SCHOOL OF MATHEMATICS AND STATISTICS

MAST30022 Decision Making **Semester 2, 2021**

Assignment 2 Solutions

1. (a) From the given payoff bi-matrix we have

$$\boldsymbol{A} = \begin{bmatrix} 9 & -1 \\ 7 & 10 \end{bmatrix},$$

and $L = \max\{-1, 1\} = 1 = U = \min\{9, 1\}$. Therefore a_{22} is a saddle point and the optimal (pure) strategy for Player 1 is $\boldsymbol{x} = (0, 1)$, and the optimal security level is $u^* = 1$.

For Player 2

$$\boldsymbol{B}^T = \begin{bmatrix} -2 & 4 \\ 1 & 10 \end{bmatrix},$$

and $L = \max\{-2, 1\} = 1 = U = \min\{1, 10\}$. Therefore b_{12} is a saddle point and the optimal (pure) strategy for Player 2 is $\boldsymbol{y} = (0, 1)$, and the optimal security level is $v^* = 1$.

Using these strategies the expected payoff for Player 1 is

$$\boldsymbol{xAy}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & -1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1.$$

For Player 2 the expected payoff is

$$\boldsymbol{xBy}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 10.$$

(b) The strategy pair $(\boldsymbol{x}, \boldsymbol{y}) = ((0, 1), (0, 1))$ is in equilibrium since in the bi-matrix

$$\begin{bmatrix} (9, -2) & (-1, 1) \\ (7, 4) & (1, 10) \end{bmatrix},$$

the pair $(1, 10)$ has the first entry the maximum in the column, and the second entry the maximum in the row.

(c) For $\boldsymbol{x} = (x_1, 1 - x_1) \in X$ and $\boldsymbol{y} = (y_1, 1 - y_1) \in Y$

$$\begin{aligned} \boldsymbol{x}A\boldsymbol{y}^T &= \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} 9 & -1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ 1 - y_2 \end{bmatrix} \\ &= (4y_1 - 2)x_1 + 6y_1 + 1, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{x}B\boldsymbol{y}^T &= \begin{bmatrix} x_1 & 1 - x_1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} y_1 \\ 1 - y_2 \end{bmatrix} \\ &= (3x_1 - 6)y_1 - 9x_1 - 10. \end{aligned}$$

For fixed y_1,

$$\text{if } y_1 < 1/2, \text{ then } \boldsymbol{x}A\boldsymbol{y}^T \text{ is maximised when } x_1 = 0,$$
$$\text{if } y_1 = 1/2, \text{ then } \boldsymbol{x}A\boldsymbol{y}^T \text{ is maximised for any } 0 \le x_1 \le 1,$$
$$\text{if } y_1 > 1/2, \text{ then } \boldsymbol{x}A\boldsymbol{y}^T \text{ is maximised when } x_1 = 1.$$

For $0 \le x_1 \le 1$, $\boldsymbol{x}B\boldsymbol{y}^T$ is maximised when $y_1 = 0$.
The best reply curves for Players 1 and 2 are shown in Figure 1.

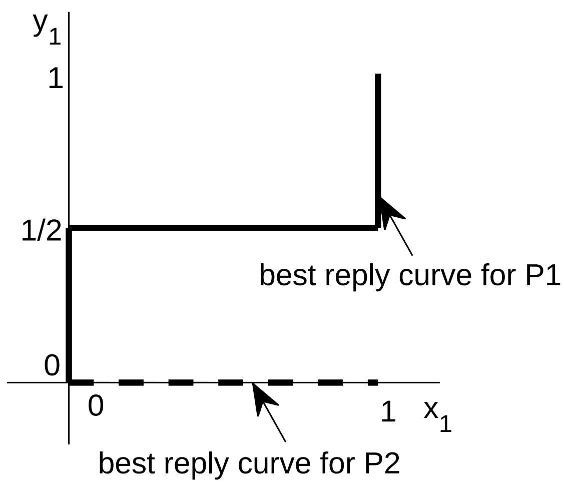

Figure 1: Best reply curves for Players 1 and 2.

