# ECOM20001
# Econometrics 1

## Lecture Note 5
## Single Linear Regression - Testing

A/Prof David Byrne
Department of Economics
University of Melbourne

Stock and Watson: Chapter 5, Sections 5.1-5.4 and part of 5.5

# Summary of Key Concepts

- Visual evidence
- Hypothesis testing with the regression model
  - 3 steps to hypothesis tests
  - testing and statistical software
  - class size and test score example
- Confidence intervals for regression model slope $\beta_1$ and intercept $\beta_0$
- Dummy variables
- Heteroskedasticity and Homoskedasticity

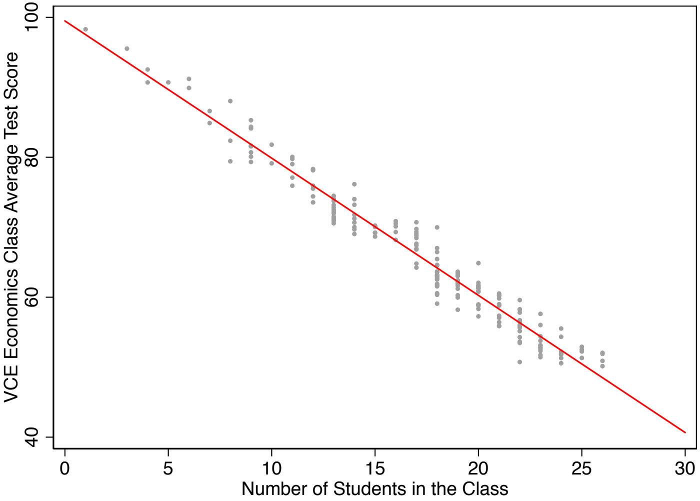
# Using Data to Evaluate Claims

- Politician says:

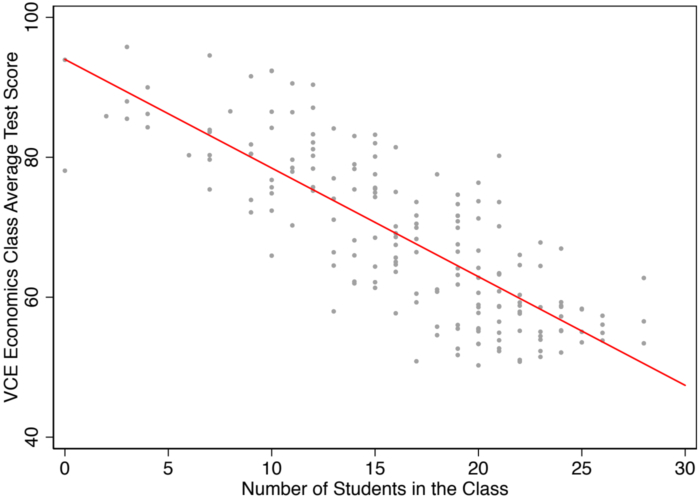
  *"There's no problem with schools with unequal class sizes! Class size doesn't affect students' test scores!"*

- How do we use data and regression model to evaluate this claim scientifically?

