Exercise booklet Sm2 2018

Linear Algebra (University of Melbourne)

THE UNIVERSITY OF MELBOURNE

School of Mathematics and Statistics

Semester 2, 2018

MAST10007 Linear Algebra

Exercise Booklet

The exercises in this booklet are for you to complete on your own during the semester.

It is very important to keep up to date with these exercises during semester!

Answers to most questions are provided. However, for questions that involve writing a proof you should ask your tutor or lecturer for help (if you need to).

In the weekly tutorials you will be working on separate tutorial sheets.

AR§1.1 References like this refer to the textbook: *Elementary Linear Algebra, Applications Version* by Anton and Rorres, 11th ed.

Exercises marked with this symbol are possibly a bit more difficult. If you get stuck, ask your tutor or lecturer.

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Topic 1: Linear equations

1.1 – Linear equations and row operations (AR §1.1–1.2)

1. *Linear Equations.* Which of the following systems of equations are linear?

(a)
$$\begin{aligned} x_1 - 3x_2 &= x_3 - 4 \\ x_4 &= 1 - x_1 \\ x_1 + x_4 + x_3 - 2 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 3x - xy &= 1 \\ x + 2xy - y &= 0 \end{aligned}$$

(c)
$$\begin{aligned} y &= x - 1 \\ x &= 1 - y \end{aligned}$$

(d)
$$\begin{aligned} x^2 &= y \\ x + y &= 1 \end{aligned}$$

2. *Row Operations.* Use row operations to solve the following linear systems of equations:

(a)
$$\begin{aligned} x + z &= 0 \\ x + y &= 0 \\ y + z &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 2x + 3y &= 1 \\ x + y &= 1 \end{aligned}$$

(c)
$$\begin{aligned} 4x - 2y &= 5 \\ -6x + 3y &= 1 \end{aligned}$$

(d)
$$\begin{aligned} x - y + z &= 1 \\ 2x + y - z &= 3 \\ 3x + 2y - 3z &= 2 \end{aligned}$$

(e)
$$\begin{aligned} x - 4y &= 1 \\ -2x + 8y &= -2 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 + 3x_2 + 3x_3 &= 1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 0 \\ - x_2 - x_3 + 2x_4 &= 1 \end{aligned}$$

1.2 – Row-echelon form and reduced row-echelon form (AR §1.2)

3. *Identifying Matrix Forms.* Which of the following matrices are in (i) row-echelon form (ii) reduced row-echelon form?

(a) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 4 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

4. *Linear Systems.* Solve the systems of linear equations whose augmented matrices can be reduced to the following row-echelon forms.

(a) $\left[\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

5. *More Linear Systems.* Solve the following systems of linear equations by reducing to (i) row-echelon form (ii) reduced row-echelon form.

(a)
$$\begin{aligned} x_1 - 2x_2 &= 1 \\ -2x_1 + 4x_2 &= -2 \end{aligned}$$

(b)
$$\begin{aligned} w &= x + y + z \\ w &= 2x - 3y + z - 1 \\ w &= -x + y - 2z + 2 \\ w &= 4x - 3y + 4z \end{aligned}$$

(c)
$$\begin{aligned} v - 2w + z &= 1 \\ 2u - v \qquad - z &= 0 \\ 4u + v - 6w + z &= 3 \end{aligned}$$

(d)
$$\begin{aligned} 4x_1 + 4x_2 - x_3 &= 3 \\ x_1 - 3x_2 + 4x_3 &= 13 \\ 2x_1 + x_2 + x_3 &= 5 \\ 3x_1 - 2x_2 + 4x_3 &= 17 \end{aligned}$$

1.3 – Consistent and inconsistent systems (AR §1.2)

6. *Consistency.* Determine the conditions on a, b, c so that the system has a solution:

(a) $$\begin{aligned} x + 2y - 3z &= a \\ 3x - y + 2z &= b \\ x - 5y + 8z &= c \end{aligned}$$

(b) $$\begin{aligned} x + 2y + 4z &= a \\ 2x + 3y - z &= b \\ 3x + y + 2z &= c \end{aligned}$$

7. *Existence of Solutions.* Determine the values of the constant k for which the system has (i) no solutions, (ii) a unique solution and (iii) an infinite number of solutions. Find the solutions when they exist.

(a) $$\begin{aligned} 2x + 3y + z &= 11 \\ x + y + z &= 6 \\ 5x - y + 11z &= k \end{aligned}$$

(b) $$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_3 &= 2 \\ 2x_2 + kx_3 &= k \end{aligned}$$

(c) $$\begin{aligned} x + y + 2z &= 9 \\ x - y + z &= 2 \\ 4x + 2y + (k-22)z &= k \end{aligned}$$

8. Use row reduction to decide whether the system has (i) no solution vector, (ii) a unique solution, or (iii) more than one solution. Solve the systems where possible.

(a) $$\begin{aligned} 3x - 2y + 4z &= 3 \\ x - y + z &= 7 \\ 4x - 3y + 5z &= 1 \end{aligned}$$

(b) $$\begin{aligned} x + 2y - z &= -1 \\ 2x + 7y - z &= 3 \\ -3x - 12y + z &= 0 \end{aligned}$$

(c) $$\begin{aligned} 3x - 4y + z &= 2 \\ -5x + 6y + 10z &= 7 \\ 8x - 10y - 9z &= -5 \end{aligned}$$

(d) $$\begin{aligned} 2x - 3y + 5z &= 10 \\ 4x + 7y - 2z &= -5 \\ 2x - 4y + 25z &= 31 \end{aligned}$$

9. Using row echelon reduction, find the general solution to the following system of equations:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 &= 3 \\ x_1 + x_2 + x_3 - x_4 &= 6 \\ x_1 - x_2 + 3x_3 + 5x_4 &= -12 \\ 4x_1 + x_2 + 7x_3 + 5x_4 &= -3 \end{aligned}$$

10. Determine the values of k for which the system of linear equations has (i) no solution vector, (ii) a unique solution vector, (iii) more than one solution vector (x, y, z):

(a) $$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$

(b) $$\begin{aligned} 2x + (k-4)y + (3-k)z &= 1 \\ 2y + (k-3)z &= 2 \\ x - 2y + z &= 1 \end{aligned}$$

(c) $$\begin{aligned} x + 2y + kz &= 1 \\ 2x + ky + 8z &= 3 \end{aligned}$$

(d) $$\begin{aligned} x - 3z &= -3 \\ 2x + ky - z &= -2 \\ x + 2y + kz &= 1 \end{aligned}$$

For the cases (b) and (c) find the solutions.

11. Determine the conditions on a, b, c so that the system has a solution:

(a) $$\begin{aligned} x + 2y - 3z &= a \\ 3x - y + 2z &= b \\ x - 5y + 8z &= c \end{aligned}$$

(b) $$\begin{aligned} x - 2y + 4z &= a \\ 2x + 3y - z &= b \\ 3x + y + 2z &= c \end{aligned}$$

Find the solutions when they exist.

Applications and other problems

12. *Interpolation.* Determine the values for a, b and c for which the parabola $y = ax^2 + bx + c$ passes through the points:

(a) $(0, -3), (1, 0)$ and $(2, 5)$

(b) $(-1, 1), (1, 9)$ and $(2, 16)$

13. *Moving in Circles.* The equation of an arbitrary circle in the x-y plane can be written in the form

$$x^2 + y^2 + ax + by + c = 0$$

where a, b, c are real constants. Find the equation of the unique circle that passes through the three points $(-2, 7), (-4, 5), (4, -3)$.

14. *The Traveler.* A traveler who just returned from Europe spent:

For housing: \$30/day in England, \$20/day in France, \$20/day in Spain

For food: \$20/day in England, \$30/day in France, \$20/day in Spain

For incidental expenses: \$10/day in each country.

The traveler's records of the trip indicate a total of \$340 spent for housing, \$320 for food, \$140 for incidental expenses while travelling in these countries. Calculate the number of days spent in each country or show that the records must be incorrect.

15. *Think.* Frank's, Dave's and Phil's ages are not known but are related as follows: The sum of Dave's and Phil's ages is 13 more than Frank's. Frank's age plus Phil's age is 19 more than Dave's. If the sum of their ages is 71, how old are Frank, Dave and Phil?

16. *Homogeneous versus non-homogeneous linear systems.* A *homogeneous linear system* is a system of linear equations where each equation is equal to 0. Otherwise, the system is *non-homogeneous*. This exercise illustrates the relationship between the solution to a non-homogemeous linear system and the solution of the associated homogeneous linear system. Consider the following non-homogeneous system (in x_1, x_2, x_3, x_4):

$$\begin{array}{ccccccc} x_1 & & - & x_3 & + & x_4 & = & 1 \\ & x_2 & - & x_3 & - & x_4 & = & 2 \end{array} \qquad (*)$$

(a) Calculate the solutions y_1, y_2, y_3, y_4 to the associated homogeneous linear system

$$\begin{array}{ccccccc} y_1 & & - & y_3 & + & y_4 & = & 0 \\ & y_2 & - & y_3 & - & y_4 & = & 0 \end{array}$$

(b) Show that $x_1^* = 1, x_2^* = 2, x_3^* = 0, x_4^* = 0$ is a (particular) solution to $(*)$.

(c) Show that $x_1 = y_1 + x_1^*, x_2 = y_2 + x_2^*, x_3 = y_3 + x_3^*, x_4 = y_4 + x_4^*$ is the (general) solution to $(*)$.

It is always the case that the solution to a linear system can be obtained from the solution to the associated homogeneous linear system by adding a particular solution.

Topic 2: Matrices and Determinants

2.1–2.2 – Matrix notation and operations (AR §1.3–1.4)

17. *Matrix Sizes.* Suppose that A, B, C, and D are matrices with the following sizes:

$$\begin{array}{cccc} A & B & C & D \\ 2\times 3 & 1\times 3 & 2\times 2 & 2\times 1 \end{array}$$

Determine which of the following expressions are defined. For those which are defined, give the size of the resulting matrix.

(a) CA (b) BD (c) DB

(d) CDA^T (e) $CA+DB$ (f) AB^TD^T+C

18. Give 3 by 3 examples of the following:

(a) a diagonal matrix; that is, a matrix A with $A_{ij}=0$ if $i\neq j$.

(b) a scalar matrix; that is, a diagonal matrix A with $A_{ii}=A_{jj}$ for all i,j.

(c) a symmetric matrix; that is, a matrix A with $A_{ij}=A_{ji}$.

(d) an upper triangular matrix; that is, a matrix A with $A_{ij}=0$ if $i>j$.

19. Given

$$A=\begin{bmatrix} 1 & y & 0 \\ -7 & 2 & 3 \end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix} 1 & 3 & 0 \\ x & 2 & 3 \end{bmatrix}$$

determine the values of x and y for which $A=B$.

20. *Matrix operations.* Let

$$A=\begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix} \qquad B=\begin{bmatrix} 0 & 4 & -2 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Find

(a) $A+B$ (b) $2A-3B$ (c) $A-\lambda I \quad (\lambda\in\mathbb{R})$

(d) A^T (e) AB (f) BA

21. Let

$$A=\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 3 \end{bmatrix} \quad B=\begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad\text{and}\quad C=\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$$

Calculate (where possible) $A+B$ and $A+C$.

22. *More Matrix Algebra.* Use matrix algebra to evaluate AB, BC, A^TC^T and $(CA)^T$ given that

$$A=\begin{bmatrix} 1 & 3 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{bmatrix} \quad C=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

23. Evaluate the following matrix products:

(a) $\begin{bmatrix} 3 & 4 & 2 \\ 1 & 3 & 6 \\ 7 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 7+i & 6 & -4+3i \end{bmatrix}\begin{bmatrix} 0 \\ 1-i \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 7 & 6 & -4 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ (f) $\begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$

24. *Matrix Properties.* Consider the following matrices:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 3 & -1 \\ -2 & 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

Verify that

(a) $A(BC) = (AB)C$
(b) $(AB)^T = B^T A^T$
(c) $A(B + C) = AB + AC$

25. Use trial and error to find 2 by 2 examples of the following:

(a) a non-zero matrix A with $A^2 = 0$;
(b) a matrix B with real entries and with $B^2 = -I$;
(c) matrices C and D with no zero entries but with $CD = 0$;
(d) matrices E and F with $EF = -FE$ but $EF \neq 0$.

26. *Compatibility of Matrix Products.*

(a) Show that if the matrix products AB and BA are both defined, then AB and BA are square matrices.
(b) Show that if A is an $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

27. *Commuting Matrices.* Find all matrices A that commute with the following matrices

(a) $C = \begin{bmatrix} 1 & -1 \\ 5 & -4 \end{bmatrix}$ (b) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

that is, find all matrices A satisfying $AC = CA$ and all matrices A satisfying $AD = DA$.

28. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

Show that $A^{-1} = -\frac{1}{3}(A^2 - 2A - 4I)$.

29. Verify that

$$\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

30. Suppose that a 2 by 2 matrix A satisfies $AB = BA$ for every 2 by 2 matrix B. Set

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $AB = BA$ when B is either of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

What does this say about a, b, c, d ? Show that a matrix A which satisfies the above must be a scalar matrix.

31. Let A be a square matrix satisfying $A^2 = A$ and let B be any matrix of the same size. Show that $(AB - ABA)^2 = 0$

32. Following the algorithm described in lectures, reduce the following matrices to row echelon form, and then to reduced row echelon form. Keep a record of the elementary row operations you use.

(a) $\begin{bmatrix} 4 & -8 & 16 \\ 1 & -3 & 6 \\ 2 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2+i & 2+i & 5 & 6+i \\ 1-2i & 1-2i & -2+i & 2-i \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 6 \\ 1 & 0 & -3 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & 2 & 7 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -4 & 5 \\ -2 & 2 & -5 & 4 \end{bmatrix}$

2.3 – Matrix inverses (AR §1.4)

33. *Matrix Inverse.* Find the inverse, if it exists, of the given matrix. If the matrix has no inverse, state this fact.

(a) $\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

34. 2×2 *Inverse.* Use row operations to show that the inverse of the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the matrix

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided that $ad - bc \neq 0$.

