MAST30001 Stochastic Modelling – 2018

Assignment 2

If you haven't already, please complete the Plagiarism Declaration Form (available through the LMS) before submitting this assignment.

Don't forget to staple your solutions, and to print your name, student ID, tutorial time and day, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 19 October, 2018 by 4pm** in the appropriate assignment box at the north end of Peter Hall Building (near Wilson Lab).

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. Assume that orders arrive to a packing and shipping warehouse according to a Poisson process with rate 1 per minute. Let $(N_t)_{t\geq 0}$ denote the counting process of the number of orders received in the next t minutes.

   (a) What is the the chance there is at least one order in the next hour?

   (b) What is the chance there is at least one order in the next hour, and all of the orders in the hour arrive within 30 minutes of each other?

   Assume that the warehouse has 3 main areas, and each order is independently and uniformly assigned to an area when it arrives. For $i = 1, 2, 3$, let $(M_t^{(i)})_{t\geq 0}$ denote the counting process of orders for each of the areas.

   (c) What is the chance that in the next 3 minutes, the first area receives exactly 2 orders, the second receives 2, and the third receives 0?

   (d) Given that the first area has 2 orders in the next 3 minutes, what is the chance that there are no orders in the other two areas before the time of the first order of the first area?

   Ans.

   (a) $\mathbb{P}(N_{60} \geq 1) = 1 - \mathbb{P}(N_{60} = 0) = 1 - e^{-60}$.

   (b) Let A be the event that all of the orders in the hour arrive within 30 minutes of each other. We condition on $T_1 \sim \text{Exp}(1)$, the time of the first arrival and use the law of total probability:

$$
\begin{aligned}
\mathbb{P}(A, T_1 < 60) &= \int_0^{60} P(A|T_1 = t)e^{-t}dt \\
&= \int_0^{30} P(A|T_1 = t)e^{-t}dt + \int_{30}^{60} P(A|T_1 = t)e^{-t}dt \\
&= \int_0^{30} P(N_{60} - N_{T_1+30} = 0|T_1 = t)e^{-t}dt + \int_{30}^{60} e^{-t}dt \\
&= \int_0^{30} P(N_{60} - N_{t+30} = 0)e^{-t}dt + e^{-30} - e^{-60} \\
&= \int_0^{30} e^{-(30-t)}e^{-t}dt + e^{-30} - e^{-60} \\
&= 30e^{-30} + e^{-30} - e^{-60},
\end{aligned}
$$

where the third equality is because if $T_1 = t < 30$, then A is the same as to $N_{60} - N_{T_1+30} = 0$, and if $T_1 = t > 30$, then A is automatically satisfied. The fourth equality uses independent increments, and the rest is straightforward.

(c) We can think of generating $(M_t^{(i)})_{t\geq 0}$ by first thinning with probability 1/3 to get the two independent processes $(M_t^{(1)}, N(t) - M_t^{(1)})_{t\geq 0}$. Then we thin $(N(t) - M_t^{(1)})_{t\geq 0}$ with probability 1/2 to get the processes $(M_t^{(2)}, N(t) - M_t^{(1)} - M_t^{(2)})_{t\geq 0} = (M_t^{(i)})_{t\geq 0}$. Thus the three processes are i.i.d. Poisson processes with rate 1/3, and so

$$\mathbb{P}(M_3^{(1)} = 2, M_3^{(2)} = 2, M_3^{(3)} = 0) = \mathbb{P}(M_3^{(1)} = 2)\mathbb{P}(M_3^{(2)} = 2)\mathbb{P}(M_3^{(3)} = 0) = \frac{e^{-3}}{4}.$$

(d) Given $M_3^{(1)} = 2$, the times of the orders are i.i.d. uniform on the interval $(0, 3)$, and thus the time of the first order $T_1^{(1)}$ has density $(2/9)(3 - u)$ on $0 < u < 3$. By conditioning on the time of the first order and using the independence of the processes, we find that the probability we want is

$$\begin{aligned}
\frac{2}{9}\int_0^3 & \mathbb{P}(M_u^{(2)} + M_u^{(3)} = 0 | T_1^{(1)} = u, M_3^{(1)} = 2)(3 - u)du \\
&= \frac{2}{9}\int_0^3 e^{-2u/3}(3 - u)du \\
&= \frac{1}{2}\left(1 + e^{-2}\right).
\end{aligned}$$

2. Customers arrive to a queuing system according to a Poisson process with rate 4 per hour. There are two servers in the system, each having service times exponentially distributed with rates 3 and 2 per hour (respectively). If both servers are free, an arriving customer chooses a server uniformly at random and goes immediately into service; if one server is free, an arriving customer goes immediately into service with the free server; if both servers are busy, an arriving customer joins a queue.

