Topic 1: Stationarity and covariance functions

This topic discusses properties of stationary random fields and their covariance functions. In particular, we consider

- Several different models of **stationary fields**.
- **Isotropy**.
- **Mean functions**.
- Some **properties and attributes** of covariance functions.
- **Nugget effect.**
- **Gaussian random fields**.
- Several examples of **isotropic models**.

Stationarity

Let $X_t, t \in T$, be a random field. There are three types of stationarity:

- **Strict stationarity**. The joint probability distribution of the data depends only on the relative positions of the sites at which the data were taken, i.e. the joint distribution of

$$(X_{t_1}, X_{t_2}, ..., X_{t_m})$$

  is the same as

$$(X_{t_1+v}, X_{t_2+v}, ..., X_{t_m+v})$$

  for any m spatial points $t_1, ..., t_m \in T$ and any space shift $v \in T$.

  If it is not assumed in advance, in most practical applications it is difficult to check that a random field is strictly stationary as it requires studying of an infinite number of joint distributions. Therefore, next concepts are more applicable in data analysis.

- **Weak stationarity** (second-order stationarity, homogeneity).
  - the mean is constant, $m(t) \equiv \beta_0$
  - the covariance at two sites depends only on the sites relative positions:

$$\boxed{C(s,t) = Cov(X_s, X_t) = C(s-t)}$$

- **Intrinsic stationarity**.
  - the mean is constant, $m(t) \equiv \beta_0$
  - the $Var[X_s - X_t]$ depends only on the sites relative positions $s-t$, i.e. there is a function γ, called a **semivariogram,** such that:

$$\boxed{Var[X_s - X_t] = 2\gamma(s-t).}$$

A second-order stationary random field with the covariance C is intrinsically stationary, with the semivariogram

$$\boxed{\gamma(\mathbf{h}) = C(\mathbf{0}) - C(\mathbf{h}).}$$

The converse is not true in general.

Mean functions

It is reasonable to assume that close sites should have similar means but sites that are far apart need not. This kind of **local stationarity**, rather than global stationarity, leads to the postulation of a continuous, relatively smooth (but non-constant 0 function for the mean).

- In many cases suitable mean functions are polynomials, i.e

  $$m(x, y) = \beta_0 + \beta_1 x + \beta_2 y$$

  or

  $$m(x, y) = \beta_0 + \beta_1 x + \beta_2 y + \beta_{11} x^2 + \beta_{12} xy + \beta_{22} y^2$$

  We can write the mean as $m(\mathbf{s}; \boldsymbol{\beta})$ to emphasize the dependence of unknown parameters $\boldsymbol{\beta}$.
- We can also model the mean function with a non-parametric approach (splines).

Covariance functions

Issues such as models' interpretability and computational efficiency suggest to consider models that have the following properties:

1. **Isotropy:** The covariance between any two measurements depends only on the Euclidean distance between their corresponding locations:

$$\boxed{C(\mathbf{h}) = C(\|\mathbf{h}\|) \quad \text{for all vectors } \mathbf{h}.}$$

2. **Geometric anisotropy:** There exists a matrix $\mathbf{A}$ such that

$$\boxed{C(\mathbf{h}) = C([\mathbf{h}'\mathbf{A}\mathbf{h}]^{1/2})}$$

$\mathbf{A}$ controls a rotation and stretching of the axis.

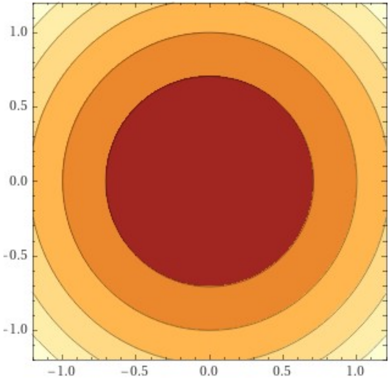


Figure: Isotropy

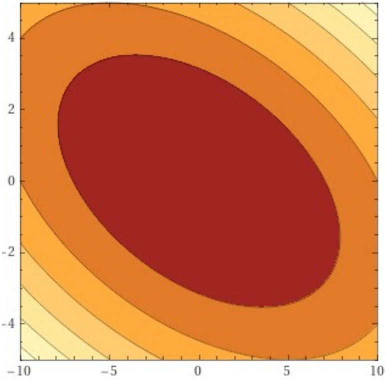


Figure: Anisotropy

3. **Separability:** A covariance function C is said to be separable if

$$\boxed{C(\mathbf{h}) = C(h_1, h_2) = C_1(h_1)C_2(h_2)}$$

for two covariance functions C_1 and C_2.

