ELSEVIER

Physics Reports 362 (2002) 1–62

PHYSICS REPORTS

www.elsevier.com/locate/physrep

Two-dimensional turbulence: a physicist approach

Patrick Tabeling

Laboratoire de Physique Statistique, Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris, France

Received June 2001; editor: I. Procaccia

Contents

Abstract

Much progress has been made on two-dimensional turbulence, these last two decades, but still, a number of fundamental questions remain unanswered. The objective of the present review is to collect and organize the available information on the subject, emphasizing on aspects accessible to experiment, and outlining contributions made on simple flow configurations. Whenever possible, open questions are made explicit. Various subjects are presented: coherent structures, statistical theories, inverse cascade of energy, condensed states, Richardson law, direct cascade of enstrophy, and the inter-play between cascades and coherent structures. The review offers a physicist's view on two-dimensional turbulence in the sense that experimental facts play an important role in the presentation, technical issues are described without much detail, sometimes in an oversimplified form, and physical arguments are given whenever possible. I hope this bias does not jeopardize the interest of the presentation for whoever wishes to visit the fascinating world of Flatland. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 47.27.Ak; 47.27.Eq; 05.45.−a

PII: S0370-1573(01)00064-3

1. Introduction

Several motivations or prospects may lead to get involved in studies on two-dimensional turbulence. One motivation, probably shared by all the investigators, is that hydrodynamic turbulence, despite its practical interest, its omnipresence, and the series of assaults it has been submitted to for decades, is still with us as the most important unsolved problem of classical mechanics. 'Unsolved' means that, for the simplest cases we may conceive, we are unable to infer the statistical characteristics of a turbulent system from first principles. It is not a matter of being unable to perform a detailed calculation. The structure of the solution is at the moment beyond the reach of existing theoretical approaches. Inspiration has not lacked for devising clever theoretical methods, but for long, success did not come and in this respect, hydrodynamic turbulence looks as a cemetery for a number of ideas that turned out to be outrageously successful in other areas of condensed matter physics. At the moment, the situation perhaps evolves, further to recent breakthroughs made on the turbulent dispersion problem. In particular, we now have, for the passive dispersion problem, a complete knowledge of turbulent solutions for particular (idealized) cases. An issue is to what extent these ideas may be used to tackle the hydrodynamic turbulent problem. The least one can say is that the issue is open at the moment.

Another motivation is related to geophysical flows. Flows in the atmosphere and in the ocean develop in thin rotating stratified layers, and it is known that rotation, stratification and confinement are efficient vectors conveying two-dimensionality. Experimental devices developed in laboratories typically use these ingredients to prepare acceptably two-dimensional flows. The extent to which two-dimensional turbulence represents atmospheric or oceanographic turbulence is however not straightforward to circumvent, and this question has been discussed in several places [77,29,118]. In the simplest cases, pure two-dimensional equations must be amended by the addition of an extra-term, characterizing the effect of Coriolis forces, to represent physically relevent situations. This term generates waves which radiate energy, a mechanism absent in pure two-dimensional systems. In more realistic cases, topography, thermodynamics, stratification, must be incorporated in the analysis, and indeed full two-dimensional approximation hardly encompasses the variety of phenomena generated by these additional terms. Nonetheless, purely two-dimensional flows provide useful guidance to understand some important aspects of geophysical flows. A remarkable (although rather isolated) example is the trajectories of the tropical hurricanes. For long, two-dimensional pointwise vorticity models have been used, leading to fairly acceptable predictions. In the oceans, the concept of coherent structures, discovered in early studies of two-dimensional turbulent flows, is frequently substantiated. A context where concepts of two-dimensional turbulence appear as particularly relevant is the dispersion of chemical species in the atmosphere, and tracers in the oceans. The structure of the ozone concentration field around the polar vortex can be accounted for if filamentary processes, similar to those prevailing in two-dimensional turbulent flows, are assumed to take place. For this particular aspect, evoking the presence of waves do not provide clues while two-dimensional turbulence offers an interesting framework, allowing to construct acceptable representations [71].

A third motivation is linked to the three-dimensional turbulence problem. The recent period has shown the existence of a common conceptual framework between two- and three-dimensional turbulence. Phenomena, such as cascades, coherent structures, dissipative processes, filamentation mechanisms take place in both systems. On the other hand, turbulence is simpler to represent and easier to compute in two than in three dimensions and physical experiments provide much more information when the flow is thoroughly visualized that when it is probed with a single sensor. Therefore, operating in Flatland may be instructive to test ideas or theories, prior to considering them in the three-dimensional world. Note that three-dimensional turbulence is not three dimensional in all respects; an example is the worms, which populate the dissipative range: they are long tubes and thus can be tentatively viewed as to two-dimensional objects. One may also add that two-dimensional turbulence is helpful for understanding turbulence in general. Turbulence is a general phenomenon, involving nonlinear dynamics and broad range of scales, including as particular cases two- and three-dimensional fluid turbulence, but also superfluid turbulence. Nonlinear Shrödinger turbulence, scalar turbulence of any sort, i.e. passive or active, low or high Prandtl numbers, 'burgulence' (i.e. the particular turbulence produced in Burgers equation), to mention but a few. Across these subjects, a common conceptual framework exists, and two-dimensional turbulence can obviously be used to understand several aspects of the general problem.

If we trace back the history, it seems that it took a long time to realize that flows can be turbulent in two dimensions. For long, two-dimensional turbulence was thought impossible, frozen by too many conservation laws. Batchelor, in his influential textbook on turbulence [8], overlooked the possibility that turbulent cascades may exist in 2D. In the presentations of turbulence, it was not uncommon to stress that two-dimensional flows are inhospitable to turbulent regimes.

The situation changed in the sixties, further to two contributions, given independantly by Kraichnan and Batchelor. Batchelor reconsidered his views on two-dimensional turbulence, and proposed that cascades may develop in the plane, in a way analogous to the direct energy cascade in three dimensions, with the enstrophy playing the role of the energy [9]. Kraichnan [67] discovered the double cascade process and explained that turbulence may be compatible with the presence of several conservation laws. These contributions stimulated research in the field, by suggesting that something can be learnt on turbulence in general from investigations led on two-dimensional systems.

Still, the development of the field was considered as difficult, since it was generally considered that turbulent two-dimensional flows could not be achieved in the laboratory, and it was uncertain that the development of the computers could provide, within a reasonable range of time, reliable and accurate data on such systems. Now the situation has drastically changed, since on both of these aspects, we witnessed considerable developments. Numerics exploded in the seventies, and succeeded to produce a considerable flux of reliable data. Several phenomena were discovered, among them the coherent structures, which deeply modified our view on turbulence. For long the available data in Flatland was computationally based; this is no more true today. The experimental effort, starting in the seventies with pioneering experiments, could be pushed forward, thanks to the availability of image processing techniques. In some cases, the physical experiment could offer conclusions difficult to draw out from numerics.

We now are in a situation where the field of two-dimensional turbulence has reached a sort of maturity, in the sense that numerical and experimental techniques are in position to convey valuable and relevant information concerning most of the questions we may ask on the problem. This is not true as soon as complications, such as wave propagation, or topographic effects are added. A source of progress is certainly expected from the theory, which at the moment, is unable to answer the simplest questions. In several cases, the theory seems to be stuck, like a bee in a honey pot, and conceptual breakthroughs appear as the only chance to move forward. Specific issues such as the role of coherent structures in turbulent systems, the intermittency effects in the inverse cancade, the ergodic issue in free systems, are example of problems which seem to raise considerable difficulties.

In this context, it seemed to me that it could be interesting to review the knowledge accumulated during the last decades on the subject. At the moment, the results are spread in various papers and for those who wish to get in touch with the field, it would take time to collect the most relevant information. Two-dimensional turbulence has already been reviewed by several authors. Kraichnan and Montgomery [69] emphasized on statistical theories, giving a complete account of the theoretical efforts made until 1980. A more recent review on the subject, written in a pleasant pedagogical form, is given by Miller et al. [89]. Lesieur [77] reviewed different subjects, emphasizing on spectral descriptions. Frisch [43] wrote a swift review, which emphasizes many of the far-reaching issues still open in the field. A short account on recent progress in two-dimensional turbulence has been given by Nelkin [94]. The presentation I am proposing here can be viewed as an extension and an update of Frisch's review. Some material was unavailable at that time, at theoretical, numerical and physical levels and I hope the present review will bring useful complements.

As a physicist, I will put emphasis on the experiment, and on the theoretical efforts dedicated to explaining phenomena within the reach of the observation. This certainly tends to confine the view on the subject. Two-dimensional turbulence is a field of investigation for theorists, who need to develop their own approach without being necessarily constrained by the experimental facts. I decided to warn the reader of this personal bias by making an announcement at the first line of the paper, i.e. in the title. I will not make any description of closure models such as EDQNM, or test field model, despite their practical interest. The reader will find a presentation of these aspects in [77]. Also cascade models, such as shell model, will not be discussed. I hope these biases and limitations do not outrageously reduce the scope of the review, and that the present paper will be of interest for a broad range of scientists interested in turbulence, for whatever reason.

2. Exact results

There exist several exact results, important to know for understanding two-dimensional turbulent flows. They are quoted in papers by Batchelor [9], reviews by Rose and Sulem [121], Kraichnan and Montgomery [69], or books by Lesieur [77], and Frisch [43]. One starts with the Navier–Stokes equations which read:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla\mathbf{p} + \mathbf{f}_{\mathrm{ext}} + \nu\Delta\mathbf{u} \;, \tag{1}$$

$$\nabla \cdot \mathbf{u} = \mathbf{0} \tag{2}$$

in which $\mathbf{u}$ is the velocity, $\mathbf{p}$ the pressure, $\mathbf{f}_{\mathrm{ext}}$ is the external forcing, and ν is the kinematic viscosity. One usually takes a system of coordinates (x, y) for the plane of flow and z for the coordinate normal to the plane.

In the inviscid case ($\nu = 0$), and without forcing ($\mathbf{f}_{\text{ext}} = 0$), these equations are the Euler equations; in two dimensions, it has been possible to prove that the Euler equations are regular in a bounded domain, and, if the vorticity is initially bounded, the solution exists and is unique (see [121]). This implies in particular that there are no singularities at finite time for the velocity, a possibility which, at the moment, is not ruled out in three dimensions.

By taking the curl of these equations, one gets an equation for the vorticity $\omega = (\nabla \mathbf{x} \mathbf{u})_{\mathbf{z}}$ in the form

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = \frac{\partial \omega}{\partial t} + \mathbf{u}\nabla\omega = \mathbf{g} + \nu\Delta\omega \; , \tag{3}$$

where $\mathbf{g}$ is the projection of the curl of the forcing, along the axis normal to the plane. Let us stress that this equation holds in two dimensions, and not in three. In three dimensions, the governing equation for the vorticity has a different form: an additional term, called vortex stretching, must be added. Sticking to the two-dimensional case, one finds that, in the inviscid limit ($\nu = 0$), and with a curl free forcing ($\mathbf{g} = 0$), the vorticity is conserved along the fluid particle trajectories, a result known as Helmhotz theorem. In two-dimensional inviscid flows, the fluid particles keep their vorticity for ever. In three-dimensional turbulence, it is not so: the additional vortex stretching term in the vorticity equation amplifies, in the average, vorticity along the trajectories, leading to the formation of small intense filaments. Freely evolving two-dimensional turbulence cannot generate small intense vortices. This difference, between two and three dimensions, is fundamental to underline.

A quantity of fundamental interest in turbulence is the kinetic energy per unit of mass, defined by

$$E = \tfrac{1}{2}\langle \mathbf{u}^2 \rangle \; , \tag{4}$$

where the brackets mean statistical averaging, i.e. the average of a large number of independant realizations. It is usually considered that in homogeneous or spatially periodic systems, taking spatial averages is equivalent to taking statistical average. This may not be always true, but we do not address this issue in this section. In 2D, and in homogeneous or periodic systems, the evolution equation for the energy reads:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}\mathbf{u}^2}{\mathrm{d}t} = -\,\nu Z \; , \tag{5}$$

where $Z = \langle \omega^2 \rangle$ is the enstrophy. This quantity is governed by the following equation (still in homogeneous or periodic systems):

$$\frac{\mathrm{D}Z}{\mathrm{D}t} = -\,\nu\langle (\nabla\omega)^2 \rangle \; . \tag{6}$$

The right-hand term involves vorticity gradients. In two-dimensional turbulence, vorticity patches, distorted by the background velocity, generate thin filaments, and thus work at enhancing the vorticity gradients. These gradients increase up to a level where viscosity come into play to smooth out the field. A stationary state may be reached, such that the dissipation $\nu\langle (\nabla\omega)^2 \rangle$ becomes a constant, independent of the viscosity. This defines a ‘dissipation anomaly’, similar to the three-dimensional case [65,95], but dealing with the enstrophy, instead of the energy. The concept of dissipation anomaly is fundamental in turbulence, and it turns out to operate both in two and three dimensions. In two dimensions, the enstrophy is forced to decrease with time, while in three-dimensional turbulence, it vigorously increases with time as long as viscous effects remain unimportant. Using the energy equation, one infers that in two-dimensional homogeneous turbulence, the energy is essentially conserved as viscosity tends to zero (from the positive side). This again contrasts with three-dimensional turbulence, where, according to Kolmogorov law, the energy decreases at a constant rate independant of viscosity. This difference between the decay properties of the different integrals led Kraichnan and Montgomery [69] to introduce a distinction between *robust* (such as the energy) and *fragile* (like the enstrophy) invariants.

One consequence (probably the most striking) of this discussion is that two-dimensional systems are unable to dissipate energy at small scales. Said differently, there is no viscous sink of energy at small scale in 2D turbulence. As a consequence, there cannot be a direct energy cascade in two dimensions. Where does the energy go? We will see later that it goes to large scales, and eventually gets dissipated by friction on walls in contact with the fluid. The situation is different for the enstrophy. Enstrophy can dissipate at small scales as the viscosity tends to zero, generating the dissipation anomaly mentioned above. Therefore, there can be a direct enstrophy cascade. According to these remarks, it is often considered that an analogy can be made between two and three dimensions if one treats the enstrophy as the energy, and the vorticity of the velocity. This analogy is helpful for a quick inspection of the phenomena; it is nonetheless misleading in many respects, since, as we will see, the energy and enstrophy cascades have different qualitative properties.

In the past, it seemed to be an impression that, on considering the inviscid limit, it could be argued that two-dimensional turbulence cannot exist. The argument was based on the remark that in the inviscid case, two-dimensional flows are constrained by an infinity of invariants. Quantities such as

$$Z_n = \langle \omega^n \rangle \tag{7}$$

are conserved for any n, in statistically homogeneous systems. There is also an infinite number of invariants in three dimensions, since, according to Kelvin's theorem, the circulation along any closed loop is conserved. However, since the loops adopt increasingly intricate shapes with time, it may be suggested that such invariants will not yield effective constraints [100]. In two dimensions, there are many more conservation laws, which moreover can be expressed as constraints along fixed boundaries. It is thus legitimate to ask whether, in the huge phase spaces representing two-dimensional flows, there remain enough degrees of freedom to sustain turbulent regimes. In the labyrinth of constraints, is it certain that even These could find a pathway leading to two-dimensional turbulence? The answer is known: the study of (inviscid) point vortex systems shows that the number of degrees of freedom linearly increases with the number of vortices; therefore, from the viewpoint of the constraints, chaotic and turbulent regimes can easily develop in such systems, provided the number of vortices is large. Moreover, in a number of situations, viscosity destroys almost all the invariants, releasing most of the constraints the system would be subjected to if the flow were inviscid. The main issues raised by the existence of an infinite number of invariants deal with the cascades. Are cascades compatible with the abundance of invariants in two dimensions? Can two-dimensional systems be turbulent in the same way as in three dimensions? We will see later that the answer to this question is yes, i.e. cascades develop in two dimensions, so that, from this respect, there is no point contrasting two- and three-dimensional systems.

An interesting limiting case of two-dimensional systems (already referred to) is when the vorticity is point-wise. This approximation is extremely helpful for understanding the dynamics of two-dimensional systems, turbulent or not, with coherent structures. Pointwise approximation has been shown to be acceptable, for some range of time, in a number of situations. The equations governing systems of point vortices can be found in text-books, and their dynamics has been remarkably discussed by Aref [3]. Many situations can be solved exactly. From the equations, one can prove that dipoles propagate, pairs of like sign vortices undergo rotational motion, three vortices systems are solvable, and chaos may appear as soon as the number of point vortices is larger than or equal to four. These results are useful for analyzing the dynamics of coherent structures in turbulent systems. The case of many vortices may receive a rigorous statistical treatment, using the tools of statistical mechanics, a topic we will return to later.

There exist also interesting exact results, obtained for forced turbulence, with prescribed forcings, and possessing virtues of isotropy and homogeneity. The first one corresponds to the case where turbulence is forced at large scales compared to the scales we are considering. It reads

$$D_{\mathrm{L}}(r) - 2\nu\frac{\mathrm{d}S_2}{\mathrm{d}r} = -\frac{4}{3}\eta r \ , \tag{8}$$

where D_{L} is the longitudinal vorticity correlation, defined by

$$D_{\mathrm{L}}(r) = \langle (u_{\mathrm{L}}(x) - u_{\mathrm{L}}(x+r))(\omega(\mathbf{x}) - \omega(\mathbf{x}+\mathbf{r}))^2 \rangle \tag{9}$$

in which u_{L} represents the velocity component along the separation vector $\mathbf{r}$; $S_2(r)$ is the structure function of order two defined by

$$S_2(r) = \langle (\omega(x) - \omega(x+r))^2 \rangle \tag{10}$$

and η is the enstrophy dissipation rate, defined by

$$\eta = \nu \langle (\nabla\omega)^2 \rangle \ . \tag{11}$$

It is the analog of Corrsin–Yaglom [90] relation for passive tracers.

Another relation corresponds to the inverse situation, i.e. the forcing holds on a scale much smaller than those we address. In this case, one can derive an equation constraining the third order structure function [90]. This equation reads:

$$F_3(r) = \tfrac{3}{2}\varepsilon \ . \tag{12}$$

where $S_3(r)$ is the longitudinal third order structure function, defined by

$$F_3(r) = \langle (u_{\mathrm{L}}(x) - u_{\mathrm{L}}(x+r))^3 \rangle \tag{13}$$

and ε is the dissipation, defined by

$$\varepsilon = \nu \langle (\omega)^2 \rangle \ . \tag{14}$$

This is the two-dimensional analog of the Kolmogorov relation [90]. Compared to the three-dimensional case, there is a change in the prefactor ($\frac{3}{2}$ instead of $-\frac{4}{5}$ [103]); more importantly this prefactor is positive, indicating that in two dimensions, the energy transfers proceed, in the average, from small to large scales. It turns out that the above relations (the one on enstrophy fluxes, and the other on energy transfers), have played a limited role in our understanding of turbulence; perhaps their deficiency is to provide little information on the system when coherent structures (we will come back to this notion in the next section) are present, i.e. in situations which focused most of the research efforts along the years [96].

3. Coherent structures

Two-dimensional turbulence has provided a remarkable context for the study of 'coherent structures' and the interplay with the classical cascade theories. In the two-dimensional context, the very first observations of coherent structures, inferred from direct inspections of numerically computed vorticity fields, traces back to the seventies. In two-dimensional jets, Aref and Siggia [2] noticed that remarkable vortical structures survived for long periods of time, in conditions where turbulence can be considered as fully developed (see Fig. 1). These numerical observations, appearing after the Brown and Roshko [17] experiment, shed new light on the intimate structure of turbulence, and one may say that today, research in turbulence is still challenged by these early observations. Later, in a turbulent wake produced in a soap film. Couder [23] observed structures — vortices and dipoles — keeping their coherence for long periods of time, and apparently forming *the* basic elements of the wake (see Fig. 2). These observations were largely confirmed by the subsequent numerical and physical studies carried out on the subject [25,85]. McWilliams gave evidence of the formation of vortices, whose lifetimes are hundred times their internal enstrophy, a period exceeding by far any expectation based on traditional ideas of turbulence. Legras et al. [72] showed several long-lived structures, including a superb long-lived tripole in a forced two dimensional turbulent system (see Figs. 3 and 4).

Coherent structures can often be visualized directly on the vorticity field, as long-lived objects of generally circular topology, with a simple structure, 'clearly' distinguishable from the background within which they evolve. Coherent structure is nonetheless a loose concept and attempts have been made to provide sharp definitions, useful for their systematic identification prior to the characterization of their properties. There are several issues raised in proposing a

Fig. 1. Simulation of a two-dimensional jet showing the formation of long-lived coherent structures.

Fig. 2. Two-dimensional Von Karman street, formed behind a cylinder dragged in a soap film.

Fig. 3. Typical vorticity field of a free-decay experiment, obtained several vortex turn-around times after the decay process starts.

Fig. 4. Tripole obtained in a simulation of freely decaying turbulence.

definition of coherent structures. For single vortices, it may be difficult to assume the internal profile be smooth since in some situations — during vortex merging for instance — vortices develop a complex intricate structure and their profile can be extremely corrugated, especially at high Reynolds numbers. Proposing a vorticity threshold criterium for identifying structures may also raise difficulties at high Reynolds numbers, since Helmoltz theorem stipulates the fluid particles keep their vorticity level wherever they move.

McWilliams analyzed these issues and proposed an empirical procedure of identification of the vortices [86]. The vortex census proposed by him includes several steps:

1. *identify extremas* (using a threshold criterion),
2. *identify interior and boundary regions* (defining a threshold for the boundary and check it has a circular topology),
3. *eliminate redundancy* (checking the structures are separate),
4. *test vortex shape* (there should be no strong departure from axisymmetry).

Another way of identifying structures is based on the use of Weiss's criterion. This criterion is discussed in detail by Basdevant et al. [6]. Weiss criterion is based on the remark that the evolution equation for the vorticity gradient satisfies the following equation:

$$\frac{\mathrm{d}\nabla\omega}{\mathrm{d}t} + A\nabla\omega = 0 \tag{15}$$

in which A is the velocity gradient tensor. The eigen values λ associated to A are calculated as the roots of the following equation:

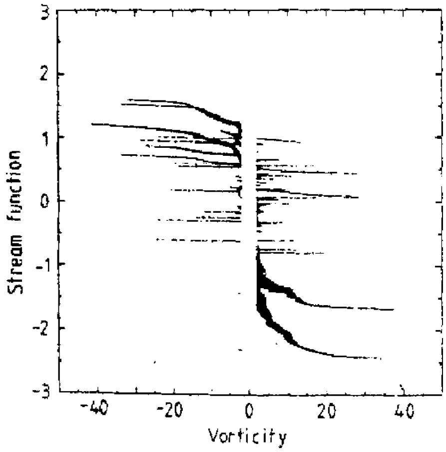
$$\lambda^2 = \tfrac{1}{4}(\sigma^2 - \omega^2) = 4Q , \tag{16}$$

where σ is the strain. Then assuming the velocity gradient tensor varies slowly with respect to the vorticity gradient, Weiss derived the following criterion: in regions where Q is positive, strain dominates; the vorticity gradient is stretched along one eigenvector, and compressed along the other. In regions where Q is negative, rotation dominates, and the vorticity gradients are merely subjected to a local rotation. Thus, selecting strongly negative Q allows capturing regions occupied by stable strong vortices, and this offers a way for identifying coherent structures, which has the advantage of incorporating some dynamics. In Ref. [6], the hypothesis on which Weiss's criterion relies is assessed; the relevance of the criterion for analyzing the dynamics of the flow is shown, although the criterion is strictly valid only near the vortex core and the hyperbolic points. Progress made to propose a more rigorous criterion, useful for handling dispersion phenomena, has recently been offered by Hua and Klein [55]. One may also mention a method, linked to the Weiss criterion, but based on the pressure, for identifying coherent structures [70]. The method is based on the remark that the Laplacian of the pressure is proportional to the Weiss discriminant Q. Therefore, by looking at the pressure field, one may identify regions dominated either by strong rotation or by high strains.

