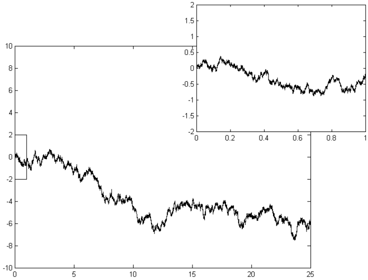
Complementary mathematical topics

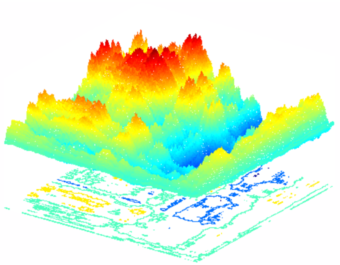
These topics will not be tested in STM5001 assignments. They are given for students interested in the mathematical background and justifications of models considered in this and previous weeks lectures.

Wiener process.

The Wiener process is a stochastic process named in honor of Norbert Wiener. It is also known as Brownian motion. Brownian motion is named after the Scottish botanist Robert Brown who first described the random motions of pollen grains suspended in water. In 1905 Einstein, unaware of the existence of earlier investigations about Brownian motion, obtained a mathematical derivation of this process from the laws of physics.
It has numerous applications in statistics, applied mathematics, economics, quantitative finance, evolutionary biology, physics, etc. Some examples of Brownian motion are pollen in water, smoke in a room, pollution in a river. Brownian motion often used in finance as a model for path of an asset price over time.



10
8
6
4
2
0
-2
-4
-6
-8
-10
0
5
10
15
20
25
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.2
0.4
0.6
0.8
1



The Wiener process is a continuous-time stochastic process which is characterized by the properties:

- $W(0) = 0$;
- $W(t)$ is Gaussian, $t \in [0, 1]$;
- if $t_1 < t_2 < \ldots < t_n$, then $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent;
- $E(W(t)) = 0$;
- $E\left(W(t) - W(s)\right)^2 = t - s, t \geq s$.

The covariance function of the Wiener process is

$$\begin{aligned} C(t,s) &= E(W(t)W(s)) = E[W(t) - W(s) + W(s)]W(s) = E\left[W(t) - W(s)\right]W(s) \\ &+ E(W^2(s)) = E\left[W(t) - W(s)\right]E(W(s)) + E(W^2(s)) = s, \quad t \geq s. \end{aligned}$$

Therefore

$$\boxed{C(t,s) = \min(t,s).}$$

Karhunen-Loève expansion of the Wiener process.

Now we derive the Karhunen-Loève expansion of the Wiener process:

$$\lambda\psi(t) = \int_0^1 C(t,s)\psi(s)ds,$$

$$\lambda\psi(t) = \int_0^1 \min(t,s)\psi(s)ds = \int_0^t s\psi(s)ds + t\int_t^1 \psi(s)ds.$$

Therefore $\psi(0) = 0$ if $t = 0$.
By the differentiation one obtains

$$\lambda\psi'(t) = t\psi(t) + \int_t^1 \psi(s)ds - t\psi(t) = \int_t^1 \psi(s)ds.$$

Thus, $\psi'(1) = 0$ if $t = 1$.

Taking the second derivative one obtains

$$\lambda\psi''(t) = -\psi(t) \quad \Rightarrow \qquad \psi(t) = C_1 \cos\left(\frac{t}{\sqrt{\lambda}}\right) + C_2 \sin\left(\frac{t}{\sqrt{\lambda}}\right).$$

From the initial conditions it follows that

$$\begin{aligned} \psi(0) &= 0 \Rightarrow C_1 = 0, \quad \psi(t) = C_2 \sin\left(\frac{t}{\sqrt{\lambda}}\right); \\ \psi'(1) &= 0 \Rightarrow \psi'(1) = \frac{C_2}{\sqrt{\lambda}} \cos\left(\frac{1}{\sqrt{\lambda}}\right) = 0 \Rightarrow \frac{1}{\sqrt{\lambda_n}} = \frac{\pi}{2} + n\pi, n = 0, 1, \ldots \end{aligned}$$

Hence,

$$\begin{aligned} \lambda_n &= \frac{1}{\left(\frac{\pi}{2} + n\pi\right)^2}, \\ \psi_n(t) &= C_2 \sin\left(\left(\frac{\pi}{2} + n\pi\right) t\right), \; n = 0, 1, \ldots \end{aligned}$$

Finality we find C_2 as

$$
\begin{aligned}
1 &= \int_0^1 \psi_n^2(t)dt = C_2^2 \int_0^1 \sin^2\left(\left(\frac{\pi}{2}+n\pi\right)t\right)dt \\
&= C_2^2 \int_0^1 \frac{1-\cos((\pi+2n\pi)t)}{2}dt = \frac{C_2^2}{2} \\
\Rightarrow\quad & C_2 = \sqrt{2}.
\end{aligned}
$$

Hence $\psi_n(t) = \sqrt{2}\sin\left(\left(\frac{\pi}{2}+n\pi\right)t\right)$.

Therefore, the Karhunen-Loève expansion of the Wiener process is

$$
\boxed{W(t) = \sqrt{2}\sum_{n=0}^{\infty} \frac{\sin\left(\left(\frac{\pi}{2}+n\pi\right)t\right)}{\frac{\pi}{2}+n\pi}\xi_n,}
$$

where $\xi_n \sim N(0,1)$.