THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assignments 7 - 9, 2020

School of Mathematics and Statistics

MAST 20018 Discrete Maths and Operations Research

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 7 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).

Instructions to Students

- This assignment is worth 30 points (in the 0-100 point scale for the assignment components). The exam will follow the same format of this assignment. The instructions below are those that you will see in the exam.
- During writing time you may only interact with the device running the Zoom session with supervisor permission. The screen of any other device must be visible in Zoom from the start of the session.
- If you have a printer, print the exam one-sided. If you cannot print, download the exam to a second device, which must then be disconnected from the internet.
- Write your answers in the boxes provided on the exam that you have printed. If you are unable to answer the whole question in the answer space provided then you can append additional handwritten solutions to the end after the 7 numbered pages. If you do this you MUST make a note in the correct answer space or page for the question, warning the marker that you have appended additional remarks at the end.
- If you have been unable to print the exam write your answers on A4 paper. The first page should contain only your student number, the subject code and the subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.
- Assemble all exam pages in correct page number order and the correct way up. Add any extra pages with additional working at the end. Use a mobile phone scanning application to scan all pages to a single PDF file. Scan from directly above to reduce keystone effects. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Submit your PDF file to the Canvas Assignment corresponding to this exam using the Gradescope window. Before leaving Zoom supervision, confirm with your Zoom supervisor that you have Gradescope confirmation of submission.

Question 1

Ozzy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the data of the problem:

	Tonnes of raw material per tonne of		
	Exterior paint	*Interior paint*	*Availability (tonnes/day)*
Raw material $M1$	6	4	24
Raw material $M2$	1	2	6
Profit per tonne ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than one tonne. Also, the maximum daily demand of interior paint is 2 tonnes. Ozzy Mikks wants to determine the optimum product combination of interior and exterior paints that maximises the total daily profit. Formulate, but do not solve, the problem as an LP. Clearly indicate the meaning of your variables.

x_1 = # tonnes produced of exterior paint
x_2 = # ~~tonnes~~ tonnes produced of interior paint
(1 pt)

Max $5x_1 + 4x_2$ (1 pt)

s.t.

$6x_1 + 4x_2 \leq 24$
$x_1 + 2x_2 \leq 6$
(1 pt)

$-x_1 + x_2 \leq 1$ (1 pt)

$x_2 \leq 2$ (1 pt)

$x_1, x_2 \geq 0$ (1 pt)

Question 2

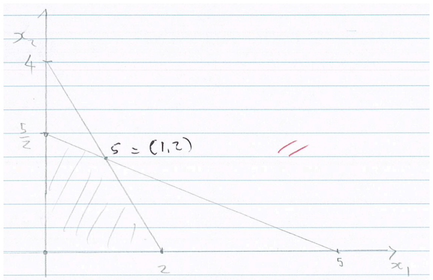
Consider the linear program:

$$\max z = 2x_1 + 3x_2$$

such that

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) Find all basic feasible solutions of the LP using the graphical method.



BFSs are extreme points.

$\therefore$ $(0, 5/2)$, $(1,2)$, $(2,0)$, $(0,0)$.

(b) Convert the LP from (a) into canonical form.

Same o.f.

Constraints:

$$2x_1 + x_2 + x_3 = 4$$

$$x_1 + 2x_2 + x_4 = 5$$

$$x_1, x_2 \geq 0$$

(c) How would you use your answer in (b) to find the basic feasible solutions without using the graphical method? Note: just state the method here; do not recalculate the basic feasible solutions.

Select all pairs of independent columns of the constraint matrix. Set the variables associated with the other columns equal to zero. Solve the resultant set of equations.

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Question 3

Consider the following LP:

$$\max z = x_1 - x_2 + 2x_3$$

such that

$$\begin{aligned} x_1 + x_2 + 3x_3 &\leq 15 \\ 2x_1 - x_2 + x_3 &\leq 2 \\ -x_1 + x_2 + x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Let x_4, x_5 and x_6 denote the slack variables for the respective constraints. After applying the Simplex method, a portion of the final tableau is as follows:

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4				1	-1	-2	
x_3				0	$1/2$	$1/2$	
x_2				0	$-1/2$	$1/2$	
z				0	$3/2$	$1/2$	

Use the algebra of the Simplex method in order to complete the tableau. Note, you are NOT required to complete the RHS column.