35. *A Matrix Equation.*

(a) Show that if a square matrix A satisfies

$$A^2 - 4A + 3I = 0$$

then

$$A^{-1} = \frac{1}{3}(4I - A).$$

(b) Verify these relations in the case that

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

36. *Another Matrix Inverse.*

(a) Find the inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 \\ 4 & 2 & 4 & 3 \\ 6 & 3 & 3 & 5 \end{bmatrix}$$

(b) Check your answer by matrix multiplication.

(c) Use your answer to part (a) to solve the system

$$\begin{aligned} 2x + y + z + w &= 3 \\ 2x + 3y + 2z + 2w &= 5 \\ 4x + 2y + 4z + 3w &= 6 \\ 6x + 3y + 3z + 5w &= 9 \end{aligned}$$

37. *Back to Linear Systems.* Write the following systems of linear equations in the form $A\mathbf{x} = \mathbf{b}$. Then by finding the inverse of A solve the systems.

(a) $\begin{aligned} 2x - 3y &= 3 \\ 3x - 5y &= 1 \end{aligned}$

(b) $\begin{aligned} x - 3y + 4z &= 4 \\ 2x + 2y \quad &= 0 \\ y - 2z &= 2 \end{aligned}$

(c) $\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 0 \\ -x_2 + 3x_3 &= 0 \end{aligned}$

38. Explain why the following matrix is *not* an elementary matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

2.4 – Rank

39. *Rank.*

(a) Determine the rank of the matrix:

(i) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -6 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(b) Let

$$A = \begin{bmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & 1 & 1 \end{bmatrix}$$

Determine rank A in the cases

(i) $k = 1$ (ii) $k = -2$ (iii) $k \neq 1$ or -2

2.5 – Determinants (AR §2.1–2.3)

40. *Basic Determinants.* Evaluate the determinant of the following matrices using cofactors:

(a) $\begin{bmatrix} 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 4 & 5 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 1 \\ 3 & -7 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ (f) $\begin{bmatrix} a & ab \\ b & a^2 + b^2 \end{bmatrix}$

41. *Elementary Operations.* Let

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1$$

Find the following determinants:

(a) $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$ (b) $\begin{vmatrix} a & -b & c \\ d & -e & f \\ g & -h & i \end{vmatrix}$ (c) $\begin{vmatrix} d & e & f \\ 3g & 3h & 3i \\ a & b & c \end{vmatrix}$

(d) $\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix}$ (e) $\begin{vmatrix} a & b & c \\ d+a & e+b & f+c \\ g-2a & h-2b & i-2c \end{vmatrix}$

42. *More Determinants.* Use row operations to evaluate the determinant of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 4 & 13 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 4 & 5 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{3} & \frac{3}{5} & \frac{2}{5} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{2} \end{bmatrix}$

43. *Vanishing Determinants.* Evaluate the determinants of the following matrices. For what values of the variables are the matrices invertible?

(a) $\begin{bmatrix} x & 2x & -3x \\ x & x-1 & -3 \\ 0 & 0 & 2x-1 \end{bmatrix}$ (b) $\begin{bmatrix} \lambda-1 & 0 & 0 & 0 \\ 2 & 0 & \lambda+1 & 0 \\ 1 & \lambda-2 & 0 & 0 \\ 2 & 3 & 9 & \lambda+2 \end{bmatrix}$

(c) $\begin{bmatrix} k & k+1 & k+2 \\ k+3 & k+4 & k+5 \\ k+6 & k+7 & k+8 \end{bmatrix}$

44. *Properties of Determinants.* Verify by direct calculation that for the arbitrary 2×2 matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the following are true:

(a) $\det A = \det(A^T)$ (b) $\det(A^{-1}) = \dfrac{1}{\det A}$

In the second part of this question you should assume that $\det A \neq 0$.

45. *Determinant of a Block Matrix.* Verify by direct calculation the following formula for the evaluation of the determinant of a certain block diagonal matrix:

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} \begin{vmatrix} 3 & 2 \\ -3 & 2 \end{vmatrix}$$

46. *Idempotent Matrices.* A matrix P is called idempotent if $P^2 = P$. If P is idempotent and $P \neq I$ show that $\det P = 0$. Give an example of such a matrix (other than the zero matrix).

47. *Eigenthings.*

(a) Determine the values of the parameter λ for which $\det(A - \lambda I) = 0$ where

(i) $A = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

(b) Find all solutions $\mathbf{v}$ of the linear system

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

for each A and each λ you found in part (a).

Topic 3: Euclidean Vector Spaces

3.1–3.2 – Vectors and dot product in $\mathbb{R}^n$ (AR §3.1–3.2)

48. *Vectors and Scalars.* Let $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = (2, 0, 1)$, $\mathbf{w} = (-1, -2, 3)$. Find
    (a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{w} - \mathbf{v}$ (c) $\mathbf{u} - 5\mathbf{v} + 2\mathbf{w}$
    (d) $\|\mathbf{w}\|$ (e) $d(\mathbf{w}, \mathbf{v})$ (f) $\|\mathbf{w} - \mathbf{v}\|$
    (g) $\|\mathbf{w}\| + \|\mathbf{v}\|$ (h) $\|5(\mathbf{w} - \mathbf{v})\|$ (i) $\mathbf{v} \cdot \mathbf{w}$

49. For the given vectors $\boldsymbol{a}$, $\boldsymbol{b}$ find $\boldsymbol{a} + \boldsymbol{b}$, $5\boldsymbol{a} - 4\boldsymbol{b}$, $\boldsymbol{a} \cdot \boldsymbol{b}$, $\|\boldsymbol{a}\|$, and $\|\boldsymbol{a} - \boldsymbol{b}\|$.
    (a) $\boldsymbol{a} = (-2, 6, 1)$, $\boldsymbol{b} = (3, -3, -1)$
    (b) $\boldsymbol{a} = 3\boldsymbol{i} - 4\boldsymbol{j} + 21\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} - 5\boldsymbol{k}$
    (c) $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = -\boldsymbol{j} + \boldsymbol{k}$

50. Find the following scalar products:
    (a) $(1, 1, 1) \cdot (2, 1, -3)$ (b) $(2, 1, 1) \cdot (1, -3, 7)$ (c) $(\sqrt{2}, \pi, 1) \cdot (\sqrt{2}, -2, 3)$

51. *Angle Between Vectors.* Determine the angle between the following pairs of vectors:
    (a) $2\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k}$
    (c) $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

52. Find the angle between the following pairs of vectors:
    (a) $(1, 0, 0)$, $(0, 0, 4)$ (b) $(1, -1, 0)$, $(0, 1, 1)$ (c) $(2, -2, 2)$, $(-1, 0, 2)$

53. *The Hexagon.* ABCDEF is a regular hexagon with centre O. If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$, express in terms of $\mathbf{a}$ and $\mathbf{c}$ the vectors $\overrightarrow{OB}$, $\overrightarrow{FB}$, $\overrightarrow{DA}$, $\overrightarrow{AC}$ and $\overrightarrow{EA}$.

54. *Similar Triangles.* Use vectors to show that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

55. *Division by Ratio.*
    (a) Find the point which divides the line segment connecting the point A with position vector $(1, 2, 3)$ and the point B with position vector $(-1, 4, 2)$ in the ratio $2:3$.
    (b) Let ABC be a triangle and let D divide the segment BC in the ratio $3:5$. Express the vector $\overrightarrow{AD}$ as a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

3.3 – Cross product in $\mathbb{R}^3$ (AR §3.5)

56. *Dot and Cross Products.* Let $\mathbf{u} = (3, -1, 4)$, $\mathbf{v} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{w} = (-1, 1, 2)$.
    Find, if they exist:
    (a) $\mathbf{u} \cdot \mathbf{v}$ (b) $(3\mathbf{u}) \cdot (-2\mathbf{v})$ (c) $\mathbf{u} \times \mathbf{v}$
    (d) $(\mathbf{v} \times \mathbf{u}) \cdot (-\mathbf{w})$ (e) $\mathbf{v} \times 2\mathbf{u}$ (f) $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$
    (g) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (h) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ (i) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

57. *Distributive Laws.* If $\mathbf{u} = (3, -5, 1)$, $\mathbf{v} = (0, 8, -1)$ and $\mathbf{w} = (1, -3, 2)$, verify the distributive laws for dot and cross products:
    (a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
    (b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

58. Prove the following properties of the vector product:
   (a) $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
   (b) $(\alpha\boldsymbol{a} + \beta\boldsymbol{b}) \times \boldsymbol{c} = \alpha\boldsymbol{a} \times \boldsymbol{c} + \beta\boldsymbol{b} \times \boldsymbol{c}$
   (c) $\boldsymbol{a} \times (\alpha\boldsymbol{b} + \beta\boldsymbol{c}) = \alpha\boldsymbol{a} \times \boldsymbol{b} + \beta\boldsymbol{a} \times \boldsymbol{c}$
   (d) $\boldsymbol{a} \times \boldsymbol{a} = \mathbf{0}$

59. *Proving Vector Identities.* For any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w} \in \mathbb{R}^3$, show that:
   (a) $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = -\boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{w})$
   (b) $(\mathbf{u} - \mathbf{v}) \cdot [(\mathbf{v} \times \mathbf{w}) + (\mathbf{w} \times \mathbf{u})] \;=\; 0$
   (c) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \;=\; \mathbf{v}\,(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}\,(\mathbf{u} \cdot \mathbf{v})$

60. Use question 59 to show that $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})\,(\boldsymbol{b} \cdot \boldsymbol{c})$.
   Deduce *Lagrange's identity:* $\|\boldsymbol{a} \times \boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2\,\|\boldsymbol{b}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

3.4 – Geometric applications, lines and planes (AR §3.3–3.4)

61. *Orthogonal and Parallel.* Find the values of x such that the following pairs of vectors are (i) orthogonal and (ii) parallel.
   (a) $(x, 1-2x, 3)$ and $(1, -x, 3x)$
   (b) $(x, x, -1)$ and $(1, x, 6)$

62. *More Vectors.*
   (a) Find two unit vectors parallel to the vector $\mathbf{u} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$.
   (b) Write the vector $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in the form $\mathbf{v} + \mathbf{w}$ where $\mathbf{v}$ is a vector parallel to $\mathbf{z} = \mathbf{i} + \mathbf{j}$ and $\mathbf{w}$ is perpendicular to $\mathbf{z}$.

63. *Unit Vectors.* Find two unit vectors orthogonal to both vectors:
   (a) $(1, -1, 1)$ and $(0, 4, 4)$
   (b) $\mathbf{i} + \mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

64. *Area.*
   (a) Find the area of the parallelogram determined by the given vectors:
      (i) $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
      (ii) $\mathbf{a} = (1, 4, -2)$, $\mathbf{b} = (1, 1, -1)$
   (b) Find the area of the triangle which has vertices:
      (i) $\mathbf{a} = 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
      (ii) $A(1, 1, 2)$, $B(2, 2, 1)$, $C(2, 3, 2)$

65. Find the area of a triangle with the vertices $(1, -1, 2)$, $(-2, 1, 1)$, $(1, 2, 3)$.
   Find a unit vector orthogonal to the plane of this triangle.

66. *Volume.*
   (a) Find the volume of the parallelepiped:
      (i) determined by the vectors $\mathbf{a} = (1, 0, 6)$, $\mathbf{b} = (2, 3, -8)$, $\mathbf{c} = (8, -5, 6)$
      (ii) with adjacent edges $\overrightarrow{\mathrm{PQ}}$, $\overrightarrow{\mathrm{PR}}$, $\overrightarrow{\mathrm{PS}}$ where $P(1, 1, 1)$, $Q(2, 0, 3)$, $R(3, 1, 7)$, $S(3, -1, -2)$
   (b) Find the volume of the tetrahedron with vertices $P(-1, 2, 0)$, $Q(2, 1, -3)$, $R(1, 0, 1)$, $S(3, -2, 3)$

67. *Lines.*
   Write down the equation for the following lines in both vector and cartesian form:
   (a) the line passing through $P(2, 1, -3)$ and parallel to $\mathbf{v} = (1, 2, 2)$
   (b) the line through $P(2, -3, 1)$ and parallel to the x-axis
   (c) the line passing through the points $P(2, 0, -2)$ and $Q(1, 4, 2)$
   (d) the line through $P(2, 4, 5)$ and perpendicular to the plane $5x - 5y - 10z = 2$

68. *More Lines.* Determine whether the lines L_1 and L_2 are parallel, intersecting or skew (not parallel or intersecting). If they intersect, find the point of intersection. Let the parameters $s, t \in \mathbb{R}$.

(a) $L_1 : x = -6t,\ y = 1 + 9t,\ z = -3t$ and $L_2 : x = 1 + 2s,\ y = 4 - 3s,\ z = s$

(b) $L_1 : x = 1 + t,\ y = 1 - t,\ z = 2t$ and $L_2 : x = 2 - s,\ y = s,\ z = 2$

(c) $L_1 : \dfrac{x-4}{2} = \dfrac{y+5}{4} = \dfrac{z-1}{-3}$ and $L_2 : x - 2 = \dfrac{y+1}{3} = \dfrac{z}{2}$

69. *Planes.* Find the equations of the following planes in both cartesian and (vector) parametric form:

(a) the plane through the point $(1, 4, 5)$ and perpendicular to the vector $(7, 1, 4)$

(b) the plane through the point $(6, 5, -2)$ and parallel to the plane $x + y - z + 1 = 0$

(c) the plane through the origin and the points $(1, 1, 1)$ and $(1, 2, 3)$

(d) the plane that passes through the point $(1, 6, -4)$ and contains the line

$$x = 1 + 2t,\ y = 2 - 3t,\ z = 3 - t \text{ where } t \in \mathbb{R}$$

70. *Co-What?*

(a) Show that three points A, B and C are collinear if and only if $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \mathbf{0}$. Are the points $A(1, 2, 3)$, $B(3, 1, 0)$ and $C(9, -2, -9)$ collinear? If yes, find the equation of the line containing these points.

