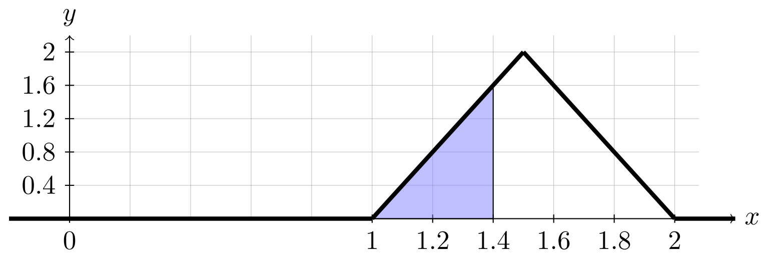
STM4PSD – Workshop 5 Solutions

1. (a) From the graph, we can see that $f(x) \geqslant 0$ for all values of x, and the area under the graph is $\frac{1}{2} \times 1 \times 2 = 1$. So $f(x)$ satisfies both required conditions to be a probability density function.

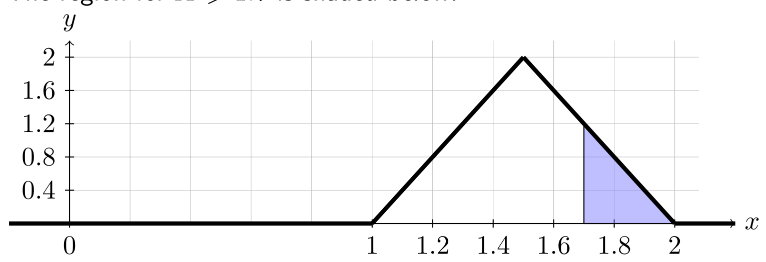
   (b) We have $f(1.2) = 0.8$ and $f(1.6) = 1.6$, so $X \approx 1.6$ is more likely as $f(1.6) > f(1.2)$.

   (c) i. The region for $X \leqslant 1.4$ is shaded below.

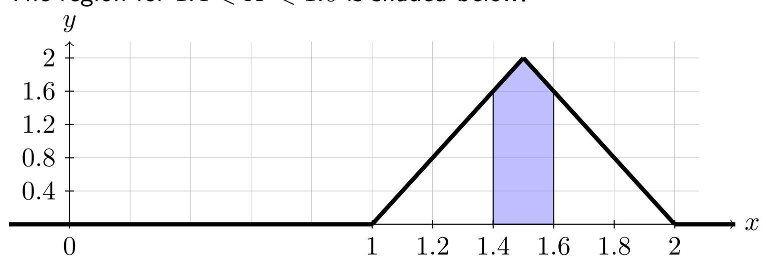
   The area of the triangle is $\frac{1}{2} \times 0.4 \times 1.6 = 0.32$, so $P(X \leqslant 1.4) = 0.32$.

   ii. The region for $X \geqslant 1.7$ is shaded below.

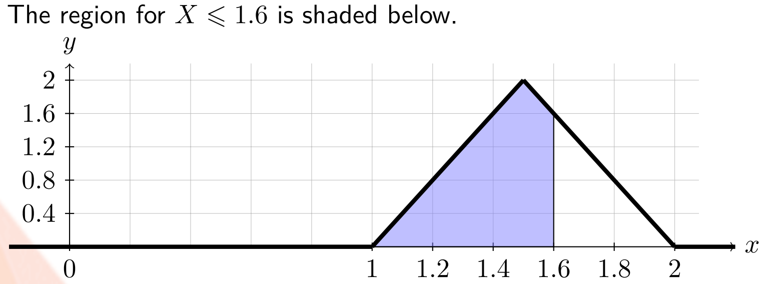
   The area of the triangle is $\frac{1}{2} \times 0.3 \times 1.2 = 0.18$, so $P(X \geqslant 1.7) = 0.18$.

   iii. The region for $1.4 \leqslant X \leqslant 1.6$ is shaded below.

   There are several ways to calculate this area. Here we use the fact that the shaded is composed of a rectangle with width 0.2 and height 1.6, plus a triangle with width 0.2 and height 0.4. The total area is then $0.2 \times 1.6 + \frac{1}{2} \times 0.2 \times 0.4 = 0.36$, and so $P(1.5 \leqslant X \leqslant 1.6) = 0.36$.

   iv. The region for $X \leqslant 1.6$ is shaded below.

   Using the fact that the total area under the graph is 1, the area of the shaded region is equal to $1 - \frac{1}{2} \times 0.4 \times 1.6 = 1 - 0.32 = 0.68$, so $P(X \leqslant 1.6) = 0.68$.

   (d) If $1 \leqslant x \leqslant 1.5$, then the appropriate region is a triangle with base length $(x-1)$ and height $(4x-4)$, so in that case we would have $P(X \leqslant x) = \frac{1}{2}(x-1)(4x-4) = (x-1)(2x-2) = 2(x-1)^2$.

