Problem set

Collected by Charl Ras

1. For a 24-hour cafe, the following number of servers are required daily:

Table 1: Minimum number of servers.

Time interval	Minimum number of servers
0200-0600	4
0600-1000	8
1000-1400	10
1400-1800	7
1800-2200	12
2200-0200	6

Each server works eight consecutive hours per day, starting at 2am, 6am, 10am, 2pm, 6pm or 10pm. The objective is to find the smallest number of servers required to meet the minimum server requirements specified in Table 1. Formulate this problem as a linear programming model. (You are not required to solve this linear program.)

Solution:
We define the following 8-hourly shifts:
Shift 1: 0200 to 1000
Shift 2: 0600 to 1400
Shift 3: 1000 to 1800
Shift 4: 1400 to 2200
Shift 5: 1800 to 0200
Shift 6: 2200 to 0600
Let x_i be the number of servers working shift $i \in \{1, 2, 3, 4, 5, 6\}$.
The linear program is shown below:

$$\min x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

s.t.

$$
\begin{aligned}
x_1 + x_6 &\geq 4 \\
x_1 + x_2 &\geq 8 \\
x_2 + x_3 &\geq 10 \\
x_3 + x_4 &\geq 7 \\
x_4 + x_5 &\geq 12 \\
x_5 + x_6 &\geq 6 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

2. Two alloys A and B are made from four different metals I, II, III, IV according to the following specification:

Alloy	Specifications
A	At most 80% of I
	At most 30% of II
	At least 50% of IV
B	Between 40% and 60% of II
	At least 30% of III
	At most 70% of IV

The four metals are extracted from three different ores whose constituents percentage of these metals, maximum available quantity, and cost per tonne are tabulated as follows:

		Constituents (%)					
Ore	Maximum quantity (tonnes)	I	II	III	IV	Others	Cost/tonne
1	1,000	20	10	30	30	10	$30
2	2,000	10	20	30	30	10	$40
3	3,000	5	5	70	20	0	$50

Assuming that the selling prices of alloys A and B are $200 and $300 per tonne and all alloys produced will be sold, formulate the problem as a linear programming model that will maximise the profit of producing the alloys. You MUST ONLY use the following set of decision variables in your formulation: x_{ij}, the number of tonnes of ore $i \in \{1, 2, 3\}$ used in alloy $j \in \{A, B\}$. (You are not required to solve this linear program.)

Solution:

$$
\begin{aligned}
\max \quad & (200 \times 0.6 - 30)x_{1A} + (200 \times 0.6 - 40)x_{2A} + (200 \times 0.3 - 50)x_{3A} \\
& +(300 \times 0.7 - 30)x_{1B} + (300 \times 0.8 - 40)x_{2B} + (300 \times 0.95 - 50)x_{3B}
\end{aligned}
$$

s.t.

$$\begin{aligned}
x_{1A} + x_{1B} &\leq 1000 \\
x_{2A} + x_{2B} &\leq 2000 \\
x_{3A} + x_{3B} &\leq 3000 \\
0.2x_{1A} + 0.1x_{2A} + 0.05x_{3A} &\leq 0.8(0.6x_{1A} + 0.6x_{2A} + 0.3x_{3A}) \\
0.1x_{1A} + 0.2x_{2A} + 0.05x_{3A} &\leq 0.3(0.6x_{1A} + 0.6x_{2A} + 0.3x_{3A}) \\
0.3x_{1A} + 0.3x_{2A} + 0.2x_{3A} &\geq 0.5(0.6x_{1A} + 0.6x_{2A} + 0.3x_{3A}) \\
0.1x_{1B} + 0.2x_{2B} + 0.05x_{3B} &\geq 0.4(0.7x_{1B} + 0.8x_{2B} + 0.95x_{3B}) \\
0.1x_{1B} + 0.2x_{2B} + 0.05x_{3B} &\leq 0.6(0.7x_{1B} + 0.8x_{2B} + 0.95x_{3B}) \\
0.3x_{1B} + 0.3x_{2B} + 0.7x_{3B} &\geq 0.3(0.7x_{1B} + 0.8x_{2B} + 0.95x_{3B}) \\
0.3x_{1B} + 0.3x_{2B} + 0.2x_{3B} &\leq 0.7(0.7x_{1B} + 0.8x_{2B} + 0.95x_{3B}) \\
x_{ij} &\geq 0, \ \forall i \in \{1,2,3\}, j \in \{A,B\}
\end{aligned}$$

Note that the alloy specification constraints are redundant since all decision variables are non-negative. You are still required to write these constraints down and then establish their redundancies.

You may also show (analytically) that it is not feasible to produce any Alloy B. But this does not mean that the LP is infeasible, since we can still produce Alloy A.

3. Recall that a linear program is formulated only with continuous decision variables. When some of the variables can only take integer values, this is called a *mixed-integer linear program.* Consider the following mixed-integer linear program with three decision variables:

$$\min z = 5x_1 + x_2 - 12x_3$$

s.t.

$$\begin{aligned}
x_1 + 2x_2 + 2x_3 &\leq 10 \\
4x_1 + x_2 - 2x_3 &\geq 4 \\
2x_1 - 3x_2 - 6x_3 &\geq 0 \\
x_1, x_2 &\geq 0 \\
x_3 &\in \{0,1,2\}
\end{aligned}$$

   (a) Sketch the feasible region when $x_3 = 0$, and determine the optimal values to x_1 and x_2 graphically.
   (b) Sketch the feasible region when $x_3 = 1$, and determine the optimal values to x_1 and x_2 graphically.
   (c) Sketch the feasible region when $x_3 = 2$, and determine the optimal values to x_1 and x_2 graphically.

(d) From the graphical solutions, determine the optimal solution to the mixed-integer linear program, and state its optimal objective value.

