

# MAST30025: Linear Statistical Models

## Solution to Week 11 Lab

1. We study the effect of various breeds and diets on the milk yield of cows. A study is conducted on 9 cows and the following data obtained:

Breed	Diet		
	1	2	3
1	18.8	16.7	19.8
	21.2		23.9
2	22.3	15.9	21.8
		19.2	

- (a) Express this as a two-factor model with no interaction in matrix form.

**Solution:**  $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ , where

$$\mathbf{y} = \begin{bmatrix} 18.8 \\ 21.2 \\ 16.7 \\ 19.8 \\ 23.9 \\ 22.3 \\ 15.9 \\ 19.2 \\ 21.8 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

and  $\boldsymbol{\varepsilon}$  is as expected.

- (b) Express this as a two-factor model with interaction in matrix form.

**Solution:**  $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ , where

$$\mathbf{y} = \begin{bmatrix} 18.8 \\ 21.2 \\ 16.7 \\ 19.8 \\ 23.9 \\ 22.3 \\ 15.9 \\ 19.2 \\ 21.8 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \xi_{11} \\ \xi_{12} \\ \xi_{13} \\ \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{bmatrix}$$

and  $\boldsymbol{\varepsilon}$  is as expected.

- (c) Express the hypothesis that there is no interaction in terms of your parameters. Eliminate any redundancies.

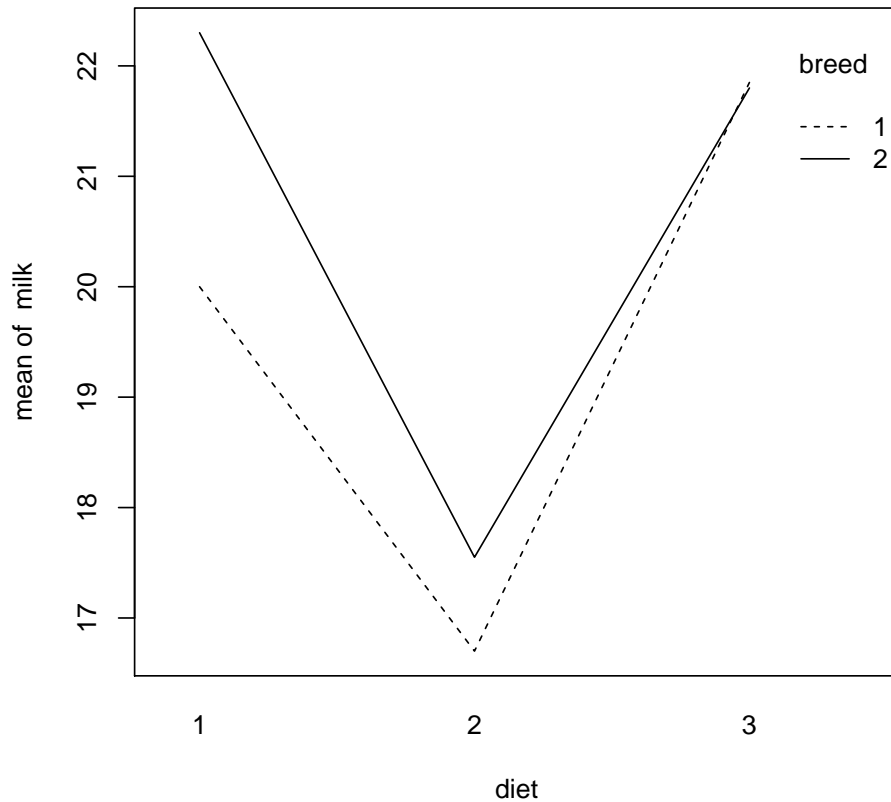
**Solution:** We know that we require  $(a-1)(b-1) = 2$  hypotheses, so we take the obviously non-redundant hypotheses

$$\begin{aligned} (\xi_{11} - \xi_{12}) - (\xi_{21} - \xi_{22}) &= 0 \\ (\xi_{11} - \xi_{13}) - (\xi_{21} - \xi_{23}) &= 0. \end{aligned}$$

- (d) Input this data into R. Plot an interaction plot between breed and diet.

