Discrete-Time Markov Chains

We are frequently interested in applications where we have a sequence $X_1, X_2, \cdots$ of outputs (which we model as random variables) in discrete time. For example,

- DNA: A (adenine), C (cytosine), G (guanine), T (thymine).
- Texts: X_j takes values in some alphabet, for example $\{A, B, \cdots, Z, a, \cdots\}$.
  - Developing and testing compression software.
  - Cryptology: codes, encoding and decoding.
  - Attributing manuscripts.

Independence?

Is it reasonable to assume that neighbouring letters are independent?

- Text $T = \{a_1 \cdots a_n\}$ of length n.
- Let $n_\ell = \#\{i \le n : a_i = \ell\}$, $n_{\ell j} = \#\{i \le n-1 : a_i a_{i+1} = \ell j\}$.
- Assuming that T is random, we expect $n_\ell / n \sim P(\text{letter} = \ell)$ and $n_{\ell j}/n \sim P(\text{two letters} = \ell j)$.
- If letters were independent, we have $P(\text{two letters} = \ell j) = P(\text{letter} = \ell) P(\text{letter} = j)$ so we would expect that $n_{\ell j}/n \approx n_\ell / n \times n_j / n$.
- However, let $\ell = j = a$, $P(\text{letter} = a) \approx 0.08$, but aa is very rare.

We conclude that assuming independence does not lead to a good model for text.

The Markov Property

The Markov property embodies a natural first generalisation to the independence assumption. It assumes a special one-step dependence on memory. Specifically, for all Borel sets B,

$$P(X_{n+1} \in B | X_n = x_n, X_{n-1} = x_{n-1}, \cdots) = P(X_{n+1} \in B | X_n = x_n)$$

Discrete-Time Markov Chains

A random sequence $\{X_n, n \geq 0\}$ with a countable state space (usually $\{1, 2, \cdots\}$ or $\{0, 1, 2, \cdots\}$) forms a DTMC if

$$P(X_{n+1} = k | X_n = j, X_{n-1} = x_{n-1}, \cdots, X_0 = x_0) = P(X_{n+1} = k | X_n = j).$$

This enables us to write

$$P(X_{n+1} = k | X_n = j) = p_{jk}(n).$$

Furthermore, we commonly assume that the transition probabilities $p_{jk}(n)$ do not depend on n, in which case the DTMC is called homogeneous (more precisely time homogeneous) and we write $p_{jk}(n) = p_{jk}$.

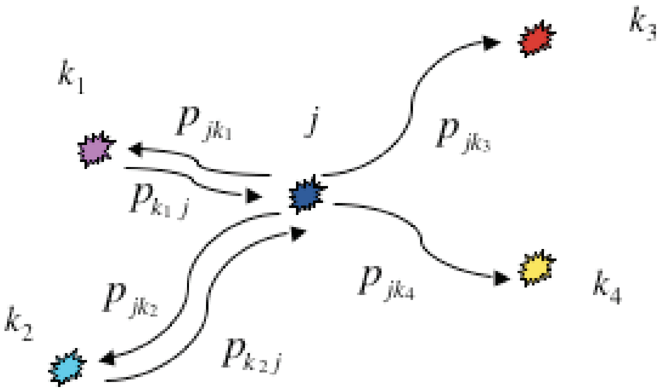
Discrete-Time Markov Chains

For a homogeneous DTMC, we define the transition matrix to be a matrix with rows and columns corresponding to the states of the process and whose jkth entry is p_{jk}. So

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix}.$$

Discrete-Time Markov Chains

We can associate a directed graph with a DTMC by letting the nodes correspond to states and putting in an arc jk if $p_{jk} > 0$.



Discrete-Time Markov Chains

For a transition matrix of a DTMC:

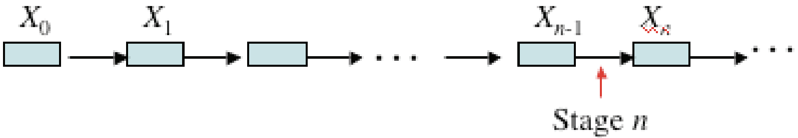
- Each entry is ≥ 0.
- Each row sums to 1.

Any square matrix having these two properties is called a stochastic matrix.

Discrete-Time Markov Chains

Examples:

- If the $\{X_n\}$ are independent and identically-distributed random variables with $P(X_i = k) = p_k$, what is the transition matrix of the DTMC?
- A communication system transmits the digits 0 and 1. At each time point, there is a probability p that the digit will not change and prob $1 - p$ it will change.



Discrete-Time Markov Chains

- Suppose that whether or not it rains tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability p and if it does not rain today, then it will rain tomorrow with probability q. If we say that the process is in state 0 when it rains and state 1 when it does not rain, then the above is a two-state Markov chain.
- A simple random walk. Let a sequence of random variables $\{X_n\} \in \mathbb{Z}$ be defined by $X_{n+1} = X_n + Y_{n+1}$, where $\{Y_n\}$ are independent and identically-distributed random variables with $P(Y_n = 1) = p$, $P(Y_n = -1) = 1 - p$.

