1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = x_1x_2 + x_2x_3 - x_1x_3 - x_2^2.$$

   (a) [2 marks] Show that all irregular points are elements of the 0-level set of f.

   (b) [2 marks] Show that $\boldsymbol{\gamma}_i : \mathbb{R} \to \mathbb{R}^3$, $i = 1, 2$, given by

$$\boldsymbol{\gamma}_1(t) = (t-1, t, t+1), \qquad \boldsymbol{\gamma}_2(s) = (s, 2s, \frac{1}{s} + 2s)$$

   are curves in the 1-level set of f.

   (c) [1 mark] Determine the point $\mathbf{p}$ where the curves from (b) intersect.

   (d) [3 marks] Show that at the point $\mathbf{p}$, from (c), the tangent vectors to the curves, from (b), are independent and orthogonal to the gradient of f at $\mathbf{p}$.

2. Consider the set-constraint problem:

$$\begin{array}{ll} \text{maximise} & (x_1-2)^2 + (x_2-2)^2 \\ \text{subject to} & \mathbf{x} \in \Omega = \{\mathbf{x} :\ x_1 \geq 0,\ x_2 \leq 4, \text{ and } x_2 \geq x_1^2\} \end{array}$$

   (a) [5 marks] Let $\mathbf{p} = (2, 4)^T$. Determine and draw in one diagram: the level set of the objective function through $\mathbf{p}$, normal vectors to the active constraints at $\mathbf{p}$, and the feasible set Ω.

   (b) [3 marks] Describe the set of feasible directions at $\mathbf{p}$ using the normal vectors you found in (a). State whether $(-1, 0)^T$ is feasible.

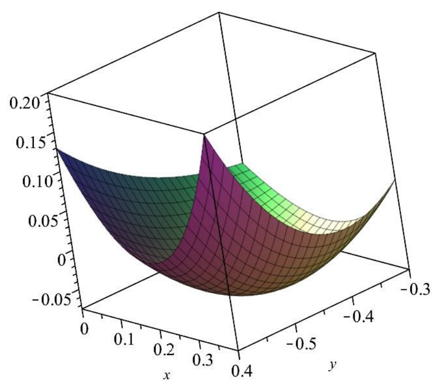
   (c) [2 marks] Is the FONC satisfied at $\mathbf{p}$? Justify your answer.

   (d) [1 mark] State whether the point $\mathbf{p}$ is a local maximiser. No reasons required.

3. [7 marks] Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(\mathbf{x}) = (x_1 - x_2)^4 + 5x_1x_2 + x_1.$$

Its graph



shows there is a minimiser close to $\mathbf{q} = (0.3, -0.5)^T$ (this is the only thing that is relevant about the graph). Perform a gradient method, with one iteration of Newton's method to do the line search, to get one step closer to the minimiser. Verify that the point you obtain is closer to the minimiser.

4. [5 marks] Solve the LP problem

$$\begin{aligned} \text{minimise} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & 2x_1 - x_2 \leq 2 \\ & 2x_2 - x_1 \geq 2 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

by sketching the feasible region and drawing some level sets of the objective function. State the minimum and the corner at which the minimum occurs.

5. Consider the LP problem

$$\begin{aligned} \text{maximise} \quad & z = 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 2 \\ & x_1 + 2x_2 \leq 4 \\ & 3x_1 - 2x_2 \leq 6 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

   (a) [3 marks] By introducing slack variables and artificial variables as appropriate, and using the 2-phase method, write down the augmented matrix for the first phase.

   Indicate the entry to pivot on in this matrix.

   (b) [3 marks] Solve the first phase using the simplex method, and write down the augmented matrix for the second phase. Indicate the entry to pivot on in this matrix.

   Note: You do not have to solve the second phase.

6. [4 marks] Use a Lagrange multiplier to show that if there is a point $(x, y)^T$ on the graph of $y = \ln(x)$ which is closest to $\mathbf{0}$ (in the Euclidean norm), then it satisfies $y = -x^2$.

7. Consider the nonlinear problem

$$\begin{aligned} \text{minimise} \quad & z = x_1(x_1 + 1) + x_2 + x_3^2 \\ \text{subject to} \quad & x_1 - x_2 - x_3 = 0, \\ & x_1 + x_2 + x_3 \leq 0. \end{aligned}$$

   (a) [1 mark] Write down the relevant KKT condition.

   (b) [3 marks] Assuming that the inequality constraint is active, write down and solve the linear system which arises from (a). Decide whether the solution gives rise to a local minimum.

   (c) [5 marks] Assuming that the inequality constraint is inactive, write down and solve the linear system which arises from (a). Decide whether the solution gives rise to a minimum.