Q4.

we know that under the non-slack columns we need $A_B^{-1}A$ and with $c_B A_B^{-1} A - c$ in the z-row.

The A comes from $Ax \le b$.

$$\therefore A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A_B = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Then A_B^{-1} occurs under the slack variables.

$$\therefore A_B^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1 & -1 & -2 \end{bmatrix}$$ (note, swapped rows).

$$\therefore A_B^{-1} \cdot A = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 0 & \frac{1}{2} & 1/2 \\ 1 & -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

For z-row. Note $c_B = (c_2, c_3, c_4)$

$$= (-1, 2, 0)$$

$$c = (1, -1, 2) \cancel{0,0,0}$$

$$\therefore c_B A_B^{-1} A - c$$

$$= (-1, 2, 0) \begin{bmatrix} -3/2 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - (1, -1, 2) \cancel{0}$$

$$= (5/2, -1, 2) - (1, -1, 2)$$

$$= (3/2, 0, 0)$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	1 ~~[illegible]~~	0	0				
x_3	1/2	0	1				
x_2	-3/2	1	0				
z	3/2	0	0				

Question 4

There are four items I_1, I_2, I_3, I_4 to be distributed among three people A, B, C using the method of sealed bids. The bids are given by the following table.

bids	A	B	C
I_1	\$10,000	\$4,000	\$7,000
I_2	\$2,000	\$1,000	\$4,000
I_3	\$500	\$1,500	\$2,000
I_4	\$800	\$2,000	\$1,000

(a) Extend the table to include the fair share of each player, and their contribution to the pot.

bids	A	B	C
I_1	\$10,000	\$4,000	\$7,000
I_2	\$2,000	\$1,000	\$4,000
I_3	\$500	\$1,500	\$2,000
I_4	\$800	\$2,000	\$1,000
Total	\$13,300	\$8,500	\$14,000
Fair Share	\$4,433	\$2,833	\$4,667
Items Won	I_1	I_4	I_2, I_3
Pot	\$5,567	−\$833	\$1,333

(b) Compute the excess in the pot.

The pot contains an excess of \$6,067 (so each player gets \$2,022)

(c) Summarize the allocation to each player.

A gets I_1, and pays $-5567 + 2,022 = \$3,545$ cash. B gets I_4, and receives $833 + 2,022 = \$2,855$ cash. C gets I_2, I_3, and receives $-1333 + 2,022 = \$689$ cash. Note that it is the rounding that causes the error in total deficits and total excess of cash.

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(d) Show, in detail and with reference to every player, that this allocation is envy free.

In the eyes of person A, they receive \$6,455 in monetary value. They value B's allocation as $\$800 + \$2,855 = \$3,655$ and C's allocation as $\$2,500 + \$689 = \$3,139$. So A is envy-free.

In the eyes of person B, they receive \$4,855 in monetary value. They value A's allocation as $\$4,000 - \$3,545 = \$455$ and C's allocation as $\$2,500 + \$689 = \$3,189$. So B is envy-free.

In the eyes of person C, they receive \$6,689 in monetary value. They value A's allocation as $\$7,000 - \$3,545 = \$3,455$ and B's allocation as $\$1,000 + \$2,855 = \$3,855$. So B is envy-free.

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Question 5

It has been suggested that range voting should replace the preferential voting system currently used in Australian elections.

(a) Describe the method of range voting.

Voters score candidates continuously between 0 and 1.
The candidate with the highest score is elected.

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(b) What are the benefits of range voting?

Satisfies IIA, monotonicity, Condorcet.
Addresses the problem of 2-party domination.

(c) What would be the drawbacks of implementing range voting in Australia?

Harder to implement, especially with fractional scores
Does not satisfy majority criterion

End of Exam