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Inertial Ranges in Two-Dimensional Turbulence

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Two-dimensional turbulence has both kinetic energy and mean-square vorticity as inviscid constants of motion. Consequently it admits two formal inertial ranges, $E(k) \sim \epsilon^{2/3}k^{-5/3}$ and $E(k) \sim \eta^{2/3}k^{-3}$, where ϵ is the rate of cascade of kinetic energy per unit mass, η is the rate of cascade of mean-square vorticity, and the kinetic energy per unit mass is $\int_0^\infty E(k)\, dk$. The $-\frac{5}{3}$ range is found to entail backward energy cascade, from higher to lower wavenumbers k, together with zero-vorticity flow. The -3 range gives an upward vorticity flow and zero-energy flow. The paradox in these results is resolved by the irreducibly triangular nature of the elementary wavenumber interactions. The formal -3 range gives a nonlocal cascade and consequently must be modified by logarithmic factors. If energy is fed in at a constant rate to a band of wavenumbers $\sim k_i$ and the Reynolds number is large, it is conjectured that a quasi-steady-state results with a $-\frac{5}{3}$ range for $k \ll k_i$ and a -3 range for $k \gg k_i$, up to the viscous cutoff. The total kinetic energy increases steadily with time as the $-\frac{5}{3}$ range pushes to ever-lower k, until scales the size of the entire fluid are strongly excited. The rate of energy dissipation by viscosity decreases to zero if kinematic viscosity is decreased to zero with other parameters unchanged.

1. INTRODUCTION

THE vorticity of each fluid element is an inviscid constant of motion in two-dimensional incompressible flow. Therefore the mean-square vorticity as well as the kinetic energy per unit mass are inviscid constants in two-dimensional isotropic turbulence. A number of authors have studied two-dimensional turbulence theoretically, and it is recognized that the vorticity constraint has profound effects on inertial energy transfer.[1-9] In contrast to the predominantly one-way flow of energy familiar in three dimensions, transfer upward in wavenumber must be accompanied by comparable or greater downward transfer.

A principal reason for exploring two-dimensional turbulence has been the possible application to intermediate-scale meteorological flows. Another motivation is that two-dimensional flows are more easily simulated on digital computers than three-dimensional flows and may therefore be a valuable testing ground for dynamical theories. The present study grew out of an investigation of the approach of a weakly coupled boson gas to equilibrium below the Bose–Einstein condensation temperature[10]. There is a fairly close dynamical analogy in which the number density and kinetic-energy density of the bosons play the respective roles of kinetic-energy density and squared vorticity. (The flow of vorticity into small scales in two-dimensional turbulence is then analogous to the flow of kinetic energy into high wavenumbers during the formation of the boson condensate.)

The present paper is limited to analysis and discussion of general properties of energy and vorticity cascade in two dimensions which can be displayed by direct use of the Navier–Stokes equation and the inviscid conservation laws. No use is made of closure approximations. However, the Lagrangian-history direct-interaction approximation, which yields Kolmogorov's similarity cascade in three dimensions,[11] preserves the vorticity constraint in two dimensions and appears to yield the principal dynamical features inferred in the present paper. That closure may thereby be useful in exploring the interaction between the predominantly two-dimensional intermediate-scale motions and the three-dimensional small-scale turbulence in meteorological flows.

According to the two-dimensional Navier–Stokes equation, the interaction of each triad of wavenumbers k, p, q individually conserves both energy and squared vorticity. In order to separate off questions involving the localness of energy transfer, suppose that all triad interactions for which the smallest of the three wavenumbers is less than, say,

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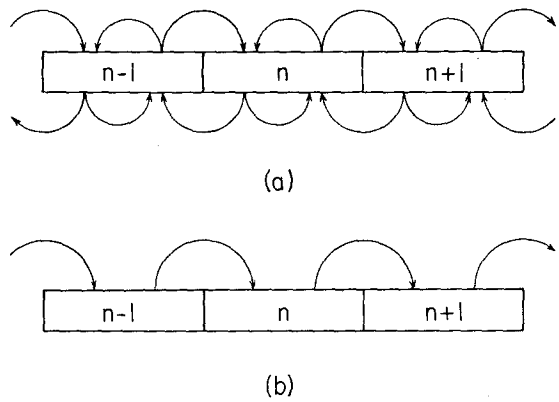

