MAST30025: Linear Statistical Models
Assignment 1 S1 2021

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Question 1 Solution:

Part a:

A^2 =A^3

Suppose A is a square matrix is (real and) symmetric then its eigenvalues are all real, and its eigenvalues are orthogonal.

***Theorem* 2.3**

Proof:

Take A to be a square matrix, n x n. First we diagonalise A,i.e., find P such that.

=

$$D = P^T AP$$

$$= \begin{bmatrix} \lambda_1 & .... & 0 \\ ...... & .... & .. \\ 0 & .... & \lambda_k \end{bmatrix}$$

where $\lambda_1,\ \lambda_2, ...,\ \lambda_k$ *are the eigenvalues of A.*

Since P is orthogonal both P and P^T $are\ non-singular$,

$$r(P^T AP) = r(p^T A) = r(A)$$

Because P^T AP *is diagonal* $r(P^T AP)$ *is the number of non zero eigenvalues of A.*

But we wanted to prove ***Theorem* 2.2**

that A any symmetric matrix is idempotent. Which has eigenvalues of $\lambda =$ 0 or $\lambda = 1$.

The eigenvalues of idempotent matrices are always either

$\lambda = 0$ or $= 1$.

$A^2 = \lambda^2$x

Multiplying by A!!!

A^3x $= A^2\ \lambda$x $= \lambda\ A^2$x $= \lambda^3$ x

(λ^3 - λ^2)$\boldsymbol{x} = 0$

By definition, x$\neq$ 0,

λ^3 - $\lambda^2 = 0$
$\lambda^2(\lambda$ - 1$) = 0$

Therefore there are two values with eigenvalues of 0 and one eigenvalue of 1! satisfies this theorem that A is idempotent!

Part b:

$$A = A^3$$

A^3x = A λx = λ Ax = λ^3 x

Using the same theorem from the previous it has eigenvalues of 0,1 and -1. Since we care that A has to be positive semi-definite. Which has an eigenvalue of -1. Which does not satisfy Theorem 2.2! A is not idempotent!

Question 2 Solution:
Theorem 2.4

There exists a matrix **P** which diagonalises A_1,....,A_m.

$$P^T A_i P = D_i$$

and

$$P^T A_j P = D_j$$

We take A_i and A_j to be k x k matrices first we diagonalizes A_i, A_j, i.e. find P such that,

$D_i = P^T$ A_i P = $\begin{bmatrix} \lambda_1 & ...... & 0 \\ .... & \lambda_2 & ...... \\ 0 & .... & \lambda_k \end{bmatrix}$

for i = 1,.....,k

$D_j = P^T$ A_j P = $\begin{bmatrix} \lambda_1 & ...... & 0 \\ .... & \lambda_2 & ...... \\ 0 & .... & \lambda_m \end{bmatrix}$

for j = 1,.....,m

Proof:

P$^T A_i A_j P = (P^T A_i P)(P^T A_j P) = (P^T A_j P)(P^T A_i P) = P^T A_j A_i P$

Pre-multiply by P and post-multiply by P^T to get $A_i A_j = A_j A_i$.

Question 3 Solution:

Pre Proof Using Theorem 2.3

For any matrix A

$$r(A) = r(A^T) = r(A^T A) = tr(A)$$

A = $\begin{bmatrix} | & | & ...... & | & .... & | \\ a_1 & a_2 & .... & ap & .... & an \\ | & | & ...... & | & .... & | \end{bmatrix}$

Given A matrix with dimensions n x p with p independent columns.

Let x$_1, x_2, x_3, ..., x_k$ *the basis for column space of A.*

Definition of basis every column vector of A is a linear combination of the column vectors of x.

$$a_1 = b_1 x_1 + b_2 x_2 + ... + b_k x_k$$

Definition of linear combination

where b is scalar

B = $\begin{bmatrix} --b_1-- \\ --b_2-- \\ | \\ --bp-- \end{bmatrix}$

$$\begin{bmatrix} | & | & ...... & | & .... & | \\ a_1 & a_2 & .... & ap & .... & an \\ | & | & ...... & | & .... & | \end{bmatrix} = \begin{bmatrix} | & | & ...... & | & .... & | \\ x_1 & x_2 & .... & xp & .... & an \\ | & | & ...... & | & .... & | \end{bmatrix} \begin{bmatrix} --b_1-- \\ --b_2-- \\ | \\ --bp-- \end{bmatrix}$$

$$A = XB$$

$$A^T = (XB)^T = B^T X^T$$

r(A) $\leq r(A^T)$ *or* $r(A) \geq r(A^T)$ *to satisfy*!

$$r(A) = r(A^T) = r(A^T A) = tr(A) = p$$

Since P is orthogonal both P and P^T *arenon−singular. Therefore we need to sum up the diagonal el*

$$r(A) = r(P^T AP) = tr(P^T AP) = tr(PP^T A) = tr(A) = p$$

Because D = P$^T AP$ *is diagonal* $r(P^T AP)$ *is the number of nonzero values of* A!

