Decision Making
Part 2: 2-person zero-sum games

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Topics in this part

- Notation
- Saddle points: saddle points, equilibrium v.s. saddle, security levels, lower and upper values
- Mixed strategies: mixed strategies, expected payoffs, security levels, optimal security levels, equilibrium pair of mixed strategies
- Fundamental theorem of matrix game theory: fundamental theory, recipe
- Computational techniques: saddle, dominance elimination, 2×2 formulae, graphical method, linear programming
- 2-person constant-sum games

References:

P. Morris, Introduction to Game Theory, 1994, Chapters 2–4.
W. L. Winston, Operations Research: Appl. & Alg., 4th ed., 2004, Sections 14.1–14.3.

Notation for 2-person zero-sum games

Notation

We use the following notation for 2-person zero-sum games in normal form.

- Two players are called I and II, or Bob and Alice.
- Player I has m strategies, denoted by $a_1, a_2, \ldots, a_m$ and called the pure strategies for player I.
- Player II has n strategies, denoted by $A_1, A_2, \ldots, A_n$ and called the pure strategies for player II.
- Pure strategies are in contrast with mixed strategies to be discussed later.
- v_{ij} denotes the payoff to player I if he uses a_i and player II uses A_j, $1 \leq i \leq m, 1 \leq j \leq n$.
- Since the game is zero-sum, the payoff to player II is equal to $-v_{ij}$ if the same pair of strategies (a_i, A_j) is played.

A 2-person zero-sum game is defined by the payoff matrix of player I:

$$
\begin{array}{cc|ccccc}
 & & & & \text{Player II} & & \\
 & & A_1 & \ldots & A_j & \ldots & A_n \\
\hline
 & a_1 & v_{11} & \ldots & v_{1j} & \ldots & v_{1n} \\
 & \vdots & \vdots & & \vdots & & \vdots \\
\text{Player I} & a_i & v_{i1} & \ldots & v_{ij} & \ldots & v_{in} \\
 & \vdots & \vdots & & \vdots & & \vdots \\
 & a_m & v_{m1} & \ldots & v_{mj} & \ldots & v_{mn}
\end{array}
$$

The game is determined uniquely by this matrix up to permutation of rows and columns.

The payoff matrix of player II is the negative of the matrix above.

Denote the payoff matrix of player I by:

$$V = \begin{bmatrix} v_{11} & \dots & v_{1j} & \dots & v_{1n} \\ \vdots & & \vdots & & \vdots \\ v_{i1} & \dots & v_{ij} & \dots & v_{in} \\ \vdots & & \vdots & & \vdots \\ v_{m1} & \dots & v_{mj} & \dots & v_{mn} \end{bmatrix}$$

- We call V the payoff matrix of the corresponding 2-person zero-sum game.
- Sometimes we simply say V is a 2-person zero-sum game.
- The rows of V are indexed by the pure strategies of player I (row player) and the columns are indexed the pure strategies of player II (column player).
- The payoff matrix of player II is equal to $-V$.
- Player I favours larger entries of V, and player II favours smaller entries of V.

Example 1. A 2-person zero-sum game with payoff matrix:

$$V = \begin{bmatrix} 2 & -4 & 25 \\ 6 & 10 & -20 \end{bmatrix}$$

What is the "best" pair of strategies?

Player I favours (a_1, A_3) with payoff 25.

Player II favours (a_3, A_3) with payoff 20.

Note: player I and II have conflicting interests, since both would like to have their maximum possible payoff.

Saddle points

Equilibria v.s. saddle points

As a principle we assume that the players tend to use strategy pairs in equilibrium.

By definition a pair (a_{i^*}, A_{j^*}) of strategies is in equilibrium if and only if

$$v_{ij^*} \leq v_{i^*j^*} \text{ for every } 1 \leq i \leq m$$

and

$$-v_{i^*j} \leq -v_{i^*j^*} \text{ for every } 1 \leq j \leq n.$$

Definition 1. An entry $v_{i^*j^*}$ of the payoff matrix V is called a saddle point if

$$\min_{1 \leq j \leq n} v_{i^*j} = v_{i^*j^*} = \max_{1 \leq i \leq m} v_{ij^*},$$

that is, $v_{i^*j^*}$ is the smallest entry in its row and the largest entry in its column **simultaneously**.

From this definition, we have:

Lemma 1. For any 2-person zero-sum game with matrix $V = (v_{ij})_{m \times n}$, a pair (a_{i^*}, A_{j^*}) of strategies is an equilibrium **if and only if** $v_{i^*j^*}$ is a saddle point.

If V has a saddle point $v_{i^*j^*}$, then there are good reasons to believe the two players will stay with (a_{i^*}, A_{j^*}) and so the game is "solved".

Example 2. Equilibrium in a 2-person zero-sum game.

Not every 2-person zero-sum game has saddle points.

Example 3. A 2-person zero-sum game with no saddle point.

Theorem 1. If v_{kl} and $v_{k'l'}$ are saddle points, then $v_{kl'}$ and $v_{k'l}$ are also saddle points and

$$v_{kl} = v_{k'l'} = v_{kl'} = v_{k'l}.$$

Proof.

Security levels

Definition 2. Let $V = (v_{ij})_{m\times n}$ be a 2-person zero-sum game. Call

$$s_i := \min_{1\le j\le n} v_{ij}, \qquad i = 1, 2, \ldots, m$$

the security level for Player I associated with strategy a_i. Similarly, call

$$S_j := \max_{1\le i\le m} v_{ij}, \qquad j = 1, 2, \ldots, n$$

the security level for Player II associated with strategy A_j.

Player I receives no less than s_i if he plays a_i, and Player II loses no more than S_j if she plays A_j.