To check internal coherence, an attractive method is to determine, in the frame of reference of the supposedly coherent structure [97], the scatter plot (also called the coherence plot), i.e. the relation between the vorticity ω and the stream function ψ. For a stationary solution, vorticity is a function (not necessarily single valued) of the stream function. Thus, the scatter plot shows a well defined line for a stationary structure, and randomly distributed points if the structure undergoes strong temporal changes. The existence of well defined lines, which do not evolve for long periods of times, provides a direct visualization of the internal coherence of the so-called 'coherent structures'. An example of a scatter plot, obtained in a decaying experiment [10], is shown in Fig. 5. Tracking such curves in a turbulent field may be a method for identifying coherent structures; however, since one must work in a frame of reference moving and rotating with the structure at hand, this technique is difficult to implement in practice.

Another technique for identifying coherent structures, based on wavelet decomposition, has been advocated by Farge [35] these last years. It consists of decomposing the field into orthonormal wavelet coefficients, and selecting coefficients larger than some threshold. Coherent structures correspond to the selected coefficients, the rest forming the background. Interesting properties have been shown. In the cases considered, the background vorticity is found to be normally distributed, offering a favorable situation for theoretical analysis. Also much of the statistics of the field is captured by only a few wavelets, which may be interesting for computational purposes.

Determining the stability of isolated vortical structures is a central issue and a difficult theoretical problem even for the simplest morphologies. There are some mathematical results, partly reviewed by Dritschel [30]. Circular patches of uniform vorticity are nonlinearly stable; this is important to mention, because the stability of isolated circular vortices opens a pathway for the formation of coherent structures in two-dimensional turbulence. It also marks a difference



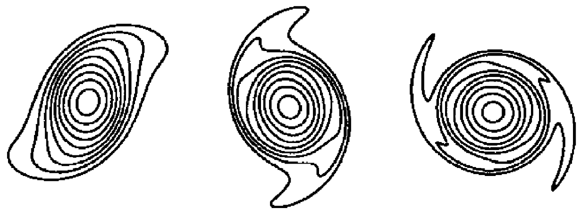


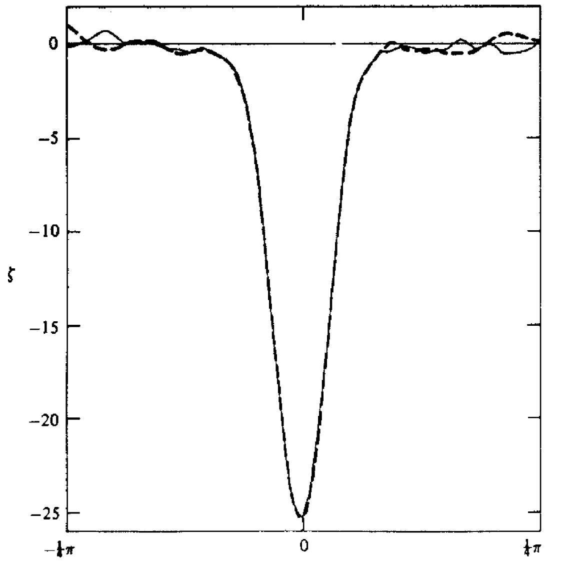
Fig. 5. Scatter plot (or 'coherence plot') obtained in a decaying experiment by Benzi et al. The presence of well defined branches reveal the existence of coherent structures, characterized by the same ω–ψ relationship.

Fig. 6. Several phases of the axisymmetrization process, transforming an initially elliptical vortex into a circular one.

between two and three dimensions, since the three dimensional equivalent of a two-dimensional vortex — the Burgers vortex — , is observed to be unstable with respect to three dimensional perturbations. In two dimensions, vortices are intrinsically stable, and the emergence of coherent structures is easier than in 3D. Elliptical vorticity patches, however, are not stable. The physical mechanism at work is better seen on initially elliptical vortices, as shown in Fig. 6 [57]. Due to the difference in rotation rate between the center and the edges, the edges of the vortex is subjected to filamentation, and after a while, the initially elliptical vortex adopts a circular shape. The overall picture of the monopolar stability has been confirmed experimentally [40]. More complicated morphologies are mostly out of reach of theoretical analysis, and one must rely on experimental evidence to determine whether a structure is stable or not. In this framework, isolated dipoles and tripoles (such as the one of Fig. 2) are reputed to be stable while pairs of like sign vortices are unstable against perturbations leading to vortex merging.

The internal vorticity profiles of the coherent structures are mostly controlled by the initial conditions and therefore can hardly be universal (Fig. 7 shows such example of such a profile, extracted from [85]). In contrast, the shapes of the coherent structures (more specifically the vortices) tend to be circular after a few turnaround times, under the action of the aforementioned axisymmetrization process [87].

The dynamics of a small number of vortices has been studied analytically, numerically and experimentally for several decades. A classification of vortex interactions, involving up to four elements, has been offered, in view of providing information helpful for the description of larger vortex systems, prone to sustain fully turbulent regimes [30]. In many cases (i.e. as distances between vortex cores exceeds by far the vortex sizes), the dynamics of vortex patches can be described by using point vortex approximation. The point vortex approximation has been used in a number of cases to study basic phenomena, such as the inverse cascade of energy and the free



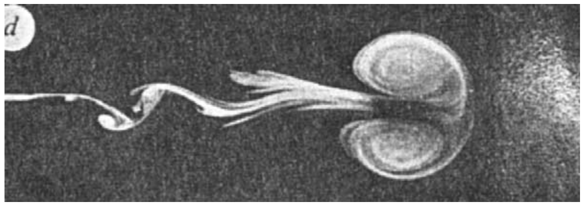
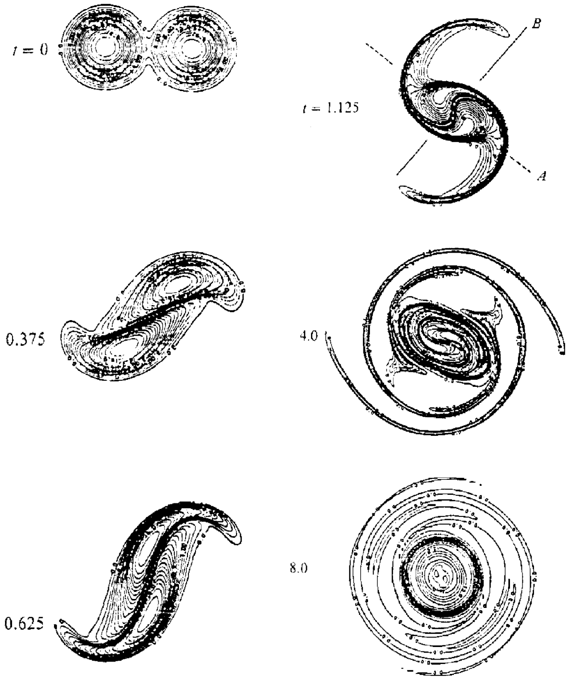


Fig. 7. An example of an averaged profile of a set of vortices, with negative vorticity, in a freely decaying system.

Fig. 8. A dipole formed in a stratified fluid system, loosing fluorescein behind it as it moves forward.

decay of turbulence [126,10]. The simplest structure involving more than one vortex is the Lamb dipole; the pointwise vortex approximation can be used to estimate their propagation velocity; however, by using this approximation, one does not accurately reproduce the dynamics of the saddle point, located at the rear of the dipolar structure, and across which leaks of vorticity or dye (if the vortices carry dye) can occur [134]. Several aspects of the dipolar structure have been investigated in detail in Refs. [38,39] (Fig. 8).

Clearly, some vortex interactions cannot be represented by using pointwise vortices. An example is the merging of like sign vortices, which plays a fundamental role in the free decay of two-dimensional turbulence, since it drives the formation of increasingly large structures, the most striking characteristics of freely decaying regimes. Vortex merging has been studied along different approaches in the past, and we may say today it is a subtle, rather well documented process. It is subtle because it involves a broad range of scales: as early numerical simulations suggested, vorticity filaments, rapidly lost by the action of viscosity, must be produced for merging to occur. The phenomenon thus involves an interplay between small and large scales. In the particular case of the merging of two identical vortices, it is remarkable that we now have an exact solution of the Navier–Stokes equations [136]; from a mathematical point of view, one thus may consider the problem is solved. However, it is not straightforward to dig out the physical content of the solution, and it is worth considering several concepts devised along the years, for discussing the phenomenon on a physical basis. First, why do vortices merge? Melander et al. [87] propose an instability mechanism, they called "axisymmetrization process" (because the merging will axisymmetrize the system formed by the vortex pair), driven by vorticity filament formation. They showed that whenever two like-sign vortices are separated



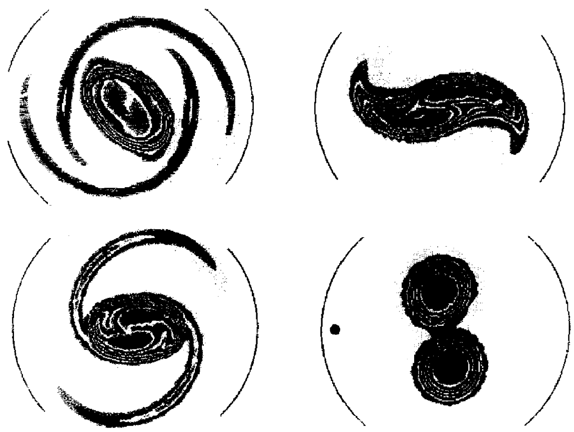


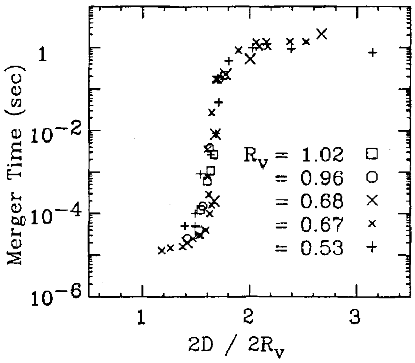
Fig. 9. The six steps of the merging process shown in the paper of Melander et al. (1988).

Fig. 10. Experiment, carried out in a plasma, showing the detail of the merging process.

by less than a critical distance, they merge. Far from each other, they do not interact; as they get close, they strain each other, developing thin filamentation along their edges, and as a result — for energetic or mechanical reasons — the cores get closer and closer, eventually forming a single structure (see Figs. 9 and 10). Merging is completed by the smoothing action of viscosity. The notion of critical distance is not straightforward to understand, but it represents well the observations (see Fig. 11). For equal vortices, of circular shapes and uniform vorticity, the critical distance is estimated to be 1.6 time the radius of each vortex. In practice however, merging of unequal vortices is more frequent than the one of equal vortices. One may also mention an appealing description, based on statistical premises, viewing vortex merging as the expression of a maximum entropy principle. I will come back to this theory later.

On the experimental side, we now have detailed analysis of the merging process: by using magnetically confined columns of electrons, Driscoll et al. [36] confirmed many of the features discussed above, in particular the existence of a critical distance; this notion appears through a plot showing that the merging time tends to diverge as the intervortical distance approaches a critical value, consistent with earlier estimates.

Another process involved in the decay phase is vortex break-up, which causes the elimination of the weakest vortices, under the action of the strain exerted by the others. This process is also rather well documented. Idealized cases, with isolated patches of uniform vorticity have been worked out by Moore and Saffman [93] and Kida [63]. Legras and Dritschel [73] showed



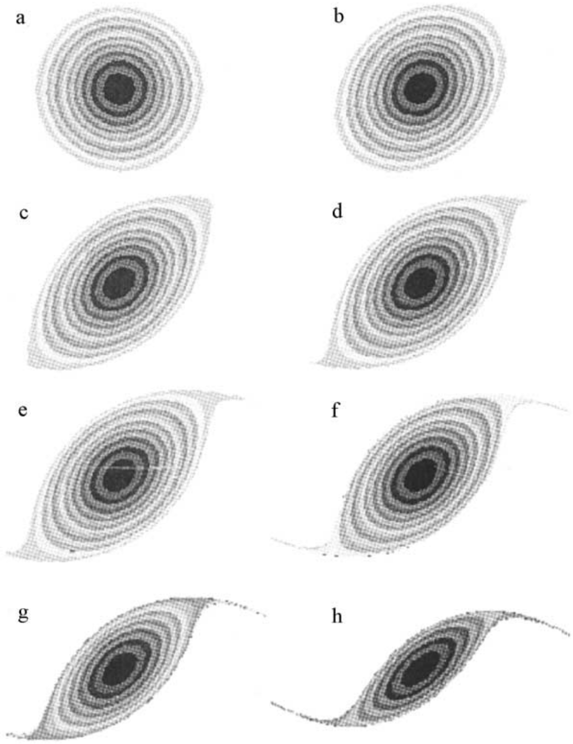


Fig. 11. Experiment showing the divergence of merging time as the intervortical distance increases (by Driscoll et al., 1991).

Fig. 12. Numerical experiment showing that a vortex, subjected to a strain, undergoes distortion, and, above some threshold, break-up.

that vortex breaking is controlled by a critical ratio of the strain rate over the core vorticity of the vortex, and this ratio was found weakly dependent on the vorticity profile and, provided the Reynolds number is large, independent of the viscosity (Mariotti et al. [80]) (see Fig. 12). The main lines of the analysis have been confirmed in physical experiments [109,108,132] (see Fig. 13). Recent progress on the subject has been achieved by Legras et al. [74]. One may also mention an analytical study of the effect of the background strain on a population of vortices, well confirmed numerically by Jimenez et al. [58].

Other processes are involved in turbulent systems, as pointed out by McWilliams [86]: lateral viscous spreading of the vorticity, translation through mutual advection, and higher order interactions between vortices; all contribute to reinforce the diversity of two-dimensional dynamics, and the reader may refer to the references listed in the present review, which include descriptions of these phenomena (Figs. 14 and 15 depict two steps of the decay of a population of 1000 vortices showing the formation of increasingly large structures, according to McWilliams [86]).

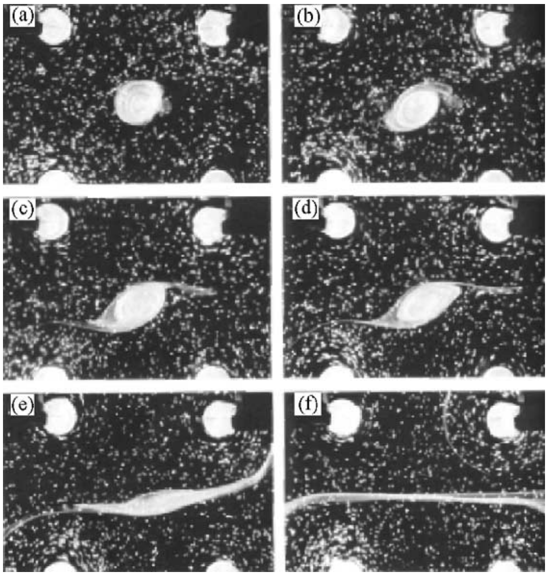


Fig. 13. Experiment, carried out in a stratified fluid, showing the break-up process.

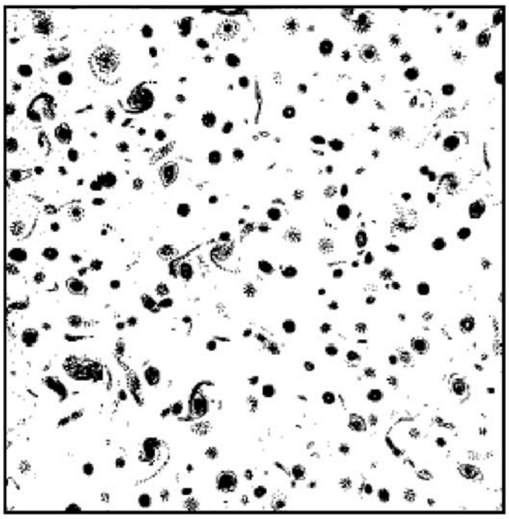


Fig. 14.

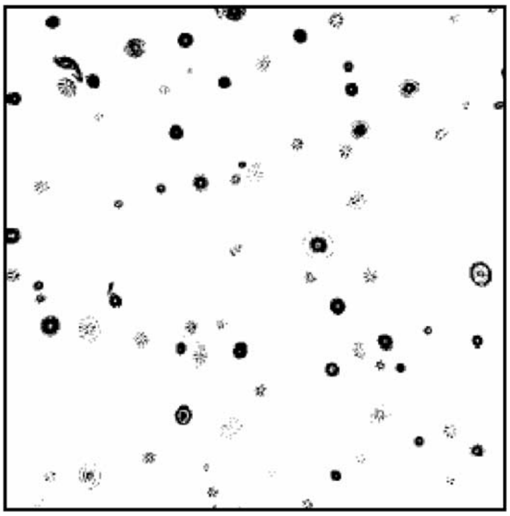


Fig. 15.

4. Statistics of the coherent structures in decaying two-dimensional turbulence

So much emphasis has been put on the free decay of two-dimensional turbulence that it sometimes stands as *the* two-dimensional turbulence problem. This is unjustified in my opinion; we will see in this review that freely decaying systems define a specific physical situation, which shares common features but also displays strong differences with the forced case. The main difference comes from the fact that coherent structures spontaneously pop up in freely decaying systems, whereas they can be inhibited or destroyed in forced systems. In three dimensions, as far as statistics are concerned, there is essentially no distinction between forced and free situations; in two dimensions, this is not so, and in general, a distinction must be made.

Briefly stated, the problem of the free decay of two-dimensional turbulence is to determine how abundant populations of vortices freely evolves with time at large Reynolds numbers. The vortices may initially be spread at random in the plane, or form a perfect crystal, have Gaussian profiles or be of uniform vorticity. In all the documented cases, the system evolves towards a state where coherent structures dominate the large scales of the flow, and apparently universal features show up. The crude aspects of the process were known in the sixties, but the first detailed analysis was performed in 1990, by McWilliams [86]: McWilliams succeeded to disentangle the few elementary mechanisms at work: he showed that in the decay phase, the weakest vortices are destroyed by the break-up process, while the strongest ones merge with other partners, of weaker or comparable strength; these mechanisms generate a refinement of the vorticity distribution (see Fig. 16). The vorticity distribution thus sharpens as the system decays, but in this process, the most probable vorticity level was observed to remain constant. Driven by a sequence of merging events, the system gradually and unavoidably evolves towards a state where vortices become fewer and larger. Throughout the decay regime, thin vorticity filaments

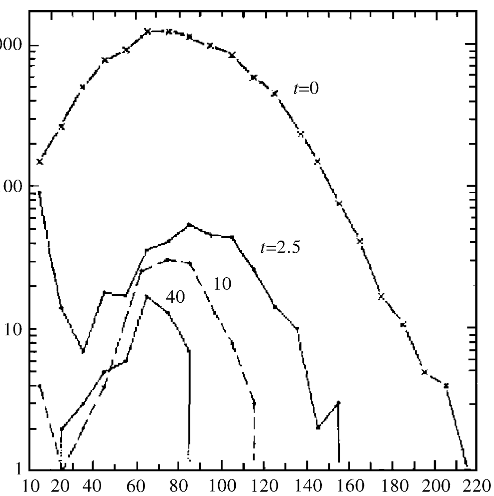


Fig. 16. Vorticity distributions plotted at various times, from $t = 0$ and 40; the distributions change, but the location of the maximum does not appreciably shift during the decay process (same paper as above).

keep being produced, either during merging events, or in the break-up process, developing a vorticity background. At moderate Reynolds number (such as the one of Ref. [86]), they rapidly disappear from the landscape, but at larger Reynolds number, they may nucleate additional small vortices, driven by Kelvin–Helmholtz type instability. Eventually, in current numerical and physical experiments, one typically reaches a state where a pair of large counter-rotating vortices are left in the system, forming the so-called “final dipole”. The reader may refer to Ref. [86] for more quantitative aspects.

The problem of the free decay of turbulence is rich, and several issues are embedded in it. What is the structure of the small scales, i.e. those smaller than the initial injection length? Are general cascade concepts relevant to discuss them? What is the ultimate state of the flow, appearing at the end of the decay process? Here, I restrict myself to presenting the dynamical evolution of the large scales of the system, i.e. those dominated by coherent structures, a problem on which much has been learnt these last years. The other issues will be discussed later.

The early theoretical analysis of the decay problem, proposing a statistical description, can be found in a remarkable paper by Batchelor [9]; the paper contains the important result that the decay process must be selfsimilar in time, a result which influenced all the theoretical approaches developed on the problem. The theory, which does not particularly specify the range of scale it addresses, is expressed in the spectral space. Batchelor’s approach has been reinterpreted in the real space, and for the large scale range, by Carnevale et al. [21]; this interpretation is interesting to consider for offering a consistent presentation of the free-decay process. If one constructs an expression for the number of vortices per unit of area, one must write the following expression:

$$N = f(E, t) ,$$

where t is time and E is the energy per unit mass and area, the only invariant surviving, at large Reynolds numbers, in neutral (i.e. with zero mean vorticity) populations [98]. Dimensional analysis further leads to the result that the density of vortices ρ decreases as

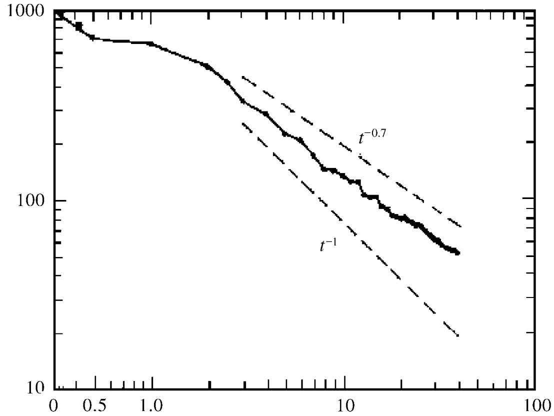
$$\rho \sim E^{-1} t^{-2}$$

[9]. The same argument applied to the vortex size, a, and the intervortex spacing, r shows they must be proportional to time, implying the system expands. This approach has been considered as a cornerstone for a long time. However, with the advent of powerful computers, unacceptable deviations appeared and we had to move towards other paradigms.

The first deviations were revealed by McWilliams [86] analysis, mentioned in the beginning of the section. The paper is a numerical study of a population of one thousand Gaussian vortices, located at random in a square with periodic boundary conditions. The number of vortices, the size and the intervortical distances were measured and the results are shown on Figs. 17 and 18. One could see well defined power laws, but the exponents were found clearly incompatible with Batchelor’s theory.

The exponent for the vortex density for instance, was found equal to -0.7, a value incompatible with Batchelor’s expectation (leading to -2); a similar comment can be made for the size (growing as $t^{0.2}$ instead of t) and the mean separation (increasing as $t^{0.4}$ instead of t).

To account for this observation, Carnevale et al. [21] offered the idea that another invariant must be dug out, and this gave rise to the ‘universal decay theory’ [101]. More specifically,



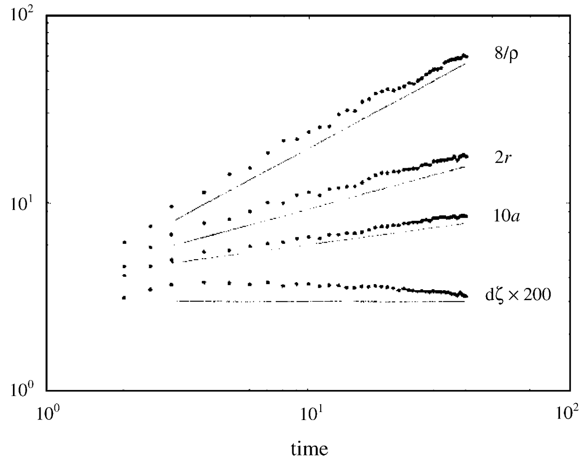


Fig. 17. Decay law for the number of vortices, found by McWilliams [86]; the results show evidence for the existence of a power law, with an exponent close to −0.70.

Fig. 18. Power laws obtained for the vortex density ρ, size a, intervortical distance r and extremal vorticity ζ.

they proposed to take the extremum vorticity of the system ω_{ext} as the missing ghost invariant. There is no rigorous justification for this assumption, but a physical plausibility, supported by numerical observation. Indeed in the inviscid limit, the maximum vorticity level, as any other levels, is conserved. But in the problem we address here, it is not clear the Euler equations provide an acceptable approximation. The constancy of ω_{ext} is physically justified by the fact that the strong vortices undergo two types of situations; either they wander in the plane or they merge; in none of these situations, their internal circulation has an opportunity to substantially decrease. Therefore, it is plausible, on a physical basis, to consider the extremum vorticity as a constant. This argument is supported by the observation that the most probable vorticity level remains (approximately) constant during the decay phase [86].

