Joint Distribution of BM.

$E[B_t]=0$ . . $B_t \sim N(0,t)$. & independent increment on disjoint intervals

$Cov(B_t, B_s) = E[B_t B_s] = E[(B_t - B_s) B_s] + E[B_s^2]$

$t>s$. $= E[(B_t - B_s)]\ E[B_s] + E[B_s^2] = Var(B_s) = s$

Slide 15: ep. Standard model.

price of stock t hours $S_t = e^{\sigma B_t + \mu t}$.

here we simple ex: $\mu t = 0$. $S_t = e^{\sigma B_t}$ · $S_0 = 1$

(if initial price k. $S_t = k \cdot e^{\sigma B_t + \mu t}$)

① $P(S_8 > 1 \mid S_4 = e^{4\sigma}) = P(e^{B_8 \sigma} > 1 \mid e^{B_4 \sigma} = e^{4\sigma})$

$= P(B_8 > 0 \mid B_4 = 4)$

$= P(B_8 - B_4 > -4) = P(2Z > -4) = P(Z > -2)$. $Z \sim N(0,1)$

$B_8 - B_4 \to N(0,4)$

q: stock $e^{4\sigma}$ halfway through 8h ⇒ a working day, what's the chance the stock will be worth more than its initial price at the end of the day?

② If at the end of day worth $e^{4\sigma}$.
what's the chance $ stock at the middle of day > original price

$P(S_4 > 1 \mid S_8 = e^{4\sigma})$

$= P(B_4 > 0 \mid B_8 = 4)$

$Cov(B_t, B_s) = \min\{t, s\}$

(B_4, B_8) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 4 & 4 \\ 4 & 8 \end{pmatrix}$

$\frac{B_8}{\sqrt{8}} = Z_1 = \frac{4}{\sqrt{8}}$

condition distr $X_2 \mid X_1 = x$. given $B_8 = 4$.

$\frac{B_4}{\sqrt{4}} = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2 = \frac{1}{\sqrt{2}} \sqrt{2} + \sqrt{1 - \frac{1}{2}}\, Z_2$

$P(B_4 > 0 \mid B_8 = 4) = P(1 + \frac{\sqrt{2}}{2} Z_2 > 0) = P(Z_2 > -\sqrt{2})$

Density of Z: $\dfrac{\exp\{(z_1^2+z_2^2+\dots+z_k^2)/2\}}{(2\pi)^{\frac{k}{2}}}$

$X = \Sigma^{\frac{1}{2}} \cdot Z + \mu$.

$X = RZ + \mu$

$\det(\Sigma^{\frac{1}{2}}) = \sqrt{\det(\Sigma)} > 0$. $\det(R) = \sqrt{\det(\Sigma)} = \det(\Sigma^{\frac{1}{2}})$

$W = aV + b \quad a>0$.

V has density $g_V(V)$.

$V = \frac{W-b}{a} \Rightarrow$ the density of W $g_W(w) = f_V(\frac{w-b}{a}) \cdot \frac{1}{a}$

$P(W \le w) = P(aV+b \le w) = \int_{av+b \le w} g_V(s)\, ds$

$s = \frac{u-b}{a}$. $ds = \frac{1}{a} du$.

$\text{Pr} = \int_{u \le w} g_V(\frac{u-b}{a}) \cdot \frac{1}{a} du = \int_{-\infty}^{w} \underbrace{g_V(\frac{u-b}{a}) \frac{1}{a}}_{\text{density of } W} du$.

$X = AZ + \mu$. A inversible.

Z has density $f_Z(z)$.

Density of $\tilde{X}$?

$Z = A^{-1}(X-\mu)$

$f_X(x) = f_Z(A^{-1}(x-\mu)) \cdot \det(A^{-1}) = \dfrac{f_Z(A^{-1}(x-\mu))}{\det(A)}$

$f_Z(z) = \dfrac{\exp(-z^T z/2)}{(2\pi)^{\frac{k}{2}}}$

$A = \Sigma^{\frac{1}{2}}$: $f_X(x) = \exp\{-(\Sigma^{-\frac{1}{2}}(x-\mu))^T \cdot (\Sigma^{-\frac{1}{2}}(x-\mu))/2\} / (2\pi)^{\frac{k}{2}} \sqrt{\det}$

$A = R$. $f_X(x) = \dfrac{\exp\{-(\Sigma^{-\frac{1}{2}}(x-\mu))^T \cdot (\Sigma^{-\frac{1}{2}}(x-\mu))/2\}}{(2\pi)^{\frac{k}{2}} \cdot \det(\Sigma^{\frac{1}{2}})} \Rightarrow \det R$.

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