Recall that $v_{i^*j^*}$ is called a saddle point if

$$\min_{1\leq j\leq n} v_{i^*j} = v_{i^*j^*} = \max_{1\leq i\leq m} v_{ij^*}.$$

Definition 3. (Equivalent definition of a saddle point) An entry $v_{i^*j^*}$ of V is a saddle point if and only if

$$s_{i^*} = v_{i^*j^*} = S_{j^*}.$$

Example 4.

Lower and upper values

Player I favours rows with maximum security level, and player II favours columns with minimum security level.

Definition 4. Define

$$L := \max_{1 \le i \le m} s_i = \max_{1 \le i \le m} \min_{1 \le j \le n} v_{ij}$$

and

$$U := \min_{1 \le j \le n} S_j = \min_{1 \le j \le n} \max_{1 \le i \le m} v_{ij}.$$

Call L and U the lower value and upper value of the game V, respectively.

Example 5. (Security levels and lower and upper values of a game)

Theorem 2. For any 2-person zero-sum game,

$$L \leq U.$$

Proof.

Example 6. (A 2-person zero-sum game with $L < U$)

Payoff matrix:

$$V = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$$

Example 7. (A 2-person zero-sum game with $L = U$)

Payoff matrix:

$$V = \begin{bmatrix} 3 & 2 & 10 \\ 4 & 1 & 5 \\ 8 & -1 & -2 \end{bmatrix}$$

Theorem 3. For any 2-person zero-sum game V,

(a) if V has a saddle point $v_{i^*j^*}$, then

$$L = s_{i^*} = v_{i^*j^*} = S_{j^*} = U;$$

(b) if $L = U$, then V has a saddle point.

In summary, $L = U$ holds if and only if V has a saddle point.

Proof.

Corollary 1. For any 2-person zero-sum game V, the following conditions are equivalent:

(a) there exists a pair of strategies (a_{i^*}, A_{j^*}) in equilibrium;

(b) V has a saddle point $v_{i^*j^*}$;

(c) $L = U$.

Moreover, if one of these conditions holds, then

$$L = s_{i^*} = v_{i^*j^*} = S_{j^*} = U.$$

Definition 5. If the strategy pair (a_{i^*}, A_{j^*}) in an equilibrium of the zero-sum game V, then $v_{i^*j^*}$ is called the value of the game.

Example 8. Solve the 2-person zero-sum game with payoff matrix below. That is find saddle points, if any; find the value of the game; state the strategies the players should use, based on the philosophy given earlier.

$$V = \begin{bmatrix} 4 & 5 & 5 & 8 \\ 6 & 7 & 6 & 9 \\ 5 & 7 & 5 & 4 \\ 6 & 6 & 5 & 5 \end{bmatrix}.$$

Example 9. For the 2-person zero-sum game with payoff matrix below, find the values of x for which there is a saddle point. Solve the game for these values of x.

$$V = \begin{bmatrix} x & 8 & 3 \\ 0 & x & -9 \\ -5 & 5 & x \end{bmatrix}.$$

Mixed strategies

Mixed strategies

If there is no saddle-point, then it is impossible to find a pair of pure strategies in equilibrium.

What should we do in this case?

We may think of playing the game repeatedly.

We allow the players to randomly choose their strategies with a view to maximising their expected payoff.

Definition 6. A mixed strategy for Player I is a vector

$$\mathbf{x} = (x_1, \ldots, x_m)$$

such that $x_i \geq 0$ for all i and $x_1 + \cdots + x_m = 1$.

A mixed strategy for Player II is a vector

$$\mathbf{y} = (y_1, \ldots, y_n)$$

such that $y_j \geq 0$ for all j and $y_1 + \cdots + y_n = 1$.

In words, x_i is the probability that Player I uses strategy a_i, and y_j is the probability that Player II uses strategy A_j.

Let

$$X := \{(x_1, \ldots, x_m) : x_i \geq 0 \text{ for all } i,\ \ x_1 + \cdots + x_m = 1\}$$

denote the set of mixed strategies for Player I.

Let

$$Y := \{(y_1, \ldots, y_n) : y_j \geq 0 \text{ for all } j,\ \ y_1 + \cdots + y_n = 1\}$$

be the set of mixed strategies for Player II.

Assumption 1. We assume that the two players choose their mixed strategies independently.

Example 10. (Mixed strategies and expected payoff)
Consider the 2-person zero-sum game with payoff matrix

$$V = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}.$$

Calculate the expected payoff if player I plays strategy $\mathbf{x} = (0.6, 0.4)$ and player II plays $\mathbf{y} = (0.3, 0.7)$.

If one coordinate of $\mathbf{x} = (x_1, \ldots, x_m)$ is equal to 1, say, $x_i = 1$, and all other coordinates are 0, then Player I uses a_i all the time and so $\mathbf{x}$ is identical to the pure strategy a_i.

For instance, $\mathbf{x} = (0, 0, 1, 0, 0)$ can be identified with a_3.

If one coordinate of $\mathbf{y} = (y_1, \ldots, y_n)$ is equal to 1, say, $y_j = 1$, and all other coordinates are 0, then $\mathbf{y}$ is the pure strategy A_j, i.e. Player II applies A_j all the time.

Therefore, pure strategies can be viewed as special mixed strategies with exactly one non-zero coordinate.

Probability of achieving $\mathbf{v_{ij}}$

Mixed strategies define a probability distribution on $\{v_{ij} : i = 1, \ldots, m, j = 1, \ldots, n\}$ with the probability of v_{ij} being $x_i y_j$, because all $x_i y_j \geq 0$ and $\sum_{i=1}^m \sum_{j=1}^n x_i y_j = (\sum_{i=1}^m x_i)(\sum_{j=1}^n y_j) = 1 \times 1 = 1$.

