THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2020

School of Mathematics and Statistics

MAST20018 Discrete Maths and Operations Research

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 18 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).

Instructions to Students

- There are 10 questions with marks as shown. The total number of marks available is 100.
- During writing time you may only interact with the device running the Zoom session with supervisor permission. The screen of any other device must be visible in Zoom from the start of the session.
- If you have a printer, print the exam one-sided. If you cannot print, download the exam to a second device, which must then be disconnected from the internet.
- Write your answers in the boxes provided on the exam that you have printed. If you are unable to answer the whole question in the answer space provided then you can append additional handwritten solutions to the end after the 18 numbered pages. If you do this you MUST make a note in the correct answer space or page for the question, warning the marker that you have appended additional remarks at the end.
- If you have been unable to print the exam write your answers on A4 paper. The first page should contain only your student number, the subject code and the subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.
- Assemble all exam pages in correct page number order and the correct way up. Add any extra pages with additional working at the end. Use a mobile phone scanning application to scan all pages to a single PDF file. Scan from directly above to reduce keystone effects. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Submit your PDF file to the Canvas Assignment corresponding to this exam using the Gradescope window. Before leaving Zoom supervision, confirm with your Zoom supervisor that you have Gradescope confirmation of submission.

Question 1 (10 marks)

A student wants to do well in life and he understands that the best way to do so is by obtaining good marks in his exams for Semester 2 2020. He is taking 4 subjects: A, B, C and D. He estimates that his mark on the exam for a given subject is proportional to the number of hours he studies for that exam, with the mark capped at 100. The constants of proportionality, p, for each subject are given by $p_A = 3$, $p_B = 5$, $p_C = 7$ and $p_D = 10$ for subjects A, B, C and D, respectively. This means, for example, that if he studies 20 hours for subject A, he will get a mark equal to $20 \times p_A = 60$ but if he studies 40h for subject A he will get 100 ($40 \times p_A = 120$, capped at 100).

(a) The student is planning on studying a total of 40h for his four exams. Define appropriate variables and formulate a linear programming model with the goal of maximising the sum of the marks the student obtains in the exams.

(b) Subjects require a minimum of 50 points in the exam for a pass. Using the same variables as in (a), propose constraints to ensure that the model will only accept solutions that ensure that the student pass in all his subjects.

(c) The student realises that he actually wants to maximise his WAM and not the raw sum of his marks. At the University website, he reads: "The WAM provides an indication of overall academic performance in each course that a student has undertaken, reflecting both the numeric mark (45 or 87, for example) and the number of credit points of each subject you complete. It is calculated progressively as your subject results are added in my.unimelb, meaning a 25 point subject will be weighted double that of a 12.5 point subject in your WAM."
Subjects A and B are 25 point subjects while subjects C and D are 12.5 point subjects. Change the model to reflect the student's new objective.

(d) Stressed with life, the student decides that all he wants is to pass the exams, with the minimum possible effort. Write a model that ensures the student passes all subjects while minimising the hours he needs to study.

Question 2 (10 marks)

Consider the following linear programming model, parametrised on a parameter a:

$$\max z = x_1 + x_2$$

$$\begin{aligned} x_1 - x_2 &\geq -4 \\ x_1 &\leq a \\ x_i &\geq 0, \quad i = 1, 2 \end{aligned}$$

(a) Write the problem in canonical form.

(b) Draw the feasible space for $a = 4$, enumerate all basic feasible solutions and all basic infeasible solutions. Which of these solutions is optimal and what is its objective value?

(c) Write the dual of the presented model.

(d) Draw the feasible space of the dual problem in (c) and use the graphical method on this dual model to obtain the optimal solution for all values of parameter a in the range $-\infty \leq a \leq \infty$.

Question 3 (10 marks)

Use the Phase-I method of the Simplex algorithm to show that the problem below has no feasible solutions.

$$\max z = x_1 + x_2$$

$$\begin{array}{rcl} x_1 - x_2 & \geq & 2 \\ x_1 + x_2 & \leq & 1 \\ x_i & \geq & 0, \quad i = 1, 2 \end{array}$$

Question 4 (10 marks)

Consider the following linear program:

$$\max z = x_1 + 2x_2$$

$$\begin{aligned} 2x_1 - x_2 - x_3 &\geq -2 \\ x_1 - x_2 + x_3 &\geq -1 \\ x_i &\geq 0, \quad i = 1, 2, 3 \end{aligned}$$

(a) Use the Simplex algorithm to verify that this problem is unbounded.

(b) Use your solutions in (a) to obtain a feasible solution for the problem with objective value at least 300. (You must use your results from (a). Trial and error solutions or solutions with no clear explanations will not be considered correct.)

