MAST30025: Linear Statistical Models

Solutions to Week 8 Lab

1. Consider a dataset where each sample has a response and two factors, which have 2 and 3 possible levels respectively. We take 2 samples from each possible combination of factor levels. We model this with a less than full rank model with one parameter for the overall mean, and one parameter for each level of each factor which adjusts the overall mean additively. Write down the linear model in both equation and matrix form.

   Solution: We denote the response variable from the kth sample from the combination of factors with the first factor at level i and the second factor at level j to be y_{ijk}. We also denote the overall mean by μ, and parameters corresponding to each factor by τ_i for the ith level of factor 1, and β_j for the jth level of factor 2. The linear model is

$$y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk},$$

   for $i = 1, 2$, $j = 1, 2, 3$, and $k = 1, 2$.

   Equivalently, $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where

$$\mathbf{y} = \begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{131} \\ e_{132} \\ e_{211} \\ e_{212} \\ e_{221} \\ e_{222} \\ e_{231} \\ e_{232} \end{bmatrix}.$$

2. Let

$$A = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}.$$

   (a) Show that $r(A) = 2$.

   (b) Find two distinct conditional inverses for A.

   Solution: It is easy to see that the third row of A is the second row minus 3 times the first row, but the first two rows are linearly independent. Therefore $r(A) = 2$.

   The inverse of the top left 2×2 minor of A is

$$\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix},$$

   from which we obtain the following conditional inverse

$$A^c = \begin{bmatrix} 7 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Alternatively, we can use the bottom left 2×2 minor of A, resulting in the conditional inverse

$$A^c = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{7}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. Show that $A = A(A^TA)^cA^TA$. You may use the result that if $A^TA = 0$, then $A = 0$. (*Hint: Consider the matrix* $A - A(A^TA)^cA^TA$.)

Solution:

$$\begin{aligned}
&(A - A(A^TA)^cA^TA)^T(A - A(A^TA)^cA^TA) \\
&= A^TA - A^TA[(A^TA)^c]^TA^TA - A^TA(A^TA)^cA^TA + A^TA[(A^TA)^c]^TA^TA(A^TA)^cA^TA \\
&= A^TA - A^TA(A^TA)^cA^TA - A^TA + A^TA(A^TA)^cA^TA \\
&= 0.
\end{aligned}$$

Therefore $A - A(A^TA)^cA^TA = 0$ and $A = A(A^TA)^cA^TA$.

4. It is known that toxic material was dumped into a river that flows into a large salt-water commercial fishing area. We are interested in the amount of toxic material (in parts per million) found in oysters harvested at three different locations in this area. A study is conducted and the following data obtained:

Site 1	Site 2	Site 3
15	19	22
26	15	26

(a) Write down the linear model in matrix form.

Solution: $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where

$$\mathbf{y} = \begin{bmatrix} 15 \\ 26 \\ 19 \\ 15 \\ 22 \\ 26 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{31} \\ \varepsilon_{32} \end{bmatrix}.$$

(b) Write down the normal equations.

Solution: $X^TX\mathbf{b} = X^T\mathbf{y}$, where

$$X^TX = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}, \quad X^T\mathbf{y} = \begin{bmatrix} 123 \\ 41 \\ 34 \\ 48 \end{bmatrix}.$$

(c) Find a conditional inverse for X^TX.

Solution:

$$(X^TX)^c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

(d) Find a solution for the normal equations.

Solution:

$$\mathbf{b} = (X^TX)^cX^T\mathbf{y} = \begin{bmatrix} 0 \\ 20.5 \\ 17 \\ 24 \end{bmatrix}.$$

5. In a manufacturing plant, filters are used to remove pollutants. We are interested in comparing the lifespan of 5 different types of filters. Six filters of each type are tested, and the time to failure in hours is given in the dataset (on the website) `filters` (in `csv` format).

(a) Use the `read.csv` function to read the data. Then convert the `type` component into a factor.

Solution:

```
> filters <- read.csv("../data/filters.csv")
> filters$type <- factor(filters$type)
```

(b) Construct X and $\mathbf{y}$ matrices for this linear model.

