

MAST30001 Stochastic Modelling

Tutorial Sheet 9

1. A rent-a-car washing facility can wash one car at a time. Cars arrive to be washed according to a Poisson process with rate 3 per day and the service time to wash a car is exponential with mean $7/24$ days. It costs the company \$150 per day to operate the facility and the company loses \$10 per day for each car tied up in the washing facility. The company can increase the rate of washing to get down to a mean service time of $1/4$ days at the additional cost of \$ C per day. What's the largest C can be for this upgrade to make economic sense?

Ans. We can model the number of cars in the wash as an $M/M/1$ queue with arrival rate $\lambda = 3$ and current service rate $\mu = 24/7$ and so with stationary distribution geometric with parameter $1 - 21/24 = 1/8$ having expectation 7. The companies current cost per day is

$$150 + 10 \times 7 = 220.$$

If the company pays C dollars per day to increase their service rate to 4, then similarly their new cost per day will be

$$150 + C + 10 \times 3 = 180 + C.$$

Thus they should spend no more than 40 dollars per day to increase their service rates.

2. Customers arrive at a bank according to a Poisson process rate λ . The bank's service policy is that
- if there are fewer than 4 customers in the bank, then there is 1 teller,
 - if there are 4 – 9 customers, there are 2 tellers,
 - if there are more than 9 customers, there are 3 tellers.

Tellers' service times are independent and exponentially distributed with rate μ . Model the number of customers in the bank as a birth and death chain and determine for what values of λ and μ there is stable long run behavior and for these parameters compute the steady state distribution. [*Hint: This is similar to the analysis of the $M/M/a$ queue done in lecture.*]

Ans. We can model the number of customers in the bank as a birth-death process with birth rates $\lambda_i = \lambda$, $i = 0, 1, \dots$ and death rates

$$\mu_j = \begin{cases} \mu & j = 1, 2, 3, \\ 2\mu & j = 4, \dots, 9, \\ 3\mu & j > 9. \end{cases}$$

For $K_j = \prod_{\ell=1}^j \lambda_{\ell-1} / \mu_{\ell}$ and writing $\rho = \lambda / \mu$,

$$K_j = \begin{cases} \rho^j & j = 1, 2, 3, \\ 2^{3-j} \rho^j & j = 4, \dots, 9, \\ 2^{-6} 3^{9-j} \rho^j & j > 9, \end{cases}$$

and the system is stable if and only if $\sum_{j \geq 1} K_j < \infty \iff \sum_{j \geq 10} K_j < \infty$ which is the same as $\rho/3 < 1$; that is $\rho < 3$. In this case, the stationary distribution is given by

$$\pi_j = \pi_0 K_j$$

and since the probabilities have to sum to one we have

$$\pi_0^{-1} = \frac{1 - \rho^4}{1 - \rho} + \frac{\rho^4(1 - (\rho/2)^6)}{2 - \rho} + \frac{\rho^{10}}{2^6(3 - \rho)}.$$

3. (M/M/ ∞ queue) Assume that in a queuing system customers arrive according to a rate λ Poisson process, customers are always served immediately (for example, customers making purchases on the internet), and the service time of a customer is exponential with rate μ , independent of arrival times and other service times.
- (a) Model this queue as a birth-death chain and write down its generator.
 - (b) Describe the long run behaviour of the chain.
 - (c) When the queue is in stationary (i.e., after its been running a long time), what is the expected number of customers in the system, number of customers in the queue, number of busy servers, and service time for an arriving customer?
 - (d) Let X_t be the number of customers in the system (including those being served) at time t and set $X_0 = 0$. What is $E[X_t]$? [Hint: if $m(t) = E[X_t]$, consider $m'(t)$.] You should check your formula makes sense as t tends to infinity.

Ans.

(a) The number of customers in the system is a birth-death chain that has birth rate λ in every state. If the chain is in state i , there are i customers each getting served at rate μ so the total rate of service is $i\mu$, and this is the death rate. Its generator has for $i \geq 1$

$$a_{ii+1} = \lambda, \quad a_{ii-1} = \mu i, \quad a_{ii} = -(\lambda + \mu i).$$

And $a_{01} = -a_{00} = \lambda$.

