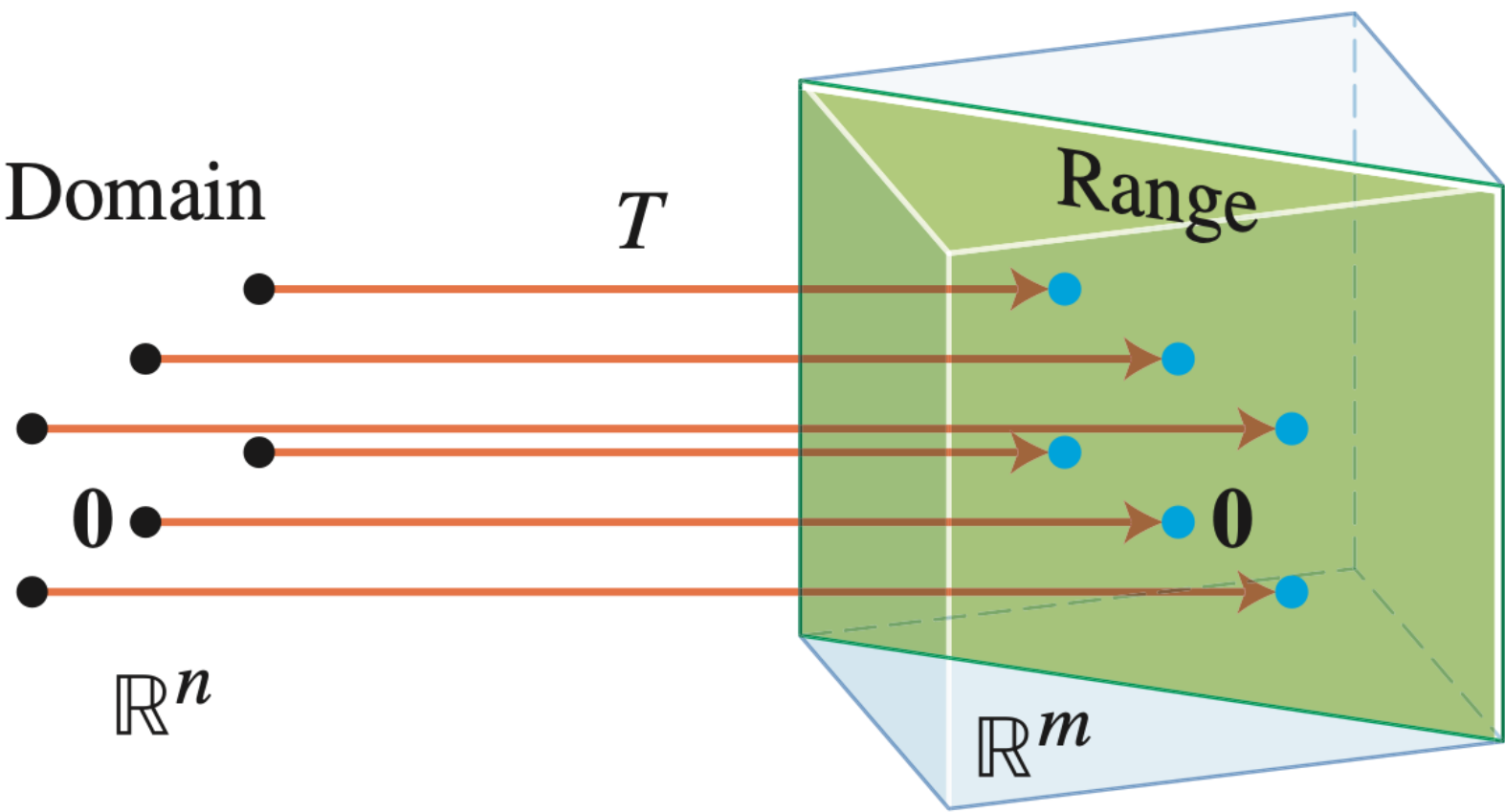
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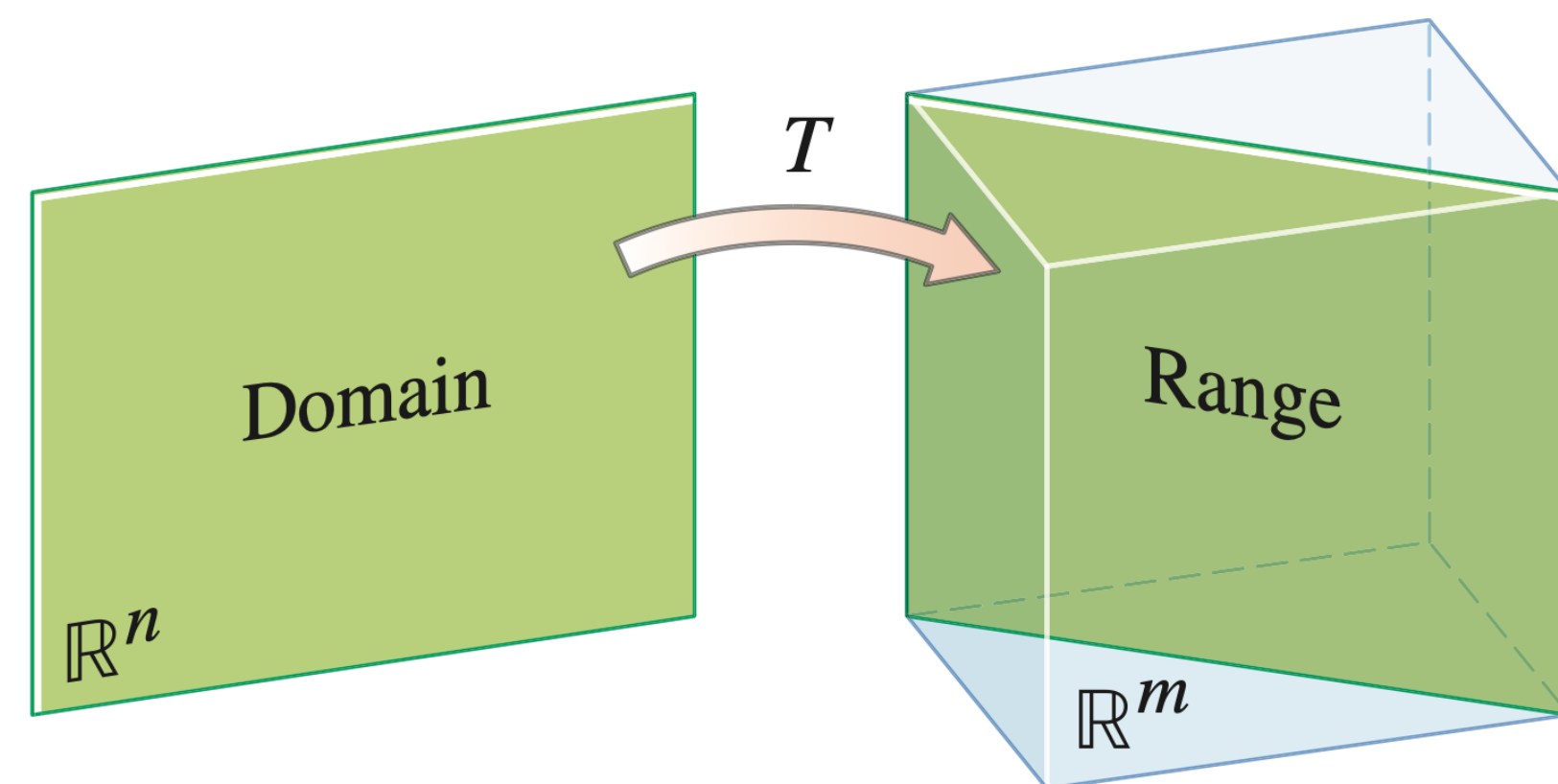


