MAST30025_2021_SM1 MAST30025 assignment 1

Michael Le

TOTAL POINTS

38 / 40

QUESTION 1

7 pts

1.1 5 / 5

1.2 1 / 2

QUESTION 2

2 4 / 4

QUESTION 3

3 4 / 4

QUESTION 4

10 pts

4.1 2 / 2

4.2 2 / 2

4.3 3 / 3

4.4 2 / 3

- You must use theorem 3.5 directly. i.e. what is the dist. of x/sigma^2?

QUESTION 5

15 pts

5.1 2 / 2

5.2 2 / 2

5.3 2 / 2

5.4 2 / 2

5.5 2 / 2

5.6 2 / 2

5.7 3 / 3

MAST30025: Linear Statistical Models Assignment 1 S1 2021

Michael Le (998211)

March 24, 2021

Question 1 Solution:

Part a:

A^2 =A^3

Suppose A is a square matrix is (real and) symmetric then its eigenvalues are all real, and its eigenvalues are orthogonal.

***Theorem* 2.3**

Proof:

Take A to be a square matrix, n x n. First we diagonalise A,i.e., find P such that.

=

$$D = P^T AP$$

$$= \begin{bmatrix} \lambda_1 & \dots\dots\dots & 0 \\ \dots\dots & \dots\dots\dots & .. \\ 0 & \dots\dots\dots & \lambda_k \end{bmatrix}$$

where $\lambda_1,\ \lambda_2, \dots\dots\dots,\ \lambda_k$ *are the eigenvalues of A.*

Since P is orthogonal both P and P^T *are non − singular,*

$$r(P^T AP) = r(p^T A) = r(A)$$

Because P^T *AP is diagonal* $r(P^T AP)$ *is the number of non zero eigenvalues of A.*

But we wanted to prove ***Theorem* 2.2**

that A any symmetric matrix is idempotent. Which has eigenvalues of $\lambda =$ 0 or $\lambda = 1$.

The eigenvalues of idempotent matrices are always either

$\lambda = 0$ or $= 1$.

$A^2 = \lambda^2$x

Multiplying by A!!!

A^3x $= A^2\ \lambda$x $= \lambda\ A^2$x $= \lambda^3$ x

$(\lambda^3$ - $\lambda^2)\boldsymbol{x} = 0$

By definition, x$\neq$ 0,

λ^3 - $\lambda^2 = 0$
$\lambda^2(\lambda$ - $1) = 0$

Therefore there are two values with eigenvalues of 0 and one eigenvalue of 1! satisfies this theorem that A is idempotent!

Part b:

$$A = A^3$$

A^3x = A λx = λ Ax = λ^3 x

Using the same theorem from the previous it has eigenvalues of 0,1 and -1. Since we care that A has to be positive semi-definite. Which has an eigenvalue of -1. Which does not satisfy Theorem 2.2! A is not idempotent!

Question 2 Solution:
Theorem 2.4
There exists a matrix **P** which diagonalises A_1**,....,**A_m.

$$P^T A_i P = D_i$$

and

$$P^T A_j P = D_j$$

We take A_i and A_j to be k x k matrices first we diagonalizes A_i, A_j, i.e. find P such that,

$D_i = P^T$ A_i P $= \begin{bmatrix} \lambda_1 & ...... & 0 \\ .... & \lambda_2 & ...... \\ 0 & .... & \lambda_k \end{bmatrix}$

for i = 1,.....,k

$D_j = P^T$ A_j P $= \begin{bmatrix} \lambda_1 & ...... & 0 \\ .... & \lambda_2 & ...... \\ 0 & .... & \lambda m \end{bmatrix}$

for j = 1,.....,m

Proof:
P$^T A_i A_j P = (P^T A_i P)(P^T A_j P) = (P^T A_j P)(P^T A_i P) = P^T A_j A_i P$

Pre-multiply by P and post-multiply by P^T to get $A_i A_j = A_j A_i$.

Question 3 Solution:
Pre Proof Using Theorem 2.3
For any matrix A

$$r(A) = r(A^T) = r(A^T A) = tr(A)$$

A = $\begin{bmatrix} | & | & \ldots\ldots & | & \ldots. & | \\ a_1 & a_2 & \ldots. & a_p & \ldots. & a_n \\ | & | & \ldots\ldots & | & \ldots. & | \end{bmatrix}$

Given A matrix with dimensions n x p with p independent columns.
Let $x_1, x_2, x_3, \ldots\ldots, x_k$ *the basis for column space of A.*
Definition of basis every column vector of A is a linear combination of the column vectors of x.

