Decision Making
Part 5: Preference Orders and Group Decision

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Topics in this part

- Binary relation: binary relation, properties of binary relations, Pareto and lexicographic orders, maximal/minimal elements, greatest/least elements, graphical representation, $P_{\max}(A)$, $P_{\min}(A)$, $L_{\max}(A)$, $L_{\min}(A)$

- Preference order: strict preference order, indifference relation, weak preference order

- Group decision and social choice: group decision problem, constitution

- Arrow's impossibility theorem: Arrow's axioms, Arrow's Impossibility Theorem, discussion

Reference:
S. French, Decision theory: an introduction to the mathematics of rationality, Ellis Horwood, 1986, Sections 3.1–3.5, 8.1–8.2

Preference order

Why orders?

In a typical decision making problem, the decision maker chooses the "most preferable alternative" from a (finite or infinite) set of alternatives. The set of alternatives is usually finite in this part.

To make a decision the decision maker should have a preference order on the set of alternatives, for otherwise it does not make sense to talk about the "most preferable" alternative.

So let us spend some time to discuss orders.

Binary relations

Definition 1. Let A be a set. Let

$$A \times A := \{(a, b) \mid a, b \in A\}$$

be the Cartesian product of A with itself.

Any subset of $A \times A$ is called a binary relation on A.

Definition 2. Let R be a binary relation on A (that is, $R \subseteq A \times A$). If $(a, b) \in R$, we write aRb and say that a has relation R with b.

Example 1. (binary relation)

Let A={2,3,4,5,6,7,8,9,10}. We say that a is a multiple of b if there is an integer $k \neq 1, -1$ such that $a = kb$. We can formally define this binary relation R by

$$R = \{(4,2), (6,2), (6,3), (8,2), (8,4), (9,3), (10,2), (10,5)\}.$$

Definition 3. A binary relation R on a set A is said to be

- transitive if, for any $a, b, c \in A$, $(aRb$ and $bRc) \implies aRc$;
- reflexive if, for any $a \in A$, aRa;
- comparable or complete if, for any $a, b \in A$, either aRb or bRa or both hold;
- symmetric if, for any $a, b \in A$, $aRb \implies bRa$;
- asymmetric if, for any $a, b \in A$, $aRb \implies \neg bRa$;
- antisymmetric if, for any $a, b \in A$, $(aRb$ and $bRa) \implies a = b$.

Definition 4. A binary relation R on a set A is called

- a strict order if it is transitive and asymmetric;
- a weak order if it is transitive and comparable;
- an equivalence relation if it is reflexive, transitive, and symmetric;
- a partial order if it is reflexive, transitive, and antisymmetric;
- a linear order if it is a comparable partial order.

A set equipped with a partial order is called a partially ordered set (poset). Posets are important objects of study in mathematics, especially in discrete mathematics and combinatorics.

Strict and weak orders are most relevant to decision making.

Example 2. (strict order)

Let

$$A = \mathbb{R}^2 = \{\mathbf{x} = (x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}.$$

Define

$$\theta = \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in A,\ a_1 = b_1,\ a_2 > b_2\}.$$

Then θ is strict order.

Example 3. (weak order)

Let

$$A = \mathbb{R}^2 = \{\mathbf{x} = (x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}.$$

Define

$$\hat{\theta} = \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in A,\ a_1 + a_2 \geq b_1 + b_2\}.$$

Then $\hat{\theta}$ is a weak order on A.

Example 4. (equivalence relation)

Let

$$A = \mathbb{R}^2 = \{\mathbf{x} = (x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}.$$

Define

$$\bar{\theta} = \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in A,\ a_2 = b_2\}.$$

Then $\bar{\theta}$ is an equivalence relation on A.

Pareto order

Definition 5. Let $n \geq 1$ be an integer. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n, \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, define

$$\mathbf{a} P \mathbf{b} \quad \text{if and only if } a_i \geq b_i \text{ for all } i = 1, \ldots, n.$$

This binary relation P on $\mathbb{R}^n$ is called the Pareto order on $\mathbb{R}^n$.

Alternatively, $P = \{(\mathbf{a}, \mathbf{b}) : \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, a_i \geq b_i \text{ for all } i = 1, \ldots, n\}$.

Any subset $A \subseteq \mathbb{R}^n$ inherits this Pareto order but restricted to A.

Example 5. (Pareto order)

Four students, Alice (a), Bob (b), Catherine (c), and David (d) are assessed on three criteria (research, grades, progress). The results are depicted in the table below.

	Research	Grades	Progress
a	9	9	8
b	9	9	7
c	8	10	9
d	8	9	6

Use the Pareto order to compare the four students.

Lemma 1. The Pareto order on any subset of $\mathbb{R}^n$ is a partial order, i.e. it is

- transitive;
- reflexive;
- antisymmetric.