The next step is to estimate various quantities of the problem. The following equations are written

$$Z \sim \rho\omega_{\text{ext}}^2 a^2 , \tag{17}$$

$$E \sim \rho\omega_{\text{ext}}^2 a^4 \tag{18}$$

in which Z is the enstrophy per unit area and E is the energy per unit mass and area. The estimate for the energy neglects logarithmic contributions; this delicate omission is discussed in some detail in Ref. [139]. Thus, using the above relations, along with the constant extremum vorticity assumption, and further hypothesing power laws, Carnevale et al. [21] obtained the following laws for the density of vortices ρ, the vortex radius a, the mean separation between vortices r, the velocity u of a vortex, the total enstrophy Z, and the kurtosis Ku of the vorticity distribution:

$$\rho \sim l^{-2}\left(\frac{t}{T}\right)^{-\xi}, \qquad a \sim l\left(\frac{t}{T}\right)^{\xi/4},$$
$$r \sim l(t/T)^{\xi/2}, \qquad u \sim \sqrt{E}\,,$$
$$Z \sim T^{-2}\left(\frac{t}{T}\right)^{-\xi/2}, \qquad Ku \sim \left(\frac{t}{T}\right)^{\xi/2} \tag{19}$$

in which length l and time scale T are defined by

$$l = \omega_{\text{ext}}^{-1}\sqrt{E}, \qquad T = \omega_{\text{ext}}^{-1}\,. \tag{20}$$

The exponent ξ is not determined by this theory, but, once it is fixed, all the power law exponents of the problem are set. Numerical studies, either of the full Navier–Stokes equations or of point-vortex models, have consistently obtained power laws, with values of ξ ranging between 0.71 and 0.75 [21,139,86].

Concerning the physical experiments, early investigations [24,53,46] revealed the essential qualitative features of the decay process, but could not provide precise determinations of whatever exponent involved in the problem. Owing to the progress made in digital image processing, the situation changed in the last decade and now, accurate measurements are currently performed on two-dimensional decaying systems [102]. One may refer to a study of decay regimes performed in electromagnetically driven flows, which yielded reliable data, in good quantitative agreement with the numerical simulation [47]. Examples of such data are shown in Figs. 19–21. In this case, the decay process stretches over a period of time hardly exceeding 25 s, representing 10 initial turnaround times. Despite this limitation, scaling regimes take place rather neatly, and the related exponents could be quite accurately determined. Agreement with the aforementioned numerical studies was obtained, and this reinforced the consistency and robustness of the "universal decay theory" framework.

ξ thus appeared as a crucial exponent and several theoretical assaults have been given to determine its value [116,49,133,1]. Pomeau proposed a derivation that yields $\xi = 1$, arguing for lowering corrections [116]. On the other hand, using a probabilistic method to describe the motion of vortices in an external strain-rotation field, it has been suggested that the value of ξ depends on initial conditions [49]. In a related context, the 2D ballistic agglomeration of hard spheres with a size–mass relation mimicking the energy conservation rule for vortices, the value $\xi = 0.8$ is derived under mean-field assumptions [133]. Further, in another possibly related context, that of Ginzburg–Landau vortex turbulence, the value $\xi = \frac{3}{4}$ has been proposed [1].

Recently, progress was made on the characterization of the decay process [47]. The mean square displacements σ_v^2 of the vortices, the mean free path λ, the collision time τ, have been measured in a physical experiment, and the following power laws have been found (see Figs. 22 and 23):

$$\sigma_v^2 \sim t^{\nu} \quad \text{with } v \sim 1.3 \pm 0.1,$$
$$\lambda \sim t^{0.45\pm0.1}, \qquad \tau \sim t^{0.57\pm0.1}. \tag{21}$$

The vortices thus do not undergo Brownian motion in the plane, but possess a hyperdiffusive motion, with an exponent equal to 1.3. On physical grounds, it is clear that this dispersion law,

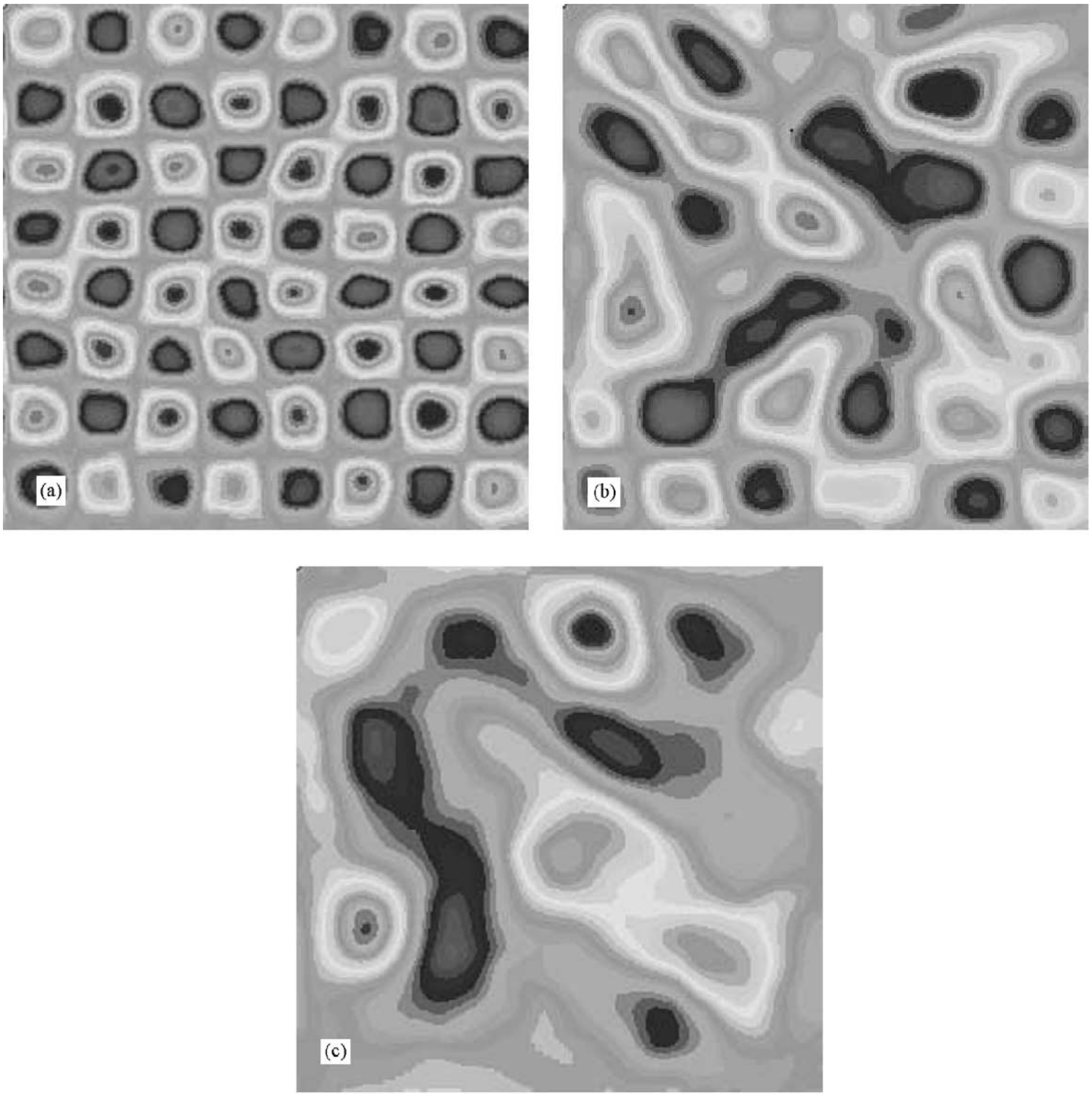
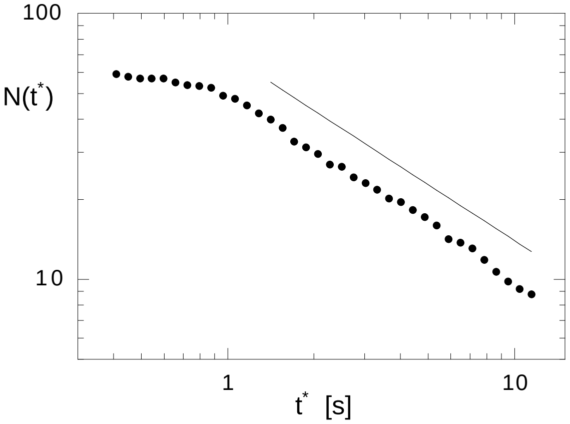


Fig. 19. Three steps of the decay process observed in a physical experiment, in a thin layer of electrolyte, where the vortices are driven by electromagnetical forces.

and the related exponent (we call v) are linked to the decay problem. The authors of Ref. [47] argue that a relation between the mean square displacement exponent v and the decay exponent, ξ, holds; it is expressed by the following formula:

$$v = 1 + \tfrac{3}{4}\xi \,. \tag{22}$$

This law is obtained by using standard relations of kinetic gas theory (assumed to be applicable for the problem at hand). In Ref. [47], the above relation appears well supported by the



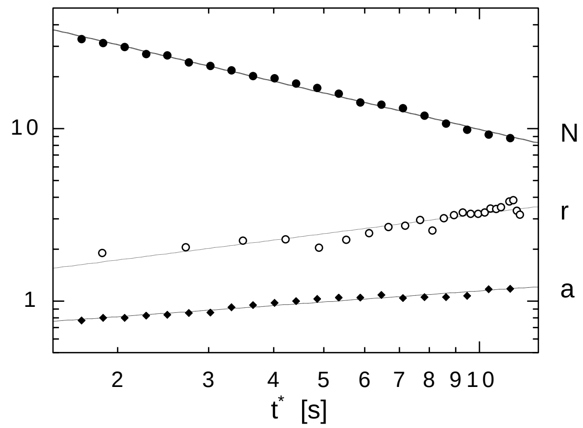
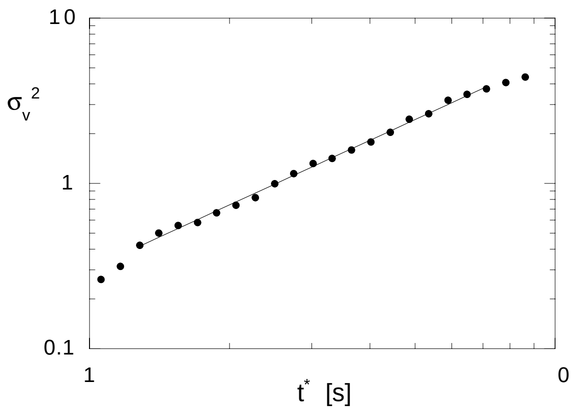


Fig. 20. Number of vortices with time obtained in the physical experiment.

Fig. 21. Additional data, showing the vortex size, and the intervortical distance with time, in the physical experiment.



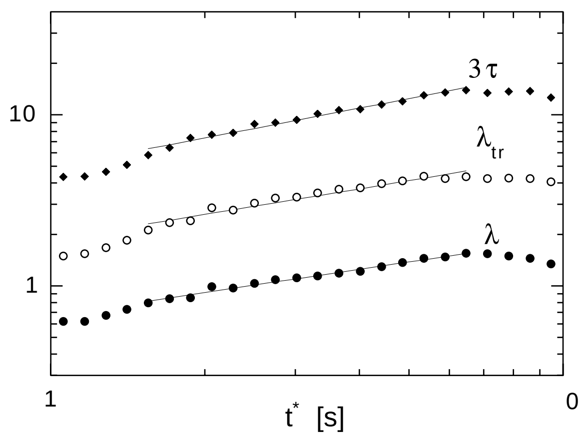


Fig. 22. Temporal evolution of the squared averaged distance between vortices, obtained in a physical experiment.

Fig. 23. Temporal evolution of various quantities, characterizing the evolution of a system of vortices: collision time, mean free paths.

experiment. According to this, both the dispersion and the decay problem seem to be the two faces of the same coin and one may ask which of them is the more amenable to theoretical understanding.

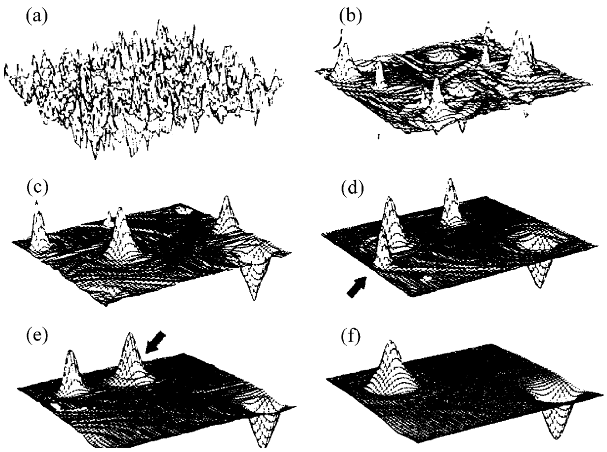
The presentation emphasizes on exponents, since their existence was implicitly admitted in most of the recent work made on the problem. However, one may consider that the situation is not fully settled; recent work revealed log periodic oscillations on the temporal decay of the number of vortices [48]; the authors further argue complex exponents must be used to represent the underlying processes. On the other hand, numerical results obtained by Dritschel [31], using vortex surgery technique, challenged the existence of power law behavior. There are several consequences of this result, one of them being that power laws may break down at extremely large Reynolds number. On the experimental side, the "universal decay theory" is generally considered as fitting well the experimental observations; however, one must mention a recent work [135] which suggests that in the experiment, the walls may have a sizeable effect in the determination of the exponents.

Concerning the decay problem, we are thus left at the present time with an elegant phenomenological theory ("universal decay theory") which turns out to represent consistent sets of numerical and experimental observations. In this framework, despite several attempts, the fundamental exponent governing the process is still unexplained. On the other hand, moving further away from phenomenology would probably be desirable, but at the moment, solving the decay problem from first principles appears as a formidable challenge theorists do not seem to rush out to tackle.

5. Equilibrium states of two-dimensional turbulence

As we discussed above, freely decaying two-dimensional turbulence evolves towards a state where eventually large structures arise, coexisting with a featureless turbulent background, formed by a collection of short lived vortex filaments. This has been shown for the turbulence decay problem, involving initially abundant vortex populations, and this is also true for systems initially composed of a few structures. Examples are displayed on Figs. 24–26, or can be found in [144,2,24,84–86,11,134,53,137]. One sees systems undergoing complicated evolutions, until eventually a steady state takes place, characterized by the presence of a few structures. The final patterns are not universal — their shapes sensitively depend on the boundaries and the initial conditions. The question is whether theory can account for them.

From the observations, it is appealing to consider that the final state is a "vorticity melasse", resulting from the mixing of the many elementary vorticity patches defining the initial state; along this line of thought, one is tempted to develop an analogy with thermodynamics, so as to interpret the final state as an equilibrium state, which would maximize some entropy. This intuitive idea is, however, not straightforward to substantiate. It has been first formalized by Onsager, for point vortices systems, i.e. in a context where the analogy could be worked out rigorously [107]. The Hamiltonian structure of the problem and the presence of a Liouville theorem (given by Helmhotz theorem) allowed to calculate equilibrium states, in the same way as in statistical physics. For the problem at hand, the canonical variables are the cartesian coordinates of the positions of the point vortices. Onsager found various possibilities, depending on the initial conditions. Among them, negative temperature states, which arise as a consequence of the finiteness of the phase space; those states are in form of clusters of like signed vortices, coexisting with regions where weak vortices are 'free to roam practically at random'. In this problem, entropy is gained by randomizing weak vorticity regions, and clusters are formed so as to conserve the total energy. This theory brought an illuminating way to account for the spontaneous formation of coherent structures in 2D flows, as repeatedly revealed experimentally and numerically. Onsager's theory conveyed new concepts in hydrodynamics and inspired, for four decades, a considerable number of contributions [69]. The theory was further



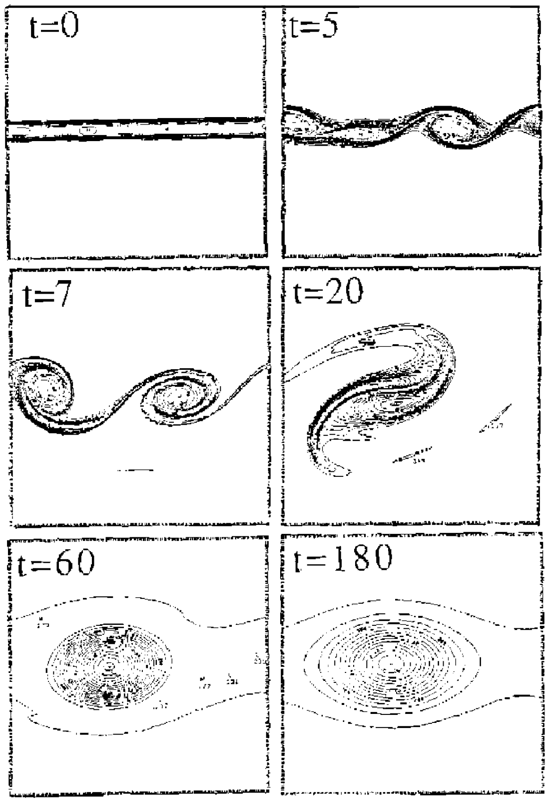


Fig. 24. Evolution of a system of vortices obtained by Matthaeus et al. (1991), showing the formation of the "final dipole".

Fig. 25. Evolution of a vorticity strip towards the formation of a chain of vortices (after Sommeria et al., 1991).

extended to magnetohydrodynamic flows and plasmas, and attempts were made to use it for the three-dimensional problem. This interest is probably motivated by the possible relevance of this approach to hydrodynamic turbulence but also — as quoted in Kraichnan and Montgomery's review [69] — by the fact that we deal with a "bizarre and instructive statistical mechanics", which makes it interesting in its own right.

There is a number of theoretical issues involved in Onsager's theory, related to the ergodicity assumption, the existence of a critical point, the pair collapses, the high and low wave-number limits, and they are reviewed in detail in Ref. [69]; also we may mention an excellent, more recent, pedagogical review by Miller et al. [89]; we comment here on the point vortices approximation, which affects the practical interest of the theory — I mean its relevance to explain observations. As noted by Pomeau [115] point vortex systems may provide a theoretical framework for superfluid turbulence, where vorticity is quantized into filaments, 1 A in diameter, with a prescribed circulation, but for ordinary fluids, the discrete approximation is constraining in the sense that it precludes the possibility to compare the theory with the experiment. The problem

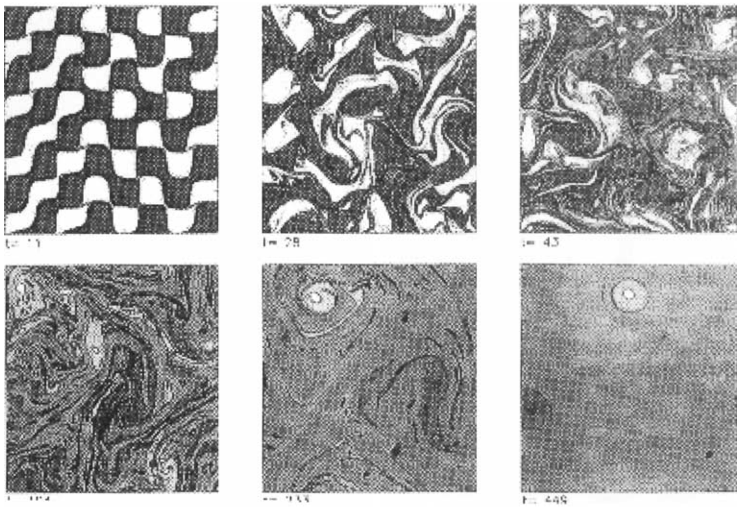


Fig. 26. Evolution of a vorticity pattern towards the formation of a dipole, in a periodic box (after Segre et al.).

of working with discrete vortices is that one has an infinite number of possibilities to construct a given initial continuous vorticity field [92]; as a consequence, by using Onsager's theory, one gets an infinity of possible equilibrium states; there is thus no way to make a quantitative contact between this approach and ordinary fluids, and a qualitative comparison is in practice extremely limited.

The desire to cross the bridge between discrete and continuous vorticity fields has stimulated a number of works, well reviewed by Kraichnan and Montgomery Refs. [69] and Miller et al. [89] and more recently by Brands [16] and Segre et al. [127]. Assuming negative temperature, and a two-level initial vorticity field, Montgomery and Joyce [91] proposed to define the vorticity as a local average of the pointwise vorticity field, under the assumptions that the vortex number density slowly varies with the position. The set of equilibrium states found under this assumption is governed by the equation

$$\nabla^2\psi + \gamma^2 \sinh(\beta' a\psi) = 0 \,, \tag{23}$$

where ψ is the equilibrium stream function, γ is a constant and β is a free parameter, corresponding to a temperature. Interestingly, this theory leads to a universal relation between the locally averaged vorticity and the stream-function, in form of a sinh function. This sinh function turns out to fit well the actual ultimate states observed in a number of experiments.

Another approach has been to work in discrete truncated Fourier spaces. The truncated system is Hamiltonian and has a Liouville theorem, so that one may calculate statistical equilibrium states, using Fourier coefficients as the conjugate variables. In this approach, the equilibrium state is found as the gravest mode (i.e. with the lowest wave-number) compatible with the boundary conditions. Formally, at equilibrium, the vorticity is found proportional to the stream-function. This proposal has been called the 'minimum enstrophy principle' since, in this problem, the equilibrium state turns out to minimize the enstrophy at constant energy [76]. A related concept is the 'selective decay', stipulating that freely evolving two-dimensional systems decrease their enstrophy at a rate much higher than the energy. The theory is appealing in its simplicity, and qualitatively reproduces well the trends observed in decaying systems. However, the linear relationship between vorticity and stream-function is generally not observed. In many cases, sinh functions or hyperbolic functions characterize the scatter plot of the final states. The theory can also lead to unphysical predictions. The minimum enstrophy approach is discussed in several places and the reader can refer to Kraichnan–Montgomery's review [69] to have a detailed account of this work.

Ten years ago, a statistical theory, tackling the discretization problem in a different way, has been worked out [88,120]. The key step has been to introduce a mesoscopic scale over which the stream function could be treated as uniform, and a local determination of the statistical ensembles could be achieved. In this context, the authors [88,120] calculated analytically the maximum entropy states, under the constraints of conserving the energy and the vorticity distribution, i.e. only the invariants which survive after coarse graining. The statistical theory provides, as a result, a relation between the averaged vorticity field $\omega(\mathbf{x})$ and the stream function $\psi(\mathbf{x})$ which fully define the equilibrium state. At variance with the previous theories, there is no free parameter in the solution; except in special cases, the form of the relation vorticity — stream function (the so-called scatter plot), requires an iterative procedure to be determined. Here, we present the essential steps of the calculation leading to the result. The final equilibrium for a distribution of discrete vorticity levels a_i of vorticity, is found by maximizing a mixing entropy defined by

$$S = -\int \sum_i p_i(\mathbf{x}) \ln(p_i(\mathbf{x}))\, d\mathbf{x} ,$$

where $p_i(\mathbf{x})$ is the probability of finding the level a_i at the location $\mathbf{x}$. By introducing the Lagrange parameters β and μ_i, respectively, related to the conservation of the energy E and initial distribution of vorticity A_i, and prescribing that the first order variation of S vanishes, one finds that the equilibrium vorticity field $\omega(\mathbf{x})$ and stream function $\psi(\mathbf{x})$ are given by

$$\omega(\mathbf{x}) = \sum_i a_i p_i(\mathbf{x}) = \sum_i a_i \frac{\exp(-\beta a_i \psi(\mathbf{x}) - \mu_\mathbf{i})}{\mathbf{Z}(\psi)} , \tag{24}$$

$Z(\psi)$ being the normalization factor of the probability distribution. To compute the equilibrium spatial vorticity distribution, one needs to solve the following set of equations:

$$E = \frac{1}{2} \int \omega\psi\, d\mathbf{x} , \tag{25}$$

$$A_i = \int p_i(\mathbf{x})\, d\mathbf{x} , \tag{26}$$

$$\omega = -\Delta\psi , \tag{27}$$

associated with the boundary condition $\psi = 0$. In the above equations, the set of A_i, prescribed by the initial conditions, is known.

