Linear programming

Recall that a linear program is an optimisation problem where the objective function is a linear function and the constraints are all linear inequalities. When just two variables are involved, the feasible region can be easily sketched and the visual method for solving them can be applied.

1. Consider the following linear programs:

(a)
$$\begin{aligned} \text{maximise} \quad & z = x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \leqslant 3 \\ & x_2 \leqslant 2 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

(b)
$$\begin{aligned} \text{maximise} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leqslant 4 \\ & x_1 - x_2 \leqslant 2 \\ & x_2 - x_1 \leqslant 1 \\ & x_1 + x_2 \geqslant 1 \\ & x_2 \leqslant 2 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

(c)
$$\begin{aligned} \text{maximise} \quad & z = x_2 - 2x_1 \\ \text{subject to} \quad & x_2 - x_1 \leqslant 1 \\ & x_1 - x_2 \leqslant 1 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

For each of them, sketch the feasible region and plot some level sets of the objective function. Then, determine all optimal solutions, if there are any.

Canonical form and basic solutions

Before we learn the algorithm for solving linear programs, we need to understand the method for converting a system of linear inequalities into a system of linear equations, and when this is in canonical form. Recall that slack variables are introduced by replacing an inequality of the form $a_1x_1 + \dots a_nx_n \leqslant b$ with the equation $a_1x_1 + \dots a_nx_n + s = b$, where s is a new variable introduced for each inequality constraint.

An augmented matrix $\mathbf{A}$ with m rows is in canonical form if there is a subset of columns equal to the standard basis for $\mathbb{R}^m$; i.e., the set of vectors (in any order)

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ \dots, \ \mathbf{e}_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

2. For each of the augmented matrices below, decide if it is in canonical form. Then, for those that are in in canonical form, write down the basic variables and the associated basic solution. Assume that column i corresponds to the variable x_i, and that the column corresponding to the objective function has been omitted.

(a) $\left(\begin{array}{ccccc|c} 10 & 0 & 3 & 1 & 1 & 35 \\ 5 & 1 & -3 & 0 & 1 & 15 \\ 5 & 0 & -1 & 0 & 6 & 45 \end{array}\right)$
(b) $\left(\begin{array}{ccccc|c} 0 & -3 & 3 & 2 & -2 & 10 \\ 1 & -2 & 5 & 0 & 1 & 5 \\ 1 & 2 & 1 & 1 & 1 & 10 \end{array}\right)$

(c) $\left(\begin{array}{cccccc|c} 8 & 0 & 0 & 1 & 6 & -30 & 125 \\ 8 & 0 & 1 & 0 & 12 & 9 & 44 \\ 4 & 1 & 0 & 0 & -2 & 7 & 2 \\ -3 & 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right)$
(d) $\left(\begin{array}{cccccc|c} 8 & 0 & 0 & 1 & 6 & -30 & 125 \\ 8 & 0 & 1 & 0 & 12 & 9 & 44 \\ 4 & 1 & 0 & 0 & -2 & 7 & 2 \\ -3 & 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right)$

3. By introducing slack variables, express each linear program of Question 1 in standard form. Then construct the initial matrix for that linear program. Which of these matrices are canonical?

Basic solutions

When we meet the simplex algorithm, we will see that it works by moving from one corner of the feasible region to another, increasing the value of the objective function at each step. In matrix form, the corner points correspond to *basic solutions*, which are obtained by applying row operations that keep the matrix in canonical form. The basic solution are obtained by setting each non-basic variable to zero.

4. Consider the following linear program:

$$\begin{aligned} \text{maximise} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leqslant 4 \\ & x_1 + 2x_2 \leqslant 4 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

You are given the following information: after introducing slack variables x_3 and x_4, the basic solutions are

$$(0,0,0,0), \quad (0,2,0,2), \quad (2,0,2,0), \quad (\tfrac{4}{3}, \tfrac{4}{3}, 0, 0)$$

(a) Using this information, determine the optimal solution.

(b) What if the problem was to minimise z instead of maximise?

(c) What if the objective function was instead $z = x_1 - x_2$?

5. Consider the linear program (c) from Question 1.

(a) Starting with the initial matrix constructed in Question 3, apply row operations to bring the matrix into canonical form with basic variables x_1 and x_4.

(b) Starting with the initial matrix constructed in Question 3, apply row operations to bring the matrix into canonical form with basic variables x_2 and x_3. Then, apply row operations to this matrix to bring it into canonical form with basic variables x_1 and x_2.

6. Perform the row operations of Question 5 in MATLAB.