Exercise: prove the above lemma.

Lexicographic order

Definition 6. Let $n \geq 1$ be an integer. For
$\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n, \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$, define

$$\mathbf{a} L \mathbf{b}$$

if and only if either

$$\mathbf{a} = \mathbf{b}$$

or

$$a_k > b_k$$

where k is the smallest subscript i for which $a_i \neq b_i$.

This binary relation L is called the lexicographic order on $\mathbb{R}^n$.

Any subset $A \subseteq \mathbb{R}^n$ inherits this lexicographic order but restricted to A.

Example 6. (Lexicographic order)

Four students, Alice (a), Bob (b), Catherine (c), and David (d) are assessed on three criteria (research, grades, progress). The results are depicted in the table below.

	Research	Grades	Progress
a	9	9	8
b	9	9	7
c	8	10	9
d	8	9	6

Use the Lexicographic order to compare the four students.

Lemma 2. The lexicographic order on any subset of $\mathbb{R}^n$ is a partial order as well as a weak order. In other words, it is

- transitive;
- reflexive;
- antisymmetric;
- comparable.

Proof.

Maximal/minimal elements

Definition 7. Let R be a transitive binary relation on a set A.

An element a of A is called a maximal element of A with respect to R if, for any $b \in A$,

$$bRa \Longrightarrow aRb.$$

An element a of A is called a minimal element of A with respect to R if, for any $b \in A$,

$$aRb \Longrightarrow bRa.$$

Greatest/least elements

Definition 8. Let R be a transitive binary relation on a set A.

An element a of A is called a greatest element (or maximum element) of A with respect to R if, for all $b \in A$,

$$aRb.$$

An element a of A is called a least element (or minimum element) of A with respect to R if, for all $b \in A$,

$$bRa.$$

Lemma 3. Let R be a transitive binary relation on a set A.

- Any greatest element of A w.r.t. R is also a maximal element.
- Any least element of A w.r.t. R is also a minimal element.
- If R is also antisymmetric, then A has at most one greatest element and at most one least element.

Proof.

Lemma 4. Let R be a **transitive** and **antisymmetric** binary relation on a **finite non-empty** set A.

- A has at least one maximal element and at least one minimal element w.r.t. R.

- If in addition R is comparable, then A has exactly one greatest element and exactly one least element, and they are also the unique maximal and minimal elements, respectively.

Proof.

A maximal element of $A \subseteq \mathbb{R}^n$ w.r.t. the Pareto order is called a Pareto maximal element of A. The set of Pareto maximal elements of A is denoted by $P_{\max}(A)$.

$P_{\max}(A)$ is exactly the Pareto boundary of A as used in cooperative 2-person games.

Pareto minimal elements and $P_{\min}(A)$ are interpreted similarly.

Corollary 1. Let A be a **finite non-empty** subset of $\mathbb{R}^n$. Then

$$P_{\max}(A) \neq \emptyset, \;\; P_{\min}(A) \neq \emptyset.$$

Corollary 2. Let A be a **finite non-empty** subset of $\mathbb{R}^n$. Then

- A has exactly one greatest element w.r.t. the lexicographic order;
- A has exactly one least element w.r.t. the lexicographic order.

We denote these two elements by $L_{\max}(A)$ and $L_{\min}(A)$ respectively.

Example 7. ($P_{\max}(A)$, $P_{\min}(A)$, $L_{\max}(A)$, and $L_{\min}(A)$)

Four students, Alice (a), Bob (b), Catherine (c), and David (d) are assessed on three criteria (research, grades, progress). The results are depicted in the table below.

	Research	Grades	Progress
a	9	9	8
b	9	9	7
c	8	10	9
d	8	9	6

Determine $P_{\max}(A)$, $P_{\min}(A)$, $L_{\max}(A)$, and $L_{\min}(A)$. Determine the least/greatest elements with respect to the Pareto order and with respect to the lexicographic order if there exists any.

Graphical representation

We may represent a binary relation R on a finite set A by a directed graph. The vertices of the graph represent elements of A, and there is an edge from a to b iff aRb. (If aRa then we draw a loop from a to a.)

Box diagram/Boolean matrix

We may represent a binary relation R on a finite set A by a box diagram or a so-called Boolean matrix. A box diagram/Boolean matrix consists of a matrix where the entries are given by $\times$'s and blanks. A $\times$ in the matrix at the (i, j)'th position indicates that $a_i R a_j$, where $a_i, a_j \in A$.

