MAT4MDS — Practice 8 Worked Solutions

Model Answers to Practice 8

Question 1. For $U(x, y) = x^a y^{1-a}$,

$$\frac{\partial U}{\partial x} = a x^{a-1} y^{1-a} = a\left(\frac{x}{y}\right)^{a-1} \qquad \frac{\partial U}{\partial y} = (1-a) x^a y^{1-a-1} = (1-a)\left(\frac{x}{y}\right)^{a-1}$$

Question 2. By the quotient rule

$$\frac{\partial f}{\partial x} = \frac{a(cx+dy)-c(ax+by)}{(cx+dy)^2} = \frac{(ad-bc)y}{(cx+dy)^2} = 0$$

$$\frac{\partial f}{\partial y} = \frac{b(cx+dy)-d(ax+by)}{(cx+dy)^2} = \frac{(bc-ad)y}{(cx+dy)^2} = 0$$

(Or note that $f(x, y) = \frac{ax+by}{cx+dy} = \frac{\frac{a}{c}(cx+dy)+(b-\frac{ad}{c})y}{cx+dy} = \frac{a}{c}$ which is a constant.)

Question 3. Let $f(x, y) = x^2 e^{2y+x} - \frac{x}{y}$.

(a) The first partial derivatives are

$$\frac{\partial f}{\partial x} = 2xe^{2y+x} + x^2 e^{2y+x} - \frac{1}{y} \qquad \frac{\partial f}{\partial y} = 2x^2 e^{2y+x} + \frac{x}{y^2}$$

(b) The second partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = (2e^{2y+x} + 2xe^{2y+x}) + (2xe^{2y+x} + x^2 e^{2y+x}) = (2 + 4x + x^2)e^{2y+x}$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = 4xe^{2y+x} + 2x^2 e^{2y+x} + \frac{1}{y^2}$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = 4xe^{2y+x} + 2x^2 e^{2y+x} + \frac{1}{y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 4x^2 e^{2y+x} - \frac{2x}{y^3}$$

(c) We notice that $\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$.

Question 4. Evaluating the required derivatives at $(1,-1)$:

$$f(1,-1)=e^{-1}+1 \qquad \frac{\partial f}{\partial x}(1,-1)=3e^{-1}+1 \qquad \frac{\partial f}{\partial y}(1,-1)=2e^{-1}+1$$

$$\frac{\partial^2 f}{\partial x^2}(1,-1)=7e^{-1} \qquad \frac{\partial^2 f}{\partial x\,\partial y}(1,-1)=6e^{-1}+1 \qquad \frac{\partial^2 f}{\partial y^2}(1,-1)=4e^{-1}+2$$

Then

$$T^2_{(1,-1)}f(x,y)=e^{-1}+1+(3e^{-1}+1)(x-1)+(2e^{-1}+1)(y+1)+\frac{7e^{-1}}{2}(x-1)^2$$

$$+(6e^{-1}+1)(x-1)y+1)+(2e^{-1}+1)(y+1)^2$$

Question 5.

(a) A function of two variables has $2^3=8$ third partial derivatives. Some of these are unique: $\frac{\partial^3 f}{\partial x^3}$ and $\frac{\partial^3 f}{\partial y^3}$ and the other six come in two sets:

$$\frac{\partial^3 f}{\partial x^2\,\partial y}=\frac{\partial^3 f}{\partial x\,\partial y\,\partial x}=\frac{\partial^3 f}{\partial y\,\partial x^2} \quad \text{and} \quad \frac{\partial^3 f}{\partial y^2\,\partial x}=\frac{\partial^3 f}{\partial y\,\partial x\,\partial y}=\frac{\partial^3 f}{\partial x\,\partial y^2}$$

(b)

$$T^3_{(a,b)}f(x,y)-T^2_{(a,b)}f(x,y))=\frac{1}{3!}\left\{\frac{\partial^3 f}{\partial x^3}(a,b)(x-b)^3\right.$$

$$\left.+3\frac{\partial^3 f}{\partial x^2\,\partial y}(a,b)(x-a)^2(y-b)+3\frac{\partial^3 f}{\partial y^2\,\partial x}(a,b)(x-a)(y-b)^2+\frac{\partial^3 f}{\partial y^3}(a,b)(y-b)^3\right\}$$

(c) For a function of three variables, $g(x,y,z)$ there are 3 first partial derivatives. There are $3^3=9$ second partial derivatives, but these are not all unique. The mixed derivatives are equal in pairs:

$$\frac{\partial^2 g}{\partial x\,\partial y}=\frac{\partial^2 g}{\partial y\,\partial x} \qquad \frac{\partial^2 g}{\partial x\,\partial z}=\frac{\partial^2 g}{\partial z\,\partial x} \qquad \frac{\partial^2 g}{\partial z\,\partial y}=\frac{\partial^2 g}{\partial y\,\partial z}$$

The linear approximation is

$$T^1_{(a,b,c)}g(x,y,z)=g(a,b,c)$$

$$+\frac{\partial^2 g}{\partial x\,\partial y}(a,b,c)(x-a)(y-b)+\frac{\partial^2 g}{\partial x\,\partial z}(a,b,c)(x-b)(z-c)+\frac{\partial^2 g}{\partial y\,\partial z}(a,b,c)(y-b)(z-c)$$