MAST30022 Decision Making
2021
Tutorial 3

1. **(PS2-8)** This question relates to the following known result:

> If one player of a zero-sum two-person game employs a fixed strategy, then the opponent has an optimal counter strategy that is pure. So, if Player I KNOWS Player II's strategy, Player I can maximize their expected payoff by using a pure strategy, and vice versa.

We explore the above statement in this question for a particular case.

(a) (i) Find the expected payoff, $\mathbb{E}(\boldsymbol{x}, \boldsymbol{y})$, to Player I if Player I uses mixed strategy $\boldsymbol{x} = (x_1, 1 - x_1), 0 \le x_1 \le 1$, whilst Player II uses mixed strategy $\boldsymbol{y} = (y_1, 1 - y_1), 0 \le y_1 \le 1$, in the 2-person zero-sum game with payoff matrix

$$\begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix}.$$

(ii) Now think of y_1 as being fixed and find the value(s) of x_1 that maximises $\mathbb{E}(\boldsymbol{x}, \boldsymbol{y})$ for fixed y_1. Hence write down the strategy(ies) for Player I that maximises $\mathbb{E}(\boldsymbol{x}, \boldsymbol{y})$. Deduce that in all cases, for this example, Player I can use a pure strategy to get the maximum $\mathbb{E}(\boldsymbol{x}, \boldsymbol{y})$, given that Player I knows Player II's strategy. (If you cannot see what to do, try putting $y_1 = 0.2$, for example, and see what happens. Then try and generalise.)

(b) Suppose that Player I knew that Player II is to use a mixed strategy $(0.2, 0.3, 0.5)$ in the 2- person zero-sum game with matrix below

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Work out the three expected payoffs if Player I uses a pure strategy and hence deduce the best strategy for Player I.

Solution

(a) (i) The expected payoff to Player I is

$$\begin{aligned}
\mathbb{E}(\boldsymbol{x}, \boldsymbol{y}) &= \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 5y_1 \\ 2 - 3y_1 \end{bmatrix} \\
&= 5x_1y_1 + (1 - x_1)(2 - 3y_1) \\
&= 8x_1y_1 - 2x_1 - 3y_1 + 2.
\end{aligned}$$

(ii) Write $\mathbb{E}(\boldsymbol{x},\boldsymbol{y}) = (8y_1 - 2)x_1 - 3y_1 + 2$, which is linear in x_1.
If $y_1 < 1/4$ then the maximum of $\mathbb{E}(\boldsymbol{x},\boldsymbol{y})$ occurs when $x_1 = 0$. Hence the optimal strategy will be $\boldsymbol{x} = (0,1)$.
If $y_1 > 1/4$ then the maximum of $\mathbb{E}(\boldsymbol{x},\boldsymbol{y})$ occurs when $x_1 = 1$. Hence the optimal strategy will be $\boldsymbol{x} = (1,0)$.
If $y_1 = 1/4$ then both pure strategies are optimal, as are any mixed strategies for Player I.

(b) Since

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.3 \\ -0.1 \end{bmatrix},$$

the optimal strategy for Player I is $\boldsymbol{x} = (0,1,0)$ with corresponding payoff 0.3.

2. **(PS2-23)**

(a) Show that, if (a_i, A_i) and (a_j, A_j) both give saddles in a 2-person zero-sum game, then (a_i, A_j) also gives a saddle, where a_i, a_j are pure strategies for Player I and A_i, A_j are pure strategies for Player II.

(b) Part (a) is a special case of the following general result: If both $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ are equilibrium pairs in a two-person zero-sum game, then so are $(\boldsymbol{x}_1, \boldsymbol{y}_2)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_1)$, where $\boldsymbol{x}_1, \boldsymbol{x}_2$ are mixed strategies for Player I and $\boldsymbol{y}_1, \boldsymbol{y}_2$ are mixed strategies for Player II. Prove this result. Prove also that $\mathbb{E}(\boldsymbol{x}_1, \boldsymbol{y}_1) = \mathbb{E}(\boldsymbol{x}_1, \boldsymbol{y}_2)$.

Solution

(a) **Proof:** This was proved in a lecture (see Slide 10, 2-person zero-sum games).

(b) Since both $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ are equilibrium pairs, for any $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$,

$$\boldsymbol{x}\boldsymbol{V}\boldsymbol{y}_1^T \le \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_1^T \le \boldsymbol{x}_1 V \boldsymbol{y}^T \tag{1}$$

$$\boldsymbol{x}\boldsymbol{V}\boldsymbol{y}_2^T \le \boldsymbol{x}_2 V \boldsymbol{y}_2^T \le \boldsymbol{x}_2 V \boldsymbol{y}^T. \tag{2}$$

In particular,

$$\begin{aligned} \boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_1^T &\le \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_1^T \le \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_2^T \\ \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_2^T &\le \boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_2^T \le \boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_1^T. \end{aligned}$$