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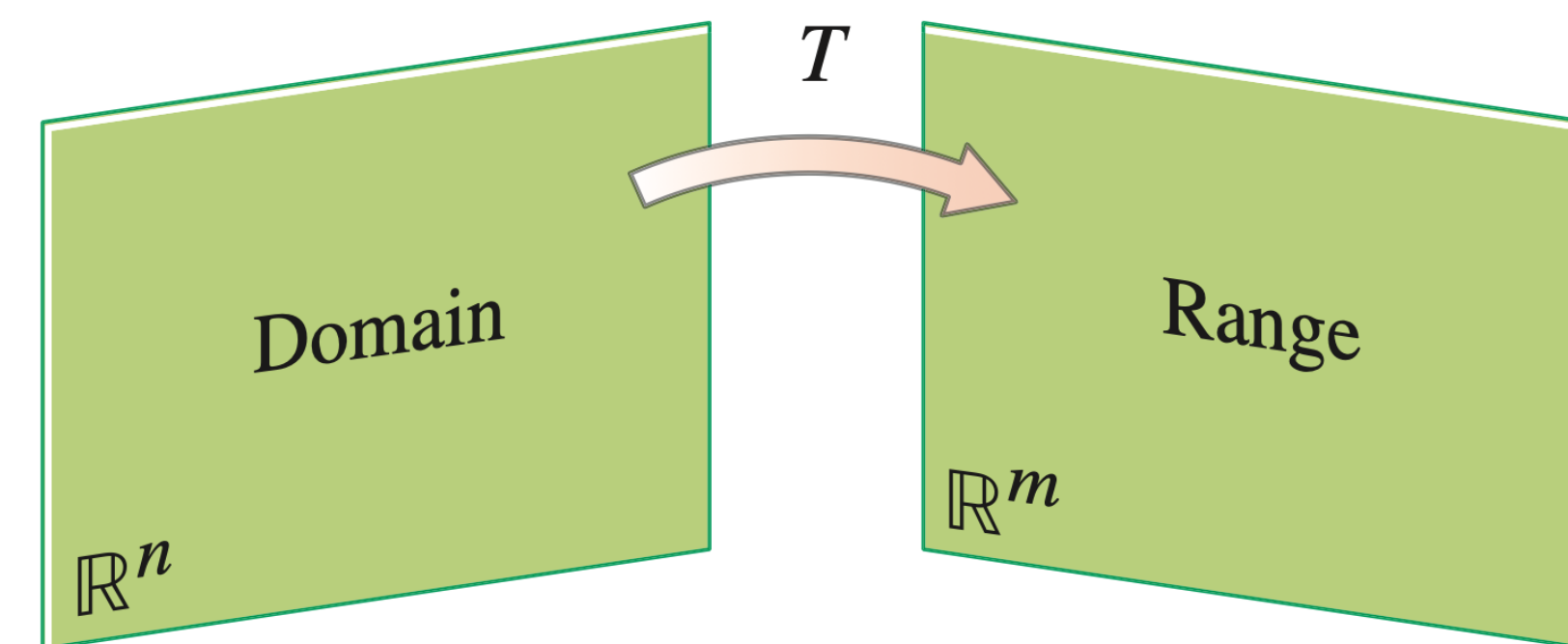


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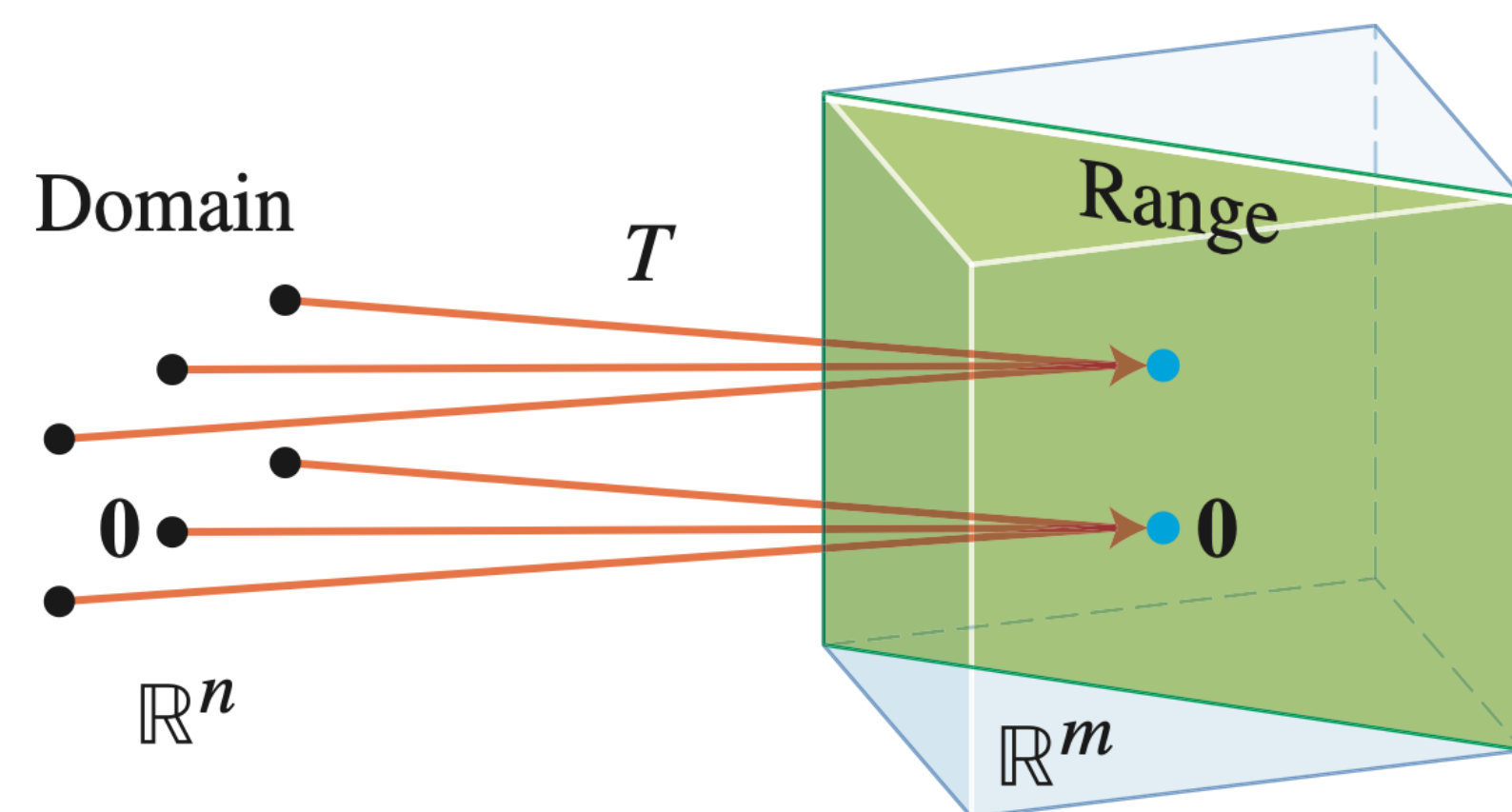
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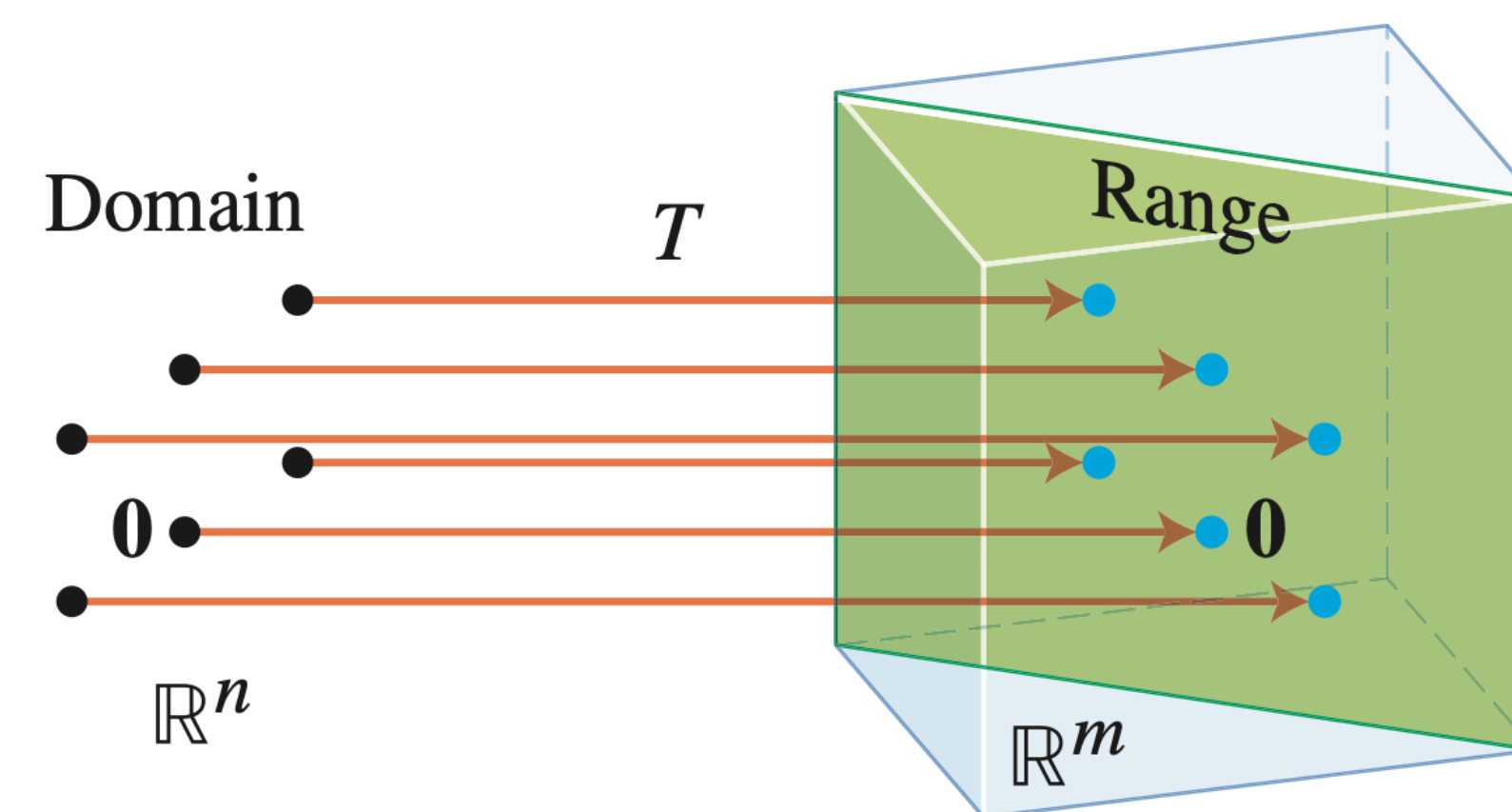
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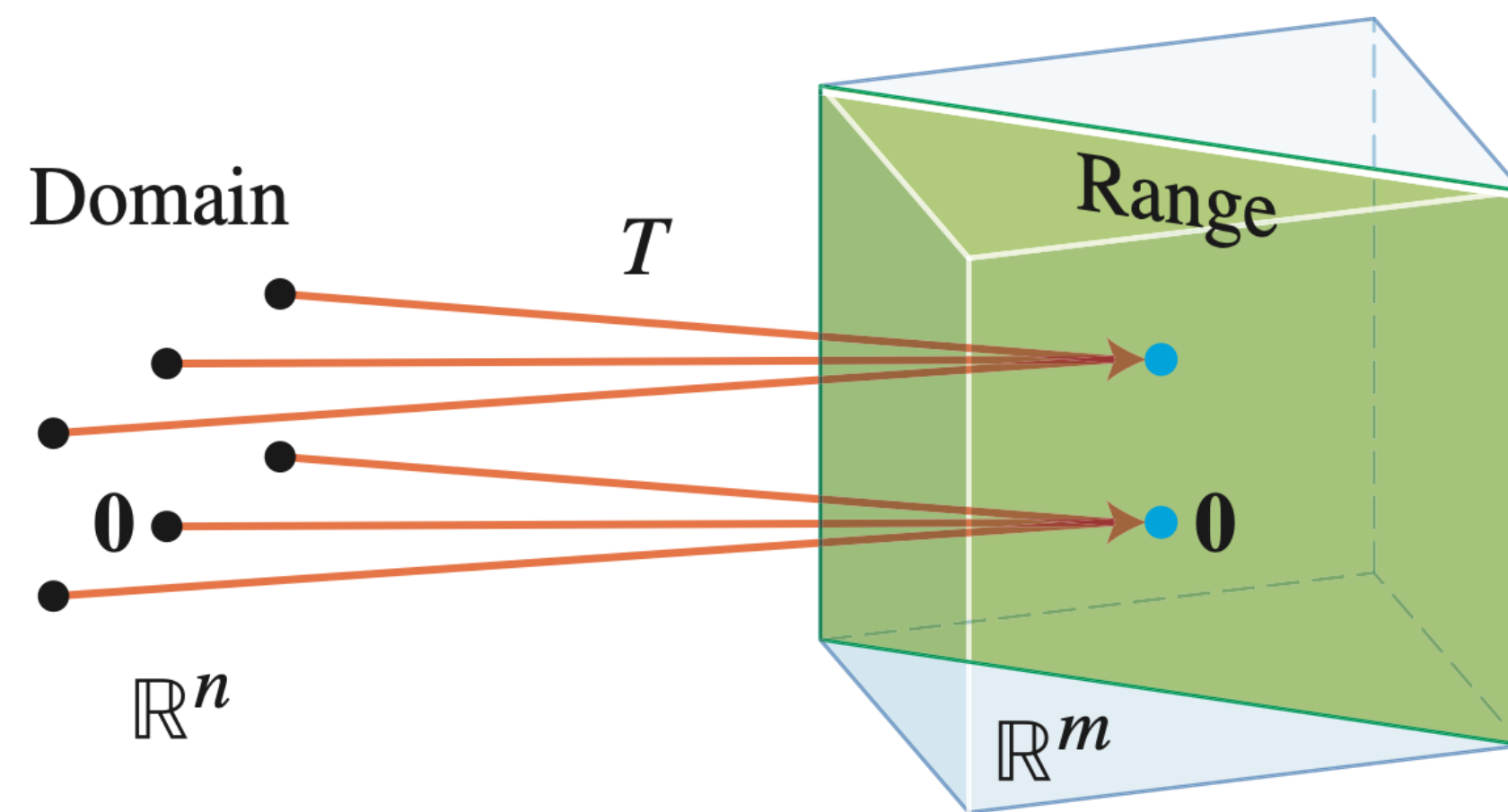


