FORMULA SHEET FOR MAST30030 Applied Mathematical Modelling

Vector Identities

$$\nabla\cdot(\phi\,\mathbf{q}) = (\nabla\phi)\cdot\mathbf{q} + \phi\nabla\cdot\mathbf{q} \qquad \nabla\times(\phi\,\mathbf{q}) = (\nabla\phi)\times\mathbf{q} + \phi\nabla\times\mathbf{q}$$

$$\nabla\cdot(\nabla\times\mathbf{q}) = 0 \qquad \nabla\times(\nabla\phi) = \mathbf{0} \qquad \nabla\times(\nabla\times\mathbf{q}) = \nabla(\nabla\cdot\mathbf{q}) - \nabla^2\mathbf{q}$$

$$\nabla\times(\mathbf{p}\times\mathbf{q}) = \mathbf{p}(\nabla\cdot\mathbf{q}) - \mathbf{q}(\nabla\cdot\mathbf{p}) + (\mathbf{q}\cdot\nabla)\mathbf{p} - (\mathbf{p}\cdot\nabla)\mathbf{q}$$

$$\nabla(\mathbf{p}\cdot\mathbf{q}) = (\mathbf{p}\cdot\nabla)\mathbf{q} + (\mathbf{q}\cdot\nabla)\mathbf{p} + \mathbf{p}\times(\nabla\times\mathbf{q}) + \mathbf{q}\times(\nabla\times\mathbf{p})$$

Polar Coordinates

For *cylindrical polar coordinates* σ, φ, z, with z measured along the axis of the cylinder, σ the distance from the axis of the cylinder, and φ the azimuthal angle:

$$\nabla f = \hat{\boldsymbol{\sigma}}\,\frac{\partial f}{\partial\sigma} + \hat{\boldsymbol{\varphi}}\,\frac{1}{\sigma}\frac{\partial f}{\partial\varphi} + \hat{\mathbf{z}}\,\frac{\partial f}{\partial z} \qquad \nabla\cdot(u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}) = \frac{1}{\sigma}\frac{\partial}{\partial\sigma}(\sigma u) + \frac{1}{\sigma}\frac{\partial v}{\partial\varphi} + \frac{\partial w}{\partial z}$$

$$\nabla\times(u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}) = \left\{\frac{1}{\sigma}\frac{\partial w}{\partial\varphi} - \frac{\partial v}{\partial z}\right\}\hat{\boldsymbol{\sigma}} + \left\{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial\sigma}\right\}\hat{\boldsymbol{\varphi}} + \left\{\frac{1}{\sigma}\frac{\partial}{\partial\sigma}(\sigma v) - \frac{1}{\sigma}\frac{\partial u}{\partial\varphi}\right\}\hat{\mathbf{z}}$$

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial f}{\partial\sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

For *spherical polar coordinates* r, θ, φ, with r the distance from the origin, θ the colatitudinal angle and φ the azimuthal angle:

$$\nabla f = \hat{\mathbf{r}}\,\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\boldsymbol{\varphi}}\,\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}$$

$$\nabla\cdot(u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi}$$

$$\nabla\times(u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}}) = \left\{\frac{\partial}{\partial\theta}(w\sin\theta) - \frac{\partial v}{\partial\varphi}\right\}\frac{\hat{\mathbf{r}}}{r\sin\theta} + \left\{\frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi} - \frac{\partial}{\partial r}(rw)\right\}\frac{\hat{\boldsymbol{\theta}}}{r} + \left\{\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial\theta}\right\}\frac{\hat{\boldsymbol{\varphi}}}{r}$$

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Stream Functions and Potentials

If $\nabla\times\mathbf{q} = \mathbf{0}$ in a simply-connected domain, then $\mathbf{q} = \nabla\phi$.

If $\nabla\cdot\mathbf{q} = 0$, then $\mathbf{q} = \nabla\times\mathbf{A}$.

In two-dimensional flow, if $\nabla\cdot\mathbf{q} = 0$, then $\mathbf{q} = \nabla\times(\psi\,\hat{\mathbf{k}})$, where $\hat{\mathbf{k}}$ is the unit basis vector associated with the z direction, and ψ is independent of z.

In axisymmetric three-dimensional flow, if $\nabla\cdot\mathbf{q} = 0$, then with Λ and χ independent of the azimuthal angle φ,

$$\mathbf{q} = \nabla\times\left(\frac{\Lambda}{\sigma}\,\hat{\boldsymbol{\varphi}}\right) + \chi\,\hat{\boldsymbol{\varphi}} = \left\{\frac{1}{r^2\sin\theta}\frac{\partial\Lambda}{\partial\theta}\right\}\hat{\mathbf{r}} - \left\{\frac{1}{r\sin\theta}\frac{\partial\Lambda}{\partial r}\right\}\hat{\boldsymbol{\theta}} + \chi\,\hat{\boldsymbol{\varphi}}.$$

The rate-of-strain tensor

The rate-of-strain tensor $\mathbf{e}$ is related to the velocity field $\mathbf{q}$ by the equation $\mathbf{e} = \frac{1}{2}\{\nabla\mathbf{q}+(\nabla\mathbf{q})^{\mathrm{T}}\}$, where $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of the tensor $\mathbf{A}$.

For *cylindrical polar coordinates* σ, φ, z, if $\mathbf{q} = u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}$, the components of $\mathbf{e}$ are

$$
\begin{aligned}
e_{\sigma\sigma} &= \frac{\partial u}{\partial \sigma} \\
e_{\varphi\varphi} &= \frac{1}{\sigma}\frac{\partial v}{\partial \varphi} + \frac{u}{\sigma} \\
e_{zz} &= \frac{\partial w}{\partial z} \\
e_{\sigma\varphi} &= e_{\varphi\sigma} = \frac{\sigma}{2}\frac{\partial}{\partial \sigma}\left(\frac{v}{\sigma}\right) + \frac{1}{2\sigma}\frac{\partial u}{\partial \varphi} \\
e_{\sigma z} &= e_{z\sigma} = \frac{1}{2}\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial w}{\partial \sigma} \\
e_{z\varphi} &= e_{\varphi z} = \frac{1}{2\sigma}\frac{\partial w}{\partial \varphi} + \frac{1}{2}\frac{\partial v}{\partial z}
\end{aligned}
$$

For *spherical polar coordinates* r, θ, φ, if $\mathbf{q} = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}}$, the components of $\mathbf{e}$ are

$$
\begin{aligned}
e_{rr} &= \frac{\partial u}{\partial r} \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \\
e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{u}{r} + \frac{v\cot\theta}{r} \\
e_{r\theta} &= e_{\theta r} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{2r}\frac{\partial u}{\partial \theta} \\
e_{r\varphi} &= e_{\varphi r} = \frac{1}{2r\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{w}{r}\right) \\
e_{\theta\varphi} &= e_{\varphi\theta} = \frac{\sin\theta}{2r}\frac{\partial}{\partial \theta}\left(\frac{w}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial v}{\partial \varphi}
\end{aligned}
$$

These formulae may also be used to deduce the polar coordinate representations of the linear strain tensor in the linear theory of elasticity.