THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 2 Assessment, 2016

School of Mathematics and Statistics

MAST30001 Stochastic Modelling

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 9 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- This paper has **7 questions**. Attempt as many questions, or parts of questions, as you can. The number of marks allocated to each question is shown in the brackets after the question statement. There are **100 total marks** available for this examination. Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

1. (a) Analyse the state space $S = \{1, 2, 3, 4\}$ for each of the three Markov chains given by the following transition matrices. That is, classify each state as essential or not, transient or positive recurrent or null recurrent, and find the period if it is periodic.

   i.
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

   ii.
$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   iii.
$$\begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

   (b) For the Markov chain given by the transition matrix in part (a) iii above, discuss the long run behaviour of the chain including deriving long run probabilities.

   (c) For the Markov chain given by the transition matrix in part (a) iii above, find the expected number of steps taken for the chain to first reach state 3 given the chain starts at state 4.

[14 marks]

Ans.

(a) (6 marks) Since all of these Markov chains are finite, any essential communicating classes are positive recurrent and non-essential classes are transient.

i. The chain is irreducible with period 3.

ii. There are three communicating classes: $\{1, 3\}, \{4\}, \{2\}$. The first two are essential and the last non-essential. All classes are aperiodic due to the presence of loops.

iii. The chain is irreducible and aperiodic because of loops.

(b) (4 marks) The chain is irreducible, aperiodic, and positive recurrent and so the long run probabilities are given by the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ satisfying

$$\pi \begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} = \pi.$$

Solving shows $\pi = (1/8, 1/2, 1/8, 1/4)$.

(c) (4 marks) We perform a first step analysis. Let e_i be the expected time to reach state 3 given the chain starts at state i. Then first step analysis implies

$$\begin{aligned} e_1 &= 1 + \frac{2}{3}e_4, \\ e_2 &= 1 + \frac{2}{3}e_2 + \frac{1}{3}e_4, \\ e_4 &= 1 + \frac{1}{2}e_1 + \frac{1}{2}e_2, \end{aligned}$$

and so $e_4 = 18$.

2. Let $(N_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ be independent Poisson processes with rates λ and μ. Your answers to the questions below should be simple and tidy formulas in terms of λ and μ.

   (a) What is the expected time of the first arrival of N that occurs after the tenth arrival of K?

   (b) What is the chance that the number of arrivals of K in the interval $(2, 5)$ is at least two?

   (c) What is the expected time until the first arrival of either N or K?

   (d) What is the expected time between the first arrivals of N and K?

   (e) What is the expected number of arrivals of N between the second and third arrival of K?

   (f) Given there are 10 total arrivals of N and K in the interval $(0, 1)$, what is the chance that exactly 5 of these are from N and 5 are from K?

   (g) Given that $N_{10} = 3$, what is the chance that $N_3 = 1$?

   (h) Given that $N_{10} + K_{10} = 3$, what is the chance that $N_3 = 1$?

[20 marks]

Ans.

(a) (2 marks) Due to the memoryless property of the exponential, the time until the next arrival of the N process after any arrival of the K process is exponential rate λ, which has mean $1/\lambda$. The tenth arrival of the K process has mean $10/\mu$ and so the total time until the first arrival of N after the tenth arrival of K is

$$\frac{10}{\mu} + \frac{1}{\lambda}.$$

(b) (2 marks) $K_5 - K_2$ is Poisson mean 3μ and so

$$P(K_5 - K_2 \geq 2) = 1 - e^{-3\mu}(1 + 3\mu).$$

(c) (2 marks) By superposition or that minimums of independent exponentials are exponential, the time until the first arrival of $N + K$ is exponential rate $\lambda + \mu$ with mean $1/(\lambda + \mu)$.