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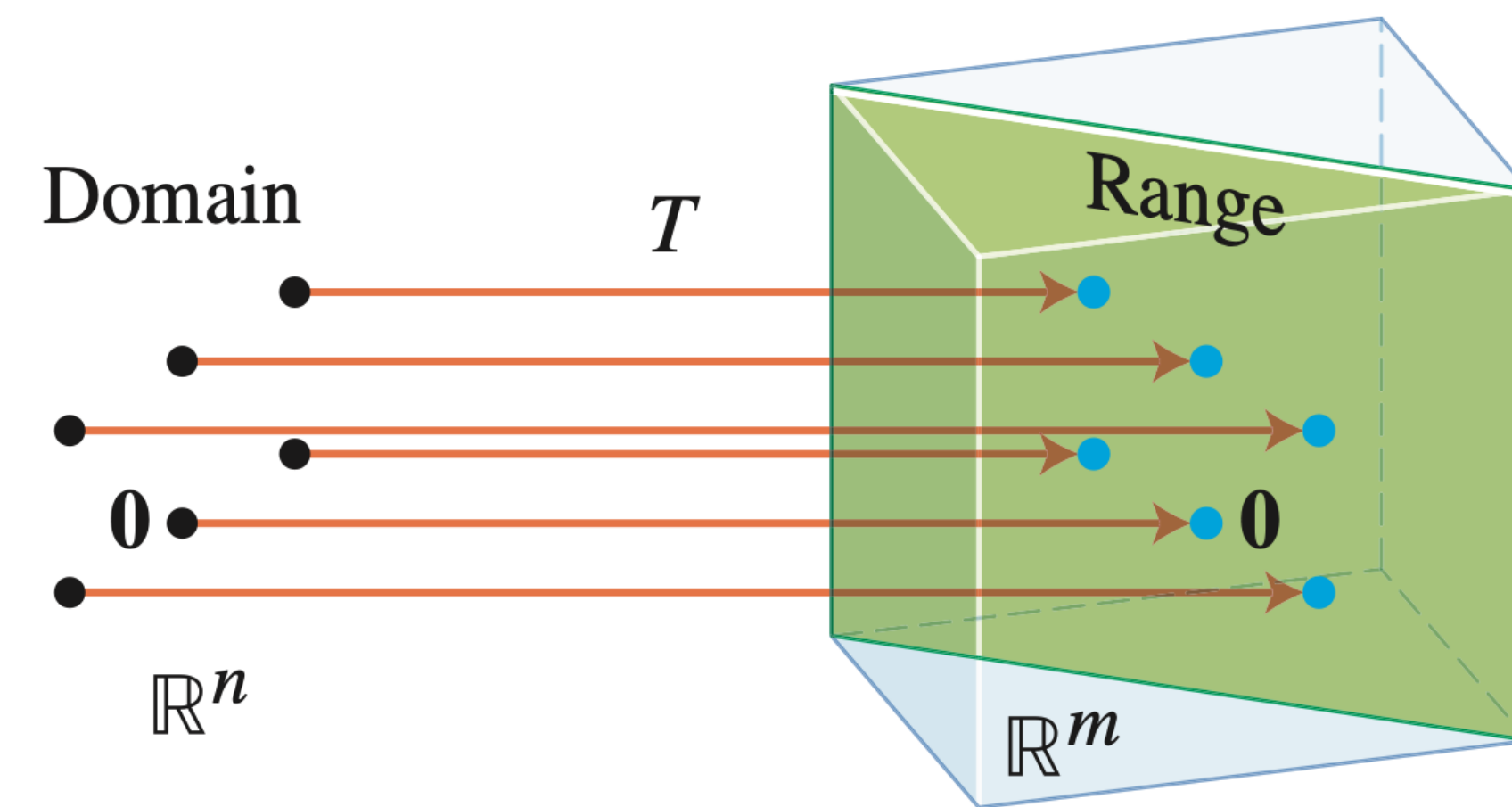


T is one-to-one

One-to-One (Pictorially)



T is *not* one-to-one



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- » A has a pivot position in every row

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- » A has a pivot position in every column

How To: One-to-One and Onto

Question. Show that the linear transformation T is one-to-one/onto.