FIG. 1. Part (a) represents the cascade of excitation through the wavenumber spectrum by means of elementary triad interactions. Part (b) represents a pair-interaction cascade, which is a valid simplification in three dimensions but not in two dimensions.

one-half the largest are arbitrarily eliminated from the dynamical equations. Divide the wavenumber range into half-octave segments, so that all the triad interactions left in the equations either connect nearest-neighbor segments or are contained entirely within a single segment. Fig. 1(a) illustrates the segmentation and the way in which the triad interactions connect nearest neighbors. In order for both energy and squared vorticity to be conserved, the net transfer by each triad interaction must either be out of the middle wavenumber into both smallest and largest wavenumbers, or vice versa. The arrows in Fig. 1(a) are arbitrarily drawn for the case of outflow from the middle wavenumber.

In three dimensions, the triad interactions can be likened with some success to pair interactions [Fig. 1(b)]. This is inadmissible in two dimensions because pair interactions cannot transfer both energy and squared vorticity conservatively between unequal wavenumbers.

There is no simple general relation between energy and squared-vorticity transfer. Segment n in Fig. 1(a) is connected to the lower-wavenumber segment $n - 1$ by two kinds of triad interactions: those with a pair of wavenumbers in n and those with a pair of wavenumbers in $n - 1$. The former transfer squared vorticity and kinetic energy per unit mass from $n - 1$ to n in ratios $< 2k_n^2$, where k_n is the wavenumber which separates the segments. The latter transfer these quantities in ratios $> 2k_n^2$. The net rates of energy transfer per unit mass ϵ and squared-vorticity transfer η from below k_n to above k_n depend on the signs and relative strengths of the two kinds of interactions. For example, if the interactions having a single wavenumber in n are sufficiently strong compared to those having a single wavenumber in $n - 1$, it is possible for η to be positive while simultaneously ϵ is negative.

An important inference can be made for similarity cascades where a k-independent total contribution to ϵ is made by all triads whose ratio of largest to smallest wavenumber falls below some arbitrary limit. The triple moments can be chosen to construct such ranges at a given instant. Whether they are self-preserving is another matter. By similarity, the vorticity cascade rate must have the form $\eta = 2Ak^2\epsilon$ with A k-independent. But the rate itself must be independent of k. Otherwise, the outflow of squared vorticity from each segment would not equal the inflow, which would violate vorticity conservation since with k-independent ϵ the rate-of-change of excitation intensity is instantaneously zero at each k. The only possible resolution is $A \equiv 0$. That is, the rate of squared-vorticity cascade is identically zero in a similarity cascade where ϵ is independent of k. This is corroborated by formal analysis in Sec. 2.

The roles of energy and squared vorticity are interchangeable in the preceding argument. If there is a similarity range with k-independent contribution to η from the triad interactions whose ratio of largest to smallest wavenumber falls within some limit, then the contribution of those triads to ϵ must be identically zero within the range. This means that two kinds of putative inertial-transfer similarity ranges must be investigated: energy-transfer and vorticity-transfer ranges. Kolmogorov's assumption that the energy spectrum $E(k)$ depends only on k and ϵ leads to

$$E(k) = C\epsilon^{2/3}k^{-5/3} \tag{1.1}$$

in two dimensions as well as in three. Here $E(k)$ is defined so that the mean kinetic energy per unit mass is $\int_0^\infty E(k)\,dk$, and C is a constant whose value can depend on the dimensionality. The alternate assumption that the squared-vorticity spectrum $2k^2E(k)$ depends on only η and k yields

$$E(k) = C'\eta^{\frac{2}{3}}k^{-3}, \tag{1.2}$$

where C' is another constant. In Sec. 2 it is shown that (1.1) and (1.2) each satisfy both conservation laws. Necessary conditions for the physical realizability of these similarity ranges are that the transfer processes be sufficiently local in wavenumber when all triads are admitted. This is discussed in Sec. 4 where it is noted that the -3 range fails by logarithmic factors to be sufficiently local and therefore must be modified by factors with logarithmic k dependence.