That is,

$$\boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_1^T \le \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_1^T \le \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_2^T \le \boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_2^T \le \boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_1^T.$$

Hence

$$\boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_1^T = \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_1^T = \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_2^T = \boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_2^T,$$

which is the value of the game. This together with (1) and (2) implies that, for any $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$,

$$\boldsymbol{x}\boldsymbol{V}\boldsymbol{y}_2^T \le \boldsymbol{x}_2\boldsymbol{V}\boldsymbol{y}_2^T = \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_2^T = \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}_1^T \le \boldsymbol{x}_1\boldsymbol{V}\boldsymbol{y}^T.$$

Hence $(\boldsymbol{x}_1, \boldsymbol{y}_2)$ is an equilibrium pair. Similarly, one can show that $(\boldsymbol{x}_2, \boldsymbol{y}_1)$ is an equilibrium pair. Moreover, from the proof above we have

$$\mathbb{E}(\boldsymbol{x}_1, \boldsymbol{y}_1) = \boldsymbol{x}_1 \boldsymbol{V} \boldsymbol{y}_1^T = \boldsymbol{x}_1 \boldsymbol{V} \boldsymbol{y}_2^T = \mathbb{E}(\boldsymbol{x}_1, \boldsymbol{y}_2).$$

3. **(PS2-32)** Solve the following 2-person zero-sum game.

$$\boldsymbol{V} = \begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix}$$

Solution

Step 1: Check for saddle points:

$$(s_1, s_2, s_3) = (-3, -3, -4), (S_1, S_2, S_3) = (3, 3, 3)$$

Since

$$L = -3 < 3 = U,$$

there is no saddle point.

Step 2: Check for dominance – None.

Step 3: To ensure that the value of the game is strictly positive, we consider the following modified game:

$$\boldsymbol{V}' = \boldsymbol{V} + 5 = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 2 & 8 \end{bmatrix}$$

Note that $c = 4$ will also do since rows 1 and 2 will be strictly positive. A smaller value of c usually makes the calculations a bit simpler as the denominators in the fractions are smaller when applying the simplex method.

LP for Player I

$\min 1/v_1' = x_1' + x_2' + x_3'$

s.t.

$8x_1' + 2x_2' + x_3' \geq 1$

$4x_1' + 8x_2' + 2x_3' \geq 1$

$2x_1' + 4x_2' + 8x_3' \geq 1$

$x_1', x_2', x_3' \geq 0$

LP for Player II

$\max 1/v_2' = y_1' + y_2' + y_3'$

s.t.

$8y_1' + 4y_2' + 2y_3' \leq 1$

$2y_1' + 8y_2' + 4y_3' \leq 1$

$y_1' + 2y_2' + 8y_3' \leq 1$

$y_1', y_2', y_3' \geq 0$

We solve the LP problem for Player II.

	y_1'	y_2'	y_3'	y_4'	y_5'	y_6'	RHS
$\boldsymbol{y_4'}$	8^*	4	2	1	0	0	1
y_5'	2	8	4	0	1	0	1
y_6'	1	2	8	0	0	1	1
z	$\mathbf{-1}$	-1	-1	0	0	0	0

	y'_1	y'_2	y'_3	y'_4	y'_5	y'_6	RHS
y'_1	1	$1/2$	$1/4$	$1/8$	0	0	$1/8$
y'_5	0	7	$7/2$	$-1/4$	1	0	$3/4$
$\boldsymbol{y'_6}$	0	$3/2$	$31^*/4$	$-1/8$	0	1	$7/8$
z	0	$-1/2$	$-\mathbf{3/4}$	$1/8$	0	0	$1/8$

	y'_1	y'_2	y'_3	y'_4	y'_5	y'_6	RHS
y'_1	1	$14/31$	0	$4/31$	0	$-1/31$	$3/31$
$\boldsymbol{y'_5}$	0	$196^*/31$	0	$-6/31$	1	$-14/31$	$11/31$
y'_3	0	$6/31$	1	$-1/62$	0	$4/31$	$7/62$
z	0	$-\mathbf{11/31}$	$\mathbf{0}$	$7/62$	0	$3/31$	$13/62$

	y'_1	y'_2	y'_3	y'_4	y'_5	y'_6	RHS
y'_1	1	0	0	$1/7$	$-1/14$	0	$1/14$
y'_2	0	1	0	$-3/98$	$31/196$	$-1/14$	$11/196$
y'_3	0	0	1	$-1/98$	$-3/98$	$1/7$	$5/49$
z	0	0	0	$5/49$	$11/196$	$1/14$	$45/196$

From the final simplex tableau we have

$$\max 1/v'_2 = 45/196$$

and so $v'_2 = 196/45$, and that

$$(y'_1, y'_2, y'_3) = (1/14, 11/196, 5/49)$$

is an optimal solution to the LP above.