(d) (3 marks) Let S, T be the first arrivals of K, N. Given $S < T$ ($T < S$) then due to the memoryless property of the exponential, the time between first arrivals is distributed exponential rate λ (μ) with mean $1/\lambda$ ($1/\mu$). The probability $S < T$ is $\mu/(\lambda + \mu)$, so the expected time between these arrivals is

$$\frac{\mu}{\lambda(\lambda + \mu)} + \frac{\lambda}{\mu(\lambda + \mu)} = \frac{\lambda^2 + \mu^2}{\lambda\mu(\lambda + \mu)}.$$

(e) (3 marks) If we let $(M_t)_{t\geq 0}$ be a Poisson rate $\lambda + \mu$ and then thin with probabilities $\lambda/(\mu + \lambda)$ and $\mu/(\lambda + \mu)$ into two independent processes distributed as (N, K), then the number of points of the N process between the second and third points of the K process is geometric with parameter $\mu/(\mu + \lambda)$ having mean λ/μ.

Alternatively, the time T between the second and third arrival of K is exponential μ and so the number of arrivals of N in this interval is Poisson mean λT which has mean $\lambda E[T] = \lambda/\mu$.

(f) (2 marks) As in part (e), if we view (N, K) as coming from a thinned M, then given $M_1 = 10$, the number of arrivals thinned to the N process is binomial with parameters 10 and $\lambda/(\mu + \lambda)$. The chance this binomial is five is

$$\binom{10}{5}\left(\frac{\lambda}{\lambda+\mu}\right)^5\left(\frac{\mu}{\lambda+\mu}\right)^5.$$

(g) (3 marks) Given $N_{10} = 3$, the positions of these 3 arrivals in the interval $(0, 10)$ are uniform and independent. Thus the chance any particular one lands in $(0, 3)$ is $3/10$ and since there are three points, the number landing in $(0, 3)$ is binomial with parameters 3 and $3/10$. The probability this binomial is equal to one is $3 \cdot 7^2 \cdot 3/10^3 = 0.441$.

(h) (3 marks) Again we look at the M process and thin it to see N and K. The chance any particular point ends up thinned to N and then in the interval $(0, 3)$ is $\frac{\lambda}{\lambda+\mu}\frac{1}{10}$ and each independently. Thus, given $M_{10} = 3$, the number of points of N landing in $(0, 3)$ is binomial with parameters 3 and $\frac{\lambda}{\lambda+\mu}\frac{3}{10}$ and the probability this binomial is 1 is

$$3\left(\frac{3\lambda}{10(\lambda+\mu)}\right)\left(1-\frac{3\lambda}{10(\lambda+\mu)}\right)^2.$$

3. A company with a subscription to a large cloud computing facility places jobs according to a rate λ Poisson process. If a job arrives and there is at least one server free, then the job is served immediately and the server becomes busy. If a job arrives and there are no servers free, then the job is placed in a queue and served in the order it was received. The subscription only allows for at most three jobs in the queue, and any jobs arriving when the queue is full are rejected. In addition, servers from the facility become free according to a rate μ Poisson process. If there are no customers waiting for service when a server becomes free, then it is placed in a queue of servers waiting for jobs. There is no cap on the number of servers that can queue. Note that it is not possible to have both jobs and servers waiting in queues simultaneously. Let X_t be the number of servers minus the number of jobs in the system at time t.

   (a) Model $(X_t)_{t\geq 0}$ as a continuous time Markov chain: specify its state space and generator matrix A.
   (b) Derive a condition on λ, μ so that the chain is ergodic.

   For the remainder of the problem, assume that $\lambda = 5$ and $\mu = 2$.

   (c) Describe the long run behaviour of the chain.
   (d) What long run proportion of the time are there servers waiting for jobs?
   (e) What long run proportion of jobs are rejected due to the queue being full?
   (f) What is the long run average number of jobs in the queue?
   (g) What is the long run average waiting time for a job arriving in the system?

[22 marks]

Ans.

(a) (4 marks) The state space of the chain is $\{-3, -2, -1, 0, \ldots\}$. If $X_t = k \geq -2$, then the system stays fixed until either a new server gets free or a new customer arrives. The former case occurs at rate μ and then $k \mapsto k + 1$ and the latter case occurs at rate λ and then $k \mapsto k - 1$. If $X_t = -3$, then the queue is full and the only thing that can happen is for a new server to get free in which case $-3 \mapsto -2$. Thus the generator A has entries

$a_{i,i+1} = \mu$ for $i \geq -3$, $a_{i,i-1} = \lambda$ for $\imath \geq -2$ and diagonal entries $a_{i,i} = -(\lambda+\mu)$ for $i \geq -2$ and $a_{-3,-3} = -\mu$.