The intersection of the two best reply curves is at $(0,0)$. Therefore, $(\boldsymbol{x}^*, \boldsymbol{y}^*) = ((0,1),(0,1))$ is the only equilibrium solution, the same one that was found in part (b).

(d) (i) $\boldsymbol{x}^*\boldsymbol{B} = [w, w]$ and $x_1 + x_2 = 1$ gives the following system of linear equations

$$\begin{aligned} -2x_1 + 4x_2 - w &= 0 \\ x_1 + 10x_2 - w &= 0 \\ x_1 + x_2 &= 1. \end{aligned}$$

Solving the system of equations gives $\boldsymbol{x}^* = (2, -1)$ and $w = -8$.

(ii) $\boldsymbol{A}\boldsymbol{y}^{*T} = [z, z]^T$ and $y_1 + y_2 = 1$ gives the following system of linear equations

$$\begin{aligned} 9y_1 - y_2 - z &= 0 \\ 7y_1 + y_2 - z &= 0 \\ y_1 + y_2 &= 1. \end{aligned}$$

Solving the system of equations gives $\boldsymbol{y}^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $z = 4$.

(iii) The pair $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is not an equilibrium solution since $\boldsymbol{x}^*$ is not a valid mixed strategy.

2. (a) (i) The cooperative payoff set is $C = \text{conv}\{(9,-2),(1,-1),(7,4),(10,1)\}$. It is shown in Figure 2.

(ii) The Pareto boundary is the two line segments joining $(1,10)$ to $(7,4)$ and $(7,4)$ to $(9,-2)$. That is,

$$PB(C) = \{(u,v)|v = -u+11, 1 \leq u \leq 7, v = -3u+25, 7 < u \leq 9\}.$$

It is shown in Figure 2.

(iii) From question 1(a) the optimal security level pair is $(u^*,v^*) = (1,1)$. Therefore, using the status quo point $(u^*,v^*) = (1,1)$ the negotiation set is is the two line segments joining $(1,10)$ to $(7,4)$ and $(7,4)$ to $(8,1)$. That is,

$$NS(C) = \{(u,v)|v = -u+11, 1 \leq u \leq 7, v = -3u+25, 7 < u \leq 8\}.$$

It is shown in Figure 2.

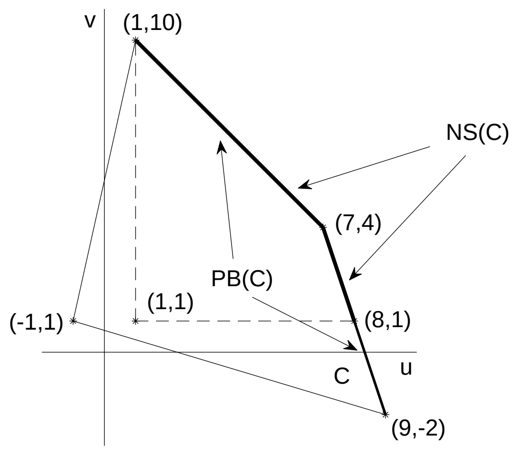

Figure 2: Cooperative payoff set, Pareto boundary, and negotiation set.

(b) Since the status quo point is $(u^*,v^*) = (1,1)$ there is clearly exits $(u,v) \in C$ such that $u > u^*$ and $v > v^*$. By Nash's theorem the Nash solution $(\underline{u},\underline{v})$ is the unique solution to

$$\max_{(u,v)\in NS(C)} (u-u^*)(v-v^*) = \max_{(u,v)\in NS(C)} (u-1)(v-1).$$

Since $NS(C)$ is in two parts we need to maximise over the two sets.

For the line segment joining $(1,10)$ to $(7,4)$

$$\begin{aligned}\max_{(u,v)\in NS(C)} (u-1)(v-1) &= \max_{1\leq u\leq 7} (u-1)(-u+10) \\ &= \max_{1\leq u\leq 7} -u^2+11u-10.\end{aligned}$$

Letting $g(u) = -u^2+11u-10 \implies g'(u) = -2u+11 = 0 \implies u = \frac{11}{2} \in [1,7]$ and $g\left(\frac{11}{2}\right) = 20.25$.

