THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2021

School of Mathematics and Statistics

MAST3022 Decision Making

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 29 pages (including this page) with 9 questions and 140 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier and blank loose-leaf paper.
- One double sided A4 page of notes (handwritten or printed).
- No calculators are permitted. No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.
- You must not write your answers on an iPad or other electronic device.
- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- Working and reasoning **must** be given to obtain full credit. Give clear and concise explanations.
- If you are writing answers on the exam or masked exam and need more space, use blank paper. Note this in the answer box, so the marker knows.
- If you are only writing on blank A4 paper, the first page must contain only your student number, subject code and subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

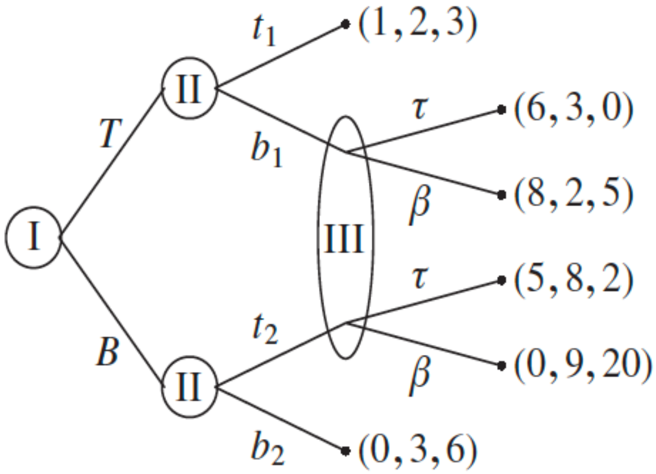
Scanning and Submitting

- **You must not leave Zoom supervision to scan your exam.** Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4.
- Submit your scanned exam as a single PDF file and carefully review the submission in Gradescope. Scan again and resubmit if necessary. Do not leave Zoom supervision until you have confirmed orally with the supervisor that you have received the Gradescope confirmation email.
- **You must not submit or resubmit after having left Zoom supervision.**

Question 1 (13 marks)

Consider the three-player game in extensive form depicted below.

The circles or ellipses enclosing the players labels represent their information sets.



(a) Is the game one of perfect information, or imperfect information?

Justify your answer.

(b) How many plays of the game are there?

Justify your answer.

(c) Write down the strategies of each player.

(d) Give the normal form of the game.

(e) Find all the equilibria in pure strategies, if any exist.

Question 2 (13 marks)

Consider the two-person zero-sum game with payoff matrix

$$\boldsymbol{V} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 2 & -1 \\ 4 & 0 & 2 \end{bmatrix}.$$

(a) Is the value of the game negative?

Justify your answer.

(b) To solve the game using the linear programming method, 4 can be added to each entry of $\boldsymbol{V}$.

Explain why this is necessary.

(c) Could a smaller value than 4 be added to each entry of $\boldsymbol{V}$ in order to solve the game using the linear programming method?

Justify your answer.

If so, what is the smallest number that acheives this?

(d) After adding 4 to each entry of $\boldsymbol{V}$ write down a suitable linear program from *Player 2's point of view* that will solve the game.

(e) When solving the linear program in Part (d), the final simplex mathod tableau is depicted below.

BV	y'_1	y'_2	y'_3	y'_4	y'_5	y'_6	RHS
y'_4	$-13/3$	0	0	1	$-7/12$	$-3/8$	$1/24$
y'_2	0	1	0	0	$1/4$	$-1/8$	$1/8$
y'_3	$4/3$	0	1	0	$-1/6$	$1/4$	$1/12$
z	$1/3$	0	0	0	$1/12$	$1/8$	$5/24$

From this tableau find the value of the original game and the optimal mixed strategies for each player.

Question 3 (30 marks)

(a) Consider a two-person non-zero-sum non-cooperative game with payoff matrix $\boldsymbol{A}$ for Player 1 and payoff matrix $\boldsymbol{B}$ for Player 2.

Let v^* be the optimal security level for Player 2.

If $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an equilibrium strategy pair, prove that $v^* \leq \boldsymbol{x}^* \boldsymbol{B} \boldsymbol{y}^{*T}$.

(b) Consider the 2-person non-zero-sum non-cooperative game with payoff bi-matrix

$$\begin{bmatrix} (-3,3) & (-2,2) & (1,2) \\ (7,5) & (8,1) & (1,5) \\ (0,3) & (-4,6) & (-1,2) \end{bmatrix}.$$

(i) Find, by inspection, any pure equilibrium strategy pair(s) $(\boldsymbol{x}^*, \boldsymbol{y}^*)$.

Give the corresponding payoff vector(s).

(ii) By first checking for any saddle points, calculate v^*.