Solutions:

(a) When $x_3 = 0$, the linear program becomes

$$\min 5x_1 + x_2$$

$$\text{s.t.}$$

$$\begin{aligned} x_1 + 2x_2 &\leq 10 \\ 4x_1 + x_2 &\geq 4 \\ 2x_1 - 3x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The feasible region is shown in Figure 1.

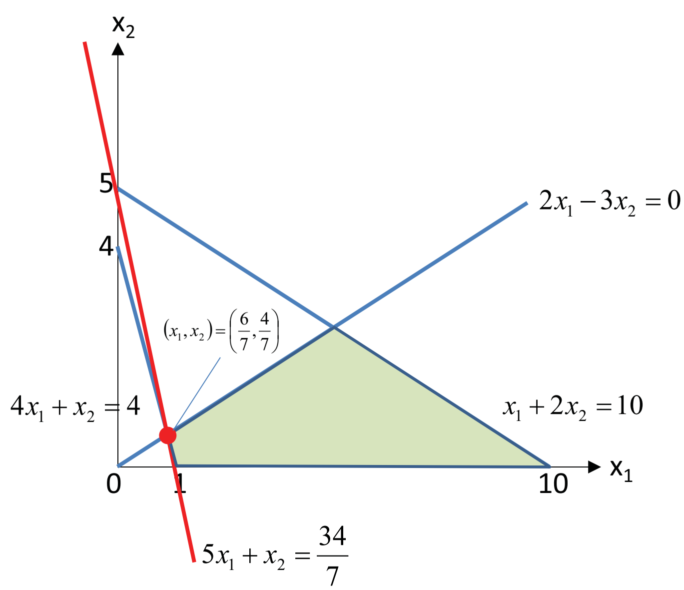

Figure 1:

The optimal solution for this modified LP is $(x_1, x_2) = (\frac{6}{7}, \frac{4}{7})$ with objective value $\frac{34}{7}$.

(b) When $x_3 = 1$, the linear program becomes

$$\min 5x_1 + x_2 - 12$$

$$\text{s.t.}$$

$$\begin{aligned} x_1 + 2x_2 &\leq 8 \\ 4x_1 + x_2 &\geq 6 \\ 2x_1 - 3x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The feasible region is shown in Figure 2.

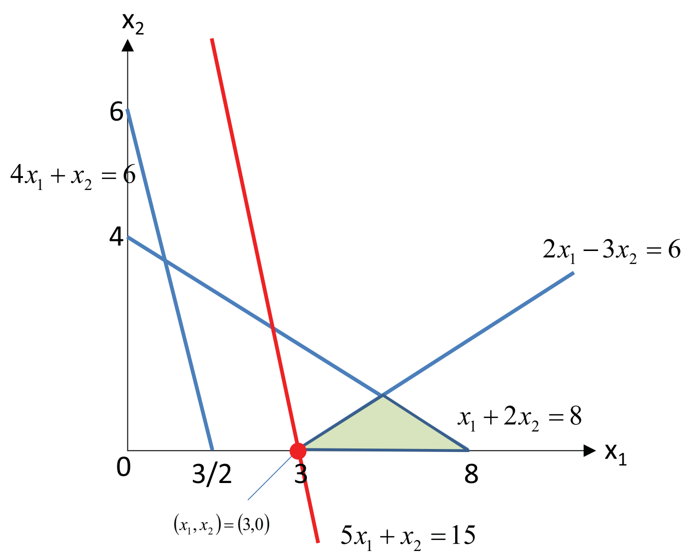

Figure 2:

The optimal solution for this modified LP is $(x_1, x_2) = (3, 0)$ with objective value 3.

(c) When $x_3 = 2$, the linear program becomes

$$\min 5x_1 + x_2 - 24$$

s.t.

$$\begin{aligned} x_1 + 2x_2 &\leq 6 \\ 4x_1 + x_2 &\geq 8 \\ 2x_1 - 3x_2 &\geq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The feasible region is shown in Figure 3.
The optimal solution for this modified LP is $(x_1, x_2) = (6, 0)$ with objective value 6.

(d) From the graphical solutions:
When $x_3 = 0$, $(x_1, x_2) = (\frac{6}{7}, \frac{4}{7})$ is optimal. So $z(x_3 = 0) = \frac{34}{7} = 4\frac{6}{7}$.
When $x_3 = 1$, $(x_1, x_2) = (3, 0)$ is optimal. So $z(x_3 = 1) = 15 - 12 = 3$.
When $x_3 = 2$, $(x_1, x_2) = (6, 0)$ is optimal. So $z(x_3 = 2) = 30 - 24 = 6$.

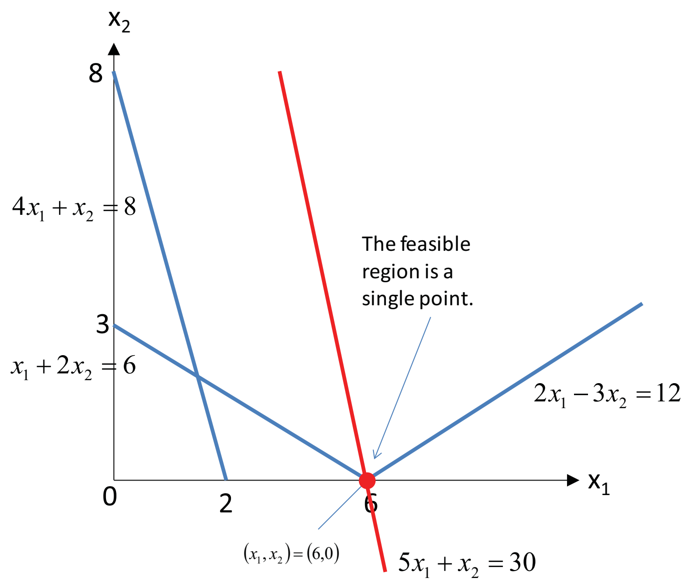

Figure 3:

We are minimizing the objective function, so the minimum of the options above is the optimal solution. Therefore, the optimal solution for the mixed-integer linear program is $(x_1, x_2, x_3) = (3, 0, 1)$ with $z = 3$.