$$a_1 = b_1 x_1 + b_2 x_2 + \ldots\ldots + b_k x_k$$

Definition of linear combination
where b is scalar

B = $\begin{bmatrix} - - b_1 - - \\ - - b_2 - - \\ | \\ - - b_p - - \end{bmatrix}$

$$\begin{bmatrix} | & | & \ldots\ldots & | & \ldots. & | \\ a_1 & a_2 & \ldots. & a_p & \ldots. & a_n \\ | & | & \ldots\ldots & | & \ldots. & | \end{bmatrix} = \begin{bmatrix} | & | & \ldots\ldots & | & \ldots. & | \\ x_1 & x_2 & \ldots. & x_p & \ldots. & a_n \\ | & | & \ldots\ldots & | & \ldots. & | \end{bmatrix} \begin{bmatrix} - - b_1 - - \\ - - b_2 - - \\ | \\ - - b_p - - \end{bmatrix}$$

$$A = XB$$

$$A^T = (XB)^T = B^T X^T$$

r(A) $\leq r(A^T)$ *or* $r(A) \geq r(A^T)$ *to satisfy*!

$$r(A) = r(A^T) = r(A^T A) = tr(A) = p$$

Since P is orthogonal both P and P^T are non-singular. Therefore we need to sum up the diagonal elements of $P^T AP$, *so we need to sum up its trace*!

$$r(A) = r(P^T AP) = tr(P^T AP) = tr(PP^T A) = tr(A) = p$$

Because D = P^TAP is diagonal r($P^T AP$) *is the number of nonzero values of A*!
But A is idempotent so its takes eigenvalues between 0 or 1. To Prove Theorem 2.7! We need only the identity matrix to allow A^TA to be positive definite!

***Using Theorem* 2.7**

***Proof* ($\Leftarrow$) :**

We want A$^T A$ *to be symmetric*

and have all the eigenvalues to be strictly positive to prove $A^T A$ is a positive definite matrix!

we know r($A^T A$) $=$ p is a p x p matrix so it has to be a full rank matrix, p!

Let,λ_1, λ_2,.....,$\lambda_p > 0$ be the eigenvalues of A^TA for every x and for each eigenvalue has to have a value of 1.

for z $=$ P^Tx $=$ $(z_1, ..., z_p)^T$

$$x^T(A^T A)x = x^T PDP^T x = z^T Dz = \sum_{i=1}^{p} z_i^2 \lambda_i$$

since $\lambda_i = 1$!

$$= \sum_{i=1}^{p} z_i^2$$

> 0

Thus $A^T A$ is positive definite as required!

***Proof* ($\Rightarrow$) :**

Suppose $A^T A$ is positive definite let x_i be its normalised i-th eigenvector then,

$$x_i^T(A^T A)x_i = \lambda_i x_i^T x_i = \lambda_i$$

From theorem 2.3 we want $A^T A$ to be symmetric and idempotent. We want the eigenvalues to be 0 or 1. This case all of the eigenvalues must equal to 1.

$\lambda_i = 1 > 0$

So, the eigenvalues of $A^T A$ are strictly positive as required!!

Question 4 Solution:

Part a:

Given information:

Let,

x_1,x_2,$x_3$$\sim$ $(MVN(\mu$,$\sigma^2))$ be a sequence of independent normal random variables,

$$\mu = \bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$\boldsymbol{x}^{\boldsymbol{T}} = (x_1, x_2, x_3)^T$

Supposed to be x^T as noted!

$\boldsymbol{y}$ = (x$_1$ - ,x_2- ,x_3-)^T

To solve A from:

$$\boldsymbol{y} = A\boldsymbol{x}$$

x $\sim N(\mu \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}) =$

$\begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \end{bmatrix} = \text{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$= \frac{1}{3} [1 \quad 1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= \frac{1}{3} [1 \quad 1 \quad 1]\boldsymbol{x}$

Then,

$\boldsymbol{y} = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \bar{x} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} \frac{1}{3}[1,1,1]x \\ \frac{1}{3}[1,1,1]x \\ \frac{1}{3}[1,1,1]x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} [1,1,1] \\ [1,1,1] \\ [1,1,1] \end{bmatrix} \text{x}$

$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} [1,1,1] \\ [1,1,1] \\ [1,1,1] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} [1,1,1] \\ [1,1,1] \\ [1,1,1] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{x}[\boldsymbol{I} - \frac{1}{3} \begin{bmatrix} [1,1,1] \\ [1,1,1] \\ [1,1,1] \end{bmatrix}]$

A $= [\boldsymbol{I} - \frac{1}{3} \begin{bmatrix} [1,1,1] \\ [1,1,1] \\ [1,1,1] \end{bmatrix}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} [1,1,1] \\ [1,1,1] \\ [1,1,1] \end{bmatrix}] = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Part b: Finding the rank of A

Proof: that there is a linear combination for any columns?

