MAST30025: Linear Statistical Models
Assignment 1 S1 2021

Michael Le (998211)

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Question 1 Solution:

Part a:

A^2 =A^3

Suppose A is a square matrix is (real and) symmetric then its eigenvalues are all real, and its eigenvalues are orthogonal.

***Theorem* 2.3**

Proof:

Take A to be a square matrix, n x n. First we diagonalise A,i.e., find P such that.

=

$$D = P^T AP$$

$$= \begin{bmatrix} \lambda_1 & .... & 0 \\ ...... & .... & .. \\ 0 & .... & \lambda_k \end{bmatrix}$$

where $\lambda_1,\ \lambda_2, ...,\ \lambda_k$ *are the eigenvalues of A.*

Since P is orthogonal both P and P^T *are non − singular,*

$$r(P^T AP) = r(p^T A) = r(A)$$

Because P^T AP *is diagonal* $r(P^T AP)$ *is the number of non zero eigenvalues of A.*

But we wanted to prove ***Theorem* 2.2**

that A any symmetric matrix is idempotent. Which has eigenvalues of $\lambda =$ 0 or $\lambda = 1$.

The eigenvalues of idempotent matrices are always either

$\lambda = 0$ or $= 1$.

$A^2 = \lambda^2$x

Multiplying by A!!!

A^3x $= A^2$ λx $= \lambda$ A^2x $= \lambda^3$ x

(λ^3 - λ^2)$\boldsymbol{x} = 0$

By definition, x$\neq$ 0,

λ^3 - $\lambda^2 = 0$
$\lambda^2(\lambda$ - 1$) = 0$

Therefore there are two values with eigenvalues of 0 and one eigenvalue of 1! satisfies this theorem that A is idempotent!

Part b:

$$A = A^3$$

A^3x = A λx = λ Ax = λ^3 x

Using the same theorem from the previous it has eigenvalues of 0,1 and -1. Since we care that A has to be positive semi-definite. Which has an eigenvalue of -1. Which does not satisfy Theorem 2.2! A is not idempotent!

Question 2 Solution:
Theorem 2.4

There exists a matrix **P** which diagonalises A_1**,....,**A_m.

$$P^T A_i P = D_i$$

and

$$P^T A_j P = D_j$$

We take A_i and A_j to be k x k matrices first we diagonalizes A_i, A_j, i.e. find P such that,

$D_i = P^T$ A_i P $= \begin{bmatrix} \lambda_1 & \ldots\ldots & 0 \\ \ldots\ldots\ldots & \lambda_2 & \ldots\ldots \\ 0 & \ldots\ldots\ldots & \lambda_k \end{bmatrix}$

for i = 1,.....,k

$D_j = P^T$ A_j P $= \begin{bmatrix} \lambda_1 & \ldots\ldots & 0 \\ \ldots\ldots\ldots & \lambda_2 & \ldots\ldots \\ 0 & \ldots\ldots\ldots & \lambda m \end{bmatrix}$

for j = 1,.....,m

Proof:

P$^T A_i A_j P = (P^T A_i P)(P^T A_j P) = (P^T A_j P)(P^T A_i P) = P^T A_j A_i P$

Pre-multiply by P and post-multiply by P^T to get $A_i A_j = A_j A_i$.

Question 3 Solution:

Pre Proof Using Theorem 2.3

For any matrix A

$$r(A) = r(A^T) = r(A^T A) = tr(A)$$

A = $\begin{bmatrix} | & | & ...... & | & .... & | \\ a_1 & a_2 & .... & ap & .... & an \\ | & | & ...... & | & .... & | \end{bmatrix}$

Given A matrix with dimensions n x p with p independent columns.

Let $x_1, x_2, x_3, ..., x_k$ *the basis for column space of A.*

Definition of basis every column vector of A is a linear combination of the column vectors of x.

$$a_1 = b_1 x_1 + b_2 x_2 + ... + b_k x_k$$

Definition of linear combination

where b is scalar

B = $\begin{bmatrix} -- b_1 -- \\ -- b_2 -- \\ | \\ -- bp -- \end{bmatrix}$

$$\begin{bmatrix} | & | & ...... & | & .... & | \\ a_1 & a_2 & .... & ap & .... & an \\ | & | & ...... & | & .... & | \end{bmatrix} = \begin{bmatrix} | & | & ...... & | & .... & | \\ x_1 & x_2 & .... & xp & .... & an \\ | & | & ...... & | & .... & | \end{bmatrix} \begin{bmatrix} -- b_1 -- \\ -- b_2 -- \\ | \\ -- bp -- \end{bmatrix}$$

$$A = XB$$

$$A^T = (XB)^T = B^T X^T$$

r(A) $\leq r(A^T)$ *or* $r(A) \geq r(A^T)$ *to satisfy*!

$$r(A) = r(A^T) = r(A^T A) = tr(A) = p$$

Since P is orthogonal both P and P^T *arenon−singular. Therefore we need to sum up the diagonal el*

$$r(A) = r(P^T AP) = tr(P^T AP) = tr(PP^T A) = tr(A) = p$$

Because D = $P^T AP$ *is diagonal* $r(P^T AP)$ *is the number of nonzero values of* A!

But A is idempotent so its takes eigenvalues between 0 or 1. To Prove Theorem 2.7! We need only the identity matrix to allow $A^T A$ *to be positive definite*!

Using Theorem 2.7

Proof **(⇐) :**

We want A$^T A$ *to be symmetric*

and have all the eigenvalues to be strictly positive to prove $A^T A$ is a positive definite matrix!

we know r($A^T A$) = p is a p x p matrix so it has to be a full rank matrix, p!

Let,λ_1, λ_2,.....,$\lambda p > 0$ be the eigenvalues of $A^T A$ for every x and for each eigenvalue has to have a value of 1.

for z = P^Tx = $(z_1, ..., zp)^T$

$$x^T (A^T A)x = x^T PDP^T x = z^T Dz = \sum_{i=1}^{p} z_i^2 \lambda_i$$

since $\lambda_i = 1$!

$$= \sum_{i=1}^{p} z_i^2$$

> 0

Thus $A^T A$ is positive definite as required!

Proof **(⇒) :**

Suppose $A^T A$ is positive definite let x_i be its normalised i-th eigenvector then,

$$x_i^T (A^T A)x_i = \lambda_i x_i^T x_i = \lambda_i$$

From theorem 2.3 we want $A^T A$ to be symmetric and idempotent.We want the eigenvalues to be 0 or 1. This case all of the eigenvalues must equal to 1.

$\lambda_i = 1$>0

So, the eigenvalues of $A^T A$ are strictly positive as required!!