Both experiment and general statistical-mechanical considerations indicate that the energy cascade through the inertial range is from lower to higher wavenumbers in three-dimensional turbulence. The intensity at high wavenumbers is suppressed by viscosity and it is natural to expect a net transfer toward these wavenumbers from the strongly excited low wavenumbers. In two dimensions, the vorticity constraint drastically changes matters. A given triad interaction spreads the excitation in wavenumber space if it gives a net flow out of the middle wavenumber into the small and large wavenumbers. The reverse flow concentrates the excitation. Spreading of the excitation by the triad interactions would seem to be the more plausible state of affairs. Some supporting evidence is presented in Sec. 3. If the triad interactions do spread the excitation in wavenumber space, then it is proved in Sec. 3 that the $-\frac{5}{3}$ range yields $\epsilon < 0$; that is, the energy cascade is *downward* in wavenumber. The -3 range under the same condition yields vorticity cascade upward in wavenumber ($\eta > 0$).

If the directions of cascade are as just described, the $-\frac{5}{3}$ range could serve to remove energy from an input range of wavenumbers down toward zero wavenumber, while the -3 range could carry vorticity up to the dissipation range. Thus both ranges could exist simultaneously. This conjecture is discussed in Sec. 4.

2. FORMAL ANALYSIS OF THE SIMILARITY RANGES

Let the flow be confined in a cyclic box of side D and expand the velocity field in Fourier series so that the incompressible Navier–Stokes equation becomes

$$(\partial/\partial t + \nu k^2)u_i(\mathbf{k}) = -ik_m(\delta_{ij} - k_ik_j/k^2) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p})u_m(\mathbf{q}), \qquad (2.1)$$

where ν is the kinematic viscosity. In the limit $D \to \infty$ (necessary for strict isotropy),

$$E(k) = \pi k U(k), \quad U(k) = (D/2\pi)^2\langle|\mathbf{u}(\mathbf{k})|^2\rangle, \qquad (2.2)$$

where $\langle\ \rangle$ denotes ensemble average and the mean kinetic energy per unit mass is $\int_0^\infty E(k)\,dk$. $U(k)$ measures the intensity of excitation per mode. The energy balance equation is

$$\begin{gathered}(\partial/\partial t + 2\nu k^2)E(k) = T(k), \\ T(k) = \frac{1}{2}\int_0^\infty \int_0^\infty T(k, p, q)\,dp\,dq, \\ T(k, p, q) = T(k, q, p),\end{gathered} \qquad (2.3)$$

where

$$\begin{gathered}T(k, p, q) = 2\pi k \operatorname{Im}\{(2\pi/|\sin(\mathbf{p}, \mathbf{q})|)(D/2\pi)^4 \\ \cdot(k_m\,\delta_{ij} + k_j\,\delta_{im})\langle u_i^*(\mathbf{k})u_j(\mathbf{p})u_m(\mathbf{q})\rangle\} \\ (\mathbf{k} = \mathbf{p} + \mathbf{q},\ k = |\mathbf{k}|,\ p = |\mathbf{p}|,\ q = |\mathbf{q}|), \\ T(k, p, q) = 0 \\ \text{(if } k, p, q \text{ cannot form the sides of a triangle).}\end{gathered} \qquad (2.4)$$

To obtain (2.4) from (2.1), use

$$\sum_{\mathbf{p}} \to (D/2\pi)^2 \iint d^2p \qquad (D \to \infty)$$

and

$$\iint d^2p = 2\pi \iint |\sin(\mathbf{p}, \mathbf{q})|^{-1}\,dp\,dq.$$

Detailed conservation of energy and squared vorticity for each triad interaction is expressed by

$$\begin{gathered}T(k, p, q) + T(p, q, k) + T(q, k, p) \equiv 0, \\ k^2T(k, p, q) + p^2T(p, q, k) + q^2T(q, k, p) \equiv 0\end{gathered} \qquad (2.5)$$

which can be verified from (2.4) with the use of incompressibility and plane-triangle identities. The over-all conservation laws

$$\int_0^\infty T(k)\,dk = 0, \qquad \int_0^\infty k^2T(k)\,dk = 0 \qquad (2.6)$$

follows from (2.5). Conversely, (2.6) implies (2.5), since (2.6) must hold for states in which only a single triad of wavevectors have nonzero amplitudes at an instant so that the instantaneous $T(\ )$ vanishes outside the triad. By (2.5),

$$\begin{gathered}T(p, q, k)/T(k, p, q) = (q^2 - k^2)/(p^2 - q^2), \\ T(q, k, p)/T(p, q, k) = (k^2 - p^2)/(q^2 - k^2), \\ T(k, p, q)/T(q, k, p) = (p^2 - q^2)/(k^2 - p^2)\end{gathered} \qquad (2.7)$$

so that only one of the $T(\ ,\ ,\ )$ associated with a given triad interaction is linearly independent.