Solution. (one approach) Find the matrix which implements T and see if it has a pivot in every column/row.

Warning: this is not the only way. Always try to think if you can solve it using *any* of the perspectives

Example: both 1-1 and onto

Rotation about the origin:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

why? :

Example: 1-1, not onto

Lifting:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$$

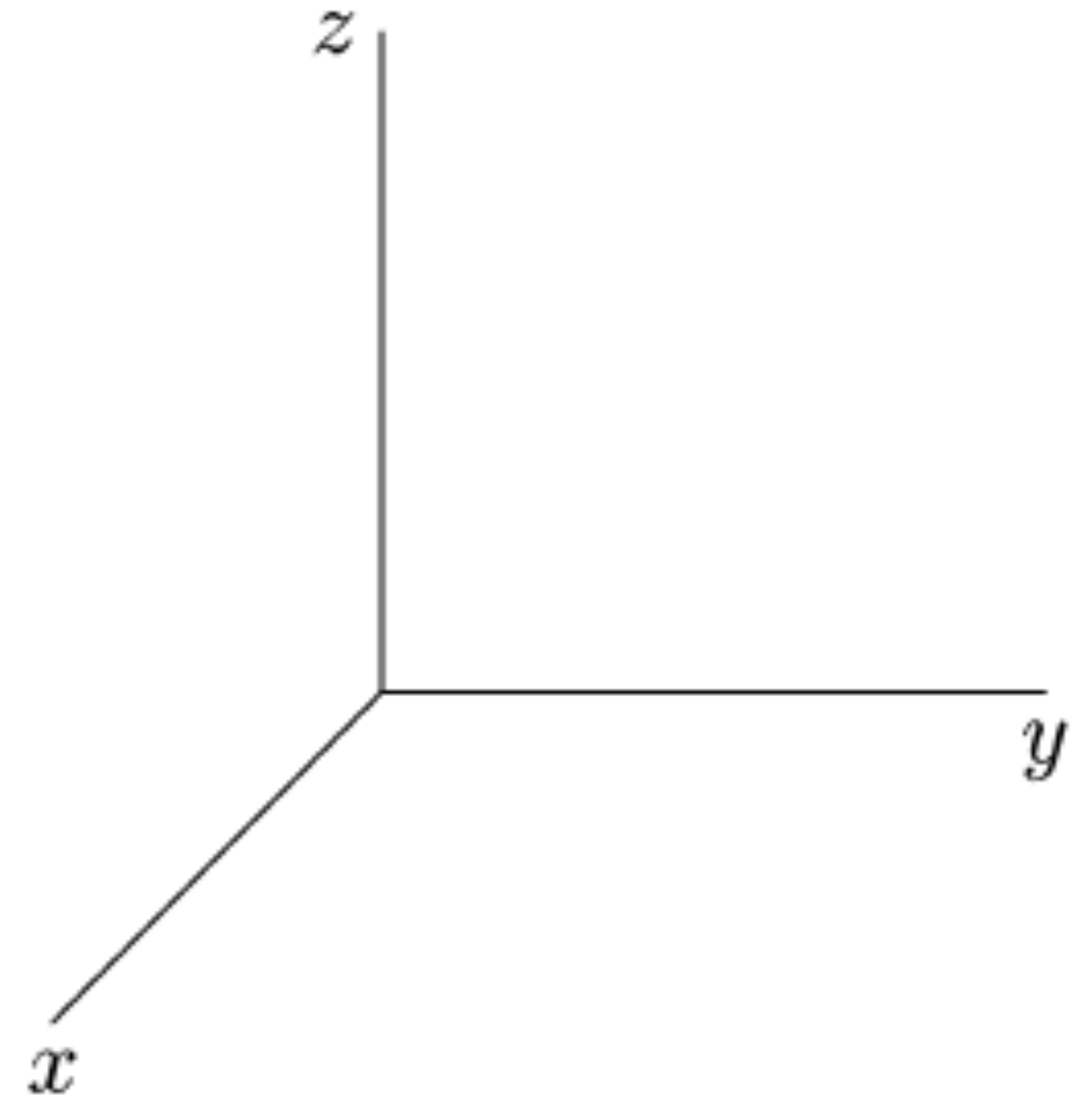
why? :

Example: onto, not 1-1

Projection from \mathbb{R}^3 to \mathbb{R}^2 .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

why? :

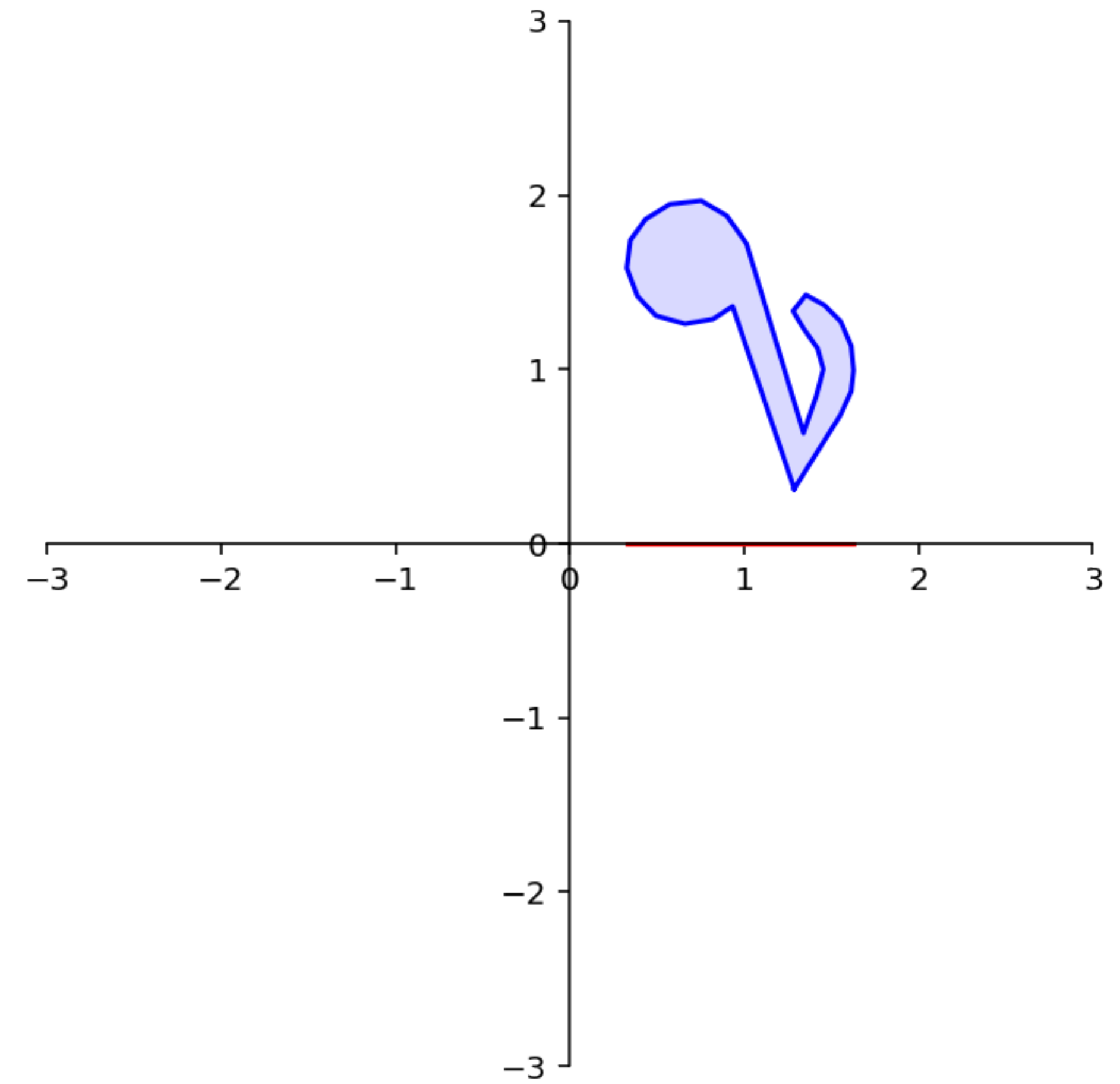


Example: not 1-1, not onto

Projection onto the x_1 -axis:

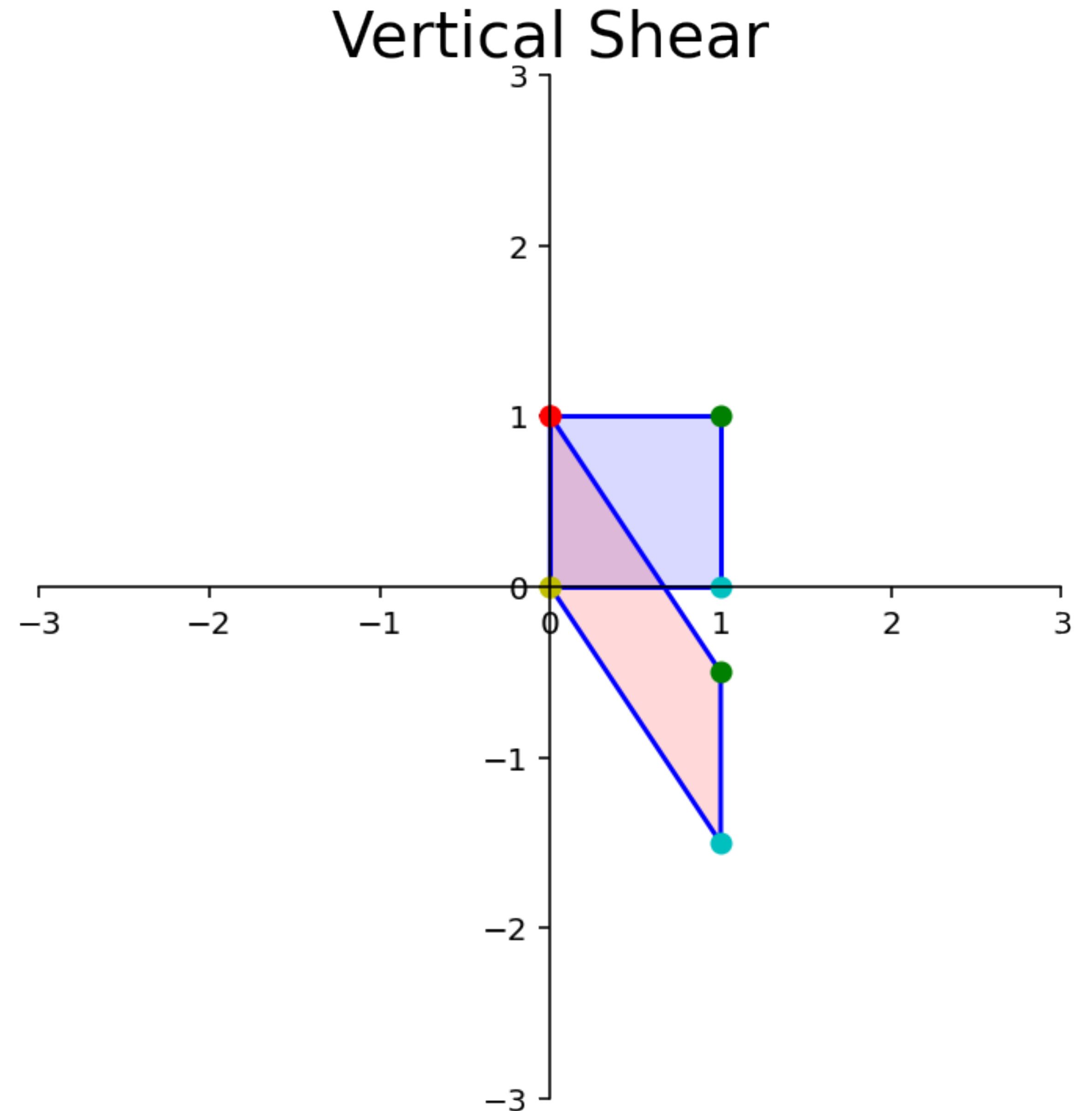
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

why? :

