School of Mathematics and Statistics
MAST30030
Applied Mathematical Modelling

Problem Sheet 7.
Navier-Stokes equations

Question 1

Consider the flow

$$\mathbf{u} = \frac{\Omega a^2}{\sigma}\mathbf{e}_\phi, \sigma \geq a$$

outside a rotating cylinder. What is wrong with the following argument?

'The flow is irrotational so the viscous forces vanish so there is no torque on the cylinder'.

Question 2

Show that the normal viscous stress at a stationary solid boundary must vanish for an incompressible Newtonian fluid. You may assume the boundary is flat but this is unnecessary.

Hint: use the continuity equation.

Question 3

For unidirectional flow down an annular pipe with inner surface $\sigma = a$ and outer surface $\sigma = 2a$, find the steady flow profile $w(\sigma)$ caused by an axial pressure gradient $\frac{dp}{dz} = -G$ and determine the stress vector acting at both inner and outer surfaces. Also find the volume flow rate through the pipe.

Question 4

Suppose viscous fluid occupies the region $\sigma \leq a$ within a circular cylinder of radius a and suppose that the cylinder and fluid are rotating with uniform angular velocity Ω so that

$$u_\phi = \Omega\sigma, \sigma \leq a, t = 0$$

The cylinder is suddenly brought to rest. Use separation of variables to show that the motion is given by

$$u_\phi(\sigma, t) = -2\Omega a \sum_{n=1}^{\infty} \frac{J_1(\lambda_n \sigma/a)}{\lambda_n J_0(\lambda_n)} \exp(-\lambda_n^2 \nu t/a^2)$$

where λ_n is the n^{th} positive real root of $J_1(x)$. Given that the first term gives the dominant rate of decay, estimate the 'spin-down' time of a cup of tea/coffee and compare with your observations. Interpret in terms of vorticity difusion.

Hint: you only need to consider the azimuthal component of the Navier-Stokes equation in cylindrical coordinates.

Question 5

A cylindrical rod of radius a is being pulled upward with velocity U after it has been coated with a liquid film of thickness h. The coated liquid is draining downwards due to gravity. Show that the velocity profile across the film is

$$u_z = U + \frac{g}{4\nu}(\sigma^2 - a^2 - 2(a+h)^2 \ln(\sigma/a))$$

You need to start from the Navier-Stokes equations in cylindrical cordinates.

Question 6

i. A liquid is in the annular space between two vertical cylinders of radii $\kappa R, R$, and the liquid is open to the atmosphere at the top. Show that when the inner cylinder rotates with angular velocity Ω_i and the outer cylinder is fixed, the free liquid surface has the shape

$$z_R - z = \frac{1}{2g}(\frac{\kappa^2 R\Omega_i}{1-\kappa^2})^2(\xi^{-2} + 4\ln\xi - \xi^2)$$

where z_R is the height of the liquid at the outer cylinder wall and $\xi = \sigma/R$.

ii. Repeat part i but with the inner cylinder fixed and the outer cylinder rotating with angular velocity Ω_o. Show that the shape of the free liquid surface is

$$z_R - z = \frac{1}{2g}(\frac{\kappa^2 R\Omega_0}{1-\kappa^2})^2[(\xi^{-2} - 1) + 4\kappa^{-2}\ln\xi - \kappa^{-4}(\xi^2 - 1)]$$

Question 7

Find the velocity field (there is only an azimuthal component) for fluid confined between two concentric *spheres* of radius $\kappa R, R$ where $\kappa < 1$, rotating with angular velocities Ω_i, Ω_o respectively.

Question 8

(a) Show that, for a sphere of radius a rotating with angular velocity $\Omega\hat{\mathbf{k}}$, the points on the surface have velocity $\mathbf{U} = \Omega a \sin\theta \mathbf{e}_\phi$, in spherical polar coordinates.

(b) 'Guided by the form of the boundary condition', consider a velocity field of the form

$$\mathbf{u} = f(r)\sin\theta\mathbf{e}_\phi$$

Show that such a flow satisfies the continuity equation for incompressible flow and express $\nabla \times \nabla \times \boldsymbol{\omega}$ in terms of f (and its derivatives).

(c) Hence show that the velocity field outside a sphere of radius a rotating with angular velocity $\Omega\hat{\mathbf{k}}$ in a quiescent fluid under creeping flow conditions is

$$\mathbf{u} = \frac{\Omega a^3}{r^2}\sin\theta\mathbf{e}_\phi$$

(d) Hence show that the torque acting on the sphere has magnitude $8\pi\mu\Omega a^3$.