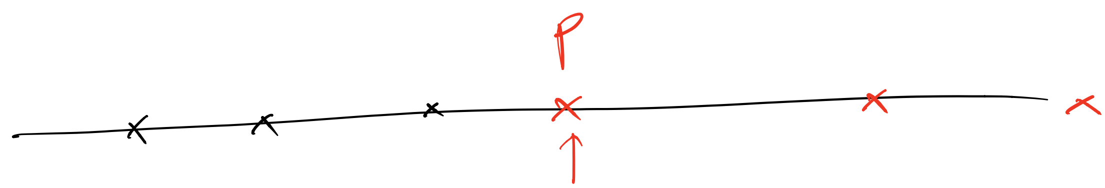
Q1.

N_t : # yeast microb. flat by
up to time t

$$N_t \sim PP(2)$$



(a) $P(N_3 = 4) = P(\text{Poi}(6) = 4)$

(b) Poi. thinning

$$M_t \sim PP(2p)$$

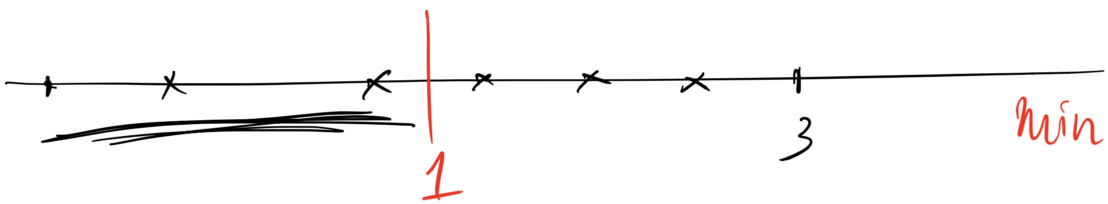
M_t : # microb join the culture

$$P(M_3 = 4)$$

(c) $P(M_3 \geq 2 \mid N_3 = 7)$

$M_t \mid N_t = n \sim Bin(n, p)$

(d) $P(N_1 = 3 \mid N_3 = 7) =$ ____



→ · given # poisson pts, the location of pts iid.
$\sim Unif(0, t)$

→ $N_1 \mid N_3 = 7 \sim Bin(7, \frac{1}{3})$.

conditional exp.

(e) $P(M_t = 4, N_t - M_t = 3)$

use indep. M_t, $N_t - M_t$

$\uparrow$

numbers don't join the culture

$N_t - M_t \sim PP(2(1-p))$

· K_t ~ # numbers of 2nd kind that float by.

$K_t \sim PP(1)$

L_t: # numbers of 2nd kind joins the culture.

$L_t \sim PP(q)$

(f) $P(N_3 + M_3 = 4)$

$N_t + M_t \sim PP(1+2)$

(g) $P(K_3 + M_3 = 4)$

$\vdots$

Q6. $Y_t = \sum_{j=1}^{N_t} X_j$

$N_t \sim PP(\underbrace{r\log(\cancel{p})}_{= \lambda})$

$\rightarrow E(e^{\theta Y_t}) = \varphi_t(\theta)$

$= E\{\exp\{\theta \sum_{j=1}^{N_t} X_j\}\}$.

$= E[E\{\exp\{\theta \sum_{j=1}^{N_t} X_j\} | N_t\}]$.

use X_j indep

$= E[\prod_{j=1}^{N_t} E(e^{\theta X_j})]$

X_j same distn

$= E[(E[e^{\theta X_j}])^{N_t}]$ ← (2)

pgf of Poi(λt)

$$\varphi_X(\theta) = E(e^{\theta X_1})$$

$$= \sum_{j \geq 1} e^{\theta j} \frac{(1-p)^j}{j \log(\frac{1}{p})}$$

$$= \frac{1}{\log(\frac{1}{p})} \sum_{j \geq 1} \frac{(e^{\theta}(1-p))^j}{j}$$

McClaurin series

$$\log(1-x) = -\sum_{j \geq 1}^{\infty} \frac{x^j}{j},$$

$|x| < 1$

Take $x = e^{\theta}(1-p)$

Assume $|e^{\theta}(1-p)| < 1$

$$\to \varphi_X(\theta) = \frac{-\log(1-e^{\theta}(1-p))}{\log(\frac{1}{p})} \quad (1)$$

plug (1) into (2):

$$E(e^{\theta Y_t}) = E\left[\left(\frac{-\log(1-e^{\theta}(1-p))}{\log(\frac{1}{p})}\right)^{N_t}\right]$$