The mean rate of transfer of kinetic energy per unit mass from wavenumbers below k to those above is $\Pi(k) = \int_k^\infty T(k')\,dk'$. By (2.5),

$$\Pi(k) = \frac{1}{2}\int_k^\infty dk' \int_0^k \int_0^k T(k', p, q)\,dp\,dq - \frac{1}{2}\int_0^k dk' \int_k^\infty \int_k^\infty T(k', p, q)\,dp\,dq. \qquad (2.8)$$

The first term on the right-hand side is the total rate of gain in the range $k' > k$ due to triad interactions with $p, q < k$, while the second term is the total rate of loss in the range $k' < k$ due to triads with $p, q > k$. These two classes of triad interactions are mutually exclusive and exhaust the interactions which contribute to net energy transfer across k. Similarly, the mean rate of transfer of squared vorticity from below k to above k is

$$Z(k) = \int_k^\infty (k')^2\, dk' \int_0^k \int_0^k T(k', p, q)\, dp\, dq - \int_0^k (k')^2\, dk' \int_k^\infty \int_k^\infty T(k', p, q)\, dp\, dq. \quad (2.9)$$

Assume that the double and triple moments at the instant considered satisfy the similarity laws

$$E(ak)/E(k) = a^{-n}, \qquad T(ak, ap, aq)/T(k, p, q) = a^{-(1+3n)/2}, \quad (2.10)$$

where a is an arbitrary scaling factor and n is so far undetermined. The scaling of $T(k, p, q)$ in (2.10) is the same as that of $[E(k)]^{\frac{3}{2}}k^{-\frac{1}{2}}$ (which has the same dimensions) and corresponds to a independence of the appropriately defined triple-correlation coefficients of the distribution of the Fourier amplitudes in the neighborhoods of the wavenumber arguments.

Note that $\int_0^k dp \int_0^k dq$ is equivalent to $2 \int_0^k dp \int_0^p dq$ in the first term on the right-hand side of (2.8) because of the symmetry of $T(k', p, q)$, while $\int_k^\infty dp \int_k^\infty dq$ in the second term is equivalent to $2 \int_k^\infty dp \int_p^\infty dq$. Set $p = k/u$, $q = pv$, $k' = pw$ in the first term and $p = k/u$, $k' = pv$, $q = pw$ in the second term. Note that $\int_1^\infty du \int_u^\infty dw$ is equivalent to $\int_1^\infty dw \int_1^w du$ in the first term and $\int_0^1 du \int_0^u dv$ is equivalent to $\int_0^1 dv \int_v^1 du$ in the second term. Use (2.10) with $a = k/u$, and finally use (2.7) to obtain

$$\Pi(k) = k^{(5-3n)/2} \int_0^1 dv \int_1^\infty dw\, W_1(v, w, n) T(1, v, w), \quad (2.11)$$

where

$$W_1(v, w, n) = -(w^2 - v^2)^{-1}\left[(1 - v^2) \int_1^w u^{(3n-7)/2}\, du - (w^2 - 1) \int_v^1 u^{(3n-7)/2}\, du\right]. \quad (2.12)$$

Repeating the procedure for $Z(k)$ gives

$$Z(k) = 2k^{(9-3n)/2} \int_0^1 dv \int_1^\infty dw\, W_2(v, w, n) T(1, v, w), \quad (2.13)$$

where

$$W_2(v, w, n) = -(w^2 - v^2)^{-1} \cdot\left[(1 - v^2)w^2 \int_1^w u^{(3n-11)/2}\, du - (w^2 - 1)v^2 \int_v^1 u^{(3n-11)/2}\, du\right]. \quad (2.14)$$

Equations (2.11) and (2.13) express $\Pi(k)$ and $Z(k)$ as integrals over contributions from all the possible *shapes* of the triangles formed by k', p, q in (2.8) and (2.9). Since $v < 1$ and $w > 1$, each pair of values v, w corresponds uniquely to a particular triangle shape. By definition, $T(1, v, w)$ is zero if $1, v, w$ cannot form a triangle. The W factors give the weights of the contributions of the different triangle shapes and arise from integration over triangle *size*.