Hence for the original game the value is

$$v = v_2 = v'_2 - 5 = 196/45 - 5 = -29/45$$

and the optimal strategy for Player II is

$$\begin{aligned}(y_1, y_2, y_3) &= (196/45)(y'_1, y'_2, y'_3)\\ &= (14/45, 11/45, 4/9).\end{aligned}$$

The optimal strategy for Player I is

$$\begin{aligned}(x_1, x_2, x_3) &= (196/45)(x'_1, x'_2, x'_3)\\ &= (196/45)(5/49, 11/196, 1/14)\\ &= (4/9, 11/45, 14/45).\end{aligned}$$

In other words,

$$((4/9, 11/45, 14/45), (14/45, 11/45, 4/9))$$

is an equilibrium pair of mixed strategies for the original game.

4. **(PS2-33)** Solve the 2-person zero-sum game with the payoff matrix below using the graphical method, possibly in combination with other methods (eg. dominance). State explicitly the value of the game and an optimal strategy pair.

$$\begin{bmatrix} 3 & 2 & 1 & 8 & 3 \\ 0 & 2 & 3 & 7 & 1 \end{bmatrix}$$

Solution

Step 1: Check for saddle point.

$$L = \max\{1, 0\} = 0,\ U = \min\{3, 2, 3, 8, 3\} = 2.$$

Since $L < U$ there is no saddle point.

Step 2: Check for dominance.

Column 4 is dominated by all other columns, and Column 5 is dominated by Column 1. Deleting Columns 4 and 5 we get

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

We now have a 2×3 game, which can be solved by the graphical method.

$$\begin{aligned} v_1 &= \max_{\boldsymbol{x} \in X} \min_{1 \le j \le 3} \boldsymbol{x}\boldsymbol{V}_{.j} \\ &= \max_{\boldsymbol{x} \in X} \min \{\boldsymbol{x}\boldsymbol{V}_{.1}, \boldsymbol{x}\boldsymbol{V}_{.2}, \boldsymbol{x}\boldsymbol{V}_{.3}\} \\ &= \max_{\boldsymbol{x} \in X} \min \{3x_1, 2x_1 + 2x_2, x_1 + 3x_2\} \\ &= \max_{0 \le x_1, x_2 \le 1, x_1 + x_2 = 1} \min \{3x_1, 2x_1 + 2x_2, x_1 + 3x_2\} \\ &= \max_{0 \le x_1 \le 1} \min \{3x_1, 2, 3 - 2x_1\}. \end{aligned}$$

From the diagram the maximin occurs at the intersection of the lines $z = 3x_1$ and $z = 3 - 2x_1$. Solving

$$3x_1 = 3 - 2x_1$$

we get $x_1^* = 3/5$. Hence $x_2^* = 1 - x_1^* = 2/5$ and $\boldsymbol{x}^* = (3/5, 2/5)$, and $v = v_1 = 3 \times (3/5) = 9/5$.

Since the optimal strategy for Player I is from Columns 1 and 3 only, the optimal strategy for Player II must be from the following 2×2 game using these columns:

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

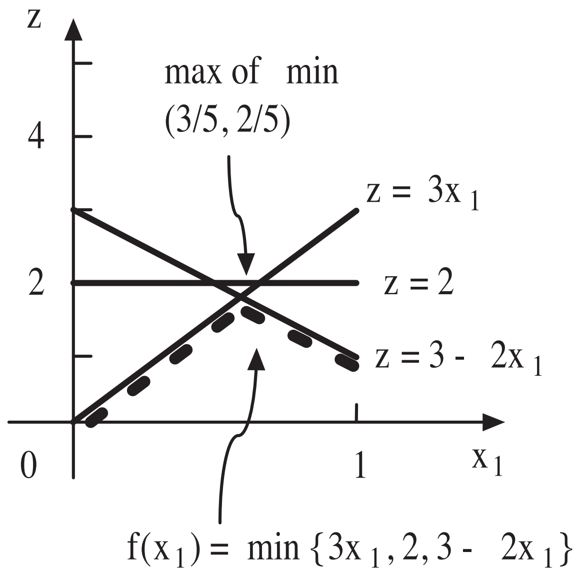

Figure 1: PS2-33

This 2×2 game has no saddle point. So we can use the 2×2 formulae:

$$r = 3 + 3 - 1 - 0 = 5, v = (3 \times 3 - 1 \times 0)/5 = 9/5$$

$$\boldsymbol{x}^* = ((3-0)/5, (3-1)/5) = (3/5, 2/5)$$
$$\boldsymbol{y}^* = ((3-1)/5, (3-0)/5) = (2/5, 3/5).$$

[Note that we get the same $\boldsymbol{x}^*$ and v as above. Why?]

Conclusion: For the original game, the value is $v = 9/5$ and an optimal strategy pair is

$$\boldsymbol{x}^* = (3/5, 2/5)$$
$$\boldsymbol{y}^* = (2/5, 0, 3/5, 0, 0).$$

[Remember to report the result in terms of the original game.]