(iii) For the pure equilibrium strategy pair(s) found in Part (b)(i), verify that $v^* \leq \boldsymbol{x}^* \boldsymbol{B} \boldsymbol{y}^{*T}$.

(c) Consider the two-person non-zero-sum cooperative game with payoff bi-matrix

$$\begin{bmatrix} (1,4) & (2,3) & (3,0) \\ (1,0) & (4,0) & (0,4) \end{bmatrix}.$$

Use $(u_0, v_0) = (1, 1)$ as the status quo point.

Find

(i) the cooperative payoff set C.

Write your answer as the convex hull of a minimum number of points.

(ii) the Pareto boundary $PB(C)$.

You are required to give the expression for $PB(C)$ as the union of sets of the form

$$\left\{(u, v) \in \mathbb{R}^2 : v = au + b, c \leq u \leq d\right\}.$$

(iii) the negotiation set $NS(C)$.

You are required to give the expression for $NS(C)$ as the union of sets of the form

$$\left\{(u, v) \in \mathbb{R}^2 : v = au + b, c \leq u \leq d\right\}.$$

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(iv) Draw a graph indicating (u_0, v_0), C, $PB(C)$, and $NS(C)$ clearly.

(v) Using the status quo point $(u_0, v_0) = (1, 1)$, determine the unique Nash solution $(\underline{u}, \underline{v})$ to the game (that is, the unique point in C that satisfies Nash's Axioms).

(vi) To achieve the Nash solution that you found in Part (v), which pure strategy pair(s) should the players use?

With which probabilities should the players apply the pair(s)?

Question 4 (16 marks)

Let $v \in \mathrm{TU}^N$ where $N = \{1, 2, \ldots, n\}$.

Denote by $d(v)$ the number of dummy players.

Consider the solution concept $\boldsymbol{\psi}$, defined for $i \in N$, by

$$\psi_i(v) \;=\; \begin{cases} v(\{i\}) + \dfrac{v(N) - \displaystyle\sum_{k=1}^{n} v(\{k\})}{n - d(v)}, & \text{if } i \text{ is not a dummy player,} \\ \qquad v(\{i\}), & \text{if } i \text{ is a dummy player} \end{cases}$$

Prove that $\boldsymbol{\psi}$ satisfies

(a) the dummy player property.

(b) efficiency.

(c) symmetry.

You may assume without proof that if two players are symmetric then they are either both dummy players, or both not dummy players.

(d) Does the solution concept $\boldsymbol{\psi}$ give an alternative characterisation of the Shapley value? Justify your answer.

Question 5 (20 marks)

Let $A = \mathbb{R}^2$. Let θ be the binary relation on $A \times A$ given by

$$\theta \;=\; \{(\boldsymbol{a}, \boldsymbol{b}) \in A \times A : a_1 \geq b_1,\, a_2 \leq b_2\}\,.$$

(a) Justify your answers below with proofs or counterexamples.

To answer any question part, you may use any results from previous question parts.

Is θ

(i) a strict order?

(ii) a weak order?

(iii) an equivalence relation?

(iv) a partial order?

(v) a linear order?

(b) Let

$$B = \{(1,3),(2,1),(2,3),(3,2)(4,1),(5,3)\}.$$

Using a Boolean matrix, find all maximal, minimal, greatest and least elements of B with respect to θ, if they exist.

(c) Let P and L denote the Pareto and lexicographic orders on $\mathbb{R}^n$ where n is a positive integer, respectively.

Show that, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x}P\boldsymbol{y} \Longrightarrow \boldsymbol{x}L\boldsymbol{y}$.

Question 6 (11 marks)

Consider the decision making problem under strict uncertainty with decision table given below. For $i = 1, 2, 3$, and $j = 1, 2, 3$, the table contains the payoffs if action a_i is taken when the state of the world is θ_j.

		state		
		θ_1	θ_2	θ_3
	a_1	2	12	3
action	a_2	8	10	0
	a_3	3	10	5

(a) Which action(s) should be taken in order to satisfy Savage's minimax regret criterion?

(b) For every $\alpha \in [0, 1]$, rank the three actions in order of preference according to Hurwicz's α-criterion.

(c) Using the information in the decision table above, show that Laplace's expected value criterion does not obey the axiom of *independence of column duplication.*

Question 7 (11 marks)

An art dealer's client is willing to buy the painting *The Circle of Light* for \$5,000. The dealer can buy the painting today for \$4,300 (and make a profit of \$700) or can wait a day and buy the painting tomorrow (if it has not been sold) for \$3,600. The dealer may also wait another day and buy the painting (if it is still available) for \$2,500. At the end of the third day, the painting will no longer be available for sale. Each day, there is a 0.7 probability that the painting will be sold.

The goal for the art dealer is to maximise her expected profit.