4. Consider the following linear program (LP):

$$\max f(x_1, x_2)$$

$$\text{s.t.}$$

$$\begin{aligned} 12x_1 + 5x_2 &\leq 60 \\ -x_1 + x_2 &\geq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

where $f(x_1, x_2)$ is a linear function of x_1 and x_2.

(a) Show that the LP's feasible region is convex. (NOTE: a graphical demonstration is not sufficient here. Marks will not be awarded for direct quotes of theorems stated in the lecture notes — you are required to show it via the use of basic definitions.)

(b) If the LP has *one* optimal solution of the form $(x_1, x_2) = (0, a)$, suggest one possible objective function for the LP and state its optimal solution. The function you suggest must have (i) non-zero coefficients for both x_1 and x_2, and (ii) opposite signs for the coefficients of x_1 and x_2, i.e. if the coefficient of x_1 is positive, then the coefficient of x_2 must be negative or vice versa.

(c) If the LP has *more than one* optimal solution, suggest one possible objective function for the LP and state all optimal solutions.

(d) Let $f(x_1, x_2) = x_1 + x_2$. If you are allowed to remove one functional constraint, which functional constraint would you remove so that the LP is unbounded?

(e) Convert the LP to canonical form.

(f) Write down the maximum number of basic solutions that this problem can have.

(g) Let $f(x_1, x_2) = x_1 + x_2$. By evaluating the objective function at each basic solution, determine the optimal solution to the problem.

(h) Verify your solution in Part (g) graphically.

Solutions:

(a) Let the LP's feasible region be $F = \{x_1, x_2 \in \mathbb{R} : 12x_1 + 5x_2 \leq 60; -x_1 + x_2 \geq -2; x_1 \geq 0; x_2 \geq 0\}$.

Let $y = (y_1, y_2)$ and $z = (z_1, z_2) \in F$. Then $y_1, y_2, z_1, z_2 \geq 0$,

$$12y_1 + 5y_2 \leq 60$$

$$-y_1 + y_2 \geq -2$$

and

$$12z_1 + 5z_2 \leq 60$$

$$-z_1 + z_2 \geq -2$$

We need to show that for all $0 \leq \lambda \leq 1$, $w \in F$, where $w = \lambda y + (1-\lambda)z$. Recall that a subset $C \subseteq \mathbb{R}^n$ is convex if and only if for any two points $y, z \in C$ and any $\lambda \in [0,1]$, the point $\lambda y + (1-\lambda)z \in C$.

For $i = 1, 2$, $w_i = \lambda y_i + (1-\lambda)z_i \geq 0$.

Also

$$\begin{aligned} 12w_1 + 5w_2 &= 12(\lambda y_1 + (1-\lambda)z_1) + 5(\lambda y_2 + (1-\lambda)z_2) \\ &= \lambda(12y_i + 5y_2) + (1-\lambda)(12z_1 + 5z_2) \\ &\leq \lambda 60 + (1-\lambda)60 \\ &= 60 \end{aligned}$$

Similarly, we can show that $-w_1 + w_2 \geq -2$. Therefore w satisfies the conditions necessary to be in F. Since this is true for any choice of $y, z \in F$ and $0 \leq \lambda \leq 1$, we conclude that F, the LP's feasible region, is convex.

(b) $f(x_1, x_2) = -x_1 + x_2$ and optimal solution is $(x_1, x_2) = (0, 12)$ with $f = 12$.

(c) $f(x_1, x_2) = 12x_1 + 5x_2$ and the optimal solutions are $(x_1, x_2) = (\frac{70}{17}(1-\lambda), 12\lambda + \frac{36}{17}(1-\lambda))$ for all $0 \leq \lambda \leq 1$.
(Another possible solution is $f(x_1, x_2) = x_1 - x_2$ and the optimal solutions are $(x_1, x_2) = (2\lambda + \frac{70}{17}(1-\lambda), \frac{36}{17}(1-\lambda))$ for all $0 \leq \lambda \leq 1$.)

(d) Remove $12x_1 + 5x_2 \leq 60$.

(e) The LP in canonical form is

$$\max f(x_1, x_2)$$

$$\text{s.t.}$$

$$\begin{aligned} 12x_1 + 5x_2 + x_3 \qquad &= 60 \\ x_1 - x_2 \qquad + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(f) There are four variables in the canonical form and a basic solution is defined by putting two of them equal to zero and solving for the other two. Therefore the maximum number of basic solutions is

$$\binom{4}{2} = 6.$$

(g) The basic solutions are given in the following table

basic variables	basic solution	feasible?	value of the objective function
x_3, x_4	(0,0,60,2)	yes	0
x_2, x_4	(0,12,0,14)	yes	12
x_1, x_4	(5,0,0,-3)	no	-
x_2, x_3	(0,-2,70,0)	no	-
x_1, x_3	(2,0,36,0)	yes	2
x_1, x_2	$(\frac{70}{17},\frac{36}{17},0,0)$	yes	$\frac{106}{17}$

The optimal solution is $(x_1, x_2) = (0, 12)$. The objective value is 12.

(h) The graphical solution is shown in Figure 4. This is consistent with the result found in part (g).