(b) Show that four points A, B, C and D are coplanar if and only if

$$\overrightarrow{\mathrm{AB}} \cdot (\overrightarrow{\mathrm{AC}} \times \overrightarrow{\mathrm{AD}}) = 0.$$

Are the points $A(1, 1, 1)$, $B(2, 1, 3)$, $C(3, 2, 1)$ and $D(4, 2, 3)$ coplanar? If yes, find the equation of the plane containing these points.

71. *Intersections.*

(a) Find the point of intersection of the line $\mathbf{r}(t) = (2, 1, 1) + t(-1, 0, 4);\ t \in \mathbb{R}$ with the plane $x - 3y - z = 1$.

(b) Find the point of intersection of the line $x = 1 + t, y = 2t, z = 3t;\ t \in \mathbb{R}$ with the plane $x + y + z = 1$.

72. *Angles.* Find the angle between:

(a) the lines $x - 3 = 2 - y,\ z = 1$ and $x = 7,\ y - 2 = z - 5$

(b) the planes $2x + y + 3z = 0$ and $3x - 2y + 4z - 4 = 0$

(c) the line $x = 2t - 7, y = 4t - 6, z = t - 5;\ t \in \mathbb{R}$ and a vector normal to the plane $x + 2y - 4z = 0$

73. If the line ℓ is given by the equations $2x - y + z = 0$, $x + z - 1 = 0$, and if M is the point $(1, 3, -2)$, find a Cartesian equation of the plane:

(a) passing through M and ℓ;

(b) passing through M and orthogonal to ℓ.

Topic 4: General Vector Spaces

4.1–4.2 – Vector space axioms and examples (AR §4.1)

74. Determine whether or not the given set is a vector space under the usual operations. If it is not a vector space, list all properties that fail to hold.

    (a) The set of all 2×3 matrices whose second column consists of 0's.

    (b) The set of all (real) polynomials with positive coefficients.

    (c) The set of all (real valued) continuous functions with the property that the function is 0 at every integer, for example $f(x) = \sin(\pi x)$.

75. Let V be the set of positive real numbers, that is, $V = \{x \in \mathbb{R} : x > 0\}$. Define the operations of vector addition $\oplus$ and scalar multiplication $\odot$ as follows:

$$\begin{aligned} x \oplus y &= xy && \text{for all } x, y \in V \\ k \odot x &= x^k && \text{for all } k \in \mathbb{R}, \text{ and } x \in V \end{aligned}$$

Show that, equipped with these operations, V forms a real vector space. What is the zero vector ? What is the (additive) inverse of a vector $x \in V$?

76. *Consequences.* Let V be a vector space with scalars $\mathbb{F}$. Prove the following consequences of the vector space axioms:

    (a) $\lambda \mathbf{0} = \mathbf{0}$ for all $\lambda \in \mathbb{F}$
    (b) $0\boldsymbol{v} = \mathbf{0}$ for all $\boldsymbol{v} \in V$
    (c) $(-1)\boldsymbol{v} = -\boldsymbol{v}$ for all $\boldsymbol{v} \in V$

    Note. You can not assume that the vector space is a subspace of $\mathbb{R}^n$, so saying, for example, that $\lambda \times (0, \ldots, 0) = (\lambda \times 0, \ldots, \lambda \times 0) = (0, \ldots, 0)$ is not sufficient.

4.3 – Linear combinations (AR §4.3)

77. *Linear Combinations.* Which of the following are linear combinations of $\boldsymbol{u} = (0, -2, 2)$ and $\boldsymbol{v} = (1, 3, -1)$?

    (a) $(2, 2, 2)$ (b) $(0, 4, 5)$

78. Let $\boldsymbol{u} = (1, 0, -1)$ and $\boldsymbol{v} = (-2, 1, 1)$.

    (a) Write $(-1, 2, -1)$ as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

    (b) Show that $(-1, 1, 1)$ cannot be written as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

    (c) For what value of c is the vector $(1, 1, c)$ a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$?

79. In this question let $\mathcal{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ be vectors in $\mathbb{R}^3$ and let A be the 3×5 matrix with the ith column given by the vector $\boldsymbol{v}_i$. Suppose that the reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Are the following sets linearly dependent or independent? If linearly dependent, express one vector as a linear combination of the others.

(a) $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ (b) $\{\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ (c) $\{\boldsymbol{v}_1, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ (d) $\{\boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$

80. Is $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ a linear combination of the matrices

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} \ ?$$

81. Express the polynomial $-9-7x-15x^2$ as a linear combination of $p_1 = 2+x+4x^2$, $p_2 = 1-x+3x^2$, and $p_3 = 3+2x+5x^2$.

4.4 – Linear dependence and independence (AR §4.3)

82. *Linear Independence.* Determine whether or not the following sets of vectors are linearly independent:

(a) $\{(2,-3,1,-5),\ (0,1,2,2),\ (1,-2,3,0)\}$
(b) $\{(1,0,2,-3),\ (0,-4,1,1),\ (2,2,0,-1),\ (1,-2,-1,3)\}$

83. Determine whether the following sets are linearly dependent or linearly independent.

(a) $\{(1,2),(0,2),(1,0),(-1,1)\}$
(b) $\{(1,2),(3,-1)\}$
(c) $\{(1,0,1),(-1,1,0),(0,1,1))\}$
(d) $\{(2,0,0,0),(2,1,0,0),(-1,3,-2,0),(1,-2,4,-3)\}$

84. Which of the following sets of vectors in $\mathbb{C}^3$ are linearly independent?

(a) $\{(1-i,1,0),\ (2,1+i,0),\ (1+i,i,0)\}$
(b) $\{(1,0,-i),\ (1+i,1,1-2i),\ (0,i,2)\}$
(c) $\{(i,0,2-i),\ (0,1,i),\ (-i,-1-4i,3)\}$

85. Show that the vectors $(1,a,a^2),(1,b,b^2),(1,c,c^2)$ are linearly independent if a,b,c are distinct (i.e., $a \neq b$, $a \neq c$ and $b \neq c$).

86. Determine whether or not the given set is linearly independent. If the set is linearly dependent, write one of its vectors as a linear combination of the others.

(a) $\{1, 1+x, 1+x+x^2\}$ in $\mathcal{P}_2$
(b) $\{1+x^2, 1+x+2x^2, x+x^2\}$ in $\mathcal{P}_2$
(c) $\left\{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$ in $M_{2,2}$
(d) $\left\{\begin{bmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}-2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\right\}$ in $M_{3,3}$

4.5 – Subspaces (AR §4.2)

87. For each of the following subsets of $\mathbb{R}^2$ sketch the set, then determine whether it is (i) closed under addition, (ii) closed under scalar multiplication, (iii) a subspace of $\mathbb{R}^2$.

(a) $A = \{(x,y) : y \geq 0\}$
(b) $B = \{(x,y) : x = y\}$
(c) $C = \{(x,y) : x^2+y^2 \leq 1\}$
(d) $D = \{(x,y) : xy = 0\}$.

88. Decide which of the following are subspaces of $\mathbb{R}^3$. Explain your answers.

(a) $A = \{(a,b,0) \in \mathbb{R}^3 : a,b \in \mathbb{R}\}$
(b) $B = \{(a,b,c) \in \mathbb{R}^3 : 2a-3b+5c = 4\}$
(c) $C = \{(a,b,c) \in \mathbb{R}^3 : 2a-3b+5c = 0\}$
(d) $D = \{(a_1,a_2,a_3) \in \mathbb{R}^3 : a_1 \geq 0\}$
(e) $E = \{(a-b, a+b, 2a) \in \mathbb{R}^3 : a,b \in \mathbb{R}\}$

89. Show that the following sets of vectors are subspaces of $\mathbb{R}^m$.

(a) The set of all linear combinations of the vectors $(1,0,1,0)$ and $(0,1,0,1)$ (of $\mathbb{R}^4$).
(b) The set of all vectors of the form $(a,b,a-b,a+b)$ (of $\mathbb{R}^4$).
(c) The set of all vectors (x,y,z) such that $x+y+z=0$ (of $\mathbb{R}^3$).

90. Show that the following sets of vectors are *not* subspaces of $\mathbb{R}^m$.

    (a) The set of all vectors whose first component is 2.

    (b) The set of all vectors *except* the vector $\mathbf{0}$.

    (c) The set of all vectors the sum of whose components is 1.

91. *General Subspaces.* Use the subspace theorem to decide which of the following are real vector spaces with the *usual* operations.

    (a) The set of all real polynomials of any degree.

    (b) The set of real polynomials of degree $\leq n$.

    (c) The set of real polynomials of degree exactly n.

    (d) The set of real polynomials p with $p(0) = 0$.

    (e) The set of real polynomials p with $p(0) = 1$.

    (f) The set of all differentiable functions.

    (g) The set of all solutions of the differential equation $y'' - 3y' + 2y = 0$.

92. Determine whether or not the given set is a subspace of $M_{2,2}$:

    (a) The set of all 2×2 matrices, the sum of whose entries is zero.

    (b) The set of all 2×2 matrices whose determinant is zero.

93. Determine whether or not the given set is a subspace of $M_{n,n}$, the space of all square matrices of size n.

    (a) The diagonal matrices of order n.

    (b) The matrices of order n with trace equal to 0.
    (The *trace* of a square matrix is the sum of the diagonal elements.)

94. *Intersections of subspaces.* Let H and K be subspaces of a vector space V. Prove that the intersection $K \cap H$ is a subspace of V.

95. *Complex matrix spaces.* Decide which of the following are *complex* vector spaces with the *usual* matrix operations. (Use the subspace theorem and the fact that the set of *all* 2×2 matrices with complex entries is a complex vector space.)

    (a) All complex 2×2 matrices $\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$ with z_1 and z_2 real.

    (b) All complex 2×2 matrices with $z_1 + z_4 = 0$.

4.5 – Spanning sets (AR §4.2)

96. Determine whether the given set spans the given vector space.

    (a) In $\mathbb{R}^2$: $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$.

    (b) In $\mathbb{R}^3$: $\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} \right\}$.

97. Determine which of the following sets span $\mathbb{R}^3$.

    (a) $\{(1, 2, 3), (-1, 0, 1), (0, 1, 2)\}$

    (b) $\{(-1, 1, 2), (3, 3, 1), (1, 2, 2)\}$

98. Find spanning sets for the following subspaces of $\mathbb{R}^3$:

    (a) $\{(2a, b, 0) : a, b \in \mathbb{R}\}$
    (b) $\{(a + c, c - b, 3c) : a, b, c \in \mathbb{R}\}$
    (c) $\{(4a + d, a + 2b, c - b) : a, b, c, d \in \mathbb{R}\}$

99. Find a set that spans the subspaces given in question 89.

100. Determine whether or not each set in question 86 spans the indicated vector space.

4.7 – Bases and dimension (AR §4.2–4.5)

101. In each part determine whether or not the given set forms a basis for the indicated (sub)space.

    (a) $\{(1, 2, 3), (-1, 0, 1), (0, 1, 2)\}$ for $\mathbb{R}^3$
    (b) $\{(-1, 1, 2), (3, 3, 1), (1, 2, 2)\}$ for $\mathbb{R}^3$
    (c) $\{(1, -1, 0), (0, 1, -1)\}$ for the subspace of $\mathbb{R}^3$ consisting of all (x, y, z) such that $x + y + z = 0$.
    (d) $\{(1, 1, 0), (1, 1, 1)\}$ for the subspace of $\mathbb{R}^3$ consisting of all (x, y, z) such that $y = x + z$.

102. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

    (a) $\{(1, 0, 0), (2, 2, 0), (3, 3, 3)\}$
    (b) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$

103. Find a basis for and the dimension of the subspace of $\mathbb{R}^n$ spanned by the following sets.

    (a) $\{(0, 1, -2),\ (3, 0, 1),\ (3, 2, -3)\}\quad (n = 3)$
    (b) $\{(1, 3),\ (-1, 2),\ (7, 6)\}\quad (n = 2)$
    (c) $\{(-1, 2, 0, 4),\ (3, 1, -1, 2),\ (-5, 3, 1, 6),\ (7, 0, -2, 0)\}\quad (n = 4)$

104. For each of the following sets choose a *subset* which is a basis for the subspace spanned by the set. Then express each vector that is not in the basis as a linear combination of the basis vectors.

    (a) $(1, 2, 0, -1),\ (2, -1, 2, 3),\ (-1, -11, 6, 13),\ (4, 3, 2, 1)$
    (b) $(0, -1, -3, 3),\ (-1, -1, -3, 2),\ (3, 1, 3, 0),\ (0, -1, -2, 1)$
    (c) $(1, 2, -1),\ (0, 3, 4),\ (2, 1, -6),\ (0, 0, 2)$

    [*Suggestion: Write down the given vectors as columns.*]

105. In each part explain why the given statement is true "by inspection."

    (a) The set $\{(1, 0, 3), (-1, 1, 0), (1, 2, 4), (0, -1, -2)\}$ is linearly dependent.
    (b) The set $\{(1, -1, 2), (0, 1, 1)\}$ does not span $\mathbb{R}^3$.
    (c) If the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ of vectors in $\mathbb{R}^4$ is linearly independent, then it spans $\mathbb{R}^4$.
    (d) The set $\{(0, 1, -1, 0), (0, -1, 2, 0)\}$ is linearly independent, and so it spans the subspace of $\mathbb{R}^4$ of all vectors of the form $(0, a, b, 0)$.

106. Determine whether or not each set in question 86 is a basis for the indicated vector space.

107. Find the dimension of the given vector space:

    (a) The subspace of $M_{2,2}$ consisting of all diagonal 2×2 matrices.
    (b) The subspace of $M_{2,2}$ consisting of all 2×2 matrices whose diagonal entries are zero.
    (c) The subspace of $\mathcal{P}_3$ consisting of all polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ with $a_2 = 0$.

108. Determine whether or not the given set is a basis for $\mathbb{C}^3$ (as a $\mathbb{C}$-vector space).

(a) $\{(i, 0, -1), (1, 1, 1), (0, -i, i)\}$ (b) $\{(i, 1, 0), (0, 0, 1)\}$

109. Find a basis for the subspace of $\mathbb{C}^3$ of all vectors of the form (z_1, z_2, z_3) satisfying $z_1 + z_2 + z_3 = 0$.

