Poisson processes

The Poisson distribution

Note: the word Poisson is a French surname, pronounced "pwah-sonn".

A discrete random variable X follows a Poisson distribution with rate λ if its probability mass function is given by

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

The Poisson distribution is typically used for counting the *number of events* in a *fixed timespan*. For example:

- The number of customers arriving at a store in an hour.
- The number of emails arriving in a day.

When the number of events follows this distribution, we call it a **Poisson process**.

Poisson processes

The definition of a Poisson process is given in the reading notes. But the simplest way to think about it is that the rate λ tells you the average number of events in a certain unit of time.

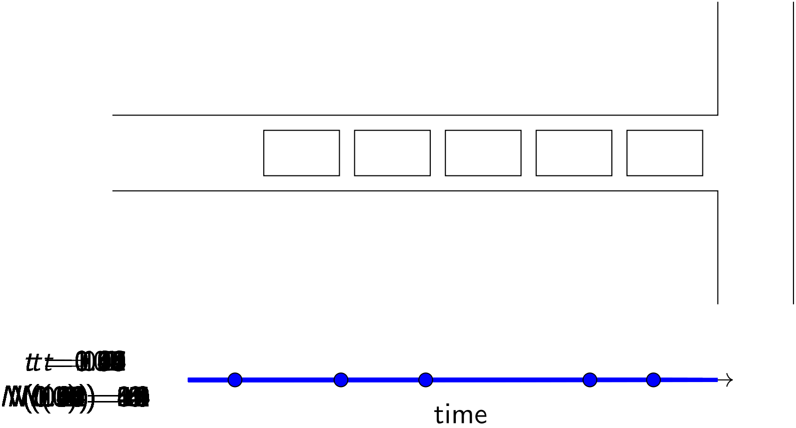
- $\lambda = 5$ could mean 5 events per hour, or 5 events per minute, etc. Therefore context is important.

Another way to think about a Poisson process is in terms of *time between events*. In a Poisson process, the time between events is exponentially distributed with rate λ.

Notation: $N(t)$ is the number of events in the interval $[0, t]$.

Poisson processes

Suppose cars arrive at an intersection according to a Poisson process with a rate $\lambda = 5$ cars per minute.



The time between events is Exp(5) distributed.

Properties

Recall that $N(t)$ denotes the number of events in the timespan $[0, t]$.

For a Poisson process, $N(t) \sim \text{Pois}(\lambda t)$, which implies:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!},$$

and $E[N(t)] = \text{Var}[N(t)] = \lambda t$.

Example

For the example on the previous slide, determine:

- The probability that no cars arrive in the first 2 minutes.
- The probability that at least 1 car arrives in the first 30 seconds.

Lack-of-memory property

The lack-of-memory property (also known as the memoryless property) says that:

the number of events in an interval depends only on the interval's width

Example

For the previous example,

- What is the probability that at least one car arrives between 4:30PM and 4:32PM?
- Given that no cars have arrived for the first minute, what is the probability that no cars arrive for the two minutes after that?

Time between events

We noted earlier that, for a Poisson process, the time between events is exponentially distributed. E.g., if $\lambda = 5$, then:

- The time until the first event follows an Exp(5) distribution.
- The time between the first and second event follows an Exp(5) distribution.
- The time between the second and third event follows an Exp(5) distribution.
- And so on...

This is not the same as the time *of* that event.

The time <u>of</u> event k follows a Gamma$(k, \frac{1}{\lambda})$ distribution.