Discrete-Time Markov Chains

The n-step transition probabilities $P(X_{m+n} = j | X_m = i)$ of a homogeneous DTMC do not depend on m. (Why?)

For $n = 1, 2, \cdots$, we denote them by

$$p_{ij}^{(n)} = P(X_{m+n} = j | X_m = i).$$

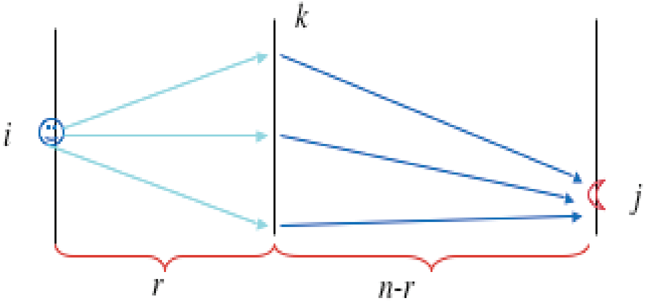
It is also convenient to use the notation

$$p_{ij}^{(0)} := \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

Discrete-Time Markov Chains

The Chapman-Kolmogorov equations show how we can calculate the $p_{ij}^{(n)}$ from the p_{ij}. For $n = 1, 2, \cdots$ and any $r = 1, 2, \cdots, n$,

$$p_{ij}^{(n)} = \sum_{k} p_{ik}^{(r)} p_{kj}^{(n-r)}.$$



Discrete-Time Markov Chains

If we define the n-step transition matrix as

$$P^{(n)} = \begin{pmatrix} p_{11}^{(n)} & p_{12}^{(n)} & \ddots & \ddots \\ p_{21}^{(n)} & p_{22}^{(n)} & p_{23}^{(n)} & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

then the Chapman-Kolmogorov equations can be written in the matrix form

$$P^{(n)} = P^{(r)} P^{(n-r)}$$

with $P^{(1)} = P$. By mathematical induction, it follows that

$$P^{(n)} = P^n,$$

the nth power of P.

Discrete-Time Markov Chains

How do we determine the distribution of a DTMC?
We have

- the initial distribution $\boldsymbol{\pi}^0 = (\pi_1^0, \ldots, \pi_m^0)$, where $\pi_j^0 = P(X_0 = j)$, for all j, and
- the transition matrix P.

In principle, we can use these and the Markov property to derive the finite dimensional distributions, although the calculations are frequently intractable.

For $k \geq 1$ and $t_1 < \cdots < t_k \in Z_+$,

$$\begin{aligned} & P(X_{t_1} = x_1, X_{t_2} = x_2, \cdots, X_{t_k} = x_k) \\ = \quad & [\boldsymbol{\pi}^0 P^{t_1}]_{x_1} [P^{t_2 - t_1}]_{x_1 x_2}, \ldots, [P^{t_k - t_{k-1}}]_{x_{k-1} x_k}. \end{aligned}$$

Example

Suppose $P(X_0 = 1) = 1/3$, $P(X_0 = 2) = 0$, $P(X_0 = 3) = 1/2$, $P(X_0 = 4) = 1/6$ and

$$P = \begin{pmatrix} 1/4 & 0 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}.$$

- Find the distribution of X_1,
- Calculate $P(X_{n+2} = 2 | X_n = 4)$, and
- Calculate $P(X_3 = 2, X_2 = 3, X_1 = 1)$.

Discrete-Time Markov Chains

Fundamental questions that we quite often want to ask are

- What proportion of time does the chain spend in each state in the long run?
- Or does this even make sense?

The answer depends on the classification of states.

Discrete-Time Markov Chains

Here are some definitions.

- State k is accessible from state j, denoted by $j \to k$, if there exists an $n \geq 1$ such that $p_{jk}^{(n)} > 0$. That is, there exists a path $j = i_0, i_1, i_2, \cdots, i_n = k$ such that $p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} > 0$.
- If $j \to k$ and $k \to j$, then states j and k communicate, denoted by $j \leftrightarrow k$.
- State j is called non-essential if there exists a state k such that $j \to k$ but $k \not\to j$.
- State j is called essential if $j \to k$ implies that $k \to j$.
- A state j is an absorbing state if $p_{jj} = 1$. An absorbing state is essential but essential states do not have to be absorbing.

Example

Draw a transition diagram and then classify the states of a DTMC with transition matrix

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

A state j which is such that $j \not\leftrightarrow j$ is called ephemeral. Ephemeral states usually don't add anything to a DTMC model and we are going to **assume that there are no such states**.
With this assumption, the communication relation $\leftrightarrow$ has the properties

- $j \leftrightarrow j$ (reflexivity),
- $j \leftrightarrow k$ if and only if $k \leftrightarrow j$ (symmetry), and
- if $j \leftrightarrow k$ and $k \leftrightarrow i$, then $j \leftrightarrow i$ (transitivity).

