1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = x_1x_2 + x_2x_3 - x_1x_3 - x_2^2.$$

(a) [2 marks] Show that all irregular points are elements of the 0-level set of f.
The gradient of f, $\nabla f = (x_2 - x_3, x_1 + x_3 - 2x_2, x_2 - x_1)$, vanishes when $x_1 = x_2 = x_3$, and we have $f(x_3, x_3, x_3) = 0$.

(b) [2 marks] Show that $\boldsymbol{\gamma}_i : \mathbb{R} \to \mathbb{R}^3$, $i = 1, 2$, given by

$$\boldsymbol{\gamma}_1(t) = (t-1, t, t+1), \qquad \boldsymbol{\gamma}_2(s) = (s, 2s, \frac{1}{s} + 2s)$$

are curves in the 1-level set of f. We have $(t-1)t + t(t+1) - (t-1)(t+1) - t^2 = 1$ as well as $2s^2 + 2s(\frac{1}{s} + 2s) - s(\frac{1}{s} + 2s) - (2s)^2 = 1$.

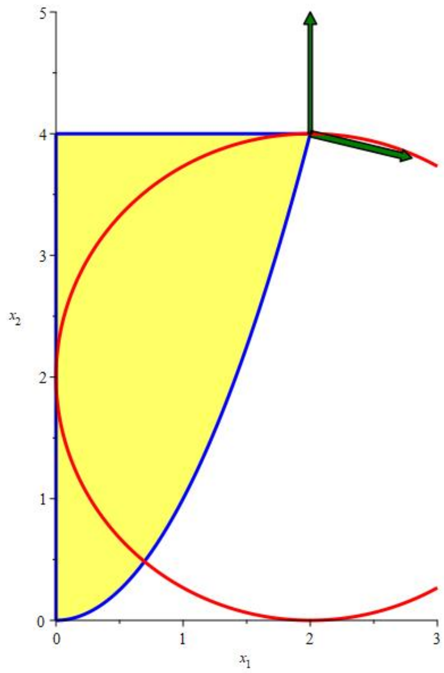
(c) [1 mark] Determine the point $\mathbf{p}$ where the curves from (b) intersect.
We need to satisfy $t - 1 = s$, $t = 2s$ which implies $s = 1, t = 2$ and then $t + 1 = \frac{1}{s} + 2s = 3$ is also satisfied, so $\mathbf{p} = (1, 2, 3) = \boldsymbol{\gamma}_1(2) = \boldsymbol{\gamma}_2(1)$.

(d) [3 marks] Show that at the point $\mathbf{p}$, from (c), the tangent vectors to the curves, from (b), are independent and orthogonal to the gradient of f at $\mathbf{p}$.
The tangent vectors are $t_1 = (1, 1, 1)$ and $t_2 = (1, 2, -\frac{1}{s^2} + 2)\mid_{s=1} = (1, 2, 1)$ which are independent as they are not a multiple of each other. The gradient, see (a), at $\mathbf{p}$ is $(2 - 3, 1 + 3 - 4, 2 - 1) = (-1, 0, 1)$, orthogonal to both t_1, t_2 as $-1 + 1 = 0$.

2. Consider the set-constraint problem:

$$\begin{array}{rl} \text{maximise} & (x_1 - 2)^2 + (x_2 - 2)^2 \\ \text{subject to} & \mathbf{x} \in \Omega = \{\mathbf{x} : \ x_1 \geq 0, \ x_2 \leq 4, \ \text{and } x_2 \geq x_1^2\} \end{array}$$

(a) [5 marks] Let $\mathbf{p} = (2, 4)^T$. Determine and draw in one diagram: the level set of the objective function through $\mathbf{p}$, normal vectors to the active constraints at $\mathbf{p}$, and the feasible set Ω.
The 4-level set is a circle with radius 2 and center (2,2). A normal to the first active constraint is $(0, 1)$. The gradient of $x_1^2 - x_2$ at $\mathbf{p}$, $(2x_1, -1)^T = (4, -1)$ is normal to the curve at $\mathbf{p}$.