(c) Derive the expression for the reduced cost of a variable, starting with the equation known as 'general solution of the system': $x_B = A_B^{-1}b - A_B^{-1}A_N x_N$. Explain the meaning of the reduced cost expression you obtain.

Question 5 (10 marks)

Answer the following questions. Use only the space provided.

(a) "The basic variables in a basic feasible solution found by the Simplex algorithm are always zero". Is this statement true or false? Justify your answer.

(b) Consider the polyhedron defining the feasible space of a linear program. How can you know if two vertices in this polyhedron are adjacent?

(c) "If an extreme point of a linear program has a better objective value than all of its adjacent extreme points, then this extreme point must be optimal". Is this statement true or false? Justify.

(d) "If there is a non-basic variable with zero reduced cost in the optimal Simplex tableau, then the problem is unbounded". Is this statement true or false? Justify.

(e) "If a problem is infeasible, then it's dual must be infeasible ". Is this statement true or false? Justify.

Question 6 (10 marks)

Consider the following problem:

$$\max z = c_1 x_1 + 2x_2 + x_3 + x_4$$

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 &\leq b_1 \\ 2x_1 + 3x_2 \qquad + 4x_4 &\leq 12 \\ 3x_1 + x_2 + 2x_3 \qquad &\leq 18 \\ x_i &\geq 0, \quad 1 \leq i \leq 4 \end{aligned}$$

Consider $c_1 = 1$ and $b_1 = 8$. After adding slack variables x_5, x_6 and x_7 and solving by the Simplex method, we obtain the final tableau below:

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_3	4/9	0	1	−1/9	1/3	−1/9	0	4/3
x_2	2/3	1	0	4/3	0	1/3	0	4
x_7	13/9	0	0	−10/9	−2/3	−1/9	1	34/3
z	7/9	0	0	14/9	1/3	5/9	0	28/3

(a) For cost coefficient c_1, calculate the range of values for which the above solution remains optimal.

(b) For parameter b_1, compute the range of values for which the above solution remains optimal.

Question 7 (10 marks)

In this question, 3 players A, B and C seek to divide a cake.

(a) Suppose A cuts the pieces according to the lone divider method. What are the valuations of the three pieces from A's viewpoint?

(b) Suppose the valuations of players B and C of the three pieces are $(2/3, 1/6, 1/6)$ and $(2/3, 1/12, 1/4)$ respectively. Which piece(s) is (are) acceptable to B and C? Justify your answer.

(c) Suppose piece 1 is given to player A. Use cut and choose to determine the allocation of portions of pieces 2 and 3 to B and C, with B the cutter.

(d) Show that the final allocation among the 3 players is **not** envy free.

Question 8 (10 marks)

Student council is holding elections for president. There are 4 candidates (labelled A,B,C and D for convenience). The preference schedule for the election is:

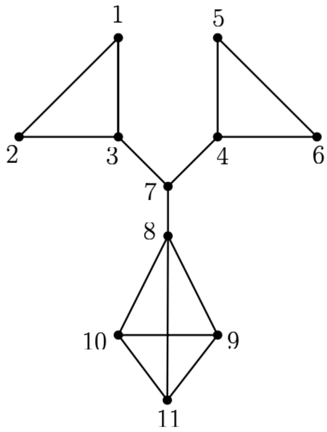
Number of voters	120	50	40	90	60	100
1st choice	C	B	D	A	A	D
2nd choice	D	C	A	C	D	B
3rd choice	B	A	B	B	C	A
4th choice	A	D	C	D	B	C

(a) Find the winner of the election using the Borda count committee scoring method.

(b) Find the winner of the election using the extended Smith set method, which ensures (at least a weak) condorcet winner.

Question 9 (10 marks)

Consider the following graph:



(a) What is node centrality? Compute the node centrality metric known as 'degree centrality' for vertices 1, 7 and 11.

(b) State four possible real-world applications for the theory of node centrality in complex networks.

(c) Construct the dendogram for the following graph using the Girvan-Newman algorithm. Note: vertices have been labelled from 1 to 11. All steps of the algorithm must be shown, but explicit calculations of edge-betweenness centrality are not necessary.

Question 10 (10 marks)

Consider the following 4×4 matrix.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Draw the bipartite graph corresponding to the matrix. Clearly label the nodes of your bipartite graph if the columns of the matrix are $y_1, \ldots, y_4$ and the rows are $x_1, \ldots, x_4$

(b) Find a matching in your bipartite graph from (a) which includes all vertices of maximum degree.

(c) Repeat the process in (b) in order to find a set of four disjoint matchings in your bipartite graph.

End of Exam — Total Available Marks = 100