A relation that satisfies these properties is known as an equivalence relation.

Discrete-Time Markov Chains

Consider a set S whose elements can be related to each other via any equivalence relation $\Leftrightarrow$.

Then S can be partitioned into a collection of disjoint subsets $S_1, S_2, S_3, \ldots S_M$ (where M might be infinite) such that $j, k \in S_m$ implies that $j \Leftrightarrow k$.

So the state space of a DTMC is partitioned into communicating classes by the communication relation $\leftrightarrow$.

Discrete-Time Markov Chains

- An essential state cannot be in the same communicating class as a non-essential state.

This means we can further divide the communicating class partition into

- $S_1^n, S_2^n, S_3^n, \ldots S_{M_n}^n$ of non-essential communicating classes and
- $S_1^e, S_2^e, S_3^e, \ldots S_{M_e}^e$ of essential communicating classes.

If the DTMC starts in a state from a non-essential communicating class S_m^n then once it leaves, it never returns.
If the DTMC starts in a state from a essential communicating class S_m^e then it can never leave.
Definition:
If a DTMC has only one communicating class (i.e., all states communicate) then it is called an irreducible DTMC.

Example

Classify the states of the DTMC with

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.15 & 0.45 & 0.15 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise

Classify the states of the DTMC with

$$P = \begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 & + \\ 0 & + & 0 & + & 0 & 0 & + \\ + & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & + & + & 0 \\ 0 & 0 & + & 0 & 0 & 0 & + \end{pmatrix}$$

Discrete-Time Markov Chains

Now let's revisit the random walk example where

$X_0 = 0,$

$X_{n+1} = X_n + Y_{n+1},$

$\{Y_n\}$ are i.i.d. $P(Y_n = 1) = p$ and $P(Y_n = -1) = 1 - p = q$.

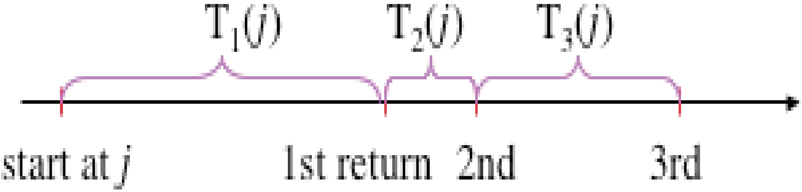
This DTMC is irreducible and so all states are essential. However,

- if $p > q$, then $E(X_n - X_0) = n(p - q) > 0$, so X_n will 'drift to infinity', at least in expectation.
- For each fixed state j, with probability one, the DTMC will visit j only finitely many times.

A state's long run essentialness is not captured by being "essential" in this case: we need a further classification of the states.

Recurrence and Transience of States

Let $X_0 = j$ and $T_i(j)$ be the time between the ith and $(i-1)$st return to state j. Then $T_1(j), T_2(j), \ldots$ are independent and identically distributed random variables.



Recurrence and Transience of States

Our further classification relies on calculating the probability that the DTMC returns to a state once it has left:

$$f_j = P(X_n = j \text{ for some } n > 0 | X_0 = j) = P(T_1(j) < \infty | X_0 = j).$$

The state j is said to be recurrent if $f_j = 1$ and transient if $f_j < 1$.

Characterizing Recurrence

If the DTMC starts in a recurrent state j then, with probability one, it will eventually re-enter j. At this point, the process will start anew (by the Markov property) and it will re-enter again with probability one. So the DTMC will (with probability one) visit j infinitely-many times.

If the DTMC starts in a transient state j then there is a probability $1 - f_j > 0$ that it will never return. So, letting N_j be the number of visits to state j after starting there, we see that N_j has a geometric distribution.

Specifically, for $n \geq 0$,

$$P(N_j = n | X_0 = j) = P(T_1(j) < \infty, \cdots, T_n(j) < \infty, T_{n+1}(j) = \infty).$$

This is equal to $(1 - f_j)f_j^n$, which implies that $E(N_j | X_0 = j) = \frac{f_j}{1-f_j}$.

Characterizing Recurrence

If

$$q_j = \sum_{n=1}^{\infty} E[I(X_n = j)|X_0 = j] = \sum_{n=1}^{\infty} p_{jj}^{(n)},$$

then $q_j = E[\sum_{n=1}^{\infty} I(X_n = j)|X_0 = j] = E(N_j|X_0 = j) = \frac{f_j}{1-f_j}$, so

$$f_j = \frac{q_j}{1 + q_j}.$$

It follows that state j is recurrent if and only if $q_j = \infty$.

[Read: j is recurrent if and only if the expected number of returns to state j is infinite.]

