Q1. a) x and $\log x$ are both C^1 on $x > 0$

$\Rightarrow f$ is C^1 for $x > 0$

$f'(x) = \log x + 1 \qquad f''(x) = \frac{1}{x} > 0$ for $x > 0$

$\Rightarrow f$ is strictly convex and unimodal.

b) $r^n < 2\epsilon\,(b-a)$

$(0.618)^7 < 2\times 0.05 / (2.1 - 0.1)$

$\Rightarrow$ Golden section search requires $n+1 = 8$ f-calculation

The length of the final interval is $(0.618)^7 (2.1-0.1)$

≈ 0.0689.

c) $(b-a)/F_n < 2\epsilon \quad \Rightarrow F_n > 20 \Rightarrow n = 7$

$p = b - \frac{F_{n-1}}{F_n}(b-a) = 2.1 - \frac{13}{21}\cdot 2 \approx 0.862$

$q = a + \frac{F_{n-1}}{F_n}(b-a) = 0.1 + \frac{13}{21}\cdot 2 \approx 1.338$

$f(p) = -0.128 < f(q) = 0.390$

$\Rightarrow b = 1.338$

The interval becomes $[0.1, 1.338]$ after 1 iteration.

d) $f'(x) = \log x + 1$ is an increasing function.

$f'(a) < 0$ and $f'(b) > 0$. $\Rightarrow$ Can apply false position method.

$f'(a) = -1.303, \quad f'(b) = 1.742$

$p = f'(a) / (-f'(b) + f'(a)) \cdot (b-a) + a \approx 0.956.$

$f'(p) = 0.955 > 0$

The interval becomes $[0.1, 0.956]$ with the current estimate 0.956.

e) $f'(x_0) = 1.693, \quad f''(x_0) = 0.5.$

The intercept of the tangent line of $f'(x)$ with $y=0$ is $-1.386 < 0$, out of the domain.

So we can't apply Newton's method.

Q2. a) $\nabla f(x) = \begin{bmatrix} 3x_1^2 + 4x_1 - 4 \\ 3x_2^2 - 8x \end{bmatrix} = 0$

$\Rightarrow \quad x_1 = -2$ or $2/3$

$x_2 = 0$ or $8/3$

Four stationary points: $(-2, 0)$, $(2/3, 0)$, $(-2, 8/3)$, $(2/3, 8/3)$

b) $\nabla^2 f(x) = \begin{bmatrix} 6x_1 + 4 & 0 \\ 0 & 6x_2 - 8 \end{bmatrix}$

At $(-2, 0)$, $\nabla^2 f(x) = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} < 0$

symmetric and negative definite as $\lambda = -8$.

$(-2, 0)$ is a local maximum.

At $(2/3, 0)$, $\nabla^2 f(x) = \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}$

indefinite as $\lambda = \pm 8$.

$(2/3, 0)$ is a saddle point.

At $(-2, 8/3)$, $\nabla^2 f(x) = \begin{bmatrix} -8 & 0 \\ 0 & 8 \end{bmatrix}$

indefinite as $\lambda = \mp 8$

$(-2, 8/3)$ is a saddle point.

At $(2/3, 8/3)$, $\nabla^2 f(x) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} > 0$

symmetric and positive-definite as $\lambda = 8$

$(2/3, 8/3)$ is a local minimum.

c) At $(\cancel{2/3}, 0)$, $\nabla^2 f(x)$ has $\lambda = \pm 8$.

For $\lambda = 8$, an eigenvector is $[1,0]^T$. f increases along direction $[1,0]^T$.

For $\lambda = -8$, an eigenvector is $[0,1]^T$. f decreases along direction $[0,1]^T$.

