# MAT4MDS — Practice 2

## Functions

| Topics 2 and 3 covered: |
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| • representing functions in formulae (rules) and graphs<br>• using transformations to produce new graphs from known graphs<br>• composition of functions<br>• inverse functions. |

An important function that is used to model data is the cumulative distribution function of the Pareto distribution:

$$F : [a, \infty) \to \mathbb{R} \quad F(x) = 1 - \left(\frac{a}{x}\right)^b$$

Here $b$ is called the shape parameter and $b > 0$. As well as being the minimum input value, $a > 0$ is called the scale parameter. Questions 1–4 all relate to this function.

**Question 1.**

(a) What is $F(a)$?
(b) As $x$ gets very large, what will happen to the graph of $F$? (Answer from the rule.)

**Question 2.** We are now going to construct the graph of $F$.

(a) First sketch the graph of $x^{-b}$, for $x \geq 1$. How would this be different if $b$ were increased?
(b) Now sketch the graph of $\left(\frac{x}{a}\right)^{-b}$ for $x \geq a$. How would this be different if $a$ were increased?
(c) Reflect your graph in the $x$-axis to obtain the graph of $-\left(\frac{x}{a}\right)^{-b}$.
(d) What is the final step needed to complete the graph of $F(x)$? (Identify it and then perform it.)

Check that your graph fulfils the two behaviours found in Question 1.

**Question 3.** The function $F$ (from above) can be written as a function composition in more than one way.

(a) Let $f(x) = \frac{a}{x}$. Identify a function $g$ such that $g\big(f(x)\big) = F(x)$.
(b) Let $h(x) = x^b$. Identify a function $k$ such that $k\big(h(x)\big) = F(x)$.

**Question 4.**

(a) Consider your final graph for $F$ in Question 2. How do you know that $F^{-1}$ will exist? What input values will it take (i.e. its domain)? What output values will it give (i.e. its range)?
(b) By reflecting in the line $y = x$, sketch the graph of $F^{-1}$.
(c) Following the model in Example 2.4.3 in the reading on inverse functions, write $y = F^{-1}(x)$ and use the fact that $F(F^{-1}(x)) = x$ to find the rule for $F^{-1}$.

**Question 5.** The Pareto distribution is an example of a heavy-tailed distribution. (Extreme events have a higher probability than in the exponential distribution.) Another such distribution is the Cauchy distribution, the probability density function of which behaves like:

$$f : \mathbb{R} \to \mathbb{R} \;\; f(x) = \frac{1}{1+x^2}.$$

(a) Identify two functions $h$ and $k$, such that $f(x)$ can be written as the composition $h(k(x))$.
(b) Sketch the graph of $k$, where $k(x) = 1 + x^2$. Use this to help you sketch the graph of $f$.
(c) Does $f(x)$ have an inverse? What about the following related function?

$$g : (-\infty, 0] \to \mathbb{R} \;\; g(x) = \frac{1}{1+x^2}.$$

(d) Find the inverse function $g^{-1}$ using the method in Question 4(c). Also sketch the graph of $g^{-1}$.

**Question 6.** Consider this restriction of the Gaussian curve:

$$f : [0, \infty) \to \mathbb{R} \;\; f(x) = e^{-x^2}.$$

Find an expression for the rule of $f^{-1}$.