Where A is symmetric and idempotent!

$$A^2 = \frac{1}{9}\begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \text{A}$$

````
```{r}
A = matrix(c(2,-1,-1,-1,2,-1,-1,-1,2)/3,3,3)

```
````

```
           [,1]       [,2]       [,3]
[1,]  0.6666667 -0.3333333 -0.3333333
[2,] -0.3333333  0.6666667 -0.3333333
[3,] -0.3333333 -0.3333333  0.6666667
```

```
# Finding rank of A

```{r}
rankMatrix(A)[1]
```

  [1] 2
```

Each column all added up together gives us 0!.

x$_1 + x_2 + x_3 = 0$

Can be written as,

x$_1 = -x_2 - x_3$

That are linearly dependant and similar for x_2 and x_3.

Using Theorem 2.3:

Hence r(A) = tr(A) = 2

Part c: Computing E[$y^T y$]

Using Theorem 3.5:

E[y$^T Ay] = tr(AV)$+μ^TAμ

since A = I,

= tr(V) + μ^T μ

V = vary = varAx = AvarxA^T

since A is symmetric and idempotent!!

var(x_i) = $\begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$

V = $\frac{1}{3}\begin{bmatrix} 2\sigma^2 & -1 & -1 \\ -1 & 2\sigma^2 & -1 \\ -1 & -1 & 2\sigma^2 \end{bmatrix}$

$\mu = E[y] = E[Ax] = AE[x]$

$=\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} = 0$

E[$y^T y$] $= tr(\frac{1}{3}\begin{bmatrix} 2\sigma^2 & -1 & -1 \\ -1 & 2\sigma^2 & -1 \\ -1 & -1 & 2\sigma^2 \end{bmatrix}) + 0$

$= 2\sigma^2$

Part d:

Using Theorem 3.5:

Proof:

Assuming that A is idempotent and has rank k. Because it is symmetric, it can be diagonalised. Let the (orthogonal) diagonalising matrix be P.

D = P^T AP = $\begin{bmatrix} \lambda_1 & ...... & 0 \\ .... & \lambda_2 & ...... \\ 0 & .... & \lambda_k \end{bmatrix}$

since A is symmetric and idempotent, all eigenvalues are either 0 or 1. We know from definition:

$$tr(A) = r(A) = k$$

A $=\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

A^2 = A = $\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

from Part 4b, we find out the rank and trace of matrix A we found in Part 4a. Is also is the same number of degrees of freedom for the chi squared distribution.

$$tr(A) = r(A) = 2$$

Therefore, A must have two eigenvalues of 1 and one eigenvalue of 0.

Using Theorem 3.5 and Corollary 3.7:

with our non central parameter λ!

$\lambda = \frac{1}{2}\mu^T A\mu$

$= \frac{1}{2} \begin{bmatrix} \mu & \mu & \mu \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix}$

$= 0$

$\iff : if\ and\ only\ if$

E[y] $=$ E[$\begin{bmatrix} x_1 - \mu \\ x_2 - \mu \\ x_3 - \mu \end{bmatrix}$]

Since x$_1, x_2$ *and* x_3 *is identically independently distributed*! *and taking* the expectation of the expectation is the expectation itself!

E[y] $=$ E[$\begin{bmatrix} \mu - \mu \\ \mu - \mu \\ \mu - \mu \end{bmatrix}$] $= 0$

NOTE: $\mu = \bar{x}$

In which case,

$\frac{y^T y}{\sigma^2}$

is just the sum of two independent standard normal's. This is just an ordinary (central) chi squared distribution χ_2^2.

with expectation of 2 and variance of 4 with 2 degrees of freedom. In which A is symmetric and idempotent!

Question 5 Solution:

Part a: Computing y,X,β and ϵ

$$\boldsymbol{y} = \begin{bmatrix} 27.3 \\ 42.7. \\ 38.7 \\ 4.5 \\ 23.0 \\ 166.3 \\ 109.7 \\ 80.1 \\ 150.7 \\ 20.3 \\ 189.7 \\ 131.3 \\ 404.2 \\ 149 \end{bmatrix}. \quad \boldsymbol{X} = \begin{bmatrix} 1 & 13.1 \\ 1 & 15.3 \\ 1 & 25.8 \\ 1 & 1.8 \\ 1 & 4.9 \\ 1 & 55.4 \\ 1 & 39.3 \\ 1 & 26.7 \\ 1 & 47.5 \\ 1 & 6.6 \\ 1 & 94.7 \\ 1 & 61.1 \\ 1 & 135.6 \\ 1 & 47.6 \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \\ \epsilon_{10} \\ \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \end{bmatrix}$$