Lemma 2. If Player I uses $\mathbf{x} = (x_1, \ldots, x_m)$ and Player II uses $\mathbf{y} = (y_1, \ldots, y_n)$, then for every pair $1 \leq i \leq m, 1 \leq j \leq n$, the probability that Player I receives v_{ij} is equal to $x_i y_j$, i.e.

$$\Pr(\text{player I receives payoff } v_{ij}) = x_i y_j.$$

Proof. This follows from the assumption that the two players choose $\mathbf{x}$ and $\mathbf{y}$ independently. $\square$

Expected payoff

Corollary 2. If Player I uses $\mathbf{x} = (x_1, \ldots, x_m)$ and Player II uses $\mathbf{y} = (y_1, \ldots, y_n)$, then the expected payoff to Player I is given by

$$\mathbf{E}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j v_{ij}.$$

We can express this as

$$\mathbf{E}(\mathbf{x}, \mathbf{y}) = \mathbf{x} V \mathbf{y}^T = (x_1, \ldots, x_m) \begin{bmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{m1} & \ldots & v_{mn} \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Example 11. (Expected payoff)
Consider the 2-person zero-sum game with payoff matrix

$$V = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}.$$

Calculate the expected payoff if player I plays strategy $\mathbf{x} \in X$ and player II plays $\mathbf{y} \in Y$.

Security levels

Definition 7. For any fixed $\mathbf{x} \in X$, call

$$s(\mathbf{x}) := \min_{\mathbf{y} \in Y} \mathbf{x} V \mathbf{y}^T$$

the security level for Player I associated with $\mathbf{x}$.

Similarly, for any fixed $\mathbf{y} \in Y$, call

$$S(\mathbf{y}) := \max_{\mathbf{x} \in X} \mathbf{x} V \mathbf{y}^T$$

the security level for Player II associated with $\mathbf{y}$.

$s(\mathbf{x})$ is the minimum expected payoff of Player I if he uses $\mathbf{x}$.

$S(\mathbf{y})$ is the maximum expected loss of Player II if she uses $\mathbf{y}$.

Example 12. (Security levels)
Consider the 2-person zero-sum game with payoff matrix

$$V = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}.$$

Determine the security level $s(x)$ of player I in case $\mathbf{x} = (0.6, 0.4)$ and the security level $S(y)$ of player II in case $\mathbf{y} = (0.3, 0.7)$.

Given $V = (v_{ij})_{m\times n}$, let

$$V_{i\cdot} = (v_{i1}, \dots, v_{in}),$$

be the ith row of V and

$$V_{\cdot j} = \begin{pmatrix} v_{1j} \\ \vdots \\ v_{mj} \end{pmatrix}$$

be the jth column of V.

Note that

$$\mathbf{x}V_{\cdot j} = \sum_{i=1}^{m} x_i v_{ij} = x_1 v_{1j} + \cdots + x_m v_{mj} \ \text{ (dot product)}$$

$$V_{i\cdot}\mathbf{y}^T = \sum_{j=1}^{n} v_{ij} y_j = v_{i1} y_1 + \cdots + v_{in} y_n \ \text{ (dot product)}$$

Theorem 4. For any $\mathbf{x} \in X$ and $\mathbf{y} \in Y$,

$$s(\mathbf{x}) = \min_{1 \le j \le n} \mathbf{x} V_{\cdot j}$$

$$S(\mathbf{y}) = \max_{1 \le i \le m} V_{i \cdot} \mathbf{y}^T$$

Proof.

Corollary 3. If $\mathbf{x} = (0, \ldots, \overbrace{1}^{i}, \ldots, 0)$ is the pure strategy a_i, then

$$s(\mathbf{x}) = s_i.$$

If $\mathbf{y} = (0, \ldots, \overbrace{1}^{j}, \ldots, 0)$ is the pure strategy A_j, then

$$S(\mathbf{y}) = S_j.$$

These agree with our earlier notions for pure strategies.
Proof.

Optimal security levels

Definition 8. Define

$$v_1 := \max_{\mathbf{x} \in X} s(\mathbf{x})$$

$$v_2 := \min_{\mathbf{y} \in Y} S(\mathbf{y})$$

Call v_1 and v_2 the optimal security levels for Players I and II respectively.

If $\mathbf{x}^* \in X$ achieves v_1 (i.e. $v_1 = s(\mathbf{x}^*)$), then $\mathbf{x}^*$ is called an optimal mixed strategy for Player I. If $\mathbf{y}^* \in Y$ achieves v_2 (i.e. $v_2 = S(\mathbf{y}^*)$), then $\mathbf{y}^*$ is called an optimal mixed strategy for Player II.

$$v_1 = \max_{\mathbf{x} \in X} s(\mathbf{x}) = \max_{\mathbf{x} \in X} \min_{\mathbf{y} \in Y} \mathbf{x} V \mathbf{y}^T = \max_{\mathbf{x} \in X} \min_{1 \le j \le n} \mathbf{x} V_{\cdot j}$$

$$v_2 = \min_{\mathbf{y} \in Y} S(\mathbf{y}) = \min_{\mathbf{y} \in Y} \max_{\mathbf{x} \in X} \mathbf{x} V \mathbf{y}^T = \min_{\mathbf{y} \in Y} \max_{1 \le i \le m} V_{i \cdot} \mathbf{y}^T$$

Example 13. (Finding optimal strategies, graphical method)
Consider the 2-person zero-sum game with payoff matrix

$$V = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}.$$

Find an optimal mixed strategy for Player I and an optimal mixed strategy for Player II. What are the values of the optimal security levels of Players I and II, respectively?

Example (cont.)

Example (cont.)

Fundamental theorem of matrix game theory

Equilibrium

Definition 9. A pair of mixed strategies $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ is said to be in equilibrium if, for any $\mathbf{x} \in X$ and $\mathbf{y} \in Y$,

$$\mathbf{E}(\mathbf{x}, \mathbf{y}^*) \leq \mathbf{E}(\mathbf{x}^*, \mathbf{y}^*) \leq \mathbf{E}(\mathbf{x}^*, \mathbf{y})$$

that is,

$$\mathbf{x} V {\mathbf{y}^*}^T \leq \mathbf{x}^* V {\mathbf{y}^*}^T \leq \mathbf{x}^* V \mathbf{y}^T.$$

Interpretation:

On average Player I receives the most by playing $\mathbf{x}^*$, given that Player II uses $\mathbf{y}^*$.

