MAST30001 Stochastic Modelling

Tutorial Sheet 5

1. A Markov chain has transition matrix

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

   (a) Analyze the state space of the chain (reducibility, periodicity, null/pos recurrence, etc).
   (b) Find the stationary distribution of the chain.
   (c) If the initial state of the chain is uniformly distributed, find $\lim_{n\to\infty} P(X_n = 2)$.

Ans.

(a) The chain is irreducible and finite so it's positive recurrent and since $p^{(2)}_{2,2} = p^{(3)}_{2,2} = 1/2$ state 2 is aperiodic and since period is a class property the chain is aperiodic.

(b) Solve $\pi P = \pi$ for P the transition matrix to find

$$\pi = [2/5, 2/5, 1/5].$$

(c) Since the chain is aperiodic, irreducible and positive recurrent it is ergodic and

$$\lim_{n\to\infty} P(X_n = 2) = 2/5.$$

2. A Markov chain has transition matrix

$$\begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix}$$

Analyze the state space (reducibility, periodicity, null/pos recurrence, etc), and discuss the chain's long run behavior.

Ans. We first determine the communicating classes. Let P be the transition matrix above with (i,j) entry p_{ij}. Since $p_{13}, p_{31} > 0$, states 1 and 3 are in the same communicating class and moreover from state 1 the chain can only stay put or go to state 3 and from state 3 the chain can only stay put or go to state 1. So $\{1,3\}$ is a communicating class which is essential since the chain cannot leave it. The same arguments apply for $\{2,4\}$ which is also an essential communicating class. State 5 must be in its own communicating class (since the communicating classes partition the state space), and since from state 5 the chain can move to the essential communicating class $\{1,3\}$, $\{5\}$ is an inessential class. So the chain is reducible since it has more than one communicating class.

Since all states have a "loop", they are all aperiodic and hence each communicating class is aperiodic.

Since the chain has a finite state space, all states in an essential communicating class (i.e., states $1-4$) are positive recurrent and state 5, being in an inessential communicating class is transient (started from state 5, the probability of not returning to state 5 eventually is 1).

To study the long run behaviour of the chain, we note first that if the chain starts in state 5, then it will eventually end up in the communicating class $\{1,3\}$. So we only need to study the long run behaviour of the essential communicating classes $\{1,3\}$ and $\{2,4\}$. Since the chain restricted to these classes is aperiodic, irreducible, and positive recurrent, it is ergodic. So if Q is the transition matrix of one of these restricted chains, then $\lim_{n\to\infty} Q^n_{ij} = \pi_j$ for a probability distribution π and this distribution must be the unique stationary distribution of the chain restricted to the communicating class. [π_j is also the limiting proportion of time the chain spends in state j.]

The chain restricted to the class $\{1,3\}$ has transition matrix

$$Q = \begin{pmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{pmatrix},$$

so the long run distribution $\pi^{(1)}$ satisfies $\pi Q = \pi$ and $\pi_1 + \pi_2 = 1$ and solving these equations for π we find

$$\pi = (9/17, 8/17).$$

A similar argument for the class $\{2,4\}$ yields that its long run distribution is

$$\rho = (3/5, 2/5).$$

To summarize, if the chain starts in the states any of the states $1,3$ or 5 then it eventually ends up in the communicating class $\{1,3\}$ and spends about $9/17$ of the time in state 1 and the rest in state 3. If the chain starts in states 2 or 4, then the chain in the long run spends about $3/5$ of the time in state 2 and the rest in state 4.

3. For $i = 0, \ldots, N$ let $0 < p_i = 1 - q_i < 1$ and let the Markov chain (X_n) with state space $\{0, \ldots, N\}$ have transition matrix

$$\begin{pmatrix} q_0 & p_0 & 0 & 0 & 0 & \cdots & 0 \\ q_1 & 0 & p_1 & 0 & 0 & \cdots & 0 \\ 0 & q_2 & 0 & p_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & q_{N-2} & 0 & p_{N-2} & 0 \\ 0 & \cdots & 0 & 0 & q_{N-1} & 0 & p_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & q_N & p_N \end{pmatrix}.$$

(a) Analyze the state space of the chain (reducibility, periodicity, null/pos recurrence, etc).

(b) If the chain starts at state 0, describe the long run behavior of the chain.

(c) If the chain starts at a uniformly chosen state, describe the long run behavior of the chain.

(d) Given the chain is in state 2 what is the expected number of steps until the chain returns to state 2?

Ans.

(a) The chain is irreducible since all states can communicate with each other (since $p_i \neq 0, 1$). States 0 and N have a "loop" and so are aperiodic. Since periodicity is a class property, all states are aperiodic; the chain is aperiodic. The chain is positive recurrent because it is an irreducible Markov chain on a finite state space.

