MAT4MDS — Practice 6

DIFFERENTIATION

function	$f(x)$	x^r $(r \neq 0)$	constant	e^x	$\log_e(x)$
derivative	$f'(x)$	rx^{r-1}	0	e^x	$\frac{1}{x}$

Question 1. Using basic derivatives from the table above, and the differentiation rules of Reading 5.2, differentiate the following:

(a) $f(x) = x^2 - 2x + 3$.

(b) $g(x) = (x^2 + 1)^3$.

(c) $f(x) = x^2 e^x$.

(d) $y = x^3 \log_e(x), x \in (0, \infty)$.

(e) $y = \frac{2x-3}{3x+1}$.

(f) $f(x) = (3x + 2)\ln(3x + 2), x \in \left(-\frac{2}{3}, \infty\right)$.

(g) $y = \frac{x^2+\sqrt{x}}{x}$, $x \in (0, \infty)$. (Think about this!)

Question 2.

(a) Beginning from the definition $f\left(f^{-1}(x)\right) = x$, and using the chain rule, find an expression for the derivative of f^{-1} in terms of the derivative of f.

(b) Using the result of (a), find the derivative of $g(x) = \log_e(x)$, using the fact that $(e^x)' = e^x$.

(Note: do not use the derivative for log given in the table above; you are finding this result.)

Question 3. In a video this week, the extended product rule was derived. Using a similar method, find the extended chain rule. That is, find a general expression for the derivative of: $y = f\left(g\left(h(x)\right)\right)$.

The Second Derivative test

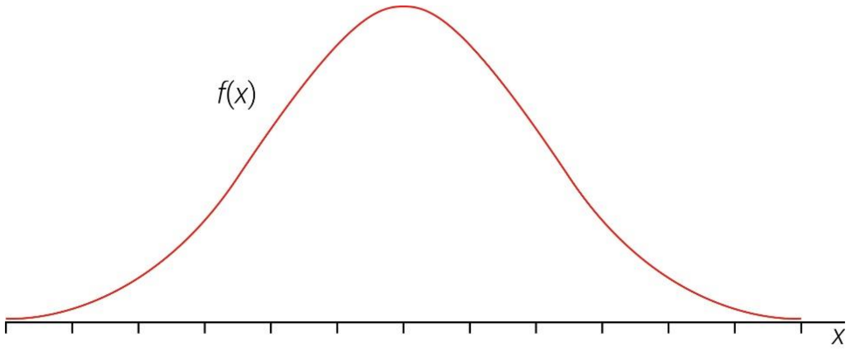
Suppose that $f''(x_0) > 0$,

(i) If $f''(x_0) > 0$, then x_0 is a local minimum point.

(ii) If $f''(x_0) < 0$, then x_0 is a local maximum point.

(iii) If $f''(x_0) = 0$, the test is inconclusive. (This point may be a local maximum, a local minimum or a point of inflection.)

Question 4.



The classic "bell shape" of the Gaussian probability density function $f(x)$ comes from its changing curvature (i.e. its varying slope).

(a) On the diagram, using your judgement, mark where you think the curve changes from being like part of a bowl to being like a hill, and where it changes back to being like part of a bowl. (These are the two types of curvature: concave up (bowl-like) and concave down (hill-like).)

(b) Find the second derivative $f''(x)$ for the Gaussian probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

using the chain rule to find $f'(x)$, and then product rule as well to find $f''(x)$.

(c) Where is the stationary point of this function? What kind of point is it?

(d) Curvature changes at points where $f''(x) = 0$. Find the two values of x such that $f''(x) = 0$.

(e) The horizontal axis in the diagram above has spacing $\frac{\sigma}{2}$. Mark the three x-values found in (c) and (d) on this axis.

Question 5.

(i) For each function in Question 1, find the second derivative.

(ii) Which functions from Question 1 change curvature on their domain? Which functions do not change curvature?

(iii) Which functions from Question 1 have stationary points, and for what values of x do these occur?

(iv) Classify the stationary points found in (iii).

(v) Use this information to make a sketch of each function.