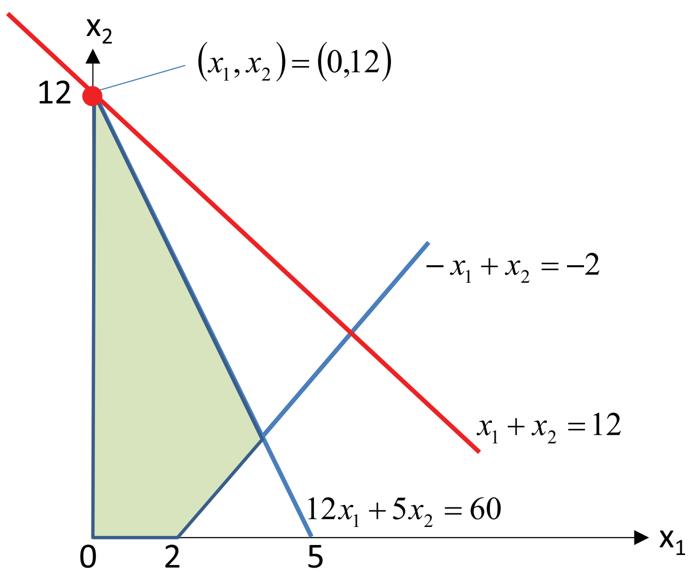

Figure 4:

5. Consider the following problem (denoted Problem A):

$$\max_{\{x,z\}} f(x) + z$$
$$s.t.$$
$$ax + bz \leq H$$
$$x, z \geq 0$$

where $f(x)$ is a piecewise continuous linear function given by

$$f(x) = \begin{cases} M_1 x, & 0 \leq x \leq B_1 \\ M_1 B_1 + M_2(x - B_1), & B_1 \leq x \leq B_2 \end{cases}$$

and $M_1 > M_2 \geq 0$.

If we let $x = y_1 + y_2$, where $y_1 \in [0, B_1]$ and $y_2 \in [0, B_2 - B_1]$, Problem A can be reformulated and solved as a linear program . The linear program, denoted LP(A), is

$$\begin{gathered}\max_{\{y_1,y_2,z\}} \; M_1y_1 + M_2y_2 + z \\ s.t. \\ 0 \leq y_1 \leq B_1 \\ 0 \leq y_2 \leq (B_2 - B_1) \\ a(y_1 + y_2) + bz \leq H \\ z \geq 0\end{gathered}$$

Let y_1^*, y_2^* and z^* be the optimal solution to LP(A).

SHOW that LP(A) correctly solves for Problem A, i.e. show that if $y_1^* < B_1$, then $y_2^* = 0$. (*Hint: you can prove this by contradiction.*)

Solution:
We will prove this by contradiction.

Suppose (y_1^*, y_2^*, z^*) is optimal and $y_1^* < B_1$, and $y_2^* > 0$. The optimal objective value is $w^* = M_1y_1^* + M_2y_2^* + z^*$.

Define $\varepsilon = \min\{B_1 - y_1^*, y_2^*\}$ and we know $\varepsilon > 0$ since $y_1^* < B_1$, and $y_2^* > 0$.

Let $(\hat{y}_1, \hat{y}_2, \hat{z})$ be a new solution such that the value of y_1^* is increased by ε and y_2^* is decreased by ε, as follows:

$$\begin{gathered}\hat{y}_1 = y_1^* + \varepsilon \\ \hat{y}_2 = y_2^* - \varepsilon \\ \hat{z} = z^*\end{gathered}$$

(NOTE: if $\varepsilon = B_1 - y_1^*$, i.e. $B_1 - y_1^* \geq y_2^*$, then $\hat{y}_1 = B_1$ and $\hat{y}_2 = y_2^* + y_1^* - B_1 \geq 0$; else if $\varepsilon = y_2^*$, i.e. $B_1 - y_1^* > y_2^*$, then $\hat{y}_1 = y_1^* + y_2^* < B_1$ and $\hat{y}_2 = 0$ — these are exactly the solutions when LP(A) correctly solves for Problem A.)

Clearly $(\hat{y}_1, \hat{y}_2, \hat{z})$ is feasible since

$$0 \leq \hat{y}_1 \leq B_1 \quad (\textit{due to the way } \varepsilon \textit{ is defined})$$

$$0 \leq \hat{y}_2 \leq (B_2 - B_1) \quad (\textit{due to the way } \varepsilon \textit{ is defined})$$

$$\hat{y}_1 + \hat{y}_2 = y_1^* + \varepsilon + y_2^* - \varepsilon = y_1^* + y_2^* = x \quad (\textit{hence } a(\hat{y}_1 + \hat{y}_2) + bz \leq H)$$

$$\hat{z} \geq 0$$

The objective value for $(\hat{y}_1, \hat{y}_2, \hat{z})$ is

$$\begin{aligned} M_1\hat{y}_1 + M_2\hat{y}_2 + \hat{z} &= M_1(y_1^* + \varepsilon) + M_2(y_2^* - \varepsilon) + z^* \\ &= M_1 y_1^* + M_2 y_2^* + z^* + (M_1 - M_2)\varepsilon \\ &= w^* + (M_1 - M_2)\varepsilon \end{aligned}$$

We assumed earlier w^* is the optimal solution. But since $(M_1 - M_2)\varepsilon > 0$, the new solution $(\hat{y}_1, \hat{y}_2, \hat{z})$ yields an objective value which is larger than (y_1^*, y_2^*, z^*). This contradicts with our earlier assumption that (y_1^*, y_2^*, z^*) is optimal. So it must be that $y_1^* < B_1$ and $y_2^* = 0$.

