MAST30013 – Techniques in Operations Research

Semester 1, 2021

Tutorial 6 Solutions

1. The Lagrange function is

$$L(\boldsymbol{x},\boldsymbol{\eta}) \;=\; x_1x_2+\eta_1\left(x_1^2+x_2^2-1\right).$$

The Lagrange condition is

$$\boldsymbol{\nabla}_{\boldsymbol{x}}L(\boldsymbol{x},\boldsymbol{\eta}) \;=\; \begin{pmatrix} x_2+2\eta_1x_1 \\ x_1+2\eta_1x_2 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Rearranging the first equation we get $x_2=-2\eta_1x_1$. Substituting this into the second equation gives $x_1-4\eta_1^2x_1=0 \implies x_1=0$ or $\eta_1=\pm\frac{1}{2}$. If $x_1=0$ then $x_2=0$ by the first equation, which violates the constraint. If $\eta_1=\frac{1}{2}$ then $x_1=-x_2$, and the constraint gives $(x_1,x_2)^T=\left(1/\sqrt{2},-1/\sqrt{2}\right)$, $(-1/\sqrt{2},1/\sqrt{2})$. Similarly, $\eta_1=-\frac{1}{2}$ leads to $(x_1,x_2)^T=(1/\sqrt{2},1/\sqrt{2}),(-1/\sqrt{2},-1/\sqrt{2})$

We now check the constraint qualifications for each stationary point. The Jacobian is

$$\boldsymbol{\nabla}h(\boldsymbol{x}) \;=\; \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}.$$

Now,

$$\boldsymbol{\nabla}h(\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}) \;=\; \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{\nabla}h(\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}) \;=\; \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{\nabla}h(\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}) \;=\; \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{\nabla}h(\begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}) \;=\; \begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

All right hand side matrices have rank 1, so the constraint qualifications hold for all stationary points.

The Hessian is

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}}L(\boldsymbol{x},\boldsymbol{\eta}) \;=\; \begin{pmatrix} 2\eta_1 & 1 \\ 1 & 2\eta_1 \end{pmatrix}.$$

Now,

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}}L(\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}) \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}}L(\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}) \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}}L(\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}) \;=\; \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}}L(\begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}) \;=\; \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

We now need to use the Jacobian and determine the descent directions which maintain feasibility, that is, find $\boldsymbol{d} = (d_1, d_2)^T$ such that $\boldsymbol{\nabla} h(\boldsymbol{x}^*)\boldsymbol{d} = 0$. Now,

$$
\begin{aligned}
\boldsymbol{\nabla} h(\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix})^T \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0 \\
\Longrightarrow \begin{pmatrix} \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0 \\
\Longrightarrow \sqrt{2}d_1 - \sqrt{2}d_2 &= 0 \\
\Longrightarrow \boldsymbol{d} &= \begin{pmatrix} d_1 \\ d_1 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla} h(\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix})^T \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0 \\
\Longrightarrow \begin{pmatrix} -\sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0 \\
\Longrightarrow -\sqrt{2}d_1 + \sqrt{2}d_2 &= 0 \\
\Longrightarrow \boldsymbol{d} &= \begin{pmatrix} d_1 \\ d_1 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla} h(\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix})^T \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0 \\
\Longrightarrow \begin{pmatrix} \sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0 \\
\Longrightarrow \sqrt{2}d_1 + \sqrt{2}d_2 &= 0 \\
\Longrightarrow \boldsymbol{d} &= \begin{pmatrix} d_1 \\ -d_1 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla} h(\begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix})^T \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0 \\
\Longrightarrow \begin{pmatrix} -\sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0 \\
\Longrightarrow -\sqrt{2}d_1 - \sqrt{2}d_2 &= 0 \\
\Longrightarrow \boldsymbol{d} &= \begin{pmatrix} d_1 \\ -d_1 \end{pmatrix}.
\end{aligned}
$$

For $\boldsymbol{x}^* = (1/\sqrt{2}, -1/\sqrt{2})^T$ and $(1/\sqrt{2}, -1/\sqrt{2})^T$,

$$
\begin{pmatrix} d_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_1 \end{pmatrix} = 4d_1^2 > 0.
$$

For $\boldsymbol{x}^* = (1/\sqrt{2}, 1/\sqrt{2})^T$ and $(-1/\sqrt{2}, -1/\sqrt{2})^T$,

$$
\begin{pmatrix} d_1 & -d_1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} d_1 \\ -d_1 \end{pmatrix} = -4d_1^2 < 0.
$$

In conclusion, the second-order sufficiency condition implies that the two minima are $(1/\sqrt{2}, -1/\sqrt{2})^T$ and $(1/\sqrt{2}, -1/\sqrt{2})^T$. In addition, we can also conclude that the two maxima are $(1/\sqrt{2}, 1/\sqrt{2})^T$ and $(-1/\sqrt{2}, -1/\sqrt{2})^T$.