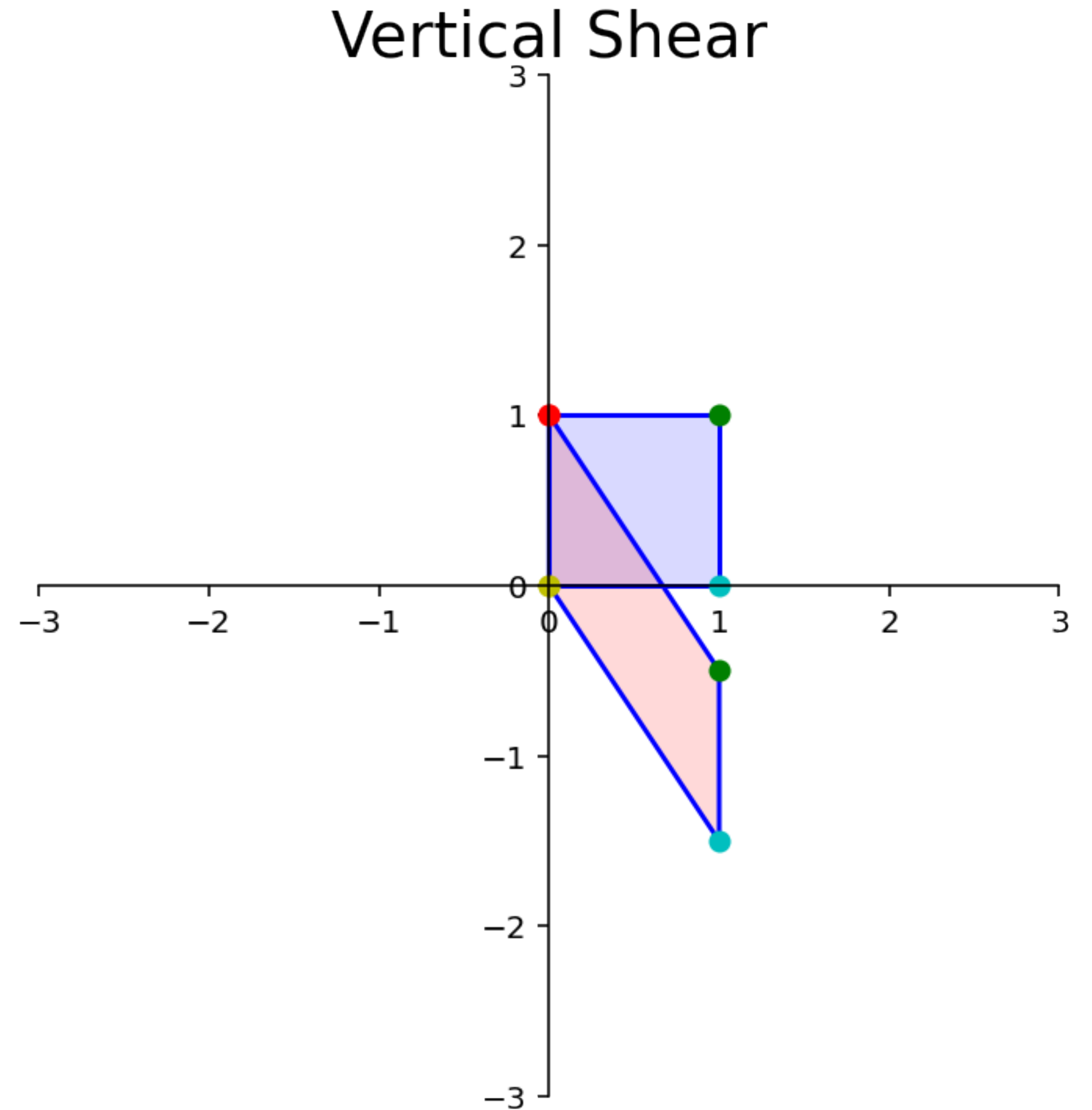


Question

*Is vertical shearing
a 1-1 transformation?
Justify your answer.*



Answer: Yes

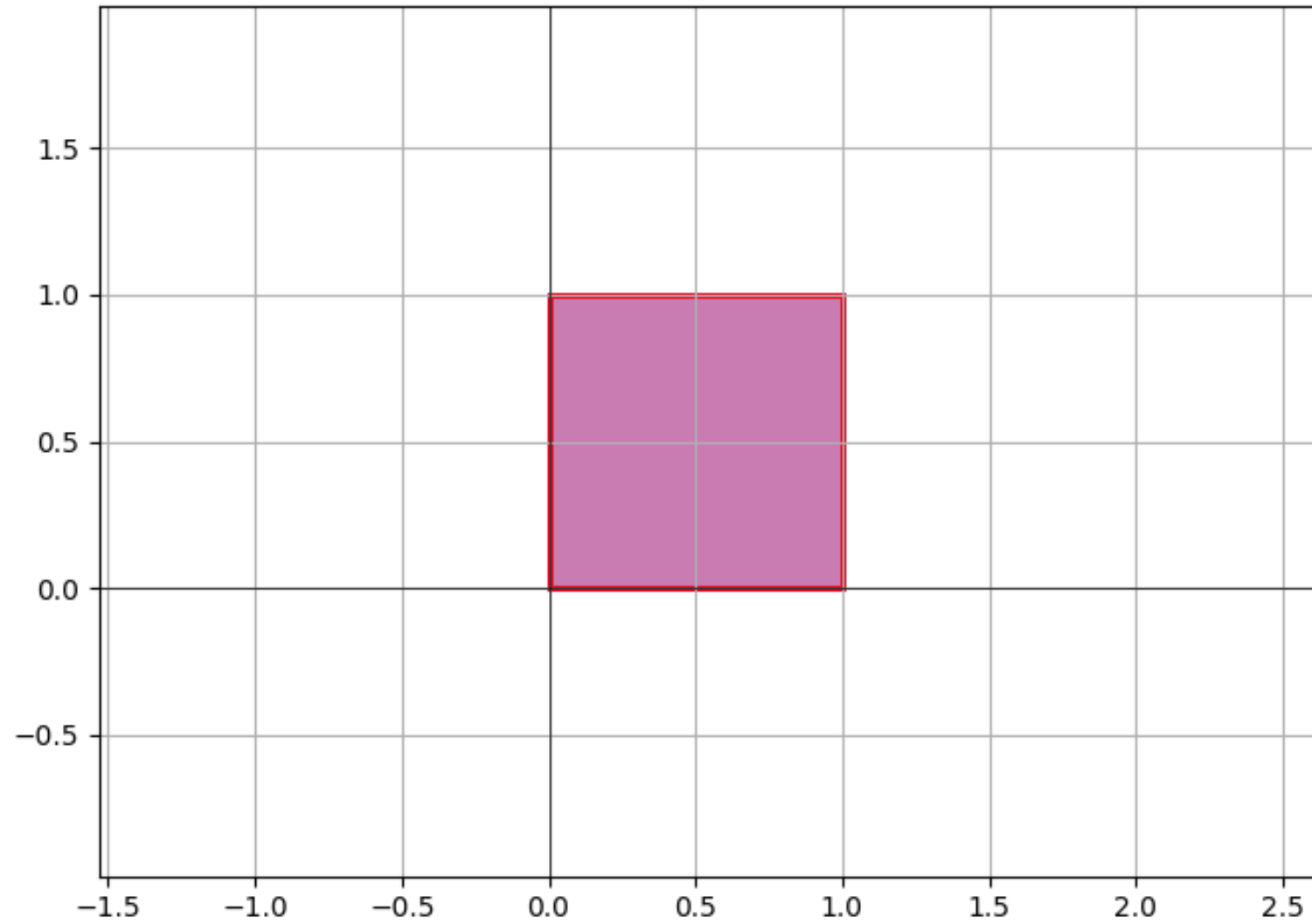


(moving on)

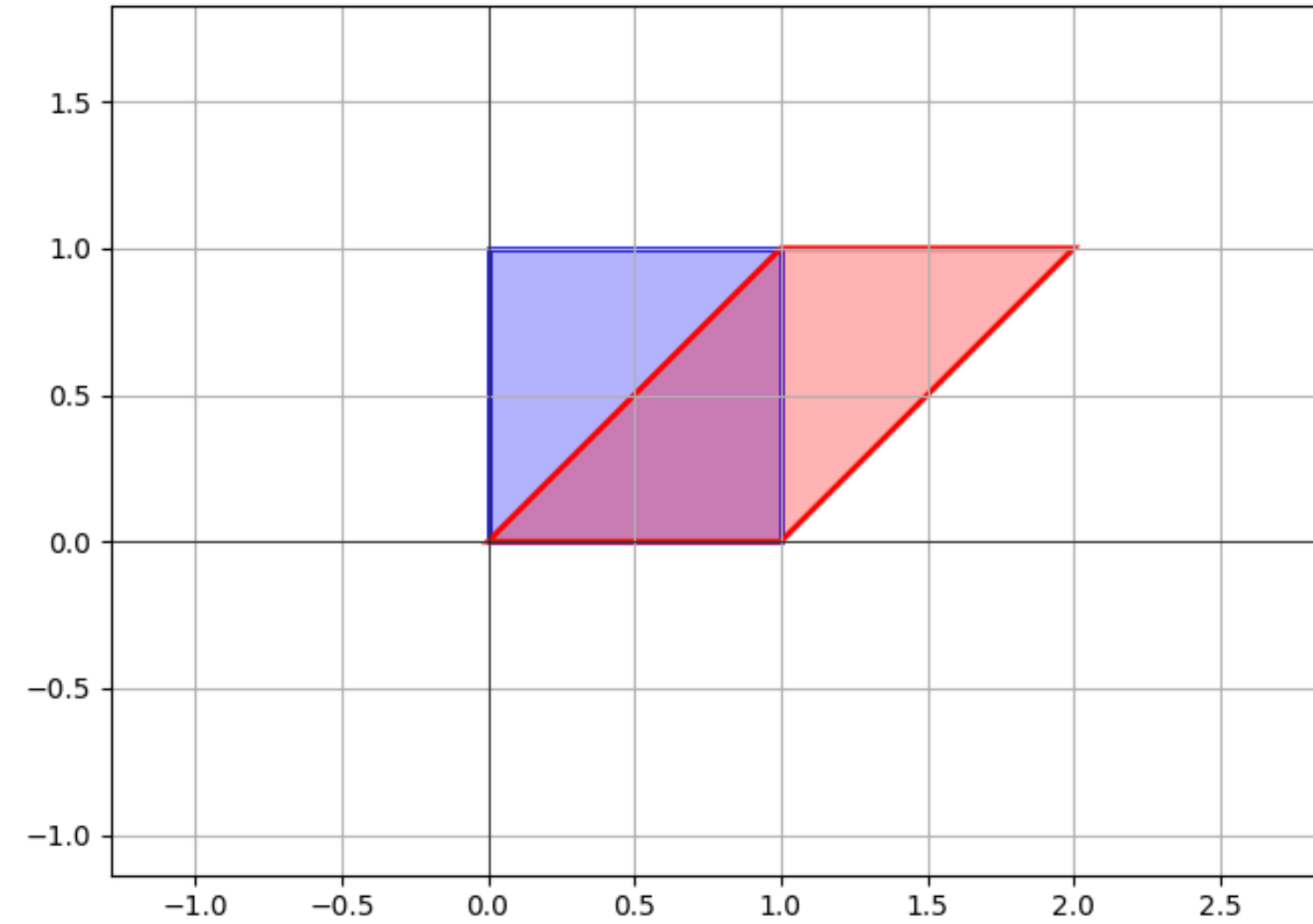
Composing Linear Transformations

Shearing and Reflecting (Geometrically)