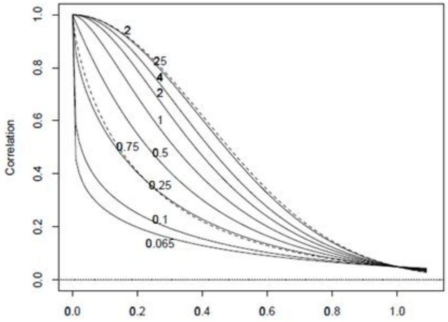
Separable covariance functions are often used when the spatio-temporal covariance structure factors into a purely spatial and a purely temporal component. It allows for simple and computationally efficient estimation and inference.

Attributes of homogeneous isotropic covariance functions

We illustrate attributes by using the **Matern class** of covariance functions:

$$C(h) = \sigma^2 \frac{1}{\Gamma(\nu)} \left(\frac{\theta h}{2}\right)^{\nu} 2K_{\nu}(\theta h), \quad \nu > 0, \theta > 0,$$

where $K_{\nu}(h)$ is modified Bessel function of the third kind of order ν.



The plot shows $C(h)$ for different of ν when other parameters are fixed.

- **Scale parameter or variance**. $C(0) = \sigma^2$. Thus, σ^2 is the variance of the random field.
- **Correlation scale parameter**. θ controls how the covariance changes.
- **Shape or smoothing parameter**. For the Matern model ν controls the *shape* of the function.
- **Range.** The distance beyond which the covariance function is equal to 0. It does not exist for the Matern class.
- **Effective range** The distance beyond which the covariance function does not exceed $0.04 \times$ variance. The Matern models has effective ranges; the cosine does not.
- **Continuity at 0**. Many covariance functions are continuous at 0. This is not a required property, but it has implications for the behavior of the random field.

Nugget effect

Theoretically, at zero separation distance (lag $= 0$), the semivariogram value is 0. However, at small separation distances, the sample semi- variogram often is discontinuous and exhibits a nugget effect, which is a value greater than 0.

The nugget effect can be attributed to measurement errors or spatial sources of variation at distances smaller than the sampling interval. Measurement errors occur because of the error in measuring devices.

The total error can be presented as

$$\boxed{\varepsilon(s) = \varepsilon_{DE}(s) + \varepsilon_{NE}(s)}$$

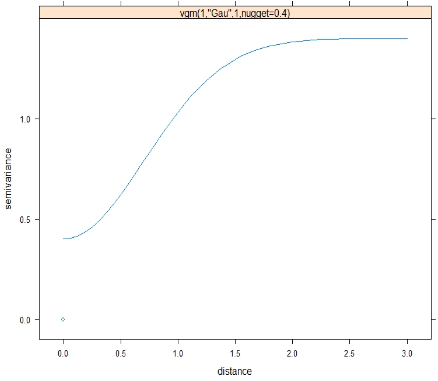
where

- $\varepsilon_{DE}(s)$ is the error dues to data variability at site s,
- $\varepsilon_{NE}(s)$ is what remains of the residual at s after subtracting the first error.

The covariance for ε is then represented as

$$\boxed{\gamma(\mathbf{h}) = \begin{cases} 0, & \text{if } \mathbf{h} = 0 \\ \theta_0 + \gamma_{DE}(\mathbf{h}) & \text{if } \|\mathbf{h}\| > 0, \end{cases}}$$

where γ_{DE} is the covariance for ε_{DE}. θ_0 is called the **nugget effect**.



New covariance models

From basic covariance models, one can construct more complex models using the following rules:

- If C_1 and C_2 are valid covariances, then so is $C(\cdot) \equiv C_1(\cdot) + C_2(\cdot)$.
- If C_0 is a valid covariance and $b > 0$, then $C(\cdot) \equiv b \cdot C_0(\cdot)$ is a valid covariance.
- If C_1 and C_2 are valid covariances, then so is $C(\cdot) \equiv C_1(\cdot) \cdot C_2(\cdot)$.
- A valid isotropic covariance function in $\mathbb{R}^{d_2}$ is a valid isotropic covariance function in $\mathbb{R}^{d_1}$, where $d_2 > d_1$. The converse is not true.

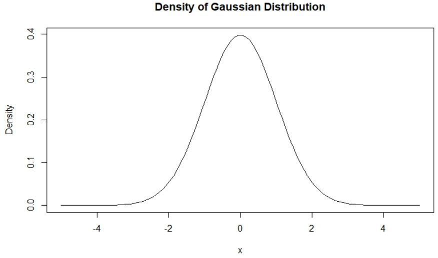
Gaussian random fields

Definition 1

A real-valued random variable X is said to be **Gaussian (or normally distributed)** *if it has the probability density function*

$$\boxed{\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad x \in \mathbb{R},}$$

for some $m \in \mathbb{R}$ and $\sigma > 0$.



