1. [2+2+2+2=8 marks] Let $f = \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = \frac{e^{2x_1x_2}}{(x_1-2)(x_2+1)},$$

which has partial derivatives

$$\frac{\partial f}{\partial x_1} = \frac{e^{2x_1x_2}(2x_1x_2 - 4x_2 - 1)}{(x_1-2)^2(x_2+1)}, \qquad \frac{\partial f}{\partial x_2} = \frac{e^{2x_1x_2}(2x_1x_2 + 2x_1 - 1)}{(x_1-2)(x_2+1)^2}$$

(a) Show that the point $\mathbf{p} = (1, -\frac{1}{2})$ is stationary. Is the FOSC satisfied?

Substituting $x_1 = 1, x_2 = -\frac{1}{2}$ in the partial derivatives yields 0. The FOSC is satisfied.

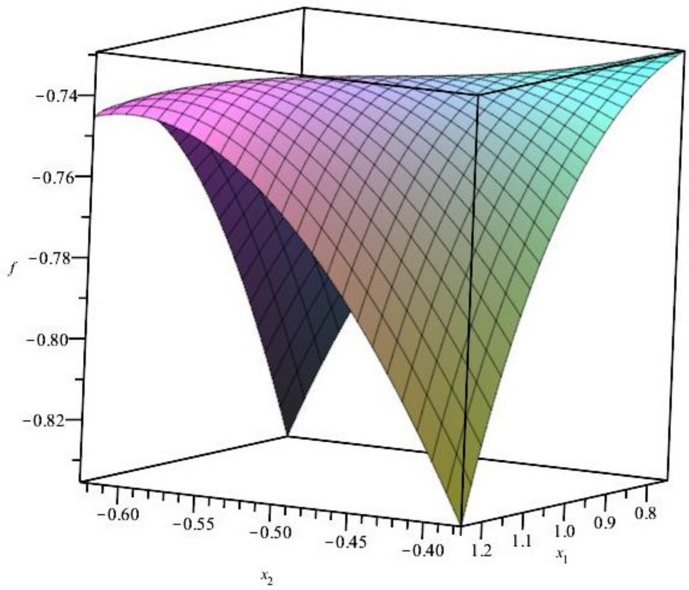
(b) Given that the Hessian at $\mathbf{p}$ is given by

$$H = -\frac{2}{e}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Is the SONC satisfied? Is the SOSC satisfied?

The determinant of the matrix is 0, the matrix is negative semi-definite. Hence, the SONC is satisfied, the SOSC is not satisfied.

(c) Looking at the graph below, the point seems to be on an edge.



What is the direction $\mathbf{d}$ of the edge at $\mathbf{p}$? Choose $\mathbf{d} = (a, b)$ with a, b coprime integers and $a > 0$.

This direction is in the kernel of the Hessian, i.e. $\mathbf{d} = (2, -1)$.

(d) Substituting $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ in $f(\mathbf{x})$ we obtain

$$f(t) = -\frac{2\,e^{-(2t+1)^2}}{(2t-1)^2}$$

which has derivatives

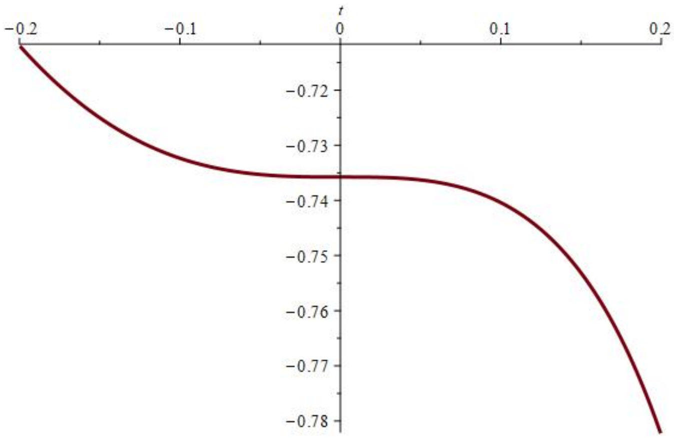
$$f'(t) = \frac{32\,\mathrm{e}^{-(1+2t)^2}t^2}{(-1+2t)^3}$$
$$f''(t) = -\frac{64\,\mathrm{e}^{-(1+2t)^2}t\,(8t^3 - t + 1)}{(-1+2t)^4}$$
$$f'''(t) = \frac{64\,\mathrm{e}^{-(1+2t)^2}\,(128t^6 - 48t^4 + 48t^3 + 1)}{(-1+2t)^5}$$

Write down the third order Taylor series for $f(t)$ about $t = 0$. Can you use the Taylor series to classify the stationary point $\mathbf{p}$?