But A is idempotent so its takes eigenvalues between 0 or 1. To Prove Theorem 2.7! We need only the identity matrix to allow A$^T A$ *to be positive definite*!

Using Theorem 2.7

Proof **(⇐) :**

We want A$^T A$ *to be symmetric*

and have all the eigenvalues to be strictly positive to prove $A^T A$ is a positive definite matrix!

we know r($A^T A$) = p is a p x p matrix so it has to be a full rank matrix, p!

Let,λ_1, λ_2,....,$\lambda p > 0$ be the eigenvalues of $A^T A$ for every x and for each eigenvalue has to have a value of 1.

for z = P^Tx = $(z_1, ..., zp)^T$

$$x^T(A^T A)x = x^T PDP^T x = z^T Dz = \sum_{i=1}^{p} z_i^2 \lambda_i$$

since $\lambda_i = 1$!

$$= \sum_{i=1}^{p} z_i^2$$

> 0

Thus $A^T A$ is positive definite as required!

Proof **(⇒) :**

Suppose $A^T A$ is positive definite let x_i be its normalised i-th eigenvector then,

$$x_i^T (A^T A)x_i = \lambda_i x_i^T x_i = \lambda_i$$

From theorem 2.3 we want $A^T A$ to be symmetric and idempotent.We want the eigenvalues to be 0 or 1. This case all of the eigenvalues must equal to 1.

$\lambda_i = 1$>0

So, the eigenvalues of $A^T A$ are strictly positive as required!!

Question 4 Solution:

Part a:

Given information:

Let,

x_1,x_2,$x_3$$\sim$ ($N(\mu$,σ^2)) be a sequence of independent normal random variables,

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$\boldsymbol{x^T} = (x_1, x_2, x_3)^T$

Supposed to be x^T as noted!

$\boldsymbol{y}$ = (x$_1$ - ,x_2- ,x_3-)^T

To solve A from:

$$\boldsymbol{y} = A\boldsymbol{x}$$

$\begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

A $= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Where A is symmetric and idempotent!

Part b: Finding the rank of A

Proof: that there is a linear combination for any columns?

```{r}
A = matrix(c(2,-1,-1,-1,2,-1,-1,-1,2)/3,3,3)

```

```
           [,1]       [,2]       [,3]
[1,]  0.6666667 -0.3333333 -0.3333333
[2,] -0.3333333  0.6666667 -0.3333333
[3,] -0.3333333 -0.3333333  0.6666667
```

```
# Finding rank of A

```{r}
rankMatrix(A)[1]
```

  [1] 2
```

Each column all added up together gives us 0!.
$x_1 + x_2 + x_3 = 0$
Can be written as,
$x_1 = -x_2 - x_3$
That are linearly dependant and similar for x$_2andx_3$
Hence r(A) = 2

Part c: Computing E[$y^T y$]

Finding E[$\boldsymbol{y^T y}$]

$$= E[(\frac{2x_1 - x_2 - x_3}{3}, \frac{-x_1 + 2x_2 - x_3}{3}, \frac{-x_1 - x_2 + 2x_3}{3}) \begin{bmatrix} \frac{2x_1 - x_2 - x_3}{3} \\ \frac{-x_1 + 2x_2 - x_3}{3} \\ \frac{-x_1 - x_2 + 2x_3}{3} \end{bmatrix}]$$

$= E[(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 (x_3 - \bar{x})^2)]$
$= E[(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 (x_3 - \bar{x})^2)]$

$= \text{E}[\sum_{i=1}^{3}(x_i - \bar{x})^2]$
$= \text{E}[\sum_{i=1}^{3} x_i^2 - 2x_i\bar{x} + \bar{x}^2]$
$= \text{E}[\sum_{i=1}^{3} x_i^2 - n\bar{x}^2]$
Since we have 3 x's that are random variables!!
$= \text{E}[\sum_{i=1}^{3} x_i^2 - 3\bar{x}^2]$
$= \text{E}[(\sum_{i=1}^{3} x_i^2) - 3\bar{x}^2]$
since x_1, x_2 *and* x_3 *are identical independent distributions*!!
$= \text{(3-1)}\sigma^2 = 2\sigma^2$

Assuming that the sample variance is unbiased! and we can imply $\lambda = 0$! *Following similarly to Theorem* 3.2. *for the Non−central distribution*!