Communication classes are either recurrent or transient

Now assume that state j is recurrent and $j \leftrightarrow k$. There must exist s and t such that $p_{jk}^{(s)} > 0$ and $p_{kj}^{(t)} > 0$. Then

$$\begin{aligned} p_{jj}^{(s+n+t)} &= P(X_{s+t+n} = j | X_0 = j) \\ &\geq P(X_{s+t+n} = j, X_{s+n} = k, X_s = k | X_0 = j) \\ &= p_{jk}^{(s)} p_{kk}^{(n)} p_{kj}^{(t)}. \end{aligned}$$

Similarly $p_{kk}^{(s+n+t)} \geq p_{kj}^{(t)} p_{jj}^{(n)} p_{jk}^{(s)}$ and so, for $n > s + t$,

$$\alpha p_{jj}^{(n-s-t)} \leq p_{kk}^{(n)} \leq p_{jj}^{(n+s+t)} / \alpha$$

where $\alpha = p_{jk}^{(s)} p_{kj}^{(t)}$. So the series $\sum_{n=1}^{\infty} p_{kk}^{(n)}$ must diverge because $\sum_{n=1}^{\infty} p_{jj}^{(n)}$ diverges, and we conclude that state k is also recurrent.

We can refer to communication classes as recurrent or transient.

If the Markov chain is irreducible, then all states are either recurrent or transient and so it's appropriate to refer to the chain as either recurrent or transient.

The Random Walk

Let $X_{n+1} = X_n + Y_{n+1}$ where $\{Y_n : n \geq 1\}$ are independent and identically-distributed random variables with $P(Y_n = 1) = p$ and $P(Y_n = -1) = 1 - p = q$.

We can compute the m-step transition probabilities from state j to itself by observing that these probabilities are zero if m is odd and equal to

$$\binom{2n}{n} p^n q^n$$

if $m = 2n$.

The Random Walk

Stirling's formula $n! \approx \sqrt{2\pi n}n^n e^{-n}$ gives us the fact that

$$p_{jj}^{(2n)} \approx \frac{(4pq)^n}{\sqrt{n\pi}},$$

and the series $\sum_{n=1}^{\infty} p_{jj}^{(2n)}$

- diverges if $p = q = 1/2$, so the DTMC is recurrent,
- converges if $p \neq q$ (compare to geometric series), so the DTMC is transient.

Periodicity

The random walk illustrates another phenomenon that can occur in DTMCs - periodicity.

Definition: State j is periodic with period $d \geq 1$ if $\{n \geq 1 : p_{jj}^{(n)} > 0\}$ is non-empty and has greatest common divisor d.

If state j has period 1, then we say that it is aperiodic.

Examples

- The random walk has period $d = 2$ for all states j.
- What is the period of the DTMC with

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}?$$

- Find the period for the DTMC with

$$P = \begin{pmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

States in a communicating class have same period

Assume that state j has period d_j and $j \leftrightarrow k$. Then, as before, there must exist s and t such that $p_{jk}^{(s)} > 0$ and $p_{kj}^{(t)} > 0$. We know straightaway that d_j divides $s + t$ since it is possible to go from j to itself in $s + t$ steps.

Now take a path from k to itself in r steps. If we concatenate our path from j to k in s steps, this r step path, and our path from from k to j in t steps, we have an $s + r + t$ step path from j to itself. So d_j divides $s + r + t$ which means that d_j divides r. So the d_j divides the period d_k of k.

Now we can switch j and k in the argument to conclude that d_k divides d_j which means that $d_j = d_k$, and all states in the same communicating class have a common period.

Discrete-Time Markov Chains

The arguments on the preceding slides bring us to the following theorem, which discusses some solidarity properties of states in the same communicating class.

Theorem: In any communicating class S_r of a DTMC with state space S, the states are

- either all recurrent or all transient, and
- either all aperiodic or all periodic with a common period $d > 1$.
- If states from S_r are periodic with period $d > 1$, then $S_r = S_r^{(1)} \cup S_r^{(2)} \cup \cdots \cup S_r^{(d)}$ where the DTMC passes from the subclass $S_r^{(i)}$ to $S_r^{(i+1)}$ with probability one at a transition (given it stays in the communicating class).

Discrete-Time Markov Chains

Examples:
$d = 4$**:**

	$S_r^{(1)}$	$S_r^{(2)}$	$S_r^{(3)}$	$S_r^{(4)}$
$S_r^{(1)}$	0		0	0
$S_r^{(2)}$	0	0		0
$S_r^{(3)}$	0	0	0	
$S_r^{(4)}$		0	0	0

How do we analyse a DTMC?

- Draw a transition diagram.
- Consider the accessibility of states, then divide the state space into essential and non-essential states.
- Define the communicating classes, and divide them into recurrent and transient communicating classes.
- Decide whether the classes are periodic.