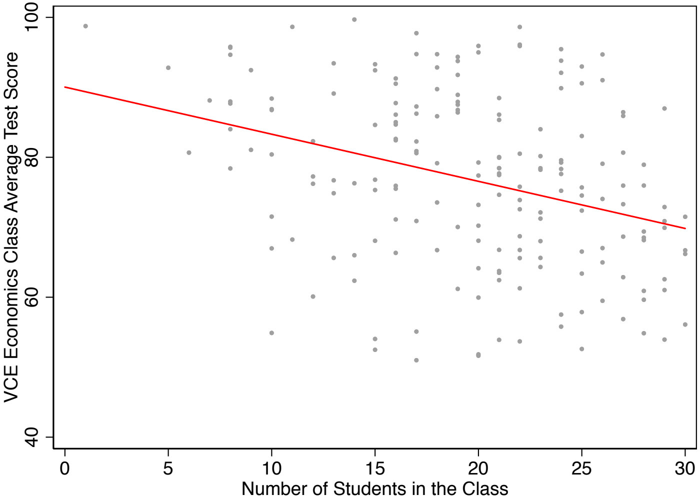
# Insanely compelling visual evidence



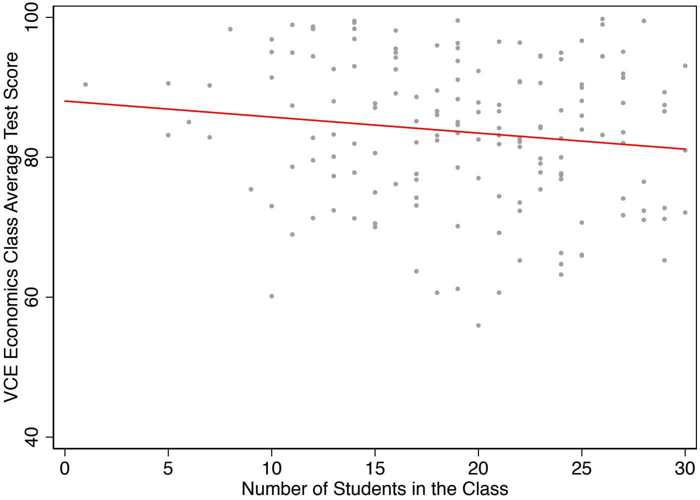
# Compelling visual evidence



# Less compelling visual evidence



# No visual evidence



# Hypothesis Testing

- While visual evidence is useful, we would like to have a more systematic way of drawing conclusions based on data
- Whenever claims are made of the sort that "$X$ affects $Y$", we can use regressions, hypothesis testing, and confidence intervals to formally evaluate a claim using data

# Hypothesis Testing

- Recall our regression model is:

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

  and we estimate it using data $(X_i, Y_i)$'s $i = 1, 2, \ldots, n$, which is an $n$ observation random sample from the population
- Determining whether "$X$ affects $Y$" or not boils down to determining whether $\hat{\beta}_1$, is statistically different from 0
- Everything we covered regarding hypothesis testing, statistical significance, and confidence intervals about sample means $\bar{Y}$ carries over directly to testing slopes of regression lines $\hat{\beta}_1$

# Hypothesis Testing

- More formally, we wish to test the following null hypothesis $H_0$ against an alternative $H_1$:

$$H_0 : \beta_1 = \beta_{1,0} \quad \text{vs.} \quad H_1 : \beta_1 \neq \beta_{1,0}$$

- Notice that we continue to focus strictly on two-sided hypothesis tests: $\beta_1 \neq \beta_{1,0}$ can mean that $\beta_1$ is either much smaller or much larger than $\beta_{1,0}$
- In practice, we conduct the hypothesis test in 3 steps, like we did with testing hypotheses about the sample mean

# 3 Steps for Testing Hypotheses About $\beta_1$

1. Compute the OLS estimate $\hat{\beta}_1$ and its standard error, $SE(\hat{\beta}_1)$, which has the following formula:

$$SE(\hat{\beta}_1) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}}$$

where

$$\hat{\sigma}^2_{\hat{\beta}_1} = \frac{1}{n} \times \frac{\frac{1}{n-2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \hat{u}_i^2}{\left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2\right]^2}$$

2. Compute the t-statistic:

$$t^{act} = \frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)}$$

3a. Compute the p-value

$$\text{p-value} = 2\Phi(-|t^{act}|)$$

where $\Phi$ is the cumulative density of the normal distribution.

# 3 Steps for Testing Hypotheses About $\beta_1$

(3a continued) Letting $\alpha$ be the level of significance of the test, we reject the null $H_0 : \beta_1 = \beta_{1,0}$ if

$$\text{p-value} < \alpha$$

where typical values of $\alpha$ are 0.10, 0.05, 0.01

3b. We can equivalently use our t-statistic and critical values from the normal distribution to conduct the hypothesis test.

We reject the null $H_0 : \beta_1 = \beta_{1,0}$ in favour of $H_0 : \beta_1 \neq \beta_{1,0}$ if

$$|t^{act}| > t^{\alpha}_{crit}$$

where recall:

- $t^{\alpha}_{crit} = 1.65$ if $\alpha = 0.10$
- $t^{\alpha}_{crit} = 1.96$ if $\alpha = 0.05$
- $t^{\alpha}_{crit} = 2.58$ if $\alpha = 0.01$

# Hypothesis Testing and Statistical Software

- In its regression package, R automatically reports the $\hat{\beta}_1$, $SE(\hat{\beta}_1)$, and the t-statistic and p-value for the test of:

$$H_0 : \beta_1 = 0 \text{ vs. } H_1 : \beta_1 \neq 0$$

- That is, the test of the null hypothesis of no relationship between $X$ and $Y$ versus the alternative hypothesis that there exists a relationship (either positive or negative) between $X$ and $Y$
- This is by far the most popular hypothesis test employed in practice: testing whether a relationship exists or not
- So in practice, we use statistics software like R to compute standard errors and do hypothesis testing of this sort