**Solution:**

```
> milk <- data.frame(milk=c(18.8,21.2,16.7,19.8,23.9,22.3,15.9,19.2,21.8),
+                    diet=factor(c(1,1,2,3,3,1,2,2,3)),
+                    breed=factor(c(1,1,1,1,1,2,2,2,2)))
> with(milk, interaction.plot(diet, breed, milk))
```



- (e) Test for the presence of interaction.

**Solution:**

```
> imodel <- lm(milk ~ breed * diet, data=milk)
> anova(imodel)
```

Analysis of Variance Table

Response: milk

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
breed	1	0.174	0.1742	0.0312	0.8710
diet	2	36.204	18.1018	3.2460	0.1777
breed:diet	2	1.874	0.9372	0.1681	0.8527
Residuals	3	16.730	5.5767		

There is clearly no interaction.

- (f) What is the degrees of freedom used for the interaction test?

**Solution:** We use 2 and 3 degrees of freedom.

- (g) From the interaction model, what is the estimated amount of milk produced from breed 2 and diet 3?

**Solution:**

```
> imodel$coeff
```

```

      (Intercept)      breed2      diet2      diet3 breed2:diet2 breed2:diet3
      20.00         2.30      -3.30         1.85        -1.45        -2.35
> c(1,1,0,1,0,1)%*%imodel$coeff
      [,1]
[1,] 21.8

```

- (h) Fit an additive model. What is the estimated amount of milk produced from breed 2 and diet 3 now?

**Solution:**

```

> amodel <- lm(milk ~ breed + diet, data=milk)
> amodel$coeff
      (Intercept)      breed2      diet2      diet3
      20.422222     1.033333     -3.844444     1.066667
> c(1,1,0,1)%*%amodel$coeff
      [,1]
[1,] 22.52222

```

- (i) Test the hypothesis (under the additive model) that the 2nd and 3rd diets are equivalent in terms of milk produced.

**Solution:**

```

> library(car)
> linearHypothesis(amodel, c(0,0,1,-1),0)

```

Linear hypothesis test

Hypothesis:

diet2 - diet3 = 0

Model 1: restricted model

Model 2: milk ~ breed + diet

```

      Res.Df    RSS Df Sum of Sq      F Pr(>F)
1         6 52.000
2         5 18.604  1    33.396 8.9752 0.03024 *
---

```

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

We reject this hypothesis at a 5% level.

- (j) Find a 95% confidence interval, under the additive model, for the amount of milk produced from breed 2 and diet 3. Use both matrix calculations and the `estimable` function from the `gmodels` package.

**Solution:** Using the design matrix:

```

> library(MASS)
> library(Matrix)
> n <- 9
> X <- matrix(0,n,6)
> X[,1] <- 1
> X[cbind(1:n,as.numeric(milk$breed)+1)] <- 1
> X[cbind(1:n,as.numeric(milk$diet)+3)] <- 1
> y <- milk$milk
> XtXc <- ginv(t(X) %*% X)
> b <- XtXc %*% t(X) %*% y
> r <- rankMatrix(X)
> s2 <- sum((y - X %*% b)^2)/(n - r)
> t <- c(1,0,1,0,0,1)
> mu23 <- t(t) %*% b
> width <- qt(.975, n - r)*sqrt(s2 * t(t) %*% XtXc %*% t)
> c(mu23 - width, mu23, mu23 + width)

```

```
[1] 18.82634 22.52222 26.21811
```

Alternatively we can use `estimable`. Note that like `linearHypothesis`, `estimable` requires that you express the estimated quantity in terms of the estimates R uses:

```
> library(gmodels)
> estimable(amodel, c(1,1,0,1), conf.int=0.95)

      Estimate Std. Error  t value DF      Pr(>|t|) Lower.CI Upper.CI
(1 1 0 1) 22.52222    1.437762 15.66477  5 1.927104e-05 18.82634 26.21811
```

- (k) Find the same confidence interval under the interaction model.

**Solution:**

```
> estimable(imodel, c(1,1,0,1,0,1), conf.int=0.95)

      Estimate Std. Error  t value DF      Pr(>|t|) Lower.CI Upper.CI
(1 1 0 1 0 1)    21.8    2.361497  9.231434  3 0.002689148 14.28466 29.31534
```

- (l) Why is the second interval wider than the first?