Exercises

- Analyse the DTMC with $P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.
- Consider a DTMC with $P = \begin{pmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Example

Analyse the Markov chain with states numbered 1 to 5 and with one-step transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Finite State DTMCs have at least one recurrent state

Recall that a state j is transient if and only if

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} = \sum_{n=1}^{\infty} E[I(X_n = j)|X_0 = j] < \infty.$$

This means that the DTMC visits j only finitely-many times (with probability one), given that it starts there.

Let S be the set of states, and $f_{j,k}$ be the probability that the DTMC ever visits state k, given that it starts in state j.

Finite State DTMCs have at least one recurrent state

If all states $k \in S$ are transient, then it must be the case that

$$\sum_{n=1}^{\infty} p_{jj}^{(n)} + \sum_{k \neq j} f_{j,k} \sum_{n=0}^{\infty} p_{kk}^{(n)} < \infty.$$

However, the left hand side of the inequality equals

$$\begin{aligned} \sum_{k \in S} \sum_{n=1}^{\infty} E[I(X_n = k) | X_0 = j] &= \sum_{n=1}^{\infty} \sum_{k \in S} E[I(X_n = k) | X_0 = j] \\ &= \sum_{n=1}^{\infty} 1 \to \infty \end{aligned}$$

which is a contradiction, and so at least one state must be recurrent.
It follows that if a finite-state DTMC is irreducible, then all states are recurrent.

Recurrence in Infinite State DTMC

When a communicating class has infinitely many states, the above line of argument does not work:

$$\sum_{k \neq j} f_{j,k} \text{ may be infinite.}$$

And it shouldn't: random walk with $p > 1/2$ has all states transient.

Recurrence in Infinite State DTMC

In order to be able to tell whether a class is recurrent, we need to be able to calculate the probability of return for at least one state.

Let's label this state 0 and denote by $f_{j,0}$ the probability that the DTMC ever reaches state 0, given that it starts in state j. Then we see that the sequence $\{f_{j,0}\}$ satisfies the equation

$$f_{j,0} = p_{j0} + \sum_{k \neq 0} p_{jk} f_{k,0}. \qquad (\ddagger)$$

Solving the equation (‡)

We illustrate how to solve this equation by

Example: Consider a random walk on the nonnegative integers:

$$p_{j,j+1} = p = 1 - p_{j,j-1}, \quad \text{for } j > 0,$$

and

$$p_{0,1} = p = 1 - p_{0,0}.$$

(And $p_{ij} = 0$ otherwise.) Equation (‡) says that for $j > 1$,

$$f_{j,0} = pf_{j+1,0} + (1-p)f_{j-1,0}$$

and, for $j = 0, 1$,

$$f_{j,0} = pf_{j+1,0} + (1-p).$$

Solving the equation (‡)

The first equation is a second-order linear difference equation with constant coefficients.
These can be solved in a similar way to second-order linear differential equations with constant coefficients, which you learned about in Calculus II or accelerated Mathematics II.
Recall that, to solve

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0,$$

we try a solution of the form $y = y(t) = e^{\lambda t}$ to obtain the Characteristic Equation

$$a\lambda^2 + b\lambda + c = 0.$$

Solving the equation (‡)

If the characteristic equation has distinct roots, λ_1 and λ_2, the general solution has the form

$$y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.$$

If the roots are coincident, the general solution has the form

$$y = Ae^{\lambda_1 t} + Bte^{\lambda_1 t}.$$

In both cases, the values of the constants A and B are determined by the initial conditions.

Solving the equation (‡)

The method for solving second-order linear difference equation with constant coefficients is similar. To solve

$$au_{j+1} + bu_j + cu_{j-1} = 0,$$

we try a solution of the form $u = m^j$ to obtain the Characteristic Equation

$$am^2 + bm + c = 0.$$

Solving the equation (‡)

If this equation has distinct roots, m_1 and m_2, the general solution has the form

$$y = Am_1^j + Bm_2^j.$$

If the roots are coincident, the general solution has the form

$$y = Am_1^j + Bjm_1^j.$$

The values of the constants A and B need to be determined by boundary equations, or other information that we have.

Solving the equation (‡)

Back to the **Example:** The characteristic equation of

$$f_{j,0} = pf_{j+1,0} + (1-p)f_{j-1,0}$$

is

$$pu^2 - u + (1-p) = 0$$

which has roots $u = 1$ and $u = (1-p)/p$.
If $(1-p)/p \neq 1$, the general solution for $j \geq 1$ is of the form

$$f_{j,0} = A + B\left(\frac{1-p}{p}\right)^j.$$

Solving the equation (‡)

If $(1-p)/p > 1$, then the general solution is

$$f_{j,0} = A + B\left(\frac{1-p}{p}\right)^j.$$

Similarly, if $(1-p)/p = 1$, the general solution is of the form

$$f_{j,0} = A + Bj.$$

In either case, these can only be probabilities if $B = 0$ and then notice

$$A = f_{1,0} = pf_{2,0} + (1-p) = pA + (1-p),$$

so $A = 1$. This makes sense because $p \leq 1/2$ and so we have a neutral or downward drift.