The third order Taylor series is

$$f(t) \sim -\frac{2}{\mathrm{e}} - \frac{32}{3\mathrm{e}}t^3 + \cdots .$$

So on the edge the function behaves like a cube power of t, i.e. it goes up to the left and down to the right.



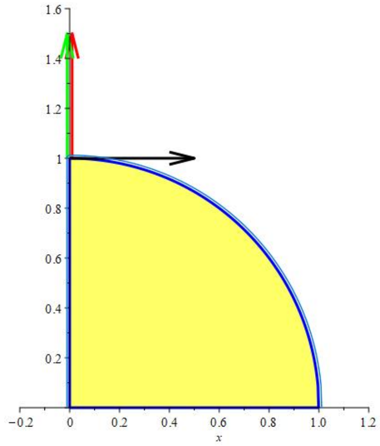
Therefore, $\mathbf{p}$ is not a maximum (just a nice point to sit down and enjoy the view).

2. [4+3+2+1=10 marks] Consider the set-constraint problem:

$$\begin{array}{rl} \text{maximize} & x_2^2 - x_1^3 \\ \text{subject to} & \mathbf{x} \in \Omega = \{\mathbf{x} :\ x_1 \geq 0,\ x_2 \geq 0,\ \text{and } x_1^2 + x_2^2 \leq 1\} \end{array}$$

(a) Let $\mathbf{p} = (0,1)^T$. Determine and draw in one diagram: the gradient of the objective function at $\mathbf{p}$, normal vectors to the active constraints at $\mathbf{p}$, and the feasible set Ω.

At $\mathbf{p}$ the gradient of the objective function is $(-3(0)^2, 2 \cdot 1)^T \propto (0,1)$. The two active constraints are $x_1 \geq 0$ and $x_1^2 + x_2^2 \leq 1$. A normal to the first active constraint is $\nabla x_1 = (1,0)$. A normal to the second active constraint is $\nabla(x_1^2 + x_2^2) = (0,2) \propto (0,1)$. The gradient of the objective function points in the same direction as the normal to the second constraint (drawn in red and green), the normal to the first constraint is drawn in black.



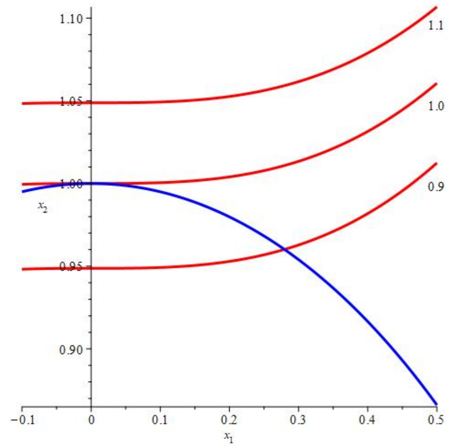
(b) Describe the set of feasible directions at $\mathbf{p}$ using the normal vectors you found in (a). State whether $(1,0)^T$ is feasible.

The set of feasible directions is $\{\mathbf{d} \in \mathbb{R}^2 \setminus \{\mathbf{0}\} : d_1 \geq 0, d_2 < 0\}$. The vector $(1,0)^T$ is not feasible.

(c) Is the FONC satisfied at $\mathbf{p}$? Justify your answer.

Yes. According to the FONC, if $\mathbf{p}$ is a local maximiser then $\nabla f(\mathbf{p})^T \mathbf{d} \leq 0$ for feasible $\mathbf{d}$. We have $\nabla f(\mathbf{p})^T = (0,1)$ and so we need $d_2 \leq 0$, which is the case for all feasible $\mathbf{d}$.

(d) The c-level sets with $c \in \{0.9, 1, 1.1\}$, and one of the active constraints, are given below.



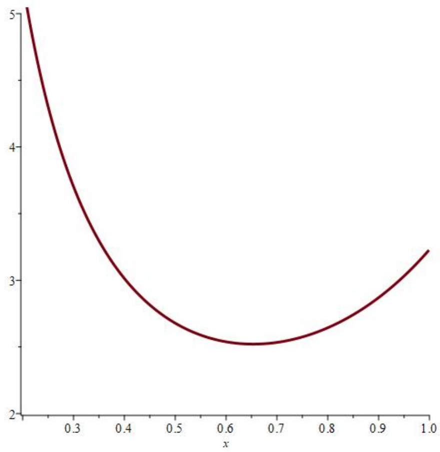
State whether the point $\mathbf{p}$ is a local maximiser. No reasons required.

It is a local maximiser.

3. [2+2+2=6 marks] Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \frac{e^{x^2}}{\sin(x)}$$

Its graph



shows there is a minimum between $a = .6$ and $c = .7$. Let $\rho = \frac{3-\sqrt{5}}{2} \approx 0.381966$.