(b) Since the chain is irreducible, aperiodic, and positive recurrent, it is ergodic. So if P is the transition matrix above, then $\lim_{n\to\infty} P_{ij}^n = \pi_j$ for a probability distribution π and this distribution must be the unique stationary distribution of the chain restricted to the communicating class. [π_j is also the limiting proportion of time the chain spends in state j.]

So π solves $\pi P = \pi$ which implies that for $1 \leq i \leq N-1$,

$$\pi_i = p_{i-1}\pi_{i-1} + q_{i+1}\pi_{i+1},$$

and

$$\begin{aligned}\pi_0 &= q_0\pi_0 + q_1\pi_1,\\ \pi_N &= p_{N-1}\pi_{N-1} + \pi_N p_N.\end{aligned}$$

Also the entries of π must sum to one in order to be a probability distribution. We know from arguments above that these equations have a unique solution π so small cases lead us to guess the formula for $i = 1, \ldots, N$,

$$\pi_i = \pi_0 \prod_{j=1}^{i} \frac{p_{j-1}}{q_j} =: \pi_0 A_i,$$

and

$$\pi_0 = \left(1 + \sum_{i=1}^{N} A_i\right)^{-1}.$$

And at this point it only must be checked that this formula satisfies the equations above (check this!).

(c) Because the chain is ergodic, the initial distribution doesn't affect the long run behaviour of the chain. So the answer from (b) holds here.

(d) From lecture we know this expectation is equal to π_2^{-1} with π_2 as in (b).

4. A possum runs from corner to corner along the top of a square fence. Each time he switches corners, he chooses among the two adjacent corners, choosing the corner in the clockwise direction with probability $0 < p < 1$ and the corner in the counter-clockwise direction with probability $1-p$. Model the possum's movement among the corners of the fence as a Markov chain, analyze its state space (reducibility, periodicity, recurrence, etc), and discuss its long run behavior.

Ans. Numbering the fence posts $1, 2, 3, 4$ in a circular way we can model the possum's movements as a Markov chain with transition matrix

$$\begin{pmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{pmatrix}.$$

The chain is irreducible since all states communicate and it's positive recurrent since the state space is finite. Its period is two and viewing the chain at times a multiple of two induces ergodic chains on $\{1,3\}$ and $\{2,4\}$ in the usual way. These chains each have long run stationary distribution the uniform distribution (since columns sum to one).

5. A machine produces two items per day. The probability that an item is *not* defective is p, with all items produced independently, and defective items are thrown away immediately. The demand for items is one per day, and any demand not met by the end of the day is lost, while any extra item is stored. Let X_n be the number of items in storage just before the beginning of the nth day.

   (a) Model X_n as a Markov chain, draw its transition diagram and compute its transition probabilities.

   (b) When is the Markov chain ergodic? Compute the limiting distribution when it exists.

   (c) Suppose it costs \$$c$ to store an item for one night and \$$d$ for every demanded item that cannot be supplied. Compute the long run cost of the production facility when the chain is ergodic.

   Ans.

   (a) The number of items in storage increases by 1 if two non-defective items are produced, it stays constant if one non-defective item is produced and decreases by one if both items are defective and there are items in storage; these probabilities only depend on the current number of items in storage and so X_n is a Markov chain with transition probabilities for $j \geq 1$:

$$p_{j,j+1} = p^2, \;\; p_{j,j} = 2p(1-p) \;\; p_{j,j-1} = (1-p)^2,$$

   and $p_{0,1} = 1 - p_{0,0} = p^2$.

   (b) The chain is irreducible and aperiodic (loops) and so we only need to determine when

$$\pi P = \pi$$

   has a probability vector π solution. We need to solve the equations

$$\pi_0 = (1-p^2)\pi_0 + (1-p)^2\pi_1$$

   and for $j \geq 1$

$$\pi_j = (1-p)^2\pi_{j+1} + 2p(1-p)\pi_j + p^2\pi_{j-1}.$$

   Working up from $j = 0$ and guessing the formula, a check shows that the solution must satisfy

$$\pi_j = \left(\frac{p^2}{(1-p)^2}\right)^j \pi_0.$$

   And also $\sum_{j=0}^{\infty} \pi_j = 1$, so the chain is ergodic only if $p/(1-p) < 1$, that is, if $p < 1/2$, and in this case the limiting distribution is geometric:

$$\pi_j = \left(\frac{p^2}{(1-p)^2}\right)^j (1 - p^2/(1-p)^2).$$

(c) To incur a cost of \$d, we need there to be no items in storage and to produce two defective items the following day; the long run proportion of days this happens when the chain is ergodic equals $\pi_0(1-p)^2 = (1-2p)$. We also incur a per day cost of cX_n, which over the long run is $c\mathbb{E}[X_n] \to cp^2/(1-2p)$ and so the per day cost of the facility is

$$d(1-2p) + cp^2/(1-2p).$$