It is possible to find the structure of the solutions to this system, in configurations of great simplicity. For instance, the case of two-dimensional flows in a closed rectangular domain (with

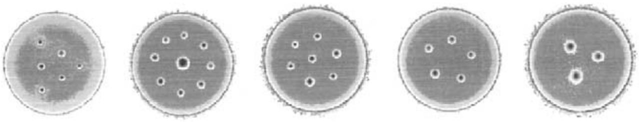


Fig. 27. Final states obtained in a plasma experimented by Fine et al. They are in the form of regular lattices of vortices. The dissipation is low enough to investigate thousands of turnaround times, i.e. more than the most conservative numerical simulations.

periodic or rigid boundary conditions), with initial uniform vorticity can be solved. The solution is [131]

$$\omega = -\Delta\psi = \frac{\mu \sinh(\alpha + \alpha'\psi)}{1 + \mu \cosh(\alpha + \alpha'\psi)}$$

in which all the parameters, i.e. α, α', μ can be calculated using the initial conditions (actually the task is not simple). There is no free parameter. It is also worth noting that the sinh Poisson is recovered in the limit of dilute vorticity. It has thus been suggested the sinh Poisson is a particular case of a more general framework.

Nonetheless, the resolution of the system in more realistic cases, along with the determination of the constants in situations such as the previous one, are not straightforward and for years, the numerical pathway for obtaining equilibrium states was inefficient; this difficulty was solved by the advent of ingenious numerical procedures [140]; the procedure devised by Whitaker and Turkington [140] is iterative and involves the following steps:

1. Compute a set of Lagrange parameters from the vorticity field and the related stream function form (25) and (26).
2. Determine the corresponding probability distribution according to Eq. (24).
3. Solve (27) for the new calculated vorticity field.
4. Return to first step.

The procedure is iterated until the convergence of all parameters is completed. The reader may refer to [59] for examples of applications (other examples can be found in [26,89]). In practice, the result is insensitive to the precise choice of n.

This theory could be confronted by experiments; it turned out to provide, in several case, predictions agreeing, even at some quantitative level, with the observation: for example, the cases of the jet and the mixing layer, yields good consistency between the theory and the simulation [130,131]. The theory predicts the existence of phase transitions, involving symmetry breakings, as the energy or geometrical factors are varied. Some of them have been found experimentally, consistently with the theory, in a two-dimensional circular mixing layer, [27]. The theory predicts the merging of two-like sign vortices, which is observed, provided they are closer than some critical distance (we will come back to this point later); some deviations have been pointed out in a series of experiments using plasmas [56], while acceptable agreement was shown in another case [82]. It would, however, be incorrect to consider that agreement is the rule. In several cases, it appeared that the theory misses some important features found experimentally. In [37], some observed states — such as vortex crystals of Fig. 27 — appear incompatible with the theory, as worked out in the original papers. An experimental study, done on the decay of

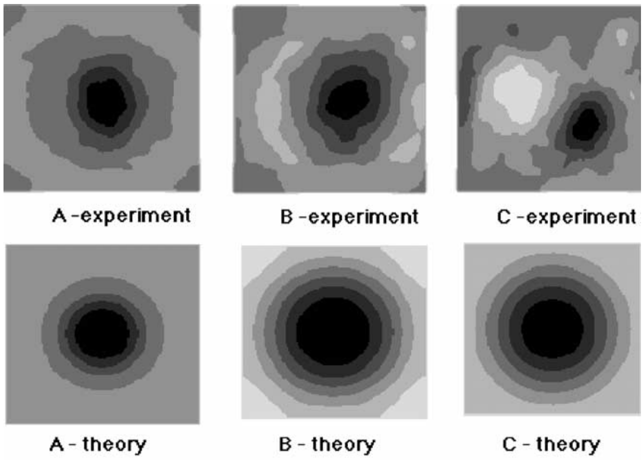


Fig. 28. Final states obtained in a physical experiment, and detailed comparison with the theoretical predictions of Robert-Sommeria.

systems of vortices, and comparing the prediction with the observation on a quantitative basis, obtained discrepancies between theory and experiment — in some cases at qualitative level [82,60] (see Fig. 28). A striking confrontation has been carried out by Segre and Kida [127]. Oscillatory states have been found as final states in a number of situations (examples are shown in Fig. 29). This is incompatible with the theory, which expects the final state to be stationary in all cases; in the same study, it is shown that the final state is not uniquely determined by the vorticity distribution and the energy level: two different two-levels vorticity patterns, with approximately the same energy have been observed to lead to different final states; this is also incompatible with the theory.

Whether we like it or not, the existence of situations where the theory fails to predict the ultimate state obviously generates a number of questions. There are technical issues, but the most critical question is ergodicity. Is is legitimate to assume ergodicity? Does the system investigate all the accessible phase space with equal probability, or can it be trapped for long in a part of it? Is the notion of equilibrium state relevant for interpreting experiments? These points have been discussed in several papers in considerably more detail and in more technical terms than herein (see [69,89,115,16]). The issue can be illustrated by the two like-sign vortices problem; in this case, statistical theory (I could say all statistical theories) predict the vortices to merge, whereas experiments show that above a critical distance, the vortices remain separated for ever ('ever' means thousands of turn-around times); in the framework we consider here, we may interpret this observation as an inhibition to reach an equilibrium state. Why can it be so? Pomeau [115] pointed out that the origin of randomness in 2D flow is far weaker than in ordinary statistical systems: as the system evolves, at finite Reynolds number, the dynamical role of the vorticity filaments gets weaker, whereas the large structures tend to form stable circular regions. The small scale fluctuations thus tend to collapse and consequently, the possibility to reach an equilibrium state, as in ordinary statistical mechanics, becomes questionable. One may ask whether increasing the Reynolds number favors ergodicity. At the moment, there is no evidence it can

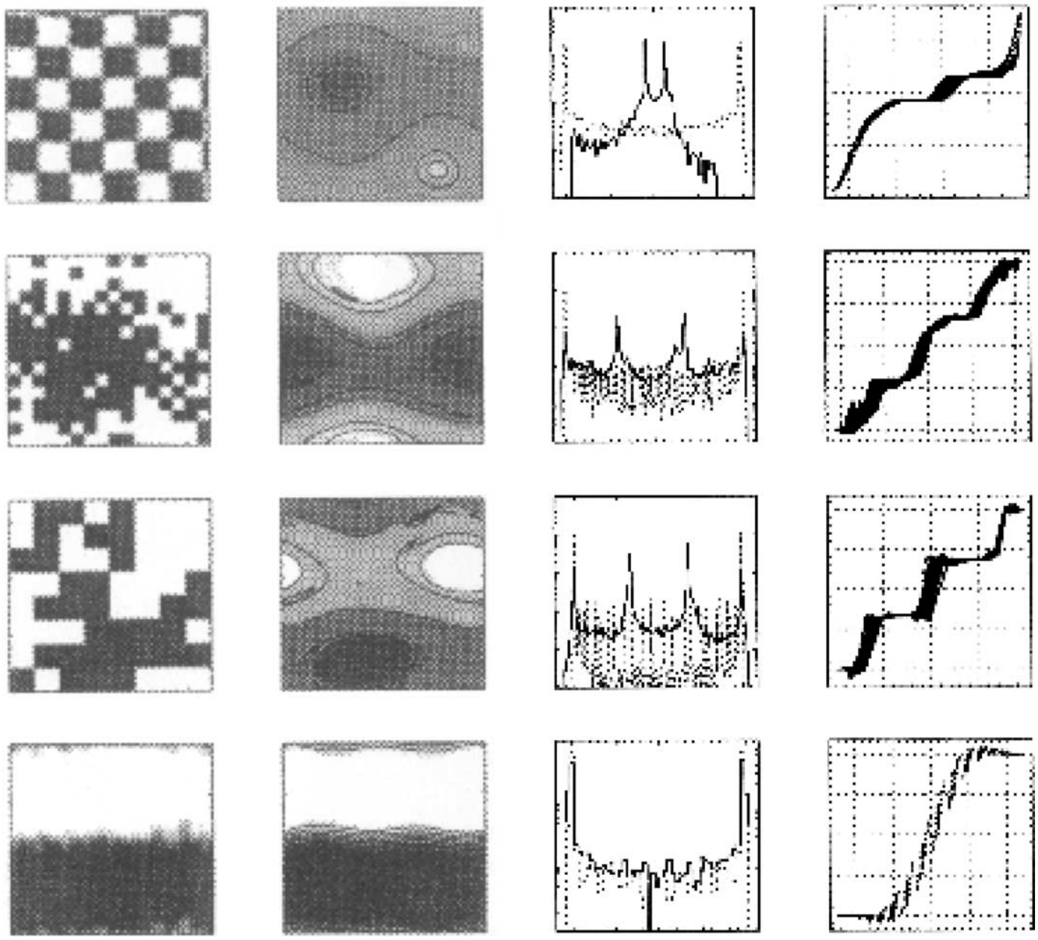


Fig. 29. Oscillatory states found as the final states in simulations initiated with two-levels vorticity patterns. The right figures are the vorticity distributions and the scatter plots. The nonstationary nature of the state is made visible by the thickness of the curve in the scatter plot.

be so. It appears that in two-dimensional systems, there is no mechanism ensuring ergodicity in general, and assuming ergodicity seems to be in too many cases a misleading premise. A discussion on ergodicity from a more dynamical prospect has been given by Weiss and McWilliams [138]; these authors underline the existence of dynamical pathways, which control the decay phase. After the decay phase is achieved, the fluctuations have declined, and the system ceases to evolve. According to this view, the final dipole is seen as the completion of a selfsimilar process, rather than an equilibrium state resulting from the mixing of the vorticity patches. The relevance of this approach has been underlined in a physical experiment [82]. Note finally that one can find cases where ergodicity does not hold [138].

Several criteria have been offered to determine the validity of the ergodicity premise [92,16]. One may perhaps propose another one, based on physical considerations, and expressed in terms of a dimensionless parameter P defined by

$$P = E/L^2 Z_0 \ ,$$

where L is the box size confining the flow and Z_0 is the initial enstrophy; this parameter is the natural dimensionless quantity one may introduce to discuss the problem (see for instance [115,89,19]). As P is small the system must gain much entropy to reach an equilibrium state, and this involves substantial mixing, difficult to achieve in two dimensions. On the other hand, when P is high, the system does not need to gain much entropy to reach equilibrium, because the energy level is high, and the number of accessible states is low; thus in this case, moderate mixing may be sufficient for the system to approach an equilibrium state and we may expect that statistical theory provides useful predictions. These rough ideas have been illustrated for vortex arrays [83]. Just to fix the ideas, one finds that deviations appear as P gets smaller than 5×10^{-3}; this may define a cross-over between two situations, one favorable for equilibrium states to be reached, the other unfavorable. A possible paradox of this situation is that statistical theory works better when the entropy tends to become irrelevant.

Recently, several modifications have been brought to improve and extend the original theoretical framework. The idea is to define bubbles within which ergodicity holds and outside of which the flow is deterministic [20]. A similar idea was proposed earlier by Driscoll, in view of proposing a more realistic version of minimum enstrophy theory [56]. This adjustment allows to improve the consistency between theory and experiment, at the expense of introducing a substantial complication.

To conclude, one may say that at the moment, the statistical approaches have not been proven yet to offer a more than qualitative framework for the interpretation of experimental observations. It seems that we are in a situation where unresolved issues are hanging, and the theory has to face formidable difficulties to make further progress. Do these theories correctly indicate the direction of the arrow of time? This is not ascertained in general. Nonetheless, the conceptual framework is appealing and certainly deserves stronger interest than now in the fluid dynamics community.

6. Inverse energy cascade

The concept of the inverse energy cascade traces back to Kraichnan [67] who first proposed that energy and enstrophy can cascade in two dimensions, a possibility often ruled out before him. In two dimensions, one has the conservation of the vorticity of each fluid parcel in the inviscid limit. These conservation laws imply the existence of two quadratic invariants: the energy and the enstrophy. These two constraints led Kraichnan to propose the existence of two different inertial ranges for two-dimensional turbulence [67]: one with constant energy flux extending from the injection scale toward larger scales and one with constant enstrophy flux extending from the injection scale down to the viscous scale. The first one is known as the inverse energy cascade, and the other as the enstrophy cascade.

The argument leading to this double cascade picture is better given in the Fourier space because the relation between energy and enstrophy is straightforward; it unfortunately does not find a simple expression in the real space, making it hard to explain physically in accurate terms.

The argument can be found in several places [77] and I restrict myself to provide the context. Let us consider a triad of wave numbers $\mathbf{k}, \mathbf{p}, \mathbf{q}$, associated to energy densities $E(\mathbf{k}), E(\mathbf{p}), E(\mathbf{q})$, and enstrophy densities $Z(\mathbf{k}), Z(\mathbf{p}), Z(\mathbf{q})$; the interaction conserves the energy and the enstrophy, and one must have the following relations:

$$E(\mathbf{k}) = \mathbf{E}(\mathbf{p}) + \mathbf{E}(\mathbf{q}) ,$$

$$k^2 E(\mathbf{k}) = \mathbf{p}^2 \mathbf{E}(\mathbf{p}) + \mathbf{q}^2 \mathbf{E}(\mathbf{q}) . \tag{28}$$

From the study of the above equations, it can be shown that energy is transferred preferentially towards small wave-numbers while enstrophy is transferred preferentially to large wave-numbers [99]. The argument must be repeated to successive triads to draw out the conclusion that energy flows towards large scale while enstrophy flows towards small scales (see Ref. [77] for a detailed presentation). This is the double cascade process proposed by Kraichnan. In this description, the only invariant involving the vorticity is the enstrophy. It is assumed that the others, such as the set of Z_n introduced in Section 2, are irrelevant, but to-date no justification is provided for this assumption. This assumption is satisfied in the low wave-number range, simply by the fact that the conservation laws dealing with vorticity are irrelevant in this range. The situation in the high wave-number range has been discussed by Eyink [33].

In this section, I focus on the inverse energy cascade. For the inverse cascade, Kraichnan predicted the existence of a selfsimilar range of scales in which the energy spectrum scales as $k^{-5/3}$. Its form reads

$$E(k) = C' \varepsilon^{2/3} k^{-5/3} . \tag{29}$$

In which C' is the Kolmogorov–Kraichnan constant. In Kraichnan's view, this cascade cannot be stationary: a sink of energy at large scales is required to reach a stationary state and, in a pure two-dimensional context, there is no candidate. Kraichnan conjectured that in finite systems, the energy will condensate in the lowest accessible mode, in a process similar to the 'Einstein–Bose condensation'. We will come to this process later. It remains that, as far as the inverse cascade is concerned, Kraichnan's view leads to considering it as a transient state, and in this respect, this energy cascade clearly differs from the three-dimensional case.

It has been realized recently that there are physically realizable ways to provide an energy sink at large scales. By working with thin fluid layers over flat plates, or using magnetic fields, one introduces a friction effect which can remove the energy on large scales. The friction effect, external to the two-dimensional framework, can be parametrized by adding a term proportional to the velocity in the Navier–Stokes equations, in the form $-\rho\beta\mathbf{u}$, where β is a parameter characterizing the amplitude, ρ is the fluid density and $\mathbf{u}$ is the local velocity; in the geophysical context, such an additional term is called the 'Rayleigh friction'. By using Kolmogorov type estimates, and comparing the nonlinear terms and the linear friction in this equation, one generates a new scale, whose expression reads:

$$l_{\mathrm{D}} = (\varepsilon/\beta^3)^{1/2} . \tag{30}$$

This scale — we may call it the 'dissipative scale' in a sense similar to the Kolmogorov scale in three dimensions — represents a balance between the transfers across the cascade and the transfers towards the energy sink. At scales smaller than l_{D}, the energy is essentially transferred

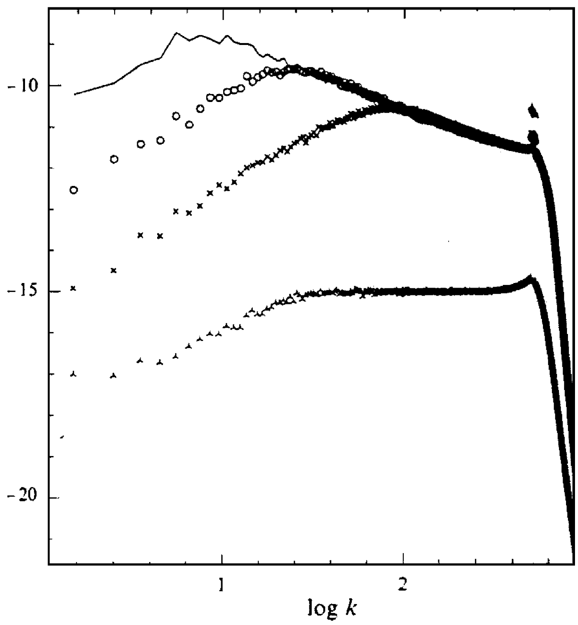


Fig. 30. Energy spectrum by Smith and Yakhot, for the inverse cascade.

through a cascade process while at scales larger than l_D, it is essentially burned. Therefore, we have here a way to extract energy at large scales, and this stabilizes the cascade.

The inverse cascade, as a transient process, was soon observed in several numerical simulations. The early observations were made by Lilly [78] were suggestive, and further simulation of Frisch and Sulem [42], led to first convincing confirmations of the spectral law. Later work consistently confirmed these observations; in particular one may mention the simulation of Smith and Yakhot [141,142], who displayed the expected spectral law over two decades, along with several crucial characteristics of the phenomenon, such as the constancy of the energy transfer in the inertial range (see (31)) (Figs. 30 and 31). From these studies, the Kraichnan Kolmogorov constant could be measured, and the best estimate to-date, provided by Smith and Yakhot, leads to

$$C' \approx 7 \, . \tag{31}$$

In the physical experiments, the strategy has been to inject at the smallest available scale so as to favor the development of a broad inertial range. The first observation of the inverse cascade in such conditions traces back to Sommeria [129]; in this experiment, two-dimensionality is maintained by applying a magnetic field. He observed Kolmogrov type spectra by probing the velocity field along a line of electrodes, therefore, supporting Kraichnan's conjecture [129] (see Fig. 32). In this case, at variance with the aforementioned numerical studies, the cascade, stabilized by a magnetohydrodynamic friction is a stationary process. Recently, Paret et al. [111], by using thin stratified layers of electromagnetically forced electrolytes, observed an inverse energy cascade with a neat $k^{-5/3}$ scaling (see Fig. 33); the measurement of the full

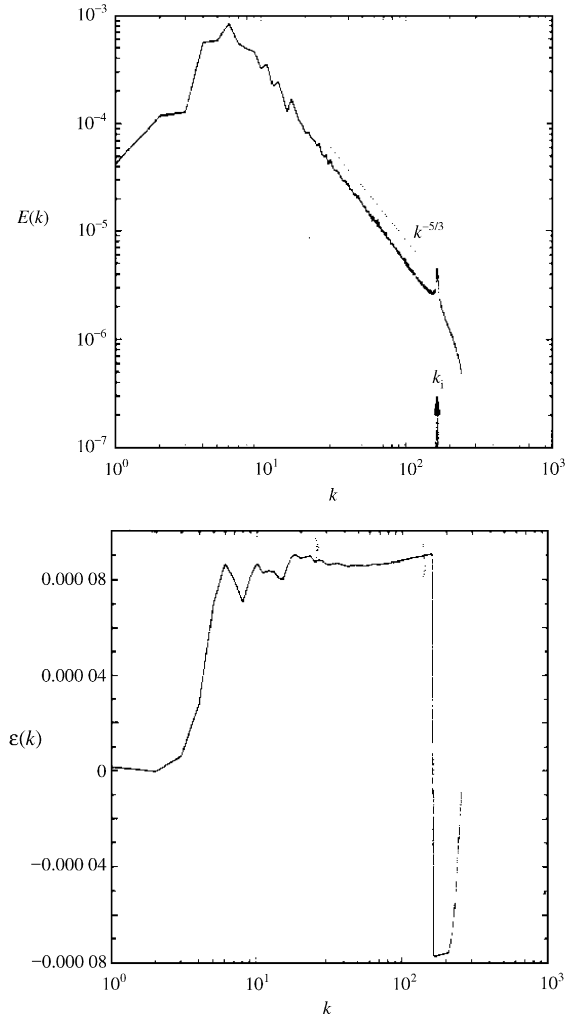


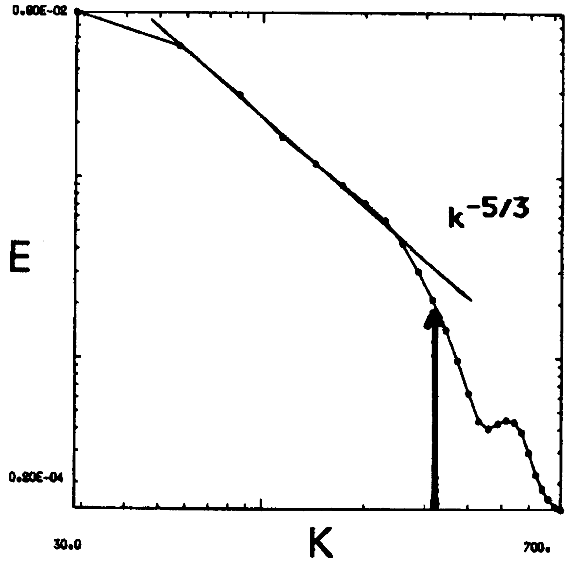
Fig. 31. Energy transfer by Smith and Yakhot, in the inverse cascade.

velocity field moreover allowed to check statistical isotropy (see Fig. 34) and homogeneity. They measured a Kolmogorov constant consistent with numerical estimates. Here again, the cascade was made stationary by the action of a friction exerted onto the fluid by the bottom plate. Finally, it is interesting to mention a recent observation of the inverse cascade, using soap films [122]. In this experiment, the authors could observe the entire process, i.e. both the inverse energy and the direct enstrophy cascades (see Fig. 35).

These observations were further extended by a 2048^2 numerical simulation by Boffeta et al. [12], in which a linear forcing is added so as to force statistical stationarity. In particular, the Kolmogorov relation mentioned in Section 2

$$S_3(r) = \frac{3}{2}\varepsilon r$$

could be accurately obtained. This is shown in Fig. 36.



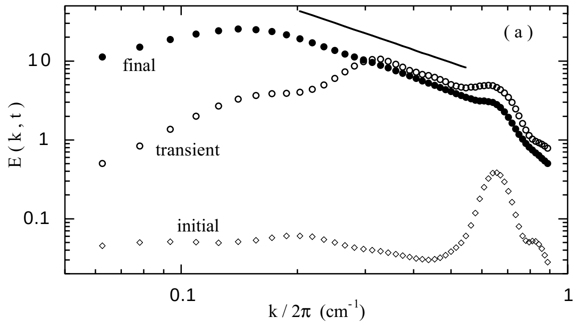


Fig. 32. Energy spectrum determined along a linear array of electrodes in a magnetohydrodynamic experiment.

Fig. 33. Energy spectrum obtained in a physical experiment where the flow is driven by electromagnetic forces.

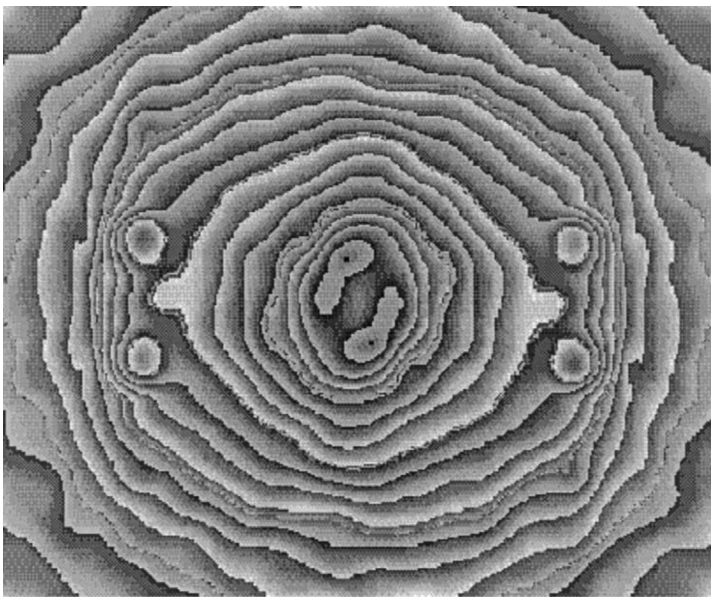
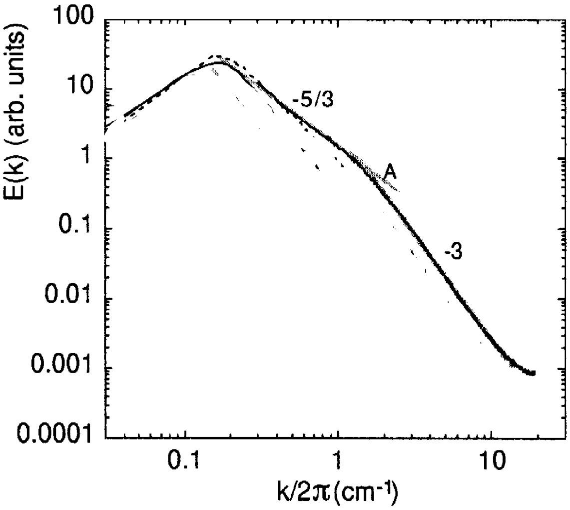


Fig. 34. Two-dimensional energy spectrum obtained in a physical experiment where the flow is driven by electromagnetic forces.