Alternative method:
Theorem 3.5:
$\text{E}[\text{y}^T Ay] = tr(AV) + \mu^T \text{A}\mu$
since A = I,
$= \text{tr(V)} + \mu^T\ \mu$
$\text{V} = \text{vary} = \text{varAx} = \text{AvarxA}^T$
since A is symmetric and idempotent!!
$$\text{var(x}_i) = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$
$$\text{V} = \frac{1}{3}\begin{bmatrix} 2\sigma^2 & -1 & -1 \\ -1 & 2\sigma^2 & -1 \\ -1 & -1 & 2\sigma^2 \end{bmatrix}$$
$\mu = E[y] = E[Ax] = AE[x]$
$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} = 0$$
$$\text{E}[\text{y}^T y] = tr(\frac{1}{3}\begin{bmatrix} 2\sigma^2 & -1 & -1 \\ -1 & 2\sigma^2 & -1 \\ -1 & -1 & 2\sigma^2 \end{bmatrix}) + 0$$
$= 2\sigma^2$
Part d:
Using Theorem 3.5:
Proof:

Assuming that A is idempotent and has rank k. Because it is symmetric, it can be diagonalised. Let the (orthogonal) diagonalising matrix be P.

D = P^T AP = $\begin{bmatrix} \lambda_1 & ...... & 0 \\ .... & \lambda_2 & ...... \\ 0 & .... & \lambda_k \end{bmatrix}$

since A is symmetric and idempotent, all eigenvalues are either 0 or 1. We know from definition:

$$tr(A) = r(A) = k$$

A $=\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

A^2 = A = $\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

from Part 4b, we find out the rank and trace of matrix A we found in Part 4a. Is also is the same number of degrees of freedom for the chi squared distribution.

$$tr(A) = r(A) = 2$$

Therefore, A must have two eigenvalues of 1 and one eigenvalue of 0.

<u>Using Theorem 3.5 and Corollary 3.7:</u>

with our non central parameter λ!

$\lambda = \frac{1}{2}\mu^T A\mu$

$= \frac{1}{2}\begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \mu & \mu & \mu \end{bmatrix}$

$= 0$

$\iff : if\ and\ only\ if$

E[y] = E[$\begin{bmatrix} x_1 - \mu \\ x_2 - \mu \\ x_3 - \mu \end{bmatrix}$]

Since x$_1, x_2$ *and* x_3 *is identically independently distributed*! *and taking* the expectation of the expectation is the expectation itself!

E[y] = E[$\begin{bmatrix} \mu - \mu \\ \mu - \mu \\ \mu - \mu \end{bmatrix}$] = 0

NOTE: $\mu = \bar{x}$

In which case,

$\frac{y^T y}{\sigma^2}$

is just the sum of two independent standard normal's. This is just an ordinary (central) chi squared distribution $\chi^2{}_2$.

with expectation of 2 and variance of 4.

MAST30025 Assignment 1 S1 2021 (3/3)

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Question 5 Solution:

Part a: Computing y,X,β and ϵ

$$\boldsymbol{y} = \begin{bmatrix} 27.3 \\ 42.7. \\ 38.7 \\ 4.5 \\ 23.0 \\ 166.3 \\ 109.7 \\ 80.1 \\ 150.7 \\ 20.3 \\ 189.7 \\ 131.3 \\ 404.2 \\ 149 \end{bmatrix}. \quad \boldsymbol{X} = \begin{bmatrix} 1 & 13.1 \\ 1 & 15.3 \\ 1 & 25.8 \\ 1 & 1.8 \\ 1 & 4.9 \\ 1 & 55.4 \\ 1 & 39.3 \\ 1 & 26.7 \\ 1 & 47.5 \\ 1 & 6.6 \\ 1 & 94.7 \\ 1 & 61.1 \\ 1 & 135.6 \\ 1 & 47.6 \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \\ \epsilon_{10} \\ \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \end{bmatrix}$$

