Exponential families

Learning goals

- Be able to show whether a given distribution is an exponential family.
- Be able to compute mean and variance of random variables belonging to exponential families.
- Understand the interpretation of the variance function and be able to compute the variance function for exponential families.

Exponential families

Y comes from an exponential family if it has density/mass function of the form

$$f(y; \theta, \phi) = \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right]$$

- θ is the *canonical parameter* (captures location)
- ϕ is the *dispersion parameter* (captures scale)

Example: normal

$Y \sim N(\mu, \sigma^2)$

$$
\begin{aligned}
f(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} \\
&= \exp\left[\frac{y\mu - \mu^2/2}{\sigma^2} - \tfrac{1}{2}\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right)\right] \\
&= \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right]
\end{aligned}
$$

where $\theta = \mu$, $\phi = \sigma^2$, and

$$
\begin{aligned}
b(\theta) &= \theta^2/2 \\
a(\phi) &= \phi \\
c(y, \phi) &= -\tfrac{1}{2}\left(\frac{y^2}{\phi} + \log(2\pi\phi)\right)
\end{aligned}
$$

Example: Poisson

$Y \sim \text{pois}(\lambda)$

$$\begin{aligned} f(y) &= e^{-\lambda}\lambda^y/y! \text{ for } y = 0, 1, 2, \ldots \\ &= \exp\left[y \log \lambda - \lambda - \log y!\right] \\ &= \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right] \end{aligned}$$

where $\theta = \log \lambda$, $\phi = 1$, and

$$\begin{aligned} b(\theta) &= e^{\theta} \\ a(\phi) &= \phi \\ c(y, \phi) &= -\log y! \end{aligned}$$

Example: binomial

$Y \sim \text{bin}(m, p)$ for known m (not a parameter)

$$f(y) = \binom{m}{y} p^y (1-p)^{m-y} \text{ for } y = 0, 1, \ldots, m$$

Lab problem in the week 3.

Other examples of exponential families are the gamma and the inverse Gaussian.

Exponential family: mean and variance

Lemma If Y is from an exponential family then

$$\begin{aligned} \mathbb{E}\,Y &= b'(\theta) \\ \mathrm{Var}\,Y &= b''(\theta)a(\phi) \end{aligned}$$

[Proof] Exercise.

Exponential family: variance function

Let $\mu = \mathbb{E}\, Y$ and write

$$\mathrm{Var}\, Y = v(\mu) a(\phi)$$

(so $v = b'' \circ (b')^{-1}$). v is called the *variance function*

Examples:

normal	$v(\mu) = 1$
Poisson	$v(\mu) = \mu$
binomial	$v(\mu) = \mu(1 - \mu/m)$

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