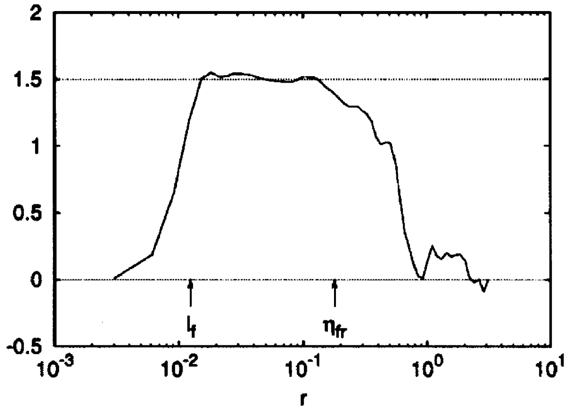


Fig. 35. Two-dimensional energy spectrum obtained in a soap film.

Fig. 36. Kolmogorov relation observed in the inverse energy cascade.

The physical mechanism involved in the inverse cascade deserves detailed analysis, because it substantiates a phenomenon which, in three dimensions is difficult to grasp. A typical vorticity field is displayed in Fig. 37(a) together with the corresponding stream function field [Fig. 37(b)]. The recirculation zones on the stream function map display a broad range of sizes but the vortices of the vorticity field are rather small in size. This is confirmed by the distribution of vortex sizes displayed in Fig. 38. The vortex sizes are essentially confined around the injection scale. This point was previously recognized by Maltrud and Vallis [79]. Since there is no vorticity patch greater than the injection scale, the conservation laws constraining the vorticity field take the form of a tautology, in the range of scale where the inverse cascade grows: zero equals zero. This is the way the system gets rid of the infinity of invariants which might have severely constrained it.

The absence of vortices on scales above the injection scale is certainly the most striking difference between decaying and forced two-dimensional turbulence. It is sometimes considered that the inverse cascade is made of a sequence of merging of like sign vortices as is the case in decaying turbulence. However, would it be so, the distribution of vortex sizes would be broad, at variance with the observation. Such merging events actually may occur, but they are rare. The inverse cascade is rather an aggregation process, driving the formation of large clusters of like sign vortices, which sustain large eddies conveying the energy towards larger scales. This picture is illustrated by the inspection of the stream function [Fig. 37(b)]: closed streamlines define large scale regions containing several like sign vortices (for example, the five "white" vortices in the upper-right corner of Fig. 37(a) are all enclosed within closed streamlines). Dynamically, the aggregation progress is observed during transient regimes. Fig. 39 displays, at 4 different times in the transient regime, the regions of positive (white) and negative (black)

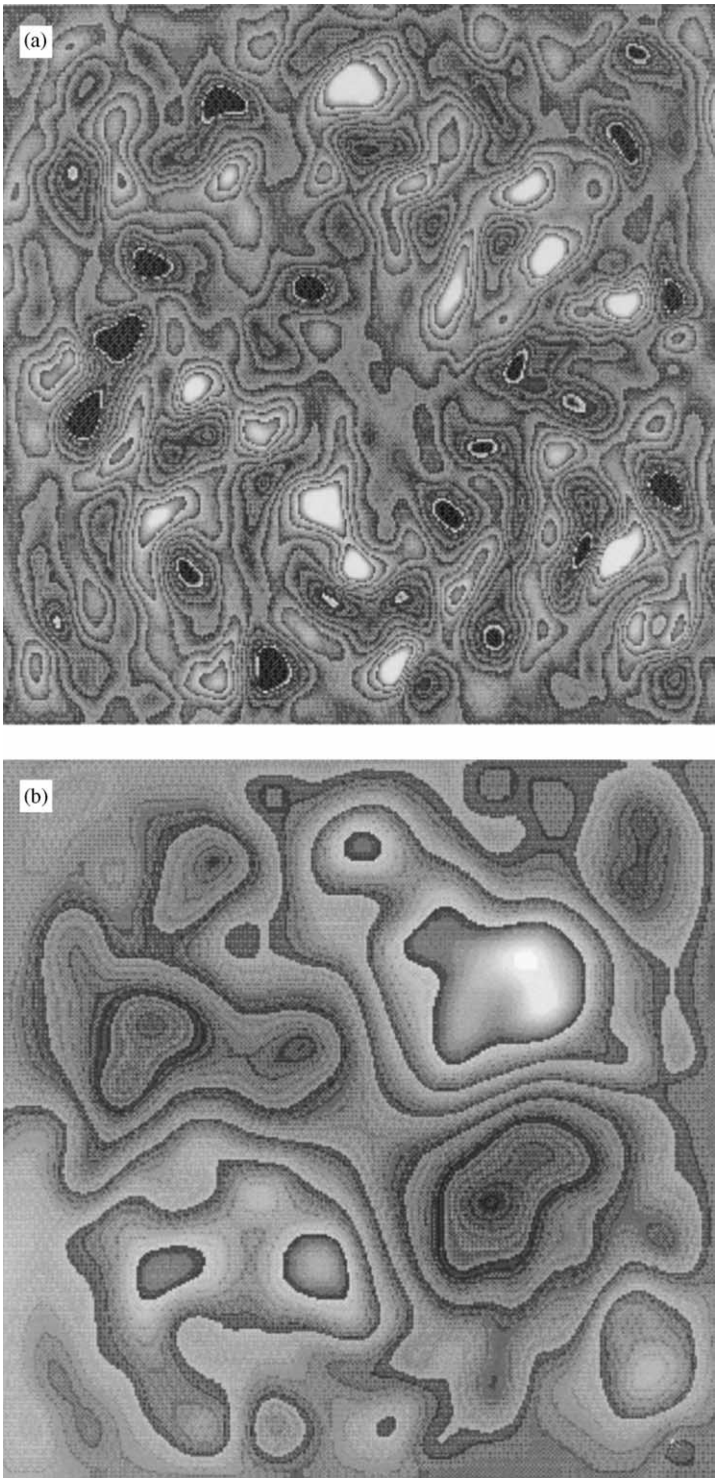


Fig. 37. Typical vorticity and streamfunction fields in the inverse cascade.

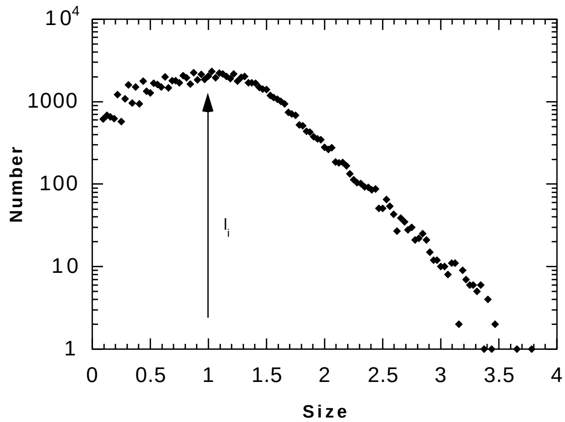


Fig. 38. Distribution of the vortex size in the inverse cascade.

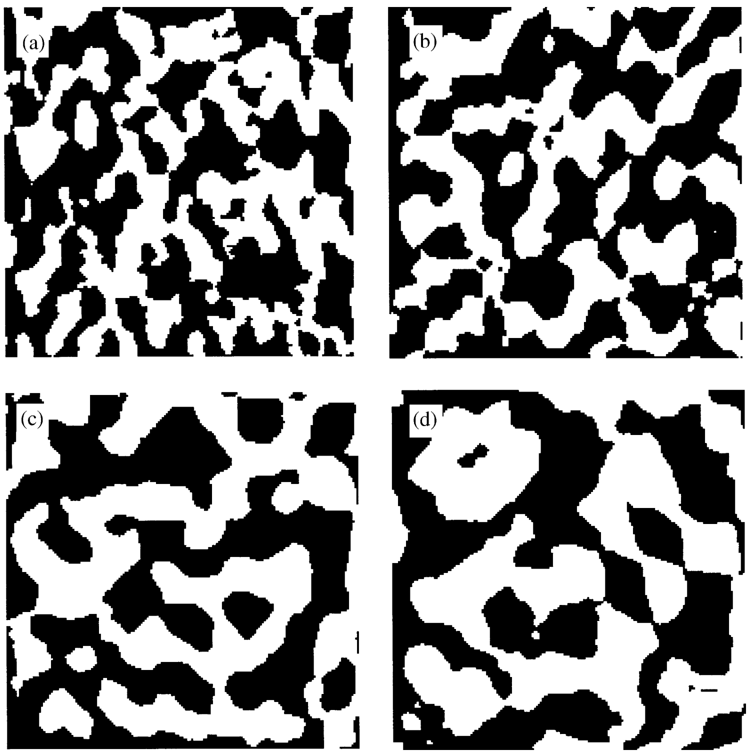
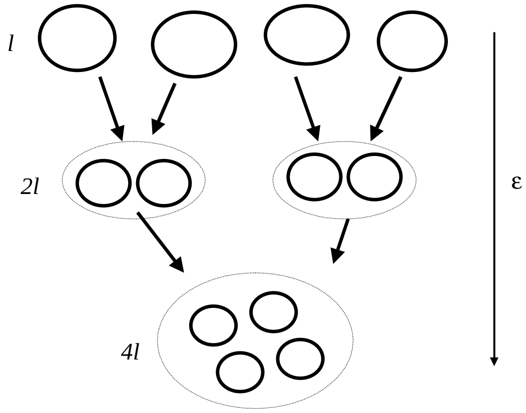


Fig. 39. Evolution of the vorticity field after the energy has been switched on, showing transfers towards larger scales in the inverse cascade.



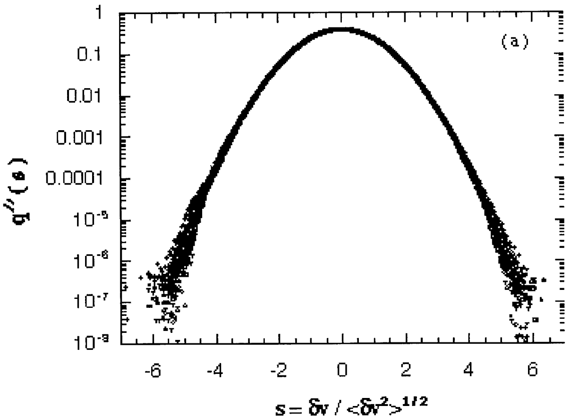


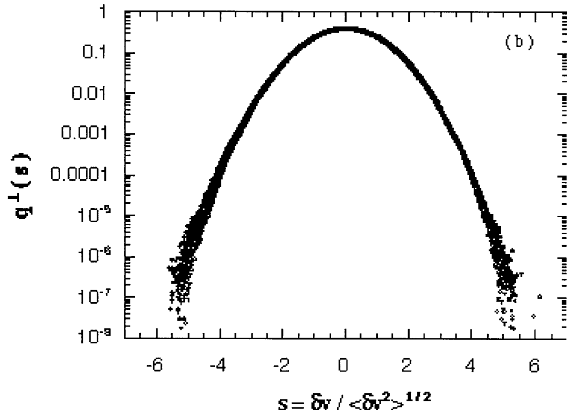
Fig. 40. Schematic view of the inverse cascade, represented in a way similar to Richardson cascade, the actual aggregation process replacing the hypothetical break-up event. In this picture, the vortices have the same sign; to represent the whole flow, a similar picture must be added, representing the vortices of the other sign.

Fig. 41. PDFs of the longitudinal velocity increments at various scales in the inertial range.

vorticity: it can be seen that, as the cascade builds up, the vorticity tends to seggregate and form larger and larger patches of the same sign.

We thus may propose the following picture for the inverse cascade. Vortices are continuously nucleated by the forcing; these vortices have sizes on the injection scale and they have a finite lifetime, equal to $1/\beta$, i.e. controlled by the friction against the bottom wall. Soon after they are formed, they wander in the plane, get distorted and strained by the action of neighboring vortices, and thus tend, in the average, to aggregate with other vortices, so as to conserve energy. The aggregation process is gradual. Small clusters are rapidly formed, and clusters of large sizes take time to be formed. In the internal clock of the vortices, it becomes extremely difficult to form new clusters beyond $1/\beta$, since the vortices have consumed most of their kinetic energy. The maximum cluster size is thus limited by the finite lifetime of the vortices. Here the cascade process can be substantiated by a selfsimilar build-up of clusters of increasing sizes. Kolmogorov arguments may be used in this context, with the understanding that the physical scale corresponds to the eddy size, i.e. the vortex cluster dimension, and the hypothetical eddy breaking is replaced by the observed vortex aggregation process. This description is illustrated as a cartoon in Fig. 40.

The search for deviations from Kolmogorov scaling in the inverse cascade is recent. The reason is that for several decades, numerical simulations could produce only unsteady cascades, and in such a context, analyzing high order moments was uncertain. The study of Ref. [111] and the numerical work of Bofetta et al. [12] could grasp this problem and provide reliable data on intermittency effects. Intermittency can be characterized using several methods: it can be signaled by a nonlinear dependence of the scaling exponents ζ_n on the order n, by a scale dependence of the normalized moments of the PDF or by a dependence of the PDF shapes on the separation r. The plots of Figs. 41–43 review the three aspects, both in the experiment of Ref. [111] and the numerical work of Ref. [12].



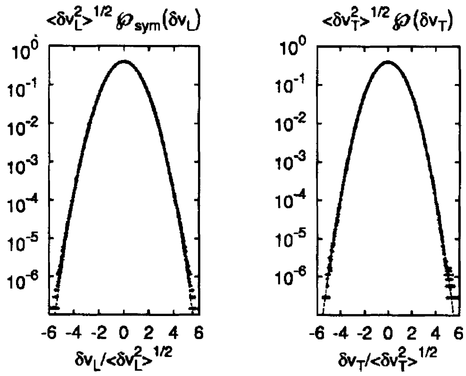
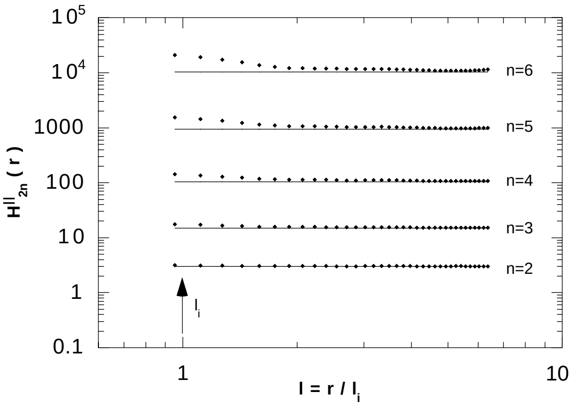


Fig. 42. PDFs of the transverse velocity increments at various scales in the inertial range.

Fig. 43. PDFs of the longitudinal and transverse velocity increments at various scales in the inertial range.



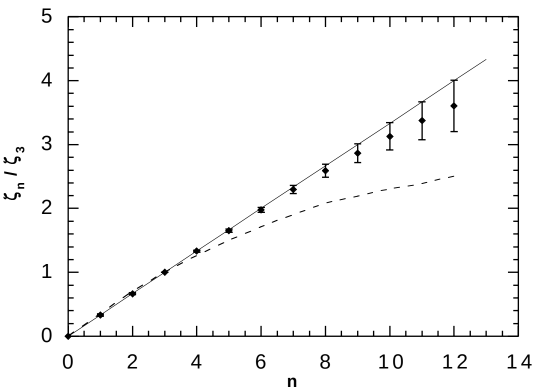
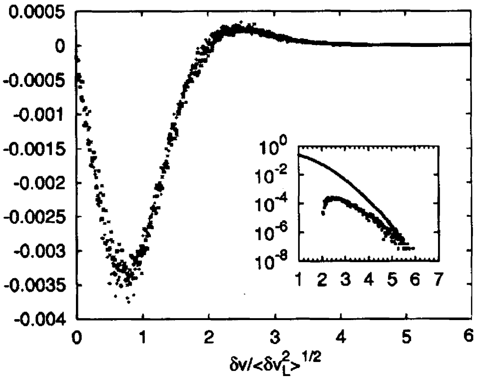


Fig. 44. Hyperflatnesses of the longitudinal velocity increments at various orders.

Fig. 45. Structure function exponents of the longitudinal velocity.

Distributions of the longitudinal and transverse velocity increments, displayed above, show they are close to Gaussian at all scales. They are slightly asymmetric, as they should be, if energy transfers are at work but this is a weak effect. A recent estimate for the skewness of the pdfs of the longitudinal velocity increments, leads to values on the order of 0.03; also, hyperflatnesses of the distributions are almost scale independent and have values close to the Gaussian limit, as shown in Fig. 44.



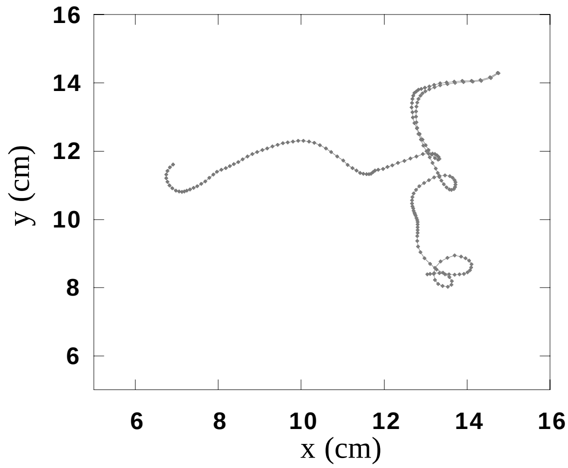


Fig. 46. Deviations from Gaussianity in the inverse cascade, according to Bofetta et al. (1999).

Fig. 47. Two particles released in the flow, in the inverse cascade regime.

The longitudinal structure function exponents of order n ζ_n are determined by the following expression:

$$S_n^{(L)}(r) = \langle|\delta v_L(r)|^n\rangle \sim r^{\zeta_n}.$$

In the above expression, $\delta v_L(r)$ is the increment along the direction of the vector $\mathbf{r}$; in Ref. [?], the symbol || is used rather than letter L to refer to longitudinal quantities. The exponents ζ_n have thus been measured both in the experiment and the simulations, and some data are displayed in Fig. 45. They hardly depart from the Kolmogorov line. We may thus conclude there is no substantial intermittency in the inverse cascade.

The proximity to Gaussianity raised a hope that the problem be more amenable to theoretical analysis than the three-dimensional problem. However, as pointed out by Bofetta et al. [12], 'the quasiGaussian behavior is moot, as deviations are intrinsically entangled to the dynamical process of inverse energy cascade' (Fig. 46). These authors measured the odd moments of the distributions of the velocity increments, and obtained that they increase quite vigorously with the order, which indicates that the departure from Gaussianity cannot be a small parameter. One may also mention a recent work given by Yakhot [143], who takes advantage of the quasiGaussianity of the distributions to construct a representation of the pressure terms, and provide a modelling of the inverse cascade.

To conclude, the inverse cascade presents, compared to the direct one, several features which perhaps suggest it may be an easier problem to tackle: in particular, there is no dissipative anomaly, i.e. the viscosity can be taken as zero without harm. Moreover, the observed statistics is close to Gaussianity; this property may tentatively be exploited to devise a perturbative scheme, based on the smallness of the skewness of the velocity increment distributions. Nonetheless, several attempts have shown the difficulty of the problem remains formidable, and at the moment, the absence of intermittency in the inverse cascade does not seem much easier to explain than the presence of intermittency in the standard three-dimensional cascade.

7. Dispersion of pairs in the inverse cascade

The dispersion of pairs is on the border of the scope of this paper since it bears issues related to the problem of passive scalar dispersion. However, it enlightens, from the Lagrangian view point some of the aspects of the inverse cascade, and this is the reason why I include it in this review.

The subject starts with the empirical proposal of Richardson [119], leading to a mean squared separation growing as the third power of time; this result has long served as a backbone for the analysis of dispersion processes in the atmosphere and the ocean [90,114]. Richardson law has further been reinterpreted in the framework of Kolmogorov theory [105,106], and the concept of effective diffusivity on which it relies reassessed by Batchelor [7]. Batchelor and Richardson approaches lead to the same scaling law for the pair mean squared separation, but provide strongly different expressions for the underlying distributions [50]. Kraichnan further reanalyzed the problem in the context of LHDI closure approximation [66] and more recently, a reinterpretation of the t^3 law, based on Levy walks, was proposed by Shlesinger et al. [125].

Richardson law has nonetheless received little experimental support for long, owing to the difficulty of performing Lagrangian measurements in turbulent flows. Existing experimental data, bearing on limited statistics and weakly controlled flows, show exponents lying in the range 2–3 [90,114]. The law has first been convincingly observed in a two-dimensional inverse cascade [5]. Recently, progress has been made, and we now have detailed information on crucial quantities, such as the pairwise separation distributions, and the Lagrangian correlation. At the moment, this data can be found in several studies; I focus here on results obtained in the physical experiment of Jullien et al. [61].

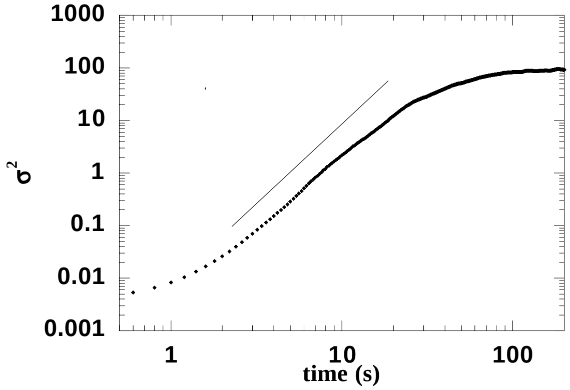
According to this work, the way the pairs are dispersed does not evoke a Brownian process (see Fig. 47): the released particles stay close to each other for some time, then separate out violently, wandering increasingly far from each other. At variance with Brownian motion, the pair separation is not a progressive process, but rather involves sequences of quiet periods and sudden bursts. Also, inspection of the pair trajectories suggests that internal correlations may persist.

The distribution of the separations are shown in Fig. 48, in a rescaled form. One sees long tails, in form of stretched exponentials. The following fit was proposed for the distributions $q(s,t)$:

$$q(s,t) = \sigma p(\sigma s,t) = A \exp(-\alpha s^{\beta}) \tag{32}$$

with $\alpha \sim 2.6$ and $\beta = 0.50$.

The separation velocity distributions also deserve a discussion since they are the Lagrangian counterpart of the velocity increment distributions of Fig. 42, which, as we previously saw, are close to being normal. So far they have not been published. They are displayed in Figs. 49 and 50, in a rescaled form.



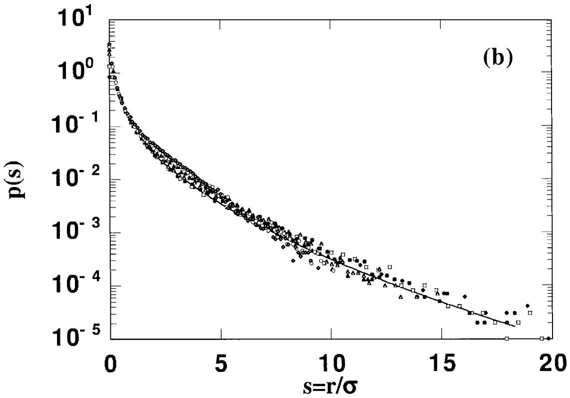
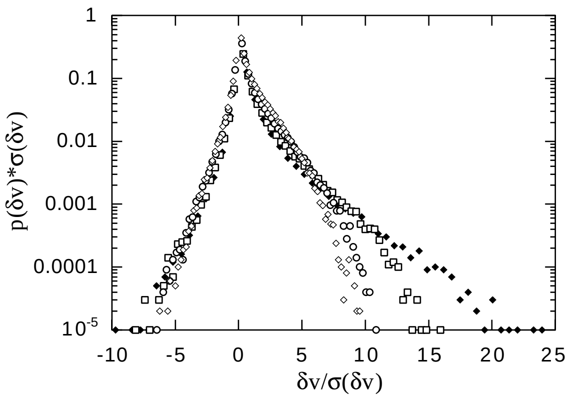


Fig. 48. Variance of the separation for a pair of particles; note that numerical evidence for the t-cube law was obtained earlier, in a pioneering work by Babiano et al. (1990).

