THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 1 Assessment, 2019

School of Mathematics and Statistics

MAST30013 Techniques in Operations Research

Writing time: 2 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- No handwritten or print materials may be brought into the exam venue.
- Approved calculators are allowed.
- Students will be provided with a 6-page formula sheet.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions.
- There are 6 questions with marks as shown. The total number of marks available is 80.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

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Question 1 (15 marks)

Consider the function

$$f(x) = e^x + x \log x.$$

where $x > 0$ and $\log x$ denotes the natural logarithm.

(a) Show that f is continuous and unimodal for $x > 0$.

(b) With a tolerance of $\epsilon = 0.005$ and an initial interval of $[0.1, 0.2]$, what is the least number of f-calculations used when applying the Golden Section search to minimise f? What is the length of the final interval?

(c) Perform one iteration of the Fibonacci search with the same tolerance and initial interval as in (b). Write down the interval in which the minimum is contained after one iteration.

(d) Can the method of false position be used to find the minimum of f with initial interval $[0.1, 0.2]$? If so perform one iteration and write down the interval. Otherwise, explain why not.

(e) Is it possible to apply Newton's method in finding the minimum of f on the interval $[0.1, 0.2]$, starting from $x_0 = 0.2$? If so, perform one iteration. Otherwise, explain why not.

Question 2 (10 marks)

Consider the function f given by

$$f(x_1, x_2, x_3) = x_1^3 + x_2^2 + x_3^2 + 6x_1x_2 - 8x_2 - 8x_3.$$

(a) Using the first-order necessary condition, find all stationary points of f.

(b) Using the second-order sufficiency condition, determine whether the stationary points are local minimums, local maximums, or saddles points. You may use the fact that the roots of the equation $r^3 - 28r^2 + 100r - 24 = 0$ are $r \in \{0.47, 2.0, 25.53\}$ and the roots of the equation $r^3 - 16r^2 + 52r + 24 = 0$ are $r \in \{-0.81, 2.0, 14.81\}$.

(c) If you found a local minimum in (b), is it also a global minimum? Justify your answer.

Question 3 (15 marks)

Consider the unconstrained nonlinear program

$$\min f(x_1, x_2) = \frac{4}{3}x_1^3 - x_1 x_2^2 + 3x_2^2 - 8x_2.$$

(a) Perform one iteration of the steepest descent method starting at the point $x^0 = (0, 2)^\top$ to find x^1.

(b) Perform one iteration of Newton's method starting at the point $x^0 = (0, 2)^\top$. Leave your answer, namely x^1, in terms of the step-length t.

(c) Find the BFGS direction for f at the point $x^0 = (0, 2)^\top$ with $H_0 = 2I_2$, where I_2 is the 2×2 identity matrix.

(d) Prove that, in the steepest descent method, we always have d^k normal to d^{k+1}, where d^k is the descent direction at the kth step, $k \geq 0$.

Question 4 (15 marks)

Consider the constrained nonlinear program

$$\min f(x_1, x_2, x_3) = 2x_1^3 + 3x_2^2 + 3x_3^2$$

$$\text{s.t.} \qquad x_1 + x_2 + x_3 = 0$$

$$x_1 \geq 3$$

(a) Write down the Lagrangian of the program.

(b) Find the KKT point(s).

(c) Find the critical cone(s) of the KKT point(s).

(d) Check the second-order sufficient condition at the KKT point(s) and determine if the KKT point(s) is/are locally minimum.

(e) If the equality constraint in the program is changed to $x_1 + x_2 + x_3 = 1$, approximately how much would the function value change compared to the minimum found in (d).

Question 5 (10 marks)

Consider the constrained nonlinear program

$$\min f(x_1, x_2) = x_1^4 - 2x_1^2 + x_2^2$$
$$\text{s.t.} \quad x_1 \geq 0$$
$$x_2 \geq 3$$

(a) Write down the l_2 penalty function $P_k(x_1, x_2)$ with penalty parameter $\alpha_k = k$.

(b) Calculate $\nabla P_k(x_1, x_2)$ and show that the stationary points for P_k only occur when $x_1 \geq 0$ and $x_2 < 3$.

(c) Solve $\nabla P_k(x_1^k, x_2^k) = 0$ to find the stationary point(s) $x^k = (x_1^k, x_2^k)$ of P_k.

(d) Solve the program by finding the limit $x^* = \lim_{k\to\infty}(x_1^k, x_2^k)$.

Question 6 (15 marks)

Consider the constrained nonlinear program

$$\min f(x_1, x_2) = x_1^2 - 2x_1x_2 + 2x_2^2 - 14x_2$$
$$\text{s.t.} \quad x_1^2 + x_2^2 \leq 5$$
$$x_1 - x_2 \geq 1$$

(a) Show that the above nonlinear program is a convex program.

(b) Write down the Lagrangian for the nonlinear program.

(c) The KKT point for the program is $(x^*, \lambda^*) = ((2, 1), (2, 10))$. Verify that the KKT point satisfies the Lagrangian Saddle Inequalities.

(d) Explain why (x^*, λ^*) is the only KKT point.

(e) Write down the Wolfe dual of this program.

End of Exam—Total Available Marks = 80