Solution:

```
> y <- filters$life
> X <- matrix(0,30,6)
> X[,1] <- 1
> for (i in 1:5) { X[filters$type==i,i+1] <- 1 }
```

(c) Using the algorithm given in the lecture slides, find a conditional inverse for $X^T X$.

Solution:

```
> XtXc <- matrix(0,6,6)
> XtXc[2:6,2:6] <- solve((t(X) %*% X)[2:6,2:6])
> XtXc
     [,1]      [,2]      [,3]      [,4]      [,5]      [,6]
[1,]    0 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
[2,]    0 0.1666667 0.0000000 0.0000000 0.0000000 0.0000000
[3,]    0 0.0000000 0.1666667 0.0000000 0.0000000 0.0000000
[4,]    0 0.0000000 0.0000000 0.1666667 0.0000000 0.0000000
[5,]    0 0.0000000 0.0000000 0.0000000 0.1666667 0.0000000
[6,]    0 0.0000000 0.0000000 0.0000000 0.0000000 0.1666667
```

(d) Use `ginv` to find another conditional inverse for $X^T X$.

Solution:

```
> library(MASS)
> (XtXc2 <- ginv(t(X) %*% X))
           [,1]        [,2]        [,3]        [,4]        [,5]        [,6]
[1,] 0.02314815  0.00462963  0.00462963  0.00462963  0.00462963  0.00462963
[2,] 0.00462963  0.13425926 -0.03240741 -0.03240741 -0.03240741 -0.03240741
[3,] 0.00462963 -0.03240741  0.13425926 -0.03240741 -0.03240741 -0.03240741
[4,] 0.00462963 -0.03240741 -0.03240741  0.13425926 -0.03240741 -0.03240741
[5,] 0.00462963 -0.03240741 -0.03240741 -0.03240741  0.13425926 -0.03240741
[6,] 0.00462963 -0.03240741 -0.03240741 -0.03240741 -0.03240741  0.13425926
```

(e) Verify that $X(X^T X)^c X^T$ is the same for your two conditional inverses.

Solution:

```
> sum((X%*%XtXc%*%t(X)-X%*%XtXc2%*%t(X))^2)
[1] 8.060017e-31
```

(f) Find two solutions for the normal equations.

Solution:

```
> (b <- XtXc %*% t(X) %*% y)
         [,1]
[1,]   0.0000
[2,] 249.1667
[3,] 187.5000
[4,] 166.0000
[5,] 357.3333
[6,] 361.3333
> (b2 <- XtXc2 %*% t(X) %*% y)
          [,1]
[1,] 220.22222
[2,]  28.94444
[3,] -32.72222
[4,] -54.22222
[5,] 137.11111
[6,] 141.11111
```

(g) Express one of your solutions in terms of the other.

Solution: $\mathbf{b}_2 = \mathbf{b} + (I - (X^TX)^cX^TX)\mathbf{b}_2$.

```
> diag(6) - XtXc %*% t(X) %*% X
```

```
     [,1]         [,2]         [,3]         [,4]         [,5]         [,6]
[1,]    1 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
[2,]   -1 1.110223e-16 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
[3,]   -1 0.000000e+00 1.110223e-16 0.000000e+00 0.000000e+00 0.000000e+00
[4,]   -1 0.000000e+00 0.000000e+00 1.110223e-16 0.000000e+00 0.000000e+00
[5,]   -1 0.000000e+00 0.000000e+00 0.000000e+00 1.110223e-16 0.000000e+00
[6,]   -1 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 1.110223e-16
```

(h) Write down a form for all solutions to the normal equations.

Solution: All solutions to the normal equation are of the form $\mathbf{b} + (I - (X^TX)^cX^TX)\mathbf{z}$ for arbitrary $\mathbf{z}$.

6. Show that $A^TA = 0 \Rightarrow A = 0$.

Hence show that $AB = AC \Leftrightarrow A^TAB = A^TAC$.

Solution: We showed that $A\mathbf{x} = 0 \Leftrightarrow A^TA\mathbf{x} = 0$ in lab sheet 2. Thus if $A^TA = 0$ then $A\mathbf{x} = 0$ for all $\mathbf{x}$, and so $A = 0$.

To see that $AB = AC$ when $A^TAB = A^TAC$, just consider $(AB - AC)^T(AB - AC)$.