Fig. 49. Rescaled pdfs of the separations, using Richardson scaling for the abcissa, and forcing the distributions to have their variance equal to 1.



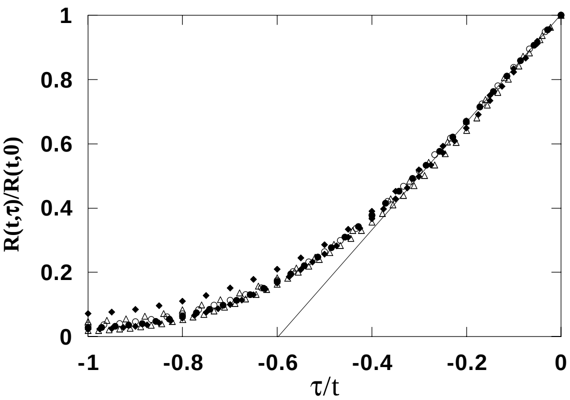


Fig. 50. Pdfs of the velocity of separation of pairs released in the flows, rescaled in the same way as the previous figure; each symbol corresponds to a given time. Temporal selfsimilarity applies in a restricted range of values of the separation velocity. Deviations from selfsimilarity are visible at large velocity separations.

Fig. 51. Lagrangian correlations of the separation for pairs of particles, rescaled as indicated in the text.

They are strongly asymmetric and have long stretched exponential tails. They hardly share anything common with Gaussian curves. Their asymmetry can be understood by the fact that in the average, the pairs separate, so that the probability to have positive separation velocity should be higher than negative ones. Their overall shape contrasts with the plots of Fig. 42, and illustrate the extent to which Eulerian and Lagrangian statistics may differ.

Another aspect of the problem is the persistence of correlations. We introduce here the Lagrangian correlation of separations defined by

$$R(t,\tau) = \langle \delta\mathbf{r}(\mathbf{0}).\delta\mathbf{r}(\tau)\rangle$$

in which $\delta\mathbf{r}(\mathbf{t})$ is the position of the separation vector at time 0, *for a pair released in the system at time — t*. The renormalized Lagrangian correlation of the separations $R(t,\tau)/R(t,0)$ are shown in Fig. 51, at different times, in a temporal range covering the inertial domain of the inverse cascade. Here, despite turbulence is stationary, we do not expect the Lagrangian correlations to be time independent quantities, owing to the fact that in the Lagrangian framework, the pair separation is a transient process. This makes a crucial difference between the corresponding eulerian quantities, which are time independent in stationary turbulence. As time grows, we effectively find the correlation curve broadens, and the time beyond which the particles decorrelate raises up. However, the set of curves collapses well onto a single curve, by renormalizing their maximum to unity and using the rescaled time τ/t, where t is the time spent since they have been released. This is shown in Fig. 51. This suggests that the general form for the Lagrangian correlation function reads:

$$R(t,\tau)/R(t,0) = f(\tau/t), \quad -t \leqslant \tau \leqslant 0 ,$$

where f is a dimensionless function. This result can be obtained by applying Kolmogorov arguments to Lagrangian statistics. There is, therefore, in this problem, a single underlying correlation function, for all the inertial domain. The corresponding physical Lagrangian correlation time τ_c, estimated from Fig. 51, is

$$\tau_c \approx 0.60t$$

which underlines the persistence of correlations throughout the separation process. At any time, the pairs remember more than half of their history, considering their history starts once they are released.

Here is thus some of the available data on the pair dispersion in the inverse cascade. In the dispersion process, temporal selfsimilarity, persistence of internal correlations, and non-Gaussianity prevails at all stages. The Richardson dispersion problem and the inverse cascade are intimately related, and probably solving one will allow to get clues for the other. For the Richardson problem, we do not understand the origin of nonGaussianity beyond the remark that, due to the presence of long range correlations, there is no reason to expect the central limit theorem to apply.

8. Condensed states

In Section 6, the situations we described corresponded to cases where the dissipation scale l_d is smaller than the box size confining the flow. What happens when l_d exceeds the system size? In this case, a new phenomenon takes place, leading to the formation of a condensed state, characterized by a scale comparable to the system size. As we already pointed out, this effect was anticipated by Kraichnan [67]; in his paper, he compared the phenomenon to the Bose–Einstein condensation, a terminology which remains. The analogy is justified: in Bose–Einstein condensation, below a certain temperature, the system collapses onto a fundamental

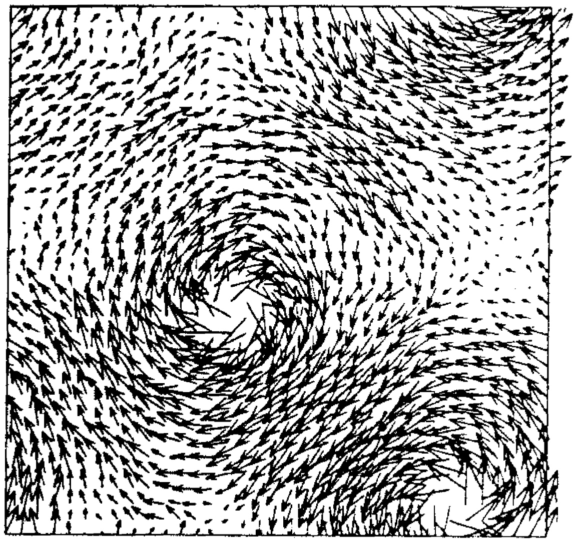


Fig. 52. Vorticity field in the condensed regime, showing the formation of a peak of vorticity, calculated by Smith et al. (1994).

Fig. 53. Streamfunction in the condensed regime, showing that the flow is dominated by large scale eddies, according to Smith et al. (1994).

state, compatible with the boundaries of the system. In the phenomenon we discuss here, in Fourier space, the energy piles up onto the smallest wave-number, compatible with the boundaries. The phenomenon is intuitive as soon as one accepts that energy is continuously dragged towards large scales. In such a situation, the way the energy is eventually burned depends on the system at hand. In practice, in physical experiments, energy is dissipated by friction along the walls limiting the system.

We now have several observations of the condensation process. The phenomenon has been first observed in physical experiments by Sommeria [129], and later by Paret et al. [112]; numerically, the observation has been made by Hossain et al. (see Ref. [54]) and more recently by Smith et al. [141,142]. We show typical numerical and physical results in Figs. 52–55. In the physical experiments, carried out in square boxes, the condensed regimes are dominated by a global rotation, whose sign erratically fluctuates in time. With periodic boundary conditions, Smith and Yakhot [141,142] observed the formation of a dipole, whose size is a fraction of the spatial period of the system. To analyze the system, it is interesting to look at the energy spectra. A typical energy spectrum for the condensate regime is shown in Fig. 56: the spectrum displays a sharp bump at large scales. One may perhaps speculate that the tendency to form a k^{-3} spectrum on the flank of the bump has some link with an enstrophy cascade [104].

In the condensation regime, there is an interesting phenomenon of vorticity intensification, first identified numerically by Smith and Yakhot [141,142], and later confirmed experimentally by Paret et al. [112]. The instantaneous vorticity fields display a single central vortex with a small intense core around which the mean global rotation is driven. A vorticity profile along a

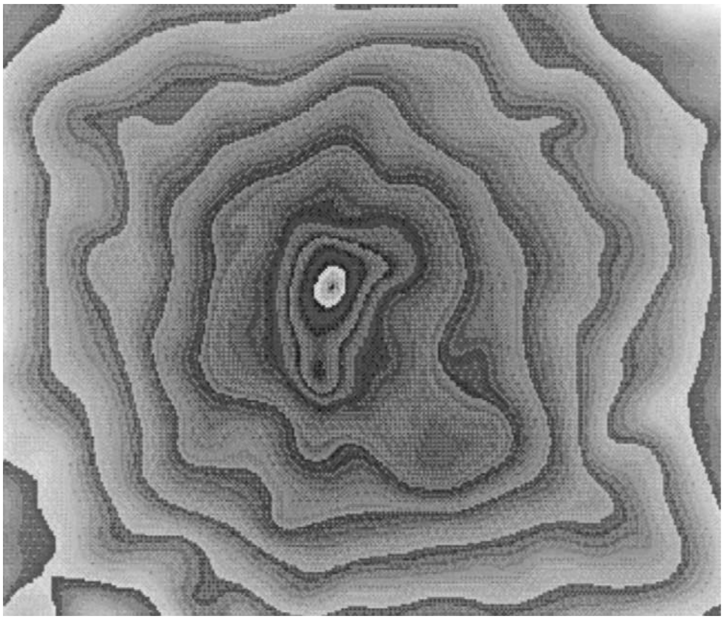
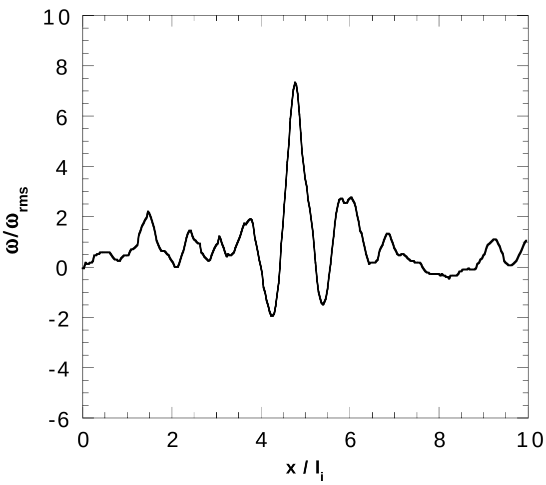


Fig. 54. Vorticity field observed in a physical experiment in the condensation regime; a localized vortex sits at the center of the box; this has been observed by Paret et al. (1996).



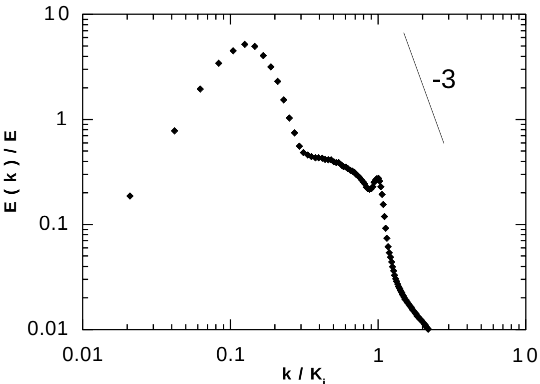


Fig. 55. Vorticity profile showing the peak vorticity in the condensation regime (Paret et al., 1996).

Fig. 56. Energy spectrum in the condensation regime, obtained in a physical experiment (Paret et al., 1996).

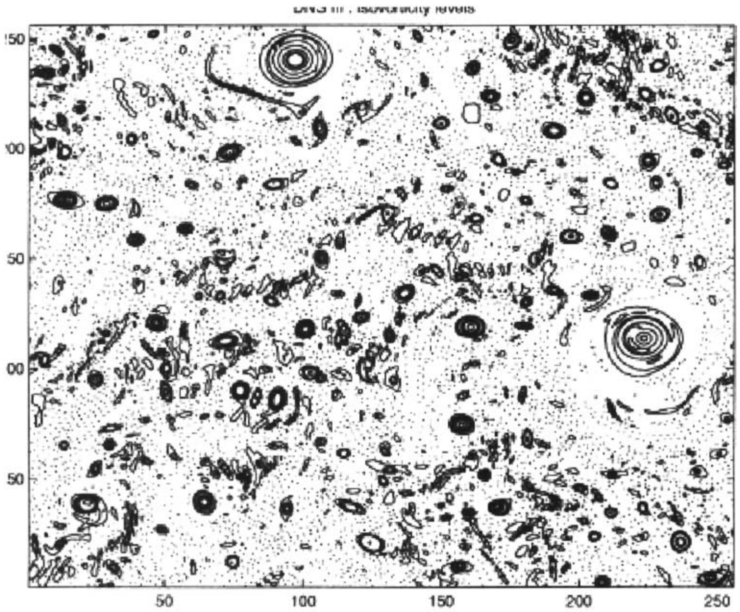


Fig. 57. Vorticity field in the condensation regime (Dubos et al., 2001).

line passing through this vortex is displayed in Fig. 55, reveals this peculiar structure. It can be seen that the central vortex is much stronger than the fluctuations on the sides. The existence of a sharp peak of vorticity is even clearer in the numerical simulation (see Fig. 52). Such a pattern is not understood at the moment.

Concerning the statistical characterization of the condensed regime, some progress has been made recently [32]. As we discuss above, in this regime, the realizations of the vorticity fields are extremely nonhomogeneous: rare, intense, peaked vortical patches coexist with low vorticity background (taking a loose definition of intermittency, it is often said that in such a case, the vorticity field is 'intermittent'; however, we will not use this word here, since the intermittency we refer to in this review involves a dependence on scale, not just a spottiness of the quantity at hand). From the point of view of turbulence, and more specifically of the scaling, it has been shown recently that there is no substantial intermittency in this regime. This is displayed in Fig. 58, where we see the velocity increment distributions of a condensed state, whose vorticity field is represented in Fig. 57. At variance with the inverse cascade regime, the distributions are nonGaussian, but they collapse fairly well at all scales, showing the system is nonintermittent (from the scaling point of view).

9. Enstrophy cascade

As mentioned previously, the existence of the enstrophy cascade was proposed by Kraichnan [67], as part of a double cascade process, taking place in a forced two-dimensional system. At about the same time, in the context of freely-decaying turbulence, Batchelor [9] proposed that

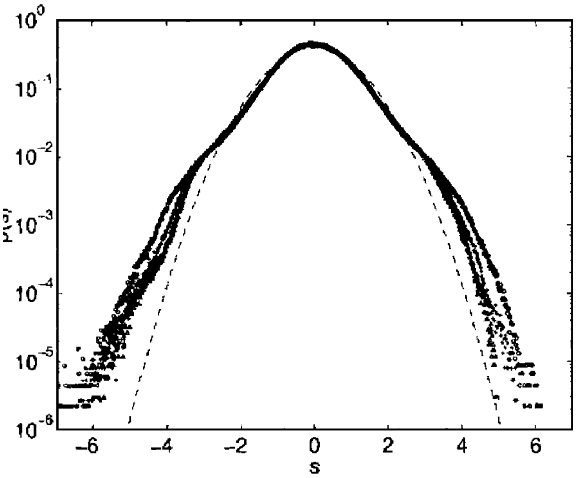


Fig. 58. Pdfs of the velocity increments in the condensation regime (Dubos et al., 2001).

Fig. 59. Lines of isovorticity in a freely decaying turbulence regime, where, for some range of time, an enstrophy cascade has been observed.

enstrophy may cascade towards small scales, in a way analogous to energy in three dimensions, in the sense that the transfer develops at constant rate across the scales. In a range of scale defined as the inertial range of the problem, the energy and the enstrophy spectrum, dimensionally constrained, have the following expression:

$$E(k) = C'\eta^{2/3}k^{-3} , \tag{33}$$

$$Z(k) = C'\eta^{2/3}k^{-1} , \tag{34}$$

where C' is the Kraichnan Batchelor constant and η is the enstrophy transfer rate.

On physical grounds, the process prevailing in the cascade is essentially the elongation of the vorticity patches, already mentioned in Section 1. The process is displayed in Figs. 59–61. Weak vorticity blobs, embedded in a random large scale strain, are elongated, and, since, for a perfect fluid, the vorticity patches conserve area, one must have a compression in the other direction, leading to the formation of thin vorticity filaments, inducing the steepening of the vorticity gradients (see Fig. 59).

In this process, the enstrophy transfer rate is controlled by the strain. The nonpassive nature of the vorticity field reflects the fact that vorticity tends to locally decrease the strain. The process stops as viscosity comes into play, leading to irreversibly diffusing vorticity. Diffusion by viscosity is the dissipative mechanism at work for this cascade; it is associated to a scale l_d, defined by the following expression:

$$l_d = \eta^{-1/6}\nu^{1/2} , \tag{35}$$

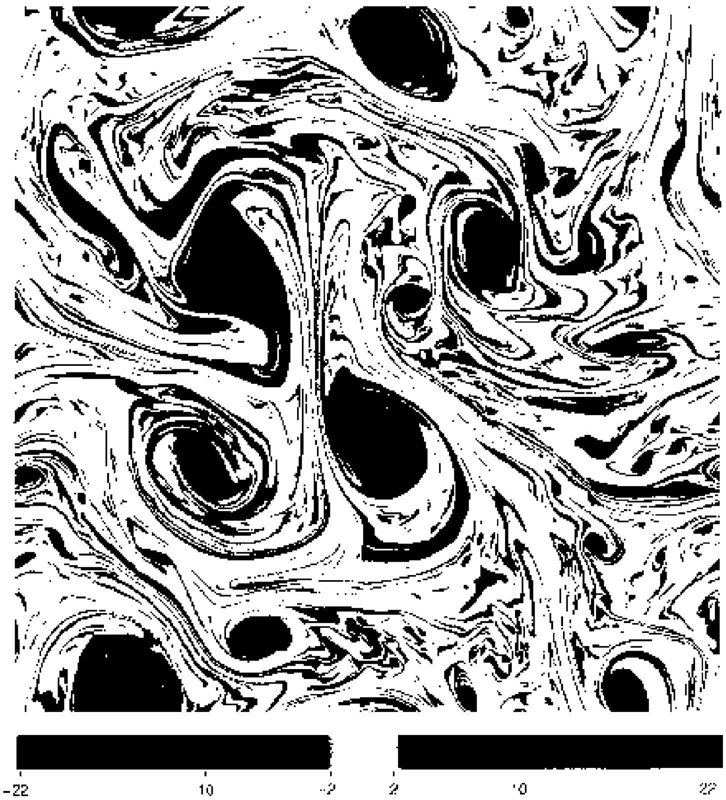


Fig. 60. Vorticity field in a decay experiment, showing the filamentation process.

where v is the kinematic viscosity of the fluid. One may infer from this result that the total number of degrees of freedom of the process scales as the Reynolds number, yielding a computationally more advantageous situation than the three dimensional energy cascade (which requires $Re^{9/4}$ degrees of freedom).

There are several issues involved in the enstrophy cascade. A first one, discussed early by Kraichnan [68] is the nonlocality of the cascade. In the enstrophy cascade, and within the inertial range, the turn-around time is equal to the rms value of the vorticity, and is therefore the same at all scales. This property has a disturbing consequence: it implies there is no internal mechanism ensuring statistical independence between large and small scales. As coined by Herring and McWilliams [52], large and small scale 'may become coherent with impunity'; the fact that small scales may directly be affected by large scales means the cascade is 'nonlocal' (in Fourier space). Physically, nonlocality originates in the fact that the large scale strain controls the formation of small scales, by continuously shaping vorticity patches into thin filaments, without being mediated by a succession or events across which decorrelation could take place. In the enstrophy cascade, the large scales are directly coupled to the small scales, i.e. the interactions cannot be assumed to be local in scale. It is conceivable that nonlocality allows nonuniversal effects to contaminate so much the inertial range that classical theory would be of

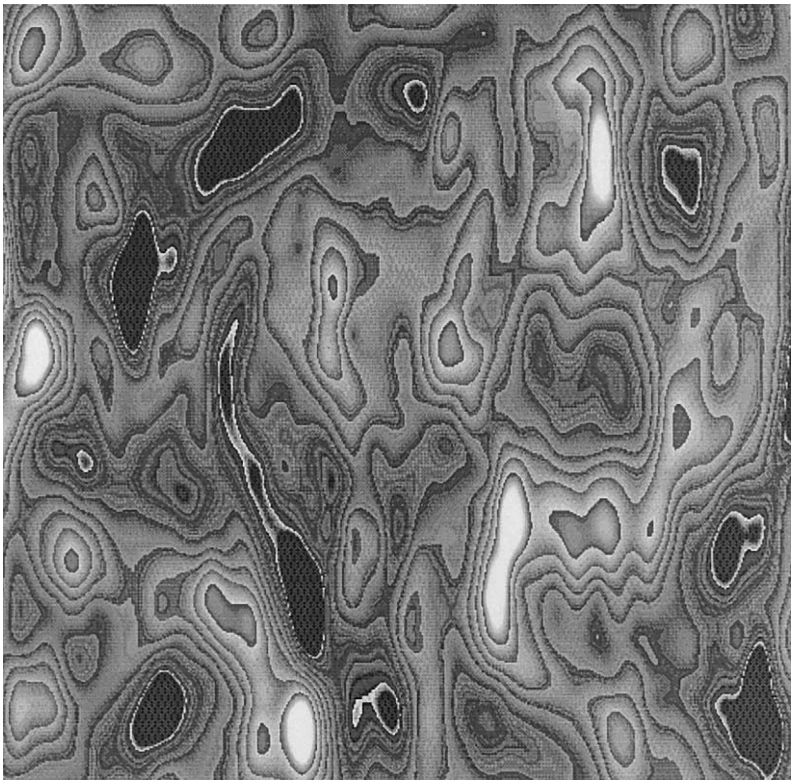


Fig. 61. Vorticity field in a physical experiment, where a k^{-3} spectrum has been observed.

little practical interest; we will see later that the contamination is limited, and in particular it is possible to define universal quantities such as the Kraichnan–Batchelor constant.

Another issue is the presence of unpleasant divergences in the classical theory: for an energy spectrum decaying as k^{-3}, the transfer rate is a logarithmic function of the wave-number, which diverges as the injection wave-number tends to zero, or — equivalently — as the Reynolds number tends to infinity; this is a concern for the internal consistency of the theory. Such divergences are called "infrared divergences". In order to remedy this difficulty, Kraichnan proposed a modified form for the spectrum:

$$E(k) = C'\eta^{2/3}k^{-3}(Ln(k/k_0))^{1/3} , \tag{36}$$

where k_0 is the injection wave-number. With such a spectrum, the divergence on the enstrophy transfer rate is suppressed; this approach can be viewed as a one-loop approximation of a more general theory which remains to be written.

Theoretical progress on the enstrophy cascade has been made recently, and one may consider that theory now sits on much more solid grounds than in the past. The theories I present below use structure functions of the vorticity, defined by the following relations:

$$S_p(r) = \langle(\omega(\mathbf{x}+\mathbf{r}) - \omega(\mathbf{x}))^{\mathrm{p}}\rangle$$

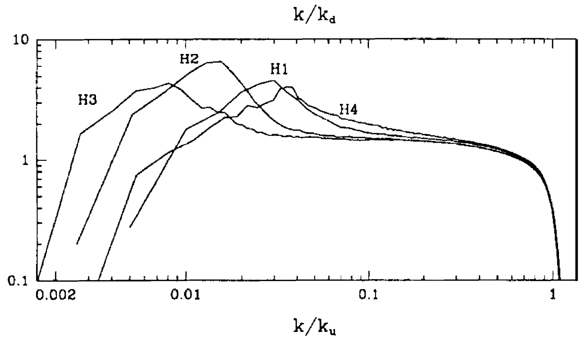
in which **x** and **r** are vectors, and r is the modulus of **r**. The brackets mean statistical average. The exponent of order p of the structure functions — called ξ_p is defined by

$$S_p(r) \approx r^{\xi p} .$$

Eyink [33] obtained exact inequalities for the set of scaling exponents ξ_p. Assuming the large scale velocity field be smooth, he showed that the structure function exponents for the vorticity field are smaller or equal to zero for $p \geqslant 3$. The analysis of Falkovich and Lebedev [34] showed that the structure functions S_p have logarithmic behavior, i.e. apart from logarithmic correction, the exponents ξ_p are essentially zero. It is interesting to note that this result has been obtained by using Lagrangian approach. The analysis showed that at variance with the energy cascade in three dimensions, the two-dimensional enstrophy cascade is nonintermittent and classical estimates "a la Kolmogorov" apply within logarithmic accuracy. The logarithmic behavior of the structure function could be found in the Batchelor theory on passive scalars, but the merit of Falkovitch and Lebedev has been to establish this result on rigorous grounds, without needing to postulate a number of quantities are uncorrelated.

