

## TUTORIAL 3 (week 4)

The Simplex Method

Input: A linear program in standard form

Output: An optimal solution

- 1: Construct canonical form and obtain a basic feasible solution
- 2: while There are negative reduced costs do
- 3: Select entering variable with most positive/negative reduced cost (Alysson's notation / slides notation).
- 4: Select leaving variable using the ratio test.
- 5: Change basis.
- 6: end while
  - 1. Solve the following LP problem by the Simplex Method.

$$\max z = x_1 + x_2 + x_3$$
$$2x_2 + x_3 \le 1$$
$$2x_1 + x_2 + 2x_3 \le 1$$
$$x_1 + 2x_2 \le 1$$
$$x_1, x_2, x_3 \ge 0$$

## **Solutions:**

We first convert the given problem to canonical form by introducing slack variables  $x_4, x_5, x_6$ :

$$\max z = x_1 + x_2 + x_3$$
$$2x_2 + x_3 + x_4 = 1$$
$$2x_1 + x_2 + 2x_3 + x_5 = 1$$
$$x_1 + 2x_2 + x_6 = 1$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

The initial tableau is:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS
$x_4$	0 (2) 1	2	1	1	0	0	1
$x_5$	(2)	1	2	0	1	0	1
$x_6$	1	2	0	0	0	1	1
$\overline{z}$	-1	-1	-1	0	0	0	0



There are three negative entries in the z-row which are equal. We can choose any one of them for the pivot column. Let us choose the first one so that  $x_1$  becomes the entering variable. The ratio test identifies  $x_5$  as the leaving variable. So the pivot entry is in the first column and second row of the tableau. Pivoting on this entry we obtain the next tableau:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS
$x_4$	0	2 1/2 (3/2)	1	1	0	0	1
$x_1$	1	1/2	1	0	1/2	0	1/2
$x_6$	0	(3/2)	-1	0	-1/2	1	1/2
$\overline{z}$	0	-1/2	0	0	1/2	0	1/2

There is only one negative entry in the z-row. So  $x_2$  is the entering variable. The ratio test identifies  $x_6$  as the leaving variable and (i = 3, j = 2) as the pivot entry. Pivoting on this entry yields the next tableau:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	RHS
$x_4$	0	0	(7/3) $4/3$ $-2/3$	1	2/3	-4/3	1/3
$x_1$	1	0	4/3	0	2/3	-1/3	1/3
$x_2$	0	1	-2/3	0	-1/3	2/3	1/3
$\overline{z}$	0	0	-1/3	0	1/3	1/3	2/3

This tableau is still not optimal since there exists a negative entry in the z-row. Greedy Rule and Ratio Test identify  $x_3$  as the entering variable and  $x_4$  as the leaving variable. Pivoting on the (i = 1, j = 3)-entry, we obtain the next tableau:

	$x_1$	$x_2$	$x_3$	$x_4$ $3/7$ $-4/7$ $2/7$	$x_5$	$x_6$	RHS
$x_3$	0	0	1	3/7	2/7	-4/7	1/7
$x_1$	1	0	0	-4/7	2/7	3/7	1/7
$x_2$	0	1	0	2/7	-1/7	2/7	3/7
$\overline{z}$	0	0	0	1/7	3/7	1/7	5/7

This tableau is optimal since there is no negative reduced costs (entries in the z-row). So we stop.

Report:

## MAST20018 - Introduction to Discrete Mathematics and Operations Research



 $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*) = (1/7, 3/7, 1/7, 0, 0, 0)$  is an optimal solution to the problem in canonical form and the optimal objective value is  $z^* = 5/7$ .

 $(x_1^*, x_2^*, x_3^*) = (1/7, 3/7, 1/7)$  is an optimal solution to the given LP problem (in standard form) with optimal value  $z^* = 5/7$ .

2. Solve the following LP problem, or identify that the problem is unbounded or infeasible, by using the Simplex Method.

$$\max z = 2x_1 + 3x_2$$

$$x_1 - 2x_2 \le 4$$

$$2x_1 + x_2 \le 18$$

$$x_2 \le 10$$

$$x_1, x_2 \ge 0$$

**Solutions:** This problem is in standard form. Convert it to canonical form by introducing slack variables  $x_3, x_4, x_5$ .

$$\max z = 2x_1 + 3x_2$$

$$x_1 - 2x_2 + x_3 = 4$$

$$2x_1 + x_2 + x_4 = 18$$

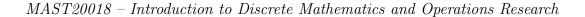
$$x_2 + x_5 = 10$$

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

The initial tableau is:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	RHS
$x_3$	1	-2 1 (1)	1	0	0	4
$x_4$	2	1	0	1	0	18
$x_5$	0	(1)	0	0	1	10
$\overline{z}$	-2	-3	0	0	0	0