2. The Lagrange function is

$$L(\boldsymbol{x}, \boldsymbol{\eta}) \;=\; 4 - x_3 + \eta_1 \left(x_1^2 + x_2^2 - 8\right) + \eta_2 \left(x_1 + x_2 + x_3 - 1\right).$$

The Lagrange condition is

$$\boldsymbol{\nabla}_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\eta}) \;=\; \begin{pmatrix} 2\eta_1 x_1 + \eta_2 \\ 2\eta_1 x_2 + \eta_2 \\ -1 + \eta_2 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The last equation gives $\eta_2 = 1$. The first two equations now give $2\eta_1 x_1 = 2\eta_1 x_2$ which implies that either $\eta_1 = 0$ or $x_1 = x_2$. If $\eta_1 = 0$ then $\eta_2 = 0$ which contradicts that fact that $\eta_2 = 1$. Substituting $x_1 = x_2$ into the first constraint gives $2x_2^2 = 8 \Longrightarrow x_2 = \pm 2$. Thus, $x_1 = \pm 2$, and using the second constraint, $x_3 = 1 - 2 - 2 = -3$ and $x_3 = 1 - (-2) - (-2) = 5$. If $x_1 = x_2 = 2$ then $\eta_1 = -\frac{1}{4}$, and if $x_1 = x_2 = -2$ then $\eta_1 = \frac{1}{4}$. The two stationary points are $(2, 2, -3)$ and $(-2, -2, 5)$.

We now check the constraint qualifications for each stationary point. The Jacobian is

$$\boldsymbol{\nabla} h(\boldsymbol{x}) \;=\; \begin{pmatrix} 2x_1 & 1 \\ 2x_2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now,

$$\boldsymbol{\nabla} h(\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}) \;=\; \begin{pmatrix} 4 & 1 \\ 4 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\nabla} h(\begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix}) \;=\; \begin{pmatrix} -4 & 1 \\ -4 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

All right hand side matrices have rank 2, so the constraint qualifications hold for both stationary points.

The Hessian is

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}} L(\boldsymbol{x}, \boldsymbol{\eta}) \;=\; \begin{pmatrix} 2\eta_1 & 0 & 0 \\ 0 & 2\eta_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now,

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}} L(\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}) \;=\; \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}} L(\begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix}) \;=\; \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We now need to use the Jacobian and determine the descent directions which maintain feasibility, that is, find $\boldsymbol{d} = (d_1, d_2, d_3)^T$ such that $\boldsymbol{\nabla} h(\boldsymbol{x}^*)\boldsymbol{d} = 0$. Now,

$$
\begin{aligned}
\boldsymbol{\nabla} h(\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}) \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} &= 0 \\
\Longrightarrow \begin{pmatrix} 4 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}^T \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} &= 0 \\
\Longrightarrow 4d_1 + 4d_2 = 0 \quad \text{and} \quad d_1 + d_2 + d_3 &= 0 \\
\Longrightarrow \boldsymbol{d} &= \begin{pmatrix} d_1 \\ -d_1 \\ 0 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla} h(\begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix}) \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} &= 0 \\
\Longrightarrow \begin{pmatrix} -4 & -4 & 0 \\ 1 & 1 & 1 \end{pmatrix}^T \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} &= 0 \\
\Longrightarrow -4d_1 - 4d_2 = 0 \quad \text{and} \quad d_1 + d_2 + d_3 &= 0 \\
\Longrightarrow \boldsymbol{d} &= \begin{pmatrix} d_1 \\ -d_1 \\ 0 \end{pmatrix}.
\end{aligned}
$$

For $\boldsymbol{x}^* = (2, 2, 3)^T$

$$
\begin{pmatrix} d_1 & -d_1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ -d_1 \\ 0 \end{pmatrix} = -d_1^2 < 0.
$$

For $\boldsymbol{x}^* = (-2, -2, 5)^T$

$$
\begin{pmatrix} d_1 & -d_1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ -d_1 \\ 0 \end{pmatrix} = d_1^2 > 0.
$$

In conclusion, the second-order sufficiency condition implies that the minimum is $(-2, -2, 5)^T$. In addition, we can also conclude that the maximum is $(2, 2, 3)^T$.