Queues

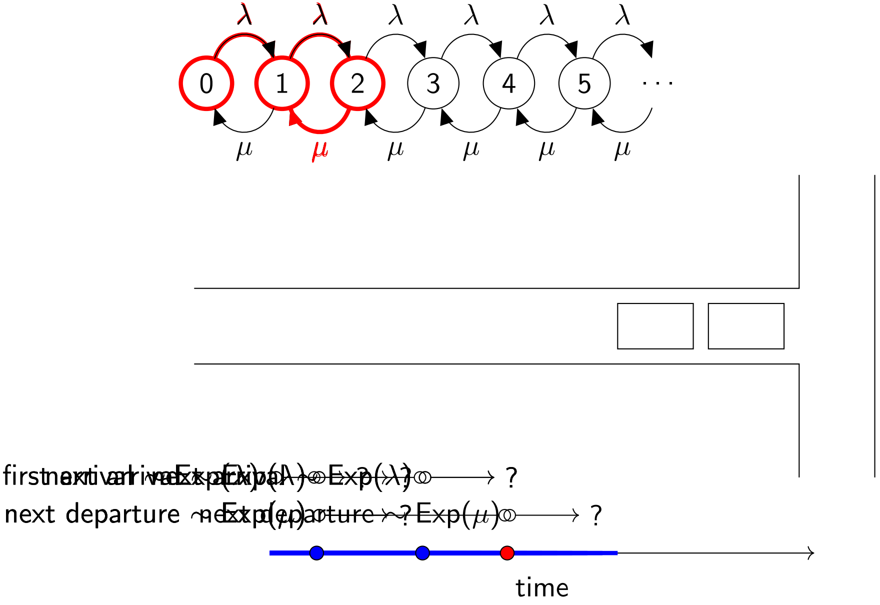
Queues

A queue is a counting process that accounts for both arrivals and departures. Our main focus in this subject will be on M/M/1 queues.

An M/M/1 queue has the following:

- an arrival rate λ,
- an assumption that time between arrivals is $\mathrm{Exp}(\lambda)$ distributed,
- a departure rate μ,
- an assumption that time to process a departure us $\mathrm{Exp}(\mu)$ distributed,
- an assumption that only one individual can be processed at a time,
- infinite capacity.

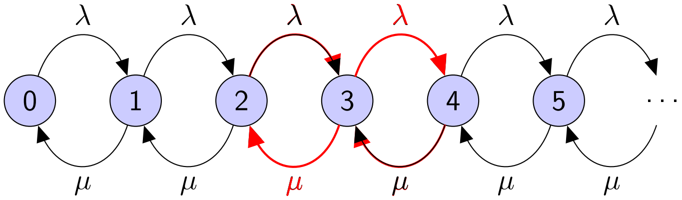
Queues


λ λ λ λ λ λ
0 1 2 3 4 5 · · ·
μ μ μ μ μ μ
first arrival ∼ Exp(λ) → ?
next arrival ∼ Exp(λ) → ?
next departure ∼ Exp(μ) → ?
next departure ∼ Exp(μ) → ?
time

Steady state

The limiting behaviour of a queue as $t \to \infty$ is called its *steady state*. In the steady state, the queue has "stabilised", and we can analyse its probabilities.

The **flow into** and **flow out of** state i are each defined based on the arrows into or out of that state.



Let P_i denote the probability that the queue is in state i.

- The flow into state 3 is $\lambda P_2 + \mu P_4$
- The flow out of state 3 is $\lambda P_3 + \mu P_3$

In a steady state, the flow into a state equals the flow out of that state. This leads to the so-called *balance equations*.

Properties of queues

For an M/M/1 queue:

- The traffic intensity is denoted by $\rho = \frac{\lambda}{\mu}$.
  - If $\rho \geqslant 1$ then the queue grows without bound.
  - If $\rho < 1$ then the queue is stable.
- The number of individuals in the queue N (in steady state) follows a Geo$(1-\rho)$ distribution, so:

$$P(N=k) = \rho^k(1-\rho), \qquad L = E(N) = \frac{\rho}{1-\rho}.$$

- The average time spent in the queue (waiting + service time) is

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}.$$

- The average waiting time before being served is

$$W_Q = \frac{\lambda}{\mu(\mu - \lambda)}$$

For other types of queue (including finite capacity queues) these properties do not necessarily hold true.

Properties of queues

Example

Consider an M/M/1 queue with arrival rate $\lambda = 5$ cars per minute and an average departure time of 20 seconds.

- What is the departure rate μ and the traffic intensity ρ?

- What does this tell you about the queue?

Properties of queues

Example

Consider an M/M/1 queue with arrival rate $\lambda = 5$ cars per minute and an average departure time of 5 seconds.

- What is the departure rate μ and the traffic intensity ρ?

- What is the average queue length?

- What is the average time before reaching the front of the queue?

- What is the average time elapsed between entering the queue and leaving the queue?