MAST20005/MAST90058: Week 3 Problems

1. (a) Let $X_1, \dots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$ where $-\infty < \mu < \infty$ and $\sigma^2 > 0$. Assume that σ^2 is known (i.e. it is a fixed, known value). Show the maximum likelihood estimator of μ is $\hat{\mu} = \bar{X}$.

   (b) A random sample $X_1, \dots, X_n$ of size n is taken from a Poisson distribution with mean $\lambda > 0$.

      i. Show the maximum likelihood estimator of λ is $\hat{\lambda} = \bar{X}$.

      ii. Suppose with $n = 40$ we observe 5 zeros, 7 ones, 12 twos, 9 threes, 5 fours, 1 five, and 1 six. What is the maximum likelihood estimate of λ?

   (c) Let $X_1, \dots, X_n$ be random samples from the following probability density functions. In each case find the maximum likelihood estimator $\hat{\theta}$.

      i. $f(x \mid \theta) = \frac{1}{\theta^2} x \exp(-x/\theta), \quad 0 < x < \infty, \quad 0 < \theta < \infty$

      ii. $f(x \mid \theta) = \frac{1}{2\theta^3} x^2 \exp(-x/\theta), \quad 0 < x < \infty, \quad 0 < \theta < \infty$

      iii. $f(x \mid \theta) = \frac{1}{2} \exp(-|x - \theta|), \quad -\infty < x < \infty, \quad -\infty < \theta < \infty$
      Hint: The last part involves minimizing $\sum_{i=1}^n |x_i - \theta|$, which is tricky. Try $n = 5$ and the sample {6.1, -1.1, 3.2, 0.7, 1.7}. Then deduce the MLE in general.

2. Consider a random sample of n observations on X having the following pmf:

x	0	1	2
$p(x)$	$1-\theta$	$3\theta/4$	$\theta/4$

   (a) Find two unbiased estimators for θ: one based on $\bar{X}$, and one based on $Z = \mathrm{freq}(0) = \sum_{i=1}^n I(X_i = 0)$ (i.e. the frequency of $x = 0$).

   (b) Compare the two estimators above in terms of their variance.

3. Let $f(x \mid \theta) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$ and let $X_1, \dots, X_n$ denote a random sample from this distribution. Note that,

$$\int_0^1 x\theta\, x^{\theta-1} dx = \frac{\theta}{\theta+1}.$$

   (a) Sketch the pdf of X for $\theta = 1/2$ and $\theta = 2$.

   (b) Show that $\hat{\theta} = -n / \left(\sum_{i=1}^n \ln X_i\right)$ is the maximum likelihood estimator of θ.

   (c) For each of the following three sets of observations from this distribution, compute the maximum likelihood estimates and the method of moments estimates.

X	Y	Z
0.0256	0.9960	0.4698
0.3051	0.3125	0.3675
0.0278	0.4374	0.5991
0.8971	0.7464	0.9513
0.0739	0.8278	0.6049
0.3191	0.9518	0.9917
0.7379	0.9924	0.1551
0.3671	0.7112	0.0710
0.9763	0.2228	0.2110
0.0102	0.8609	0.2154

   ($\sum_{i=1}^n \ln(x_i) = -18.2063$, $\sum_{i=1}^n \ln(y_i) = -4.5246$, $\sum_{i=1}^n \ln(z_i) = -10.42968$, $\sum_{i=1}^n x_i = 3.7401$, $\sum_{i=1}^n y_i = 7.0592$, $\sum_{i=1}^n z_i = 4.6368$)

4. Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution whose pdf is $f(x \mid \theta) = (1/\theta)\exp(-x/\theta)$, $0 < x < \infty$, $0 < \theta < \infty$.

   (a) Show that $\bar{X}$ is an unbiased estimator of θ.

   (b) Show that the variance of $\bar{X}$ is θ^2/n.

   (c) Calculate an estimate of θ if a random sample gave the following values:

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   3.5  8.1  0.9  4.4  0.5
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5. Let $X_1, \ldots, X_n$ be a random sample from a distribution having finite variance σ^2. Show that

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

   is an unbiased estimator of σ^2.

   Hint: Use a result from question 4(a)(i) from week 2 to derive an alternative expression for S^2 and then compute $E(S^2)$.

6. Let $X_1, \ldots, X_n$ be iid observations from $\mathrm{N}(0, \theta^2)$. Consider the estimators S^2 and $\hat{\theta}^2 = n^{-1}\sum_{i=1}^{n} X_i^2$. Show that $\hat{\theta}^2$ is unbiased and $\mathrm{var}(\hat{\theta}^2) < \mathrm{var}(S^2)$ for any $n > 1$.

7. Let $X_1, \ldots, X_n$ be iid observations from $X \sim \mathrm{N}(\mu, \sigma^2)$. Since X has a symmetric pdf, we might expect that both the sample mean $\bar{X}$ and the sample median $\hat{\pi}_{0.5}$ will be good estimators of the population mean μ.

   (a) Find the variance of $\bar{X}$.

   (b) In general, the sample median will approximately follow a normal distribution, $\hat{\pi}_{0.5} \sim \mathrm{N}\left(\pi_{0.5}, \pi/2 \times \sigma^2/n\right)$, where $\pi_{0.5}$ is the true median (we will learn more about this later in the semester). How does the variance of the sample median compare with that of the sample mean?

   (c) Are the estimators biased?

   (d) Which estimator do you expect to be more accurate?