MAT4MDS — Practice 4

Eigenvalues, eigenvectors, rank

Question 1. Consider the vectors $X = \begin{bmatrix}1\\0\end{bmatrix}$, $Y = \begin{bmatrix}3\\-4\end{bmatrix}$ and $Z = \begin{bmatrix}1\\1\end{bmatrix}$.

(a) For each vector, multiply on the left by $H = \begin{bmatrix}3 & 3\\4 & 2\end{bmatrix}$.

(b) Decide which of the given vectors are eigenvectors of H, and what the associated eigenvalue is.

For an $n \times n$ matrix, the eigenvalues are the values of λ for which
$\det(A - \lambda I_n) = 0. \qquad (*)$
This equation (*) is called the **characteristic equation** of A.

Question 2. Find the eigenvalues of $K = \begin{pmatrix}4 & 2\\-3 & -3\end{pmatrix}$ by calculating the appropriate determinant (*) and solving the resulting polynomial equation.

For a 2×2 matrix, the characteristic equation of A is
$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0 \qquad (⊞)$
where $\mathrm{trace}(A)$ is the sum of the diagonal entries of A.
For $n > 2$ it is still true that the sum of the eigenvalues is equal to the trace, and the product of the eigenvalues is equal to the determinant.

Question 3. For the matrix $L = \begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$, use (⊞) to find the eigenvalues.

Question 4.

(a) Form the matrix product $J_{n\times n}J_{n\times 1}$, where J is the all-ones matrix of the appropriate order. Hence state an eigenvalue and eigenvector of $J_{n\times n}$.

(b) What is the trace of $J_{n\times n}$? Is $J_{n\times n}$ a positive semi-definite matrix? What does this tell you about the other eigenvalues?

Question 5. Find the eigenvalues of the matrix M, where

$$M = \begin{pmatrix}2 & 1 & -1\\1 & 2 & -1\\-1 & -1 & 2\end{pmatrix}.$$

Compare the sum and product of the eigenvalues with $\mathrm{trace}(M)$ and $\det(M)$.

Question 6. For each of the matrices L, M state the rank, and whether the matrix is of full rank.

Question 7. Let A and B be invertible $n \times n$ matrices. Prove the **socks and shoes** formula: $(AB)^{-1} = B^{-1}A^{-1}$.

> **Hint:** simplify $AB(B^{-1}A^{-1})$ using associativity, then use the following property from Practice class 3:
>
> $$AB = I_n \implies A^{-1} = B. \qquad (\bigstar)$$

Question 8. Let M be an invertible matrix with eigenvalue μ. (From a result stated in the readings, we know $\mu \neq 0$ because M is invertible.)

(a) Show that M^{-1} has an eigenvalue $\frac{1}{\mu}$. (**Hint:** begin from $MX = \mu X$ and use matrix algebra.)

(b) Let A be a square matrix, of the same order as M (but not necessarily invertible). Using the multiplicative property of determinants, show that A and $M^{-1}AM$ have the same characteristic equation. What does this tell us about their eigenvalues?