# How Regression Output Looks in R

```
Call:
lm(formula = earnings ~ height, data = mydata1)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.7972 -2.1909 -0.7923  3.4421  5.0579 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -0.051174   0.338050  -0.151     0.88    
height       0.027859   0.001984  14.042   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.678 on 17868 degrees of freedom
Multiple R-squared:  0.01092,   Adjusted R-squared:  0.01086 
F-statistic: 197.2 on 1 and 17868 DF,  p-value: < 2.2e-16
```

# How Regression Output Looks in R

$Y_i$ $X_i$

```
Call:
lm(formula = earnings ~ height, data = mydata1)
```

**Regression**

```
Residuals:
    Min      1Q  Median      3Q     Max
-4.7972 -2.1909 -0.7923  3.4421  5.0579

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.051174   0.338050  -0.151     0.88
height       0.027859   0.001984  14.042   <2e-16 ***
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Multiple R-squared:  0.01092,   Adjusted R-squared:  0.01086
F-statistic: 197.2 on 1 and 17868 DF,  p-value: < 2.2e-16
```

# How Regression Output Looks in R

Test Results for Null that Intercept=0

```
Call:
lm(formula = earnings ~ height, data = mydata1)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.7972 -2.1909 -0.7923  3.4421  5.0579 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -0.051174   0.338050  -0.151     0.88    
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```

$\hat{\beta}_0$ Intercept

# How Regression Output Looks in R

```
Call:
lm(formula = earnings ~ height, data = mydata1)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.7972 -2.1909 -0.7923  3.4421  5.0579 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -0.051174   0.338050  -0.151     0.88    
height       0.027859   0.001984  14.042   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

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F-statistic: 197.2 on 1 and 17868 DF,  p-value: < 2.2e-16
```

$\hat{\beta}_1$

**Slope**

**Test Results for Null that Slope=0**

# How Regression Output Looks in R

```
Call:
lm(formula = earnings ~ height, data = mydata1)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.7972 -2.1909 -0.7923  3.4421  5.0579 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -0.051174   0.338050  -0.151     0.88    
height       0.027859   0.001984  14.042   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.678 on 17868 degrees of freedom
Multiple R-squared:  0.01092,	Adjusted R-squared:  0.01086 
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```

**Regression Fit: R-Squared and SER**

# Additional Comments on Hypothesis Testing

- Remember, fundamental to conducting hypothesis testing in this way is having a large enough $n$ so we can apply the LLN and CLT to compute p-values and critical values for a given significance level $\alpha$
- We can also conduct hypothesis tests for the intercept of the regression model of the form:

$$H_0 : \beta_0 = \beta_{0,0} \quad \text{vs.} \quad H_1 : \beta_0 \neq \beta_{0,0}$$

  where R similar reports standard errors, t-statistics, and p-values for the OLS estimate $\hat{\beta}_0$
- The testing procedure is identical as it is for testing $\beta_1$

# One-sided Alternatives

- We can conduct one-sided hypothesis tests for the slope (and intercept) coefficients exactly as we did with sample means
- For the following test:

  $$H_0 : \beta_1 = 0 \;\text{ vs. }\; H_1 : \beta_1 > 0$$

  we would compute p-value$=1 - \Phi(t^{act})$
- For the following test:

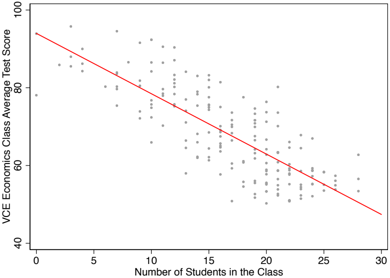
  $$H_0 : \beta_1 = 0 \;\text{ vs. }\; H_1 : \beta_1 < 0$$

  we would compute p-value$=\Phi(t^{act})$
- One-sided tests are rarely used with regression because we want to the data to tell us the sign of the slope of the regression line, not make an assumption about from the outset

# Hypothesis Testing with Class Size and Test Score Example

## Original Class Size and Test Score Dataset

- Original scatter plot of class size and test scores:




1. We can report the corresponding regression results as follows:

$$\widehat{TestScore}_i = \underset{(1.55)}{93.99} - \underset{(0.09)}{1.55}\, ClassSize_i, \quad R^2 = 0.63, SER = 7.24$$

where the numbers in parantheses () are $SE(\hat{\beta}_0)$ and $SE(\hat{\beta}_1)$