On average Player II loses the least by playing $\mathbf{y}^*$, given that Player I uses $\mathbf{x}^*$.

Remark 1. $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ is in equilibrium iff, for any $\mathbf{x} \in X$ and $\mathbf{y} \in Y$,

$$\mathbf{E}(\mathbf{x}, \mathbf{y}^*) \leq \mathbf{E}(\mathbf{x}^*, \mathbf{y}^*), \;\; -\mathbf{E}(\mathbf{x}^*, \mathbf{y}) \leq -\mathbf{E}(\mathbf{x}^*, \mathbf{y}^*)$$

That is, Player I cannot get better expected payoff by deviating from $\mathbf{x}^*$, given that Player II uses $\mathbf{y}^*$; and Player II cannot achieve smaller expected loss by deviating from $\mathbf{y}^*$, given that Player I uses $\mathbf{x}^*$.

Thus the notation of equilibrium here is consistent with our earlier usage for general games in normal form.

Exercise 1. Show that, if a pair of pure strategies is in equilibrium, then it is also in equilibrium as a pair of mixed strategies.

This guarantees that our usage of an equilibrium pair is consistent for both pure and mixed strategies.

Theorem 5. If both $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ are in equilibrium, then so are $(\mathbf{x}_1, \mathbf{y}_2)$ and $(\mathbf{x}_2, \mathbf{y}_1)$.

Proof.

Recall that an equilibrium pair of pure strategies may not exist.

Question 1.

(a) Does there exist an equilibrium pair of mixed strategies all the time?

(b) How can we construct an equilibrium pair of mixed strategies if it does exist?

Once we obtain an equilibrium pair, the game is thought as solved. Thus the major task is to find equilibrium pairs.

Fundamental theorem of matrix game theory

Theorem 6. (Fundamental Theorem) For any 2-person zero-sum game with payoff matrix $V = (v_{ij})_{m\times n}$, both players have optimal mixed strategies, that is, there exist $\mathbf{x}^* \in X$ and $\mathbf{y}^* \in Y$ such that

$$v_1 = \min_{1\leq j\leq n} \mathbf{x}^* V_{\cdot j} \ \ (= s(\mathbf{x}^*)), \ \ v_2 = \max_{1\leq i\leq m} V_{i\cdot}\mathbf{y}^{*T} \ \ (= S(\mathbf{y}^*)).$$

Moreover,

$$v_1 = v_2.$$

This surprising and important result was proved by von Neumann in 1937.

Definition 10. Call $v_1 = v_2$ the value of the game and denote it by v.

Proof. Special case: $v_{ij} > 0$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$.
(Why is this only a technicality?)

Plan: show that the problems faced by the players can be expressed as two LP problems and one is the dual of the other.

By definition of v_1 we have

$$\begin{aligned} v_1 &= \max\{s(\mathbf{x}) : \mathbf{x} \in X\} \\ &= \max_{\mathbf{x} \in X}\{\min_j\{\mathbf{x}V_{\cdot j}\}\} \end{aligned}$$

This is equivalent to

$$v_1 = \max_{u,\mathbf{x}} u$$

s.t.

$$\begin{aligned} \mathbf{x}V_{\cdot j} &\geq u, \ \ j = 1, \ldots, n \\ \mathbf{x} &\in X \end{aligned}$$

$$v_1 = \max_{u,\mathbf{x}} u$$

s.t.

$$\begin{aligned}
\mathbf{x}V_{\cdot 1} - u &\geq 0 \\
\mathbf{x}V_{\cdot 2} - u &\geq 0 \\
&\vdots \\
\mathbf{x}V_{\cdot n} - u &\geq 0 \\
x_1 + x_2 + \ldots + x_m &= 1 \\
x_1, x_2, \ldots, x_m &\geq 0
\end{aligned}$$

Since V is strictly positive, so is the optimal value of u. Thus, without loss of generality, we can restrict the analysis to positive values of u, and divide the constraints by u.

$$v_1 = \max_{u,\mathbf{x}} u$$

s.t.

$$\begin{aligned}
\frac{\mathbf{x}}{u} V_{\cdot 1} &\geq 1 \\
\frac{\mathbf{x}}{u} V_{\cdot 2} &\geq 1 \\
&\vdots \\
\frac{\mathbf{x}}{u} V_{\cdot n} &\geq 1 \\
\frac{x_1}{u} + \frac{x_2}{u} + \ldots + \frac{x_m}{u} &= \frac{1}{u} \\
\frac{x_1}{u}, \frac{x_2}{u}, \ldots, \frac{x_m}{u} &\geq 0
\end{aligned}$$

Since maximizing u is equivalent to minimizing $1/u$, the problem under consideration is equivalent to:

$$\frac{1}{v_1} = \min_{u,\mathbf{x}} \frac{1}{u}$$

s.t.

$$\begin{aligned}
\frac{\mathbf{x}}{u} V_{\cdot 1} &\geq 1 \\
\frac{\mathbf{x}}{u} V_{\cdot 2} &\geq 1 \\
&\vdots \\
\frac{\mathbf{x}}{u} V_{\cdot n} &\geq 1 \\
\frac{x_1}{u} + \frac{x_2}{u} + \ldots + \frac{x_m}{u} &= \frac{1}{u} \\
\frac{x_1}{u}, \frac{x_2}{u}, \ldots, \frac{x_m}{u} &\geq 0
\end{aligned}$$

Setting $\mathbf{x}' := \mathbf{x}/u = (x'_1, \ldots, x'_m)$, we obtain

$$\frac{1}{v_1} = \min_{u,\mathbf{x}} \frac{1}{u}$$

s.t.