One may mention that different schemes were proposed in the nineties, as alternatives to the aforementioned theoretical approaches. It is worth mentioning the conformal proposal [117], presented in the original paper as *the* theory of two-dimensional turbulence; this work did not provide outstanding clues, probably because the relevance of conformal theory to two-dimensional turbulence is limited. Other approaches, based on structural premises, were developed in the same period and before. Saffmann [123] proposed a dynamical scenario for the formation of vorticity filaments, leading to a k^{-4} spectrum. More recently, Moffatt, emphasizing the importance of spiral like structures, proposed a $k^{-11/3}$ spectrum. So far, these appealing descriptions have not been compared in detail with the experiment. It remains to assess the extent to which these structural approaches may describe the enstrophy cascade.

Concerning the comparison between theory and experiment, it turned out that the advent of large computers revealed puzzling deviations between the observation and the Batchelor–Kraichnan spectrum, especially in decaying systems [10,15,72]. A variety of energy spectral slopes, between -3 and -6 was observed, depending on the range of time under consideration, and the initial conditions. We will come back to these studies later, in a section dedicated to the relation between coherent structures and spectra. Here we focus on situations where coherent vortices are destroyed, and this leads us to concentrate on forced cases. To disrupt long-lived structures, one may adjust the forcing so as vortical structures of circular topology do not survive more than a few turn around times. This is a delicate and empirical task, and in practice, one tries different protocols and check, a posteriori, from the inspection of the vorticity field, that long-lived structures are no more there. In this context high resolution simulations [52,14,79] convincingly showed that classical behavior takes place; Herring et al. [52] were able to obtain different regimes, with and without coherent structures, and could conclude that as soon as long-lived structures, and could conclude that as soon as long-lived structures disappear from the field, classical behavior is restored. In the same spirit, Borue [13] obtained remarkable agreement with the classical theory. Figs. 62 and 63 display an enstrophy flux, constant over two decades of wave number, and a spectrum, agreeing with Kraichnan expectation, and suggesting logarithmic corrections may be within the reach of the measurement. The following estimate of



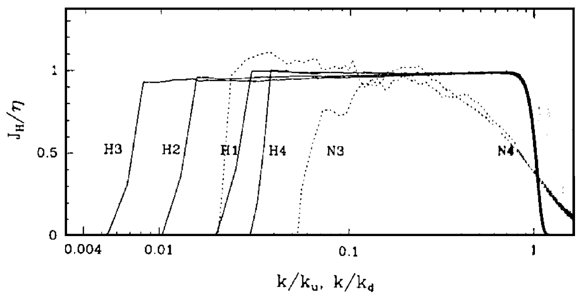


Fig. 62. Compensated spectra, obtained by multiplying the energy spectra by k^3, and for various Reynolds numbers.

Fig. 63. Energy fluxes in the spectral space, showing a constant transfer rate holds, directed towards small scales.

C' was proposed:

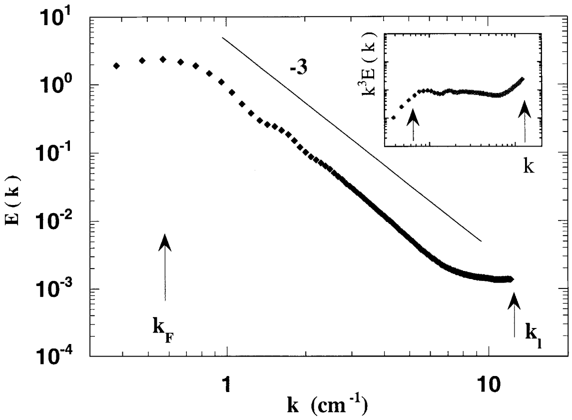
$$C' \approx 1.4 \ . \tag{37}$$

This is probably the best estimate to date.

On the (physical) experimental side, early attempts were made to observe the enstrophy cascade by measuring velocity fluctuations behind a grid towed in a mercury tank, in the presence of a strong magnetic field, which is expected to favor two-dimensionality. k^{-3} spectra were observed but interpretations, emphasizing on the role of the anisotropic Joule dissipation, or the role of the walls, were further proposed [128]; this makes the connection of this observation with the enstrophy cascade uncertain. In a recent period, soap film experiments provided evidences that enstrophy cascades may develop, consistently with classical expectations, in real systems [45,62,81,122]. More recently, Paret et al. [112], using electromagnetically driven flows in a stratified fluid layers, obtained neat k^{-3} spectra, in a system shown to be homogeneous and isotropic. The Kraichnan Batchelor constant was found consistent with numerical estimates. These experiments are displayed in Figs. 64 and 65.

It is instructive to point out the actual relation between the vorticity and the velocity field, using experimental data obtained in a context where an enstrophy cascade develops. Fig. 66 shows a cut of the velocity profiles corresponding to a vorticity field similar to Fig. 56. One shows the overall structure of the profiles displays variations over lengths on the order of 3 cm, i.e. a fraction of the box size. This substantiates the injection scale. Nonetheless, there are a few steep, localized gradients on this profile, signalling the presence of strong vorticity filaments. This is visible in Fig. 66. In the theory, we usually neglect these steep regions to estimate the strain, in a way somewhat similar to mean field approximation. One may worry that this raises an uncomfortable situation but the success of the theories discussed above, indicates this approximation is probably the first step of a selfconsistent approach.

Results on higher order structure functions, and distributions of vorticity increments were obtained in a physical experiment carried out by Paret et al. [113], which revealed the absence of intermittency in the enstrophy cascade, in good agreement with the expectations of Falkovich et al. [34] and Eyink [33]. Fig. 67 shows a set of five distributions of the vorticity increments,



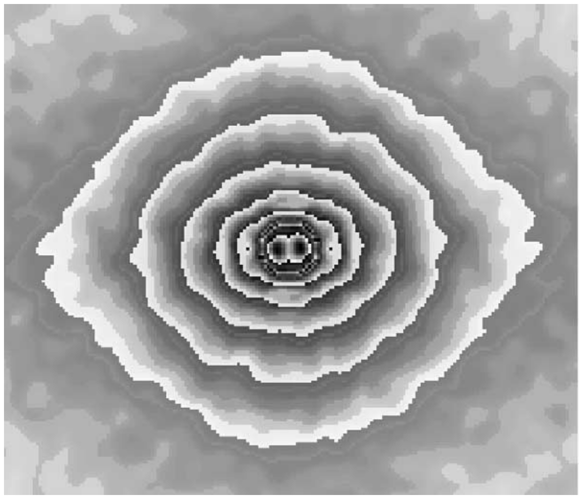
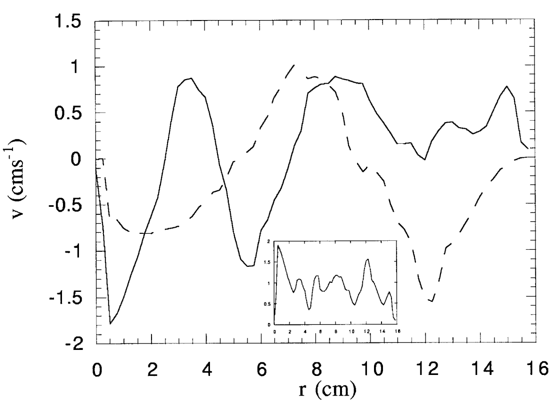


Fig. 64. Energy spectrum obtained in a physical experiment, displaying the expected k^3 spectrum.

Fig. 65. Two-dimensional energy spectrum, showing isotropy, in a physical experiment.



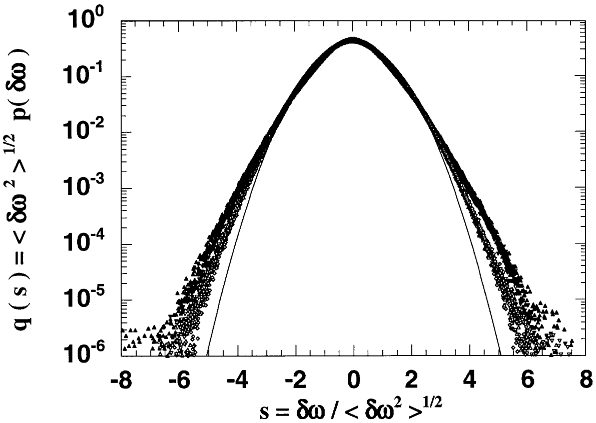
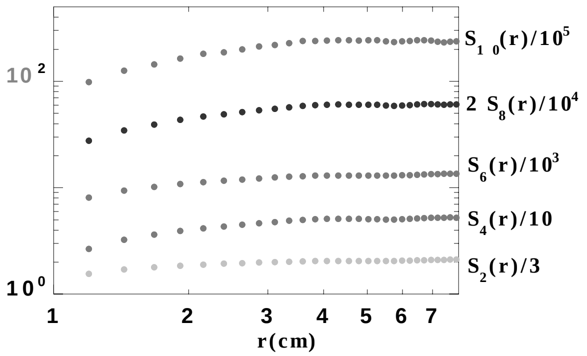


Fig. 66. Vorticity profile, taken at a given time, in the enstrophy cascade regime.

Fig. 67. Pdfs of the vorticity increments, in a physical experiment.

obtained for different inertial scales. As usual, in order to analyze shapes, the pdfs have been renormalized to impose their variance be equal to unity. The shapes of the pdfs are not exactly the same, but it is difficult to extract a systematic trend with the scale. Within experimental error, they seem to collapse onto a single curve, indicating absence of anomalous scaling, we usually associate with the absence of small scale intermittency; the analysis of the structure functions of the vorticity shown in Fig. 68 confirms this statement. Fig. 68 represents a series of vorticity



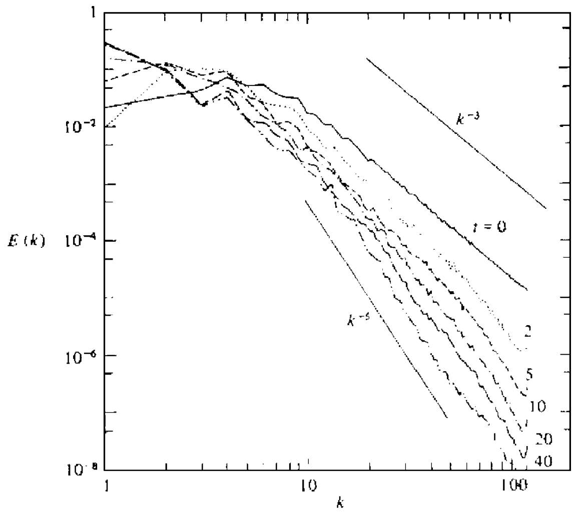


Fig. 68. Structure functions of the vorticity increments, in a physical experiment.

Fig. 69. Spectra in a freely decaying turbulent regime, at various times.

structure functions $S_p(r)$, obtained in such conditions, emphasizing the inertial domain, i.e. with r varying between 1 and 10 cm. The structure functions vary weakly with the scale, indicating the exponents are close to zero. The corresponding values fall in the range -0.05, 0.15, for p varying between 2 and 10; owing to experimental uncertainty, this is indistinguishable from zero. The result is compatible with the classical theory, for which the exponents are predicted to be zero at all orders. Concerning logarithmic deviations, such as those proposed by the theory [68,34], the experiment could not draw out any firm conclusion.

To conclude on the enstrophy cascade, one may say the theory is at a well advanced stage, even if at the moment, quantities such as the pdfs of the vorticity increments have not been determined theoretically. It seems the overall structure of the problem is documented, perhaps understood, and confrontation between theory and experiment leads to acceptable agreement. In this context, a number of observations made in soap films [81], revealing departures from classical behaviour, are challenging; they call for theoretical understanding, not necessarily in a strict two-dimensional framework.

10. Coherent structures vs. cascades

As we mentioned previously, two-dimensional turbulence has provided a remarkable context for the study of "coherent structures" and the interplay with the classical cascade theories. Describing the full statistics of a system including coherent structures probably represents at the moment one of the most challenging problems raised by two-dimensional flows. A prototype situation is the free decay of turbulence, whose large scales were previously discussed. Coherent structures are also observed in some forced systems, but the phenomenology is more compli-

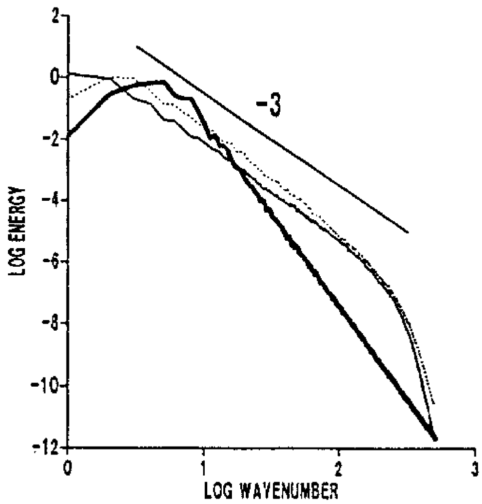


Fig. 70. Spectra in a freely decaying turbulent regime, for various times.

cated, and I focus here on freely decaying systems, on which most of the information available on the subject has been collected.

In freely decaying turbulence, one expects, from the transposition of Kolmogorov ideas, the development of energy spectra decreasing like k^{-3}. This has been shown by Batchelor [9]. Extending Kolmogorov approach [64] to the free decay problem, Batchelor predicted an energy spectrum in the form:

$$E(k) \sim t^{-2}k^{-3} . \tag{38}$$

As we mentioned earlier, substantial deviations from this spectral law were consistently observed in a number of numerical experiments (Figs. 69 and 70). The first observations of Lilly suggested k^{-3} but soon after, using improved resolution, Herring et al. (1974) [51] found a k^{-4} energy spectrum. As noted by Lesieur [77], contours of the vorticity plot already indicated the presence of what has been called later the coherent structures. Deviations from Batchelor spectral laws were further confirmed in a number of numerical simulations. In particular, a variety of slopes, typically lying between -3 and -6, depending on the range of time under consideration, and the initial conditions were obtained [124,72]. Recent work by Chasnov [22] indicates the same trends, although in his case, a proposal for the existence of a critical Reynolds number, at which the classical exponents are obtained, is made. We give here a few examples of the results.

To account for this diversity, Brachet et al. [15] proposed to split the decay regime into two steps: the first one is inviscid; it leads to the formation of isolated filaments, and is governed by the k^{-4} spectral law, proposed by Saffman [123]. In a second step, the filaments get packed and k^{-3} spectrum is recovered. This work offered an observation of the classical slope for the enstrophy cascade, along with an elegant explanation for the establishment of steeper slopes. However, this description did not encompass the considerable diversity of the spectral laws obtained in the simulations on decaying regimes. At about the same time, the relation between the presence of coherent structures and the spectral slopes was analyzed, and eventually the main origin of the deficiency of the classical theory could be figured out.

From the dynamical viewpoint, it is now clear that coherent vortices, with their stable topology, inhibit the transfers towards small scales, and thus tend to break the selfsimilarity of the process. In this view, coherent structures can be viewed as 'laminar drops in a turbulent background' as quoted in [124], or in more culinary terms, as lumps in a soup. On the other hand, the coherent structures may have complex internal structures, with sharp gradients, giving contributions to the spectrum, at all wave-numbers. Since the coherent structures tend to keep their internal vorticity at high Reynolds number, this possibility can be controlled by the initial conditions. If initial conditions include isolated cusps or fronts of vorticity with a stable topology, the contribution of the coherent structures to the spectrum can indeed be substantial throughout the decay process (note that it would be unlikely or physically hardly conceivable that vorticity cusps arise spontaneously in the decay process). This underlines the importance of the initial conditions in the spectral properties of decaying systems. The dynamical inhibition of the enstrophy cascade and possible contributions of the vortices to the spectrum are presently considered as the main causes for deviations from classical laws observed in decaying systems.

The contribution of the coherent structures on the spectral law obviously depends on their density and spatial distribution. One may stress that even if their density is small, since they retain most of the energy of the system, their contribution to the energy spectrum can be substantial. More specifically, it has been proposed that coherent structures are selfsimilar in space, i.e. their size distribution is selfsimilar [124]. Pushing the analysis further, a relation between the distributions of sizes of the coherent structures and the spectral slope has been offered by Santangelo et al. [124]. If the probability distribution of vortices as a function of their radius has the form

$$p(a) \sim a^{-\alpha}$$

then the spectral law is

$$E(k) \sim k^{\alpha-6} .$$

In this context, the spectral exponent is found controlled by the size distribution of the coherent structures, which depends on the initial conditions. In an ordinary cascade, the distribution of vortex sizes would not have any power law structure. With coherent structures, the distribution is a power law in scale, and the exponent is sensitive to the initial conditions. The possibility that the vortices have a selfsimilar structure has been further supported by numerical studies based on contour surgery technique, showing the presence of coherent structures at practically all scales of interest [30]. However, it is not clear that selfsimilarity of the coherent structures is a genuine property of freely decaying systems. The vorticity field presented in Section 2 does not offer any evidence of selfsimilarity. The existence of selfsimilarity certainly depends on the way the forcing is made, and on the range of time we consider, and the few available counter-examples leave an impression that the existence of selfsimilarity cannot be taken as a general feature of decaying two-dimensional systems.

Attempts have been further made to remove the coherent structures from the field and analyze the remaining part, i.e. that forming the background filling up the intervortical region. Although the procedure of removal incorporates some subjectiveness, the trend appears that the background display properties interpretable in terms of classical cascade theories. In particular, k^{-3} spectra are typically observed. This leads to the generally well accepted picture that decaying two-dimensional turbulence is a superposition of a classical background, with universal properties, and coherent structures, strongly dependent of the initial conditions.

In three dimensions, the coherent structures currently mentioned are located on the lower border of the spectrum, and they hardly affect the inertial range because vorticity intensifies along the energy cascade. In two dimensions, there is no such intensification process, and coherent structures may contaminate the spectrum without much effort. Coherent structures may thus deserve stronger attention in two than in three dimensions. The matter is completely different as soon as coherent structures are present at all scales. In this case, they are in position to control the statistical characteristics of the flow, in two and three dimensions as well, and this possibility is sometimes pushed forward as an alternative to the current description of turbulent systems [35].

A concept often used in two-dimensional studies is intermittency, and a confusion can be made between different definitions of intermittency. The intermittency usually referred to in turbulence studies (at least in three dimensions) is clearly presented in Frisch textbook [43]. To define the word, one operates in one dimension, and consider the flatness factor, defined by the quantity

$$F = \frac{\langle (u(x+r) - u(x))^4 \rangle}{\langle (u(x+r) - u(x))^2 \rangle^2} ,$$

where u is the variable at hand, x the coordinate, and the brackets are statistical averages. If this factor is scale dependent, with a power law, the system is called "intermittent". In the direct energy cascade in three dimensions, the exponent is negative. The flow thus becomes more and more intermittent or spotty as we go to smaller and smaller scales. In the inverse energy cascade, the only intermittency we can think of is large scale intermittency; this would imply a positive exponent for the power law of the flatness factor. This definition is neat, and fits well with the conceptual framework developed over the last decade, which emphasizes on anomalous exponents. Another definition for intermittency is often used in two-dimensional studies. It refers to the fact that the vorticity field is typically low in the intervortical region, and high within the coherent structures. This is a loose definition of intermittency, used to characterize the spottiness of the vorticity field rather than an anomalous scaling behavior. It would be more appropriate perhaps to call this intermittency 'spottiness', in order to avoid confusion. Intermittency is a special case of spottiness, since when a direct cascade is intermittent, the system becomes more and more spotty as we probe smaller and smaller scales. On the other hand, one may have situations where the system is spotty and nonintermittent (in the sense discussed above). The condensed states described in Section 8 offer an example of such a situation: here we have a spotty vorticity field but no anomalous scaling in the vorticity statistics. The flatness factor may be high in this case, but still the system is nonintermittent, because this factor does not vary with the scale.

There have been only a few attempts figuring out a framework for describing turbulent systems with coherent structures. It is certainly a difficult task. Any decomposition of the field into two parts faces the problem of handling complicated correlations. As stressed by M Farge, we are not in a situation where the two components occupy well separate ranges of scale, as is often the case in condensed matter physics. One can say at the moment, the problem is a formidable theoretical challenge.

11. Conclusions

Let us summarize a few points made in the review:

- It is well established that cascades develop in two-dimensional systems; these cascades require coherent structures to be disrupted — a rather loose concept —. These last years, a number of detailed studies of the enstrophy and inverse energy cascades have been made available, and now, one can say that the two-point statistics of these cascades are rather well documented. Concerning the enstrophy cascade, it is interesting to mention that theory is at a well advanced stage; most of the characteristics of the cascade seem to fall under the grasp of the theory.
- In contrast, the inverse cascade is much less understood. We now know that this cascade is nonintermittent (and thus, Kolmogorov theory fully applies), but, so far, no hint has been found for explaining the absence of intermittency. The statistics of the velocity increments is close to Gaussian, but this fact seems hard to exploit theoretically. Concerning the pair dispersion problem in the inverse cascade, it appears that Kolmogorov theory successfully captures several aspects of the problem, in particular the selfsimilar properties. When the dissipation length of the inverse cascade is larger than the system size, a 'condensation' regime takes place. In this regime, interesting phenomena arise, such as the formation of point vortices like structures. Theoretical understanding of these phenomena stands, at the moment, at a preliminary stage.
- When coherent structures are present, the flow dynamics may thoroughly differ from classical expectations, i.e. those obtained by using Kolmogorov type arguments. Nonclassical exponents have been repeatedly observed along the years, and it is now established they originate in the presence of long-lived structures. The influence of the coherent structure is both kinematical (by contributing to the vorticity field) and dynamical (by inhibiting the development of the enstrophy cascade). The absence of universality, the strong dependence with the initial conditions,... are characteristics which are challenging for the construction of a general theory of these systems. On the other hand, coherent structures are important to incorporate in our vision of two-dimensional systems, since they spontaneously arise, trap much energy, and (by definition) are long-lived. A situation, perhaps more favorable to theoretical analysis, is the free decay of two-dimensional turbulence. In this problem, power laws are observed, with apparently robust and possibly universal exponents. An elegant approach, mostly phenomenological, provides deep insight into the problem. A (hopefully) helpful link between this problem and the dispersion problem has been pointed out.
- Statistical theories, initiated by Onsager, represent an appealing approach to two-dimensional turbulence; it has given rise to an impressive amount of work. These last years, a method for handling continuous vorticity field has been proposed. Also, new observations have been made available, so that theoretical analysis could be confronted with experiment. Although the theory proved to be successful in a few cases, it turned out to

fail in a number of situations, at quantitative and qualitative levels. The main issue seems whether two-dimensional turbulent systems possess ergodic properties justifying a search for equilibrium states.

I hope to have shown, in this review, that two-dimensional turbulence encompasses a rich variety of phenomena; in three dimensions, there is no life for turbulence outside the direct energy cascade, while in two dimensions, there exists several types of cascades, and turbulence, in the ordinary sense, may exist without cascade at all. In this respect, the two-dimensional world is surprisingly more wealthy than the three-dimensional one. On another aspect, two-dimensional turbulence may be more pleasant to teach than three-dimensional turbulence, in the sense that the phenomena can be visualized in all detail. On the other hand, the phenomenology is more complicated, and the diversity of landscapes in Flatland may appear as a bit disorientating; however, thanks to progress made and in the recent period, visibility has been improved, and it certainly is an easier task for the traveller to find his way.

12. Uncited references

[4,28,41,44,75]

Acknowledgements

The author is grateful to M.C. Jullien, A. Babiano, C Basdevant, T. Dubos and M. Farge for valuable discussions related to this work. He is particularly indebted to Y. Pomeau and B. Legras for comments on the first version of the manuscript, and I. Procaccia and K.R. Sreenivasan for additional remarks on the latest versions.

This work has been supported by Centre National de la Recherche Scientifique, Ecole Normale Supérieure, Universités Paris 6 and Paris 7.