**Solution:** The second interval is wider than the first because we are attributing some degrees of freedom to the interaction term(s). The resulting loss in degrees of freedom for the residuals leads to greater error in our estimations.

2. We study the growth of peas when fed three different types of fertilizer. A study is conducted where the samples are divided into 6 “blocks”, corresponding to different plots of land. The data is stored in the `npk` data frame in R. This data frame contains 5 variables:

- block: label of the block of the sample
- N: indicator (0/1) for the application of nitrogen
- P: indicator (0/1) for the application of phosphate
- K: indicator (0/1) for the application of potassium
- yield: yield of peas in pounds/plot

- (a) Fit an additive linear model with all variables; then repeat without the block variables. Does the fitted model change? Are the block variables significant?

**Solution:**

```
> blockmodel <- lm(yield ~ ., data = npk)
> summary(blockmodel)
```

Call:

```
lm(formula = yield ~ ., data = npk)
```

Residuals:

```
      Min       1Q   Median       3Q      Max
-7.0000 -1.7083 -0.0833  2.2458  6.4833
```

Coefficients:

```
      Estimate Std. Error t value Pr(>|t|)
(Intercept)   53.800     2.450  21.955 8.13e-13 ***
block2         3.425     2.830   1.210 0.24483
block3         6.750     2.830   2.386 0.03068 *
block4        -3.900     2.830  -1.378 0.18831
block5        -3.500     2.830  -1.237 0.23512
block6         2.325     2.830   0.822 0.42412
N1             5.617     1.634   3.438 0.00366 **
P1            -1.183     1.634  -0.724 0.47999
K1            -3.983     1.634  -2.438 0.02767 *
```

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Residual standard error: 4.002 on 15 degrees of freedom

Multiple R-squared: 0.7259, Adjusted R-squared: 0.5798

F-statistic: 4.966 on 8 and 15 DF, p-value: 0.003761

```

> amodel <- lm(yield ~ . - block, data = npk)
> summary(amodel)

Call:
lm(formula = yield ~ . - block, data = npk)

Residuals:
    Min       1Q   Median       3Q      Max
-9.2667 -3.6542  0.7083  3.4792  9.3333

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   54.650      2.205   24.784  <2e-16 ***
N1             5.617      2.205    2.547   0.0192 *
P1            -1.183      2.205   -0.537   0.5974
K1            -3.983      2.205   -1.806   0.0859 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.401 on 20 degrees of freedom
Multiple R-squared:  0.3342,    Adjusted R-squared:  0.2343
F-statistic: 3.346 on 3 and 20 DF,  p-value: 0.0397
> anova(amodel, blockmodel)

```

#### Analysis of Variance Table

```

Model 1: yield ~ (block + N + P + K) - block
Model 2: yield ~ block + N + P + K
  Res.Df  RSS Df Sum of Sq    F Pr(>F)
1     20 583.48
2     15 240.18  5    343.3 4.2879 0.01272 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

The fitted model does not change, in the sense that the parameters corresponding to N, P and K are the same for the two models. This is because the design is *balanced* and *complete*: the overall effect of the predictors of interest are observed in each individual block. However, the blocks themselves are significant: there is a difference in the yield of each block. Because the blocks have been carefully designed, this does not affect the fitted model itself when the blocks are removed from consideration.

- (b) Fit a model with the fertilizer variables and all pairwise interaction terms. Are the interaction terms significant?

#### Solution:

```

> imodel <- lm(yield ~ (. - block)^2, data=npk)
> summary(imodel)

Call:
lm(formula = yield ~ (. - block)^2, data = npk)

Residuals:
    Min       1Q   Median       3Q      Max
-8.8917 -3.2875  0.4583  3.4000  9.7083

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   52.6750      3.0114   17.492 2.64e-12 ***
N1             9.8500      3.9429    2.498   0.023 *
P1             0.4167      3.9429    0.106   0.917
K1            -1.9167      3.9429   -0.486   0.633

```