$$\begin{aligned}
\mathbf{x}'V_{\cdot 1} &\geq 1 \\
\mathbf{x}'V_{\cdot 2} &\geq 1 \\
&\vdots \\
\mathbf{x}'V_{\cdot n} &\geq 1 \\
x_1' + x_2' + \ldots + x_m' &= \frac{1}{u} \\
x_1', x_2', \ldots, x_m' &\geq 0
\end{aligned}$$

Substituting the equality constraint for $1/u$ in the objective function, we obtain the equivalent problem

$$\frac{1}{v_1} = \min_{\mathbf{x}'} x_1' + x_2' + \ldots + x_m'$$

s.t.

$$\begin{aligned} \mathbf{x}' V_{\cdot 1} &\geq 1 \\ \mathbf{x}' V_{\cdot 2} &\geq 1 \\ &\vdots \\ \mathbf{x}' V_{\cdot n} &\geq 1 \\ x_1', x_2', \ldots, x_m' &\geq 0 \end{aligned}$$

Setting $\mathbf{b} = (1, 1, \ldots, 1)$ (m-dimensional vector) and $\mathbf{c} = (1, 1, \ldots, 1)$ (n-dimensional vector), we can rewrite the problem as

$$\frac{1}{v_1} = \min_{\mathbf{x}'} \mathbf{x}'\mathbf{b}^\top$$

s.t.

$$\begin{aligned} \mathbf{x}'V &\geq \mathbf{c} \\ \mathbf{x}' &\geq \mathbf{0} \end{aligned}$$

Repeating the process for Player II, we discover that her problem is equivalent to

$$\frac{1}{v_2} = \max_{\mathbf{y}'} \mathbf{c}\mathbf{y}'^\top$$

s.t.

$$\begin{aligned} V\mathbf{y}'^\top &\leq \mathbf{b}^\top \\ \mathbf{y}' &\geq \mathbf{0} \end{aligned}$$

Both problems are feasible and have optimal solutions. Moreover, they are the dual of each other. Thus the duality theory of LP tells us that

$$v_1 = v_2.$$

To obtain the optimal strategies from the solutions to the LP problem we have to multiply them by the optimal value of the original objective function that is v_1 or v_2.

General case: Not all entries of V are strictly positive.

Theorem 7. $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ is in equilibrium if and only if

$$s(\mathbf{x}^*) = v_1 = v_2 = S(\mathbf{y}^*).$$

Proof.

Proof.

The theorem above and the Fundamental Theorem together imply:

Corollary 4. The pair $(\mathbf{x}^*, \mathbf{y}^*)$ in the Fundamental Theorem is in equilibrium.

Summary

The following hold for any 2-person zero-sum game.

- There always exists a pair of mixed strategies $(\mathbf{x}^*, \mathbf{y}^*)$ such that $\mathbf{x}^*$ is optimal for Player I, $\mathbf{y}^*$ is optimal for Player II, and this optimal pair of mixed strategies $(\mathbf{x}^*, \mathbf{y}^*)$ is in equilibrium.

- The notion of an equilibrium pair of mixed strategies and that of an optimal pair of mixed strategies are identical.

  (This is not true for 2-person non-zero-sum games as we will see later.)

- The respective optimal security levels for the two players are equal, that is, $v_1 = v_2$, this is also called the value of the game and denoted by v.

- Solving a 2-person zero-sum game means to find an equilibrium pair of mixed strategies (or an equilibrium pair of pure strategies if it exists) together with the value $v = v_1 = v_2$ of the game.

- An equilibrium pair $(\mathbf{x}^*, \mathbf{y}^*)$ together with v_1 and v_2 can always be computed by applying the simplex method to one of the linear programming problems as described in the proof of the Fundamental Theorem of Game Theory. But, in some examples other (less laborious) methods will be available to compute an equilibrium pair.

Using the LP-method to solve a 2-person zero-sum game

Recipe: positive entries:

Let

$$\mathbf{0} = (0, \ldots, 0), \;\; \mathbf{1} = (1, \ldots, 1) \;\; \text{(both of dim } m \text{ or } n\text{)}$$

$$\mathbf{x}' = (x'_1, \ldots, x'_m), \;\; \mathbf{y}' = (y'_1, \ldots, y'_n)$$

Then

$$\mathbf{x}'\mathbf{1}^T = x'_1 + \cdots + x'_m, \;\; \mathbf{1}\mathbf{y}'^T = y'_1 + \cdots + y'_n$$

Player I	Player II
$z_1 = \min \mathbf{x}'\mathbf{1}^T$	$z_2 = \max \mathbf{1}\mathbf{y}'^T$
$\mathbf{x}'V \geq \mathbf{1}$	$V\mathbf{y}'^T \leq \mathbf{1}$
$\mathbf{x}' \geq \mathbf{0}$	$\mathbf{y}' \geq \mathbf{0}$

If $\mathbf{x}'$ and $\mathbf{y}'$ are respective optimal solutions, then $v_1 = 1/z_1 = 1/z_2 = v_2$ is the value of the game, and $\mathbf{x}^* = v_1\mathbf{x}', \;\; \mathbf{y}^* = v_1\mathbf{y}'$ is an equilibrium pair of mixed strategies.

We may apply the simplex method to either problem, and having done that, read off the reduced costs of the slack variables to find a solution of the dual.

It is easier to deal with the second LP problem since no artificial variables are required.

Example 14. (LP-method, all entries positive) Solve the 2-person zero-sum game with payoff matrix

$$V = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$$

Example (cont.)