If $n = \frac{5}{3}$, (2.11) says that $\Pi(k)$ has a value ϵ which is independent of k. If $n = 3$, (2.13) gives $Z(k)$ a value η which is independent of k. By (2.12) and (2.14),

$$\begin{aligned} W_1(v, w, \tfrac{5}{3}) &= -(w^2 - v^2)^{-1} \cdot[(1 - v^2) \ln(w) + (w^2 - 1) \ln(v)], \\ W_2(v, w, \tfrac{5}{3}) &= 0, \\ W_1(v, w, 3) &= 0, \\ W_2(v, w, 3) &= -(w^2 - v^2)^{-1} \cdot[(1 - v^2)w^2 \ln(w) + (w^2 - 1)v^2 \ln(v)]. \end{aligned} \quad (2.15)$$

Thus, for each triangle shape individually, an $n = \frac{5}{3}$ similarity range yields a k-independent energy cascade and identically-zero vorticity cascade, while an $n = 3$ similarity range yields a k-independent squared-vorticity cascade and identically-zero energy cascade.

The scaling of $E(k)$ has not been used in obtaining these results. Therefore they hold also for more general similarity ranges in which n is replaced by $n' \neq n$ in the first equation of (2.10).

3. CASCADE DIRECTIONS

$T(1, v, w)$ in (2.11) and (2.13) represents a (signed) flow into the middle wavenumber of the triad $1, v, w$ since $v < 1$ and $w > 1$. It is shown in the Appendix that $W_1(v, w, \frac{5}{3}) > 0$ and $W_2(v, w, 3) < 0$. This means that the contribution of each triangle shape to ϵ in the $-\frac{5}{3}$ range has the same sign as the flow of excitation into the middle wavenumber, while the contribution to η in the -3 range has the opposite sign. There is nothing in the conservation properties by themselves to determine the sign of $T(1, v, w)$. Indeed, if a similarity range with a given sign of $T(1, v, w)$ exists at an instant, then a range with the opposite sign is produced by reversing the velocity everywhere in space.

Physical interest attaches not to hypothetical instantaneous similarity ranges but to the possibility of quasisteady ranges which develop under the dynamical equations. In three dimensions, a hint as to the direction of cascade in the $-\frac{5}{3}$ range comes from considering the absolute statistical equilibrium which would obtain if viscosity were zero and the system were truncated by removing all degrees of freedom with k greater than some cutoff wavenumber $k_{\max}$ from the dynamical equations. The total kinetic energy per unit mass is $\frac{1}{2}\sum_{\mathbf{k}} |\mathbf{u}(\mathbf{k})|^2$, and consequently the energy spectrum in the hypothetical absolute equilibrium would have the equipartition form $U(k) = \text{const}$, or $E(k) \propto k^2$. The $-\frac{5}{3}$ spectrum means that higher wavenumbers in the inertial range are far below absolute equilibrium with lower wavenumbers and it is plausible that the dynamical interaction should act toward producing equilibrium, a state which never can be reached because the viscous dissipation provides a high-k sink.

In two dimensions, the absolute equilibrium has a more complicated structure because there are two linearly independent quadratic constants of motion. The general form of the equilibrium spectrum is

$$U(k) = 1/(\beta k^2 + \alpha), \tag{3.1}$$

where β and α are constants. This is an equipartition distribution[12] for the constant of motion

$$\sum_{\mathbf{k}} (\beta k^2 + \alpha)\,|\mathbf{u}(\mathbf{k})|^2.$$

The corresponding vorticity spectrum $2\pi k^3 U(k)$ increases monotonically with k so that most of the vorticity in equilibrium is at wavenumbers $\sim k_{\max}$. Since $k_{\max}$ can be arbitrarily high, this suggests that a tendency toward equilibrium in an actual physical flow should involve an upward flow of vorticity and therefore, by the conservation laws, a downward flow of energy. Thus if the nonlinear interaction does act toward producing equilibrium, $T(1, v, w)$ should be typically negative. A simpler and cruder statement is that $T(1, v, w)$ should be negative because that represents a statistically plausible spreading of the excitation in wavenumber: out of the middle wavenumber into the extremes.

Supporting evidence is provided by the initial growth of energy transfer in turbulence whose initial statistical distribution is Gaussian. The exact expression for this in two dimensions has been obtained by Reid[8] and Ogura.[9] It is

$$[dT(k, p, q)/dt]_0 = 2\pi^2 k^2\, d_{kpq}[2a_{kpq}U(p)U(q) - b_{kpq}U(q)U(k) - b_{kqp}U(k)U(p)]. \tag{3.2}$$

[12] R. C. Tolman, *Statistical Mechanics* (Oxford University Press, New York, 1938), p. 95.