110. Determine whether the given set of vectors spans the given vector space.

(a) In $\mathcal{P}_2$: $\{1 - x, 3 - x^2\}$.

(b) In $M_{2,2}$: $\{\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}\}$

111. Which of the following sets of vectors are bases for $\mathcal{P}_2$?

(a) $\{1 - 3x + 2x^2, 1 + x + 4x^2, 1 - 7x\}$ (b) $\{1 + x + x^2, x + x^2, x^2\}$

112. Show that the following set of vectors is a basis for $M_{2,2}$:

$$\left\{ \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \right\}$$

113. (a) Show that any four polynomials in $\mathcal{P}_2$ are linearly dependent.

(b) Show that two polynomials cannot span $\mathcal{P}_2$.

114. Prove that if V and W are three-dimensional subspaces of $\mathbb{R}^5$, then V and W must have a nonzero vector in common. (Hint: Start with bases for the two subspaces, making six vectors in all.)

115. *Symmetric matrices.* A matrix A is *symmetric* if it is equal to its transpose A^T. Let S_n be the set of all symmetric $n \times n$ matrices. Show that S_n is a subspace of $M_{n,n}$ and that $\dim(S_n) = n(n+1)/2$.

116. *Sums and intersections.* Let H and K be subspaces of a vector space V and define

$$H + K = \{h + k : h \in H \text{ and } k \in K\}.$$

(a) Show that $H + K$ is a subspace of V.

(b) Assume that H and K are each finite dimensional, and that $H \cap K = \{\mathbf{0}\}$. Show that $\dim(H + K) = \dim(H) + \dim(K)$.

117. *Wronskian.* Let f and g be differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) Show that f and g are linearly independent (as vectors in $\mathcal{F}(\mathbb{R}, \mathbb{R})$) if the determinant

$$\begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

is non-zero for some x in $\mathbb{R}$.

(b) Show that the functions $\sin x$ and $\cos x$ are linearly independent.

118. Let V be a vector space and suppose that $\mathcal{B}$ is a basis for V. Show that any vector in V can be written *uniquely* as a linear combination of elements from $\mathcal{B}$.

119. In each part find a basis for and the dimension of the indicated subspace.

(a) The solution space of the homogeneous linear system:

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & & & = & 0 \\ & & x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & - & x_2 & & & + & x_4 & = & 0 \end{array}$$

(b) The solution space of

$$\begin{array}{rcrcrcrcrcl} x_1 & - & 3x_2 & + & x_3 & & & - & x_5 & = & 0 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & & & = & 0 \\ x_1 & - & x_2 & + & x_3 & - & 2x_4 & + & x_5 & = & 0 \end{array}$$

(c) The subspace of $\mathbb{R}^4$ of all vectors of the form $(x, -y, x-2y, 3y)$.

120. In the following exercises verify that the row rank is equal to the column rank by explicitly finding the dimensions of the row space and the column space of the given matrix:

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

121. Find a basis for the column space for each matrix of question 120.

122. Find a basis for the row space for each matrix of question 120.

123. Find a basis for the solution space for each matrix of question 120.

124. Find bases for the following subspaces of $\mathbb{R}^3$.

(a) The set of vectors lying in the plane $2x - y - z = 0$.
(b) The set of vectors on the line $x/2 = y/3 = z/4$.

125. Let

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 6 & 1 & -8 \\ -9 & 3 & 15 \end{bmatrix} \quad \text{and} \quad W = \{(x,y,z) \in \mathbb{R}^3 : A \begin{bmatrix} x & y & z \end{bmatrix}^T = 3 \begin{bmatrix} x & y & z \end{bmatrix}^T \}$$

Show that W is a two dimensional subspace of $\mathbb{R}^3$, and find a basis for it.

4.8 – Coordinate vectors (AR §4.4)

126. (a) Show that the set $B = \{(-2,2,2),\ (3,-2,3),\ (2,-1,1)\}$ is a basis for $\mathbb{R}^3$.
(b) Find the vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ whose coordinates with respect to B are

$$[\boldsymbol{x}]_B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad [\boldsymbol{y}]_B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(c) For each of the following vectors find its coordinates with respect to B:

$$\mathbf{a} = (2,-1,1) \qquad \mathbf{b} = (1,0,5) \qquad \mathbf{c} = (3,-1,6)$$

127. Find the coordinate vector of $\boldsymbol{v}$ with respect to the given basis $\mathcal{B}$ for the vector space V.

(a) $\boldsymbol{v} = 2 - x + 3x^2, \mathcal{B} = \{1, x, x^2, x^3\}, V = \mathcal{P}_3$.

(b) $\boldsymbol{v} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \mathcal{B} = \{E^{ij} \mid i = 1, 2; j = 1, 2, 3\}, V = M_{2,3}$.

(Here E^{ij} is the matrix with (i, j) entry equal to 1 and other entries equal to 0.)

(c) $\boldsymbol{v} = 2 - 5x, \mathcal{B} = \{x + 1, x - 1\}, V = \mathcal{P}_1$

(d) $\boldsymbol{v} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, V is the vector space of all diagonal 2×2 matrices.

128. Use coordinate vectors to decide whether or not the given set is linearly independent. If it is linearly *dependent*, express one of the vectors as a linear combination of the others.

(a) $\{x^2 + x - 1, x^2 - 2x + 3, x^2 + 4x - 3\} \in \mathcal{P}_2$

(b) $\left\{ \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \right\}$ in $M_{2,2}$

Topic 5: Inner Product Spaces

5.1 – Definitions and computing inner products (AR §6.1)

129. *Computing inner products.*

(a) If $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ are any two 2×2 matrices, then

$$\langle U, V\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

defines an inner product on $M_{2,2}$.

Compute $\langle U, V\rangle$ if $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$ and $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$.

(b) If $p = a_0 + a_1x + a_2x^2$ and $q = b_0 + b_1x + b_2x^2$ are any two vectors in $\mathcal{P}_2$, then

$$\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$$

is an inner product on $\mathcal{P}_2$. Compute $\langle p, q\rangle$ if $p = -2 + x + 3x^2$ and $q = 4 - 7x^2$.

130. In $\mathbb{R}^2$, for $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, define $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + 3x_2y_2$. Show that $\langle \mathbf{x}, \mathbf{y}\rangle$ is an inner product on $\mathbb{R}^2$.

131. In $\mathbb{R}^2$, let $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 - x_2y_2$. Is this an inner product? If not, why not?

132. Verify that the operation

$$\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ is an inner product in $\mathbb{R}^2$.

133. Decide which of the suggested operations on $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ define an inner product:

(a) $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + 2x_2y_2 + x_3y_3$,

(b) $\langle \mathbf{x}, \mathbf{y}\rangle = x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2$,

(c) $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 - x_2y_2 + x_3y_3$,

(d) $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2$.

134. Decide which of the operations $\langle p, q\rangle$ on real polynomials $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ define inner products on $\mathcal{P}_2$:

(a) $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$

(b) $\langle p, q\rangle = a_0b_0$

(c) $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$

5.2 – Geometry from inner products (AR §6.2)

135. Let $\mathcal{P}_2$ have the inner product defined in question 129b. If $p = -2 + 3x + 2x^2$, find $\|p\|$.

136. Let $M_{2,2}$ have the inner product defined in question 129a. If $A = \begin{bmatrix} -2 & 5 \\ 3 & 6 \end{bmatrix}$, find $\|A\|$.

137. Let $\mathcal{P}_2$ have the inner product defined in question 129b. If $p = 3 - x + x^2$, $q = 2 + 5x^2$ are two points in $\mathcal{P}_2$, find the distance between them.

138. Consider $M_{2,2}$ with the inner product defined in question 129a. If $A = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$ are two points in $M_{2,2}$, find the distance between them.

139. *Two different inner products on* $\mathbb{R}^3$. For the vectors $\mathbf{x} = (1, 1, 0)$, $\mathbf{y} = (0, 1, 0)$ in $\mathbb{R}^3$ compute the norms $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ as well as the angle between $\mathbf{x}$ and $\mathbf{y}$ using the following inner products.

(a) $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2 + x_3y_3$

(b) $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + 3x_2y_2 + x_3y_3$

140. *Parallelogram law.* Prove that the following holds for all vectors $\mathbf{x}, \mathbf{y}$ in every inner product space:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

141. *Quadratic forms.* Let A be a real invertible $n \times n$ matrix. Show that

$$\langle \mathbf{x}, \mathbf{y} \rangle \equiv \mathbf{y}^T A^T A \mathbf{x} = (A\mathbf{y})^T (A\mathbf{x})$$

defines an inner product in $\mathbb{R}^n$, where $\mathbf{x}$ and $\mathbf{y}$ are column vectors in $\mathbb{R}^n$. What happens when A is not invertible?

5.3 – Cauchy-Schwarz inequality (AR §6.2)

142. By choosing appropriate vectors in the Cauchy-Schwartz inequality, prove that

$$(a_1 + \ldots + a_n)^2 \le n(a_1^2 + \ldots + a_n^2)$$

for all real numbers $a_1, \ldots, a_n$. When does equality hold?

5.4 – Orthogonality, orthonormal bases, and the Gram-Schmidt algorithm (AR §6.2–6.3)

143. In each part determine whether the given vectors are orthogonal with respect to the Euclidean inner product (i.e., the usual dot product).

(a) $\boldsymbol{u} = (-1, 3, 2)$, $\boldsymbol{v} = (4, 2, -1)$

(b) $\boldsymbol{u} = (0, 3, -2, 1)$, $\boldsymbol{v} = (5, 2, -1, 0)$

144. Let $\mathbb{R}^4$ have the Euclidean inner product, and let $\boldsymbol{u} = (-1, 1, 0, 2)$. Determine whether the vector $\boldsymbol{u}$ is orthogonal to the following vectors:
$\boldsymbol{w}_1 = (0, 0, 0, 0)$, $\boldsymbol{w}_2 = (1, -1, 3, 0)$, and $\boldsymbol{w}_3 = (4, 0, 9, 2)$.

145. Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ each with the Euclidean inner product. In each part find the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) $\boldsymbol{u} = (1, -3)$, $\boldsymbol{v} = (2, 4)$

(b) $\boldsymbol{u} = (-1, 5, 2)$, $\boldsymbol{v} = (2, 4, -9)$

146. Show that $p = 1 - x + 2x^2$ and $q = 2x + x^2$ are orthogonal with respect to the inner product in question 129b.

147. Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Which of the following matrices are orthogonal to A with respect to the inner product in question 129a?

(a) $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$

148. Show that in every inner product space: $v + w$ is orthogonal to $v - w$ if and only if $\|v\| = \|w\|$. Give a geometric interpretation of this result.

149. *Orthonormal bases.* Let $\langle \mathbf{x}, \mathbf{y} \rangle$ be an inner product on a vector space V, and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis for V. Prove:

(a) For each $x \in V$, $x = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 + \cdots + \langle x, e_n \rangle e_n$;

(b) $\langle \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n, \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n \rangle = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_n \beta_n$;

(c) $\langle x, y \rangle = \langle x, e_1 \rangle \langle y, e_1 \rangle + \cdots + \langle x, e_n \rangle \langle y, e_n \rangle$.

150. Use the results in question 149 to express the given vector as a linear combination of the vectors in the following orthonormal basis (with respect to the dot product).

$$\left\{(\frac{1}{3},-\frac{2}{3},\frac{2}{3}),(-\frac{2}{3},\frac{1}{3},\frac{2}{3}),(\frac{2}{3},\frac{2}{3},\frac{1}{3})\right\}.$$

(a) $\mathbf{x}=(1,2,3)$ (b) $\mathbf{y}=(-1,0,1)$

151. *Pythagoras's theorem.* Let $u_1,u_2,\dots,u_n$ be orthogonal vectors in an inner product space V and let $\|x\|=\sqrt{\langle x,x\rangle}$ be the norm induced by the inner product on V. Show that

$$\|u_1+u_2+\cdots+u_n\|^2=\|u_1\|^2+\|u_2\|^2+\cdots+\|u_n\|^2.$$

152. *Gram-Schmidt.* Use the Gram-Schmidt procedure to construct orthonormal bases for the subspaces of $\mathbb{R}^n$ spanned by the following sets of vectors (using the dot product):

(a) $(1,0,1,0)$, $(2,1,1,1)$, $(1,-1,1,-1)$
(b) $(2,2,-1,0)$, $(2,3,1,-2)$, $(3,4,5,-2)$
(c) $(1,-2,1,3,-1)$, $(0,6,-2,-6,0)$, $(4,-2,2,6,-4)$

153. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most two with the inner product

$$\langle p,q\rangle=\int_{-1}^{1}p(x)q(x)\,dx.$$

Obtain an orthonormal basis for $\mathcal{P}_2$ from the basis $\{1,x,x^2\}$ using the Gram-Schmidt process.

154. *Fourier series.* Show that the infinite set

$$\left\{\frac{1}{\sqrt{2\pi}},\frac{1}{\sqrt{\pi}}\sin(nx),\frac{1}{\sqrt{\pi}}\cos(nx)\;|\;\;n=1,2,\dots\right\}$$

is an orthonormal set in the vector space $C[0,2\pi]$ of real continuous functions on the interval $[0,2\pi]$ equipped with the inner product

$$\langle f,g\rangle=\int_{0}^{2\pi}f(x)g(x)\,dx$$

155. For the vector space $C[0,2\pi]$ equipped with the inner product of the preceding problem let W_k denote the subspace given by

$$W_k=\mathrm{Span}\{1,\sin(x),\cos(x),\sin(2x),\cos(2x),\dots,\sin(kx),\cos(kx)\}$$

(a) Find the element of W_2 closest to $f(x)=1+x$.
(b) Find the element of W_k closest to $f(x)=1+x$.

Note: $\int x^n\sin x\,dx=-x^n\cos x+n\int x^{n-1}\cos x\,dx$ and $\int x^n\cos x\,dx=x^n\sin x-n\int x^{n-1}\sin x\,dx$.