$\vdots$ use pgf of $Poi(\lambda t)$ and calc ...

$$= \left(\frac{p}{1-e^{\theta}(1-p)}\right)^{rt},$$

which is the mgf of $NB(rt, p)$. //

Q4. I $N \sim \text{geo}(p)$ on $\{0, 1, 2, \dots\}$

$$f_{X|N=n}(x|n) = \frac{x^n e^{-x}}{n!}$$

$$\Rightarrow \underline{f(x)} = \sum_{n=0}^{\infty} f_{X|N=n}(x|n) P(N=n)$$

$$\vdots$$

$$= \underline{pe^{-px}}$$

$$\Rightarrow X \sim \exp(p)$$

X : Arrival time of first pt of PP(p)

(I) $S \sim \exp(\lambda) \leftarrow$

$$M \mid S = s \sim \text{Poi}(s)$$

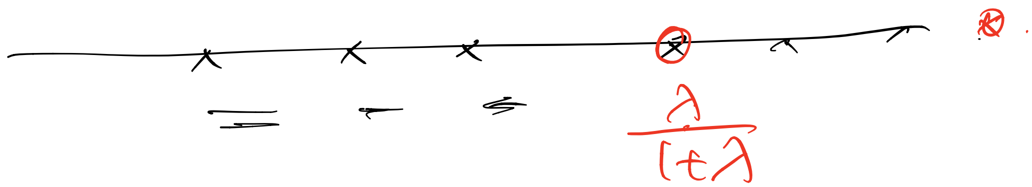
$$P(M=m) = \int_0^\infty f_S(s) \cdot \frac{e^{-\lambda s} \cdot s^m}{m!} \, ds$$

$$\vdots$$

$$= \left(\frac{\lambda}{1+\lambda}\right)\left(1 - \frac{\lambda}{1+\lambda}\right)^m$$

$\text{geo}\left(\frac{\lambda}{1+\lambda}\right)$ on $\{0, 1, 2, \dots\}$

$PP(1+\lambda)$



marking red = 'success'

M : # points until we mark the first red pt.

Thinning.

red pts up to time t

$$\sim PP\left(\frac{\lambda}{1+\lambda}\cdot(1+\lambda)\right)$$

$$\parallel$$

$$PP(\lambda)$$

(III) $\begin{cases} K \sim Poi(\mu) \text{ on } \{0,1,2,\cdots\} \\ J \mid K=k \sim Bin(k,p). \end{cases}$

$\underbrace{P(J=j)}$ $\Longrightarrow$ $J \sim Poi(\mu p)$

K : # pts up to time μ of $PP(1)$

J : # thinned pts on $(0,\mu]$.

Q2. ∘ $PP(\lambda)$.

∘ T_i : Arrival time of the i-th point.

∘ $\underbrace{f_{T_1, T_2}(t_1, t_2)}$.

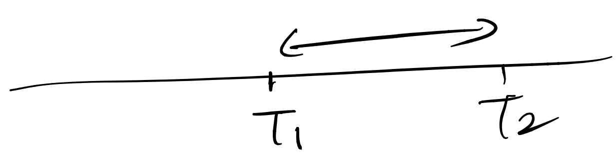
· $T_1 \sim \exp(\lambda)$

· $T_2 - T_1 \sim \exp(\lambda)$, indep of T_1

$f_{T_1, T_2}(t_1, t_2)$

$= \underbrace{f_{T_1}(t_1)} \cdot f_{T_2 | T_1}(t_2 | t_1)$.

$= \lambda e^{-\lambda t_1} \cdot \lambda e^{-\lambda(t_2 - t_1)}$.



