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The University of Melbourne
Semester 1 Assessment 2011

Department of Mathematics and Statistics
MAST10007 Linear Algebra

Reading Time: 15 minutes
Writing Time: 3 hours

This paper has: 6 pages

Identical Examination Papers: None
Common Content Papers: None

Authorized Materials:

No materials are authorized.

Calculators and mathematical tables are not permitted.

Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:

This examination consists of 12 questions.

The total number of marks is 100.

All questions may be attempted. All answers should be appropriately justified.

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) Consider the following linear system:

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & + & 4x_4 & = & 5 \\ -2x_1 & + & 4x_2 & - & 5x_3 & - & 9x_4 & = & -12 \\ -x_1 & + & 2x_2 & - & 2x_3 & - & 5x_4 & = & -7 \end{array}$$

   (i) Write down the augmented matrix corresponding to the linear system.

   (ii) Reduce the matrix to reduced row-echelon form.

   (iii) Use the reduced row-echelon form to give all solutions to the linear system.

   (b) Determine the values of k (if any) for which the following linear system has:

   (i) no solution,

   (ii) a unique solution,

   (iii) infinitely many solutions.

$$\begin{array}{rcrcrcl} x & + & 2y & + & 2z & = & -2k+1 \\ x & + & ky & + & (k+4)z & = & 0 \\ x & + & 2y & + & (k+2)z & = & 1 \end{array}$$

[10 marks]

2. Let

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 2 & 6 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Evaluate, if possible:

(a) AC

(b) A^2

(c) ABA

(d) $A + AC$

[4 marks]

3. Consider the matrix

$$M = \begin{bmatrix} -1 & -2 & -1 \\ 8 & 0 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

(a) Use cofactor expansion to calculate its determinant $\det(M)$.

(b) Find M^{-1} or explain why it does not exist.

(c) Suppose that N is a 3×3 matrix with $\det(N) = -3$. Calculate $\det(2NM^2)$.

[8 marks]

4. (a) Let L be the line in $\mathbb{R}^3$ given by the following cartesian equation:

$$\frac{2x-1}{3} = \frac{y-1}{2} = \frac{2-3z}{2}$$

(i) Write down a vector equation of the line L.

(ii) Find the Cartesian equation of the plane Π that contains the point $(2, 3, 1)$ and is perpendicular to the line L.

(b) Let $\boldsymbol{u} = (1, -2, 3)$ and $\boldsymbol{v} = (2, 0, 1)$.

(i) Calculate the cross product $\boldsymbol{u} \times \boldsymbol{v}$.

(ii) What is the area of the triangle in $\mathbb{R}^3$ having vertices at $(0, 0, 0)$, $(1, -2, 3)$ and $(2, 0, 1)$?

[8 marks]

5. (a) For each of the following, decide whether or not the given set S is a subspace of the vector space V. Justify your answers by either appealing to appropriate theorems, or providing a counter-example.

(i) $V = \mathcal{P}_2$ (all polynomials of degree at most two) and

$$S = \{a_0 + a_1 x + a_2 x^2 : |a_0| = |a_2|\}$$

(For a real number $a \in \mathbb{R}$, $|a|$ denotes the absolute value of a.)

(ii) $V = M_{2,2}$ (all 2×2 matrices) and

$$S = \left\{ A \in M_{2,2} : \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

(b) Let V be a vector space and let H and K be two subspaces of V. Show that the following set W is a subspace of V:

$$W = \{\boldsymbol{u} + \boldsymbol{v} : \boldsymbol{u} \in H, \boldsymbol{v} \in K\}$$

[10 marks]

6. (a) Determine whether or not the given set S is linearly independent in the given vector space V. If the set is linearly dependent, express one of its vectors as a linear combination of the other vectors in the set.

(i)
$$S = \left\{ \begin{bmatrix} -4 & -2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -16 & 6 \\ 0 & 10 \end{bmatrix}, \begin{bmatrix} 7 & 1 \\ 0 & -4 \end{bmatrix}, \begin{bmatrix} -6 & -2 \\ 0 & 3 \end{bmatrix} \right\}$$
$$V = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \quad \text{(all upper-triangular } 2 \times 2 \text{ matrices)}$$

(ii)
$$S = \{-1 + 2x - 2x^2 + 2x^3,\ 1 - 2x + 2x^2 - 4x^3,\ -1 + 4x - 4x^2 + x^3\}$$
$$V = \mathcal{P}_3 \qquad \text{(the vector space of all polynomials of degree at most 3)}$$

(b) Let V be a vector space and $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$.

