MAST20004 Probability

Tutorial Set 10

1. If $X \stackrel{d}{=} R(0, \frac{\pi}{2})$ and $Z = \sin X$, find $V(Z)$ and compare this with the approximate value calculated using $V(\psi(X)) \approx \psi'(\mu)^2 V(X)$.

   Solution:

$$f_X(x) = \begin{cases} \frac{2}{\pi} & x \in [0, \pi/2] \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \mathbb{E}[Z] &= \frac{2}{\pi} \int_0^{\pi/2} \sin x dx \\ &= \frac{2}{\pi} [-\cos x]_0^{\pi/2} \\ &= \frac{2}{\pi}. \end{aligned}$$

$$\begin{aligned} E[Z^2] &= \frac{2}{\pi} \int_0^{\pi/2} \sin^2 x dx \\ &= \frac{2}{\pi} \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{\pi/2} \\ &= \frac{1}{2}. \end{aligned}$$

   Therefore $V(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 = 1/2 - 4/\pi^2 = 0.0947$.
   We have $\mu = \pi/4$ and $V(X) = \pi^2/48 = 0.2056$. Therefore

$$\begin{aligned} V(\sin(X)) &\approx \frac{\psi'(\pi/4)^2 \pi^2}{48} \\ &= \frac{\cos^2(\pi/4)\pi^2}{48} \\ &= \frac{\pi^2}{96} \\ &= 0.1028 \end{aligned}$$

2. Let $N \geq 0$ be an integer-valued random variable with $\mathbb{E}[N] = a, V(N) = b^2$ and $X_1, X_2, \cdots$ be independent random variables, also independent of N, with $\mathbb{E}[X_j] = \mu$ and $V(X_j) = \sigma^2$. Using conditional expectations, compute $\text{Cov}(S_N, N)$, where $S_N = \sum_{j=1}^{N} X_j$.

   Solution:

$\text{Cov}(S_N, N) = \mathbb{E}[S_N N] - \mathbb{E}[S_N]\mathbb{E}[N]$. First $\mathbb{E}[S_N|N=n] = \mathbb{E}[S_n|N=n] = \mathbb{E}[S_n] = n\mu$, where the second last equality is due to the fact that S_n and N are independent. Therefore $E[S_N|N] = N\mu$. Now

$$\begin{aligned}\mathbb{E}[S_N] &= \mathbb{E}[\mathbb{E}[S_N|N]]\\ &= \mathbb{E}[N\mu]\\ &= a\mu.\end{aligned}$$

Also

$$\begin{aligned}\mathbb{E}[S_N N|N=n] &= \mathbb{E}[nS_n|N=n]\\ &= n\mathbb{E}[S_n|N=n]\\ &= n\mathbb{E}[S_n]\\ &= n^2\mu,\end{aligned}$$

where, again, the second last equality is from the independence between S_n and N. Therefore $\mathbb{E}[S_N N|N] = N^2\mu$. Now

$$\begin{aligned}\mathbb{E}[S_N N] &= \mathbb{E}[\mathbb{E}[S_N N|N]]\\ &= \mathbb{E}[N^2\mu]\\ &= \mu\left(V(N) + \mathbb{E}[N]^2\right)\\ &= \mu\left(b^2 + a^2\right).\end{aligned}$$

Therefore $\text{Cov}(S_N, N) = \mu\left(b^2 + a^2\right) - a^2\mu = \mu b^2$.

3. Let the random variable X have the probability generating function

$$P_X(z) = c + 0.1(1+z)^3 + 0.3z^5.$$

(a) Find the constant c.

(b) Give the distribution of X.

(c) Use the $P_X(z)$ to compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

(d) Compute the probability generating function of $Y = X + 2$.

Solution:

(a) $P_X(1) = 1$ implies that $c + 0.1(2)^3 + 0.3 = 1$ and so $c = -0.1$.

(b) $P_X(z) = 0.3z + 0.3z^2 + 0.1z^3 + 0.3z^5$, so

$$p_X(x) = \begin{cases} 0.3 & \text{if } x = 1\\ 0.3 & \text{if } x = 2\\ 0.1 & \text{if } x = 3\\ 0.3 & \text{if } x = 5\\ 0 & \text{otherwise}\end{cases}$$

(c) $P'_X(z) = 0.3(1+z)^2 + 1.5z^4$,
$P''_X(z) = 0.6(1+z) + 6z^3$,
and so $\mathbb{E}[X] = P'_X(1) = 2.7$ and $\mathbb{E}[X^2] = P''_X(1) + P'_X(1) = 9.9$.