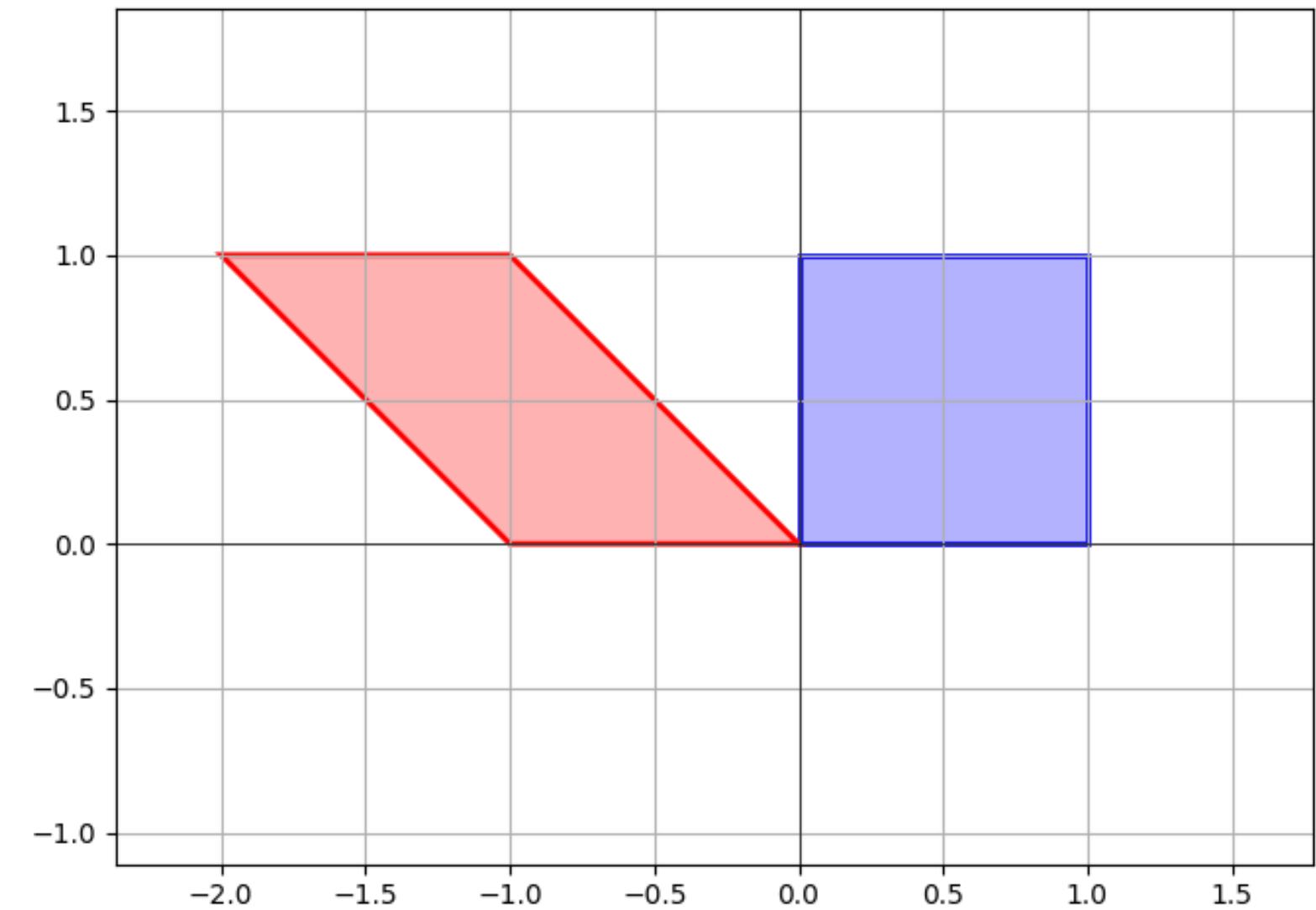
2D Matrix Transformations



2D Matrix Transformations



2D Matrix Transformations



shear



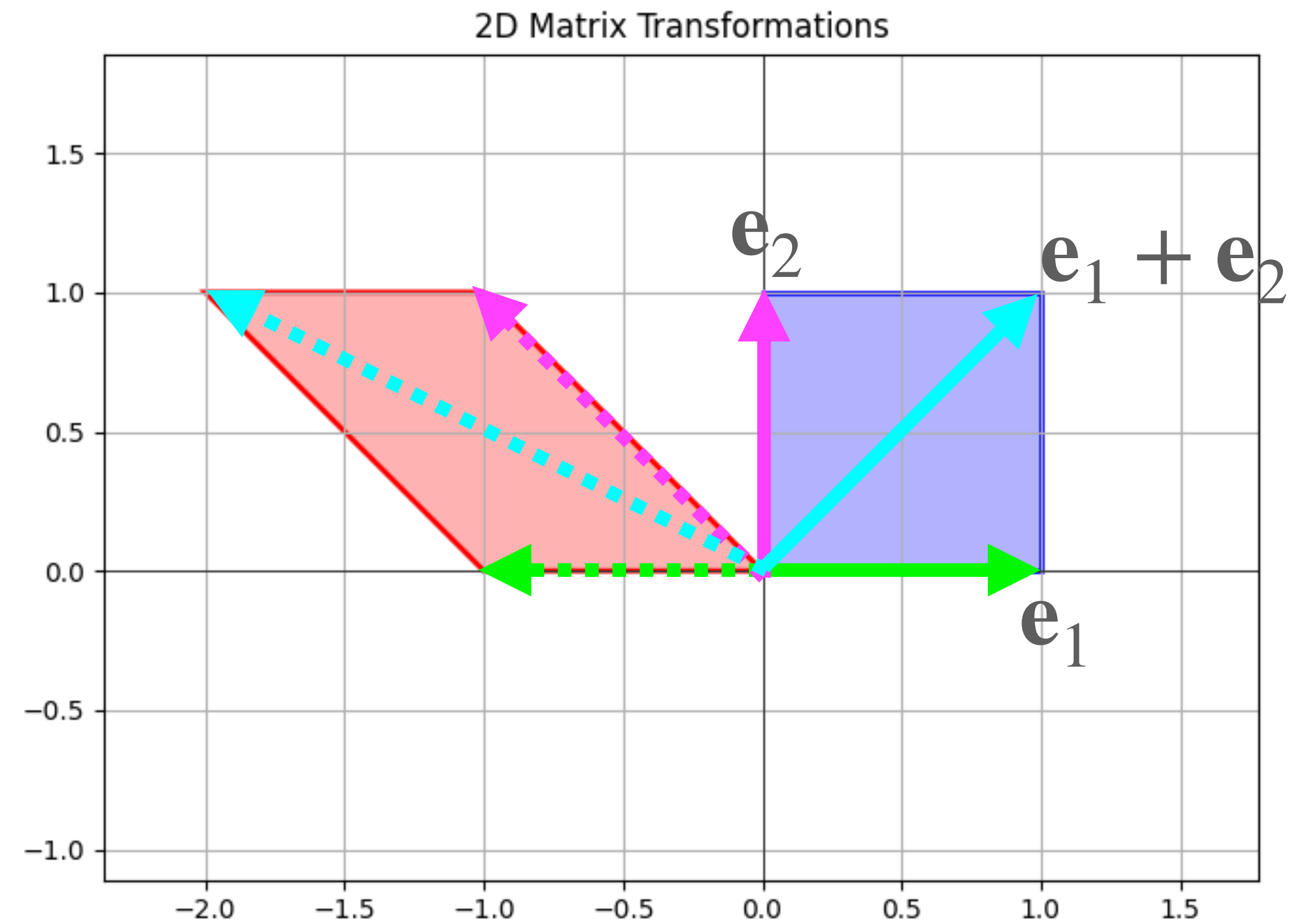
reflect

Shearing and Reflecting Matrix

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mapsto$$



Shearing and Reflecting (Algebraically)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

reflect shear

First multiply by shear matrix, then multiply
by reflection matrix

Shearing and Reflecting (Algebraically)

$$\begin{matrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ \text{reflect} & \text{shear} \end{matrix}$$

First multiply by shear matrix, then multiply
by reflection matrix

This gives us the same transformation.

Shearing and Reflecting

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \right)$$

The Key Fact

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Fact. The composition of two linear transformations is a linear transformation.

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Verify:

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Fact. The composition of two linear transformation is a linear transformation.

Verify:

This means the composition of two matrix transformation can be represented as a *single* matrix.

The Key Question

*Given two linear transformations,
how to we compute the matrix which
implements their composition?*

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Matrix Multiplication

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Shearing and Reflecting

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

General Composition (2D)

$$A \left(\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) =$$

Matrix Multiplication

Definition. For a $m \times n$ matrix A and a $n \times p$ matrix B with columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ the product AB is the $m \times p$ matrix given by

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \dots & A\mathbf{b}_p \end{bmatrix}$$

Replace each column of B with A multiplied by that column.

Tracking Dimensions

this only works if the number of columns of the left matrix matches the number of rows of the right matrix

The diagram illustrates the compatibility of matrix dimensions for multiplication. It shows three matrices with their dimensions explicitly labeled:

- Left Matrix:** A 5x3 matrix. The number of rows is labeled m (blue) and the number of columns is labeled n (red). Below it, the dimension is given as $(m \times n)$, with m in a blue box and n in a red box.
- Middle Matrix:** A 3x4 matrix. The number of rows is labeled n (red) and the number of columns is labeled k (purple). Below it, the dimension is given as $(n \times k)$, with n in a red box and k in a purple box.
- Result Matrix:** A 5x4 matrix. The number of rows is labeled m (blue) and the number of columns is labeled k (purple). Below it, the dimension is given as $(m \times k)$, with m in a blue box and k in a purple box.

The matrices are connected by an equals sign, indicating the multiplication operation. The red n in the middle matrix's row label corresponds to the red n in the left matrix's column label, showing they match.

