Setup

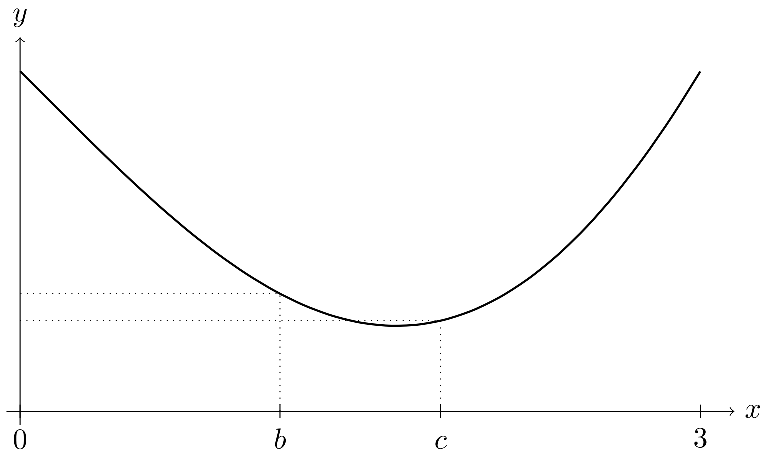
For several questions below, you will be using the function defined in this question. The functions that calculate the first and second derivatives will also be required.

1. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = e^{-x} + \tfrac{1}{2}x^2.$$
   (a) Create a MATLAB function `f` that computes $f(x)$. Then plot the graph of $f(x)$ on the interval $[0, 1]$ and observe that f is unimodal on $[0, 1]$.
   (b) Try to use the FONC to identify the minimiser exactly. What goes wrong?
   (c) Create MATLAB functions `fd` and `fdd` that compute $f'(x)$ and $f''(x)$, respectively. Then plot the graph of the function and its derivatives on the same graph, over the interval $[0, 1]$, and create a legend using the `legend` command to distinguish the different functions.

The golden section method

Recall that the golden section method starts with an interval $[a, b]$ and a function f that is unimodal on that interval. It constructs a sequence of intervals of decreasing size, each of which contains the minimiser of f. Refer to Section 3.3 of the reading materials for the description of the golden section method.

2. A unimodal function is shown below on the interval $[0, 3]$.



   The points b and c as determined by the golden section method are also marked. Suppose you applied the golden section method to this function starting on $[0, 3]$.
   (a) Determine b and c to 3 decimal places.
   (b) After the first iteration, what are the values of a and d?

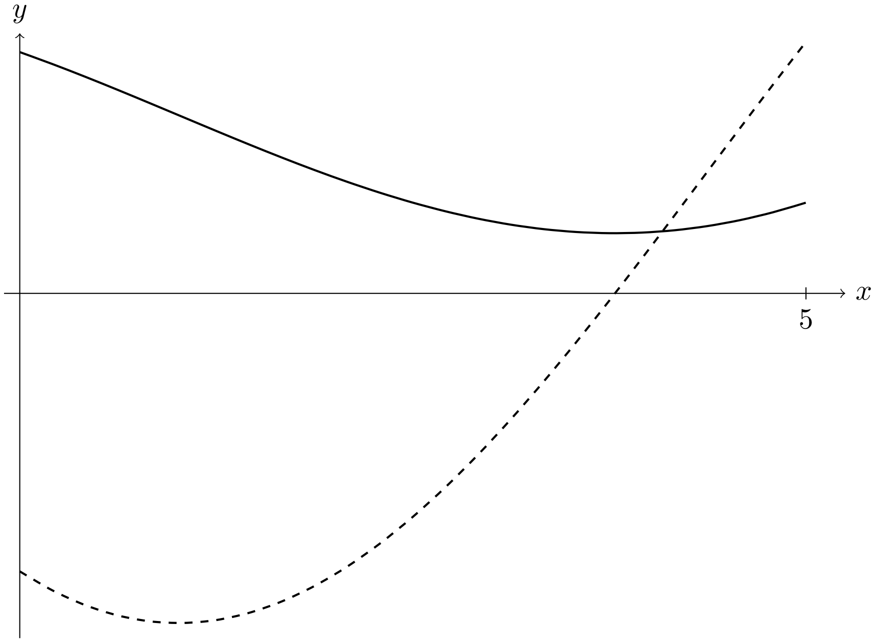
3. Consider the function f defined in Question 1.
   (a) Manually perform 2 iterations of the golden section method to f, starting with the interval $[0, 1]$. Use a calculator or MATLAB to perform the function evaluations.
   (b) Implement the golden section method in MATLAB and find a point that is within $\pm 10^{-6}$ of the minimiser, starting with the interval $[0, 1]$. Keep track of the number of iterations in your code.
   (c) Algebraically determine the number of iterations needed to find a point that is within $\pm 10^{-6}$ of the minimiser using the golden section method. Compare your answer with the number of iterations used in part (b).

4. Without negating the objective function, how could the golden section be adjusted to find a maximiser instead? What assumptions are necessary?

The bisection method

Recall that, like the golden section method, the bisection method starts with an interval $[a, b]$ and a function f that is unimodal on that interval. It also constructs a sequence of intervals of decreasing size, each of which contains the minimiser of f, but it relies on the derivative of f. Refer to Section 3.4 of the reading materials for the description of the bisection method.

5. A unimodal function and its derivative is shown below on the interval $[0, 5]$.



   Suppose you were to apply one iteration of the bisection method to this function starting on $[0, 5]$. What is the interval after one iteration?

6. Consider the function f defined in Question 1.
   (a) Manually perform 2 iterations of the bisection method to f, starting with the interval $[0, 1]$. Use a calculator or MATLAB to perform the function evaluations.
   (b) Implement the bisection method in MATLAB and find a point that is within $\pm 10^{-6}$ of the minimiser, starting with the interval $[0, 1]$. Keep track of the number of iterations in your code.
   (c) Algebraically determine the number of iterations needed to find a point that is within $\pm 10^{-6}$ of the minimiser using the bisection method. Compare your answer with the number of iterations used in part (b).

7. Without negating the objective function, how could the bisection method be adjusted to find a maximiser instead? What assumptions are necessary?

Newton's method

Unlike the previous methods, Newton's method finds a sequence of points that converge to a minimiser, rather than a sequence of intervals. However, it also depends on both the first and second derivatives of f.

8. For the same function in Question 5, starting at the point $x_0 = 2.5$, what is the value x_1 obtained after one iteration of Newton's method? *You won't be able to do this exactly. Find it approximately from the graph.*

9. Consider the function f defined in Question 1.
   (a) Manually perform 2 iterations of Newton's method to f, starting with the point $x_0 = 1$. Use a calculator or MATLAB to perform the function evaluations.
   (b) Implement Newton's method in MATLAB and find a point that is within $\pm 10^{-6}$ of the minimiser, starting with the point $x_0 = 1$.

10. Without negating the objective function, how could Newton's method be adjusted to find a maximiser instead? What assumptions are necessary?