MAST30001 Stochastic Modelling

Tutorial Sheet 4

1. Refer to Tutorial Sheet 3, Problem 3 and also assume that in Corner 2 there is a bigger spider ready to eat the little spider and in Corner 3, there is a hole leading to the outside through which the spider can escape.

   (a) If the spider starts in Corner 1, what is the chance the spider will escape before being eaten?

   (b) If the spider starts in Corner 1, what is the expected number of steps before the spider exits (either the box or this world)?

   (c) If initially the spider is dropped in the middle of the box and it chooses a corner uniformly, what is the chance the spider will escape before being eaten?

   (d) If initially the spider is dropped in the middle of the box and it chooses a corner uniformly, what is the expected number of steps before the spider exits (either the box or this world)?

Ans.

(a) The Markov chain has transition matrix equal to

$$\frac{1}{47}\begin{pmatrix} 0 & 15 & 12 & 20 \\ 15 & 0 & 20 & 12 \\ 12 & 20 & 0 & 15 \\ 20 & 12 & 15 & 0 \end{pmatrix}$$

If A is the event that the spider escapes before being eaten, then we want to find $P(A|X_0 = 1)$ and denote $x_j = P(A|X_0 = j)$ for $j = 1, 4$. Then first step analysis implies that the x_j satisfy

$$\begin{aligned} x_1 &= \frac{12}{47} + \frac{20}{47}x_4, \\ x_4 &= \frac{15}{47} + \frac{20}{47}x_1, \end{aligned}$$

and solving this we have $x_1 = 32/67$ (this number is the answer) and $x_4 = 35/67$.

(b) Similar to above, but now if e_j is the expected number of steps before the spider exits given it starts at corner j, then we need to solve

$$\begin{aligned} e_1 &= \frac{15}{47} + \frac{12}{47} + \frac{20}{47}(1 + e_4), \\ e_4 &= \frac{12}{47} + \frac{15}{47} + \frac{20}{47}(1 + e_1), \end{aligned}$$

and solving we have $e_1 = 47/27$ (the answer) and $e_4 = 47/27$.

(c) By averaging over the possible starting states and using our answers above we find

$$P(A|X_0 = \text{Uniform}(1, 2, 3, 4)) = \frac{1}{4} + \frac{1}{4}x_1 + \frac{1}{4}x_4 = \frac{1}{2}.$$

You can also see the answer is 1/2 by using symmetry.

(d) Again we average over the possible starting states and use our answers above to find the expectation we're after to be

$$\frac{1}{4}e_1 + \frac{1}{4}e_4 = \frac{47}{54}.$$

2. A simplified model for the spread of a contagion in a small population of size 5 is as follows. At each discrete time unit, two individuals in the population are chosen uniformly at random to meet. If one of these persons is healthy and the other has the contagion, then with probability 1/4 the healthy person becomes sick. Otherwise the system stays the same. If initially one person has the disease, what is the average amount of time before everyone in the population has the disease? What about if the population is of size N?

   Ans.

   As above we use first step analysis. Notice that the transition probabilities of the chain are for $i = 1, \ldots, 4$

$$p_{i,i+1} = 1 - p_{i,i} = \frac{i(5-i)}{4\binom{5}{2}}$$

   and $p_{5,5} = 1$. For e_i the expected number of steps until everyone has the disease given that the process starts from state i we have for $i = 1, 2, 3$,

$$e_i = p_{i,i}(1 + e_i) + p_{i,i+1}(1 + e_{i+1}),$$

   and for $e_4 = p_{4,4}(1 + e_4) + p_{4,5}$. Solving this recursion we have for $i = 1, \ldots, 4$

$$e_i = \sum_{j=i}^{4} \frac{1}{p_{j,j+1}}.$$

   In general if the the population is of size N, then

$$p_{i,i+1} = 1 - p_{i,i} = \frac{i(N-i)}{4\binom{N}{2}}$$

   and

$$e_i = \sum_{j=i}^{N-1} \frac{1}{p_{i,i+1}}. \tag{1}$$

   Also note that the number of steps to go from state i to $i + 1$ is distributed as a positive Geometric with parameter $p_{i,i+1}$, with mean $1/p_{i,i+1}$, which also yields the formula (1).

3. A Markov chain has transition matrix

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Analyse the state space of the chain (reducibility, periodicity, null/pos recurrence, etc).

(b) Find the stationary distribution of the chain.

(c) If the initial state of the chain is uniformly distributed, find $\lim_{n\to\infty} P(X_n = 2)$.

Ans.

(a) The chain is irreducible and finite so it's positive recurrent and since $p_{2,2}^{(2)} = p_{2,2}^{(3)} = 1/2$ state 2 is aperiodic and since period is a class property the chain is aperiodic.

(b) Solve $\pi P = \pi$ for P the transition matrix to find

$$\pi = [2/5, 2/5, 1/5].$$

(c) Since the chain is aperiodic, irreducible and positive recurrent it is ergodic and

$$\lim_{n\to\infty} P(X_n = 2) = 2/5.$$

4. A Markov chain has transition matrix

$$\begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix}$$

Analyse the state space (reducibility, periodicity, null/pos recurrence, etc), and discuss the chain's long run behaviour.