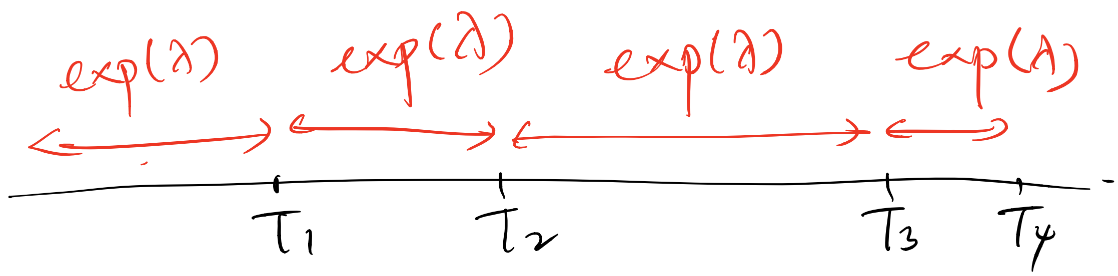
$$= \lambda^2 e^{-\lambda t_2}.$$

- $i < j \Rightarrow T_i < T_j$

$f_{T_j, T_i}(t_j, t_i)$ ✓

$\rightarrow T_i \sim \text{Gamma}(i, \lambda)$

$\rightarrow T_j - T_i \sim \text{Gamma}(j-i, \lambda)$



$$f_{T_j, T_i}(t_j, t_i) = f_{T_j \mid T_i}(t_j \mid t_i)\, f_{T_i}(t_i)$$

plug gamma densities

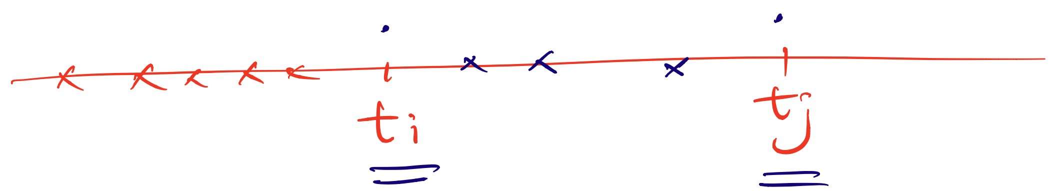
$\vdots$ some rearrangements.

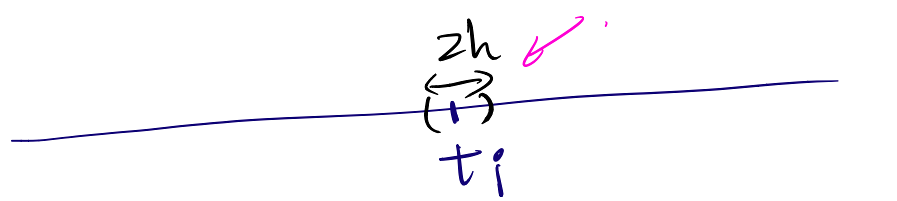
$$= \lambda^2 \underbrace{\frac{(\lambda t_i)^{i-1} e^{-\lambda t_i}}{(i-1)!}} \cdot \underbrace{\frac{(\lambda(t_j-t_i))^{j-i-1} e^{-\lambda(t_j-t_i)}}{(j-i-1)!}}$$

prob. seeing pts AT t_i and t_j.

prob. $i-1$ pts in $(0, t_i)$

prob. that $j-i-1$ pts at (t_i, t_j)





$$P(N_{t_i+h} - N_{t_i-h} > 0)$$

$$= P(Poi(2\lambda h) > 0)$$

$$\lim_{h\to 0} \frac{P(N_{t_i+h} - N_{t_i-h} > 0)}{2h}$$

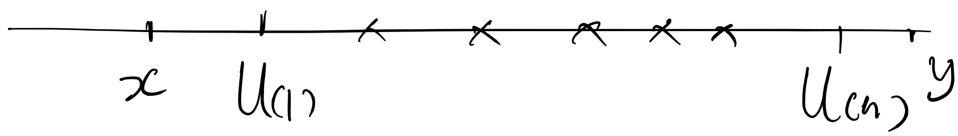
$$= \lim_{h\to 0} \frac{1 - e^{-2\lambda h}}{2h}$$

l'hôpital
$$= \lim_{h\to 0} \frac{\cancel{2}\lambda e^{-2\lambda h}}{\cancel{2}}$$

$$= \lambda$$

$$P(N_{t_i+h} - N_{t_i-h} > 0) \approx 2h \times \lambda$$

3a.



$$P(U_{(1)} > x, U_{(n)} < y)$$

$$= P(\text{all pts inside } (x-y))$$

$$= (y-x)^n$$