Solving the equation (‡)

However if $(1-p)/p < 1$, we need to work harder to obtain the solution to our problem. Let $f_{j,0}(m)$ be the probability that the DTMC moves from state j to state 0 in less than or equal to m steps. Because

$$\cup_{m=1}^{\infty}\{j \mapsto 0 \text{ in } \leq \text{m steps}\} = \{j \mapsto 0 \text{ ever}\},$$

we find $f_{j,0}(m) \nearrow f_{j,0}$.

Solving the equation (‡)

Now let $\{g_{j,0}\}$ be any nonnegative solution to

$$g_{j,0} = p_{j0} + \sum_{k \neq 0} p_{jk} g_{k,0}.$$

We show by induction that $f_{j,0}(m) \leq g_{j,0}$ for all m. Clearly this is true for $m = 1$. Assume that it is true for $m = \ell$. Then

$$\begin{aligned} f_{j,0}(\ell+1) &= p_{j0} + \sum_{k \neq 0} p_{jk} f_{k,0}(\ell) \\ &\leq p_{j0} + \sum_{k \neq 0} p_{jk} g_{k,0} \\ &= g_{j,0}. \end{aligned}$$

It follows that $f_{j,0} = \lim_{m\to\infty} f_{j,0}(m) \leq g_{j,0}$ and so $\{f_{j,0}\}$ is the minimal nonnegative solution to (‡).

Solving the equation (‡)

For the random walk with $(1-p)/p < 1$, the general solution for $j \geq 1$ was of the form

$$f_{j,0} = A + B\left(\frac{1-p}{p}\right)^j .$$

Plugging into the boundary condition

$$f_{1,0} = pf_{2,0} + (1-p),$$

The minimal nonnegative solution is for $j \geq 1$:

$$f_{j,0} = \left(\frac{1-p}{p}\right)^j .$$

and $f_{0,0}$ boundary condition implies $f_{0,0} = 2(1-p)$.

The Gambler's Ruin Problem

- Denote the initial capital of a gambler by N.
- The gambler will stop playing if he/she wins $\$M$ or loses his/her initial stake of $\$N$.
- There is a probability p that the gambler wins \$1 and a probability $1 - p$ that he/she loses \$1 on each game.
- We assume that the outcomes of successive plays are independent.
- This is a simple DTMC with a finite state space $\{-N, \ldots, M\}$ and transition probabilities $p_{j,j+1} = p$ and $p_{j,j-1} = 1 - p$ for $j \in \{-N+1, \ldots, M-1\}$, and $p_{-N,-N} = p_{M,M} = 1$.

The gambler would like to know the probability that he/she will win $\$M$ before becoming bankrupt.

The Gambler's Ruin Problem

We use (‡) to calculate the probability that the gamblers ruin DTMC hits $-N$. For $-N+1 \leq j \leq M-1$, we have

$$f_{j,-N} = pf_{j+1,-N} + (1-p)f_{j-1,-N}$$

with $f_{-N,-N} = 1$ and $f_{M,-N} = 0$.
When $p \neq 1/2$, the general solution to the first equation is again

$$f_{j,-N} = A + B\left(\frac{1-p}{p}\right)^j.$$

(For the same range of j.)

The Gambler's Ruin Problem

The upper boundary condition gives us

$$A = -B\left(\frac{1-p}{p}\right)^{M},$$

and the lower boundary condition gives us

$$B = \left(\left(\frac{1-p}{p}\right)^{-N} - \left(\frac{1-p}{p}\right)^{M}\right)^{-1},$$

So the general solution is

$$f_{j,-N} = \frac{\left(\frac{1-p}{p}\right)^{j} - \left(\frac{1-p}{p}\right)^{M}}{\left(\frac{1-p}{p}\right)^{-N} - \left(\frac{1-p}{p}\right)^{M}}.$$

The Gambler's Ruin Problem

When $p = 1/2$, the general solution to the first equation is

$$f_{j,-N} = A + Bj.$$

The upper boundary condition gives us

$$A = -BM,$$

and the lower boundary condition gives us

$$B = \frac{-1}{M+N},$$

So the general solution is

$$f_{j,-N} = \frac{M-j}{M+N}.$$

The Gambler's Ruin Problem

- The expected gain is $E(G) = M - (N + M)f_{0,-N}$

Here are some numbers:

- If $N = 90$, $M = 10$ and $p = 0.5$ then $f_{0,-N} = 0.1$.
- If $N = 90$, $M = 10$ and $p = 0.45$ then $f_{0,-N} = 0.866$.
- If $N = 90$, $M = 10$ and $p = 0.4$ then $f_{0,-N} = 0.983$.
- If $N = 99$, $M = 1$ and $p = 0.4$ then $f_{0,-N} = 0.333$.