   (e) If $1.5 \leqslant x \leqslant 2$, then, using reasoning similar to that in (c)(iv), the complementary region is a triangle with base $2-x$ and height $8-4x$, so the area of the required region is $1 - \frac{1}{2}(2-x)(8-4x) = 1 - (2-x)(4-2x) = 1 - 2(2-x)^2$; hence in this case, $P(X \leqslant x) = 1 - 2(2-x)^2$.

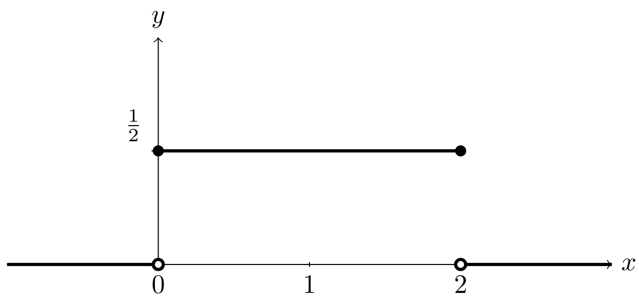
(f) Let $F(x)$ denote the cumulative distribution function for X. Then,

$$F(x) = \begin{cases} 0 & \text{if } x < 1, \\ 2(x-1)^2 & \text{if } 1 \leqslant x \leqslant 1.5, \\ 1-2(2-x)^2 & \text{if } 1.5 < x \leqslant 2, \\ 1 & \text{otherwise} \end{cases}$$

2. (a) Since $Z \sim U(0,2)$, and we have $\frac{1}{2-(0)} = \frac{1}{2}$, the formula for $f_Z(x)$ is

$$f_Z(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

The graph of $f_Z(x)$ is shown below.



(b) $P(Z \geqslant 1.5) = 1 - P(Z \leqslant 1.5) = 1 - \frac{1.5-0}{2-0} = 1 - \frac{1.5}{2} = 0.25.$

(c) Let R denote the number of generated values which are greater than 1. Note that $P(Z \geqslant 1) = 0.5$, so $R \sim \text{Bin}(6, 0.5)$.

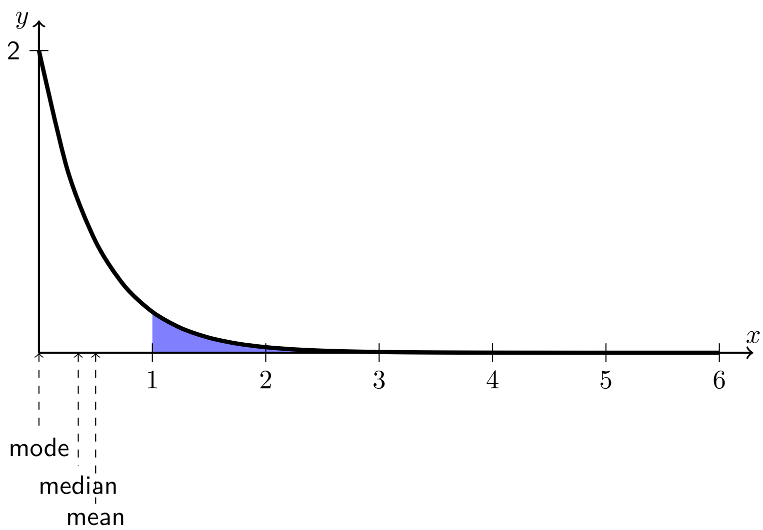
i. The probability that at least one number is greater than 1 is then

$$P(R \geqslant 1) = 1 - P(R = 0) = 1 - \binom{6}{0} 0.5^0 (1-0.5)^6 = 0.984375.$$

ii. The probability that all six are greater than 1 is

$$P(R = 6) = \binom{6}{6} 0.5^6 (1-0.5)^0 = 0.015625$$

3. (a) and (b)



Note that because of the scale, it may be hard to recognise that the shaded area is between the x-values 1 and 3.

(c) We use the probability distribution function to calculate probabilities of the form $P(X \leqslant x)$, so:

i. $P(X \leqslant 1) = 1 - e^{-2\times 1} = 1 - e^{-2}$

ii. $P(X \leqslant 3) = 1 - e^{-2\times 3} = 1 - e^{-6}$

iii.
$$\begin{aligned} P(1 \leqslant X \leqslant 3) &= P(X \leqslant 3) - P(X < 1) \\ &= P(X \leqslant 3) - P(X \leqslant 1) \qquad \text{(because } X \text{ is continuous)} \\ &= 1 - e^{-6} - (1 - e^{-2}) \\ &= e^{-2} - e^{-6}. \end{aligned}$$

(d) Mean: $E(X) = \frac{1}{2} = 0.5$.
Median: $\frac{\ln(2)}{2} \approx 0.35$
Mode: 0
The points are indicated on the graph above.