(b) The chain is irreducible and has unique stationary distribution (by directly solving $\pi A = 0$) $\pi_k = e^{-\rho} \rho^k / k!$ with $\rho = \lambda / \mu$ so it is ergodic with long run frequencies given by π .

(c) The expected number of customers in the system is just the expectation against π which is ρ , the number of customers in the queue is 0 since customers are served instantaneously, the number of busy servers is equal to the number of customers in the system which as already said has mean ρ , and since customers are served instantaneously and at rate μ , the expected time in the system of an arriving customer is $1/\mu$.

(d) Since for $k \geq 1$,

$$\frac{d}{dt} P_{0,k}(t) = \lambda P_{0,k-1}(t) - (\lambda + k\mu) P_{0,k}(t) + (k+1)\mu P_{0,k+1}(t),$$

multiplying by k and summing we find

$$m'(t) = \lambda - \mu m(t),$$

and $m(0) = 0$. This ODE has solution $m(t) = (\lambda/\mu)(1 - e^{-\mu t})$, and note that as t tends to infinity, this quantity tends to the mean of the stationary number of customers in the queue.

4. (M/G/ ∞ queue) In a certain communications system, information packets arrive according to a Poisson process with rate λ per second and each packet is processed in one second with probability p and in two seconds with probability $1 - p$, independent of the arrival times and other service times. Let N_t be the number of packets that have entered the system up to time t and X_t be the number of packets in the system (including those being served) at time t .

- (a) Is $(X_t)_{t \geq 0}$ a Markov chain? (No detailed argument is necessary here, just think about it heuristically.)
- (b) If $X_0 = 0$, what is the distribution of X_2 ?
- (c) If $X_0 = 0$, is there a “stationary” limiting distribution $\pi_k = \lim_{t \rightarrow \infty} P(X_t = k)$? If so, what is it?
- (d) If $X_0 = N_0 = 0$, what is the joint distribution of X_t and N_t ?

Ans.

(a) X_t is not a Markov chain because the chance of the chain decreasing by one in the interval $(t, t+h)$ given the value of the chain at time t also depends on the times of the arrivals in the past.

(b) If A_t are the arrivals that require one second of service, and B_t are the arrivals requiring two seconds of service, then A_t and B_t are independent Poisson processes with rates $p\lambda$ and $(1-p)\lambda$. And $X_2 = (N_2 - N_1) + B_1$; the sum of two independent Poisson variables (using independent increments) with respective means λ and $(1-p)\lambda$. So X_2 is Poisson with mean $\lambda(2-p)$.

(c) X_t only depends on the number of arrivals of the two different types in the interval $(t-2, t)$ since all arrivals previous to this time have left the system. As in part (b), we can write $X_t = (N_t - N_{t-1}) + (B_{t-1} - B_{t-2})$, and the two variables in parentheses are independent Poisson with respective means λ and $(1-p)\lambda$. So for $t \geq 2$, X_t is Poisson with mean $\lambda(2-p)$.

(d) When $0 < t \leq 1$, then $X_t = N_t$ and they're both distributed as Poisson mean t . The case $1 < t < 2$ is similar but easier than $t \geq 2$; the latter case we show here. Assuming $t \geq 2$, then as above we write $X_t = (N_t - N_{t-1}) + (B_{t-1} - B_{t-2})$ and also $N_t = X_t + (A_{t-1} - A_{t-2}) + N_{t-2}$, and note that by the comments of part (b), X_t is independent of $(A_{t-1} - A_{t-2})$ and these variables are both independent of N_{t-2} . So we can write $N_t = X_t + Y_t$, where Y_t is a Poisson variable with mean $\lambda(p+t-2)$, independent of X_t which implies that for $0 \leq j \leq n$,

$$\begin{aligned} \mathbb{P}(X_t = j, N_t = n) &= \mathbb{P}(X_t = j, Y_t = n - j) \\ &= \mathbb{P}(X_t = j) \mathbb{P}(Y_t = n - j) \\ &= e^{-\lambda t} \frac{\lambda^n}{n!} \binom{n}{j} (2-p)^j (p+t-2)^{n-j}. \end{aligned}$$