*MAST30022 Decision Making*
*2021*
*Tutorial Solutions 2*

1. **(PS2-1)**

   (a) Solve the 2-person zero-sum-game with payoff matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 2 \\ -1 & -1 & 3 & 0 \\ 1 & -2 & 5 & 1 \end{bmatrix}$$

   (You should find the saddle points, if any, and state the value of the game.)

   (b) In **any** game with the same columns as columns 2 and 3 above, would Player II ever uses the strategy represented by column 3? Explain.

**Solution**

(a) We have $L = \max\{-1, -1, -2\} = -1$ and $U = \min\{1, -1, 5, 2\} = -1$. Therefore $v_{12} = v_{22} = -1$ are saddle points and $-1$ is the value of the game.

(b) $A_3$ is strictly dominated by $A_2$, that is $2 > -1$, $3 > -1$, and $5 > -2$. Thus, no matter what Player I plays, Player II is always better off playing $A_2$ rather than $A_3$.

2. **(PS2-5)** Let

$$G_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad G_2 = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Show that if no two of $a, b, c, d$ are equal, and if the 2-person zero-sum game with payoff matrix $G_1$ has a saddle point at $(a_1, A_1)$, then the 2-person zero-sum game with payoff matrix $G_2$ has a saddle point. (In fact it can be shown that $G_2$ has a saddle point if and only if $G_1$ has a saddle point.)

**Solution**

If $g_{1,11}$ is a saddle point then $a < b$ and $a > c$.

Now, consider $G_2$.

If $d < c < a < b$ then $g_{2,12}$ is a saddle point.

If $c < d < a < b$ then $g_{2,22}$ is a saddle point.

If $c < a < d < b$ then $g_{2,22}$ is a saddle point.

If $c < a < b < d$ then $g_{2,21}$ is a saddle point.

3. **(PS2-27)** Find the range of values for $p$ and $q$ that make the entry $(a_2, A_2)$ a saddle point in the 2-person zero-sum game with payoff matrix

$$\begin{bmatrix} 3 & 4 & 5 \\ 9 & 7 & q \\ 4 & p & 6 \end{bmatrix}.$$

**Solution**

The strategy $(a_2, A_2)$ corresponds to a saddle point if and only if 7 is the smallest entry in row 2 **and** the largest entry in column 2. This is equivalent to

$$\min\{9, 7, q\} = 7 \ \textbf{ and } \ \max\{4, 7, p\} = 7,$$

or equivalently

$$q \geq 7 \ \textbf{ and } \ p \leq 7.$$

In conclusion, $(a_2, A_2)$ corresponds to a saddle point of the 2-person zero-sum game with the given payoff matrix if and only if $p \leq 7$ and $q \geq 7$.

4. **(PS2-28)** Suppose that the payoff matrix of a 2-person zero-sum game is the same as in Question 3 above. Is it possible to have **both** of $(a_2, A_2)$ and $(a_3, A_2)$ as saddle points at the same time? What about **both** of $(a_2, A_2)$ and $(a_2, A_3)$? Give reasons for your answers.

**Solution**

The strategy profile $(a_3, A_2)$ corresponds to a saddle point if and only if $p$ is the smallest entry in row 3 **and** the largest entry in column 2. That is,

$$\min\{4, p, 6\} = p \ \textbf{ and } \ \max\{4, 7, p\} = p,$$

or equivalently

$$p \leq 4 \ \textbf{ and } \ p \geq 7.$$

Since these two inequalities cannot hold simultaneously, it is impossible to have $(a_3, A_2)$ corresponding to a saddle point. Of course it is impossible to have both $(a_2, A_2)$ and $(a_3, A_2)$ corresponding to saddle points.

$(a_2, A_3)$ corresponds to a saddle point if and only if $q$ is the smallest entry in row 2 **and** the largest entry in column 3. That is,

$$\min\{9, 7, q\} = q \ \textbf{ and } \ \max\{5, q, 6\} = q,$$

or equivalently

$$6 \leq q \leq 7.$$

By Question 3, $(a_2, A_2)$ is a saddle point if and only if $p \leq 7$ and $q \geq 7$. Therefore, both $(a_2, A_2)$ and $(a_2, A_3)$ are saddle points if and only if $p \leq 7$ and $q = 7$.

5. **(PS2-29)** Find the values of $x$ for which the following 2-person zero-sum game has a saddle point, and solve the game for these cases.

$$\begin{bmatrix} -1 & 2 & 7 \\ x & 1 & 2 \\ 7 & x & 9 \end{bmatrix}$$

**Solution**

We have

$$s_1 = -1, s_2 = \min\{x, 1\}, s_3 = \min\{x, 7\}$$
$$S_1 = \max\{x, 7\}, S_2 = \max\{x, 2\}, S_3 = 9.$$

If $x < -1$, then $L = \max\{-1, x, x\} = -1$.

If $-1 \leq x < 1$, then $L = \max\{-1, x, x\} = x$.

If $1 \leq x < 7$, then $L = \max\{-1, 1, x\} = x$.

If $x \geq 7$, then $L = \max\{-1, 1, 7\} = 7$.

If $x < 2$, then $U = \min\{7, 2, 9\} = 2$.

If $2 \leq x < 7$, then $U = \min\{7, x, 9\} = x$.

If $7 \leq x < 9$, then $U = \min\{x, x, 9\} = x$.

If $x \geq 9$, then $U = \min\{x, x, 9\} = 9$.

Both $L$ and $U$ are piecewise linear functions of $x$. See Figure 1 for their graphs. From these graphs we can see that $L = U$ occurs if and only if $2 \leq x \leq 7$. Hence the game has a saddle point if and only if $2 \leq x \leq 7$.

Pairs of equilibrium strategies (also optimal strategies):

(a) $x = 2$: equilibrium strategy $(a_3, A_2)$, that is, Player I plays $a_3$ and Player II plays $A_2$; value of the game $= 2$.

(b) $2 < x < 7$: equilibrium strategy $(a_3, A_2)$; value of the game $= x$.

(c) $x = 7$: equilibrium strategies $(a_3, A_1)$ and $(a_3, A_2)$; value of the game $= 7$.

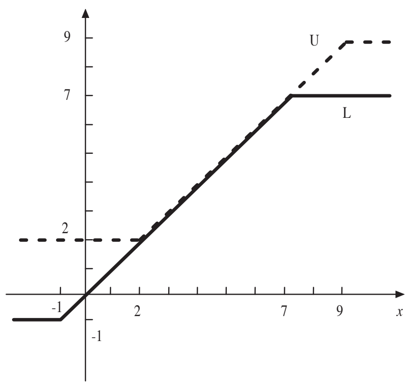


Figure 1: PS2-29