MAT4MDS – Practice 11 Worked Solutions

Model Answers to Practice 11

Question 1.

(a) $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}$.

(b) $-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)$ so that $\Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$.

(c) $10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 2^5(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 2^5 5!$

(d)

$$9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} = \frac{9!}{2^4 4 \cdot 3 \cdot 2 \cdot 1} = \frac{9!}{2^4 4!} = \frac{\Gamma(10)}{2^4\Gamma(5)}$$

Alternatively

$$\begin{aligned}
9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 &= 2^5\left(4+\frac{1}{2}\right)\left(3+\frac{1}{2}\right)\left(2+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(\frac{1}{2}\right) \\
&= \frac{2^5}{\Gamma\left(\frac{1}{2}\right)}\left(4+\frac{1}{2}\right)\left(3+\frac{1}{2}\right)\left(2+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{2^5}{\Gamma\left(\frac{1}{2}\right)}\Gamma\left(5+\frac{1}{2}\right) = \frac{2^5\Gamma\left(\frac{11}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}
\end{aligned}$$

(e)

$$\begin{aligned}
1 \cdot 3 \cdot 5 \ldots (2k-1)(2k+1) &= 2^k \ \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \ldots \frac{(2k-1)}{2} \cdot \frac{(2k+1)}{2} \\
&= \frac{2^k}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]\frac{3}{2} \cdot \frac{5}{2} \ldots \frac{(2k-1)}{2} \cdot \frac{(2k+1)}{2} \\
&= \frac{2^k}{\Gamma\left(\frac{1}{2}\right)}\Gamma\left(\frac{2k+1}{2}+1\right) = \frac{2^k}{\Gamma\left(\frac{1}{2}\right)}\Gamma\left(\frac{2k+3}{2}\right)
\end{aligned}$$

Alternatively,

$$1 \cdot 3 \cdot 5 \ldots (2k-1)(2k+1) = \frac{(2k+1)!}{2^k k!} = \frac{\Gamma(2k+2)}{2^k(k+1)!}$$

Question 2. Let $t = u^2$, so that $\frac{dt}{du} = 2u$. Then

$$\Gamma(x) = \int_0^\infty (u^2)^{x-1} e^{-u^2}\frac{dt}{du} du = 2\int_0^\infty u^{2x-2}e^{-u^2}u \ du = 2\int_0^\infty u^{2x-1}e^{-u^2} du$$

(Note that in the change of variables, the terminals do not appear to change.)

Question 3. Let $u = e^{-t}$. Then $t = -\log(u) = \log\left(\frac{1}{u}\right)$ and $\frac{dt}{du} = -\frac{1}{u}$. Hence

$$\Gamma(x) = \int_1^0 \left[\log\left(\frac{1}{u}\right)\right]^{x-1} \cdot u \cdot \left(-\frac{1}{u}\right) du$$

$$= -\int_1^0 \left[\log\left(\frac{1}{u}\right)\right]^{x-1} du = \int_0^1 \left[\log\left(\frac{1}{u}\right)\right]^{x-1} du$$

Question 4.

$$B(p,q+1) + B(p+1,q) = \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)} + \frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1)}$$

$$= \frac{\Gamma(p)q\Gamma(q) + p\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)}$$

$$= \frac{(p+q)\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} = \frac{(p+q)\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)} = B(p,q)$$

Question 5.

(a)

$$\int_{-\infty}^{\infty} y^2 f(y) dy = \frac{1}{B(\alpha,\beta)} \int_0^1 y^{2+\alpha-1}(1-y)^{\beta-1} dy$$

$$= \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

(b) Then the variance is

$$\frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \left[\frac{\alpha}{\alpha+\beta}\right]^2 = \frac{\alpha}{\alpha+\beta}\left[\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right]$$

$$= \frac{\alpha}{\alpha+\beta}\left[\frac{\alpha^2+\alpha+\alpha\beta+\beta-(\alpha^2+\alpha\beta+\alpha)}{(\alpha+\beta)(\alpha+\beta+1)}\right]$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Question 6. For $y \in [0,1]$, $f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$. First note that if $\alpha \leqslant 1$ or $\beta \leqslant 1$ the function will have its greatest value at an end point ($y = 0$ or $y = 1$). (If this is not finite, the function does not have a well-defined mode.) For other values of the parameters, the function is zero at each end, and we look for a local maximum between them. Using the product rule:

$$\begin{aligned} f'(y) &= \frac{(\alpha-1)y^{\alpha-2}(1-y)^{\beta-1}-(\beta-1)y^{\alpha-1}(1-y)^{\beta-2}}{B(\alpha,\beta)} \\ &= \frac{y^{\alpha-2}(1-y)^{\beta-2}}{B(\alpha,\beta)}[(\alpha-1)(1-y)-(\beta-1)y] \\ &= \frac{y^{\alpha-2}(1-y)^{\beta-2}}{B(\alpha,\beta)}[\alpha-1-y(\alpha+\beta-2)] \end{aligned}$$