(b) [3 marks] Describe the set of feasible directions at $\mathbf{p}$ using the normal vectors you found in (a). State whether $(-1,0)^T$ is feasible.

The set of feasible directions is $\{\mathbf{d} \in \mathbb{R}^2 : 4d_1 - d_2 < 0, d_2 \leq 0\}$. The vector $(-1,0)^T$ lies in the feasible set.

(c) [2 marks] Is the FONC satisfied at $\mathbf{p}$? Justify your answer.

Yes. According to the FONC, if $\mathbf{p}$ is a local maximiser then $\nabla f(\mathbf{p})^T\mathbf{d} \leq 0$ for feasible $\mathbf{d}$. We have $\nabla f(\mathbf{p})^T = (0,4)$ and so we need $4d_2 \leq 0$, which is the case for all feasible $\mathbf{d}$.

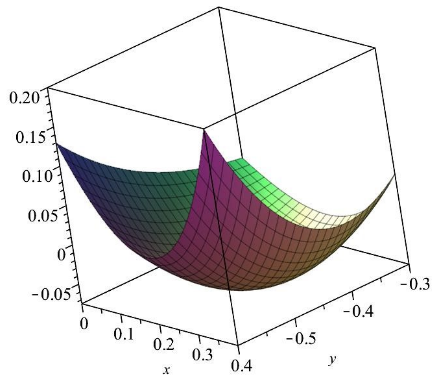
(d) [1 mark] State whether the point $\mathbf{p}$ is a local maximiser. No reasons required.

It is a local maximiser.

3. [7 marks] Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(\mathbf{x}) = (x_1 - x_2)^4 + 5x_1x_2 + x_1.$$

Its graph



shows there is a minimiser close to $\mathbf{q} = (0.3, -0.5)^T$ (this is the only thing that is relevant about the graph). Perform a gradient method, with one iteration of Newton's method to do the line search, to get one step closer to the minimiser. Verify that the point you obtain is closer to the minimiser.

A gradient method defines the next point to be

$$\hat{\mathbf{q}}(\alpha) = \mathbf{q} - \alpha\nabla f(\mathbf{q}).$$

A line search aims to find an α such that $f(\hat{\mathbf{q}}(\alpha)) < f(\mathbf{q})$. Applying Newton's method to optimise $g(\alpha) = f(\hat{\mathbf{q}}(\alpha))$, starting with $\alpha = 0$ (which corresponds to $\hat{\mathbf{q}}(0) = \mathbf{q}$), we find

$$\alpha = -\frac{g'(0)}{g''(0)} = 0.09652509652,$$

which yields

$$\hat{\mathbf{q}} = (0.3 - 0.548\alpha, -0.5 + 0.548\alpha) = (0.2471042471, -0.4471042471).$$

We have $f(\hat{\mathbf{q}}) = -0.0730504215$, which is smaller than $f(\mathbf{q}) = -0.0404$. MATLAB code which performs the above:

```
syms x1 x2 alpha
f=(x1-x2)^4+5*x1*x2+x1
gf=[diff(f,x1) diff(f,x2)]
```