Example 8. ($P_{\max}(A)$, $P_{\min}(A)$, $L_{\max}(A)$, and $L_{\min}(A)$)

Let

$$A = \{(1,2,3),(4,3,5),(3,-1,2),(4,0,8),(9,3,2)\}.$$

Draw a suitable graph representation and provide the Boolean matrix with respect to the Pareto order and the lexicographic order. Furthermore, determine

$P_{\max}(A)$, $P_{\min}(A)$, $L_{\max}(A)$, and $L_{\min}(A)$.

Preference order

Strict preference order

Let A be the set of alternatives for a decision making problem.

Let $\succ$ denote the decision maker's strict preference order on A. That is, for $a, b \in A$,

$$a \succ b$$

means that "the decision maker **strictly** prefers a over b".

If a rational decision maker holds $a \succ b$ and $b \succ c$, then he should hold $a \succ c$ as well. This means his strict preference order is transitive.

If a rational decision maker holds $a \succ b$, then he cannot hold $b \succ a$. So his strict preference order is asymmetric.

Therefore, a rational decision maker's strict preference order should be transitive and asymmetric, i.e., a strict order on A.

Example 9. (Money-pumping)

Let $A = \{a, b, c\}$. Assume that you hold

$$a \succ b,\ b \succ c,\ c \succ a.$$

Crown Casino offers the following to you: Select one element each time; if there is a "better item" left in the set, pay \$1 to get it. Repeat till you get the "best".

There is no "best" element. If you pay \$1 for getting something better than you have, you will go bankrupt quickly!

The problem is caused by the intransitivity of your preference order.

Indifference

We use the notation

$$a \sim b$$

to mean that "the decision maker is indifferent between alternatives a and b". In other words, he is equally happy to choose a or b.

For a rational decision maker,

$$a \sim a \quad \text{(reflexivity)}$$

$$a \sim b \Longrightarrow b \sim a \quad \text{(symmetry)}$$

$$a \sim b \text{ and } b \sim c \Longrightarrow a \sim c \quad \text{(transitivity)}$$

In other words, a rational decision maker's indifference relation $\sim$ should be an equivalence relation on the set of alternatives.

Weak preference order

We use the notation

$$a \succeq b$$

to mean

$$\text{either } a \succ b \text{ or } a \sim b;$$

that is, "a is at least as good as b" for the decision maker. We call this relation $\succeq$ the weak preference order of the decision maker.
We assume that a rational decision maker's weak preference order satisfies

$$a \succeq b \text{ and } b \succeq c \Longrightarrow a \succeq c \quad \text{(transitivity)}$$

$$\text{for any } a, b \in A \text{ either } a \succeq b \text{ or } b \succeq a \quad \text{(comparability)}$$

In other words, we assume that **a rational decision maker's weak preference order is a weak order on the set of alternatives**. We also assume that

$$a \sim b \Longleftrightarrow (a \succeq b \text{ and } b \succeq a)$$

$$a \succ b \Longrightarrow b \not\succeq a.$$

Group decision and social choice

Group decision problem

Problem. Given a group of individuals $1, 2, \ldots, n$ and a finite set of alternatives $A = \{a, b, c, \ldots, \}$, the group as a whole chooses the "most preferred alternative" from A by "aggregating" the preferences of the individuals on the alternatives. (Equivalently, the group as a whole decides a preference order on alternatives which reflects individuals' preferences.)

In other words, the problem is to aggregate in some way the individuals' preferences into the group preference.

The size n of the group may range from small, medium, large to very large. When n is very large (e.g. electorates with millions of voters), the problem is also called a social choice problem. Typically the set A of alternatives is not very big.

How do we aggregate individuals' preferences into a group preference?

The preference obtained for the group relies heavily on the procedure of aggregation.

Two popular rules:

Majority rule: A group should strictly prefer a to b if the majority of its members strictly prefer a to b; if there is a tie (i.e. equal members prefer a to b as prefer b to a), then the group should be indifferent between a and b.

Plurality rule: The candidate who receives the largest number of votes is elected.

Many other rules exist in practice. Some of them were discussed in the second year course "Discrete Mathematics and Operations Research".

Notation 1. We use $\succ_i$, $\succeq_i$ and $\sim_i$ to denote the strict preference order, the weak preference order, and the indifference relation of individual i, respectively. That is, $a \succ_i b$ means that the ith individual prefers a to b strictly, $a \succeq_i b$ indicates that he or she believes a is at least as good as b, and $a \sim_i b$ means that he or she is indifferent between a and b.

Denote by $\succ_g$, $\succeq_g$ and $\sim_g$ the strict preference order, the weak preference order, and the indifference relation for the group, respectively.

Example 10. (group preferences)

Consider a group of three individuals $1, 2, 3$ and three alternatives $A = \{a, b, c\}$. Suppose

$$a \succ_1 b \succ_1 c$$

$$b \succ_2 c \succ_2 a$$

$$c \succ_3 a \succ_3 b.$$

Work out the group preference relation using the simple majority rule.