Since there are negative reduced costs, we choose the most negative of them. This means we select  $x_2$  as the entering variable. The ratio test identifies  $x_5$  as the leaving variable and





(i=3,j=2) as the pivot entry. Pivoting on this entry yields:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	RHS
$x_3$	1 (2) 0	0	1	0	2	24
$x_4$	(2)	0	0	1	-1	8
$x_2$	0	1	0	0	1	10
$\overline{z}$					3	

There is only one negative reduced cost which is located in the first column. So we select  $x_1$  as the entering variable. The ratio test gives the leaving variable  $x_4$  and the pivot entry (i = 2, j = 1). Pivoting on this entry yields the next tableau:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	RHS
$x_3$	0	0	1	-1/2 $1/2$ $0$	5/2	20
$x_1$	1	0	0	1/2	-1/2	4
$x_2$	0	1	0	0	1	10
$\overline{z}$	1			1	2	38

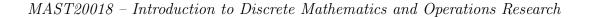
Since all reduced costs are nonnegative, this is an optimal tableau. It gives the optimal solution  $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (4, 10, 20, 0, 0)$  (to the problem in canonical form) and optimal value  $z^* = 38$ . So  $(x_1^*, x_2^*) = (4, 10)$  is an optimal solution to the given problem (in standard form).

The fact that  $x_4^* = x_5^* = 0$  indicates that  $(x_1^*, x_2^*) = (4, 10)$  satisfies the last two functional constraints with equality.

3. Solve the following LP problem, or identify that the problem is unbounded or infeasible, by using the Simplex Method. Provide a graphical interpretation.

$$\max z = 5x_1 + 3x_2$$
$$-x_1 + x_2 \le 4$$
$$x_1 - 2x_2 \le 6$$
$$x_1, x_2 \ge 0$$

**Solutions:** Convert the problem to canonical form:





$$\max z = 5x_1 + 3x_2$$
$$-x_1 + x_2 + x_3 = 4$$
$$x_1 - 2x_2 + x_4 = 6$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Initial tableau:

	$x_1$	$x_2$	$x_3$	$x_4$	RHS
$x_3$	-1	1	1	0	4
$x_4$	(1)	-2	0	1	6
$\overline{z}$	-5	-3	0	0	0

Next tableau:

	$x_1$	$x_2$	$x_3$	$x_4$	RHS
$x_3$	0	-1	1	1	10
$x_1$	1	-2	0	1	6
$\overline{z}$	0	-13	0	5	30

There is a unique negative reduced cost, namely -13, which is in the second column. When we do the ratio test we find that all entries of the coefficient matrix in the second column are non-positive. So the ratio test fails. This means that the problem is unbounded, i.e. it has no optimal solution. The objective function can take arbitrarily large value over the feasible region.

Exercise: Convince yourself of this result by using the graphical method.

4. The tableau method consists in writing the problem in a table and doing simplex iterations by doing pivot operations in this table. The basic variables can be identified in the tableau by



finding the columns of an identity matrix. The reduced costs can be found in the row z for the columns of the non-basic variables.

	$x_1$	$x_2$	$x_3$	$x_4$	RHS
$x_1$	1	1/2	1/2	0	4
$x_4$	0	1/2	-1/2	1	1
$\overline{z}$	0	0	1	0	8

(a) Identify the basic variables in the solution corresponding to the tableau above. What are their values?

(b) Identify the non-basic variables and their reduced costs. Is this solution optimal for a maximisation problem ?

(c) Who will enter the basis and who will leave in the next iteration?

**Solutions:** (a) The basic variables are indicated in the first column ( $x_1$  and  $x_4$ ) and correspond to the two columns of the identity matrix. Their values can be read directly from the last column in the tableay:  $x_1 = 4$  and  $x_4 = 1$ .

(b) The non basic-variables are  $x_2$  and  $x_3$  and their reduced costs are  $\hat{c}_2 = 0$  and  $\hat{c}_3 = 1$ . The solution of this maximisation problem is therefor not optimal.

(c) We have  $A_B = I_{m \times m}$  (we will always keep this in the tableau method) and

$$A_{NB} = \left[ \begin{array}{cc} 1/2 & 1/2 \\ 1/2 & -1/2 \end{array} \right]$$

.

We enter with variable  $x_3$  (since it is the only one with a positive reduced cost) and therefore the vecyor y is given as:

$$y = A_B^{-1} a_3 = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

and since  $A_B = I$ , this can be read directly from the tableau.

Now, we see that the maximum value we can add to the current non-basic variable  $x_3$  is given by 4/(1/2) = 8 (from the row associated with basic variable  $x_1$ . The other row in the vector y is non-positive, so we disconsider it).

This gives us the conclusion that basic variable  $x_1$  is going to leave the basis to give place to variable  $x_3$  in our basis.