$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ *becomes,*

$$\begin{bmatrix} 27.3 \\ 42.7. \\ 38.7 \\ 4.5 \\ 23.0 \\ 166.3 \\ 109.7 \\ 80.1 \\ 150.7 \\ 20.3 \\ 189.7 \\ 131.3 \\ 404.2 \\ 149 \end{bmatrix}. = \begin{bmatrix} 1 & 13.1 \\ 1 & 15.3 \\ 1 & 25.8 \\ 1 & 1.8 \\ 1 & 4.9 \\ 1 & 55.4 \\ 1 & 39.3 \\ 1 & 26.7 \\ 1 & 47.5 \\ 1 & 6.6 \\ 1 & 94.7 \\ 1 & 61.1 \\ 1 & 135.6 \\ 1 & 47.6 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \\ \epsilon_{10} \\ \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \end{bmatrix}$$

Part b: Solving the least squares estimator

```{r}
b = solve(t(X)%*%X, t(X)%*%y)
b
```

```
          [,1]
[1,] -1.233836
[2,]  2.701553
```

Part c:

```{r}
e = y - X%*%b
e #Residual errors


```

```
            [,1]
 [1,]  -6.8565106
 [2,]   2.3000724
 [3,] -29.7662361
 [4,]   0.8710405
 [5,]  10.9962256
 [6,]  17.8677893
 [7,]   4.7627957
 [8,]   9.2023660
 [9,]  23.6100596
[10,]   3.7035852
[11,] -64.9032511
[12,] -32.5310639
[13,]  39.1032233
[14,]  21.6399042
```

````
```{r}
n = 14 #sample size
p = 2 #number of parameters
SSRes = sum(e^2)
ssquared = SSRes/(n-p)
ssquared


```

 [1] 777.1528
````

Part d:

```{r}
c(1,28)%*%b
```

```
         [,1]
[1,] 74.40965
```

Part e:

```{r}
a = solve(t(X)%*%X)
a
```

```
              [,1]          [,2]
[1,]   0.163081936 -2.230009e-03
[2,]  -0.002230009  5.425812e-05
```

```{r}
H = X%*%a%*%t(X)
H


```

```{r}
z = e/sqrt(ssquared * (1 - diag(H)))
z[13]
```

```
[1] 2.104999
```

Part f:

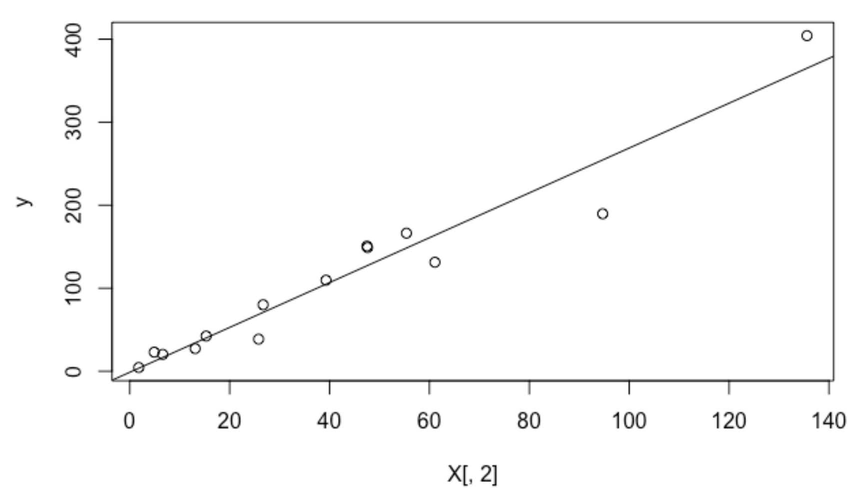
```{r}
k = 1
D = z^2 * (diag(H)/(1-diag(H))) * 1/(k+1)
D[13]
```

```
[1] 2.774008
```

Part g:

````
```{r}
plot(X[,2],y)
abline(b[1],b[2])

```
````



Full explaination: The Cook's distance certainly indicates it should be of some concern; however looking at the plot, it seems that the fit is actually okay. There is considerable evidence for heteroskedasticity — the variance increases with x (the design variable). Sea scallops has (by far) the largest x and so may be prone to a larger variance than the remaining points. The high Cook's distance therefore comes primarily from a very high leverage, rather than a bad fit to the model.

END OF ASSIGNMENT!!