Thus there is a stationary point at

$$y = \frac{\alpha-1}{\alpha+\beta-2}.$$

Question 7. Possibly by trial-and-error, try $y = \frac{1}{1+u}$. Then $\frac{dy}{du} = -\frac{1}{(1+u)^2}$ and

$1-y = 1-\frac{1}{1+u} = \frac{u}{1+u}$. Also, as $y \to 0,\ u \to \infty$ and at $y=1,\ u=0$. Then

$$B(p,q) = -\int_{\infty}^{0}\left(\frac{u}{1+u}\right)^{p-1}\left(\frac{1}{1+u}\right)^{q-1}\frac{1}{(1+u)^2}du = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}du$$

Question 8. For $y \in (0,\infty)$, $f(y) = \frac{\beta^\alpha y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)}$. First note that if $\alpha \leqslant 1$ the function will have its greatest value at $y=0$. (If this is not finite, the function does not have a well-defined mode.) For other values of the parameter, the function is zero at $y = 0$ and tends to zero as $y \to \infty$, and we look for a local maximum. Using the product rule:

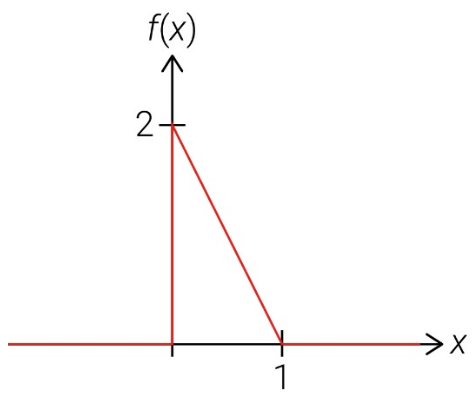
$$\begin{aligned} f'(y) &= \beta^\alpha\ \frac{(\alpha-1)y^{\alpha-2}e^{-\beta y}-\beta y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)} \\ &= \frac{\beta^\alpha y^{\alpha-2}e^{-\beta y}[(\alpha-1-\beta y]}{\Gamma(\alpha)} \end{aligned}$$

Thus there is a stationary point at

$$y = \frac{\alpha-1}{\beta}.$$

Question 9.

(a) $f(x)$ is plotted as follows.



(b) Let F denote the associated CDF, which is defined as follows:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

(i) For $x < 0$ $F(x) = \int_{-\infty}^{x} 0\, dt = 0$.

(ii) For $0 \leq x \leq 1$ we have

$$F(x) = \int_{-\infty}^{0} 0\, dt + \int_{0}^{x} (2 - 2t)\, dt = [2t - \frac{2t^2}{2}]_0^x = 2x - x^2.$$

(iii) For $x > 1$ we have

$$F(x) = \int_{-\infty}^{1} f(t)\, dt + \int_{1}^{x} f(t)\, dt = F(1) + \int_{1}^{x} 0\, dt = 2 \cdot 1 - 1^2 = 1.$$

Summarizing the above steps we obtain:

$$F(x) = \begin{cases} 0, & x < 0 \\ 2x - x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Question 10.

$$\int_{1}^{8} x^2 e^{2y+x} - \frac{x}{y} dy = \frac{1}{2} x^2 e^{2y+x} - x\log_e(y)\Big|_{y=1}^{8}$$

$$= \frac{1}{2} x^2 (e^{16+x} - e^{2+x}) - x\log_e(8) = \frac{x^2 e^x}{2}(e^{16} - e^2) - 3x\log_e(2)$$

Question 11. For $x, y \geqslant 0$:

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)dt\ ds = \int_{0}^{x} \int_{0}^{y} 6e^{-2s} e^{-3t} dtds$$

$$= \int_{0}^{x} -2e^{-2s}[e^{-3t}]_{t=0}^{y}\ ds$$

$$= \int_{0}^{x} 2(1 - e^{-3y})e^{-2s}\ ds$$

$$= -(1 - e^{-3y})[e^{-2s}]_{s=0}^{x} = (1 - e^{-3y})(1 - e^{-2x})$$

Question 12. For x or y negative, $F(x, y) = 0$. As $x \to \infty$ and $y \to \infty$, $F(x, y) \to 1$, as required.