```

N1:P1      -3.7667      4.5529 -0.827      0.420
N1:K1      -4.7000      4.5529 -1.032      0.316
P1:K1       0.5667      4.5529  0.124      0.902
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

```

Residual standard error: 5.576 on 17 degrees of freedom
Multiple R-squared:  0.3968,    Adjusted R-squared:  0.184
F-statistic: 1.864 on 6 and 17 DF,  p-value: 0.146
> anova(amodel, imodel)
Analysis of Variance Table

```

```

Model 1: yield ~ (block + N + P + K) - block
Model 2: yield ~ ((block + N + P + K) - block)^2
  Res.Df    RSS Df Sum of Sq    F Pr(>F)
1      20 583.48
2      17 528.58  3    54.898 0.5885 0.6308

```

The interaction terms are not significant.

- (c) Perform variable selection using stepwise selection with AIC, starting from the model with no interaction terms (but considering them for inclusion). What do you find?

**Solution:**

```

> (finalmodel <- step(amodel, scope=~(.-block)^2, data=npk))
Start:  AIC=84.58
yield ~ (block + N + P + K) - block

```

	Df	Sum of Sq	RSS	AIC
- P	1	8.402	591.88	82.926
<none>			583.48	84.583
+ N:K	1	33.135	550.34	85.180
+ N:P	1	21.282	562.20	85.691
- K	1	95.202	678.68	86.210
+ P:K	1	0.482	583.00	86.563
- N	1	189.282	772.76	89.326

```

Step:  AIC=82.93
yield ~ N + K

```

	Df	Sum of Sq	RSS	AIC
<none>			591.88	82.926
+ N:K	1	33.135	558.75	83.543
- K	1	95.202	687.08	84.506
+ P	1	8.402	583.48	84.583
- N	1	189.282	781.16	87.586

```

Call:
lm(formula = yield ~ N + K, data = npk)

```

```

Coefficients:
(Intercept)      N1      K1
    54.058      5.617    -3.983

```

We find that our final model includes the variables corresponding to nitrogen and potassium, but not phosphate.

- (d) What is the best treatment for peas, according to your final model? Find a 95% confidence interval for the yield of this treatment.

**Solution:** According to the final model, we should treat peas with nitrogen and not potassium (phosphate is unimportant). The confidence interval is:

```
> estimable(finalmodel, c(1,1,0), conf.int=0.95)
      Estimate Std. Error  t value DF Pr(>|t|) Lower.CI Upper.CI
(1 1 0)    59.675    1.876994 31.79286 21      0 55.77158 63.57842
```

3. Suppose that  $\mathbf{y} \sim MVN(\mu\mathbf{1}, \Sigma)$ , where

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{bmatrix}.$$

For what values of  $\rho$  are the sample mean and sample variance independent?

**Solution:** Let  $J$  be the matrix of all 1s. We can write

$$\begin{aligned}\bar{y} &= \frac{1}{n}\mathbf{1}^T\mathbf{y}, \\ s^2 &= \frac{1}{n-1}\mathbf{y}^T(I - \frac{1}{n}J)\mathbf{y}.\end{aligned}$$

Thus they are independent iff  $\mathbf{1}^T \text{Var } \mathbf{y}(I - \frac{1}{n}J) = 0$ . The LHS is

$$\begin{aligned}\mathbf{1}^T((1-\rho)I + \rho J)(I - \frac{1}{n}J) &= \mathbf{1}^T\left((1-\rho)I + \rho J - \frac{1-\rho}{n}J - \rho J\right) \\ &= (1-\rho)\mathbf{1}^T(I - \frac{1}{n}J) \\ &= 0.\end{aligned}$$

So the sample mean and variance are independent for any  $\rho$ .

4. In the one-way classification model, show that any linear combination of  $\bar{y}_1 - \bar{y}, \dots, \bar{y}_k - \bar{y}$  can be written as a linear combination of  $\bar{y}_1, \dots, \bar{y}_k$ . Does the converse hold?

**Solution:** We have

$$\sum a_i(\bar{y}_i - \bar{y}) = \sum a_i\bar{y}_i - \left(\sum a_i\right)\frac{1}{k}\sum \bar{y}_i = \sum (a_i - \bar{a})\bar{y}_i.$$

The converse only holds for contrasts.