4. (a) $k = 4$ and $\theta = 1$.

(b) $\Gamma(4) = (4-1)! = 3! = 3 \times 2 \times 1 = 6$.

(c) The formula for the density function is

$$f_X(x) = \frac{1}{1^4 \times \Gamma(4)} x^{4-1} e^{-x/1} = \frac{x^3 e^{-x}}{6}$$

(d) $E(X) = k\theta = 4 \times 1 = 4$, and $\mathrm{Var}(X) = k\theta^2 = 4 \times 1^2 = 4$.

5. (a) The probability that no cars arrive in the first two minutes is

$$P(N(2) = 0) = \frac{(3 \times 2)^0 e^{-3\times 2}}{0!} = e^{-6}.$$

(b) The probability that at least 2 cars arrive in the first 3 minutes is

$$\begin{aligned} P(N(3) \geqslant 2) &= 1 - P(N(3) < 2) \\ &= 1 - P(N(3) = 1) - P(N(3) = 0) \\ &= 1 - \frac{(3 \times 3)^1 e^{-3\times 3}}{1!} - \frac{(3 \times 3)^0 e^{-3\times 3}}{0!} \\ &= 1 - 9e^{-9} - e^{-9} = 1 - 10e^{-9} \end{aligned}$$

(c) Due to the lack-of-memory property, this is the same as the probability in part (a). So the required probability is e^{-6}.

6. (a) $\lambda = 3$ cars per minute.

(b) $\mu = 6$ cars per minute.

(c) $\rho = \frac{3}{6} = \frac{1}{2}$.

(d) $L = \frac{\rho}{1-\rho} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$ car.

(e) This is the total time spent in the queue, W, so we need

$$W = \frac{L}{\lambda} = \frac{1}{3} \text{ minutes} = 20 \text{ seconds.}$$

(f) This is the waiting time before being served, W_Q, so we need

$$W_Q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{3}{6(6-3)} = \frac{1}{6} \text{ minutes} = 10 \text{ seconds.}$$

(g) i. If $\lambda \geqslant 6$, then $\rho \geqslant 1$, meaning that in the long term, the queue will grow without bound.

ii. In terms of λ, keeping $\mu = 6$, we have

$$W = \frac{1}{\mu - \lambda} = \frac{1}{6 - \lambda}.$$

iii. This amounts to solving $W = 1$ for λ:

$$\begin{aligned} W = 1 &\implies \frac{1}{6-\lambda} = 1 \\ &\implies 6 - \lambda = 1 \\ &\implies \lambda = 5. \end{aligned}$$

Noting that increasing λ will increase the waiting time, this means that the largest permissible arrival rate is $\lambda = 5$.

7. (a) $\mu = \frac{60}{2.5} = 24$ cars per hour. $\lambda = 19$ cars per hour.

   (b) $L = \frac{\lambda}{\mu - \lambda} = \frac{19}{24-19} = 3.8$.

   (c) $W_Q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{19}{24(24-19)} = \frac{19}{120} \approx 0.158$ hours, or 9.5 minutes.

   (d) $W = W_Q + \frac{1}{\mu} = \frac{19}{120} + \frac{1}{24} = \frac{1}{5}$ hours, or 12 minutes.

   (e) i. Here we have $\rho = \frac{\lambda}{\mu} \approx 0.792$. Then, $N \sim$ Geo$(1 - 0.792)$ i.e. $N \sim$ Geo(0.208). So, the probability that at least 3 cars are in the queue is:

$$\begin{aligned} P(N \geqslant 3) &= 1 - P(N = 0) - P(N = 1) - P(N = 2) \\ &= 1 - (1 - 0.208)^0 \cdot 0.208 - (1 - 0.208)^1 \cdot 0.208 - (1 - 0.208)^2 \cdot 0.208 \\ &= 0.497 \end{aligned}$$

   This means that cars will be impeding traffic about 49.7% of the time, so the restaurant should be concerned about being fined.

   ii. To have cars impeding traffic about 30% of the time, we would need $P(N \geqslant 3) = 0.3$. So we will find the value of ρ such that $P(N \geqslant 3) = 0.3$ to make a suitable decision.
   Since $N \sim$ Geo$(1 - \rho)$, we have

$$\begin{aligned} P(N \geqslant 3) = 0.3 &\implies 1 - \rho^0(1 - \rho) - \rho^1(1 - \rho) - \rho^2(1 - \rho) = 0.3 \\ &\implies 1 - 1 + \rho - \rho + \rho^2 - \rho^2 + \rho^3 = 0.3 \\ &\implies \rho^3 = 0.3 \\ &\implies \rho = \sqrt[3]{0.3} \end{aligned}$$

   So any value of ρ less than $\sqrt[3]{0.3}$ would be best for the restaurant.