MAST30027: Modern Applied Statistics

Assignment 3 Solution 2021

1. (a) **Solution:** The posterior distribution of the precision τ is

$$
\begin{aligned}
f(\tau|x_1,\ldots,x_{100}) &\propto \prod_{i=1}^{100} P(x_i|\tau)f(\tau) \\
&\propto \tau^{100/2}\exp\left\{-\frac{\tau}{2}\sum_{i=1}^{100}(x_i-75)^2\right\}\tau^{2-1}\exp(-\tau) \\
&\propto \tau^{52-1}\exp\left\{-\left[\frac{1}{2}\sum_{i=1}^{100}(x_i-75)^2+1\right]\tau\right\}
\end{aligned}
$$

This is a kernel for Gamma(52, $\frac{1}{2}\sum_{i=1}^{100}(x_i-75)^2+1$).
Let's compute the rate parameter from the data.

```
> x = scan("assignment3_prob1.txt", what=double(0))
> rate = sum((x-75)^2)/2 + 1
> rate
[1] 1805.65
```

Thus, $\tau|x_1,\ldots,x_{100} \sim \text{Gamma}(52, 1805.65)$.

(b) **Solution:** Let $\alpha^* = 52$ and $\beta^* = 1805.65$.

$$
\begin{aligned}
p(\tilde{x}|x_1,...,x_{100}) &\propto \int p(\tilde{x}|\tau)p(\tau|x_1,...,x_{100})d\tau \\
&\propto \int \tau^{\frac{1}{2}}\exp\left\{-\frac{\tau}{2}(\tilde{x}-75)^2\right\}\tau^{\alpha^*-1}\exp(-\beta^*\tau)d\tau \\
&\propto \int \tau^{\alpha^*+\frac{1}{2}-1}\exp\left\{-\left[\beta^*+\frac{1}{2}(\tilde{x}-75)^2\right]\tau\right\}d\tau
\end{aligned}
$$

the integrand is proportional to the pdf of Gamma($\alpha^*+\dfrac{1}{2}$, $\beta^*+\dfrac{1}{2}(\tilde{x}-75)^2$)

$$
\begin{aligned}
&\propto \frac{1}{\left(\beta^*+\frac{1}{2}(\tilde{x}-75)^2\right)^{\alpha^*+\frac{1}{2}}} \\
&\propto \left[1+\frac{(\tilde{x}-75)^2}{2\beta^*}\right]^{-(\alpha^*+\frac{1}{2})} \\
&\propto \left[1+\frac{(\tilde{x}-75)^2}{2\beta^*}\right]^{-\frac{2\alpha^*+1}{2}}.
\end{aligned}
$$

Therefore we conclude that

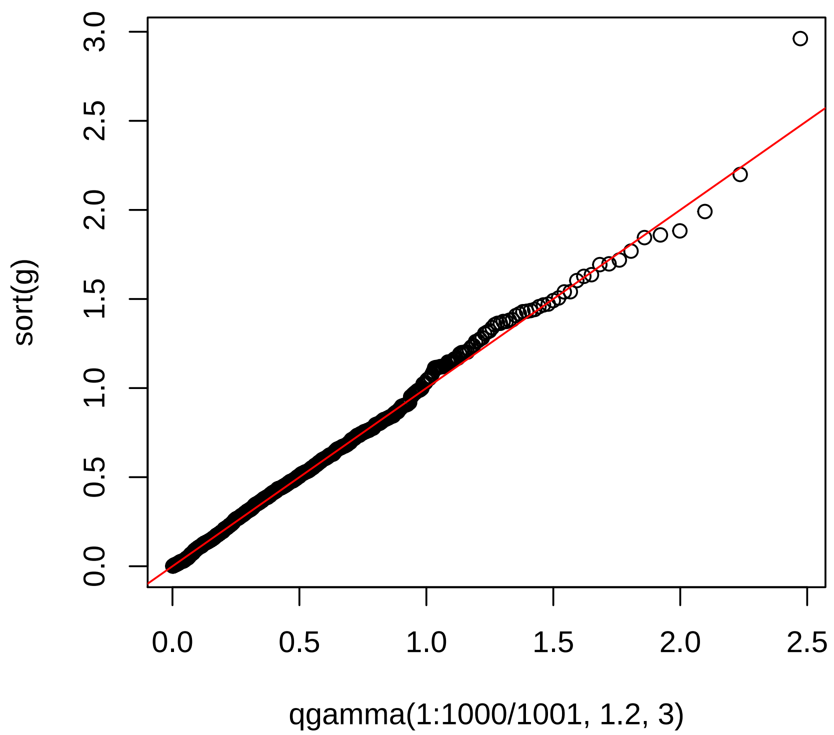
$$
\tilde{x}|x_1,...,x_{100} \sim t(\nu=2\alpha^*, a=75, b=\frac{\beta^*}{\alpha^*}) = t(\nu=104, a=75, b=34.72404).
$$

2. **Solution:** Using the fact (3), we will simulate X from $h(x)^{\alpha-1}e^{-h(x)}h'(x)/\Gamma(\alpha)$ using the modified version of the general rejection method. We will use $f(x)^* = \exp(g(x))$ and use $h(x)^* = \exp(-x^2/2)$ as an envelope. $h(x)^*$ is a kernel of N(0, 1), so we know how to simulate samples from the envelope. Then, from the results of (1) and (2), $\frac{h(X)}{\lambda}$ will follow gamma(α, λ).

Note that the range of X for the distribution $h(x)^{\alpha-1}e^{-h(x)}h'(x)/\Gamma(\alpha)$ is $[-1/c, \infty)$. Also, note that

$$
U < \frac{e^{g(x)}}{e^{-x^2/2}} \iff \log U < g(x) + x^2/2.
$$

```
> my_rgamma <- function(n, al, la) {
+   # simulate n gammas with shape al (>= 1) and rate la
+   # using the algorithm of Marsaglia & Tsang, 2000
+   if (al < 1 || la < 0) stop("invalid parameters")
+   d <- al - 1/3
+   c <- 1/sqrt(9*d)
+   y <- rep(NA, n)
+   for (i in 1:n) {
+     repeat {
+       repeat {
+         x <- rnorm(1)
+         if (c*x > -1) break
+       }
+       z <- (1 + c*x)^3
+       if (log(runif(1)) < d*log(z) - d*z + d + x^2/2) break
+     }
+     y[i] <- d*z
+   }
+   return(y/la)
+ }
> g <- my_rgamma(1000, 1.2, 3)
> plot(qgamma(1:1000/1001, 1.2, 3), sort(g))
> abline(0, 1, col="red")
```



3. **NA.**