(i) Define the span of S, $\operatorname{Span}(S)$.

(ii) Show that if $\operatorname{Span}(S) = \operatorname{Span}(\{v_1, \ldots, v_{k-1}\})$ then S is linearly dependent.

[10 marks]

7. Let

$$A = \begin{bmatrix} 2 & 1 & 4 & 0 & 1 \\ -1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & 5 \\ -1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B is the reduced row echelon form of the matrix A. Using this information, or otherwise, answer the following, giving reasons for your answers.

(a) Is A invertible? Give a reason.

(b) Write down a basis for the column space of A.

(c) Write down a basis for the row space of A.

(d) Do the rows of A span $\mathbb{R}^5$? Give a reason.

(e) Are the vectors $(2, -1, 0, -1, 0), (1, 0, 1, 1, 2), (4, -1, 2, 1, 4), (0, 1, 2, 1, 2)$ linearly independent? If not, write one of these vectors as a linear combination of the others.

(f) Find a basis for the solution space of A.

[8 marks]

8. Let $\mathcal{P}_n$ denote the vector space of all real polynomials of degree at most n in the variable x. A linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_2$ is defined by

$$T(p(x)) = p(x+1) - p(x)$$

(a) Write down $T(a + bx + cx^2 + dx^3)$ and check that this is in $\mathcal{P}_2$.

(b) Find the matrix that represents T relative to the basis $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$ and the basis $\{1, x, x^2\}$ for $\mathcal{P}_2$.

(c) Find bases for the image and kernel of T.

[8 marks]

9. Consider the bases $\mathcal{S} = \{(1,0),(0,1)\}$ and $\mathcal{B} = \{(2,1),(-1,1)\}$ for $\mathbb{R}^2$.

(a) (i) Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{S}$.

(ii) Find the transition matrix $P_{\mathcal{B},\mathcal{S}}$ from $\mathcal{S}$ to $\mathcal{B}$.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x,y) = (2x+6y, 2x+y)$.

(i) Find the matrix $[T]_{\mathcal{S}}$ of the transformation T with respect to the basis $\mathcal{S}$.

(ii) Find the matrix $[T]_{\mathcal{B}}$ of the transformation T with respect to the basis $\mathcal{B}$.

(iii) If $\boldsymbol{v} = (10, -1)$ find $[\boldsymbol{v}]_{\mathcal{B}}$ and $[T\boldsymbol{v}]_{\mathcal{B}}$.

[8 marks]

10. (a) Let $\boldsymbol{a} = (1, -2)$ and $\boldsymbol{b} = (2, 1)$. The formula

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \;=\; \langle (x_1, x_2), (y_1, y_2) \rangle \;=\; 2x_1y_1 + 2x_2y_1 + 2x_1y_2 + 3x_2y_2$$

defines an inner product on $\mathbb{R}^2$. (You do not need to prove this.)
Use this inner product to answer parts (i) to (iii) below.

(i) Are the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ orthogonal? Explain your answer.

(ii) Find the length of the vector $\boldsymbol{a}$, that is find $||\boldsymbol{a}||$.

(iii) Find the distance between $\boldsymbol{a}$ and $\boldsymbol{b}$, that is find $d(\boldsymbol{a}, \boldsymbol{b})$.

(b) Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\boldsymbol{v}_1 = (1, -1, 0, 1)$ and $\boldsymbol{v}_2 = (2, 0, 1, 1)$. Use the *dot product* on $\mathbb{R}^4$ to answer the following:

(i) Find an orthonormal basis for W;

(ii) Find the point in W closest to $\boldsymbol{c} = (0, 2, 1, 0)$.

[8 marks]

11. Let

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) Find corresponding eigenvectors of A.

(c) Find a matrix P and a matrix D such that $P^{-1}AP = D$. Check your answers by showing that $AP = PD$.

(d) Find an orthogonal matrix Q and a diagonal matrix D such that

$$D = Q^T AQ$$

(e) Consider the conic

$$3x^2 + 4xy + 3y^2 = 5.$$

Use your previous answers to find a simplified equation for the conic. Hence identify and sketch the conic using both the x-y axes and your new coordinate axes (i.e., the principal axes for the conic).

[10 marks]

12. (a) For each of the following matrices decide whether or not the matrix is diagonalizable over $\mathbb{R}$. You should justify your answers.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

(b) Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

(i) Find the eigenvalues of A.

(ii) Find a basis for each eigenspace of A.

(iii) Is the matrix A diagonalizable? Explain your answer.

[8 marks]

— END OF EXAMINATION QUESTIONS —