6. Carco has a \$150,000 advertising budget. To increase car sales, the firm is considering advertising in newspapers and on television. The more Carco uses a particular medium, the less effective is each additional ad. The number of new customers reached by newspaper ads is described by the following function:

$$f(x) = \begin{cases} 900x, & 0 \leq x \leq 10 \\ 3000 + 600x, & x \geq 10 \end{cases}$$

where x is the number of newspaper ads. On the other hand, the number of new customers reached by television ads is described by the following function:

$$g(y) = \begin{cases} 10000y, & 0 \leq y \leq 5 \\ 25000 + 5000y, & y \geq 5 \end{cases}$$

where y is the number of television ads. At most 30 newspaper ads and 15 television ads can be placed. The total number of ads placed (newspaper ads plus television ads) must be at least 25. Each newspaper ad costs \$1,500, and each television ad costs \$10,000.

(a) Formulate a linear program that will maximise the number of new customers created by Carco's advertising campaign.

(b) Write down the dual to the linear program in (a).

(c) Convert the linear program formulated in (a) to canonical form by introducing slack, surplus and artificial variables when necessary.

(d) Solve the linear program in (a) using the two-phase simplex algorithm. All workings MUST be shown (including simplex tables, row operations, ratio tests).

(e) Use your final tableau in (d) to write down the optimal solution of the dual to the linear program in (a).

Solutions:

(a) Let

x_1 = the number of newspaper ads created in the first lot of 10 ads
x_2 = the number of newspaper ads created in the second lot of 10 ads
y_1 = the number of television ads created in the first lot of 5 ads
y_2 = the number of television ads created in the second lot of 5 ads

$$\max z = 900x_1 + 600x_2 + 10000y_1 + 5000y_2$$

s.t.

$$
\begin{aligned}
x_1 &\le 10 \\
x_2 &\le 20 \\
y_1 &\le 5 \\
y_2 &\le 10 \\
3(x_1+x_2)+20(y_1+y_2) &\le 300 \\
x_1+x_2+y_1+y_2 &\ge 25 \\
x_1,x_2,y_1,y_2 &\ge 0
\end{aligned}
$$

(b) The dual of the linear program in (a) is,

$$\min w = 10u_1 + 20u_2 + 5u_3 + 10u_4 + 300u_5 - 25u_6$$

s.t.

$$
\begin{aligned}
u_1 + 3u_5 - u_6 &\ge 900 \\
u_2 + 3u_5 - u_6 &\ge 600 \\
u_3 + 20u_5 - u_6 &\ge 10000 \\
u_4 + 20u_5 - u_6 &\ge 5000 \\
u_1,u_2,u_3,u_4,u_5,u_6 &\ge 0
\end{aligned}
$$

where u_i is the dual variable associated with the i^{th} functional constraint.

(c) The LP in canonical form is,

$$\max z = 900x_1 + 600x_2 + 10000y_1 + 5000y_2$$

s.t.

$$
\begin{array}{rrrrrrrrrrrcl}
x_1 & & & & +s_1 & & & & & & & = & 10 \\
 & x_2 & & & & +s_2 & & & & & & = & 20 \\
 & & y_1 & & & & +s_3 & & & & & = & 5 \\
 & & & y_2 & & & & +s_4 & & & & = & 10 \\
3x_1 & +3x_2 & +20y_1 & +20y_2 & & & & & +s_5 & & & = & 300 \\
x_1 & +x_2 & +y_1 & +y_2 & & & & & & -s_6 & +a_1 & = & 25 \\
\multicolumn{11}{r}{x_1,x_2,y_1,y_2,s_1,s_2,s_3,s_4,s_5,s_6,a_1} & \ge & 0
\end{array}
$$

(d) In the first phase, we will minimise $w = a_1$.

	1st phase														
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**a1**	**RHS**	**Ratio Test**
R1			1	0	0	0	1	0	0	0	0	0	0	10	
R2			0	1	0	0	0	1	0	0	0	0	0	20	
R3			0	0	1	0	0	0	1	0	0	0	0	5	
R4			0	0	0	1	0	0	0	1	0	0	0	10	
R5			3	3	20	20	0	0	0	0	1	0	0	300	
R6			1	1	1	1	0	0	0	0	0	-1	1	25	
R0		w	0	0	0	0	0	0	0	0	0	0	-1	0	
	Restore canonical form.														
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**a1**	**RHS**	**Ratio Test**
R1		s1	1	0	0	0	1	0	0	0	0	0	0	10	10
R2		s2	0	1	0	0	0	1	0	0	0	0	0	20	
R3		s3	0	0	1	0	0	0	1	0	0	0	0	5	
R4		s4	0	0	0	1	0	0	0	1	0	0	0	10	
R5		s5	3	3	20	20	0	0	0	0	1	0	0	300	100
R6		a1	1	1	1	1	0	0	0	0	0	-1	1	25	25
R0	R6 + R0 -> R0	w	1	1	1	1	0	0	0	0	0	-1	0	25	
	Pivot on R1, x1														
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**a1**	**RHS**	**Ratio Test**
R1		x1	1	0	0	0	1	0	0	0	0	0	0	10	
R2		s2	0	1	0	0	0	1	0	0	0	0	0	20	20
R3		s3	0	0	1	0	0	0	1	0	0	0	0	5	
R4		s4	0	0	0	1	0	0	0	1	0	0	0	10	
R5	-3R1 + R5 -> R5	s5	0	3	20	20	-3	0	0	0	1	0	0	270	90
R6	-R1 + R6 -> R6	a1	0	1	1	1	-1	0	0	0	0	-1	1	15	15
R0	-R1 + R0 -> R0	w	0	1	1	1	-1	0	0	0	0	-1	0	15	
	Pivot on R6, x2														
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**a1**	**RHS**	**Ratio Test**
R1		x1	1	0	0	0	1	0	0	0	0	0	0	10	
R2	-R6 + R2 -> R2	s2	0	0	-1	-1	1	1	0	0	0	1	-1	5	
R3		s3	0	0	1	0	0	0	1	0	0	0	0	5	
R4		s4	0	0	0	1	0	0	0	1	0	0	0	10	
R5	-3R6 + R5 -> R5	s5	0	0	17	17	0	0	0	0	1	3	-3	225	
R6		x2	0	1	1	1	-1	0	0	0	0	-1	1	15	
R0	-R6 + R0 -> R0	w	0	0	0	0	0	0	0	0	0	0	-1	0	

In the second phase, we will drop the a_1 column from the final table in the first phase, and replace row zero with the original objective function, $z = 900x_1 + 600x_2 + 10000y_1 + 5000y_2$.