(d)

$$\begin{aligned} P_Y(z) &= \mathbb{E}[z^Y] \\ &= \mathbb{E}[z^X + 2] \\ &= z^2\mathbb{E}[z^X] \\ &= z^2 P_X(z) \\ &= 0.3z^3 + 0.3z^4 + 0.1z^5 + 0.3z^7. \end{aligned}$$

4. Let $X \stackrel{d}{=} \mathrm{R}(0,1)$ and $Y \stackrel{d}{=} \mathrm{R}(1,3)$ be independent random variables.
   (a) Compute the moment generating function $M_X(t)$ of X.
   (b) Compute the moment generating function of Y.
   (c) Compute the moment generating function of $Z = X - 2Y + 2$.
   (d) Use the moment generating function $M_X(t)$ to verify that $\mathbb{E}[X] = 1/2$ and $V(X) = 1/12$.

Solution:

(a) For $t \neq 0$,

$$\begin{aligned} M_X(t) &= \int_0^1 e^{tx} dx \\ &= \left[\frac{e^{tx}}{t}\right]_0^1 \\ &= \frac{e^t - 1}{t}. \end{aligned}$$

(b) For $t \neq 0$,

$$\begin{aligned} M_Y(t) &= \int_1^3 \frac{e^{tx}}{2} dx \\ &= \left[\frac{e^{tx}}{2t}\right]_1^3 \\ &= \frac{e^{3t} - e^t}{2t}. \end{aligned}$$

(c) For $t \neq 0$,

$$\begin{aligned} M_Z(t) &= \mathbb{E}[e^{tZ}] \\ &= \mathbb{E}[e^{t(X-2Y+2)}] \\ &= e^{2t}\mathbb{E}[e^{tX}]\mathbb{E}[e^{-2tY}] \\ &= e^{2t} M_X(t) M_Y(-2t) \\ &= \frac{-e^{2t}(e^t - 1)(e^{-6t} - e^{-2t})}{4t^2}. \end{aligned}$$

(d) For $t \neq 0$,

$$\begin{aligned}
M'_X(t) &= \frac{te^t - e^t + 1}{t^2} \\
\mathbb{E}[X] &= \lim_{t\to 0} M'_X(t) \\
&= \lim_{t\to 0} \frac{e^t + te^t - e^t}{2t} \\
&= \frac{1}{2},
\end{aligned}$$

by L'Hopital's Rule.

$$\begin{aligned}
M''_X(t) &= \frac{t^3e^t - 2t^2e^t + 2te^t - 2t}{t^4} \\
\mathbb{E}[X^2] &= \lim_{t\to 0} M''_X(t) \\
&= \lim_{t\to 0} \frac{t^3e^t + t^2e^t - 2te^t + 2e^t - 2}{4t^3} \\
&= \lim_{t\to 0} \frac{t^3e^t + 4t^2e^t}{12t^2} \\
&= \frac{1}{3},
\end{aligned}$$

again by L'Hopital's Rule. So $V(X) = 1/3 - 1/4 = 1/12$.

5. Let $Y_\lambda \overset{d}{=} \mathrm{Pn}(\lambda)$.
   (a) Write down the moment generating function of Y_λ.
   (b) Compute the moment generating function of $Z_\lambda = (Y_\lambda - \lambda)/\sqrt{\lambda}$.
   (c) Using part (b) show that $Z_\lambda \xrightarrow{d} N(0,1)$ as $\lambda \to \infty$.

Solution:

(a) For any $t \in \mathbb{R}$,

$$M_{Y_\lambda}(t) = e^{-\lambda(1-e^t)}.$$

(b) For any $t \in \mathbb{R}$,

$$\begin{aligned}
M_{Z_\lambda}(t) &= \mathbb{E}\left[e^{(Y_\lambda-\lambda)/\sqrt{\lambda}}\right] \\
&= e^{-\lambda t} M_{Y_\lambda}\left(t/\sqrt{\lambda}\right) \\
&= e^{-\sqrt{\lambda}t} e^{-\lambda\left(1-e^{t/\sqrt{\lambda}}\right)}.
\end{aligned}$$

(c) For any $t \in \mathbb{R}$,

$$\begin{aligned}
\log(M_{Z_\lambda}(t)) &= -\sqrt{\lambda}t - \lambda\left(1 - e^{t/\sqrt{\lambda}}\right) \\
&= -\sqrt{\lambda}t + \lambda\left(t/\sqrt{\lambda} + t^2/2\lambda + t^3/6\lambda^{3/2} \ldots\right) \\
&= \lambda\left(t^2/2\lambda + t^3/6\lambda^{3/2} \ldots\right).
\end{aligned}$$

So $\log(M_{Z_\lambda}(t)) \to t^2/2 \Longrightarrow M_Z(t) \to e^{t^2/2}$ as $\lambda \to \infty$. Therefore $Z_\lambda \xrightarrow{d} N(0,1)$ as $\lambda \to \infty$.