(b) (2 marks) The system is a birth-death process (albeit shifted so the boundary is at -3 rather than 0) with birth rates μ and death rates λ. Thus there is a stationary distribution only if $\mu < \lambda$.

(c) (4 marks) From lectures we know that the long run and stationary distribution of a birth-death process with constant birth and death rates is a geometric distribution. After shifting we find that the stationary distribution π has

$$\pi_i = \left(\frac{\mu}{\lambda}\right)^{i+3}\left(1 - \frac{\mu}{\lambda}\right) = (2/5)^{i+3}(3/5).$$

(d) (3 marks) Servers are waiting for jobs when $X_t \geq 1$, thus the long run proportion of time there are servers waiting for jobs is

$$\sum_{k\geq 1} \pi_k = 1 - \sum_{k=-3}^{0} \pi_k = \frac{16}{625} = 0.0256.$$

(e) (2 marks) Jobs don't enter the system when $X_t = -3$ and so the long run proportion of jobs rejected is $\pi_{-3} = 3/5 = 0.6$.

(f) (3 marks) There are no customers in the queue if $X_t \geq 0$, and there are $k = 1, 2, 3$ customers in the system if $X_t = -k$. Thus the long run proportion of customers in the queue is

$$\pi_{-1} + 2\pi_{-2} + 3\pi_{-3} = \frac{297}{125} = 2.376.$$

(g) (4 marks) If an arriving job finds the system in state $0, -1, -2$, then the the waiting time has a gamma distribution with parameters $1, 2, 3$ and rate 2 with respective means $1/2, 1, 3/2$. Otherwise the job is served immediately and the waiting time is zero. Due to PASTA, the proportion of customers that enter the system in state k is $\pi_k/(1 - \pi_{-3}) = (2/5)^{k+2}(3/5)$ and so the average waiting time of an entering job is

$$\frac{3}{10}\left(\frac{4}{25} + \frac{4}{5} + 3\right) = \frac{297}{250} = 1.188.$$

Alternatively, Little's law says that $L_q = \lambda' W$, where λ' is the arrival rate of the system and L_q is the average queue length, computed in part (d). The effective arrival rate $\lambda' = 5(1 - \pi_{-3}) = 2$, and so we find that $W = L_q/2 = (297/250) = 1.188$.

4. A certain electrical system requires a single battery. Battery lifetimes in *hours* are independent and have a gamma distribution with density $te^{-t/50}/2500$, $t > 0$. Batteries are replaced as soon as they fail.

   (a) State from formulas (or compute) the mean and variance of battery lifetimes.

   (b) On average, about how many batteries are needed over a 300 day period?

   (c) Give an interval around your estimate from (b) that will have a 95% chance of covering the true number of batteries needed over the 300 day period. Note that

   $$\frac{1}{\sqrt{2\pi}}\int_{-1.96}^{1.96} e^{-x^2/2}dx = 0.95.$$

   (d) If you inspect the electrical system at some point, what would you estimate to be the mean and variance of the length of time that the current battery has been in use?

You may find the following formula useful for this problem: for $a, b > 0$

$$\int_0^\infty t^a e^{-bt} dt = \frac{\Gamma(a+1)}{b^{a+1}}.$$

[13 marks]

Ans.

(a) (2 marks) Using formulas for gamma distributions, the inter-arrival distribution τ has

$$\mu := E[\tau] = 100, \quad \sigma^2 := Var(\tau) = 5000.$$

(b) (3 marks) For t in hours, $N_t/t \to 1/E[\tau] = 1/100$ as $t \to \infty$, and so in the first $24 \times 300 = 7200$ hours we expect to need about $7200/100 = 72$ batteries.