	2nd phase														
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**RHS**	**Ratio Test**
R1			1	0	0	0	1	0	0	0	0	0	10	
R2			0	0	-1	-1	1	1	0	0	0	1	5	
R3			0	0	1	0	0	0	1	0	0	0	5	
R4			0	0	0	1	0	0	0	1	0	0	10	
R5			0	0	17	17	0	0	0	0	1	3	225	
R6			0	1	1	1	-1	0	0	0	0	-1	15	
R0		z	-900	-600	-10000	-5000	0	0	0	0	0	0	0	
	Restore canonical form.													
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**RHS**	**Ratio Test**
R1			1	0	0	0	1	0	0	0	0	0	10	
R2			0	0	-1	-1	1	1	0	0	0	1	5	
R3			0	0	1	0	0	0	1	0	0	0	5	
R4			0	0	0	1	0	0	0	1	0	0	10	
R5			0	0	17	17	0	0	0	0	1	3	225	
R6			0	1	1	1	-1	0	0	0	0	-1	15	
R0	900R1 + R0 -> R0	z	0	-600	-10000	-5000	900	0	0	0	0	0	9000	
	Restore canonical form.													
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**RHS**	**Ratio Test**
R1		x1	1	0	0	0	1	0	0	0	0	0	10	
R2		s2	0	0	-1	-1	1	1	0	0	0	1	5	
R3		s3	0	0	1	0	0	0	1	0	0	0	5	5
R4		s4	0	0	0	1	0	0	0	1	0	0	10	
R5		s5	0	0	17	17	0	0	0	0	1	3	225	13.2352941
R6		x2	0	1	1	1	-1	0	0	0	0	-1	15	15
R0	600R6 + R0 -> R0	z	0	0	-9400	-4400	300	0	0	0	0	-600	18000	
	Pivot on R3, y1													
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**RHS**	**Ratio Test**
R1		x1	1	0	0	0	1	0	0	0	0	0	10	
R2	R3 + R2 -> R2	s2	0	0	0	-1	1	1	1	0	0	1	10	
R3		y1	0	0	1	0	0	0	1	0	0	0	5	
R4		s4	0	0	0	1	0	0	0	1	0	0	10	10
R5	-17R3 + R5 -> R5	s5	0	0	0	17	0	0	-17	0	1	3	140	8.23529412
R6	-R3 + R6 -> R6	x2	0	1	0	1	-1	0	-1	0	0	-1	10	10
R0	9400R3 + R0 -> R0	z	0	0	0	-4400	300	0	9400	0	0	-600	65000	
	Pivot on R5, y2													
Row Name	**Row operations**	**BV**	**x1**	**x2**	**y1**	**y2**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**RHS**	**Ratio Test**
R1		x1	1	0	0	0	1	0	0	0	0	0	10	
R2	(1/17)R5 + R2 -> R2	s2	0	0	0	0	1	1	0	0	0.058824	1.176471	18.23529	
R3		y1	0	0	1	0	0	0	1	0	0	0	5	
R4	(-1/17)R5 + R4 -> R4	s4	0	0	0	0	0	0	1	1	-0.05882	-0.17647	1.764706	
R5	(1/17)R5 -> R5	y2	0	0	0	1	0	0	-1	0	0.058824	0.176471	8.235294	
R6	(-1/17)R5 + R6 -> R6	x2	0	1	0	0	-1	0	0	0	-0.05882	-1.17647	1.764706	
R0	(4400/17)R5 + R0 -> R0	z	0	0	0	0	300	0	5000	0	258.824	176.471	101235	

The optimal solution is $(x_1, x_2, y_1, y_2) = (10, 1.76, 5, 8.24)$ with $z = 101,235$.

(e) The optimal solution to the dual LP is read directly from the slack/surplus variable entries of the $z-$row.

The optimal dual solution is $(u_1, u_2, u_3, u_4, u_5, u_6) = (300, 0, 5000, 0, 258.82, 176.47)$

7. A company assembles both cars and trucks. The assembly is carried out in three departments: sheet metal stamping, engine assembly and final assembly. The number of person hours required for assembling cars and trucks in each of these departments is shown below:

	sheet metal	*engine assembly*	*final assembly*
cars	50	50	25
trucks	25	50	100

Each *department* has a capacity of 200,000 person hours per year. Cars earn a profit of $6000 each, and trucks $10,000 each, for the company. Ignoring for now that the cars and trucks are integral, formulate a linear program that can be used to determine how many cars and trucks the company should manufacture each year to maximise its profit.

Solutions: Let x_1 be the number of cars manufactured each year and x_2 be the number of trucks manufactured each year. Then we can formulate the problem as

$$\max 6,000x_1 + 10,000x_2$$

such that

$$\begin{aligned} 50x_1 + 25x_2 &\leq 200,000 \\ 50x_1 + 50x_2 &\leq 200,000 \\ 25x_1 + 100x_2 &\leq 200,000 \\ x_1, x_2 &\geq 0. \end{aligned}$$

8. a. State what it means for a subset $\mathbb{C}$ of $\mathbb{R}^n$ to be convex.

   b. The feasible region $\mathbb{A}$ for a linear programming problem is given by the constraints

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 12 \\ 2x_1 + x_2 + 3x_3 &\leq 9 \\ 5x_1 - 5x_2 + x_3 &= 10 \end{aligned}$$

   with x_1, x_2 and x_3 non-negative. Show, using your definition from part (a), that the region $\mathbb{A}$ is convex.