(a) Use the point $b = a + \rho(c - a) = 0.638197$ to prove that there is a minimum between a and c.

We have $f(a) = 2.538473$, $f(b) = 2.522455$, $f(c) = 2.533796$. Because $f(b) < \min(f(a), f(c))$ there is a minimum in the interval (a, c).

(b) Let $d = b + (1 - 2\rho)(c - a) = 0.661803$. Find out whether the minimum is in the interval (a, d) or (b, c)

We have $f(d) = 2.521526 < \min(f(b), f(c))$, so the minimum is in (b, c).

(c) Update a, b, c and determine a new d. Once more, find out whether the minimum is in the interval (a, d) or (b, c).

We set $a = 0.638197$, $b = 0.661803$, $c = .7$ and calculate $d = b+(1-2\rho)(c-a) = 0.676393$. We know the values $f(a) = 2.522455$, $f(b) = 2.521526$, $f(c) = 2.533796$, and calculate $f(d) = 2.524233858$. Because $f(b) < \min(f(a), f(d)$ the minimum is in (a, d).

4. [5 marks] Solve the LP problem

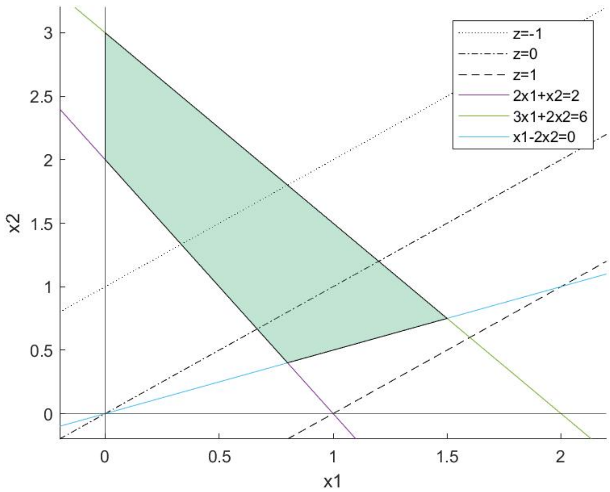
Maximise $\quad z = x_1 - x_2$

Subject to $\quad 2x_1 + x_2 \geq 2$

$$3x_1 + 2x_2 \leq 6$$
$$x_1 - 2x_2 \leq 0$$
$$\mathbf{x} \geq \mathbf{0}$$

by sketching the feasible region and drawing some level sets of the objective function. State the maximum and the corner at which the maximum occurs.

The feasible region is bounded by the three constraints, and the vertical axis. The corner points are (0,2), (0,3), (3/2,3/4), and (4/5,2/5). The -1,0,1-level set are drawn, the maximum will be a bit less than 1.



Adding the equation $3x_1 + 2x_2 = 6$ to $x_1 - 2x_2 = 0$ gives $x_1 = 3/2$. Then $x_2 = 3/2/2 = 3/4$, and we obtain that the corner at which the maximum $x_1 - x_2 = \boxed{3/4}$ occurs is (3/2,3/4).

5. [2+4+3=9 marks] Consider the ILP problem

$$\begin{aligned} \text{maximize} \quad & z = 3x_2 - x_1 \\ \text{subject to} \quad & x_2 + 2x_1 \geq 8 \\ & 3x_2 - 2x_1 \leq 3 \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \in \mathbb{Z}^2. \end{aligned}$$

(a) A lecturer, Peter, applied the simplex algorithm to the corresponding LP problem and found the canonical matrix of the optimal solution to be

$$\begin{pmatrix} 1 & 0 & \frac{3}{8} & -\frac{1}{8} & \frac{21}{8} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & \frac{3}{8} & \frac{7}{8} & \frac{45}{8} \end{pmatrix}$$

Introduce the Gomory cut which corresponds to the first row, and write it in standard form.

The Gomory cut is

$$\frac{3}{8}x_3 + \frac{7}{8}x_4 \leq \frac{5}{8}$$

or, in standard form,

$$\frac{3}{8}x_3 + \frac{7}{8}x_4 - x_5 = \frac{5}{8}$$

(b) Introduce an artificial variable x_6, write down the objective function for $w = -x_6$ (which should be maximised), give the augmented matrix for the phase 1 problem, and solve it.