[*Suggestion: Use the orthonormal basis of question 154.*]

5.5 - Orthogonal projections

156. *Orthogonal Projections.* Find the orthogonal projection of (x,y,z) onto the subspace of $\mathbb{R}^3$ spanned by the vectors

(a) $(1,\ 2,\ 2)$, $(-2,\ 2,\ -1)$; (b) $(1,\ 2,\ -1)$, $(0,\ -1,\ 2)$.

157. Let V be a finite dimensional real vector space with inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. Then the *orthogonal complement* of W is defined to be

$$W^{\perp} = \{v \in V : \langle v, w\rangle = 0 \text{ for all } w \in W\}.$$

Prove the following:

(a) $W^{\perp}$ is a subspace of V.

(b) $W \cap W^{\perp} = \{\mathbf{0}\}$.

(c) $\dim W + \dim W^{\perp} = \dim V$.

5.6 – Curve fitting (AR §6.4–6.5)

158. Find the least square polynomial of the specified degree n for the given data points.

(a) $\{(0,0),(1,0),(2,1),(3,3),(4,5)\}$, $n = 1$

(b) $\{(-2,2),(-1,1),(0,-1),(1,0),(2,3)\}$, $n = 2$

159. A maths lecturer was placed on a rack by his students and stretched to lengths $L = 1.7, 2.0$ and 2.3 metres when forces of $F = 1, 2$ and 4 tonnes were applied. Assuming Hooke's law $L = a + bF$, find his normal length a by least squares.

160. A firm that manufactures widgets finds the daily consumer demand $d(x)$ for widgets as a function of their price x is as in the following table:

x	1	1.5	2	2.5	3
$d(x)$	200	180	150	100	25

Using least squares polynomials, approximate the daily consumer demand

(a) By a linear function.

(b) By a quadratic function.

Topic 6: Eigenvalues and Eigenvectors

6.1–6.3 – Definitions and computing eigenvalues and eigenvectors (AR §5.1)

161. *Eigenobjects.* Find the eigenvalues and linearly independent eigenvectors of the following matrices:

(a) $\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -2 \\ 15 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

162. Find, by inspection, the eigenvalues of the given matrix.

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$

163. Prove that for an invertible matrix A, λ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. What relationship holds between the eigenvectors of A and A^{-1}?

164. Find the eigenvalues and linearly independent eigenvectors for the following matrices.

(a) $\begin{bmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & -8 & -12 \\ -6 & -10 & -12 \\ 6 & 10 & 13 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

165. For each matrix find all eigenvalues and a basis for each eigenspace.

(a) $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The Cayley-Hamilton Theorem

The Cayley-Hamilton Theorem asserts that every square matrix satisfies its own characteristic equation. That is, if

$$p(\lambda) = \det(\lambda I - A) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$$

is the characteristic polynomial of an $n \times n$ matrix A, then

$$a_0 I + a_1 A + \cdots + a_{n-1}A^{n-1} + A^n = \mathbf{0}$$

(where $\mathbf{0}$ is the $n \times n$ zero matrix).

166. Verify the Cayley-Hamilton Theorem for the following matrices.

(a) $\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

167. For each matrix, find a non-zero polynomial satisfied by the matrix.

(a) $\begin{bmatrix} 2 & 5 \\ 1 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

168. Use the Cayley-Hamilton Theorem to calculate the inverse of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

6.4 – Diagonalization (AR §5.2, 7.1–7.3)

169. Decide which of the matrices A in questions 161 and 164 above are diagonalizable and, if possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

170. Determine whether or not the given matrix A is orthogonal.

(a) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

171. Show that the rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

172. *Symmetric matrices.* For each symmetric matrix A below find a decomposition $A = PDP^T$, where P is orthogonal and D diagonal.

(a) $\begin{bmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

173. Let A be an orthogonal matrix. Show that $\det A = \pm 1$.

174. Prove that if A, B are orthogonal $n \times n$ matrices, then so are A^{-1} and AB.

175. *Diagonal matrix powers.* Let D be a diagonal matrix, $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_s)$ (i.e., $D_{ii} = \lambda_i$, $D_{ij} = 0$ for $i \neq j$). Prove by induction that, for each positive integer n,

$$D^n = \operatorname{diag}(\lambda_1^n, \ldots, \lambda_s^n).$$

176. *Powers.* Let A be a matrix such that $A = PDP^{-1}$, where D is diagonal. Prove that, for each positive integer n,

$$A^n = PD^nP^{-1}.$$

177. *Practical powers.* Use the results of the preceding two problems to find A^5, where A is

(a) $\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 9 & 18 & -24 \\ 7 & 20 & -24 \\ 7 & 21 & -25 \end{bmatrix}$ (c) $\frac{1}{8}\begin{bmatrix} 8 & 1 & 27 & 5 \\ 0 & 18 & 14 & -6 \\ 0 & 2 & -18 & -6 \\ 0 & 8 & 8 & -8 \end{bmatrix}$

[*Hint: The matrix has eigenvalues* $3, 2, -1$ *in (b), and* $1, 2, -1, -2$ *in (c).*]

178. Suppose the nth pass through a manufacturing process is modelled by the linear equations $x_n = A^n x_0$, where x_0 is the initial state of the system and

$$A = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

Show that

$$A^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + (\frac{1}{5})^n \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Then, with the initial state $x_0 = \begin{bmatrix} p \\ 1-p \end{bmatrix}$, calculate $\lim_{n\to\infty} x_n$.

179. Two companies, Lemon and LIME, introduce a new type of computer. At the start, their shares of the market are 60% and 40%. After a year, Lemon kept 85% of its customers and gained 25% of LIME's customers; LIME gained 15% of Lemon's customers and kept 75% of its customers. Assume that the total market is constant and that the same fractions shift among the firms every year.

    (a) Write down the market share shift as a system of linear equations.
    (b) Express the shift in matrix form and find the transition matrix A.
    (c) Find the market shares after 5 and 10 years.
    (d) Show that the market eventually reaches a steady state, and give the limit market shares.

6.5 – Conics (AR §7.3)

180. *Conics.* Name and sketch the following conics:

    (a) $x^2 + y^2 = 4$
    (b) $x^2 + 4y^2 = 1$
    (c) $3x^2 - 4y^2 = 4$
    (d) $x - 4y^2 = 0$

181. For each of the following conics, name the conic, give the direction of the principal axes, and sketch the curve (in the xy-plane).

    (a) $2x^2 - 4xy - y^2 + 8 = 0$
    (b) $5x^2 + 4xy + 5y^2 = 9$
    (c) $11x^2 + 24xy + 4y^2 - 15 = 0$

Topic 7: Linear Transformations

7.1 – General linear transformations (AR §8.1–8.3)

182. Show that each of the following maps is a linear transformation:

 (a) $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(x, y) = (2x - y, x + y)$,

 (b) $T : \mathbb{R}^3 \to M_{2,2}$ given by $T(x, y, z) = \begin{bmatrix} y & z \\ -x & 0 \end{bmatrix}$.

183. Determine whether or not the given map is a linear transformation, and justify your answer.

 (a) $F : \mathbb{R}^3 \to \mathbb{R}^2$, $F(x, y, z) = (0, 2x + y)$

 (b) $K : \mathbb{R}^2 \to \mathbb{R}^4$, $K(x, y) = (x, \sin y, 2x + y)$

184. Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be vectors in a vector space V and $T : V \to \mathbb{R}^3$ a linear transformation for which $T(\boldsymbol{v}_1) = (1, -1, 2)$, $T(\boldsymbol{v}_2) = (0, 3, 2)$, and $T(\boldsymbol{v}_3) = (-3, 1, 2)$.
 Find $T(2\boldsymbol{v}_1 - 3\boldsymbol{v}_2 + 4\boldsymbol{v}_3)$.

7.2 – Linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ (AR §4.9-4.11)

185. For the linear transformations of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by the following matrices:
 (i) Sketch the image of the rectangle with vertices $(0, 0), (2, 0), (0, 1), (2, 1)$.
 (ii) Describe the geometric effect of the linear transformation.

 (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

 (d) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ (e) $\begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}$ (f) $\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

186. *Invariant lines.* Find 1-dimensional subspaces of $\mathbb{R}^2$ invariant under the linear transformations given by the following matrices:

 (a) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$

187. *Rotations.* Show that there is no line in the real plane $\mathbb{R}^2$ through the origin which is invariant under the transformation whose matrix is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

 when θ is not an integral multiple of π. Give a geometric interpretation of this problem commenting on the case when $\theta = k\pi$ for some $k \in \mathbb{Z}$.

188. Find the matrix of the following linear transformations of $\mathbb{R}^2$.

 (a) rotation by $\frac{3\pi}{4}$ (b) rotation by $-\frac{\pi}{2}$

 (c) reflection in the line $y = x$ (d) reflection in the $x-$axis

189. In each part, find a single matrix that performs the indicated succession of operations.

 (a) Compresses by a factor of $\frac{1}{2}$ in the x-direction, then expands by a factor of 5 in the y-direction.

 (b) Reflects about $y = x$, then rotates about the origin through an angle of π.

 (c) Reflects about the y-axis, then expands by a factor of 5 in the x-direction, and then reflects about $y = x$.

190. Find the matrix that rotates a point (x, y) about the origin through
    (a) $\frac{\pi}{4}$ $(45°)$ anticlockwise (b) π $(180°)$

7.3 – Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ (AR §4.9–4.11)

The following functions are all linear transformations.
Use them in questions 191 to 193.

$K(x, y, z) = (x, x + y, x + y + z)$ $L(x, y, z) = (2x - y, x + 2y)$
$S(x, y, z) = (z, y, x)$ $T(x, y) = (2x + y, x + y, x - y, x - 2y)$

191. Find the matrix that represents each of the following linear transformations (with respect to the standard bases).
    (a) K (b) L
    (c) S (d) T

192. *Combining linear transformations.* Find the indicated linear transformation if it is defined. If it is not defined, explain why not.
    (a) $LK(= L \circ K)$ (b) $TL(= T \circ L)$ (c) S^2
    (d) $K + S$ (e) T^2

193. Find the matrix which represents those linear transformations in question 192 which exist. (Use your results from the previous questions.)

194. Find the matrices of the transformations T which orthogonally project a point (x, y, z) onto the following subspaces of $\mathbb{R}^3$. Show by two methods that each transformation is idempotent (i.e., $T \circ T = T$).
    (a) The z-axis.
    (b) The straight line $x = y = 2z$.
    (c) The plane $x + y + z = 0$.

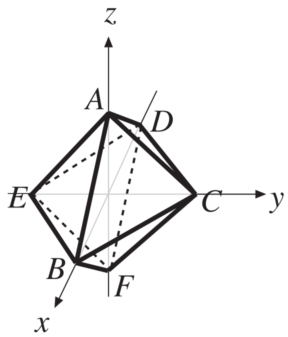
195. *Computer graphics.* One of the most important applications of linear transformations is computer graphics where we wish to view 3-dimensional objects (for example a crystal) on a 2-dimensional screen. The screen is the xy-plane. The aim is to rotate the crystal and orthogonally project it onto the xy-plane to obtain different views of it. We consider 3 possible rotations:

    - A rotation of θ round the x-axis using the matrix $R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
    - A rotation of θ round the y-axis using the matrix $R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
    - A rotation of θ round the z-axis using the matrix $R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

    The matrix used to orthogonally project the object onto the computer screen is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

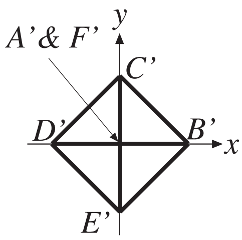
    A crystal has vertices $A : (0, 0, 1)$, $B : (1, 0, 0)$, $C : (0, 1, 0)$, $D : (-1, 0, 0)$, $E : (0, -1, 0)$, $F : (0, 0, -1)$ with edges the line segments AB, AC, AD, AE, FB, FC, FD, FE, BC, CD, DE and EB.



Projecting onto the xy-plane, we find $P(A) = A'$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In the same way $B' = (1,0)$, $C' = (0,1)$, $D' = (-1,0)$, $E' = (0,-1)$ and $F' = (0,0)$. Connecting up the line segments appropriately we find the view on the computer screen is



Draw the picture that would appear on the computer screen if:

(a) the crystal is rotated 45° around the x-axis before projection,

(b) the crystal is rotated 45° around the x-axis and then 30° around the z-axis,

(c) the crystal is rotated 45° around the x-axis, 30° around the z-axis and then −60° around the y-axis.

You might like to think about whether or not orthogonal projection is the best way to project from 3-D to 2-D.

7.4 – General Matrix Representations (AR §8.4)

196. Let $T : \mathcal{P}_2 \to \mathcal{P}_3$ denote the function defined by multiplication by x: $T(p(x)) = xp(x)$. In other words, $T(a + bx + cx^2) = ax + bx^2 + cx^3$.

(a) Show that T is a linear transformation.

(b) Find the matrix of T with respect to the standard bases $\{1, x, x^2\}$ for $\mathcal{P}_2$ and $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$.

197. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation defined by $T(p(x)) = p(2x + 1)$, that is,

$$T(a_0 + a_1x + a_2x^2) = a_0 + a_1(2x + 1) + a_2(2x + 1)^2.$$

Find $[T]_B$ with respect to the basis $B = \{1, x, x^2\}$.

198. Find the matrix A that represents the linear transformation T with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$.

(a) $T : \mathbb{R}^3 \to M_{2,2}$ given by

$$T(x, y, z) = \begin{bmatrix} y & z \\ -x & 0 \end{bmatrix}$$

where $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathcal{B}' = \{E^{ij} | i = 1, 2; j = 1, 2\}$ (i.e. the standard basis for $M_{2,2}$).

(b) $T : \mathcal{P}_3 \to \mathcal{P}_3$ given by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = (a_0 + a_2) - (a_1 + 2a_3)x^2$$

where $\mathcal{B}, \mathcal{B}' = \{1, x, x^2, x^3\}$.

