Joint Distribution of Brownian motion.

let $Z_i := B_{t_i} - B_{t_{i-1}}$. $\Rightarrow$ Z_i are independent normal function variables.

$(B_{t_1}, \ldots, B_{t_k}) = (Z_1, \sum_{i=1}^{2} Z_i, \ldots, \sum_{i=1}^{k} Z_i)$.

Def

$X = (X_1, \ldots, X_k)$ has multivariate normal distribution with parameters

μ : a k-vector called mean

Σ : a $k \times k$ positive-definite matrix $\longrightarrow$ variance / covariance matrix

if $X \overset{d}{=} \Sigma^{\frac{1}{2}} \cdot Z + \mu$.

$$X_i = \sum_{A} (\Sigma^{\frac{1}{2}})_{iA} z_A + \mu_i$$

constance, idenpent of every

$$\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}\left(\sum_{A} (\Sigma^{\frac{1}{2}})_{iA} z_A + \mu_i, \sum_{l} (\Sigma^{\frac{1}{2}})_{jl} z_l + \mu_j\right).$$

$$= \sum_{n,l=1}^{k} \underbrace{\mathrm{Cov}(Z_n, Z_l)}_{\substack{=0 \text{ if } n \neq l \\ =1 \quad n = l}} \cdot (\Sigma^{\frac{1}{2}})_{in} (\Sigma^{\frac{1}{2}})_{jl}$$

$$= \sum_{l=1}^{k} (\Sigma^{\frac{1}{2}})_{il} (\Sigma^{\frac{1}{2}})_{jl} \quad \text{\& } \Sigma^{\frac{1}{2}} \text{ is symmetric}$$

$$= \sum_{l=1}^{k} (\Sigma^{\frac{1}{2}})_{il} \cdot (\Sigma^{\frac{1}{2}})_{lj}$$

$$= (\Sigma^{\frac{1}{2}} \cdot \Sigma^{\frac{1}{2}})_{ij} \text{ entry}.$$

$$= \Sigma_{ij}.$$

$\Rightarrow$ (Σ is called covariance matrix)

if lower triangular matrix R $\exists$ such that $\Sigma = RR^T$

$\Rightarrow$ then $X \overset{d}{=} RZ + \mu$.

Brownian motion.

X_i. iid mean: 0. $\sigma^2=1$. $\Rightarrow S_n := \sum_{i=1}^{n} X_i$

we've $\lim_{n\to\infty} P(\frac{S_n}{\sqrt{n}} \le z) = \varphi z$.

More. If $t \ge 0$.

$\frac{S_{\lfloor nt \rfloor}}{\sqrt{nt}} \to N(0.1)$.

$\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} = \frac{S_{\lfloor nt \rfloor}}{\sqrt{nt}} \sqrt{t} \to \underbrace{Z}_{N(0.1)} \sqrt{t} \to N(0.t)$.

independent

$(B_{t+s} - B_t) + B_t = B_{t+s} \sim N(0, t+s)$

$B_t \to N(0.t)$.

Moment generating function $\quad Z \sim N(0,1)$

$M_t(\theta) = E[e^{\theta B_t}] = E[e^{\theta \sqrt{t} Z}]$

$= M_Z(\theta\sqrt{t})$

$M_Z(\theta\sqrt{t}) = e^{\frac{\theta^2 t}{2}}$.

$M_{B_{t+s}-B_t}(\theta) \ast M_t(\theta) = M_{t+s}(\theta)$.

$M_{B_{t+s}-B_t}(\theta) = \frac{M_{t+s}(\theta)}{M_t(\theta)} = \frac{e^{\frac{\theta^2(t+s)}{2}}}{\cancel{e^{\frac{\theta^2 t}{2}}}} = e^{\theta^2 \frac{s}{2}} \sim M_s(\theta)$

$\Rightarrow B_{t+s} - B_t \sim N(0.s)$.

$\Rightarrow$ for fixed h. $B_t^* = B_{t+h} - B_h \Rightarrow B_t^*$ is a standard Brownian motion

1) continuous. (as B_{t+h} $_{t\ge0}$ is continuous).

2) independent increment $(B_{t_1}^* - B_{s_1}^*, B_{t_2}^* - B_{s_2}^*)$

$= (B_{t_1+h} - B_{s_1+h}, B_{t_2+h} - B_{s_2+h})$

independent increments

3) $B_t^* = B_{t+h} - B_h \sim N(0.t)$.