MAST30001 Stochastic Modelling – 2014

Assignment 2

If you didn't already hand in a completed and signed Plagiarism Declaration Form (available from the LMS or the department's webpage), please do so and attach it to the front of this assignment.

Please hand in your assignment directly to me. **Don't forget** to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 24 October, 2014 at 5:10pm (end of lecture).**

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness and style count.

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1. Earthquakes in a certain town occur according to a Poisson process with rate 100 per year. There is a 1/1000 chance an earthquake will produce significant damage.

   (a) What is the chance that there will be 40 earthquakes in the next six months, 20 of which occur in the next three months?

   (b) What is the chance that there will be 40 earthquakes in the next six months, one of which will produce significant damage?

   (c) Given there are two earthquakes in one day, what is the chance that both of them occurred in the first half of the day?

   (d) Given there are two earthquakes in one day, what is the chance that exactly one of them produced significant damage and occurred in the first half of the day?

   In addition to the earthquakes, assume that typhoons strike the town according to a Poisson process with rate 1/2 per year and that there is 3/4 chance a typhoon will produce significant damage; typhoons occur independently of earthquakes.

   (e) What is the chance there are ten typhoon/earthquake events in a single year?

   (f) What is the chance that there are five typhoon/earthquake events that produce significant damage in a single year?

   Ans. Let N_t be the number of earthquakes between now and t years from now and M_t be the number that produce significant damage.

   (a) Using the independent increments property and that N_t is Poisson with mean $100t$,

$$
\begin{aligned}
P(N_{1/2} = 40, N_{1/4} = 20) &= P(N_{1/2} - N_{1/4} = 20, N_{1/4} = 20) \\
&= P(N_{1/2} - N_{1/4} = 20)P(N_{1/4} = 20) \\
&= e^{-25}\frac{25^{20}}{20!}e^{-25}\frac{25^{20}}{20!}.
\end{aligned}
$$

(b) The Poisson thinning theorem says that $(M_t)_{t\geq 0}$ and $(N_t - M_t)_{t\geq 0}$ are independent Poisson processes with rates 0.1 an 99.9. Thus

$$\begin{aligned} P(N_{1/2} = 40, M_{1/2} = 1) &= P(N_{1/2} - M_{1/2} = 39, M_{1/2} = 1) \\ &= P(N_{1/2} - M_{1/2} = 39)P(M_{1/2} = 1) \\ &= e^{-99.9/2}\frac{(99.9/2)^{39}}{39!}e^{-.05}(.05). \end{aligned}$$

(c) Given two earthquakes occur in one day, the times of the earthquakes are independent and uniformly distributed over that day. Thus the number occurring in the first half of the day is binomial with parameters 2 and 1/2 and so the chance exactly one occurs in the first half of the day is 1/4.

(d) Given two earthquakes in one day, the number of earthquakes occurring in the first half of the day and causing damage is binomial with parameters 2 and $1/2 \times 1/1000 = 1/2000$. Thus the chance exactly one earthquake produced significant damage and occurred in the first half of the day is $2(1/2000)(1999/2000)$.

(e) By the superposition theorem, L_t, the number of earthquakes and typhoon events between now and t years from now is a Poisson process with rate $1/2 + 100 = 100.5$. Thus

$$P(L_1 = 10) = e^{-100.5}\frac{(100.5)^{10}}{10!}.$$

(f) By Poisson thinning, $(K_t)_{t\geq 0}$, the number of significant damage events due to typhoons between now and t years from now is a Poisson process with rate $(1/2) \times (3/4) = 3/8$, independent of $(M_t)_{t\geq 0}$, the significant earthquake events process. Similar to (e), we use the superposition theorem to find

$$P(K_1 + M_1 = 5) = e^{-19/40}\frac{(19/40)^5}{5!}.$$

2. A system has two servers. Customers arrive according to a Poisson process with rate $\lambda = 2$ (per hour) and if Server A is free then they begin service with that server. If an arriving customer finds Server A busy but Server B free, then the customer will begin service with Server B. If both servers are busy, then arriving customers are turned away. The service times for Servers A and B are independent and exponential with rates $\mu_A = 3$ and $\mu_B = 2$ (per hour).

   (a) What is the long run proportion of time Server B busy?

   (b) What is the expected number of customers an **arriving** customer finds in the system after its been running for a long time?

   (c) What is the long run proportion of **entering** customers that receive service from Server B?

   (d) What is the average time **entering** customers spend in the system?

   (e) Let $M(t)$ be the mean number of customers turned away after the system is running for t hours. Find $\lim_{t\to\infty} M(t)/t$.

Ans. We can model the system as a continuous time Markov chain with state space $\{(0,0),(1,0),(0,1),(1,1)\}$; the first (second) coordinate is 1 if Server A (B) is working and zero otherwise. The generator is (states in dictionary order)

$$A = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu_A & -(\mu_A+\lambda) & 0 & \lambda \\ \mu_B & 0 & -(\mu_B+\lambda) & \lambda \\ 0 & \mu_B & \mu_A & -(\mu_A+\mu_B) \end{pmatrix}.$$

The system is irreducible and finite and so positive recurrent with long run stationary probabilities given by π, the unique solution to $\pi A = 0$, or (plugging in numbers now)

$$\pi = \frac{1}{55}(27, 14, 6, 8).$$

(a) Server B is busy if the system is in states $(0,1)$ or $(1,1)$, which according to the above is $14/55$ of the time.

(b) By PASTA, an arriving customer sees the system in stationary and the average number of customers in the stationary regime is

$$\frac{1}{55}(14+6+2*8) = 36/55.$$

(c) The proportion of entering customers who are served by Server B is the same as the chance an entering customer finds the system in a state where server B will serve them, state $(1,0)$. Given an arriving customer enters the system, the system is in states $(0,0),(1,0)$ or $(0,1)$ and so the chance the system is in state $(1,0)$ among these three is

$$\frac{14}{27+14+6} = 14/47.$$

(d) As in (c), an entering customer finds the system in states $(0,0),(1,0),(0,1)$ with probabilities $(7/12, 4/12, 1/12)$ and for the first and third states the customer gets served by server A with average service time $1/3$ and otherwise by Server B with average service time $1/2$. So the expected time an entering customer spends in the system is

$$\frac{1}{3}\left(\frac{33}{47}\right) + \frac{1}{2}\left(\frac{14}{47}\right) = 18/47.$$

(e) Customers are turned away when the system is in state $(1,1)$ the proportion of time the system is in this state for large t is $8/55$ and so the amount of time the system is in this state for large t is roughly $8t/55$. The expected number of arrivals of the Poisson process in this interval is $8\lambda t/55 = 16t/55$ and these are the customers turned away and so for large t, $M(t) \approx 16t/55$ and so $\lim_{t\to\infty} M(t)/t = 16/55$.