For the line segment joining $(7,4)$ to $(8,1)$

$$\begin{aligned}\max_{(u,v)\in NS(C)} (u-1)(v-1) &= \max_{7\leq u\leq 8} (u-1)(-3u+24) \\ &= \max_{7\leq u\leq 8} -3u^2+27u-24.\end{aligned}$$

Letting $g(u) = -3u^2 + 27u - 24 \Longrightarrow g'(u) = -6u + 27 = 0 \Longrightarrow u = \frac{9}{2} \notin (7, 8]$. Checking the endpoints of the interval $(7, 8]$ we have $g(7) = 18$ and $g(8) = 8$. Thus the Nash solution is $(\underline{u}, \underline{v}) = \left(\frac{11}{2}, \frac{11}{2}\right)$.

(c) The Nash solution lies on the line segment joining $(1, 10)$ to $(7, 4)$. Thus

$$\left(\tfrac{11}{2}, \tfrac{11}{2}\right) \;=\; t(1, 10) + (1 - t)(7, 4),$$

which gives $-6t + 7 = \frac{11}{2} \Longrightarrow t = \frac{1}{4}$. Thus

$$(\underline{u}, \underline{v}) \;=\; \tfrac{1}{4}(1, 10) + \tfrac{3}{4}(7, 4),$$

which corresponds to the players playing strategy pair (a_2, A_2) with probability $\frac{1}{4}$, and playing strategy pair (a_1, A_1) with probability $\frac{3}{4}$.

(d) In Question 2 the equilibrium strategy pair (a_2, A_2) corresponded to the payoff pair $(1, 10)$. Player 2 benefits by not cooperating with Player 1 in refusing to play A_1 any proportion of the time. But if Player 1 plays a_1, Player 2's best payoff would be 1 despite Player 1 losing 1. But if they play cooperatively Player 1 can exert some influence over Player 2 by convincing them to play A_1 some of the time despite receiving a lower payoff.

3. (a) (i) A TU-game (N, v) is superadditive if

$$v(S \cup T) \geq v(S) + v(T) \tag{1}$$

for any disjoint coalitions $S, T \in 2^N$. If S or T is empty, then inequality (1) is trivial. Hence v is superadditive if and only if the following inequalities hold

$$\begin{aligned}
v(\{1\}) + v(\{2\}) &\leq v(\{1,2\}) \Longleftrightarrow 1 \leq 3 \\
v(\{1\}) + v(\{3\}) &\leq v(\{1,3\}) \Longleftrightarrow 1 \leq 5 \\
v(\{2\}) + v(\{3\}) &\leq v(\{2,3\}) \Longleftrightarrow 0 \leq 3 \\
v(\{1\}) + v(\{2,3\}) &\leq v(N) \Longleftrightarrow 4 \leq \alpha \\
v(\{2\}) + v(\{1,3\}) &\leq v(N) \Longleftrightarrow 5 \leq \alpha \\
v(\{3\}) + v(\{1,2\}) &\leq v(N) \Longleftrightarrow 3 \leq \alpha.
\end{aligned}$$

Thus the minimum value of α such that v is superadditive is 5.

(ii) If $\alpha = 5$ any imputation in the core $C(v)$ must satisfy

$$\begin{aligned}
x_1 + x_2 + x_3 &= 5 && (2) \\
x_1 &\geq 1 && (3) \\
x_2 &\geq 0 && (4) \\
x_3 &\geq 0 && (5) \\
x_1 + x_2 &\geq 3 \Longrightarrow x_3 \leq 2 && (6) \\
x_1 + x_3 &\geq 5 \Longrightarrow x_2 \leq 0 && (7) \\
x_2 + x_3 &\geq 3 \Longrightarrow x_1 \leq 2. && (8)
\end{aligned}$$

(4) and (7) give $x_2 = 0$. But then (6) gives $x_1 \geq 3$ and (8) gives $x_1 \leq 2$, a contradiction. Thus $C(v) = \emptyset$.