Arrow's Theorem

Constitution

Definition 9. The voting system or mechanism whereby the preferences of the individuals in the group are aggregated into a preference of the group is called the constitution of the group.

For instance, the majority rule is a constitution.

What constitution should the group use to ensure justice, fairness, and democracy? How should we form $\succeq_g$ from $\succeq_1, \succeq_2, \ldots, \succeq_n$ appropriately?

Arrow (1951) suggested a number of conditions for a rational constitution to follow.

Arrow's axioms

Axiom 1. (Weak order) $\succeq_1, \succeq_2, \ldots, \succeq_n$ and $\succeq_g$ are weak orders (i.e. they should be transitive and comparable).

Axiom 2. (Non-trivality) There are at least two members in the group (i.e. $n \geq 2$) and there are at least three alternatives in A.

Comment: We will see what happens if there are only two alternatives.

Axiom 3. (Universal domain) $\succeq_g$ is defined for any $\succeq_1, \succeq_2, \ldots, \succeq_n$.

Comment: This requires that the constitution produces a group preference no matter what individuals' preferences are.

Axiom 4. (Binary relevance) Let $\succeq_1, \succeq_2, \ldots, \succeq_n$ be a set of preference orders over a set A of alternatives. Let $\succeq_1', \succeq_2', \ldots, \succeq_n'$ be a set of preference orders over another set A' of alternatives. (The group is the same for A and A'.) Suppose

$$\{a, b\} \subset A \cap A'.$$

Suppose further that, for any i,

$$a \succeq_i b \Longleftrightarrow a \succeq_i' b,$$

$$b \succeq_i a \Longleftrightarrow b \succeq_i' a.$$

Then the constitution produces the same group preference between a and b:

$$a \succeq_g b \Longleftrightarrow a \succeq_g' b,$$

$$b \succeq_g a \Longleftrightarrow b \succeq_g' a.$$

Axiom 5. (Pareto's principle) If every individual member in the group holds $a \succ_i b$, then the group holds $a \succ_g b$.

Comment: This means that the constitution should obey the principle of unanimity.

Axiom 6. (No dictatorship) There is no individual in the group whose preferences automatically become the preferences of the group regardless of the preferences of the other members.

Comment: This is essential for a democratic society. Nobody likes dictators except the dictators themselves and their followers.

Arrow's Impossibility Theorem

Theorem 1. There exists no constitution which allows $\succeq_g$ to be defined from $\succeq_1, \succeq_2, \ldots, \succeq_n$ in a manner which is consistent with the Axioms above.

This means that, for every constitution, there is at least one set of possible individual preferences $\succeq_1, \succeq_2, \ldots, \succeq_n$ such that the constitution of $\succeq_g$ breaks at least one of the Axioms above.

In 1972 Kenneth J. Arrow and Sir John R. Hicks won the Nobel Prize in economics for their contributions to welfare theory and social choice.

Discussion

- Arrow's Impossibility Theorem is an unpleasant result.

- It applies to both open and secret ballots.

- It applies no matter whether the individuals' preferences are weighted equally or not.

- "Pareto's principle" and "no dictatorship" are widely accepted.
- "Non-triviality" seems a reasonable hypothesis.

- **If there are only two alternatives, the majority rule satisfies all the axioms above.** So Arrow's Impossibility Theorem does not apply to this case.

- "Weak order" is controversial, but removing it does not remove impossibility.

- "Universal domain" and "binary relevance" are controversial as well.

Outline of proof of Arrow's Impossibility Theorem

Terminology: Let I be the set of all individuals. A subset V of I is called decisive for a over b if, whenever $a \succ_i b$ for all $i \in V$ and $b \succ_i a$ for all $i \notin V$, the group holds $a \succ_g b$.

A subset V of I is a minimal decisive subset if V is decisive for some a over some b, but no proper subset of V is decisive for any c over any d.

Set $D = \{V \subseteq I : V \text{ is decisive for some } a \text{ over some } b\}$.

Outline of the proof:

- Pareto's principle implies that $D \neq \emptyset$ as $I \in D$.
- "$\subseteq$" is a transitive and antisymmetric binary relation on D. Since D is finite, it contains minimal elements w.r.t. $\subseteq$. Hence minimal decisive subset exists.
- Show that a minimal decisive subset must be a single individual.
- Show that the only member j in this minimal decisive subset is a dictator.

But this contradicts Axiom 6. Hence there is no constitution which allows $\succeq_g$ to be defined in a manner which is consistent with all the six axioms. $\square$

The complete proof can be found in S. French, Decision Theory: An Introduction to the Mathematics of Rationality, Ellis Horwood, 1986, pp 286-288.