MAST30001 Stochastic Modelling

Assignment 1

Please complete and sign the Plagiarism Declaration Form (available through the LMS), which covers all work submitted in this subject. The declaration should be attached to the front of your first assignment.

Don't forget to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 9 September, 2016 by 4pm** in the appropriate assignment box at the north end of Richard Berry Building (near Wilson Lab).

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. (a) Analyse the state space $S = \{1, 2, 3, 4\}$ for each of the three Markov chains given by the following transition matrices. That is, write down the communication classes and their periods, label each class as essential or not, and as transient or positive recurrent or null recurrent.

i.
$$\begin{pmatrix} 0 & 1/8 & 0 & 7/8 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 3/8 & 0 & 5/8 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

ii.
$$\begin{pmatrix} 1/4 & 0 & 0 & 3/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 1/6 & 0 & 0 & 5/6 \end{pmatrix}.$$

iii.
$$\begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix}.$$

(b) For the Markov chain given by the transition matrix in part iii above, discuss the long run behaviour of the chain including deriving long run probabilities.

(c) For the Markov chain given by the transition matrix in part iii above, find the expected number of steps taken for the chain to first reach state 3 given the chain starts at state 1.

(d) For the Markov chain given by the transition matrix in part iii above, find the expected number of steps taken for the chain to first return to state 2 given the chain starts at state 2.

Ans.

(a) Since all of these Markov chains are finite, any essential communicating classes are positive recurrent and non-essential classes are transient.

i. The chain is irreducible with period 2.

ii. There are three communicating classes: $\{1,4\},\{3\},\{2\}$. The first two are essential and the last non-essential. All classes are aperiodic due to the presence of loops.

iii. The chain has two communicating classes: $\{1,2,3\},\{4\}$, the first essential and the second not. Both classes are aperiodic because of loops.

(b) Regardless of where the chain starts it ends up in the essential communicating class and stays forever. The long run probabilities are given by the stationary distribution $\pi=(\pi_1,\pi_2,\pi_3)$ satisfying

$$\pi\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \end{pmatrix}=\pi.$$

Solving shows $\pi=(3/19,12/19,4/19)$.

(c) We perform a first step analysis. Let e_i be the expected time to reach state 3 given the chain starts at state i. Then first step analysis implies

$$\begin{aligned} e_1 &= 1+\frac{e_1}{3}+\frac{e_2}{3}, \\ e_2 &= 1+\frac{3e_2}{4}, \end{aligned}$$

and so $e_1=7/2$.

(d) Because we're starting in the essential class, $E[T(2)|X_0=2]=1/\pi_2=19/12$.

2. A Markov chain $(X_n)_{n\geq 0}$ on $\{0,1,2,\ldots\}$ has transition probabilities for $i=1,2,\ldots$,

$$p_{i,i+1}=1-p_{i,i-1}=\frac{1}{2}\left(\frac{i+1}{i+2}\right),$$

and $p_{0,1}=1-p_{0,0}=1/4$. Note that this chain is irreducible.

(a) Is the chain transient, null recurrent, or positive recurrent?

(b) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

(c) If $T(i)=\min\{n\geq 1: X_n=i\}$, find $E[T(i)|X_0=i]$ for $i=0,1,\ldots$.

(d) If $X_0=0$, what is the chance the chain reaches state 3 before it returns to state 0?

Ans.

(a) Since the chain is irreducible, we only need to check transience, pos/null recurrence at a single state, in this case state 0. Since state 0 has a loop, the chain is irreducible and aperiodic so to determine whether the chain is positive recurrent it's enough to check whether $\pi P=\pi$ has a probability vector solution π, where P is the transition matrix of the chain. These equations are for $i=1,2,\ldots$,

$$\pi_i=p_{i-1,i}\pi_{i-1}+p_{i+1,i}\pi_{i+1},$$

and $\pi_0 = p_{0,0}\pi_0 + p_{1,0}\pi_1$. Solving recursively similar to Question 3 in Tutorial 5, we have that for $i = 1, 2, \ldots,$

$$\pi_i = \pi_0 \prod_{j=1}^{i} \frac{p_{i-1,i}}{1 - p_{i,i-1}} = \pi_0 \frac{3}{(i+1)(i+3)}.$$

For this to be a probability solution, we must have the entries of π vector sum to one so that

$$\pi_0 = \left(3 \sum_{i=0}^{\infty} \frac{1}{(i+1)(i+3)}\right)^{-1} = \frac{4}{9};$$

the sum follows from the identity

$$\frac{1}{(i+1)(i+3)} = \frac{1}{2}\left(\frac{1}{i+1} - \frac{1}{i+3}\right),$$

which makes the sum telescope. Therefore there is a probability vector solution and the chain is positive recurrent.

(b) The chain is irreducible, aperiodic, and positive recurrent, so it's ergodic with long run probabilities given by the stationary distribution π of part (a). That is, the chain spends $\pi_i = \frac{4}{3(i+1)(i+3)}$ proportion of time in state i.

(c) We know that $\mathbb{E}[T(i)|X_0 = i] = 1/\pi_i$, where π are the long run probabilities from Part (b). Thus

$$\mathbb{E}[T(i)|X_0 = i] = \frac{3(i+1)(i+3)}{4}.$$

(d) Use first step analysis. For $j = 0, 1, 2$, let f_j be the chance that the chain visits state 3 before visiting state 0 at a positive time, given it starts in state j, and note we want to compute f_0. First step analysis yields three equations:

$$\begin{aligned}
f_0 &= p_{0,1} f_1 = \frac{1}{4} f_1 \\
f_1 &= p_{1,2} f_2 = \frac{1}{3} f_2 \\
f_2 &= p_{2,3} + p_{2,1} f_1 = \frac{3}{8} + \frac{5}{8} f_1
\end{aligned}$$

which is solved by $f_2 = 9/19$, $f_1 = 3/19$, and $f_0 = 3/76 = 0.0395$.