Recipe: not all entries of $V = (v_{ij})_{m\times n}$ are positive

Choose a sufficiently large constant $c > 0$ such that all $v_{ij} + c > 0$. (Any $c > \max\{-v_{ij} : 1 \le i \le m, 1 \le j \le n\}$ will do the job.) Let

$$V' = V + c = (v_{ij} + c)_{m\times n}.$$

Player I	Player II
$z_1 = \min \mathbf{x}'\mathbf{1}^T$	$z_2 = \max \mathbf{1}{\mathbf{y}'}^T$
$\mathbf{x}'V' \ge \mathbf{1}$	$V'{\mathbf{y}'}^T \le \mathbf{1}$
$\mathbf{x}' \ge \mathbf{0}$	$\mathbf{y}' \ge \mathbf{0}$

If $\mathbf{x}'$ and $\mathbf{y}'$ are respective optimal solutions, then $v_1' = 1/z_1 = 1/z_2 = v_2'$ is the value of game V', and $\mathbf{x}^* = v_1'\mathbf{x}', \;\; \mathbf{y}^* = v_1'\mathbf{y}'$ is an equilibrium pair of mixed strategies for V'.
$(\mathbf{x}^*, \mathbf{y}^*) = (v_1'\mathbf{x}', v_1'\mathbf{y}')$ is also an equilibrium pair of mixed strategies for the original game V. And the value of game V is equal to $v_1' - c$.

Example 15. (LP-method, not all entries positive)
Consider the 2-person zero-sum game, with payoff matrix below

$$V = \begin{bmatrix} -2 & 1 & -3 \\ -1 & -1 & 2 \\ 3 & 0 & -1 \end{bmatrix}.$$

Solve this game.

Note that $L = \max\{-3, -1, -1\} = -1$ and $U = \min\{3, 1, 2\} = 1$. Since $L < U$ there is no saddle point. The value v may not be positive. In fact, since $L \leq v \leq U$, we have $-1 \leq v \leq 1$.
We add a "sufficiently large" constant, say $c = 4$, to every element of the matrix to ensure that the value is positive.

Remark 2. Adding a constant, c to every element of a payoff matrix does not change the optimal strategies, but the new matrix will have c added to its value.

Remark 3. Making one row strictly positive is sufficient to give a game with positive value v. Of course, you may add a larger c so that every entry in the resultant matrix is positive.

Example (cont.)

Let $V' = V + 2 = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 4 \\ 5 & 2 & 1 \end{bmatrix}$.

The corresponding LP problems are

Player I	Player II
$1/v_1' = \min x_1' + x_2' + x_3'$	$1/v_2' = \max y_1' + y_2' + y_3'$
$x_2' + 5x_3' \geq 1$	$3y_2 - y_3' \leq 1$
$3x_1' + x_2' + 2x_3' \geq 1$	$y_1' + y_2' + 4y_3' \leq 1$
$-x_1' + 4x_2' + x_3' \geq 1$	$5y_1' + 2y_2' + y_3' \leq 1$
$x_1', x_2', x_3' \geq 0$	$y_1', y_2', y_3' \geq 0$

We prefer Player II's formulation.

Example (cont.)
We use the Simplex method with starting with the following initial tableau with slack variables y_4', y_5', y_6'.

	y_1'	y_2'	y_3'	y_4'	y_5'	y_6'	RHS
y_4'	0	3	-1	1	0	0	1
y_5'	1	1	4	0	1	0	1
y_6'	$\mathbf{5}$	2	1	0	0	1	1
z	-1	-1	-1	0	0	0	0

	y_1'	y_2'	y_3'	y_4'	y_5'	y_6'	RHS
y_4'	0	$\mathbf{3}$	-1	1	0	0	1
y_5'	0	$\frac{3}{5}$	$\mathbf{\frac{19}{5}}$	0	1	$-\frac{1}{5}$	$\frac{4}{5}$
y_1'	1	$\frac{2}{5}$	$\frac{1}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$
z	0	$-\frac{3}{5}$	$-\frac{4}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$

Example (cont.)

	y_1'	y_2'	y_3'	y_4'	y_5'	y_6'	RHS
y_4'	0	$\mathbf{\frac{60}{19}}$	0	1	$\frac{5}{19}$	$-\frac{1}{19}$	$\frac{23}{19}$
y_3'	0	$\frac{3}{19}$	1	0	$\frac{5}{19}$	$-\frac{1}{19}$	$\frac{4}{19}$
y_1'	1	$\frac{7}{19}$	0	0	$-\frac{1}{19}$	$\frac{4}{19}$	$\frac{3}{19}$
z	0	$-\frac{9}{19}$	0	0	$\frac{4}{19}$	$\frac{3}{19}$	$\frac{7}{19}$

	y_1'	y_2'	y_3'	y_4'	y_5'	y_6'	RHS
y_2'	0	1	0	$\frac{19}{60}$	$\frac{5}{60}$	$-\frac{1}{60}$	$\frac{23}{60}$
y_3'	0	0	1	$-\frac{3}{60}$	$\frac{15}{60}$	$-\frac{3}{60}$	$\frac{9}{60}$
y_1'	1	0	0	$-\frac{7}{60}$	$-\frac{5}{60}$	$\frac{13}{60}$	$\frac{1}{60}$
z	0	0	0	$\frac{9}{60}$	$\frac{15}{60}$	$\frac{9}{60}$	$\frac{33}{60}$

We use the final tableau to compute the value of the game and an optimal strategy pair.

$\frac{1}{v'} = \frac{33}{60} \Longrightarrow v' = \frac{60}{33}$.

$\mathbf{y}' = \left(\frac{1}{60}, \frac{23}{60}, \frac{9}{60}\right) \Longrightarrow \mathbf{y}^* = \frac{60}{33}\mathbf{y}' = \left(\frac{1}{33}, \frac{23}{33}, \frac{9}{33}\right)$.

$\mathbf{x}' = \left(\frac{9}{60}, \frac{15}{60}, \frac{9}{60}\right) \Longrightarrow \mathbf{x}^* = \frac{60}{33}\mathbf{x}' = \left(\frac{9}{33}, \frac{15}{33}, \frac{9}{33}\right)$.