7.5 – Kernel, image, rank and nullity (AR §8.1–8.2, 4.7–4.8)

199. Determine whether or not $\boldsymbol{v}_1 = (-2, 0, 0, 2)$ and $\boldsymbol{v}_2 = (-2, 2, 2, 0)$ are in the kernel of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & -4 & 6 & 2 \end{bmatrix}.$$

200. Determine whether or not $\boldsymbol{w}_1 = (1, 3, 1)$ or $\boldsymbol{w}_2 = (-1, -1, -2)$ is in the image of the linear transformation given in question 199.

201. For the linear transformation given in question 199, find the nullity of T and give a basis for the kernel of T. Is the transformation injective?

202. For the linear transformation given in question 199, find the rank of T and give a basis for the image of T. Is the transformation surjective?

203. For the linear transformation T find
(i) its standard matrix, (ii) a basis for the kernel and (iii) a basis for the image.

(a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 3y \end{bmatrix}$ (b) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 + x_2 \end{bmatrix}$

(c) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ -y \\ x - y \end{bmatrix}$ (d) $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 - 6x_3 \\ -2x_1 + x_2 + 5x_3 \\ 3x_1 + 3x_2 + 6x_3 \end{bmatrix}$

204. Compute $\ker(T)$ for each linear transformation in 198. Which ones are one-to-one?

205. Compute $\mathrm{Im}(T)$ for each linear transformation in 198. Which ones are onto?

206. Let V be the vector space of all real 2×2 matrices. Let $T : V \to \mathbb{R}^2$ be the map defined by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2a_{11} - a_{21} & 2a_{12} - a_{22} \end{bmatrix}.$$

(a) Show that T is a linear transformation.

(b) Find bases for the kernel and image of T. Deduce the rank and nullity of T.

(c) Find the matrix of T with respect to the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of V and the standard basis of $\mathbb{R}^2$.

207. Let $S : \mathcal{P}_2 \to \mathcal{P}_3$ be defined as follows. For each $p(x) = a_2x^2 + a_1x + a_0$, define $S(p) = \frac{1}{3}a_2x^3 + \frac{1}{2}a_1x^2 + a_0x$. Find the matrix A that represents S with respect to the bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{B}' = \{1, x, x^2, x^3\}$ (The linear transformation S gives the *integral* of $p(x)$, with the constant term equal to zero.)

208. Use the matrix of question 207 to find the integral of $p(x) = 1 - x + 2x^2$.

209. Determine whether or not the given linear transformation is invertible. If it is invertible, compute its inverse.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (x + z, x - y + z, y + 2z)$
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (3x + 2y, -6x - 4y)$
   (c) $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ an anticlockwise rotation around the origin through an angle of θ.
   (d) $T^\theta : \mathbb{R}^2 \to \mathbb{R}^2$ a reflection in the line through the origin which forms an angle θ with the x-axis.

210. Show that the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y \\ y + z \\ z + x \end{bmatrix}$$

is invertible and find its inverse.

211. Consider the matrix

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) Evaluate $\det A(\theta)$.
   (b) Interpret geometrically the effect of multiplying a vector by $A(\theta)$.
   (c) Show that $A(\theta)A(\phi) = A(\theta + \phi)$ and interpret this result.
   (d) Use the previous part to find the inverse of $A(\theta)$. How does this compare this with the transpose $A(\theta)^T$?

7.6 – Change of basis (AR §4.6, 8.5)

212. (a) Find the transition matrix P from B to C, where

$$B = \{(1, -2, 1),\ (0, 3, 2),\ (1, 0, -1)\} \text{ and } C = \{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}.$$

   (b) Use P to find $[\boldsymbol{x}]_B$ (the coordinate vector of $\boldsymbol{x}$ with respect to B) if

      (a) $\boldsymbol{x} = (3, -2, 5)$ (b) $\boldsymbol{x} = (-2, 7, 4)$

213. Verify that the given set $\mathcal{B}$ is a basis for $\mathbb{R}^n$. Compute the change of basis matrix for each of the bases, and use it to find the coordinate vector of $\boldsymbol{v}$ with respect to $\mathcal{B}$.

   (a) $\mathcal{B} = \{(1, 2), (1, -2)\}, \boldsymbol{v} = (-1, 3), (n = 2)$
   (b) $\mathcal{B} = \{(1, 1, 1), (1, 0, 1), (-1, 1, 0)\}, \boldsymbol{v} = (3, -1, 1), (n = 3)$

214. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\mathbf{x}) = A\mathbf{x}$ where A is the given matrix.

   (a) $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$
   Find the matrix $[T]_\mathcal{B}$ that represents T with respect to the basis $\mathcal{B}$ of question 213a. Calculate the eigenvalues of A and $[T]_\mathcal{B}$.

   (b) $A = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 1 & 2 \\ -1 & -1 & 3 \end{bmatrix}$
   Find the matrix $[T]_\mathcal{B}$ that represents T with respect to the basis $\mathcal{B}$ of question 213b. Calculate the eigenvalues of A and $[T]_\mathcal{B}$.

215. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z) = (4x + y - 4z, -3x - y + 5z, x)$. Find the matrix $[T]_\mathcal{B}$ that represents T with respect to the basis $\mathcal{B}$ of question 213b. Calculate the eigenvalues of $[T]_S$ and $[T]_\mathcal{B}$.

216. Write down the matrix $[T]_B$ of T with respect to B, and compute the matrix $[T]_{B'}$ of T with respect to B', where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x_1, x_2) = (x_1 - 2x_2, -x_2)$, $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, and $\boldsymbol{u}_1 = (1, 0)$, $\boldsymbol{u}_2 = (0, 1)$, $\boldsymbol{v}_1 = (2, 1)$, $\boldsymbol{v}_2 = (-3, 4)$. Calculate the eigenvalues of $[T]_B$ and $[T]_{B'}$.

217. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has matrix

$$[T]_S = \begin{bmatrix} 13 & -4 & -5 \\ 15 & -4 & -6 \\ 18 & -6 & =7 \end{bmatrix}$$

with respect to the standard basis for $\mathbb{R}^3$. Find the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis

$$B = \{(1, 2, 1), (0, 1, -1), (2, 3, 2)\}.$$

Calculate the eigenvalues of $[T]_S$ and $[T]_B$.

218. The following change of basis arises in special relativity and is known as the *Lorentz transformation:*

$$x' = \gamma(x - vt)$$
$$t' = \gamma(\frac{-v}{c^2}x + t)$$

where v is the speed of a moving object, c is the speed of light (a constant) and $\gamma = (1 - (\frac{v}{c})^2)^{-\frac{1}{2}}$. Find the change of basis matrix $A(v)$ that converts (x', t')-coordinates to (x, t)-coordinates.

Answers for Topic 1: Linear equations

1.
(a) Linear (b) Nonlinear
(c) Linear (d) Nonlinear

2.
(a) $x = y = z = 0.$ (b) $x = 2\,,\ y = -1\,.$ (c) No solution.
(d) $x = 4/3\,,\ y = 3\,,\ z = 8/3\,.$ (e) $x = 1 + 4t\,,\ y = t;\ t \in \mathbb{R}.$
(f) $x_1 = 2 - 3t\,,\ x_2 = -1 - s + 2t\,,\ x_3 = s\,,\ x_4 = t;\ s, t \in \mathbb{R}.$

3.
(i) row-echelon form: b, c, d, e (ii) reduced row-echelon form: d, e

4.
(a) $x = -37\,,\ y = -8\,,\ z = 5\,.$ (b) No solution.
(c) $x = 4 - 2s - 3t\,,\ y = s\,,\ z = t;\ s, t \in \mathbb{R}.$

5.
(a) $x_1 = 1 + 2t\,,\ x_2 = t;\ t \in \mathbb{R}.$ (b) No Solution
(c) $u = 1/2 + s\,,\ v = 1 + 2s - t\,,\ w = s\,,\ z = t;\ s, t \in \mathbb{R}.$
(d) $x_1 = 3\,,\ x_2 = -2\,,\ x_3 = 1\,.$

6.
(a) $2a - b + c = 0$ (b) a, b and c may take any value.

7.
(a) For consistency $k = 36$. Then $x = 7 - 2z$ and $y = z - 1$ where z can have any real value.
(b) No solution if $k = 2$. If $k \neq 2$ then $x_1 = 2/(k-2)\,,\ x_2 = k/(k-2)\,,\ x_3 = (k-4)/(k-2)\,.$
(c) If $k = 29$ there are an infinite number of solutions given by $x = (11 - 3z)/2$ and $y = (7 - z)/2$, where z can take any real value. Otherwise there is the unique solution $x = 4$, $y = 3$, $z = 1$.

8. (a) and (b) no solution (c) multiple solutions: $x = -20 + 23\alpha, y = -\frac{31}{2} + \frac{35}{2}\alpha, z = \alpha$, where $\alpha \in \mathbb{R}$,
or in vector form: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 23 \\ \frac{35}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -20 \\ -\frac{31}{2} \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$

(d) unique solution: $x = 1$, $y = -1$, $z = 1$, or in vector form: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

9. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \\ 0 \\ 0 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

10.

(a) (i) $k = -2$: no solution; (ii) $k \in \mathbb{R}\backslash\{-2, 1\}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{k+2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

(iii) $k = 1$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(b) (i) $k = 2$: no solution; (ii) $k \in \mathbb{R}\backslash\{-1, 2\}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{k-2} \begin{bmatrix} k-2 \\ 1 \\ 2 \end{bmatrix}$;

(iii) $k = -1$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

(c) (i) $k = 4$: no solution; (ii) no such k; (iii) $k \in \mathbb{R}\backslash\{4\}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -k-4 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{k-4} \begin{bmatrix} k-6 \\ 1 \\ 0 \end{bmatrix}$.

(d) (i) $k = -5$: no solution; (ii) $k \in \mathbb{R}\backslash\{-5, 2\}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{k+5} \begin{bmatrix} -3k-3 \\ 4 \\ 4 \end{bmatrix}$;

(iii) $k = 2$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ -\frac{5}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$.

11.

(a) If $c = -2a + b$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 11 \\ 7 \end{bmatrix} + \frac{1}{7} \begin{bmatrix} a + 2b \\ 3a - b \\ 0 \end{bmatrix}$, $\alpha \in \mathbb{R}$

(b) For all a, b, c: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -7 & -8 & 10 \\ 7 & 10 & -9 \\ 7 & 7 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

12.

(a) $a = 1$, $b = 2$ and $c = -3$ giving $y = x^2 + 2x - 3$.

(b) $a = 1$, $b = 4$ and $c = 4$ giving $y = x^2 + 4x + 4$.

13. The equation of the circle is $x^2 + y^2 = 2x + 4y + 29$.

14. 6 days in England, 4 days in France and 4 days in Spain.

15. Frank is 29, Dave is 26 and Phil is 16.

16.

(a) $y_1 = s - t, y_2 = s + t, y_3 = s, y_4 = t$

(c) The general solution is: $x_1 = s - t + 1, x_2 = s + t + 2, x_3 = s, x_4 = t$.

Answers for Topic 2: Matrices and Determinants

17.

(a) 2x3 (b) Not defined (c) 2x3

(d) Not defined (e) 2x3 (f) 2x2

18.

(a) $\begin{bmatrix} 28 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 30 \end{bmatrix}$ (b) $\begin{bmatrix} 29 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 29 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 30 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 30 \end{bmatrix}$

19. $x = -7$, $y = 3$

20.

(a) $\begin{bmatrix} -1 & 4 & -1 \\ 5 & 0 & 5 \\ -1 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -12 & 8 \\ -5 & -5 & 0 \\ 3 & 2 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} -1-\lambda & 0 & 1 \\ 2 & -1-\lambda & 3 \\ 0 & 1 & -2-\lambda \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 3 & -2 \end{bmatrix}$

(e) $\begin{bmatrix} -1 & -4 & 3 \\ -6 & 7 & -3 \\ 5 & 1 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 8 & -6 & 16 \\ -1 & 1 & 2 \\ 1 & 1 & -3 \end{bmatrix}$

21. $A + B = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 5 \end{bmatrix}$, $A + C$ is not defined

22. $AB = \begin{bmatrix} 4 & 9 \end{bmatrix}$, $BC = \begin{bmatrix} 5 \\ 1 \\ 13 \end{bmatrix}$, $A^T C^T = (CA)^T = \begin{bmatrix} 3 & 1 \\ 9 & 3 \\ 3 & 1 \end{bmatrix}$

23.

(a) $\begin{bmatrix} 1 & 2 \\ -5 & -1 \\ 7 & 15 \end{bmatrix}$ (b) The matrices are incompatible.

(c) $[2 - 3i]$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 7 & 6 & -4 \\ 7 & 6 & -4 \end{bmatrix}$

(e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (f) $\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

24.

$$A(BC) = (AB)C = \begin{bmatrix} 14 & 21 & 7 \\ 19 & 14 & -48 \\ 15 & 26 & 55 \end{bmatrix}, \quad (AB)^T = B^T A^T = \begin{bmatrix} 8 & 19 & -3 \\ 3 & 0 & 9 \\ 4 & -5 & 2 \end{bmatrix}$$

$$AB + AC = A(B + C) = \begin{bmatrix} 10 & 6 & 2 \\ 24 & 5 & -8 \\ 0 & 15 & 16 \end{bmatrix}$$

25.

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

(d) $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

26. *Hint:* Enforce compatibility of matrix dimensions for matrix multiplication.

27.

(a) The matrices that commute with C are of the form $\begin{bmatrix} a & b \\ -5b & 5b+a \end{bmatrix}$.

(b) The only matrices that commute with D are the diagonal matrices $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

28. $A^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}$

29.

30.

31.

32. An RE form and *the* RRE form are given. Note: RE forms are not unique.

(a) REF: $\begin{bmatrix} 1 & -2 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) REF: $\begin{bmatrix} 4-3i & 4-3i & 5-10i & 8-11i \\ 0 & 0 & -10+10i & -3+11i \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 1 & 0 & 1.6+0.7i \\ 0 & 0 & 1 & 0.7-0.4i \end{bmatrix}$

(c) REF: $\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) REF: $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 6 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(e) REF: $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(f) REF: $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 6 & 21 \\ 0 & 0 & 0 & 33 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

33.