(c) (3 marks) The renewal CLT says that $N_{7200} \approx \text{Normal}(7200/\mu, 7200\sigma^2/\mu^3) = \text{Normal}(72, 36)$ and so for $t = 7200$, we expect with there is a 95% chance the number of trams that have arrived is in the interval

$$72 \pm (1.96)6 = 72 \pm 11.76.$$

(d) (5 marks) Defining the renewal times $T_k = \sum_{i=1}^k \tau_i$, we know that for large t and $A_t := t - T_{N_t}$ roughly has density

$$\frac{1 - F(t)}{\mu} = \frac{e^{-t/50}(50 + t)}{5000} \quad t > 0,$$

and now its only a matter of computing the first and second moments to find $E[A_t] \approx 75$ and $Var(A_t) \approx 4375$.

5. Let $(X_t)_{t \geq 0}$ be a continuous time Markov chain on $\{0, 1, 2, 3, 4\}$ with generator matrix

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

(a) Given the chain is in state 1 right now, what is the expected amount of time until the chain jumps to a new state?

(b) Given the chain is in state 1 right now, what is the probability that the next time the chain jumps, it jumps to state 0?

(c) Find the stationary distribution of the chain.

(d) Find a simple expression for $p_{2,2}(t) := P(X_t = 2 | X_0 = 2)$ that holds for all $t \geq 0$.

(e) More generally, assume now that $(Y_t)_{t \geq 0}$ is a continuous time Markov chain on $\{0, 1, \ldots, 2N\}$ for some $N \geq 2$ with generator matrix A having positive off-diagonal entries given by $a_{i,i+1} = a_{i,i-1} = 1$ for $i = 1, \ldots, 2N - 1$ and $a_{0,N} = a_{2N,N} = 1$. Find the stationary distribution of $(Y_t)_{t \geq 0}$.

[15 marks]

Ans.

(a) (1.5 marks) Since $a_{1,1} = -2$, the time spent in state 1 before jumping is exponential rate 2 having mean 1/2.

(b) (1.5 marks) From state 1, the chain jumps to states 0 and 2 with the same rate 1 so they chance of jumping to state 0 is 1/2 (also the same as $-a_{0,1}/a_{1,1}$).

(c) (3 marks) We need to solve $\pi A = 0$. By symmetry, $\pi_0 = \pi_4$ and $\pi_1 = \pi_3$. From here we easily reduce to three equations:

$$\begin{aligned} -\pi_0 + \pi_1 &= 0 \\ -2\pi_1 + \pi_2 &= 0 \\ 2\pi_0 + 2\pi_1 + \pi_2 &= 1, \end{aligned}$$

which is easily solved to find $\pi = (1/6, 1/6, 1/3, 1/6, 1/6)$.

(d) (5 marks) In the usual notation, we can use the forward equations $\frac{d}{dt} P^{(t)} = P^{(t)} A$ to find

$$p'_{2,2}(t) = p_{2,0}(t) + p_{2,1}(t) - 2p_{2,2}(t) + p_{2,3}(t) + p_{2,4}(t).$$

Using symmetry, we have $p_{2,0}(t) = p_{2,4}(t)$ and $p_{2,1}(t) = p_{2,3}(t)$, so the above equation reduces to

$$p'_{2,2}(t) = 2p_{2,0}(t) + 2p_{2,1}(t) - 2p_{2,2}(t).$$

Further, the symmetry and the law of total probability implies

$$2p_{2,0} + 2p_{2,1}(t) + p_{2,2}(t) = 1.$$

Combining these last two equations, we have

$$p'_{2,2}(t) = 1 - 3p_{2,2}(t),$$

which, with the boundary condition $p_{2,2}(0) = 1$, is easily solved to find

$$p_{2,2}(t) = \frac{1 + 2e^{-3t}}{3}.$$

(e) (4 marks) We use symmetry in the analogous way as above: for $j = 0, \ldots, N-1$, $\pi_j = \pi_{2N-j}$ and $p_{N,j}(t) = p_{N,N-j}(t)$ so in particular

$$\pi_N + 2\sum_{j=0}^{N-1} \pi_j = 1, \tag{1}$$

Solving $\pi A = 0$ yields the equations $\pi_1 = \pi_0$, $\pi_2 = 2\pi_1$, and for $i = 2, \ldots, N-1$,

$$\pi_{i-1} - 2\pi_i + \pi_{i+1} = 0,$$

which, combined with (1) and solving in the usual way yields that for $j = 1, \ldots, N$,

$$\pi_j = \pi_{N-j} = \frac{j}{N^2 + 2},$$

and $\pi_0 = \pi_{2N} = \pi_1$.

6. Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, 2, \ldots\}$ with transition probabilities for $i \geq 1$

$$p_{i,i+1} = 1 - p_{i,i-1} = p < 1/2,$$

and $p_{0,1} = 1 - p_{0,0} = p$. Define the random variable $T = \min\{n \geq 0 : X_n = 0\}$.

(a) Compute $E[T|X_0 = i]$ for each $i = 0, 1, 2, \ldots$, (hint: consider $T' = \min\{n \geq 1 : X_n = 0\}$).

(b) If X_0 has the stationary distribution of the chain, then find $E[T]$.

[7 marks]

Ans.

(a) (4 marks) Let $e_i := E[T|X_0 = i]$. Then $e_0 = 0$ and for $i \geq 1$, first step analysis implies

$$e_i = 1 + pe_{i+1} + (1-p)e_{i-1},$$

which is the same as

$$e_{i+1} = \frac{e_i - 1 - (1-p)e_{i-1}}{p}.$$

We can solve for e_1 by considering $T' = \inf\{n \geq 1 : X_n = 0\}$, noting that if $e'_i := E[T'|X_0 = i]$, then $e'_1 = e_1$, $e'_0 = 1/\pi_0$, where $\pi_0 := 1 - p/(1-p)$ is the stationary probability at zero, and

$$e'_0 = 1 + (1-p)e'_1,$$

so that $e_1 = e'_1 = 1/(1-2p)$. Now plugging in small values for the recursion for e_i, we see the pattern that

$$e_i = \frac{i}{1-2p},$$

and it is easy to check that this satisfies the recursion in general.

(b) (3 marks) From lectures we know that if $p < 1/2$, then the stationary distribution is geometric $1 - p/(1-p)$ and so

$$E[T] = \sum_{i \geq 0} e_i \left(\frac{p}{1-p}\right)^i \frac{1-2p}{1-p} = \frac{p}{(1-2p)^2}.$$

7. Let $0 < T_1 < T_2 < \cdots$ be the times of the arrivals of a rate λ Poisson process $(N_t)_{t\geq 0}$. For $t \geq 0$, let $Y_t = T_{N_t+1} - t$ be the time until the next arrival after t and $A_t = t - T_{N_t}$ be the time until the previous arrival before t of the Poisson process $(N_t)_{t\geq 0}$.

   (a) Find a simple expression in terms of λ and x for $P(Y_t > x)$, $x > 0$.

   (b) Find a simple expression in terms of λ and x for $P(A_t > x)$, $x > 0$.

   (c) Hence show the expected length of the inter-arrival interval containing t is

   $$\frac{2 - e^{-\lambda t}}{\lambda}.$$

   (d) Name the approximate distribution of the length of the inter-arrival interval containing t, for large values of t. (No calculation is required, but instead you can write a sentence or two to justify your answer.)

[9 marks]

Ans.

(a) (2 marks) By the memoryless property of the exponential distribution, the time until the next arrival is exponential rate λ and so $P(Y_t > x) = e^{-\lambda x}$.

(b) (2 marks) For $0 < x < t$, $A_t > x$ corresponds to $N_t - N_{t-x} = 0$, which occurs with probability $e^{-\lambda x}$. For $x \geq t$, $P(A_t > x) = 0$.

(c) (3 marks) The length of the inter-arrival interval containing t is $Y_t + A_t$. $E[Y_t] = 1/\lambda$ and $E[A_t] = \int_0^\infty P(A_t > x)dx = (1 - e^{-\lambda t})/\lambda$ and so the sum of these

$$E[Y_t + A_t] = \frac{2 - e^{-\lambda t}}{\lambda}.$$

(d) (2 marks) By independent increments, A_t and Y_t are independent and as $t \to \infty$, the distribution of A_t converges to an exponential distribution, so there sum converges to a gamma with parameters 2 and rate λ.

End of Exam