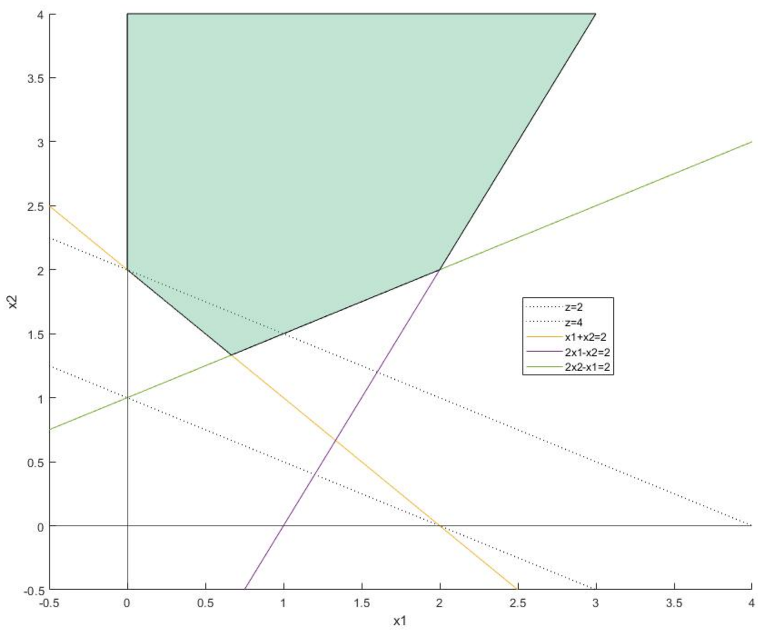
```
x1=.3
x2=-.5
s=eval(f)
h=eval(gf)
x1=x1-alpha*h(1)
x2=x2-alpha*h(2)
g=eval(f)
dg=diff(g,alpha)
ddg=diff(dg,alpha)
alpha=0
alpha=-eval(dg/ddg)
eval([x1 x2])
eval(eval(f))
```

4. [5 marks] Solve the LP problem

$$\begin{aligned} \text{minimise} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & 2x_1 - x_2 \leq 2 \\ & 2x_2 - x_1 \geq 2 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

by sketching the feasible region and drawing some level sets of the objective function. State the minimum and the corner at which the minimum occurs.

The feasible region is unbounded. Its boundary is the x_2-axis, the line $x_1 + x_2 = 2$ through (2,0) and (0,2), the line $2x_1 - x_2 = 2$ through (1,0) and (2,2), and the line $2x_2 - x_1 = 2$ through (0,1) and (2,2). Two level sets of the objective function through (0,1) and (0,2) with slope $-1/2$ are given below



Adding the equation $x_1 + x_2 = 2$ to $2x_2 - x_1 = 2$ gives $x_2 = 4/3$. By solving for x_1 and substituting into $x_1 + 2x_2$, we obtain that the corner at which the minimum $\boxed{10/3}$ occurs is $\boxed{(2/3, 4/3)}$.

5. Consider the LP problem

$$\begin{array}{rl} \text{maximise} & z = 3x_1 + 2x_2 \\ \text{subject to} & x_1 + 2x_2 \geq 2 \\ & x_1 + 2x_2 \leq 4 \\ & 3x_1 - 2x_2 \leq 6 \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

(a) [3 marks] By introducing slack variables and artificial variables as appropriate, and using the 2-phase method, write down the augmented matrix for the first phase.

Indicate the entry to pivot on in this matrix.

One would introduce slack variables x_3, x_4, x_5 and artificial variables x_6: $x_1+2x_2-x_3+x_6 = 2$, $x_1 + 2x_2 + x_4 = 4$, $3x_1 - 2x_2 + x_5 = 6$ and maximise

$$w = -x_6 = x_1 + 2x_2 - x_3 - 2.$$

The augmented matrix for the first phase is

$$\left(\begin{array}{cccccc|c} 1 & \boxed{2} & -1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 & 0 & 0 & 4 \\ 3 & -2 & 0 & 0 & 1 & 0 & 6 \\ -1 & -2 & 1 & 0 & 0 & 0 & -2 \end{array}\right)$$

(b) [3 marks] Solve the first phase using the simplex method, and write down the augmented matrix for the second phase. Indicate the entry to pivot on in this matrix.

Note: You do not have to solve the second phase.

