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Question 1.

(a) $f(x) = x^2e^x \implies f'(x) = 2xe^x + x^2e^x$ and $f'' = 2e^x + 2xe^x + 2xe^x + x^2e^x = 2e^x + 4xe^x + x^2e^x$.

This gives $f(0) = 0,\ f'(0) = 0$ and $f''(0) = 2$ so that $(T_2f)(x) = 0 + 0x + \frac{2x^2}{2!} = x^2$.

(b) Let $y = (x+1)\ln(x+1) = uv$ where $u = (x+1)$ and $v = \ln(x+1)$. Then

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v = (x+1)\frac{1}{x+1} + \ln(x+1) = 1 + \ln(x+1)$$

It follows that $\frac{d^2y}{dx^2} = \frac{1}{x+1}$. Now, when $x = 0$ we have

$$y = 1\ln(1) = 0, \quad \frac{dy}{dx} = 1 + \ln(1) = 1 \quad \text{and} \quad \frac{d^2y}{dx^2} = 1$$

so that $(T_2f)(x) = 0 + 1x + \frac{1}{2!}x^2 = x + \frac{1}{2}x^2$.

(c) Using the product rule first, $f'(x) = e^{x^2} + x \times 2xe^{x^2} = e^{x^2} + 2x^2e^{x^2}$.

We now get, using the sum, chain and product rules,

$$f''(x) = 2xe^{x^2} + 4xe^{x^2} + 2x^2 \times 2xe^{x^2} = 6xe^{x^2} + 4x^3e^{x^2}.$$

Since $f(0) = 0,\ f'(0) = 1$ and $f''(0) = 0$, we have $(T_2f)(x) = x$.

Question 2.

(a) $f(x) = f'(x) = f''(x) = f'''(x) = e^x$, so $f(0) = 1 = f'(0) = f''(0) = f'''(0)$.
This gives $(T_3f)(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

(b) $g(x) = xe^x \Rightarrow g'(x) = e^x + xe^x = (1+x)e^x, g''(x) = e^x + (1+x)e^x = (2+x)e^x$ and $g'''(x) = (3+x)e^x$, giving $g(0) = 0,\ g'(0) = 1,\ g''(0) = 2$ and $g'''(0) = 3$ so that $(T_3g)(x) = x + x^2 + \frac{x^3}{2}[= x(T_2f)(x)]$.

Question 3. (a), (d) and (f) are correct statements.

Question 4. $(T_2f)'(x) = f'(0) + f''(0)\,x$ and $(T_2f)''(x) = f''(0)$ so $(T_2f)(0) = f(0)$, $(T_2f)'(0) = f'(0)$ and $(T_2f)''(0) = f''(0)$ so the graphs are wrong because

(a) $(T_2f)(0) \neq f(0)$ (wrong value at 0), (b) $(T_2f)'(0) \neq f'(0)$ (wrong slope at 0) and

(c) $(T_2f)''(0) \neq f''(0)$ (wrong curvature at 0).

Question 5. From $f(x) = e^{-x^2}$, we obtain

$$\begin{aligned}
f'(x) &= -2xe^{-x^2} && \implies f'(0) = 0 \\
f''(x) &= [4x^2 - 2]e^{-x^2} && \implies f''(0) = -2 \\
f'''(x) &= [(4x^2-2)(-2x) + 8x]e^{-x^2} && \implies f'''(0) = 0 \\
f^{(iv)}(x) &= [12 - 24x^2 + (-2x)(12x - 8x^3)]e^{-x^2} && \implies f^{(iv)}(0) = 12
\end{aligned}$$

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Thus $(T_4 f)(x) = 1 - \dfrac{2x^2}{2!} + \dfrac{12x^4}{4!} = 1 - x^2 + \dfrac{x^4}{2}$. It appears that we could obtain the Taylor polynomial for the Gaussian by replacing x by $-x^2$ in the Taylor polynomial for e^x. (This gives a polynomial of order $2n$, from the polynomial of order n.

Question 6.

(a) Using a truncation of the geometric series, with x replaced by $-x$, we obtain

$$(T_5 g)(x) = 1 - x + x^2 - x^3 + x^4 - x^5$$

(b) We replace x by x^2 in (b) and note that we do not need any terms of order higher than x^5:

$$(T_5 h)(x) = 1 - x^2 + x^4$$

Question 7. Let $g = f'$. Then

$$(T_n f)(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$(T_n g)(x) = (T_n f')(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n+1)}(a)}{n!}(x-a)^n$$

Now differentiating $(T_n f)(x)$ we obtain

$$\begin{aligned}(T_n f)'(x) &= 0 + f'(a) + \frac{2f''(a)}{2!}(x-a) + \cdots + \frac{nf^{(n)}(a)}{n!}(x-a)^{n-1} \\ &= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}\end{aligned}$$

We can conclude that $(T_n f)'(x) = (T_{n-1} f')(x)$.