The strategy pair $(\mathbf{x}^*, \mathbf{y}^*)$ is also an equilibrium pair of mixed strategies for the original game V, and the value of the original game is

$$v = v' - 2 = \frac{60}{33} - 2 = -\frac{6}{33}.$$

Computation techniques

The following techniques can be used to solve 2-person zero-sum games.

- Saddle points
- 2×2 formulae
- Dominance elimination
- Graphical method for $2 \times n$ or $m \times 2$ games
- Linear programming (ultimate method for solving 2-person zero-sum games)

Saddle points

If V has a saddle point, then we obtain the value of the game and an equilibrium pair of **pure** strategies easily.

Example 16. (A 2-person zero-sum game with saddle points, cf. Example 7)

Payoff matrix:

$$V = \begin{bmatrix} 3 & 2 & 10 \\ 4 & 1 & 5 \\ 8 & -1 & -2 \end{bmatrix}.$$

$\mathbf{2 \times 2}$ games

Theorem 8. A 2×2 matrix $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has no saddle point (i.e. the game with payoff matrix V has no equilibrium pair of pure strategies) if and only if both a and d are larger than b and c, or both a and d are smaller than b and c, i.e.,

either

$$a > b\ ,\ a > c\ ,\ d > b\ ,\ d > c$$

or

$$a < b\ ,\ a < c\ ,\ d < b\ ,\ d < c.$$

Although this result is of theoretical value, you do not need it when testing whether a particular 2×2 matrix has a saddle point.

Proof.

By using the previous theorem, we can prove:

Theorem 9. Suppose $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has no saddle point. Denote

$$r = a + d - c - b.$$

Then the value of 2-person zero-sum game V is equal to

$$v = \frac{ad - bc}{r}$$

and an equilibrium pair of strategies is given by

$$\mathbf{x}^* = \left(\frac{d-c}{r}, \frac{a-b}{r} \right), \quad \mathbf{y}^* = \left(\frac{d-b}{r}, \frac{a-c}{r} \right).$$

Proof.
V does not have saddle points, the previous theorem implies that $r \neq 0$ and r is of the same sign as each of

$$a-b,\ a-c,\ d-b,\ d-c.$$

This ensures that $\mathbf{x}^* \in X$ and $\mathbf{y}^* \in Y$.

To show that $(\mathbf{x}^*, \mathbf{y}^*)$ is in equilibrium, it remains to prove that for any $\mathbf{x} \in X$ and any $\mathbf{y} \in Y$,

$$\mathbf{x} V \mathbf{y}^{*\top} \leq \mathbf{x}^* V \mathbf{y}^{*\top} \leq \mathbf{x}^* V \mathbf{y}^\top.$$

Example 17. (2×2 formulae)
Use the formula for solving 2×2 zero-sum games to solve the 2-person zero-sum with payoff matrix

$$V = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}.$$

Dominance elimination

Definition 11. Let $V = (v_{ij})_{m \times n}$.

- Row i of V is dominated by row k if $v_{ij} \leq v_{kj}$ for $j = 1, \ldots, n$ and $v_{ij} < v_{kj}$ for at least one j.
- Column j of V is dominated by column p if $v_{ij} \geq v_{ip}$ for $i = 1, \ldots, m$ and $v_{ij} > v_{ip}$ for at least one i.

Example 18. (dominance)
Consider the 2-person zero-sum game with the following pay-off matrix

$$V = \begin{bmatrix} 1 & -3 & -2 & 1 \\ 5 & 7 & 8 & -1 \\ 8 & 0 & -4 & -1 \end{bmatrix}.$$

Theorem 10. Let V' be the matrix obtained from $V = (v_{ij})_{m\times n}$ by eliminating dominated rows and columns successively (in any order). Let I denote the set of indices of dominated rows and J the set of indices of dominated columns. Suppose $\mathbf{x}' = (x'_i)_{i\in\{1,\ldots,m\}\setminus I}$ and $\mathbf{y}' = (y'_j)_{j\in\{1,\ldots,n\}\setminus J}$ is an equilibrium pair of strategies for V'.

Then the strategy pair $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_n)$ defined by

$$x_i := \begin{cases} x'_i & , \quad i \notin I \\ 0 & , \quad i \in I \end{cases}$$

and

$$y_j := \begin{cases} y'_j & , \quad j \notin J \\ 0 & , \quad j \in J \end{cases}$$

is an equilibrium pair of strategies for V. Moreover, the games V and V' have the same value.

Proof.

Example 19. (dominance)
Consider the 2-person zero-sum game with the following pay-off matrix

$$V = \begin{bmatrix} 1 & -3 & -2 & 1 \\ 5 & 7 & 8 & -1 \\ 8 & 0 & -4 & -1 \end{bmatrix}.$$

Solve this game.

Remarks on dominance elimination

The theorem says that any optimal strategy pair of the reduced game (i.e. with dominated rows and columns deleted) gives rise to an optimal strategy pair of the original game. The converse of this statement is not true.

It is possible that an optimal strategy pair of the original game cannot be obtained from any optimal strategy pair of the reduced game. Thus, we may lose some optimal solutions of the original game.

Graphical method for $\mathbf{2} \times n$ games

Let $V = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ v_{21} & \cdots & v_{2n} \end{bmatrix}$. Since $m = 2$, we have

$$\begin{aligned} X &= \{(x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \geq 0\} \\ &= \{(x_1, 1 - x_1) : 0 \leq x_1 \leq 1\} \end{aligned}$$

and

$$\begin{aligned} v_1 &= \max_{\mathbf{x} \in X} \min_{1 \leq j \leq n} \mathbf{x} V_{\cdot j} \\ &= \max_{0 \leq x_1 \leq 1} \min_{1 \leq j \leq n} [x_1 v_{1j} + (1 - x_1) v_{2j}] \\ &= \max_{0 \leq x_1 \leq 1} \min_{1 \leq j \leq n} [(v_{1j} - v_{2j}) x_1 + v_{2j}] \end{aligned}$$

By the Fundamental Theorem, v_1 is the value of the game, and if this is attained at x_1, then $(x_1, 1 - x_1)$ is an optimal strategy for Player I.