(iii) We first write v as a linear combination of unanimity games $v = \sum_{T\in 2^N\setminus\emptyset} c_T u_T$ with $c_T \in \mathbb{R}$ for all $T \in 2^N\setminus\emptyset$. In order to compute the coefficients c_T in the linear combination, we use the fact that

$$v(S) = \sum_{T\subseteq S} c_T(S) \qquad \text{for all } S \in 2^N,$$

and we use this relation starting with coalitions S of size $|S| = 1$, then $|S| = 2, \ldots,$ until $|S| = n$ (for $S = N$). We then have

$$\begin{aligned}
1 &= v(\{1\}) = c_1 \Longrightarrow c_1 = 1\\
0 &= v(\{2\}) = c_2 \Longrightarrow c_2 = 0\\
0 &= v(\{3\}) = c_3 \Longrightarrow c_3 = 0\\
3 &= v(\{1,2\}) = c_1 + c_2 + c_{12} \Longrightarrow c_{12} = 2\\
5 &= v(\{1,3\}) = c_1 + c_3 + c_{13} \Longrightarrow c_{13} = 4\\
3 &= v(\{2,3\}) = c_2 + c_3 + c_{23} \Longrightarrow c_{23} = 3\\
5 &= v(\{1,2,3\}) = c_1 + c_2 + c_3 + c_{12} + c_{13} + c_{23} + c_{123} \Longrightarrow c_{123} = -5,
\end{aligned}$$

and therefore

$$v \;=\; u_{\{1\}} + 2u_{\{1,2\}} + 4u_{\{1,3\}} + 3u_{\{2,3\}} - 5u_{\{1,2,3\}}.$$

Then the Shapley value $\mathbf{\Phi}(v)$ is given by $\Phi_i(v) = \sum_{T\in 2^N; i\in T} \frac{c_T}{|T|}$ for all $i \in N$ and we obtain

$$\begin{aligned}
\mathbf{\Phi}(v) &= (1,0,0) + (1,1,0) + (2,0,2) + \left(0,\tfrac{3}{2},\tfrac{3}{2}\right) + \left(-\tfrac{5}{3},-\tfrac{5}{3},-\tfrac{5}{3}\right)\\
&= \left(\tfrac{7}{3},\tfrac{5}{6},\tfrac{11}{6}\right).
\end{aligned}$$

(b) (i) If the core contains a single point then

$$\begin{aligned}
x_1 + x_2 + x_3 &= \alpha && (9)\\
x_1 &\geq 1 && (10)\\
x_2 &\geq 0 && (11)\\
x_3 &\geq 0 && (12)\\
x_1 + x_2 &= 3 \Longrightarrow x_3 = \alpha - 3 && (13)\\
x_1 + x_3 &= 5 \Longrightarrow x_2 = \alpha - 5 && (14)\\
x_2 + x_3 &= 3 \Longrightarrow x_1 = \alpha - 3. && (15)
\end{aligned}$$

Now (9) gives $3\alpha - 11 = \alpha \Longrightarrow \alpha = \frac{11}{2}$. Now $C(v) = \left(\frac{5}{2},\frac{1}{2},\frac{5}{2}\right)$.

(ii) The marginal vectors when $\alpha = \frac{11}{2}$ are given below.

σ	$m^\sigma(v)$
(1 2 3)	$\left(1, 2, \frac{5}{2}\right)$
(1 3 2)	$\left(1, \frac{1}{2}, 4\right)$
(2 1 3)	$\left(3, 0, \frac{5}{2}\right)$
(2 3 1)	$\left(\frac{5}{2}, 0, 3\right)$
(3 1 2)	$\left(5, \frac{1}{2}, 0\right)$
(3 2 1)	$\left(\frac{5}{2}, 3, 0\right)$

The Shapley value is determined by the average of all marginal vectors

$$\mathbf{\Phi}(v) = \frac{1}{6}(15, 6, 12) = \left(\frac{5}{2}, 1, 2\right).$$

(iii) The dual characteristic function $v^* : 2^N \to \mathbb{R}$ is given by

$$v^*(S) \;=\; v(N) - V(N \backslash S).$$

Specifically, with $\alpha = \frac{11}{2}$, the values of v^* are tabulated below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^*(S)$	$5/2$	$1/2$	$5/2$	$11/2$	$11/2$	$9/2$	$11/2$