$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ *becomes,*

$$\begin{bmatrix} 27.3 \\ 42.7. \\ 38.7 \\ 4.5 \\ 23.0 \\ 166.3 \\ 109.7 \\ 80.1 \\ 150.7 \\ 20.3 \\ 189.7 \\ 131.3 \\ 404.2 \\ 149 \end{bmatrix}. = \begin{bmatrix} 1 & 13.1 \\ 1 & 15.3 \\ 1 & 25.8 \\ 1 & 1.8 \\ 1 & 4.9 \\ 1 & 55.4 \\ 1 & 39.3 \\ 1 & 26.7 \\ 1 & 47.5 \\ 1 & 6.6 \\ 1 & 94.7 \\ 1 & 61.1 \\ 1 & 135.6 \\ 1 & 47.6 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \\ \epsilon_{10} \\ \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \end{bmatrix}$$

Part b: Solving the least squares estimator

$\boldsymbol{b} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$

```
```{r}
b = solve(t(X)%*%X, t(X)%*%y)
b
```
```

```
          [,1]
[1,] -1.233836
[2,]  2.701553
```

Part c:

$\boldsymbol{s^2} = \frac{\boldsymbol{SS_{Res}}}{\boldsymbol{n-p}}$

```
```{r}
e = y - X%*%b
e #Residual errors


```
```

```
            [,1]
 [1,]  -6.8565106
 [2,]   2.3000724
 [3,] -29.7662361
 [4,]   0.8710405
 [5,]  10.9962256
 [6,]  17.8677893
 [7,]   4.7627957
 [8,]   9.2023660
 [9,]  23.6100596
[10,]   3.7035852
[11,] -64.9032511
[12,] -32.5310639
[13,]  39.1032233
[14,]  21.6399042
```

````
```{r}
n = 14 #sample size
p = 2 #number of parameters
SSRes = sum(e^2)
ssquared = SSRes/(n-p)
ssquared


```
````

```
 [1] 777.1528
```

Part d:

$\boldsymbol{t^T b = [1, 28] b = b_0 + 28 b_1}$

```{r}
c(1,28)%*%b
```

```
        [,1]
[1,] 74.40965
```

Part e:

$z_i = \frac{e_i}{\sqrt{(s^2(1-H_{ii}))}}$

```{r}
a = solve(t(X)%*%X)
a
```

```
                [,1]          [,2]
[1,]  0.163081936 -2.230009e-03
[2,] -0.002230009  5.425812e-05
```

```{r}
H = X%*%a%*%t(X)
H


```

```{r}
z = e/sqrt(ssquared * (1 - diag(H)))
z[13]
```

```
[1] 2.104999
```

Part f:
$D_i = \frac{z_i^2 H_{ii}}{(k+1)(1-H_{ii})}$

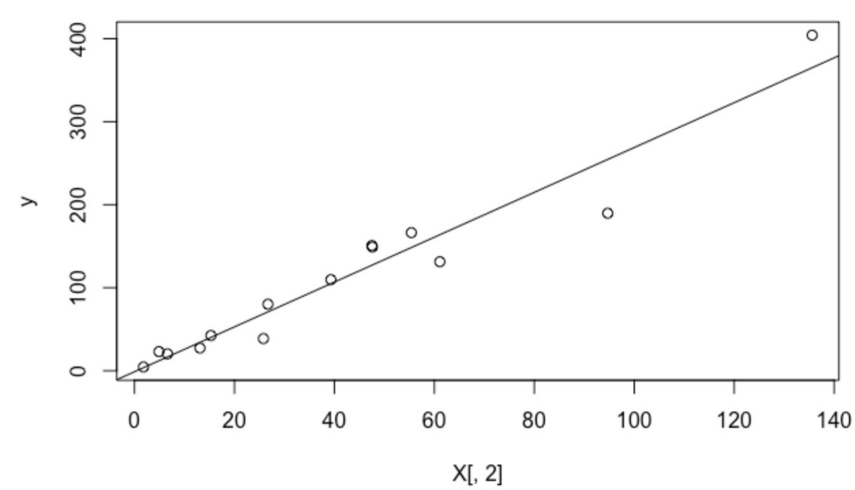
```{r}
k = 1
D = z^2 * (diag(H)/(1-diag(H))) * 1/(k+1)
D[13]
```

```
[1] 2.774008
```

Part g:

```{r}
plot(X[,2],y)
abline(b[1],b[2])

```



Full explaination: The Cook's distance certainly indicates it should be of some concern; however looking at the plot, it seems that the fit is actually okay. There is considerable evidence for heteroskedasticity — the variance increases with x (the design variable). Sea scallops has (by far) the largest x and so may be prone to a larger variance than the remaining points. The high Cook's distance therefore comes primarily from a very high leverage, rather than a bad fit to the model.

END OF ASSIGNMENT!!