Long run behaviour of DTMCs

We want to know the proportion of time a DTMC spends in each state over the long run (if this concept makes sense) which should be the same as the limiting probabilities $\lim_{n\to\infty} p_{kj}^{(n)}$.

- These will be zero for transient states and non-essential states.
- For an irreducible and recurrent DTMC, we will see that these limiting probabilities exist and are even independent of k.

Long run behaviour of DTMCs

Recall that we used $T_i(j)$ to denote the time between the ith and $(i-1)$st return to state j. We then defined state j (and hence its communicating class) to be

- transient if $T_i(j) = \infty$ with positive probability, and
- recurrent if $T_i(j) < \infty$ with probability one.

There is a further classification of recurrent states. Specifically, j is

- null-recurrent if $E[T_i(j)] = \infty$, and
- positive-recurrent if $E[T_i(j)] < \infty$.

This classification is important for the calculation of the limiting probabilities.

Examples

- The symmetric random walk with $p = q = 1/2$. For all j, $T_i(j) < \infty$ with probability one, but $E[T_i(j)] = \infty$. That is, all states are null-recurrent.
- A finite irreducible DTMC: $E[T_i(j)] < \infty$ for all j.

Long run behaviour of DTMCs

In the long run, how often does a DTMC visit a state j?

Let $\mu_j \equiv E[T_1(j)|X_0 = j] < \infty$. By the Law of Large Numbers, $T_1(j) + T_2(j) + \cdots + T_k(j) \approx \mu_j k$. So there are approximately k visits in $k\mu_j$ time-steps, and the relative frequency of visits to j is $1/\mu_j$. This leads us to

Theorem: If j is an aperiodic state in a positive recurrent communicating class: $\mu_j = E[T_1(j)|X_0 = j] < \infty$, then

$$\lim_{n\to\infty} p_{jj}^{(n)} = \frac{1}{\mu_j}.$$

In the null recurrent or transient case where $\mu_j = \infty$:

$$\lim_{n\to\infty} p_{jj}^{(n)} = 0.$$

Long run behaviour of DTMCs

Why do we need aperiodicity?

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Long run behaviour of DTMCs

Further:

Theorem: if $i \leftrightarrow j$ are aperiodic and the communication class is positive recurrent: $\mu_j = E[T_1(j)|X_0 = j] < \infty$, then

$$\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{\mu_j}.$$

Note that the right hand side doesn't have an i in it!

Example:

We can compute μ_j directly for a two state Markov chain with transition matrix

$$\begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix},$$

where $0 < p < 1$ and $0 < q < 1$.

Ergodicity and Stationarity

Definition: We call the DTMC ergodic if for all j the limit $\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}$ exists, is positive, and doesn't depend on i.

For an ergodic DTMC, with limiting distribution $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots)$,

- $\sum_j \pi_j = 1$.
- For any initial probability distribution $\boldsymbol{\pi}^0$,

$$\boldsymbol{\pi}^0 P^n \to \boldsymbol{\pi} \text{ as } n \to \infty.$$

- $\boldsymbol{\pi} P = \boldsymbol{\pi}$ and hence $\boldsymbol{\pi} P^n = \boldsymbol{\pi}$.

Any distribution satisfying the last item is called a stationary distribution for the DTMC.

Ergodicity and Stationarity

Theorem: A DTMC $\{X_n\}$ is ergodic if and only if it is irreducible, aperiodic and positive recurrent.

In this case there is a unique solution to the system of linear equations

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

with $\sum_j \pi_j = 1$ (that is, a unique stationary distribution) and moreover $\pi_j = 1/\mu_j$.

In addition, an irreducible DTMC is positive recurrent if and only if the equation $\boldsymbol{\pi} P = \boldsymbol{\pi}$ has a probability solution.

Examples

An $m \times m$ stochastic matrix P is called doubly-stochastic if all the column sums are equal to one.
If an aperiodic DTMC has a doubly-stochastic transition matrix, then we can easily verify that

$$(1/m, 1/m, 1/m, \ldots)P = (1/m, 1/m, 1/m, \ldots).$$

It follows that

$$\boldsymbol{\pi} = (1/m, 1/m, 1/m, \ldots),$$

and the stationary distribution is uniform on S.

Find a stationary distribution for

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Examples

Find a stationary distribution for

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the stationary distribution for

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{pmatrix}.$$

Random walk with one barrier

Give a criterion for ergodicity of the DTMC with state space $\{0, 1, 2, \cdots\}$ and transition matrix

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & \ddots \\ q & 0 & p & 0 & 0 & \ddots \\ 0 & q & 0 & p & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

When the DTMC is ergodic, derive its stationary distribution.

Random walk with one barrier

We saw that this DTMC is irreducible, aperiodic and recurrent when $p \leq q$.