Solutions:

a. A set $C \in R^n$ is convex if for any two points y and z in C and any $\lambda \in [0,1]$, the point $\lambda y + (1-\lambda)z \in C$.

b. Let $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3) \in A$. Then, for $i = 1, 2, 3$, $y_i \geq 0$ and $z_i \geq 0$. Furthermore

$$\begin{aligned} 2y_1 + 3y_2 + 4y_3 &\leq 12 \\ 2y_1 + y_2 + 3y_3 &\leq 9 \\ 5y1 - 5y2 + y3 &= 10 \end{aligned}$$

and

$$\begin{aligned} 2z_1 + 3z_2 + 4z_3 &\leq 12 \\ 2z_1 + z_2 + 3z_3 &\leq 9 \\ 5z_1 - 5z_2 + z_3 &= 10. \end{aligned}$$

For, $\lambda \in [0, 1]$, let $x = \lambda y + (1 - \lambda)z$. Then, for $i = 1, 2, 3$, $x_i = \lambda y_i + (1 - \lambda)z_i \geq 0$. Also

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 2(\lambda y_1 + (1 - \lambda)z_1) + 3(\lambda y_2 + (1 - \lambda)z_2) + 4(\lambda y_3 + (1 - \lambda)z_3) \\ &= \lambda(2y_1 + 3y_2 + 4y_3) + (1 - \lambda)(2z_1 + 3z_2 + 4z_3) \\ &\leq \lambda \times 12 + (1 - \lambda) \times 12 \\ &= 12. \end{aligned}$$

Similarly , we can show that $2x_1 + x_2 + 3x_3 \leq 9$ and $5x_1 - 5x_2 + x_3 = 10$. Therefore x satisfies the conditions necessary to be in A, which implies that A is convex.

9. Prove the following theorem:

   If a linear programming problem has more than one optimal solution, it must have infinitely many optimal solutions. Furthermore, the set of optimal solutions is convex.

   Solutions: Suppose we have a linear program in standard form with two optimal solutions, at the points x_1 and x_2 where $x_1 \neq x_2$. Clearly these solutions must have the same value, i.e. $z_1 = z_2$, otherwise only the greater value will represent the optimal solution. Since both x_1 and x_2 are feasible, and the feasible region is convex, then $x_3 = \lambda x_1 + (1 - \lambda)x_2$ is also feasible for any $\lambda \in [0, 1]$:

   $$x_3 = \lambda x_1 + (1 - \lambda)x_2 \tag{1}$$

   for $\lambda \in [0, 1]$. Since there are infinitely many $\lambda \in [0, 1]$, there are infinitely many choices of x_3 that satisfy (1). Furthermore, since the objective function itself is linear, then every point satisfying (1) must also be feasible. That is,

   $$f(x_3) = f(\lambda x_1 + (1 - \lambda)x_2) = \lambda f(x_1) + (1 - \lambda)f(x_2) = \lambda z_1 + (1 - \lambda)z_1 = z_1. \tag{2}$$

   Therefore x_3 is optimal and there are infinitely many choices of λ that satisfy (2). Therefore, if there is more than one optimal solution, there are infinitely many optimal solutions.

   From above, we also see that the linear combination of any two optimal solutions is also optimal. Therefore the optimal set is convex.

10. A whisky blending company has 3,000 litres of type A spirit, 5,000 litres of type B spirit and 10,000 litres of type C spirit available. From these three types of spirit the company can blend three grades of whisky, which are made up in the percentage proportions given below.

	Spirit A	*Spirit B*	*Spirit C*
Whisky grade 1	90	10	0
Whisky grade 2	30	50	20
Whisky grade 3	10	30	60

The profit per litre for selling grades 1, 2 and 3 is \$12, \$6, \$4 respectively.

a. Create a list of all the basic feasible solutions (this can be done by hand if you have a lot of spare time – however, we recommend using Matlab or a similar solver to help do this efficiently).
b. Evaluate the objective function at each *bfs* to determine the optimal solution.

Solutions:
The optimal solution is $(1538, 0, 16154, 0, 0, 308)$ with an objective function value of 83077. All generated solutions and example Matlab code to generate the solution is provided below:
The basic feasible solutions, and respective objective function values are:
Variables 1 2 6

Basic feasible solution = 0 10000 8000

Objective function value = 60000

Variables 1 3 4

Basic feasible solution = 1.0e+04 *

0 1.6667 0.1333

Objective function value = 6.6667e+04

Variables 1 3 6

Basic feasible solution = 1.0e+04 *

0.1538 1.6154 0.0308

Objective function value = 8.3077e+04

Variables 1 5 6

Basic feasible solution = 1.0e+04 *

0.3333 0.4667 1.0000

Objective function value = 40000

Variables 2 3 4

Basic feasible solution = 1.0e+04 *

0 1.6667 0.1333

Objective function value = 6.6667e+04

Variables 2 3 6

Basic feasible solution = 10000 0 8000

Objective function value = 60000

Variables 2 4 6

Basic feasible solution = 10000 0 8000

Objective function value = 60000

Variables 2 5 6

Basic feasible solution = 10000 0 8000

Objective function value = 60000

Variables 3 4 5

Basic feasible solution = 1.0e+04 *

1.6667 0.1333 0

Objective function value = 6.6667e+04

Variables 3 4 6

Basic feasible solution = 1.0e+04 *

1.6667 0.1333 0

Objective function value = 6.6667e+04

Variables 4 5 6

Basic feasible solution = 3000 5000 10000

Objective function value = 0

The other solutions are:

Variables 1 2 3

Infeasible solution = 1.0e+04 *

0.1600 -0.0400 1.6800

Objective function value = 84000

Variables 1 2 4

Infeasible solution = -200000 50000 168000

Objective function value = -2100000

Variables 1 2 5

Infeasible solution = 1.0e+04 *

-1.3333 5.0000 -1.8667

Objective function value = 140000

Variables 1 3 5

Infeasible solution = 1.0e+04 *

0.1481 1.6667 -0.0148

Objective function value = 8.4444e+04

Variables 1 4 5

Infeasible solution = Infeasible Objective function value = Does not exist

Variables 1 4 6

Infeasible solution = 50000 -42000 10000

Objective function value = 600000

Variables 2 3 5

Infeasible solution = 5000 15000 -2000

Objective function value = 90000

Variables 2 4 5

Infeasible solution = 50000 -12000 -20000

Objective function value = 300000
Variables 3 5 6
Infeasible solution = 30000 -4000 -8000
Objective function value = 120000

11. A taxi company is trying to determine the number of staff it requires to cover demand – they are most interested in keeping the number of staff to the minimum to cover projected demand. They are also interested in knowing when the drivers should start work. A driver will work for 16 hours at a time which begins at the start of one of the designated shifts. The company has identified a rough demand schedule below which is broken into 8 hour segments:

3am – 11am	11.01am – 7pm	7.01pm – 3am
2	10	14

a. Formulate the problem as a linear programming problem, making sure to order your variables in the same way as the data and ordering your constraints such that the matrix A is symmetric.

Solutions: Let x_1 be the number of staff starting work at 3am, x_2 be the number starting at 11.01am and x_3 be the number of staff starting at 7.01pm.

$$\min \quad x_1 + x_2 + x_3$$

such that

$$\begin{aligned} x_1 \phantom{{}+x_2} + x_3 &\geq 2 \\ x_1 + x_2 \phantom{{}+x_3} &\geq 10 \\ x_2 + x_3 &\geq 14 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

b. Attain canonical form for this problem.

Solutions:
Standard form:

$$\max \quad -x_1 - x_2 - x_3$$

such that

$$\begin{aligned} x_1 \phantom{{}+x_2} + x_3 - x_4 \phantom{{}-x_5-x_6} &= 2 \\ x_1 + x_2 \phantom{{}+x_3-x_4} - x_5 \phantom{{}-x_6} &= 10 \\ x_2 + x_3 \phantom{{}-x_4-x_5} - x_6 &= 14 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0. \end{aligned}$$

Canonical form:

$$\max \quad -x_1 - x_2 - x_3$$

such that

$$\begin{array}{rcl} x_1 \quad + x_3 - x_4 \quad + \underline{x}_7 + & = & 2 \\ x_1 + x_2 \quad - x_5 \quad + \underline{x}_8 & = & 10 \\ x_2 + x_3 \quad - x_6 \quad + \underline{x}_9 & = & 14 \\ x_1, x_2, x_3, x_4, x_5, x_6, +\underline{x}_7, +\underline{x}_8, +\underline{x}_9 & \geq & 0. \end{array}$$

c. Write down the dual problem to the above primal problem.

Solutions: First we transform the primal to equivalent non-standard form:

$$\max \quad -x_1 - x_2 - x_3$$

such that

$$\begin{array}{rcl} -x_1 \quad - x_3 & \leq & -2 \\ -x_1 - x_2 & \leq & -10 \\ -x_2 - x_3 & \leq & -14 \\ x_1, x_2, x_3 & \geq & 0. \end{array}$$

Now we can easily obtain the dual:

$$\min \quad -2y_1 - 10y_2 - 14y_3$$

such that

$$\begin{array}{rcl} -y_1 - y_2 & \geq & -1 \\ -y_2 - y_3 & \geq & -1 \\ -y_1 \quad - y_3 & \geq & -1 \\ y_1, y_2, y_3 & \geq & 0. \end{array}$$

d. Suppose that the final tableau for the above problem is:

Basis	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	-1	0	1	0	1	-1	4
x_2	0	1	1	0	0	-1	0	10
x_4	0	-2	0	0	1	1	-1	2
	1	1	0	0	0	0	1	-14

We notice that there are no further negative reduced costs (we are working with a maximisation problem). We have found the optimal solution! We should not forget to multiply our final objective function value by -1. This is $x = (0, 10, 4, 2, 0, 0)$ where $f(x) = 14$. That is, we require 14 workers, 10 who will start in the second shift and 4 who will start in the third shift.

Using the tableau, identify the optimal solution for the dual. Show that the solution you obtained is dual feasible.

Solutions:

We read the dual solution off the final tableau, where the coefficients for the slack variables are the solutions for the dual variables. That is, $y = (0, 0, 1)$. We must still multiply the final objective function by -1 since we solve the problem as a minimisation problem! Therefore $w = 14$ as expected. All the constraints are satisfied for this point, i.e.:

$$\begin{aligned} 0 &\geq -1 \\ 1 &\geq -1 \\ 1 &\geq -1 \\ 0, 0, 1 &\geq 0. \end{aligned}$$

We are satisfied that the optimal solution for y is feasible for the dual problem.

e. Using the complementary slackness relations, verify that you have indeed found the optimal solution.

Solutions:

The simplest way to verify the complementary slackness relations is to express the primal and dual solutions as row vectors and take the column product (since we wish to show that $xt = 0$ and $ys = 0$). We note that we present y in the form it is read from the tableau, with 'surplus' variables first and dual variables second:

$$\begin{array}{ccc cccccc} x & = & \{0, & 10, & 4, & 2, & 0, & 0\} \\ y & = & \{1, & 0, & 0, & 0, & 0, & 1\} \\ \hline x_i \times y_i & & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

The complementary slackness conditions are therefore satisfied.