References

[1] G. Huber, P. Alstrøm, Universal decay of vortex density in two dimensions, Phys. A 195 (1993) 448.
[2] H. Aref, E.D. Siggia, Vortex dynamics of the two-dimensional turbulent shear layer, J. Fluid Mech. 100 (1980) 705.
[3] H. Aref, Integrable, chaotic and turbulent vortex motion in two-dimensional flows, Annu. Rev. Fluid Mech. 15 345.
[4] A. Babiano, C. Basdevant, R. Sadourny, Structure functions and dispersion laws in two-dimensional turbulence, J. Atmos. Sci. 42 (1985) 941–949.
[5] A. Babiano, C. Basdevant, P. Le Roy, R. Sadourny, Relative dispersion in two-dimensional turbulence, J. Fluid Mech. 214 (1990) 535–557.
[6] C. Basdevant, T. Philipovitch, On the validity of the Weiss criterion in two-dimensional turbulence, Physica D 73 (1994) 17–30.
[7] G.K. Batchelor, Diffusion in field of homogeneous turbulence II: the relative motion of particles, Proc. Camb. Phil. Soc. 48 (1952) 345–362.
[8] G.K. Batchelor, Theory of homogeneous turbulence, Cambridge University Press, Cambridge, 1956.
[9] G.K. Batchelor, Computation of the energy spectrum in homogeneous two-dimensional turbulence, Phys. Fluids 12 (Suppl. II) (1969) 233–239.

[10] R. Benzi, S. Patarnello, P. Santangelo, On the statistical properties of two-dimensional decaying turbulence, Europhys. Lett. 3 (1987) 811–818.
[11] R. Benzi, S. Patarnello, P. Santangelo, Self-similar coherent structures in two-dimensional decaying turbulence, J. Phys. A: Math. Gen. 21 (1988) 1221–1237.
[12] G. Boffetta, A. Celani, M. Vergassola, Inverse energy cascade in two-dimensional turbulence: deviations from Gaussian behavior, Phys. Rev. E 61 (2000) R29.
[13] V. Borue, Spectral exponents of enstrophy cascade in stationary two-dimensional homogeneous turbulence, Phys. Rev. Lett. 71 (1993) 3967–3970.
[14] V. Borue, Inverse energy cascade in stationary two-dimensional homogeneous turbulence, Phys. Rev. Lett. 72 (1994) 1475–1478.
[15] M.E. Brachet, M. Meneguzzi, H. Politano, P.L. Sulem, The dynamics of freely decaying two-dimensional turbulence, J. Fluid Mech. 194 (1988) 333–349.
[16] H. Brands, Coherent structures and statistical mechanics in two-dimensional flows. These, University of Twente, The Netherlands, 1998.
[17] G.L. Brown, A. Roshko, On density effects and large structure in turbulent mixing layers, J. Fluid Mech. 64 (1974) 775–816.
[18] O. Cardoso, D. Marteau, P. Tabeling, Experimental study of the free decay of quasi-2D turbulence, Phys. Rev. E 49 (1994) 454.
[19] P.H. Chavanis, J. Sommeria, Equilibrium states of two-dimensional turbulence: an experimental study, J. Fluid Mech. 134 (1996) 267.
[20] P.H. Chavanis, J. Sommeria, Classification of robust isolated vortices in two-dimensional hydrodynamics, J. Fluid. Mech. 356 (1998) 259.
[21] G.F. Carnevale, J.C. McWilliams, Y. Pomeau, J.B. Weiss, W.R. Young, Evolution of vortex statistics in two-dimensional turbulence, Phys. Rev. Lett. 66 (1991) 2735–2737.
[22] J.R. Chasnov, On the decay of two-dimensional homogeneous turbulence, Phys. Fluids 9 (1997) 171–180.
[23] Y. Couder, Observation experimentale de la turbulence bidimensionnelle dans un film liquide mince, C. R. Acad. Sc. Paris 297 (1983) 641–645.
[24] Y. Couder, Two-dimensional grid turbulence in a thin liquid film, J. Physique Lett. 45 (1984) 353–360.
[25] Y. Couder, C. Basdevant, Experimental and numerical study of vortex couples in two-dimensional flow, J. Fluid Mech. 173 (1986) 225.
[26] Peilong Chen, M.C. Cross, Phase diagram for coherent vortex formation in the two-dimensional inviscid fluid in circular geometries, Phys. Rev. E 50 (1994) 2022.
[27] M.A. Denoix, Etude experimentale de structures stables en turbulence bidimensionnelle, Comparaison avec une theorie de mcanique statistique, These de doctorat. Institut National Polytechnique, Grenoble. 1992.
[28] F.V. Dolzhanskii, V.A. Krymov, D.Y. Mani, An advanced experimental investigation of quasi-two-dimensional shear flows, J. Fluid Mech. 241 (1992) 705–722.
[29] D.G. Dritschell, B. Legras, Modelling oceanic and atmospheric vortices, Physics Today (1993) 44–51.
[30] D.G. Dritschell, A general theory for two-dimensional vortex interactions, J. Fluid Mech 293 (1995) 269–303.
[31] D.G. Dritschell, Contour dynamics/surgery on a sphere, J. Comput. Phys. 77 (1988) 77–240.
[32] T. Dubos, J. Paret, A. Babiano, P. Tabeling, Study of the condensed regime, (2001) submitted.
[33] G.L. Eyink, Exact results on stationary turbulence in 2D: consequences of vorticity conservation, Physica D 91 (1996) 97–142.
[34] G. Falkovich, V. Lebedev, Nonlocal vorticity cascade in two dimensions, Phys. Rev. E 49 (1994) 1800–1803.
[35] Farge, Wavelet transforms and their applications to turbulence, Annu. Rev. Fluid Mech. 24 (1992) 395–457.
[36] K.S. Fine, C.F. Driscoll, J.H. Malmberg, T.B. Mitchell, Measurements of symmetric vortex merger, Phys. Rev. Lett. 67 (1991) 588–591.
[37] K.S. Fine, A.C. Cass, W.G. Flynn, C.F. Driscoll, Relaxation of 2D turbulence to vortex crystals, Phys. Rev. Lett. 75 (1995) 3277–3280.
[38] J.B. Flór, G.J.F. Van Heijst, An experimental study of dipolar structures in a stratified fluid, J. Fluid Mech. 279 (1994) 101–133.

[39] J.B. Flór, G.J.F. Van Heijst, R. Delfos, Decay of dipolar structures in a stratified fluid, Phys. Fluids 7 (1995) 374–383.
[40] J.B. Flór, G.J.F. Van Heijst, Stable and unstable monopolar vortices in a stratified fluid, J. Fluid Mech. 311 (1996) 257–287.
[41] B. Fornberg, A numerical study of 2-D turbulence, J. Comp. Phys. 25 (1977) 1–31.
[42] U. Frisch, P.L. Sulem, Numerical simulation of the inverse cascade in two-dimensional turbulence, Phys. Fluids 27 (1984) 1921–1923.
[43] U. Frisch, Turbulence, Cambridge University Press, Cambridge, 1995.
[44] U. Frisch, A. mazzino, A. Noullez, M. Vergassola, (1999): Lagrangian method for multiple correlations in passive scalar advection, in: Phys. Fluids, Special number for Los Alamos Conference, May, 1988, a celebration of R.H. Kraichnan.
[45] M. Gharib, P. Derango, A liquid film (soap film) tunnel to study two-dimensional laminar and turbulent shear flows, Physica D 47 (1989) 406–416.
[46] R.W. Griffiths, E.J. Hopfinger, Experiments with baroclinic vortex pairs in a rotating fluid, J. Fluid. Mech. 173 (1986) 501–518.
[47] A.E. Hansen, D. Marteau, P. Tabeling, Two-dimensional turbulence and dispersion in a freely decaying system, Phys. Rev. E 58 (1998) 7261–7271.
[48] A. Johansen, A. Hansen, D. Sornette, Punctuated vortex coalescence and discrete scale invariance in two-dimensional turbulence, Physica D 138 (2000) 302–315.
[49] X. He, On exponent of vortex number in 2D turbulence, in: Advances in Turbulence VI, Vol. 277, Kluwer Academic Publishers, Dordrecht, 1996.
[50] H.G.E. Hentschel, I. Procaccia, Relative diffusion in turbulent media: the fractal dimension of clouds, Phys. Rev. A 29 (1984) 1461–1470.
[51] J.R. Herring, S.A. Orszag, R.H. Kraichnan, D.G. Fox, Decay of two-dimensional homogeneous turbulence, J. Fluid Mech. 66 (1974) 417–444.
[52] J.R. Herring, J.C. McWilliams, Comparison of direct numerical simulation of two-dimensional turbulence with two-point closure: the effects of intermittency, J. Fluid Mech. 153 (1985) 229–242.
[53] E.J. Hopfinger, R.W. Griffiths, M. Mory, The structure of turbulence in homogeneous and stratified rotating fluids, Journal de mecanique theorique et appliquee Numero special, 1983, pp. 21–44.
[54] M. Hossain, W.H. Matthaeus, D. Montgomery, Long-time states of inverse cascades in the presence of a maximum length scale, J. Plasma Phys. 30 (1983) 479–493.
[55] B.L. Hua, P. Klein, An exact criterion for the stirring properties of nearly two-dimensional turbulence, Physica D 113 (1998) 98–110.
[56] X.P. Huang, C.F. Driscoll, Relaxation of 2D turbulence to a metaequilibrium near the minimum enstrophy state, Phys. Rev. Lett. 72 (1994) 2187–2190.
[57] J. Jimenez, Vorticity and Turbulence, Cours de l'Ecole Polytechnique, France, 2000.
[58] J. Jimenez, H.K. Moffatt, Carlos Vasco, The structure of the vortices in freely decaying two-dimensional turbulence, J. Fluid Mech. 313 (1996) 209–222.
[59] B. Jüttner, A. Thess, J. Sommeria, On the symmetry of selforganized structures in two-dimensional turbulence, Phys. Fluids 7 (1995) 2108.
[60] B. Juttner, D. Marteau, P. Tabeling, A. Thess, Numerical simulations of experiments on quasi-two-dimensional turbulence, Phys. Rev. E 55 (1997) 5479–5488.
[61] C. Jullien, J. Paret, P. Tabeling, Experimental observation of the Richardson dispersion, Phys. Rev. Lett. 55 (1997) 5479–5488.
[62] H. Kellay, X.-L. Wu, W.I. Goldburg, Experiments with turbulent soap films, Phys. Rev. Lett. 74 (1995) 3975–3978.
[63] S. Kida, Motion of an elliptical vortex in a uniform shear flow, J. Phys. Soc. Japan 50 (1981) 3517–3520.
[64] A.N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Dokl. Akad. Nauk USSR 30 (1941) 301–305.
[65] A.N. Kolmogorov, Dissipation of energy in locally isotropic turbulence, Dokl. Akad. Nauk USSR 32 (1941) 16–18.
[66] R.H. Kraichnan, Dispersion of particle pairs in homogeneous turbulence, Phys. Fluids 9 (1966) 1937–1943.

[67] R.H. Kraichnan, Inertial ranges transfer in two-dimensional turbulence, Phys. Fluids 10 (1967) 1417–1423.
[68] R.H. Kraichnan, Inertial-range transfer in two- and three-dimensional turbulence, J. Fluid Mech. 47 (1971) 525–535.
[69] R.H. Kraichnan, D. Montgomery, Two-dimensional turbulence, Rep. Prog. Phys. 43 (1980) 547–619.
[70] M. Larcheveque, Pressure field, vorticity field and coherent structures in two-dimensional incompressible turbulent flows, Theor. Comp Fluid Dynamics 5 (1993) 215–222.
[71] I am indebted to B. Legras for the remarks related to the relevance of two-dimensional approximation to geo-physical flows.
[72] B. Legras, P. Santangelo, R. Benzi, High resolution numerical experiments for forced two-dimensional turbulence, Europhys. Lett. 5 (1988) 37–42.
[73] B. Legras, D.G. Dritschel, Vortex stripping and the generation of high vorticity gradients in two-dimensional flows, Appl. Sci. Res. 51 (1993) 445–455.
[74] Private communication.
[75] C.E. Leith, Atmospheric predictability and two-dimensional turbulence, J. Atmos. Sci. 28 (1971) 145–161.
[76] C.E. Leith, Minimum enstrophy vortices, Phys. Fluids 27 (1984) 1388.
[77] M. Lesieur, Turbulence in Fluids, 2nd revised Edition, Kluwer Academic Publishers, Dordrecht, 1990.
[78] D.K. Lilly, Numerical simulation of developing and decaying two-dimensional turbulence, J. Fluid Mech. 45 (1971) 395–415.
[79] M.E. Maltrud, G.K. Vallis, Energy spectra and coherent structures in forced two-dimensional and beta-plane turbulence, J. Fluid Mech. 228 (1991) 321–342.
[80] A. Mariotti, B. Legras, D. Dritschel, Vortex stripping and the erosion of coherent structures in two-dimensional flows, Phys. Fluids 6 (1994) 3954–3962.
[81] B.K. Martin, X.-L. Wu, W.I. Goldburg, Spectra of decaying turbulence in soap film, Phys. Rev. Lett. 80 (1998) 3964–3967.
[82] D. Marteau, O. Cardoso, P. Tabeling, Equilibrium states of 2D turbulence: an experimental study, Phys. Rev. E 51 (1995) 5124.
[83] D. Marteau, Etude exprimentale du dclin de la turbulence bidimensionnelle, thse de doctorat, Universit Pierre et Marie Curie, Paris, 1996.
[84] W.H. Matthaeus, W.T. Stribling, D. Martinez, S. Oughtin, D. Montgomery, Decaying, two-dimensional, Navier–Stokes turbulence at very long times, Physica D 51 (1991) 531.
[85] J.C. McWilliams, The emergence of isolated coherent vortices in turbulent flow, J. Fluid Mech. 146 (1984) 21–43.
[86] J.C. McWilliams, The vortices of two-dimensional turbulence, J. Fluid Mech. 219 (1990) 361–385.
[87] M.V. Melander, N.J. Zabusky, J.C. McWilliams, Symmetric vortex merger in two dimensions: causes and conditions, J. Fluid Mech. 195 (1988) 303–340.
[88] J. Miller, Statistical mechanics of Euler equations in two-dimensions, Phys. Rev. Lett. 65 (1990) 2137–2140.
[89] J. Miller, P.B. Weichman, M.C. Cross, Statistical mechanics, Euler's equation and Jupiter's Red Spot, Phys. Rev. A 45 (1992) 2328.
[90] A.S. Monin, A.M. Yaglom, Statistical Fluid Mechanics, Vol. 2, The MIT press, Cambridge, 1975.
[91] D. Montgomery, G. Joyce, Statistical mechanics of "negative temperature states", Phys. Fluids 17 (1974) 1139.
[92] D. Montgomery, W.H. Matthaeus, W.T. Stribling, D. Martinez, S. Oughton, Relaxation in two dimensions and the "sinh-Poisson" equation, Phys. Fluids A 4 (1992) 3.
[93] D. Moore, P. Saffman, Structure of a line vortex in an imposed strain, in: J. Olsen, A. Goldberg, N. Rogers (Eds.), Aircraft Wake Turbulence and its Detection, Plenum, New York, 1971.
[94] M. Nelkin, Turbulence in fluids, Am. J. Phys 68 (2000) 310.
[95] The Kolmogorov paper is a usual reference for introducing dissipation anomaly. However, as pointed out by K.R. Sreenivasan (private communication), G.I. Taylor made the argument before Kolmogorov.
[96] The issue is not homogeneity and isotropy, which may indeed hold in the presence of coherent structures. The issue is the notion of statistical averaging, which may be difficult to substantiate, especially when long live structures are present. This happens for the decay problem, when one analyzes a single realization of the process.

[97] This is not easy to do. In the general case, one must remove the translational velocity and the mean rotation of the candidate.
[98] In the case of nonneutral populations, other invariants should in principle be introduced, such as the total vorticity; in the special case of circular boundaries, angular momentum conservation must be added. We shall, however, restrict ourselves to the case of neutral populations, as is traditionally done for this problem.
[99] The argument is not straightforward, and requires some calculation. Take a triad with $p < k < q$; on using the conservation equations for the energy and the enstrophy, one shows that more enstrophy is transferred on q than on p, and more energy is transferred on p than on q. By iterating the argument, one obtains the result that at large Reynolds number, enstrophy is essentially transferred towards large wave-numbers while energy is essentially transferred towards small wave-numbers. A detailed reasoning is displayed in Lesieur's book, and in the paper by Kraichnan.
[100] This remark has been made by Kraichnan, 1967.
[101] The authors seem to have no responsibility in the name given later to their theory. The name can be critized: for instance, "Universal" is perhaps a bit too strong, although several experiments, performed in different conditions, suggest the exponents are 'universal', i.e. they are the same for a broad range of different initial conditions. Also, the word 'theory' is perhaps slightly in accurate, if we consider that a true theory should be constructed from first principles, which is not the case for 'universal decay theory'. Nonetheless, the name 'universal decay theory' is useful; it allows to refer to a particular framework, using only three words, and without being misleading.
[102] There exists some confusion concerning an experimental work performed in thin layers of electrolytes, dedicated to studying the problem of the free decay of turbulence. A first series of experiments was conducted in nonstratified electrolytic layers; they are reported in Cardoso et al. ([18]); this work revealed discrepancies between universal decay theory and experiments. The origin of the discrepancy is due to the presence of three-dimensional flows which develop within the thin layer. Later, the authors suppressed these flows by working with stratified layers (see Ref. [110]). The study of the free decay, using stratified layers, is given by Hansen et al. [47], 7261–7271; in such conditions, the authors found good agreement between universal decay theory and experiment. The second series of experiment is relevant to two-dimensional flows, while the first one, contaminated by three-dimensional effects, is probably more difficult to interpret. The situation is clear; it has unfortunately been made confusing by the publication of preliminary measurements (Experimental study of freely decaying two-dimensional turbulence, P. Tabeling, S. Burkhart, O. Cardoso, H. Willaime, Phys. Rev. Lett., 67, 3772 (1991)), using nonstratified layers, and in which the authors claimed that experiments agree with universal decay theory; at that time, the errors bars were too large to draw out such a conclusion, for nonstratified layers.
[103] The general formula in d dimensions is $12/d(d+2)$.
[104] The speculation is that there is some large scale vorticity, produced by the condensation mechanism; these vorticity patches may undergo filamentation and give rise to an enstrophy cascade; this would favor the formation of a k^{-3} spectrum.
[105] A.M. Obukhov, Dokl. Akad. USSR 32 (1941) 22.
[106] A.M. Obukhov, Izv. Akad. Nauk. USSR, Ser. Georgr. i. Geofiz. 5 (1941) 453.
[107] L. Onsager, Statistical hydrodynamics, Nuevo Cimento Suppl. 6 (1949) 279–287.
[108] O. Paireau, Dynamique d'un tourbillon cisaille, interaction chimie-melange: tudes experimentales, thse de doctorat, Universite Pierre et Marie curie, Paris, 1997.
[109] O. Paireau, P. Tabeling, B. Legras, Vortex subjected to a shear: an experimental study, J. Fluid. Mech. 351 (1997) 1.
[110] J. Paret, D. Marteau, O. Paireau, P. Tabeling, Are flows electromagnetically forced in thin stratified layer two-dimensional?, Phys. Fluids 9 (1997) 3102–3104.
[111] J. Paret, P. Tabeling, Experimental observation of the two-dimensional inverse energy cascade, Phys. Rev. Lett. 79 (1997) 4162–4165.
[112] J. Paret, P. Tabeling, Intermittency in the 2D inverse cascade of energy: experimental observations, Phys. Fluids 10 (1998) 3126–3136.
[113] J. Paret, C. Jullien, P. Tabeling, Statistics of the two-dimensional enstrophy cascade, Phys. Rev. Lett. 83 (1999) 3418.

[114] F. Pasquill, F.B. Smith, Atmospheric Diffusion, 3rd Edition, Wiley, New York, 1983.
[115] Y. Pomeau, Statistical approach to 2D turbulence, in: O. Cardoso, P. Tabeling (Eds.), Turbulence: A tentative dictionary, NATO ASI Series, B: physics, Plenum Press, 1994.
[116] Y. Pomeau, Vortex dynamics in perfect fluids, J. Plasma Phys. 56 (3) (1996) 407.
[117] A.M. Polyakov, The theory of turbulence in two-dimensions, Nucl. Phys. B 396 (1993) 367–385.
[118] A. Provenzalle, Transport by coherent barotropic vortices. Ann. Rev. Fluid. Mech. (2000) to appear.
[119] L.F. Richardson, Atmospherical diffusion shown on a distance-neighbour graph, Proc. R. Soc. Lond. A 110 (1926) 709–737.
[120] R. Robert, J. Sommeria, Statistical equilibrium states for two-dimensional flows, J. Fluid Mech. 229 (1991) 291–310.
[121] H.A. Rose, P.L. Sulem, Fully developed turbulence and statistical mechanics, J. Physique 39 (1978) 441–484.
[122] M.A. Rutgers, Forced 2D turbulence: experimental evidence of simultaneous inverse energy and forward enstrophy cascades, Phys. Rev. Lett. 81 (1998) 2244–2247.
[123] P.J. Saffman, Vortex Dynamics, Cambridge University Press, Cambridge, 1995.
[124] P. Santangelo, R. Benzi, B. Legras, The generation of vortices in high-resolution, two-dimensional, decaying turbulence and the influence of initial conditions on the breaking of self-similarity, Phys. Fluids A 1 (1989) 1027–1034.
[125] M.F. Shlesinger, B.J. West, J. Klafter, Levy dynamics of enhanced diffusion: application to turbulence, Phys. Rev. Lett. 58 (1987) 1100–1103.
[126] E. Siggia, H. Aref, Point-vortex simulation of the inverse energy cascade in two-dimensional turbulence, Phys. Fluids 24 (1981) 171–173.
[127] E. Segre, S. Kida, Late states od incompressible 2D decaying vorticity fields, Fluid Dynamics Res. 24 (1998) 171–173.
[128] J. Sommeria, R. Moreau, Why, how, and when, MHD turbulence becomes two-dimensional, J. Fluid Mech. 118 (1982) 507–518.
[129] J. Sommeria, Experimental study of the two-dimensional inverse energy cascade in a square box, J. Fluid Mech. 170 (1986) 139–168.
[130] J. Sommeria, C. Staquet, R. Robert, Final equilibrium state of a two dimensional shear layer, J. Fluid. Mech. 233 (1991) 661.
[131] A. Thess, J. Sommeria, B. Jttner, Inertial organization of a two-dimensional turbulent vortex street, Phys. Fluids 6 (1994) 2417–2429.
[132] R.R. Trieling, Two-dimensional vortices in strain and shear flows, Ph.D. Thesis, Eindhoven, The Netherlands, 1996.
[133] E. Trizac, J.P. Hansen, Dynamic scaling behavior of balistic coalescence, Phys. Rev. Lett. 74 (1995) 4114; Dynamics and Growth of Particles Undergoing Balistic Coalescence, J. Stat. Phys. 82 (1996) 5/6.
[134] G.J.F. Van Heijst, J.B. Flòr, Dipole formation and collisions in a stratified fluid, Nature 340 (1989) 212–215.
[135] G.J.F. Van Heijst, et al., preprint, 2001.
[136] O. Agullo, A.D. Varga, Exact two vortices solution of Navier–Stokes equations, preprint, 1996.
[137] J. Verron, J. Sommeria, Numerical simulation of a two-dimensional experiment in magneto-hydrodynamics, Phys. Fluids 30 (1986) 732.
[138] J.B. Weiss, J.C. McWilliams, NonErgodicity of point vortices, Phys. Fluids A 3 (1991) 835.
[139] J.B. Weiss, J.C. McWilliams, Temporal scaling behavior of decaying two-dimensional turbulence, Phys. Fluids A 5 (1993) 608–621.
[140] N. Whitaker, B. Turkington, Maximum entropy states for rotating patches, Phys. Fluids 6 (1994) 3963.
[141] L.M. Smith, V. Yakhot, Bose condensation and small-scale structure generation in a random force driven 2D turbulence, Phys. Rev. Lett. 71 (1993) 352–355.
[142] L.M. Smith, V. Yakhot, Finite size effects in forced two-dimensional turbulence, J. Fluid Mech. 274 (1994) 115–138.
[143] V. Yakhot, Two-dimensional turbulence in the inverse cascade range, preprint, 2001.
[144] N.J. Zabusky, G.S. Deem, Dynamical evolution of two-dimensional unstable shear flows, J. Fluid Mech. 47 (1971) 353.