Question 8. Using $a = 1$, we obtain $f(1) = e^{-1}$ and $f'(1) = -2e^{-1}$. This gives a linear approximation

$$e^{-x^2} \approx e^{-1}(1 - 2(x-1))$$

Question 9. The values of M are:

(a) 3 (b) 4 (c) 2 (d) 1

Question 10.

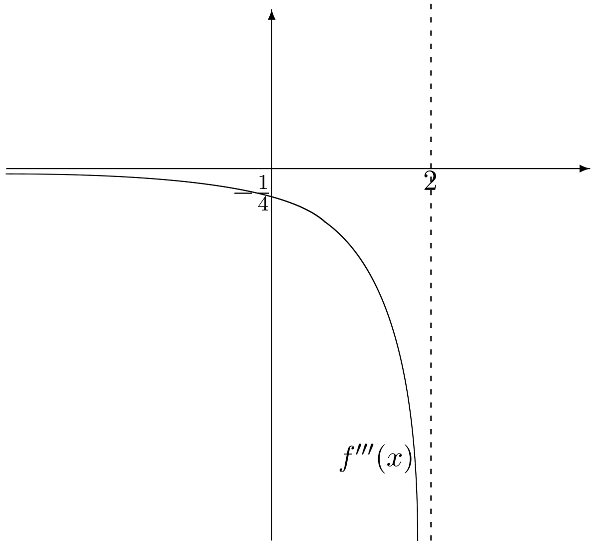
(a) $f'(x) = -\frac{1}{2-x}$, $f''(x) = -\frac{1}{(2-x)^2}$, $f'''(x) = -\frac{2}{(2-x)^3}$ so that

$f(0) = \ln(2)$, $f'(0) = -\frac{1}{2}$ and $f''(0) = -\frac{1}{4}$

giving $(T_2 f)(x) = \ln(2) - \frac{x}{2} - \frac{x^2}{8}$.

(b) $f'''(x) = -\frac{2}{(2-x)^3} \implies f^{(4)}(x) = -\frac{6}{(2-x)^4}$ which is negative wherever it is defined, so that $f'''(x)$ is a decreasing function everywhere on its domain.

As $f'''(x) < 0$ on the interval $[-1,0]$, this means that, on this interval, $f'''(x)$ is farthest from the x-axis at $x = 0$.



Now $f'''(0) = -\frac{1}{4}$ so we may choose $M = \left|-\frac{1}{4}\right| = \frac{1}{4}$. (You could also argue from the graph!)

(c) From Taylor's theorem, $|(E_2 f)(x)| \leqslant \frac{|x|^3}{4\times 3!} = \frac{|x|^3}{24}$ for all $x \in [-1, 0]$.

(d) $-1 \leqslant x \leqslant 0 \implies 0 \leqslant |x| \leqslant 1 \implies 0 \leqslant |x|^3 \leqslant 1 \implies 0 \leqslant \frac{|x|^3}{24} \leqslant \frac{1}{24}$ so that $|(E_2 f)(x)| \leqslant \frac{|x|^3}{24} \leqslant \frac{1}{24}$ for all $x \in [-1, 0]$ from (d).

Question 11.

(a) For $x > -1$, $f(x) = \ln(1+x) \implies f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{1}{(1+x)^2}$, $f'''(x) = \frac{2!}{(1+x)^3}$ and $f^{(4)}(x) = -\frac{3!}{(1+x)^4}$ giving $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$ and $f'''(0) = 2!$.

This gives $(T_3 f)(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$.

(b) For $x \in [0, 1]$, we have $f^{(4)}(x) = -\frac{3!}{(1+x)^4} \implies f^{(5)}(x) = \frac{4!}{(1+x)^5}$. Now, as $x \geqslant 0$, we have $1 + x \geqslant 1$ so that $f^{(5)}(x) > 0$. This means that $f^{(4)}$ is an increasing function. Finally, $f^{(4)}$ is negative, so it has its minimum value (and hence its greatest distance from the x-axis) on $[0, 1]$ at 0. This gives

$$-3! = f^{(4)}(0) \leqslant f^{(4)}(x) \leqslant 0,$$

so we may choose $M = 3!$

(c) From (b) and Taylor's theorem, $|(E_3 f)(x)| \leqslant M\frac{|x|^4}{4!} = \frac{|x|^4}{4}$

(d) $(T_3 f)(0.1) = 0.1 - \frac{1}{2}0.01 + \frac{1}{3}0.001 = 0.095333\ldots$ and $|(E_3 f)(0.1)| \leqslant \frac{1}{4}|0.1|^4 = 0.000025$

This means that $f(0.1)$ is 0.09533 with an error of at most 3 in the last decimal place or, accurate to 4 decimal places we have $f(0.1) = 0.0953$. Using Excel, accurate to 11 decimal places, $f(0.1) = 0.0953101798$, confirming the analysis.