Definition 2

A random vector $(X_{t_1}, ..., X_{t_n})$ is said to be **Gaussian** *if, for any non-zero numbers $\alpha_1, ..., \alpha_d \in \mathbb{R}$, the sum $\sum_{j=1}^{d} \alpha_j X_j$ is Gaussian.*

Definition 3

We can now define **a Gaussian (random) field** *to be a random field X_t on a parameter set T for which the distributions of all vectors $(X_{t_1}, ..., X_{t_n})$ are multivariate Gaussian for each $1 \leq n < \infty$ and each $t_1, ..., t_n \in T$.*

The functions

$$m(t) = EX_t$$

and

$$C(s, t) = Cov(X_s, X_t) = E(X_s - m(s))(X_t - m(t))$$

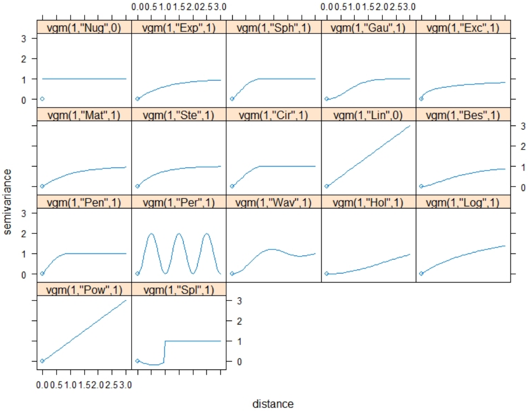
are the **mean and covariance functions** of X_t.

In fact, one can also go in the other direction as well. Given any set T, a function $m(t) : T \to \mathbb{R}$, and a positive-definite function $C : T \times T \to \mathbb{R}$ there exists a Gaussian field on T with mean function $m(t)$ and covariance function C.

The important principle: for a Gaussian random field, everything about it is determined by the mean and covariance functions.

Isotropic models.

For isotropic random fields the covariance between any two values depends only on the Euclidean distance between their corresponding locations: $C(\mathbf{h}) = C(\|\mathbf{h}\|)$ for all $\mathbf{h}$. Thus, their semivariogram $\gamma(\cdot)$ also depends only on the Euclidean distance between locations.



Isotropic fields are convenient to deal with because there are a number of widely used parametric forms for $\gamma(\cdot)$. Here are several examples:

1. **Linear:**

$$\gamma(t) = \begin{cases} 0, & \text{if } t = 0; \\ c_0 + c_1 t, & \text{if } t > 0, \end{cases}$$

where c_0 and c_1 are positive constants. The function tends to ∞ as $t \to \infty$ and so does not correspond to a stationary field.

2. **Spherical:**

$$\gamma_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ c_0 + c_1 \left\{ \frac{3}{2}\frac{t}{R} - \frac{1}{2}\left(\frac{t}{R}\right)^3 \right\} & \text{if } 0 < t \leq R, \\ c_0 + c_1 & \text{if } t \geq R. \end{cases}$$

This is valid if $d = 1,\ 2$ or 3, but for higher dimensions it fails the positive-definiteness condition.

3. **Exponential:**

$$\gamma_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ c_0 + c_1(1 - e^{-t/R}) & \text{if } t > 0. \end{cases}$$

It is simpler in functional form than the spherical case (and valid for all d) but without the finite range of the spherical form.

4. **Gaussian:**

$$\gamma_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ c_0 + c_1(1 - e^{-t^2/R^2}) & \text{if } t > 0. \end{cases}$$

5. **Exponential-power form:**

$$\gamma_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ c_0 + c_1(1 - e^{-|t/R|^p}) & \text{if } t > 0. \end{cases}$$

Here $0 < p \leq 2$. This form generalizes both the exponential and Gaussian forms.

6. **Rational quadratic:**

$$\gamma_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ c_0 + c_1 t^2/(1 + t^2/R) & \text{if } t > 0. \end{cases}$$

7. **Wave:**

$$\gamma_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ c_0 + c_1 \left\{1 - \frac{R}{t} \sin\left(\frac{t}{R}\right)\right\} & \text{if } t > 0. \end{cases}$$

The only non-monotonic example in this list.

8. **Power law:**

$$\gamma_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ c_0 + c_1 t^{\lambda} & \text{if } t > 0. \end{cases}$$

Positive-definiteness requires $0 \leq \lambda < 2$. This generalizes the linear case and is an example of a semivariogram that does not correspond to a stationary field.

9. **The Matérn class:**

$$C(h) = \sigma^2 \frac{1}{\Gamma(\nu)} \left(\frac{\theta h}{2}\right)^{\nu} 2K_{\nu}(\theta h), \quad \nu > 0, \theta > 0,$$

where $K_{\nu}(h)$ is modified Bessel function of the third kind of order ν.

It was originally suggested by Matérn in 1960, but largely neglected in favor of simpler analytic forms.

However, more recently Handcock and Stein (1993) and Handcock and Wallis (1994) demonstrated its flexibility in handling a variety of spatial data sets, including ones related to global warming.

Now, the Matérn model is implemented in various software packages and is very popular.