Here

$$\begin{aligned} b_{kpq} &= 2pk^{-1}(xy - z + 2z^3), \\ 2a_{kpq} &= b_{kpq} + b_{kqp}, \\ d_{kpq} &= k/(1 - x^2)^{\frac{1}{2}} \\ &= \text{diameter of circumscribed circle,} \end{aligned} \tag{3.3}$$

where x, y, z are the interior angles opposite the triangle sides k, p, q. The coefficients obey the identities

$$\begin{aligned} &a_{kpq} \geq 0, && d_{kpq} = d_{pqk} = d_{qkp}, \\ &k^2 b_{kpq} = p^2 b_{pkq}, && 2k^2 a_{kpq} = p^2 b_{kpq} + q^2 b_{kpq}, \end{aligned} \tag{3.4}$$

whence

$$\begin{aligned} b_{kpq}/a_{kpq} &= 2(q^2 - k^2)/(q^2 - p^2), \\ b_{kqp}/a_{kpq} &= 2(p^2 - k^2)/(p^2 - q^2). \end{aligned} \tag{3.5}$$

Now suppose

$$U(p) = (p/k)^{-r}U(k), \quad U(q) = (q/k)^{-r}U(k). \tag{3.6}$$

Equations (3.2)–(3.5) yield

$$\begin{aligned} [dT(k, p, q)/dt]_0 = 2\pi^2 k^2\, d_{kpq}a_{kpq}(pq/k^2)^{-r}[U(k)]^2 \\ \cdot\{1 - v^r(w^2 - 1)/(w^2 - v^2) \\ - w^r(1 - v^2)/(w^2 - v^2)\}, \end{aligned} \tag{3.7}$$

where $v = p/k$, $w = q/k$. It is shown in the Appendix that when $v < 1$ and $w > 1$ the curly bracket in (3.7) is >0 if $0 < r < 2$ and <0 if $r < 0$ or $r > 2$. Thus the initial growth of $T(k, p, q)$ gives a positive flow into the middle wavenumber k if r falls between the limits (0, 2) set by the extreme absolute equilibrium distributions $\beta = 0$ and $\alpha = 0$ in (3.1). If r is outside these limits, there is net flow out of k. The $-\frac{5}{3}$ and -3 similarity ranges correspond to $r = \frac{8}{3}$ and $r = 4$, both of which yield net outflow.

4. CONJECTURES ON QUASI-STEADY STATES

Are the formal $-\frac{5}{3}$ and -3 similarity ranges asymptotic limits of states which can arise physically? Suppose that an infinite fluid is excited by isotropic stirring forces confined to $k \sim k_i$, where k_i is a characteristic input wavenumber. Let the stirring forces supply energy at a steady rate ϵ and squared vorticity at a steady rate $\eta \sim 2k_i^2\epsilon$. More general ratios η/ϵ are interesting but will not be considered here. The preceding analysis suggests that if the input continues for a sufficiently long time and the Reynolds number $[E(k_i)/k_i]^{\frac{1}{2}}/\nu$ is large enough, a quasi-steady state may be set up in which an approximate -3 vorticity-transfer range carries most of the squared-vorticity input up to $k \gg k_i$, where it is dissipated by viscosity, while an approximate $-\frac{5}{3}$ energy-transfer range carries most of the energy input down toward zero wavenumber. The $-\frac{5}{3}$ range can be only quasi-steady because its lower end keeps moving down to ever-lower wavenumbers, a wavenumber $k \ll k_i$ being reached in a time $t \sim (\epsilon k^2)^{-\frac{1}{3}}$ according to energy conservation. As $t \to \infty$, the rate of transfer of squared vorticity from $k \sim k_i$ to lower wavenumbers decreases steadily toward zero and the energy-transfer range approaches the asymptotic $-\frac{5}{3}$ dynamics ever more closely.

The formal similarity ranges can represent asymptotic quasi-steady states only if the cascades are sufficiently local in wavenumber. The questions involved here are the same in two and three dimensions. Local transfer in the $-\frac{5}{3}$ range is plausible for the reasons given by Kolmogorov. The transfer is associated with the distortion of the velocity field by its own shear. The cascade $\Pi(k)$ through a given wavenumber k in a $-\frac{5}{3}$ range is expected to be negligibly affected by wavenumbers $\ll k$ because the integral $\int_0^\infty k^2 E(k)\, dk$, which measures the mean-square shear, converges at $k = 0$. It is expected to be negligibly affected by wavenumbers $\gg k$ because the vorticity associated with those wavenumbers fluctuates rapidly in space and time and gives an effective shear across distances of order k^{-1} which is small compared to the shear associated with the wavenumbers $\sim k$.