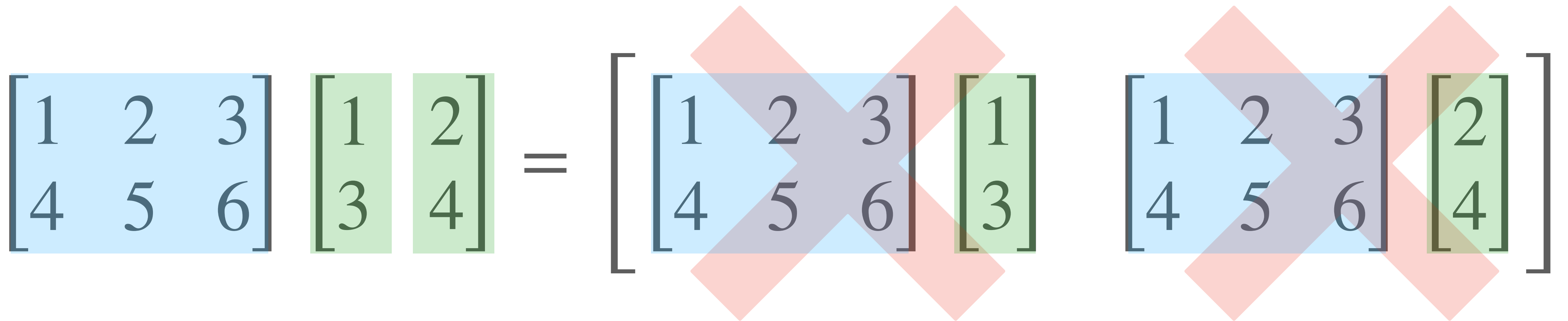
Important Note

Even if AB is defined, it may be that BA is not defined

Non-Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$$

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These are not defined.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix}$$

The Key Fact (Restated)

For any matrices A and B (such that AB is defined) and any vector \mathbf{v}

$$A(B\mathbf{v}) = (AB)\mathbf{v}$$

The matrix implementing the composition is the product of the two underlying matrices.

Row-Column Rule

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Given a $m \times n$ matrix A and a $n \times p$ matrix B , the entry in row i and column j of AB is defined above.

Example

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix with its first row highlighted in light blue.
- A 3x4 matrix with its fourth column highlighted in light red.
- A 5x4 matrix with its top-right element highlighted in light purple.

An equals sign is placed between the second and third matrices, indicating that the product of the first row of the first matrix and the fourth column of the second matrix results in the top-right element of the third matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

A pictorial representation of the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 grid of asterisks with its second row highlighted in light blue. The second matrix is a 3x4 grid of asterisks with its fourth column highlighted in light red. An equals sign follows, and then a 5x4 grid of asterisks where the element at the intersection of the second row and fourth column is highlighted in light purple, representing the result of the dot product of the two highlighted rows and columns.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix of asterisks with its third row highlighted in light blue. The second matrix is a 3x4 matrix of asterisks with its first column highlighted in light red. These two matrices are multiplied, as indicated by an equals sign, to produce a 5x4 matrix of asterisks where the element in the third row and first column is highlighted in light purple. This purple element represents the dot product of the third row of the first matrix and the first column of the second matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C . The third row of A is highlighted in light blue, the third column of B is highlighted in light red, and the resulting third row of C is highlighted in light purple. The matrices are represented as grids of asterisks (*).

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product C , which is a 5x4 matrix. The third row of A is highlighted in light blue, the third column of B is highlighted in light red, and the resulting element in the third row, fourth column of C is highlighted in light purple. The matrices are represented as grids of asterisks (*) enclosed in large square brackets.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A with its third row highlighted in light blue.
- A 3x4 matrix B with its first column highlighted in light red.
- A 5x4 matrix C with its third row highlighted in light purple.

The equation is represented as:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices:

- A 5x3 matrix A with its third row highlighted in light blue.
- A 3x4 matrix B with its third column highlighted in light red.
- The resulting 5x4 matrix AB with its third row highlighted in light purple.

The matrices are represented as follows:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C . The bottom row of A is highlighted in light blue, the first column of B is highlighted in light red, and the bottom-left element of C is highlighted in light purple. The matrices are represented as grids of asterisks (*) enclosed in large square brackets.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the row-column rule for matrix multiplication. It shows three matrices. The first matrix is a 5x3 matrix with asterisks in all cells; its bottom row is highlighted in light blue. The second matrix is a 3x4 matrix with asterisks in all cells; its third column is highlighted in light red. An equals sign follows, leading to a 5x4 matrix where the asterisks in the bottom row and third column are highlighted in light purple, representing the resulting product matrix.

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Row-Column Rule (Pictorially)

The diagram illustrates the Row-Column Rule for matrix multiplication. It shows three matrices: a 5x3 matrix A , a 3x4 matrix B , and their product, a 5x4 matrix C .

- Matrix A is a 5x3 matrix with all elements marked with asterisks (*). The bottom row is highlighted in light blue.
- Matrix B is a 3x4 matrix with all elements marked with asterisks (*). The fourth column is highlighted in light red.
- Matrix C is a 5x4 matrix with all elements marked with asterisks (*). The bottom-right element is highlighted in light purple.

The equation is represented as:

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Question

Compute $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$

short version: What is the entry in the 2nd row and 2nd column?

Answer

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$

Matrix Operations

Connection with Matrix-Vector Multiplication

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

Connection with Matrix-Vector Multiplication

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$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

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This is just vector multiplication.

Connection with Matrix-Vector Multiplication

What about when the right matrix is a single column?

$$A[\mathbf{b}_1] = [A\mathbf{b}_1] = A\mathbf{b}_1$$

This is just vector multiplication.

We can think of $[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_p]$ as collection of simultaneous matrix-vector multiplications

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

Matrix "Interface"

multiplication

what does AB mean when A and B are matrices?

addition

what does $A + B$ mean when A and B are matrices?

scaling

what does cA mean when A is matrix and c is a real number?

These should be consistent with matrix-vector interface and vector interface

Matrix Addition

$$[\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] + [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] = [(\mathbf{a}_1 + \mathbf{b}_1) \quad \dots \quad (\mathbf{a}_n + \mathbf{b}_n)]$$

Addition is done column-wise (or equivalently, element-wise)

$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3-2) & (4-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

Matrix Addition

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This is exactly the same as vector addition, but for matrices.

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

Matrix Addition and Scaling

$$c [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = [c\mathbf{a}_1 \quad c\mathbf{a}_2 \quad \dots \quad c\mathbf{a}_n]$$

Scaling and adding happen element-wise (or, equivalently, column-wise).

$$\text{e.g. } 2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix}$$

This is exactly the same as vector scaling, but for matrices.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the same size and r and s are scalars (\mathbb{R})

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + 0 = A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$r(sA) = (rs)A$$

Now we need to know/memorize these.

Algebraic Properties (Addition and Scaling)

In these properties A , B , and C are matrices of the appropriate size so that everything is defined, and r is a scalar

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A I_n$$

Now we need to know/memorize these.

Verifying $A(B + C) = AB + AC$

Matrix Multiplication is not Commutative

Important. AB may not be the same as BA

(it may not even be defined)

Question (Conceptual)

Find a pair of 2D linear transformations T_1 and T_2 such that T_1 followed by T_2 is not the same as T_2 followed by T_1 .

(also find a pair where they are the same)

Answer: Rotation and Reflection

Computational Aspects of Matrix Multiplication

Matrix Operations in Numpy

`a @ b` (matrix multiplication)

`a + b` (matrix addition)

`c * a` (matrix scaling)

We've seen this in passing, we'll be using it a lot more moving forward.

Matrix Operations in Numpy

Let `a` and `b` be 2D numpy arrays and let `c` be a floating point number.

» `a @ b` (matrix multiplication)

» `a + b` (matrix addition)

» `c * a` (matrix scaling)

We've seen these, we've used them a bit, we'll use them much more.

A Note on Complexity

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

Suppose A and B are $n \times n$ matrices.

This operations takes n multiplications and n divisions ($2n$ FLOPS total)

Repeating for each entry gives $\sim 2n^3$ FLOPS

A Note on Parallelization

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

The main part of this procedure is highly parallelizable.

A Note on Parallelization

```
a = np.array(...)  
b = np.array(...)  
prod = np.zeros([a.shape[0], b.shape[1]])  
for i in range(a.shape[0]):  
    for j in range(b.shape[1]):  
        prod[i, j] = np.dot(a[i], b[:, j])
```

The main part of this procedure is highly parallelizable.

One processor per entry gets you to $\sim 2n$ FLOPS

A Note on Libraries

There are a lot of other considerations for doing linear algebra on computers.

Best leave it to experts (or do research in the area).

LAPACK is the state of the art library for matrix operations.

numpy uses LAPACK

Summary

We can reason about matrix equations by reasoning directly about properties of linear transformations.

Matrix multiplication coincides with composition of linear transformations.

There is an algebra of matrices which is consistent with the algebra of vectors.