Denote

$$f(x_1) := \min_{1 \leq j \leq n} [(v_{1j} - v_{2j})x_1 + v_{2j}], \quad 0 \leq x_1 \leq 1.$$

The graph of $f(x_1)$ is piecewise linear with vertices where two or more linear functions intersect.

The maximum of $f(x_1)$ is attained at one of these intersecting vertices or at $x_1 = 0$ or $x_1 = 1$ (end-points).

Identifying this maximum vertex (maximin) yields x_1 and v_1, and hence an optimal $\mathbf{x}^* = (x_1, 1 - x_1)$ for Player I.

The two columns corresponding to the two lines intersecting at the maximin vertex gives rise to a 2×2 subgame. Solving gives an optimal strategy $\mathbf{y}^*$ for Player II.

In this way we obtain an equilibrium pair of strategies $(\mathbf{x}^*, \mathbf{y}^*)$.

Example 20. (graphical method)
Solve the 2-person 2×3 zero-game with payoff matrix

$$V = \begin{bmatrix} 2 & 8 & 3 \\ 6 & 0 & 2 \end{bmatrix}.$$

Graphical method for $\mathbf{m \times 2}$ games

Let $V = \begin{bmatrix} v_{11} & v_{12} \\ \vdots & \vdots \\ v_{m1} & v_{m2} \end{bmatrix}$. Then

$$\begin{aligned} Y &= \{(y_1, y_2) : y_1 + y_2 = 1, y_1, y_2 \geq 0\} \\ &= \{(y_1, 1 - y_1) : 0 \leq y_1 \leq 1\} \end{aligned}$$

and

$$\begin{aligned} v_2 &= \min_{\mathbf{y} \in Y} \max_{1 \leq i \leq m} V_{i \cdot} \mathbf{y}^T \\ &= \min_{0 \leq y_1 \leq 1} \max_{1 \leq i \leq m} [(v_{i1} - v_{i2}) y_1 + v_{i2}] \end{aligned}$$

By the Fundamental Theorem, v_2 is the value of the game, and if this is attained at y_1, then $(y_1, 1 - y_1)$ is an optimal strategy for Player II.

The rest is symmetrical to the $2 \times n$ case with $f(x_1)$ replaced by

$$g(y_1) = \max_{1 \leq i \leq m} [(v_{i1} - v_{i2}) y_1 + v_{i2}], \quad 0 \leq y_1 \leq 1.$$

Linear programming

A few remarks in order:

- Linear programming is the ultimate method for solving a 2-person zero-sum game.
- In the case when some entry $v_{ij} \leq 0$, we add a sufficiently large constant c to every entry of V so that every entry of the resultant matrix is strictly positive.
- Making one row of the new matrix strictly positive is sufficient.

See slides 57-65 for some examples.

Two-person constant-sum games

A 2-person game with payoff bi-matrix

$$
\begin{array}{ll|ccccc}
 & & & & \text{Player II} & & \\
 & & A_1 & \ldots & A_j & \ldots & A_n \\
\hline
 & a_1 & (a_{11}, b_{11}) & \ldots & (a_{1j}, b_{1j}) & \ldots & (a_{1n}, b_{1n}) \\
 & \vdots & \vdots & & \vdots & & \vdots \\
\text{Player I} & a_i & (a_{i1}, b_{i1}) & \ldots & (a_{ij}, b_{ij}) & \ldots & (a_{in}, b_{in}) \\
 & \vdots & \vdots & & \vdots & & \vdots \\
 & a_m & (a_{m1}, b_{m1}) & \ldots & (a_{mj}, b_{mj}) & \ldots & (a_{mn}, b_{mn})
\end{array}
$$

is called a constant-sum game if

$$a_{ij} + b_{ij} = c, \quad 1 \leq i \leq m, 1 \leq j \leq n$$

for some **constant** c.

Example 21. (2-person constant sum game)
In a certain time slot, two TV networks are vying for 100 million viewers. They each have the same three choices for that time slot. Surveys suggest the following numbers of viewers would tune in to each network (in millions).

	Western	Soap Opera	Comedy
Western	(35,65)	(15,85)	(60,40)
Soap Opera	(45,55)	(58,42)	(50,50)
Comedy	(38,62)	(14,86)	(70,30)

Conversion of a 2-person constant-sum game to a zero-sum game

In general, if in a 2-person game the sum of the payoffs to Players I and II is a constant c, then by subtracting $c/2$ from each coordinate of every entry of the bi-matrix we obtain a 2-person zero-sum game.

Solving this 2-person zero-sum game, we obtain an equilibrium pair of (pure or mixed) strategies for the original constant-sum game.

How do we find the corresponding expected payoff to each player?

Example 22. (solving a constant sum game)
Solve the constant-sum game with payoff matrix

$$\begin{pmatrix} (35, 65) & (15, 85) & (60, 40) \\ (45, 55) & (58, 42) & (50, 50) \\ (38, 62) & (14, 86) & (70, 30) \end{pmatrix}.$$

Summary: procedures for solving 2-person zero-sum games

- Check for saddle points.

- If there exists a saddle point, find one, say $v_{i^*j^*}$. The value of the game is $v = v_{i^*j^*}$ and (a_{i^*}, A_{j^*}) is an equilibrium pair of pure strategies. (If there are a few saddle points then we get a few equilibrium pairs, but the value of the game is unique.)

- If there is no saddle, do the following:

  1. Try to reduce the size of the matrix by dominance elimination.
  2. If the reduced game is of size 2×2, you may use the 2×2 formulae.
  3. If the reduced matrix is $2 \times m$ or $n \times 2$, use the graphical method plus the 2×2 formulae or linear programming.
  4. Otherwise use the linear programming method.
  5. Of course in all situations we can simply use the linear programming method.