The simplex method gives

$$\left(\begin{array}{ccccccc|c} 1 & \boxed{2} & -1 & 0 & 0 & 1 & 2 & \\ 1 & 2 & 0 & 1 & 0 & 0 & 4 & \\ 3 & -2 & 0 & 0 & 1 & 0 & 0 & 6 \\ -1 & -2 & 1 & 0 & 0 & 0 & -2 & \end{array}\right) \equiv \left(\begin{array}{ccccccc} \frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 2 \\ 4 & 0 & -1 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Omitting the last row and the artificial column, and adding the original objective yields the phase 2 matrix

$$\left(\begin{array}{cccccc} \boxed{\frac{1}{2}} & 1 & -\frac{1}{2} & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 4 & 0 & -1 & 0 & 1 & 8 \\ -3 & -2 & 0 & 0 & 0 & 0 \end{array}\right).$$

6. [4 marks] Use a Lagrange multiplier to show that if there is a point $(x, y)^T$ on the graph of $y = \ln(x)$ which is closest to $\mathbf{0}$ (in the Euclidean norm), then it satisfies $y = -x^2$.

The objective is to minimise $x^2 + y^2$ subject to the constraint $y - \ln(x) = 0$. So we introduce $L = x^2 + y^2 + \lambda(y - \ln(x))$. The FONC gives $2x - \frac{\lambda}{x} = 0$, and $2y + \lambda = 0$. This gives $0 = 2x^2 - \lambda = 2(x^2 + y)$, and hence $y = -x^2$.

7. Consider the nonlinear problem

$$\begin{array}{rl} \text{minimise} & z = x_1(x_1+1) + x_2 + x_3^2 \\ \text{subject to} & x_1 - x_2 - x_3 = 0, \\ & x_1 + x_2 + x_3 \leq 0. \end{array}$$

(a) [1 mark] Write down the relevant KKT condition.
With $f(\mathbf{x}) = x_1(x_1+1) + x_2 + x_3^2$, $h(\mathbf{x}) = x_1 - x_2 - x_3$, $g(\mathbf{x}) = x_1 + x_2 + x_3$, the KKT condition is the existence of multipliers λ, μ such that

$$Df(\mathbf{x}) + \lambda Dh(\mathbf{x}) + \mu Dg(\mathbf{x}) = \mathbf{0}^T,\ \ h(\mathbf{x}) = 0,\ \ \mu g(\mathbf{x}) = 0,\ \ g(\mathbf{x}) \leq 0,\ \ \mu \geq 0$$

(b) [3 marks] Assuming that the inequality constraint is active, write down and solve the linear system which arises from (a). Decide whether the solution gives rise to a local minimum.
The linear system is

$$2x_1 + 1 + \lambda + \mu, 1 - \lambda + \mu, 2x_3 - \lambda + \mu, x_1 - x_2 - x_3, x_1 + x_2 + x_3.$$

It has augmented matrix

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and solution $(\mathbf{x}, \lambda, \mu) = (0, -1/2, 1/2, 0, -1)$. As $\mu < 0$ the solution is a local maximum, not a minimum.

(c) [5 marks] Assuming that the inequality constraint is inactive, write down and solve the linear system which arises from (a). Decide whether the solution gives rise to a minimum.
The linear system is

$$2x_1 + 1 + \lambda + \mu, 1 - \lambda + \mu, 2x_3 - \lambda + \mu, x_1 - x_2 - x_3, \mu.$$

It has augmented matrix

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and solution $(\mathbf{x}, \lambda, \mu) = (-1, -3/2, 1/2, 1, 0)$. The Jacobian of the Lagrangian is $D^2\mathcal{L} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. The active tangent space is $T^a = \mathrm{Ker}(Dh)$ where $Dh = (1, -1, -1)$, so $T^a = \mathrm{Sp}((1,1,0)^T, (1,0,1)^T)$. For all $\mathbf{y} = c(1,1,0)^T + d(1,0,1)^T \in T^a \setminus \{\mathbf{0}\}$ we have

$$\begin{pmatrix} c+d & c & d \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c+d \\ c \\ d \end{pmatrix} = 2(c+d)^2 + 2d^2 > 0,$$

so $\mathbf{x} = (-1, -3/2, 1/2)$ is a minimum.