Complementary mathematical topics

These topics will not be tested in STM5001 assignments. They are given for students interested in the mathematical background and justifications of models considered in this and previous weeks lectures.

Counting Processes.

A stochastic process $\{N(t), t \geq 0\}$ is said to be a **counting process** if $N(t)$ represents the total number of "events" that occur by time t.
$N(t)$ must satisfy:

(i) $N(t) \geq 0$.

(ii) $N(t)$ is an integer value.

(iii) If $s < t$, then $N(s) \leq N(t)$

(iv) For $s < t, N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$.

Definition 1

The counting process $\{N(t), t \geq 0\}$ *is said to be a* **Poisson process** *having rate* $\lambda, \lambda > 0$, *if*

(i) $N(0) = 0$.

(ii) *The process has independent increments.*

(iii) *The number of events in any interval of length* t *is Poisson distributed with mean* λt. *That is, for all* $s, t \geq 0$:

$$P(N(t+s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad n = 0, 1, \ldots.$$

Theorem 1

Let $\{N(t), t \geq 0\}$ be a counting process and

(i) $N(0) = 0$.

(ii) *The process has stationary and independent increments.*

(iii) $P(N(h) = 1) = \lambda h + o(h)$.

(iv) $P(N(h) \geq 2) = o(h)$.

Then $N(t)$ is a Poisson process having rate λ.

Proof. Fix $u \geq 0$ and let $g(t) = E[\exp(-uN(t))]$. We derive a differential equation for $g(t)$ as follows

$$\begin{aligned} g(t+h) &= E[\exp(-uN(t+h))] = E[\exp(-uN(t)) \times \exp(-u(N(t+h) - N(t)))] \\ &= E[\exp(-uN(t))] \times E[\exp(-u(N(t+h) - N(t)))] = \\ &= g(t) \cdot E[\exp(-uN(h))]. \end{aligned}$$

The last two lines follows from the independence and stationarity of increments.

Now, assumptions (iii) and (iv) imply that $P(N(h)=0)=1-\lambda h-o(h)$. Hence, conditioning on whether $N(h)=0$ or $N(h)=1$ or $N(h)\geq 2$ yields

$$\begin{aligned} E[\exp(-uN(h))] &= 1-\lambda h+o(h)+e^{-u}(\lambda h+o(h))+o(h) \\ &= 1-\lambda h+e^{-u}\lambda h+o(h). \end{aligned}$$

Therefore, we obtain that

$$g(t+h)=g(t)(1-\lambda h+e^{-u}\lambda h)+o(h),$$

which implies that

$$\frac{g(t+h)-g(t)}{h}=g(t)\lambda(e^{-u}-1)+\frac{o(h)}{h}.$$

Letting $h\to 0$ gives

$$g'(t)=g(t)\lambda(e^{-u}-1) \quad \text{or equivalently} \quad \frac{g'(t)}{g(t)}=\lambda(e^{-u}-1).$$

Now, integrating, and using $g(0) = 1$, one obtains that

$$\log(g(t)) = \lambda t(e^{-u} - 1),$$

$$g(t) = \exp(\lambda t(e^{-u} - 1)).$$

That is the moment-generating function of $N(t)$ evaluated at $-u$.

Since, that is also the moment-generating function of a Poisson random variable with the mean λt, the result follows from the fact that the distribution of a random variable is uniquely determined by its moment-generating function.