(a) $\begin{bmatrix} 1/2 & 0 \\ 3/2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

(d) Matrix is singular (i.e., has no inverse).

34.

35.

(a)

(b)

$$A^{-1} = \frac{1}{3}(4I - A) = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

36. The inverse matrix is $\frac{1}{16}\begin{bmatrix} 19 & -4 & -2 & -1 \\ 6 & 8 & -4 & -2 \\ -4 & 0 & 8 & -4 \\ -24 & 0 & 0 & 8 \end{bmatrix}$

Hence the solution is $x = 1$, $y = 1$, $z = 0$, $w = 0$.

37.

(a) $A = \begin{bmatrix} 2 & -3 \\ 3 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. $\mathbf{x} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 2 & 0 \\ 0 & 1 & \text{[illegible]} \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$. $\mathbf{x} = \begin{bmatrix} 4 \\ -4 \\ -3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. No solution.

38. It can not be obtained from I_3 via a single row operation.

39.

(a) (i) 2 (ii) 3.

(b) (i) 1 (ii) 2 (iii) 3.

40.

(a) 3 (b) -5 (c) 15

(d) 0 (e) 120 (f) a^3

41.

(a) -1 (b) -1 (c) 3

(d) 8 (e) 1

42.

(a) 2 (b) 15 (c) $\dfrac{-1}{720}$

43.

(a) $x(x+1)(1-2x)$, invertible if $x \neq -1, 0, 1/2$.

(b) $(1-\lambda)(\lambda+1)(\lambda-2)(\lambda+2)$, invertible if $\lambda \neq \pm 1, \pm 2$.

(c) 0. Not invertible for any value of k.

44.

(a) $\det A = \det A^T = ad - bc$

(b) Hint: First calculate A^{-1}.

45. 72

46. *Hint:* If $\det(P) \neq 0$, then P is invertible. Example: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

47. The solutions for part (b) are NOT unique.

(i) $\lambda = 1$ and $\mathbf{v} = \alpha(2, -3)$, $\lambda = 2$ and $\mathbf{v} = \beta(1, -1)$.

(ii) $\lambda = 7$ and $\mathbf{v} = \alpha(1, 2, 1)$, $\lambda = 1$ and $\mathbf{v} = \beta(-2, 1, 0) + \gamma(-1, 0, 1)$.

Answers for Topic 3: Euclidean Vector Spaces

48.

(a) $(5,1,1)$ (b) $(-3,-2,2)$ (c) $(-9,-3,1)$

(d) $\sqrt{14}$ (e) $\sqrt{17}$ (f) $\sqrt{17}$

(g) $\sqrt{14}+\sqrt{5}$ (h) $5\sqrt{17}$ (i) 1

49.

(a) $(1,3,0)$; $(-22,42,9)$; -25; $\sqrt{41}$; $\sqrt{110}$.

(b) $(4,-2,16)$; $(11,-28,125)$; -110; $\sqrt{466}$; $2\sqrt{179}$.

(c) $(1,0,1)$; $(5,9,-4)$; -1; $\sqrt{2}$; $\sqrt{6}$.

50.

(a) 0 (b) 6 (c) $5-2\pi$

51.

(a) $\arccos\left(\frac{1}{5}\right)$ (b) $\arccos\left(-\frac{1}{\sqrt{3}}\right)$

(c) $\frac{\pi}{2}$

52.

(a) $\frac{\pi}{2}$ (b) $\frac{2\pi}{3}$ (c) $\arccos\left(\frac{1}{\sqrt{15}}\right)$

53. $\overrightarrow{\mathrm{OB}}=\mathbf{a}+\mathbf{c}, \overrightarrow{\mathrm{FB}}=\mathbf{a}+2\mathbf{c}, \overrightarrow{\mathrm{DA}}=2\mathbf{a}, \overrightarrow{\mathrm{AC}}=\mathbf{c}-\mathbf{a}, \overrightarrow{\mathrm{EA}}=2\mathbf{a}+\mathbf{c}$

54.

55.

(a) $\left(\frac{1}{5},\frac{14}{5},\frac{13}{5}\right)$

(b) $\frac{1}{8}(5\mathbf{b}+3\mathbf{c})$ where $\mathbf{b}=\overrightarrow{\mathrm{AB}}$ and $\mathbf{c}=\overrightarrow{\mathrm{AC}}$.

56.

(a) 4 (b) -24 (c) $(11,-7,-10)$

(d) -38 (e) $(-22,14,20)$ (f) Not Defined

(g) -38 (h) Not Defined (i) -38

57.

(a) -21 (b) $(-10,-2,20)$

58.

59.

60.

61.

(a) Orthogonal when $x=0,-\frac{9}{2}$. Parallel when $x=1$.

(b) Orthogonal when $x=2,-3$, never parallel.

62.

(a) $\pm\frac{1}{3}(1,-2,2)$ (b) $\mathbf{u}=\frac{1}{2}(\mathbf{i}+\mathbf{j})+(\frac{5}{2}\mathbf{i}-\frac{5}{2}\mathbf{j}+\mathbf{k})$

63.

(a) $\frac{\pm 1}{\sqrt{6}}(2,1,-1)$ (b) $\frac{\pm 1}{\sqrt{22}}(-3\mathbf{i}-2\mathbf{j}+3\mathbf{k})$.

64.

(a) (i) $\sqrt{29}$, (ii) $\sqrt{14}$ (b) (i) $5\sqrt{10}/2$, (ii) $\frac{1}{2}\sqrt{6}$.

65. Area $=\frac{1}{2}\sqrt{115}$; Unit vector: $\pm\frac{1}{\sqrt{115}}(5,3,-9)$

66.

(a) (i) 226 (ii) 14

(b) $\frac{2}{3}$

67.

(a) $L = (2, 1, -3) + t(1, 2, 2);\ t \in \mathbb{R}$ or $x - 2 = \frac{y-1}{2} = \frac{z+3}{2}$.

(b) $L = (2, -3, 1) + t(1, 0, 0);\ t \in \mathbb{R}$ or in parametric form $x = 2 + t,\ y = -3,\ z = 1;\ t \in \mathbb{R}$.

(c) $L = (2, 0, -2) + t(1, -4, -4);\ t \in \mathbb{R}$ or $x - 2 = \frac{-y}{4} = \frac{z+2}{-4}$.

(d) $L = (2, 4, 5) + t(1, -1, -2);\ t \in \mathbb{R}$ or $x - 2 = 4 - y = \frac{5-z}{2}$.

68.

(a) Lines are parallel.

(b) Intersect at $(2, 0, 2)$.

(c) Lines are skew.

69.

(a) $7x + y + 4z = 31$ or $\mathbf{r} = (1, 4, 5) + s(4, 0, -7) + t(0, 4, -1);\ s, t \in \mathbb{R}$

(b) $x + y - z = 13$ or $\mathbf{r} = (6, 5, -2) + s(1, 0, 1) + t(0, 1, 1);\ s, t \in \mathbb{R}$

(c) $x - 2y + z = 0$ or $\mathbf{r} = (0, 0, 0) + s(1, 1, 1) + t(1, 2, 3);\ s, t \in \mathbb{R}$

(d) $25x + 14y + 8z = 77$ or $\mathbf{r} = (1, 6, -4) + s(0, -4, 7) + t(-8, 0, 25);\ s, t \in \mathbb{R}$

70.

(a) Collinear. Line is $L = (1, 2, 3) + t(2, -1, -3);\ t \in \mathbb{R}$

(b) Coplanar. Plane is $2x - 4y - z = -3$.

71.

(a) $(13/5, 1, -7/5)$

(b) $(1, 0, 0)$

72.

(a) $\pi/3$

(b) $\arccos\left(\frac{16}{\sqrt{14}\sqrt{29}}\right)$

(c) $\arccos\left(\frac{2}{7}\right)$

73.

(a) $x - 2y - z + 3 = 0$

(b) $x + y - z - 6 = 0$

Answers for Topic 4: General Vector Spaces

74.

(a) Vector space

(b) Not a vector space (No zero vector, not closed under scalar multiplication, etc.)

(c) Vector space

75. To show that it is a vector space, you need to show that all the axioms of the definition are satisfied. The zero vector is 1. The additive inverse of x is $\frac{1}{x}$.

76. (i) Start with $\mathbf{x} + \mathbf{0} = \mathbf{x}$, multiply both sides by λ.
(ii) Start with $\lambda + 0 = \lambda$, multiply both sides by $\mathbf{x}$.
(iii) Start with $1 + (-1) = 0$, multiply both sides by $\mathbf{x}$.

77.

(a) yes (b) no

78.

(a) $3\boldsymbol{u} + 2\boldsymbol{v}$

(b) $c = -2$

79. (a) Dependent: $\boldsymbol{v}_2 = 2\boldsymbol{v}_1$ (b) Dependent: $\boldsymbol{v}_4 = -\boldsymbol{v}_1 + 3\boldsymbol{v}_3$

(c) Independent (d) Independent

80. Yes

81. $-2p_1 + p_2 - 2p_3$

82. (a) Independent (b) Dependent.

83. (a) Dependent (b) Independent

(c) Dependent (d) Independent

84. (a) Dependent (b) Independent (c) Independent

85.

86. (a) Independent (b) $1 + x + 2x^2 = (1)(1 + x^2) + (1)(x + x^2)$

(c) Independent (d) Independent

87. (a) Yes, No, No (b) Yes, Yes, Yes (c) No, No, No (d) No, Yes, No.

88. (a) Yes (b) No (c) Yes (d) No (e) Yes

89.

90. Counter-examples required.

91. (a) Yes (b) Yes (c) No (d) Yes (e) No (f) Yes (g) Yes

92. (a) Subspace (b) Not a subspace

93. (a) Subspace (b) Subspace

94.

95. (a) No (b) Yes

96. (a) Yes (b) No

97. (a) Does not span (b) Does span

98. Spanning sets obtained below are not the only possible.
(a) $\{(2,0,0),\ (0,1,0)\}$ (b) $\{(1,0,0),\ (0,-1,0),\ (1,1,3)\}$ (c) $\{(4,1,0),(0,2,?1),(0,0,1),(1,0,0)\}$

99. Note: Spanning sets obtained below are not the only ones possible.

(a) $\{(1,0,1,0),(0,1,0,1)\}$ (b) $\{(1,0,1,1),(0,1,-1,1)\}$

(c) $\{(-1,1,0),(-1,0,1)\}$

100. (a) Spans (b) Does not span

(c) Spans (d) Does not span

101. (a) Not a basis (b) Is a basis
(c) Is a basis (d) Not a basis (or even a subset)

102. (a) Yes (b) No

103. The bases given are not the only ones possible.
(a) $\{(3,0,1),\ (0,1,-2)\},\ \dim W = 2$
(b) $\{(1,0),\ (0,1)\},\ \dim W = 2$
(c) $\{(1,-2,0,-4),\ (0,7,-1,14)\},\ \dim W = 2$

104. (a) $\{u_1, u_2, u_3\};\ u_4 = 2u_1 + u_2$ (b) $\{u_1, u_2, u_4\};\ u_3 = 2u_1 - 3u_2$
(c) $\{u_1, u_2, u_4\};\ u_3 = 2u_1 - u_2$

105.
(a) Have 4 vectors in a 3-dimensional space.
(b) Two vectors cannot span a 3−dimensional subspace.
(c) Four independent vectors span a 4−dimensional space.
(d) Have two linearly independent vectors for a 2−dimensional space.

106. (a) Basis (b) Not a basis
(c) Basis (d) Not a basis

107. (a) 2 (b) 2 (c) 3

108. (a) Yes (b) No

109. $\{(-1,1,0),(-1,0,1)\}$ (Not the only possibility!)

110. (a) No (b) Yes.

111. (a) No (b) Yes

112.

113.
(a) Any 4 vectors in a 3-dimensional space are linearly dependent.
(b) No 2 vectors can span a 3-dimensional space.

114.

115.

116. (a) (b)

117. (a) (b)

118.

119.
(a) $\{(1,1,1,0),(-2,-1,0,1)\};2$
(b) $\{(-1,0,1,0,0),(3,1,0,1,0),(-2,-1,0,0,1)\};3$
(c) $\{(1,0,1,0),(0,-1,-2,3)\};2$

120.
(a) Rank = 2 (b) Rank = 1 (c) Rank = 3

121. (a) $\{(1,2),(2,1\}$ (b) $\{(1,-1)\}$
(c) $\{(1,0,1,2),(-1,1,1,-1),(3,1,0,1)\}$

122. (a) $\{(1,2,1),(2,1,-1)\}$ (b) $\{(1,0,-1)\}$
(c) $\{(1,-1,3),(0,1,1),(1,1,0)\}$

123. (a) $\{(1,-1,1)\}$ (b) $\{(1,0,1),(0,1,0)\}$
(c) basis $= \emptyset$ (= empty set)

124. (a) $\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$ (b) $\left\{\begin{pmatrix}2\\3\\4\end{pmatrix}\right\}$

125. A basis is $\{(1,3,0),(4,0,3)\}$.

126. (a) (b) $(1,1,8),\ (-4,3,1)$ (c) $\begin{bmatrix}0\\0\\1\end{bmatrix},\ \begin{bmatrix}1\\1\\0\end{bmatrix},\ \begin{bmatrix}1\\1\\1\end{bmatrix}$

127. (a) $(2, -1, 3, 0)$ (b) $(1, 2, 1, -1, 1, 2)$

(c) $(-\frac{3}{2}, -\frac{7}{2})$ (d) $(-2, 3)$

128.