MAST20004 Probability

Computer Lab 10

In this lab you

- simulate the total amount T claimed from an insurance company in one day and compare your simulation estimates against the theoretical values of $\mathbb{E}[T]$ and $V(T)$.
- investigate the accuracy of the approximation formulae for the mean and variance of a function of a random variable.

Exercise A - Simulation of insurance company total claims

Suitably modified, the **incomplete** Matlab m-file **Lab10ExA.m** will simulate the total amount claimed from an insurance company in one day. You will need to add a few lines to the program to generate the required distributions. **Lab10ExA.m** produces estimates for $\mathbb{E}[T]$ and $V(T)$ and also plots the empirical pdf for T.

Let the number of claims in one day be $N \stackrel{d}{=} \text{Pn}(10)$ and $X_1, X_2, \cdots$ be random variables representing claim amounts. We assume that N and $X_1, X_2, \cdots$ are independent, with $X_i \stackrel{d}{=} X$ (for all i) for some claim size X. Then $T = \sum_{i=1}^{N} X_i$ is the sum of a (random) number of random variables and represents the total amount claimed in one day.

1. We start with the assumption that $X_i \stackrel{d}{=} \exp(\lambda)$. Using the appropriate formulae from lectures calculate the theoretical values for $\mathbb{E}[T]$ and $V(T)$.
2. Open **Lab10ExA.m** in the m-file editor and add the code required to generate the claims. Run the program for a couple of different values of λ and compare your theoretical answers with the simulation estimates. Also comment on the shape of the empirical pdf for T.
3. Repeat this exercise for a claim distribution $X \stackrel{d}{=} R(10, 20)$.

Exercise B - Approximations for mean and variance of functions

Let X be a random variable with $\mathbb{E}[X] = \mu$ and $V(X) = \sigma^2$, and let $\psi(X)$ be a transformation of X. As we have seen, it is **not true** in general that the mean and the variance of the transformation $\psi(X)$ are equal simply to the transformations of the mean and variance of X, respectively (an important exception is $\psi(x) = ax + b$ for which $\mathbb{E}[\psi(X)] = \psi(\mathbb{E}[X])$). Often it is difficult to find the exact values of $\mathbb{E}[\psi(X)]$ and $V(\psi(X))$ (due to the fact that the integrals or sums are complicated). In lectures (refer to slides 446–448) we derived the following approximation formulae for the mean and the variance of $\psi(X)$:

$$\begin{aligned} \mathbb{E}[\psi(X)] &\approx \psi(\mu) + \frac{1}{2}\psi''(\mu)\sigma^2, \\ V(\psi(X)) &\approx \psi'(\mu)^2\sigma^2. \end{aligned} \tag{1}$$

These relations are based on the Taylor series approximations of the form

$$\psi(X) \approx \psi(\mu) + \psi'(\mu)(X-\mu) + \frac{1}{2}\psi''(\mu)(X-\mu)^2. \tag{2}$$

To help you understand these formulae and test how well they work, in this lab we will apply them and verify the results using the m-files **Lab10ExB1.m** and **Lab10ExB2.m**. **Lab10ExB1.m** plots $\psi(x)$ and its Taylor series approximations over a specified domain. **Lab10ExB2.m** simulates 'nreps' observations on $\psi(X)$ to estimate the mean and variance.

1. This first example examines the transformation that we looked at in questions 3 and 4 of tutorial 9. Let $X \stackrel{d}{=} R(0,1)$ and $Y = \sqrt{X}$ (so that $X = \psi(X)$ with $\psi(x) = \sqrt{x}$).
   (a) Write down the approximating functions $l(x)$ and $q(x)$ of $\sqrt{x}$, given by the first two and three terms on the right-hand side of (2) respectively. (Note: $l(x)$ is the tangent line approximation and $q(x)$ the quadratic approximation). By adding the appropriate code to **Lab10ExB1.m**, plot $\psi(x)$, $l(x)$ and $q(x)$ on the same graph, over an appropriate domain. Do you expect both approximations to be good?
   (b) Add the appropriate code to **Lab10ExB2.m** to produce observations on $\psi(X)$. Compare the simulation estimates with your approximations.
   (c) How do your simulation results compare to the exact values of $\mathbb{E}[Y]$ and $V(Y)$?
   (d) Repeat this exercise for $X \stackrel{d}{=} R(1,2)$. Do the approximations work better or worse? Explain.
2. Let $V = e^{-1/X}$, where $X \stackrel{d}{=} N(5,4)$. In this case, no exact values are readily available. Complete parts (a) and (b) of section 1 for this random variable.