Solve the linear equations

$$(\pi_0, \pi_1, \cdots) = (\pi_0, \pi_1, \cdots)P$$

to get $\pi_k = (p/q)^k \pi_0$.

We also need $\sum_{k \geq 0} \pi_k = 1$. The sum on the left hand side is finite only if $p < q$, in which case $\pi_0 = 1 - (p/q)$ and

$$\pi_k = [1 - (p/q)]\,(p/q)^k.$$

So there is a solution to $\boldsymbol{\pi} = \boldsymbol{\pi} P$ with $\sum_{k \geq 0} \pi_k = 1$, and hence the DTMC is ergodic, only if $p < q$, in which case

$$\mu_k = \frac{1}{(p/q)^k (1 - (p/q))}.$$

The distribution $\boldsymbol{\pi}$

For an irreducible, aperiodic and positive-recurrent DTMC, the distribution defined by $\boldsymbol{\pi}$ has a number of interpretations.
It can be seen as

- limiting,
- stationary,
- ergodic.

The distribution $\boldsymbol{\pi}$

The limiting interpretation

By definition

$$\pi_j = \lim_{n \to \infty} P_{ij}^{(n)}$$

and so π_j is the limiting probability that the DTMC is in state j. This means that so long as the DTMC has been going for quite a long time, the probability that it is in state j will be approximately π_j.

The stationary interpretation

We showed that

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

and so $\boldsymbol{\pi}$ has a stationary interpretation. If the DTMC starts with probability distribution $\boldsymbol{\pi}$ it will persist with this distribution forever.

Furthermore, the DTMC is strictly stationary in the sense that its finite-dimensional distributions are invariant. For each k, $(X_m, X_{m+1}, \ldots, X_{m+k})$ has the same distribution as $(X_0, X_1, \ldots, X_k)$, independently of m.

The distribution $\boldsymbol{\pi}$

The ergodic interpretation

This means that for sample paths of DTMC that constitute a set of probability one, the proportion of time that the process spends in state j is π_j.

This is can be formally stated as a Law of Large Numbers. For any initial distribution, the proportion of time $\sum_{i=1}^{n} I(X_i = j)/n$ that the DTMC spends in state j converges to π_j with probability one as $n \to \infty$.

Reducible DTMC

- From non-essential states, the DTMC will eventually leave forever.
- As soon as it enters a class S_r of essential states, it will stay in S_r forever.
- If the stochastic sub-matrix P_r, derived from P by restricting it to the states of S_r, is aperiodic and positive-recurrent, then this subchain is ergodic and

$$P(X_n = j | X_0 \in S_r) \begin{cases} \to \pi_j^{(r)} & j \in S_r \\ = 0 & \text{if } \; j \notin S_r. \end{cases}$$

Periodic DTMC

- If P_r is a periodic subchain with a period $d > 1$, which is recurrent with a finite expected recurrence time, then for $0 \leq k \leq d-1$, $\{X_{nd+k}|X_0 \in S_r\}$ is an ergodic DTMC with state space $S_r^{(k)}$.
- For any ℓ and $k = 0, 1, \cdots, d-1$,

$$P(X_{nd+k} = j|X_0 \in S_r^{(\ell)}) \begin{cases} \to \tilde{\pi}_j^{(r)} \text{ as } n \to \infty & \text{for } j \in S_r^{((\ell+k)(\bmod d))} \\ = 0 & \text{for } j \notin S_r^{((\ell+k)(\bmod d))} \end{cases}$$

so $\sum_{j \in S_r^{(\ell)}} \tilde{\pi}_j^{(r)} = 1$ for any ℓ.

Example

Classify the DTMC with

$$P = \begin{pmatrix} 0 & 0 & 0.1 & 0.9 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

and discuss its properties.

Good Trick

Sometimes we want to model a physical system where the future depend on part of the past. Consider following example. A sequence of random variables $\{X_n\}$ describes the weather at a particular location, with $X_n = 1$ if it is sunny and $X_n = 2$ if it is rainy on day n.
Suppose that the weather on day $n+1$ depends on the weather conditions on days $n-1$ and n as is shown below:

$$
\begin{aligned}
P(X_{n+1} = 2 | X_n = X_{n-1} = 2) &= 0.6 \\
P(X_{n+1} = 1 | X_n = X_{n-1} = 1) &= 0.8 \\
P(X_{n+1} = 2 | X_n = 2, X_{n-1} = 1) &= 0.5 \\
P(X_{n+1} = 1 | X_n = 1, X_{n-1} = 2) &= 0.75
\end{aligned}
$$

Good Trick

If we put $Y_n = (X_{n-1}, X_n)$, then Y_n is a DTMC. The possible states are $1' = (1,1)$, $2' = (1,2)$, $3' = (2,1)$ and $4' = (2,2)$. We see that $\{Y_n : \ n \geq 1\}$ is a DTMC with transition matrix

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{pmatrix}.$$