Q3. a)

$$\nabla f(x) = \begin{bmatrix} 4x_1 - 2x_2 + 9 \\ 2x_2 - 2x_1 - 3 \\ 8x_3 \end{bmatrix}$$

Steepest descent method:

$$d^0 = -\nabla f(x^0) = \begin{bmatrix} -9 \\ 3 \\ 0 \end{bmatrix}$$

$$x' = x^0 + d^0 t_1 = \begin{bmatrix} -9t \\ 3t \\ 0 \end{bmatrix}$$

$$f(x') = t(225t - 90)$$

$$f'(t) = 450t - 90 = 0 \Rightarrow t = 1/5.$$

$$x' = (-9/5, 3/5, 0)$$

b) $$t_k = \arg \frac{df(x^{k+1})}{dt} = 0$$

Given $x^{k+1} = x^k + t_k d^k$,

$$\frac{df(x^{k+1})}{dt} = \nabla f(x^{k+1})^T \cdot \frac{d(x^k + t_k d^k)}{dt}$$

$$= \nabla f(x^{k+1})^T \cdot d^k$$

$$= 0$$

$$d^{k+1} = -\nabla f(x^{k+1})$$

$$\Rightarrow (d^{k+1})^T \cdot d^k = -\nabla f(x^{k+1})^T \cdot d^k = 0$$

Therefore $d^{k+1} \perp d^k \quad \forall k$.

c). $\nabla^2 f(x) = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ symmetric and positive definite

$\nabla^2 f(x)^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}$

Newton's direction $d^0 = -\nabla^2 f(x^0)^{-1} \nabla f(x^0) = \begin{bmatrix} -3 \\ -3/2 \\ 0 \end{bmatrix}$

$x' = x^0 + t d^0$

$f(x') = 45t^2/4 - 45t/2$

$f'(t) = 0 \Rightarrow t = 1$

$x' = (-3, -3/2, 0)$

d) $\nabla f(x') = 0$ and $\nabla^2 f(x)$ is positive definite given $\lambda = 8$, $3 \pm \sqrt{5}$ all greater than 0.

$\Rightarrow x'$ is a local minimum.

Because f is a quadratic function with positive-definite Hessian, Newton's method solves the problem using exactly one step.

e) Same as the steepest descent direction in (a).

Q4. a). $L(x, \lambda, y) = (x_1 - 1)^2 + (x_2 + 1)^2 + \lambda(x_1 x_2 - 1) + y(x_1 + x_2 - 2)$

b) KKTa : $\nabla_x L(x, \lambda, y) = \begin{bmatrix} 2(x_1 - 1) + \lambda x_2 + y \\ 2(x_2 + 1) + \lambda x_1 + y \end{bmatrix} = 0$

KKT b: $\lambda \geq 0$

$x_1 x_2 \leq 1$

$\lambda(x_1 x_2 - 1) = 0$

KKT c: $x_1 + x_2 - 2 = 0$

① when $\lambda = 0$, $x_1 = 2$, $x_2 = 0$, $y = -2$.

KKT point $((2,0), 0, -2)$

② when $\lambda \neq 0$ $x_1 x_2 = 1$, $x_1 + x_2 = 2$ $\Rightarrow$ $x_1 = x_2 = 1$

$\Rightarrow$ $\lambda + y = 0$, $4 + \lambda + y = 0$ $\Rightarrow$ $4 \neq 0$ contradiction.

c) Critical cone at the KKT point.

$$C(x^*, \lambda^*) = \{ d \in \mathbb{R}^2,\ \langle \nabla g_i(x^*), d \rangle \leq 0 \quad i \in I(\cdot),\ \lambda_i^* = 0$$
$$\langle \nabla g_i(x^*), d \rangle = 0 \quad \lambda_i^* > 0$$
$$\langle \nabla h_j(x^*), d \rangle = 0 \}$$

$g(x)$: $x_1 x_2 \leq 1$ is an inactive constraint.

$\nabla h(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ $[1, 1] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0$ $\Rightarrow$ $d_1 = -d_2$

$C(x^*, \lambda^*) = \{ (d_1, -d_1),\ d_1 \in \mathbb{R} \}$

d) $\nabla_{xx} L(x^*, \lambda^*, y^*) = \begin{bmatrix} 2 & \lambda^* \\ \lambda^* & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

is symmetric and positive definite on $\mathbb{R}^2$.

The active constraint is affine.