On the other hand, the squared-vorticity spectrum in the -3 range is $\propto k^{-1}$ so that each octave below a given wavenumber k contributes the same amount to the mean-square shear and the latter diverges logarithmically toward small k. This means that transfer in the -3 range is not local when all triad interactions are admitted. It would seem plausible that when this range occurs as a quasi-steady state the power law is modified by logarithmic corrections. Subject to such corrections, the -3 range can be expected to extend up to $k \sim k_d = (\eta/\nu^3)^{\frac{1}{6}}$, at which wavenumber the integrated rate of dissipation of squared vorticity by viscosity reaches the order of η. At higher wavenumbers, $E(k)$ is expected to fall off at a faster-than-algebraic rate.

The corresponding viscous dissipation of kinetic energy is $\epsilon_d \sim \eta/k_d^2$, which implies $\epsilon_d \sim \epsilon(k_i/k_d)^2$. Thus $\epsilon_d/\epsilon \to 0$ if $\nu \to 0$, in marked contrast to three-dimensional turbulence where the energy loss becomes independent of ν as $\nu \to 0$. These considerations imply that the entire energy input ϵ is carried down toward $k = 0$ without viscous loss in the limit of infinite Reynolds number. The result is not directly applicable to meteorological flows because the constraints which render the latter two-dimensional break down at sufficiently high k.

The strict $-\frac{5}{3}$ asymptotic inertial-range law is not established beyond doubt in three-dimensional flow, and the arguments for similarity ranges in two dimensions are substantially less secure. The present paper has demonstrated some elementary consistency properties, but this does not show that the similarity ranges actually exist.

One important difference between two and three dimensions is the existence of an infinite number of local inviscid constants of motion in the former: the vorticity of each fluid element. This implies that inertial forces alone cannot produce universal statistical distributions in the similarity ranges, independent of the statistical distribution of the driving forces. In three dimensions there are also an infinite number of inviscid constants of motion: the circulations about all closed curves moving with the fluid. However a given closed curve is expected to stretch and migrate in complicated fashion through the fluid with the passage of time so that it is reasonable to expect that the circulation invariance does not impose effective constraints on n-variate distribution functions for small n. If (1.1) and (1.2) are realized in two dimensions (the latter corrected by a logarithmic-type function of k/k_i), it is to be expected that C and C' are not universal constants but depend on the character of the driving forces, whatever may be the situation in three dimensions. A further point is that the nonlocalness of transfer in the -3 range suggests in itself that cascade there is not accompanied by degradation of higher statistics in the fashion usually assumed in a three-dimensional Kolmogorov cascade. This is consistent with a picture of the transfer process as a clumping-together and coalescence of similarly signed vortices with the high-wavenumber excitation confined principally to thin and infrequent shear layers attached to the ever-larger eddies thus formed.[1,5]

In connection with the sign of ϵ in the $-\frac{5}{3}$ range, it should be noted that a positive-ϵ range extending from the input wavenumbers up to the dissipation range would be physically unrealizable. The viscous dissipation would remove squared vorticity and kinetic energy in a ratio η_d/ϵ_d which would greatly exceed the ratio at which these quantities were cascaded, since the latter ratio goes to zero as the asymptotic $-\frac{5}{3}$ structure is approached. Thus an upward-transfering $-\frac{5}{3}$ range could not exist as quasi-steady state. The inconsistency of such a range has previously been demonstrated by Lee.[2]

Suppose now that the fluid is confined to a finite domain and that the lowest wavenumber allowed by the boundary conditions is $k_0 \ll k_i$. The conjecture is offered here that after the $-\frac{5}{3}$ range reaches down to wavenumbers $\sim k_0$ the downward cascade from k_i continues and the energy delivered to the bottom of the range piles up in the mode k_0. As the energy in k_0 rises sufficiently, modification of the $-\frac{5}{3}$ range toward absolute equilibrium is expected, starting at the bottom and working up to progressively larger wavenumbers.