We bring the Gomory cut into canonical form,

$$\frac{3}{8}x_3 + \frac{7}{8}x_4 - x_5 + x_6 = \frac{5}{8},$$

so that

$$w = \frac{3}{8}x_3 + \frac{7}{8}x_4 - x_5 - \frac{5}{8}.$$

The phase 1 augmented matrix is

$$\begin{pmatrix} 1 & 0 & \frac{3}{8} & -\frac{1}{8} & 0 & 0 & \frac{21}{8} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{11}{4} \\ 0 & 0 & \frac{3}{8} & \frac{7}{8} & -1 & 1 & \frac{5}{8} \\ 0 & 0 & -\frac{3}{8} & -\frac{7}{8} & 1 & 0 & -\frac{5}{8} \end{pmatrix}.$$

Pivoting about the (3,4) element gives

$$\begin{pmatrix} 1 & 0 & \frac{3}{7} & 0 & -\frac{1}{7} & \frac{1}{7} & \frac{19}{7} \\ 0 & 1 & \frac{1}{7} & 0 & \frac{2}{7} & -\frac{2}{7} & \frac{18}{7} \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{8}{7} & \frac{8}{7} & \frac{5}{7} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

which is a normal form for the system with $w = 0$ which we can use for the phase 2 problem.

(c) Reintroduce the original objective and solve the phase 2 problem. Do you find an integer solution?

Omitting the artificial column, and reinstating the original objective the augmented matrix becomes

$$\begin{pmatrix} 1 & 0 & \frac{3}{7} & 0 & -\frac{1}{7} & \frac{19}{7} \\ 0 & 1 & \frac{1}{7} & 0 & \frac{2}{7} & \frac{18}{7} \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{8}{7} & \frac{5}{7} \\ 1 & -3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Bringing this to canonical form, by subtracting the first row and adding 3 times the second row to the last row, yields

$$\begin{pmatrix} 1 & 0 & \frac{3}{7} & 0 & -\frac{1}{7} & \frac{19}{7} \\ 0 & 1 & \frac{1}{7} & 0 & \frac{2}{7} & \frac{18}{7} \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{8}{7} & \frac{5}{7} \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}.$$

The simplex algorithm requires zero steps, and we can read of the solution $(19, 18)/7$, which is not integer valued.

6. [3+3+3+3=12 marks]

(a) Find the points on the parabola $y = 1 - x^2$ which are local extremisers with respect to the squared distance function $x^2 + y^2$.

The Lagrangian for this problem is

$$L = x^2 + y^2 + \lambda(y - 1 + x^2)$$

For (x, y) to be an extremiser, we need DL to vanish,

$$2x + 2\lambda x = 0, \quad 2y + \lambda = 0, \quad y - 1 + x^2 = 0.$$

The first equation gives $x = 0$, or $\lambda = -1$. When $x = 0$ we have $y = 1$ and hence $\lambda = -2$. When $\lambda = -1$ we have $y = \frac{1}{2}$ and hence $x = \pm\frac{1}{2}\sqrt{2}$.

(b) For each of the points you found in (a), determine the tangent space to the parabola.

The gradient of $y - 1 + x^2$ is $(2x, 1)$, and the tangent space is normal to the gradient. So at $\mathbf{p}_1 = (0, 1)$ the tangent space is $\{t(1, 0), t \in \mathbb{R}\}$, at $\mathbf{p}_2 = (-\frac{\sqrt{2}}{2}, \frac{1}{2})$ the tangent space is $\{t(1, \sqrt{2}), t \in \mathbb{R}\}$, and at $\mathbf{p}_3 = (\frac{\sqrt{2}}{2}, \frac{1}{2})$ the tangent space is $\{t(1, -\sqrt{2}), t \in \mathbb{R}\}$.

(c) Using the Hessian of the Lagrangian, classify the points you found in (a).

The Hessian of the Lagrangian is the matrix

$$\begin{pmatrix} 2(\lambda+1) & 0 \\ 0 & 2 \end{pmatrix}.$$

At $\mathbf{p}_1$ we have

$$\begin{pmatrix} t & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} = -2t^2,$$

so the Hessian is negative definite on the tangent space, and hence $\mathbf{p}_1$ is a local maximum. At $\mathbf{p}_2$ ($\mathbf{p}_3$) we have

$$\begin{pmatrix} t & (-)\sqrt{2}t \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} t \\ (-1)\sqrt{2}t \end{pmatrix} = 2t^2,$$

so the Hessian is positive definite on the tangent space, and hence $\mathbf{p}_2$ and $\mathbf{p}_3$ are local minima.

(d) Solve the nonlinear optimisation problem with inequality constraint

$$\begin{array}{rl} \text{extremise} & x^2 + y^2 \\ \text{subject to} & y \leq 1 - x^2. \end{array}$$

The Lagrange multiplier now becomes a KKT-multiplier, which needs to be positive for minimiser, and negative for maximisers. Thus $\mathbf{p}_1$ is a local maximiser, but $\mathbf{p}_2$ and $\mathbf{p}_2$ are not minimisers. This leaves us with the case $\lambda = 0$. Here we obtain $2x = 2y = 0$ and we find $(x, y) = (0, 0)$ to be a global minimum.