Therefore, $x^* = (2, 0)$ is a local minimum and $f(x^*) = 2$

e) f changes by $-y^* \Delta = -(-2) \times (-1) = -2$.

Q5. a) $P_K(x) = x_1 x_2 + \frac{K}{2}(-x_1)_+^2 + \frac{K}{2}(x_1 - x_2 - 1)^2$

b) $\nabla P_K(x) = \begin{bmatrix} x_2 - K(-x_1)_+ + K(x_1 - x_2 - 1) \\ x_1 - K(x_1 - x_2 - 1) \end{bmatrix} = 0$

If $x_1 \geq 0$, $x_2 + K(x_1 - x_2 - 1) = 0 \Rightarrow x_1^K = \frac{K}{2K-1}$, $x_2^K = \frac{K}{1-2K}$

If $x_1 < 0$, $x_2 + Kx_1 + K(x_1 - x_2 - 1) = 0 \Rightarrow x_1^K = K/(K^2+2K-1) > 0$

contradiction.

So $x^K = \left(\frac{K}{2K-1}, \frac{K}{1-2K}\right)$.

$x^* = \lim_{K\to\infty} x^K = \left(\frac{1}{2}, -\frac{1}{2}\right)$

c) $\lambda^K = K(-x_1^K)_+$

$= 0$ since $x_1^K > 0 \ \forall K$

$y^K = K(x_1^K - x_2^K - 1)$

$= + K/2K+1$

$\lambda^* = 0$

$y^* = \lim_{K\to\infty} y^K = +\frac{1}{2}$.

(y^* could be $-\frac{1}{2}$ if $h(x) = 1 - x_1 + x_2$)

Q6 a) $\nabla^2 f(x) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ $\quad 1>0, \quad 1\times 2-(-1)\times(-1)=1>0$

All ~~pri~~ leading principle minors are positive.

$\nabla^2 f(x)$ is positive definite on $\mathbb{R}^2$.

f is a quadratic function with positive definite Hessian, and hence convex. ~~All constraints~~ The inequality constraints are either convex or affine. So the NLP is a convex program.

b) $L(x,\lambda) = x_1^2/2 - x_1x_2 + x_2^2 - 7x_2 + \lambda_1(x_1^2+x_2^2-5)$
$+ \lambda_2(1-x_1+x_2)$

c) The Lagrangian Saddle Inequality is

$L(x^*,\lambda) \le L(x^*,\lambda^*) \le L(x,\lambda^*)$

$L(x^*,\lambda) = -6$

$L(x^*,\lambda^*) = -6$

$L(x,\lambda^*) = \frac{3}{2}x_1^2 + 2x_2^2 - x_1x_2 - 5x_1 - 2x_2$

$\min_x L(x,\lambda^*)$ ~~this~~ is an unconstrained NLP.

$\nabla_x L(x,\lambda^*) = \begin{bmatrix} 3x_1 - x_2 - 5 \\ 4x_2 - x_1 - 2 \end{bmatrix} = 0 \Rightarrow \begin{matrix} x_1^* = 2 \\ x_2^* = 1 \end{matrix}$

$\nabla_{xx}^2 L(x,\lambda^*) = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} > 0$ (positive definite)

~~So~~ $(2,1)$ is a global minimum given $L(x,\lambda^*)$ is convex. $L((2,1),\lambda^*) = -6$.

$\Rightarrow L(x^*,\lambda) \le L(x^*,\lambda^*) \le L(x,\lambda^*)$

d) Because the NLP is a convex program.

e) Wolfe dual is

$$\max_{x,\lambda} \quad x_1^2/2 - x_1x_2 + x_2^2 - 7x_2 + \lambda_1(x_1^2 + x_2^2 - 5) + \lambda_2(x_2 - x_1 + 1)$$

s.t.

$$\lambda_1 \geq 0$$

$$\lambda_2 \geq 0$$

$$x_1 - x_2 + 2\lambda_1 x_1 - \lambda_2 = 0$$

$$-7 - x_1 + 2x_2 + 2\lambda_2 x_2 + \lambda_2 = 0$$