Some support for the idea of energy piling up in k_0 comes from considering the absolute equilibrium ensembles (3.1). Suppose that the wavenumber range is truncated from below at k_0 and from above at a wavenumber $k_{\max}$. Let the mean energy and mean-square vorticity have specified values E and $\Omega = 2k_1^2E$. The ratio and signs of β and α in (3.1) depend on the relative values of k_0, k_1, and $k_{\max}$. In particular,

$$\alpha = 0 \quad \text{if} \quad k_1^2 = \tfrac{1}{2}(k_{\max}^2 - k_0^2)/\ln(k_{\max}/k_0), \tag{4.1}$$

$$\beta = 0 \quad \text{if} \quad k_1^2 = \tfrac{1}{2}(k_{\max}^2 + k_0^2). \tag{4.2}$$

For values of k_1^2 between (4.1) and (4.2), $\alpha > 0$, $\beta > 0$. For k_1^2 less than (4.1), $\beta > 0$, $-\beta k_0^2 < \alpha < 0$. If $k_1 - k_0 \ll k_0$, then $\beta k_0^2 + \alpha \ll \beta k_0^2$ and $E(k)$ has a sharp peak at $k = k_0$. For k_1^2 greater than (4.2), $\alpha > 0$, $-\alpha/k_{\max}^2 < \beta < 0$, and $k^2E(k)$ shows a sharp peak at $k = k_{\max}$ if $k_{\max} - k_1 \ll k_{\max}$. The values $k_1 < k_0$ and $k_1 > k_{\max}$ are impossible. These results are all for a continuous spectrum of allowed wavenumbers. When the discreteness associated with a finite fluid is taken into account, the sharp peak in $E(k)$ is modified so that the lowest mode k_0 singlehandedly carries most of the total kinetic energy if $k_1 - k_0$ is much less than the mode separation Δk of the low-lying degrees of freedom.

These results suggest that a piling up of energy in k_0 under a steady input would represent a plausible way for the wavenumbers $<k_i$ to seek an absolute statistical equilibrium of the kind that corresponds to very large E/Ω. The phenomenon is analogous to the Einstein–Bose condensation of a finite two-dimensional quantum gas.

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APPENDIX. PROOFS OF INEQUALITIES

Take $w > 1$, $0 < v < 1$ throughout this Appendix. Let

$$I(v, w) = (1 - v^2)\ln(w) + (w^2 - 1)\ln(v).$$

Then $I(1, w) = 0$, $\partial I(v, w)/\partial v = (w^2 - 1)/v - 2v\ln(w)$, and

$$[\partial I(v, w)/\partial v]_{v=1} = (w^2 - 1) - 2\ln(w) \equiv F(w).$$

Now $F(1) = 0$, $dF(w)/dw = 2w - 2/w > 0$. Therefore $w^2 - 1 > 2\ln(w)$ and, since $v^{-1} > v$, it follows that $\partial I(v, w)/\partial v > 0$, so that $I(v, w) < 0$. This establishes $W_1(v, w, \frac{5}{3}) > 0$.

Let

$$J(v, w) = (1 - v^2)w^2\ln(w) + (w^2 - 1)v^2\ln(v).$$

Then $J(1, w) = 0$,

$$\partial J(v, w)/\partial v = (w^2 - 1)v[1 + 2\ln(v)] - 2vw^2\ln(w).$$

Now $2w^2\ln(w) > w^2 - 1$ and $v > v + 2v\ln(v)$, the first inequality readily following upon differentiation and the second following from $\ln(v) < 0$. Therefore, $\partial J(v, w)/\partial v < 0$, $J(v, w) > 0$, whence $W_2(v, w, 3) < 0$.

Let

$$K(v, w, r) = 1 - v^r(w^2 - 1)/(w^2 - v^2) - w^r(1 - v^2)/(w^2 - v^2).$$

Then

$$\partial K(v, w, r)/\partial r = [v^r(w^2 - 1)\ln(v^{-1}) - w^r(1 - v^2)\ln(w)]/(w^2 - v^2),$$

considered as a function of r, is the sum of a positive term of monotonically decreasing magnitude and a negative term of monotonically increasing magnitude. Therefore it has just one zero, and it is positive as $r \to -\infty$, negative as $r \to +\infty$. Since $K(v, w, r)$ has zeroes at $r = 0$, $r = 2$, it follows that $K(v, w, r)$ is > 0 for $0 < r < 2$ and < 0 for $r < 0$ and $r > 2$. This establishes the sign of the right-hand side of (3.7).