(a) Draw a *decision tree* that models this problem.

You are required to indicate for each vertex whether it is a *decision* vertex or an *event* vertex, for each edge to which decision or event it corresponds (in the latter case, also indicate the corresponding probability), and for each leaf the corresponding profit made by the art dealer.

(b) Which strategy maximises the art dealer's expected profit?

What is the expected profit earned using this strategy?

Show all necessary working on your decision tree from Part (a).

(c) Assume that the art dealer's utility function for an expected profit of $\$x$ is $u(x) = \left(\frac{x}{100}\right)^2$.

(i) Is the art dealer risk-averse, risk-neutral, or risk-seeking?

Justify your answer.

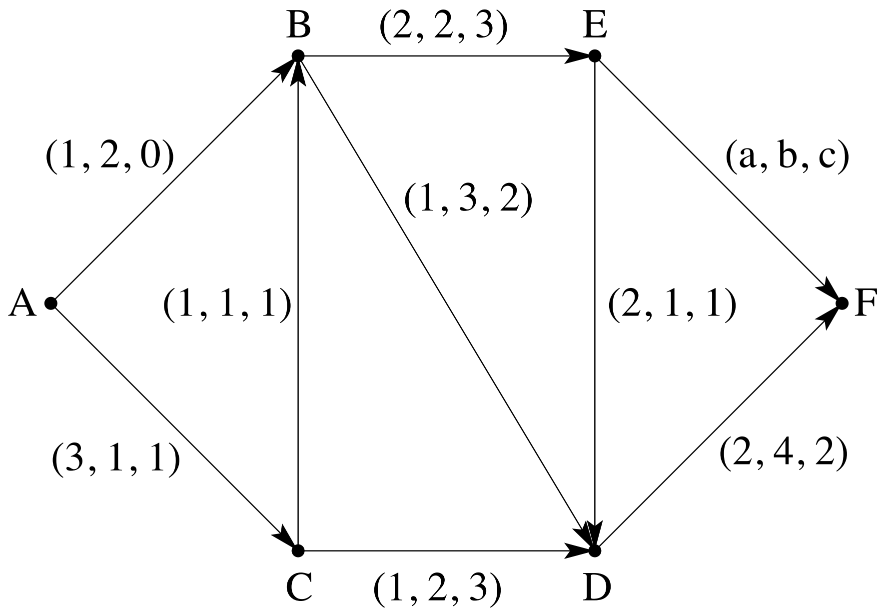
(ii) If the art dealer's goal is to maximise her expected utility, what is the optimal strategy?

Show all necessary working on a decision tree.

Question 8 (9 marks)

In the network depicted below each edge is associated with a 3-dimensional vector whose first, second, and third components represent revenue, time, and distance (in appropriate units), respectively, in traveling from the tail to the head of the edge.

The vector associated with the edge EF is (a, b, c) where $a, b,$ and c are nonnegative real numbers.



(a) Find a proper labelling for this network.

(b) Based on your proper labelling found in Part (a), find the ranges of the values of a, b, and c such that there exists exactly one *Pareto maximal path* between A and F.

Give this path and its corresponding Pareto maximal length.

Use *backward dynamic programming* **showing all necessary working**.

Continue your answer in the box below.

Question 9 (17 marks)

At the beginning of each week, a machine is in good, fair, or poor condition. If the machine is in good condition, the machine generates revenue of \$400 per week; if the machine is in fair condition, the revenue is \$300 per week; if the machine is in poor condition, the revenue is \$200 per week. A fair machine can be overhauled for \$100, and it immediately becomes a good machine. A poor machine can be replaced (immediately) with a good machine for \$300.

The probabilities by which the machine changes condition from one week to the next are given in the table below.

	good	fair	poor
good	0.6	0.2	0.2
fair	0	0.4	0.6
poor	0	0	1

(a) Formulate this problem as an *infinite horizon* Markov decision process with a discount factor $\alpha = 0.95$, where each time step corresponds to one week.

You are required to describe explicitly

(i) the state space I.

(ii) the decision set $D(i)$ for each state $i \in I$.

(iii) the transition probabilities $p_{ij}^{(k)}$ for each decision k and each pair of states i, j.

(iv) the expected rewards $r_i^{(k)}$ for each decision k and each state i.

(v) the system of value determination equations for the stationary policy δ that stipulates that the machine is not overhauled if it is in good or fair condition, but is replaced if it is in poor condition.

You are not required to solve the the system of value determination equations.

(b) (i) Write down the linear program that when solved will give the maximum expected discounted reward for each state and the optimal stationary policy.

You are not required to solve the linear program.

(ii) Explain how the optimal stationary policy is derived from the solution to the linear program in Part (b)(i).

End of Exam — Total Available Marks = 140