

connect proportion of time $\propto u_j$.

$$E\left(\frac{\sum_{i=1}^n I\{X_i=j\}}{n} \mid X_0=k\right) = \frac{\sum_{i=1}^n P(X_i=j \mid X_0=k)}{n} = \frac{\sum_{i=1}^n P_{kj}^{(i)}}{n}$$

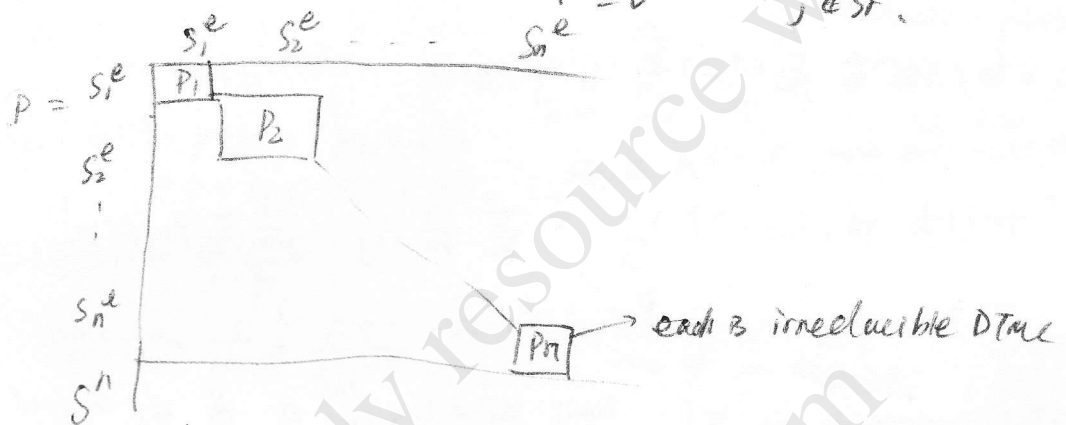
$E(I \sim) \rightarrow P$ $\rightarrow \pi_j \text{ as } n \rightarrow \infty$

reducible DTMC

If the submatrix P_r derive from P by restricting to the states of S_r is aperiodic & positive-recurrent.

then this subchain is ergodic & ^{in the long run}

$$P(X_n=j \mid X_0 \in S_r) \begin{cases} \rightarrow \pi_j^{(r)} & j \in S_r \\ = 0 & j \notin S_r \end{cases} \quad \left(\sum_{j \in S_r} \pi_j^{(r)} = 1 \right)$$



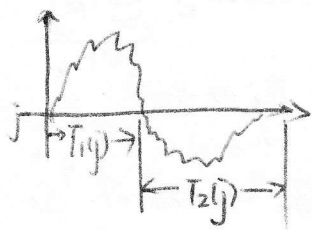
① Once enters a class S_r^e , it will stay in S_r^e forever

② From non-essential states, DTMC will eventually leave forever

ex $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

此矩阵划分成两块

$$\left(\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array} \right)$$



$$P(T_{1,j} = \infty) > 0 \iff j \text{ transient}$$

$$P(T_{1,j} < \infty) = 1 \iff j \text{ recurrent}$$

$$E[T_{1,j}] = \begin{cases} \infty & \text{null recurrent} \\ < \infty & \text{positive recurrent} \end{cases}$$

$$= u_j$$

Doubly - Stochastic

satisfy $\sum_{j=1}^m P_{ij} = \sum_{i=1}^m P_{ij} = 1$

\Rightarrow then we have $\pi = (\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots)$

for $(\frac{1}{m}, \frac{1}{m}, \dots) P)_{i,j} = \sum_k \frac{1}{m} P_{kj} = \frac{1}{m} \sum_k P_{kj} = \frac{1}{m}$

$\Rightarrow (\frac{1}{m}, \frac{1}{m}, \dots) P = (\frac{1}{m}, \frac{1}{m}, \dots)$

DTMC ergodic if $\forall j$

limit $\pi_j = \lim_{n \rightarrow \infty} P_{ij}^{(n)} \equiv \pi_j$ not depend on i .

ex1. $P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{pmatrix}$

find stationary distribution

$$\begin{cases} 0.8\pi_1 + 0.5\pi_3 = \pi_1 \\ 0.2\pi_1 + 0.5\pi_3 = \pi_2 \\ 0.5\pi_2 + 0.4\pi_4 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

ex2. $P = \begin{pmatrix} q & p & 0 & 0 & \dots \\ 0 & p & 0 & \dots \\ 0 & q & 0 & p & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

$$\pi_0 = \pi_0 q + \pi_1 q$$

$$\pi_1 = p \pi_0 + q \pi_2$$

$$\pi_j = p \pi_{j-1} + q \pi_{j+1}$$

$$\sum \pi_k = \pi_0 \sum_{j=0}^{\infty} \left(\frac{p}{q}\right)^j = \pi_0 \left(1 - \frac{p}{q}\right) = 1 \quad \pi_0 = 1 - \frac{p}{q}$$

only if $p < q \rightarrow$ combine with forward