Ans. We first determine the communicating classes. Let P be the transition matrix above with (i, j) entry p_{ij}. Since $p_{13}, p_{31} > 0$, states 1 and 3 are in the same communicating class and moreover from state 1 the chain can only stay put or go to state 3 and from state 3 the chain can only stay put or go to state 1. So $\{1, 3\}$ is a communicating class which is essential since the chain cannot leave it. The same arguments apply for $\{2, 4\}$ which is also an essential communicating class. State 5 must be in its own communicating class (since the communicating classes partition the state space), and since from state 5 the chain can move to the essential communicating class $\{1, 3\}$, $\{5\}$ is an inessential class. So the chain is reducible since it has more than one communicating class.

Since all states have a "loop", they are all aperiodic and hence each communicating class is aperiodic.

Since the chain has a finite state space, all states in an essential communicating class (i.e., states $1 - 4$) are positive recurrent and state 5, being in an inessential communicating class is transient (started from state 5, the probability of not returning to state 5 eventually is 1).

To study the long run behaviour of the chain, we note first that if the chain starts in state 5, then it will eventually end up in the communicating class $\{1, 3\}$. So we only need to study the long run behaviour of the essential communicating classes $\{1, 3\}$ and $\{2, 4\}$. Since the chain restricted to these classes is aperiodic, irreducible, and positive recurrent, it is ergodic. So if Q is the transition matrix of one of these restricted chains, then $\lim_{n\to\infty} Q^n_{ij} = \pi_j$ for a probability distribution π and this distribution must be the unique stationary distribution of the chain restricted to the communicating class. [π_j is also the limiting proportion of time the chain spends in state j.]

The chain restricted to the class $\{1,3\}$ has transition matrix

$$Q = \begin{pmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{pmatrix},$$

so the long run distribution $\pi^{(1)}$ satisfies $\pi Q = \pi$ and $\pi_1 + \pi_2 = 1$ and solving these equations for π we find

$$\pi = (9/17, 8/17).$$

A similar argument for the class $\{2,4\}$ yields that its long run distribution is

$$\rho = (3/5, 2/5).$$

To summarize, if the chain starts in the states any of the states $1,3$ or 5 then it eventually ends up in the communicating class $\{1,3\}$ and spends about $9/17$ of the time in state 1 and the rest in state 3. If the chain starts in states 2 or 4, then the chain in the long run spends about $3/5$ of the time in state 2 and the rest in state 4.

5. A machine produces two items per day. The probability that an item is *not* defective is p, with all items produced independently, and defective items are thrown away immediately. The demand for items is one per day, and any demand not met by the end of the day is lost, while any extra item is stored. Let X_n be the number of items in storage just before the beginning of the nth day.

   (a) Model X_n as a Markov chain, draw its transition diagram and compute its transition probabilities.

   (b) When is the Markov chain ergodic? Compute the limiting distribution when it exists.

   (c) Suppose it costs \$$c$ to store an item for one night and \$$d$ for every demanded item that cannot be supplied. Compute the long run cost of the production facility when the chain is ergodic.

   Ans.

   (a) The number of items in storage increases by 1 if two non-defective items are produced, it stays constant if one non-defective item is produced and decreases by one if both items are defective and there are items in storage; these probabilities only depend on the current number of items in storage and so X_n is a Markov chain with transition probabilities for $j \geq 1$:

   $$p_{j,j+1} = p^2, \;\; p_{j,j} = 2p(1-p) \;\; p_{j,j-1} = (1-p)^2,$$

   and $p_{0,1} = 1 - p_{0,0} = p^2$.

   (b) The chain is irreducible and aperiodic (loops) and so we only need to determine when

   $$\pi P = \pi$$

has a probability vector π solution. We need to solve the equations

$$\pi_0 = (1-p^2)\pi_0 + (1-p)^2\pi_1$$

and for $j \geq 1$

$$\pi_j = (1-p)^2\pi_{j+1} + 2p(1-p)\pi_j + p^2\pi_{j-1}.$$

Working up from $j = 0$ and guessing the formula, a check shows that the solution must satisfy

$$\pi_j = \left(\frac{p^2}{(1-p)^2}\right)^j \pi_0.$$

And also $\sum_{j=0}^{\infty} \pi_j = 1$, so the chain is ergodic only if $p/(1-p) < 1$, that is, if $p < 1/2$, and in this case the limiting distribution is geometric:

$$\pi_j = \left(\frac{p^2}{(1-p)^2}\right)^j (1 - p^2/(1-p)^2).$$

(c) To incur a cost of \$$d$, we need there to be no items in storage and to produce two defective items the following day; the long run proportion of days this happens when the chain is ergodic equals $\pi_0(1-p)^2 = (1-2p)$. We also incur a per day cost of cX_n, which over the long run is $c\mathbb{E}[X_n] \to cp^2/(1-2p)$ and so the per day cost of the facility is

$$d(1-2p) + cp^2/(1-2p).$$