(iv) Any imputation in the core $C(v^*)$ must satisfy

$$x_1 + x_2 + x_3 \;=\; \frac{11}{2} \tag{16}$$
$$x_1 \;\le\; \frac{5}{2} \tag{17}$$
$$x_2 \;\le\; \frac{1}{2} \tag{18}$$
$$x_3 \;\le\; \frac{5}{2} \tag{19}$$
$$x_1 + x_2 \;\le\; \frac{11}{2} \implies x_3 \ge 0 \tag{20}$$
$$x_1 + x_3 \;\le\; \frac{9}{2} \implies x_2 \ge 0 \tag{21}$$
$$x_2 + x_3 \;\le\; \frac{11}{2} \implies x_1 \ge 1 \tag{22}$$

Now, (17) and (18) give $x_1 + x_2 \le 3$ which implies that $x_3 \ge \frac{5}{2}$ by (16). Then (19) gives $x_3 = \frac{5}{2}$. Similarly, (18) and (19) give $x_2 + x_3 \le 3$ which implies that $x_1 \ge \frac{5}{2}$ by (16). Then (17) gives $x_1 = \frac{5}{2}$ and (16) gives $x_2 = \frac{1}{2}$. Thus, $C(v^*) = \left(\frac{5}{2}, \frac{1}{2}, \frac{5}{2}\right) = C(v)$.

(v) The marginal vectors for v^* are given below.

σ	$m^\sigma(v^*)$
(1 2 3)	$\left(\frac{5}{2}, 3, 0\right)$
(1 3 2)	$\left(\frac{5}{2}, 0, 3\right)$
(2 1 3)	$\left(5, \frac{1}{2}, 0\right)$
(2 3 1)	$\left(1, \frac{1}{2}, 4\right)$
(3 1 2)	$\left(3, 0, \frac{5}{2}\right)$
(3 2 1)	$\left(1, 2, \frac{5}{2}\right)$

The Shapley value is determined by the average of all marginal vectors

$$\begin{aligned} \mathbf{\Phi}(v^*) &= \tfrac{1}{6}(15, 6, 12) \\ &= \left(\tfrac{5}{2}, 1, 2\right) \\ &= \mathbf{\Phi}(v). \end{aligned}$$

(c) In general the Shapley value is *not* contained in the core. In part (a) the core was empty but we could compute the Shapley value, and in part (b) the core was not the same point as the Shapley value.

4. (a) Suppose $i \in N$ is a null player. Then, for every coalition $S \subseteq N$, $v(S \cup \{i\}) = v(S)$. In particular if $S = \emptyset$, then $v(\{i\}) = 0$.

   So, if i is a null player, for every coalition $S \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) = v(S) + v(\{i\})$. Therefore i is a dummy player.

   (b) (i) Suppose player i is a dummy player. Then, for every $S \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) = v(S) + v(\{i\})$. Therefore, since for all $i \in N$, $\psi_i(v) = v(i)$, the dummy player property is satisfied.

   (ii) Suppose players i and j are symmetric. Then, for every $S \subseteq N \backslash \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$. In particular, if $S = \emptyset$, $v(\{i\}) = v(\{j\})$. Hence, $\psi_i(v) = \psi_j(v)$, and the symmetry property is satisfied.

   (iii) Let $u, v \in \mathrm{TU}^N$. Then, for all $i \in N$, $\psi_i(u) + \psi_i(v) = u(\{i\}) + v(\{i\}) = (u + v)(\{i\}) = \psi_i(u + v)$. Therefore, the additivity property is satisfied.

   (c) The solution concept $\boldsymbol{\psi}$ is *not* an alternative characterisation for the Shapley value. It does not satisfy *efficiency*.

   Consider the game (N, v) given by $N = \{1, 2\}$, and $v(\{1\}) = v(\{2\}) = 0$, $v(\{1, 2\}) = 1$. We have $\psi_1(v) + \psi_2(v) = 0 \neq 1 = v(N)$.