   (a) What is the long run proportion of time that the rate 3 server is idle?

   (b) What is the average number of customers in the system?

   (c) What is the expected amount of time an arriving customer spends in the system before leaving?

   (d) What is the expected amount of time that an arriving customer has to wait for service?

   (e) What is the average number of customers in the queue?

Ans.

We view the system as a CTMC with states $\{0, (1, 0), (0, 1), 2, 3, \ldots\}$, where $(1, 0)$ means the first server is busy and the second is not, and $(0, 1)$ means the second is busy and not the first; note in both these cases the number of customers in the system is 1. The generator is

$$A = \begin{pmatrix} -4 & 2 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 3 & -7 & 0 & 4 & 0 & 0 & 0 & \cdots \\ 2 & 0 & -6 & 4 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 3 & -9 & 4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 5 & -9 & 4 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 5 & -9 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where it continues for $j \geq 3$ as a birth-death process. The chain is irreducible and ergodic since the maximum service rate is greater than the arrival rate.

To answer the questions, we need to determine the stationary distribution of this system. We solve $\pi A = 0$. For $j \geq 3$, the equations are

$$9\pi_j = 4\pi_{j-1} + 5\pi_{j+1},$$

which can be solved in the usual way using way (solution of the form $\pi_j = u^j$) to find for $j \geq 2$, and some constants a, b,

$$\pi_j = a + b(4/5)^j.$$

The a coefficient must be zero, since $\sum_{j \geq 3} \pi_j < 1$, so that for $j \geq 2$,

$$\pi_j = \pi_2 (4/5)^{j-2}.$$

Now for states $0, (1,0), (0,1)$, the equations are

$$\begin{aligned} 4\pi_0 &= 3\pi_{(1,0)} + 2\pi_{(0,1)}, \\ 7\pi_{(1,0)} &= 2\pi_2 + 2\pi_0, \\ 6\pi_{(0,1)} &= 3\pi_2 + 2\pi_0. \end{aligned}$$

Solving these in terms of π_2, we have

$$\begin{aligned} \pi_0 &= (3/4)\pi_2, \\ \pi_{(1,0)} &= (1/2)\pi_2, \\ \pi_{(0,1)} &= (3/4)\pi_2. \end{aligned}$$

Since $\sum_{j \geq 0} \pi_j = 1$, we find

$$\begin{aligned} \pi_0 &= 3/28, \\ \pi_{(1,0)} &= 1/14 \\ \pi_{(0,1)} &= 3/28, \\ \pi_j &= (1/7)(4/5)^{j-2}, \quad j \geq 2. \end{aligned}$$

(a) From the work above, the rate 3 server is idle in states $0, (0,1)$, in which the system spends a proportion of time equal to

$$\pi_0 + \pi_{(0,1)} = 3/14.$$

(b) The average number of customers in the system is

$$\pi_{(0,1)} + \pi_{(1,0)} + \sum_{j \geq 2} j\pi_j = \frac{125}{28}.$$

(c) By Little's law, $\lambda D = L$, where λ is the arrival rate, D is the expected time for an arriving customer to leave the system, and L is the average number of customers in the system, which we just computed. Thus

$$D = L/\lambda = \frac{125}{112}.$$

(You can also compute this directly, but it's more involved.)

(d) From the PASTA principle, customers find the system in stationary. If the system is in states $0, (1,0), (0,1)$, then the amount of time they have to wait for service is zero. If there are $j \geq 2$ people in the system, then the customer has to wait for one of the customers in service, as well as all the customers in the queue, to finish their services; there are $(j-1)$ total. The rate of "death" in these states is 5, so each of these customers takes $1/5$ of an hour on average, and so the average time to service is

$$\frac{1}{5} \sum_{j \geq 2} (j-1)(4/5)^{j-2} = \frac{5}{7}.$$

(You can also compute L_q directly as below and use Little's law.)

(e) Using the queue version of Little's law,

$$L_q = \lambda D_q = 4 \times \frac{5}{7} = \frac{20}{7}.$$

(You can also compute this directly: $L_q = \sum_{j \geq 2} (j-2)\pi_j$.)