(a) Independent

(b) Dependent; $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

Answers for Topic 5: Inner Product Spaces

129. (a) 3 (b) -29
130.
131. No; (i) $\langle(0,1),(0,1)\rangle = -1 < 0$; (ii) $\langle(1,1),(1,1)\rangle = 0$.
132.
133. (a) Yes (b) No (c) No (d) No (e.g. $\langle(0,0,1),(0,0,1)\rangle = 0$)
134. (a) Yes (b) No (c) Yes.
135. $\sqrt{17}$
136. $\sqrt{74}$
137. $3\sqrt{2}$
138. $\sqrt{105}$
139.
   (a) $\|\mathbf{x}\| = \sqrt{2}, \quad \|\mathbf{y}\| = 1, \quad \theta = \frac{1}{4}\pi$
   (b) $\|\mathbf{x}\| = 2, \quad \|\mathbf{y}\| = \sqrt{3}, \quad \theta = \frac{1}{6}\pi$
140.
141.
142.
143. (a) Yes (b) No.
144. yes, no, yes
145. (a) $-\dfrac{1}{\sqrt{2}}$ (b) 0
146. Show $\langle p,q\rangle = 0$.
147. (a) Yes (b) No
148.
149. (a) (b) (c)
150.
   (a) $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}) + 2(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}) + 3(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ (b) $\frac{1}{3}(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}) + \frac{4}{3}(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}) - \frac{1}{3}(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$
151.
152.
   (a) $\frac{1}{\sqrt{2}}(1,0,1,0), \quad \frac{1}{\sqrt{10}}(1,2,-1,2), \quad \frac{1}{\sqrt{10}}(2,-1,-2,-1)$
   (b) $\frac{1}{3}(2,2,-1,0), \quad \frac{1}{3}(0,1,2,-2), \quad \frac{1}{3}(1,0,2,2)$
   (c) $\frac{1}{4}(1,-2,1,3,-1), \quad \frac{1}{\sqrt{3}}(1,1,0,0,-1)$ – the space has dimension 2
153. $\dfrac{1}{\sqrt{2}}, \ \dfrac{\sqrt{3}}{\sqrt{2}}x, \ \dfrac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \dfrac{1}{3}\right)$
154.
155. $p_2(x) = 1 + \pi - 2\sin x - \sin 2x$ (b) $p_k(x) = 1 + \pi - 2\displaystyle\sum_{n=1}^{k} \frac{\sin nx}{n}$
156.
   (a) $\dfrac{1}{9}(5x - 2y + 4z, -2x + 8y + 2z, 4x + 2y + 5z)$
   (b) $\dfrac{1}{14}(5x + 6y + 3z, 6x + 10y - 2z, 3x + -2y + 13z)$
157. (a) (b) (c)
158.
   (a) $p(x) = -\frac{4}{5} + \frac{13}{10}x$
   (b) $p(x) = -\frac{4}{7} + \frac{1}{10}x + \frac{11}{14}x^2$
159. 1.55 metres
160.
   (a) $p(x) = 303 - 86x$
   (b) $p(x) = 173 + 62.6x - 37.1x^2$

Answers for Topic 6: Eigenvalues and Eigenvectors

161.

(a) $\lambda = 2,\ u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(b) $\lambda_1 = 1,\ u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$; $\lambda_2 = 2,\ u_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

(c) $\lambda = 2,\ u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(d) complex eigenvalues and eigenvectors: $\lambda_1 = i,\ u_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$, $\lambda_2 = -i,\ u_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

162. (a) 1, 3, 4 (b) 1, 3, 6, 10

163.

164.

(a) $\lambda_1 = 2,\ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\lambda_2 = 2,\ u_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$; $\lambda_3 = 3,\ u_3 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$

(b) $\lambda_1 = 2,\ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\lambda_2 = -3,\ u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(c) $\lambda_1 = -1,\ u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$; $\lambda_2 = 1,\ u_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$; $\lambda_3 = -2,\ u_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$

(d) $\lambda_1 = 2,\ u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$; $\lambda_2 = 1,\ u_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\lambda_3 = 1,\ u_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

165. We write $W(\lambda)$ for the eigenspace corresponding to λ. The spanning sets for $W(\lambda)$ listed below are bases.

(a) $\lambda_1 = \lambda_2 = 2,\ W(2) = \langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rangle$; $\lambda_3 = 6,\ W(6) = \langle \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rangle$

(b) $\lambda_1 = \lambda_2 = \lambda_3 = 1,\ W(1) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle$

The Cayley-Hamilton Theorem asserts that every square matrix satisfies its own characteristic equation. That is, if

$$p(\lambda) = \det(\lambda I - A) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$$

is the characteristic polynomial of an $n \times n$ matrix A, then

$$a_0 I + a_1 A + \cdots + a_{n-1}A^{n-1} + A^n = \mathbf{0}$$

(where $\mathbf{0}$ is the $n \times n$ zero matrix).

166. (a) (b)

167. (a) $t^2 + t - 11$ (b) $(t-1)(t^2 - 2t - 5)$

168. $\begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

169. **161:**

(a) not diagonalizable

(b) diagonalizable; $P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(c) not diagonalizable

(d) diagonalizable (using complex D, P);
$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$, $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

164:

(a) diagonalizable;
$P = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 3 & -6 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) not diagonalizable

(c) diagonalizable;
$P = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 3 \\ 1 & 1 & -2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(d) diagonalizable;
$P = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ -1 & -2 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

170. (a) Orthogonal (b) Not orthogonal

171.

172.

(a) $P = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $P = \frac{1}{5}\begin{bmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ -3 & 0 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$

(c) $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

173.

174.

175.

176.

177.

(a) $A^5 = PD^5P^{-1} = \begin{bmatrix} 43 & -22 \\ 22 & -12 \end{bmatrix}$ (b) $A^5 = PD^5P^{-1} = \begin{bmatrix} 1509 & 198 & -1464 \\ 1477 & 230 & -1464 \\ 1477 & 231 & -1465 \end{bmatrix}$

(c) $A^5 = PD^5P^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 31 & 357 & 155 \\ 0 & 258 & 254 & -6 \\ 0 & 62 & -318 & -186 \\ 0 & 68 & 188 & 52 \end{bmatrix}$

178. Hint: First diagonalize the matrix; eigenvalues are $1, 1/5$.
$\lim_{n\to\infty} x_n = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

179. Write x_i = Lemon's share and y_i = LIME's share after i years.

(a) $x_{i+1} = 0.85x_i + 0.25y_i$, $y_{i+1} = 0.15x_i + 0.75y_i$ with $x_0 = 0.60$ and $y_0 = 0.40$.

(b) $\begin{bmatrix} x_i \\ y_i \end{bmatrix} = A^i \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ where $A = \begin{bmatrix} .85 & .25 \\ .15 & .75 \end{bmatrix}$.

(c) $\begin{bmatrix} 0.6231 \\ 0.3769 \end{bmatrix}$, $\begin{bmatrix} 0.6248 \\ 0.3752 \end{bmatrix}$ (d) Lemon 62.5%, LIME 37.5%.

180. (a) Circle (b) Ellipse (c) Hyperbola (d) Parabola.

181.

(a) hyperbola, principal-axis directions: (-2,1), (1,2), standard form: $-3(x')^2 + 2(y')^2 = 8$

(b) ellipse, principal-axis directions: (1,-1), (1,1), standard form: $3(x')^2 + 7(y')^2 = 9$

(c) hyperbola, principal-axis directions: (-3,4), (4,3), standard form: $-5(x')^2 + 20(y')^2 = 15$

Answers for Topic 7: Linear Transformations

182. (a) (b)

183. (a) (b)

184. $(-10, -7, 6)$

185. (The constants a, b, c are assumed nonzero.)
   (a) Reflection in the line $y = x$.
   (b) Orthogonal projection onto the x-axis followed by the reflection in the line $y = x$.
   (c) Projection onto the x-axis in the direction of the line $y = -x$.
   (d) Shear parallel to the y-axis.
   (e) Expansion by factor $|b|$ along the x-axis and by $|c|$ along the y-axis, with a possible change of direction of the axis (if $b < 0$ or $c < 0$).
   (f) Rotation about 0 through $\theta = \arctan \frac{4}{3}$.

186.
   (a) $\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rangle$ and $\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$ (b) $\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix} \rangle$ and $\langle \begin{bmatrix} 1 \\ -3 \end{bmatrix} \rangle$

187. Rotation through θ.
For $\theta = k\pi$, k odd, $A_\theta = -I$ and for $\theta = k\pi$, k even, $A_\theta = I$. In both cases *all* lines are invariant.

188. (a) $\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

189. (a) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$

190. (a) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

191. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -2 \end{bmatrix}$

192.
   (a) $LK(x, y, z) = (x - y, 3x + 2y)$
   (b) $TL(x, y, z) = (5x, 3x + y, x - 3y, -5y)$
   (c) $S^2(x, y, z) = (x, y, z)$
   (d) $(K + S)(x, y, z) = (x + z, x + 2y, 2x + y + z)$
   (e) Not defined (domain $T \not\subseteq$ range T)

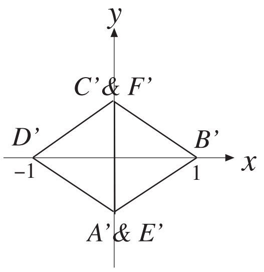
193. (a) $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -5 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

194.

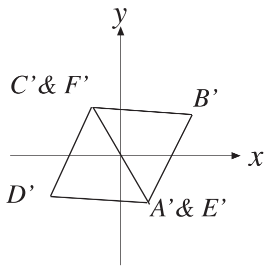
(a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\frac{1}{9}\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ (c) $\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

195.

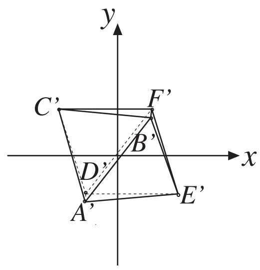
(a) A':(0,–.707)
B':(1,0)
C':(0, .707)
D':(–1,0)
E':(0,–.707)
F':(0, .707)



(b) A':(.354,–.612)
B':(.866, .500)
C':(–.354, .612)
D':(–.866,–.500)
E':(.354,–.612)
F':(–.354, .612)



(c) A':(–.436,–.612)
B':(.433, .500)
C':(–.789, .612)
D':(–.433,–.500)
E':(.789,–.612)
F':(.436, .612)



196.

(a) (b) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

197. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

198. (a) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

199. Both are.

200. $\boldsymbol{w}_1$ is not; $\boldsymbol{w}_2$ is.

201. Nullity 2; basis for $\ker(T)$ is $\{(-1,1,1,0),(-1,0,0,1)\}$; not injective

202. Rank 2; basis for $\mathrm{Im}(T)$ is $\{(1,1,2),(2,0,-4)\}$; not surjective

203.

(a) (i)$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ (ii) $\ker T = \{\mathbf{0}\}$ (iii)$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$; $\mathrm{Im}\, T = \mathbb{R}^2$

(b) (i)$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (ii) $\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$ (iii)$\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$;
$\mathrm{Im}\, T = \mathbb{R}^2$

(c) (i)$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$ (ii) $\ker T = \{\mathbf{0}\}$ (iii)$\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\right\}$

(d) (i)$A = \begin{bmatrix} 3 & -1 & -6 \\ -2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$ (ii) $\left\{\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}\right\}$ (iii)$\left\{\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}\right\}$

204.

(a) $\ker(T) = \{\mathbf{0}\}$; one to one
(b) $\ker(T) = \{-a - 2bx + ax^2 + bx^3 | a, b \in \mathbb{R}\}$; not one to one

205.

(a) $\mathrm{Im}(T) = \left\{\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R}\right\}$ (b) $\mathrm{Im}(T) = \{a + bx^2 | a, b \in \mathbb{R}\}$

206.

(a)

(b) The kernel of T has a basis $\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 2\end{bmatrix}$; the image of T has a basis $\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}$; rank$(T) = 2$, nullity$(T) = 2$

(c) $\begin{bmatrix}2 & 0 & -1 & 0\\0 & 2 & 0 & -1\end{bmatrix}$

207. $\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & \frac{1}{2} & 0\\0 & 0 & \frac{1}{3}\end{bmatrix}$

208. $x - x^2/2 + 2x^3/3$

209.

(a) $T^{-1}(x, y, z) = (\frac{3}{2}x - \frac{1}{2}y - \frac{1}{2}z, x - y, -\frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z)$

(b) Not invertible

(c) $T_\theta^{-1} = T_{-\theta} = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$

(d) T_θ is its own inverse.

210.

$$T^{-1}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}x - y + z\\x + y - z\\-x + y + z\end{bmatrix}$$

211.

(a) 1 (b) Rotation about the z-axis through θ (c)

(d) Inverse of $A(\theta) = A(-\theta) = \begin{bmatrix}\cos\theta & \sin\theta & 0\\-\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix} = A(\theta)^T$

212.

(a) $P = \begin{bmatrix}1 & 0 & 1\\-2 & 3 & 0\\1 & 2 & -1\end{bmatrix}$, $P^{-1} = \dfrac{1}{10}\begin{bmatrix}3 & -2 & 3\\2 & 2 & 2\\7 & 2 & -3\end{bmatrix}$

(b) (i) $\dfrac{1}{5}\begin{bmatrix}14\\6\\1\end{bmatrix}$ (ii) $\dfrac{1}{5}\begin{bmatrix}-4\\9\\-6\end{bmatrix}$

213.

(a) $\frac{1}{4}\begin{bmatrix}2 & 1\\2 & -1\end{bmatrix}$; $\frac{1}{4}(1, -5)$

(b) $\begin{bmatrix}1 & 1 & -1\\-1 & -1 & 2\\-1 & 0 & 1\end{bmatrix}$; $(1, 0, -2)$

214.

(a) $P^{-1}AP = \begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}$, $\lambda = -1, 1$

(b) $P^{-1}AP = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$, $\lambda = 1, 2, 3$

215. $P^{-1}AP = \begin{bmatrix}1 & 1 & 0\\0 & 0 & -1\\0 & 1 & 2\end{bmatrix}$, $\lambda = 1, 1, 1$

216. $\begin{bmatrix}1 & -2\\0 & -1\end{bmatrix}$, $\frac{1}{11}\begin{bmatrix}-3 & -56\\-2 & 3\end{bmatrix}$, $\lambda = 1, -1.$

217. $\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 2\end{bmatrix}$, $\lambda = 1, -1, 2.$

218. $\gamma\begin{bmatrix}1 & v\\v/c^2 & 1\end{bmatrix}$