# Hypothesis Testing with Class Size and Test Score Example

## Original Class Size and Test Score Dataset

- With our results in hand, we are ready to conduct steps 2 and 3 of our hypothesis test:

$$H_0 : \beta_1 = 0 \text{ vs. } H_1 : \beta_1 \neq 0$$

2. The t-statistic for the test is computed as:

$$t^{act} = \frac{-1.55 - 0}{0.09} = -17.22$$

3a. The corresponding p-value for the test statistic is:

$$\text{p-value} = 2\Phi(-|-17.22|) = 2\Phi(-17.22) = 0.00001$$

We reject the null as our p-value is less than our level of significance: $0.00001 < 0.05 = \alpha$

- there is a $0.00001 \times 100 = 0.001\%$ chance that we obtain $\hat{\beta}_1 = -1.55$ or smaller if the null $H_0 : \beta_1 = 0$ is true
- extremely unlikely that $\beta_1 = 0$ given our estimate $\hat{\beta}_1$

# Hypothesis Testing with Class Size and Test Score Example

## Original Class Size and Test Score Dataset

3b. (Alternative test using $t^{act}$ and critical values)
Using $\alpha = 0.05$, we compare $t^{act}$ to the critical value of $t_{crit}^{\alpha=0.05} = 1.96$ and find:

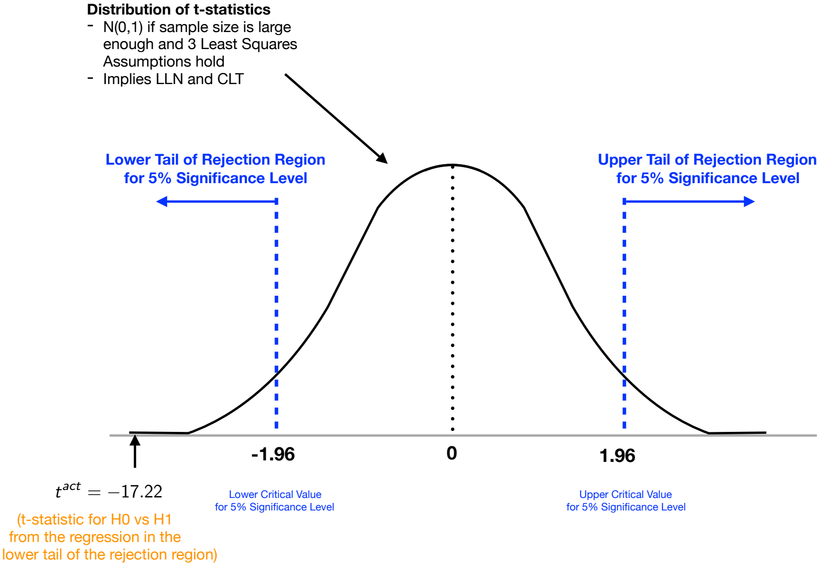
$$|t^{act}| = 17.22 > 1.96$$

which leads us to reject the null hypothesis of $H_0 : \beta_1 = 0$ in favour of the the alternative $H_1 : \beta_1 \neq 0$.

- The evidence (or hypothesis tests) does not support the politician's claim of no test score – class size relationship!

# Hypothesis Testing with Class Size and Test Score Example

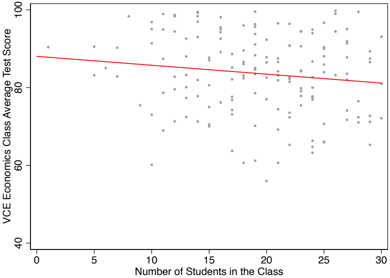
## Original Class Size and Test Score Dataset



# Hypothesis Testing with Class Size and Test Score Example

## Noisier Class Size and Test Score Dataset

- Consider the "no visual evidence" scatter plot of class size and test scores:




1. We can report the corresponding regression results as follows:

$$\widehat{TestScore}_i = \underset{(2.71)}{88.04} - \underset{(0.133)}{0.23}\ ClassSize_i, \quad R^2 = 0.018, SER = 10.34$$

where the numbers in parantheses are $SE(\hat{\beta}_0)$ and $SE(\hat{\beta}_1)$

# Hypothesis Testing with Class Size and Test Score Example

## Noisier Class Size and Test Score Dataset

- With our results in hand, we are ready to conduct steps 2 and 3 of our hypothesis test:

$$H_0 : \beta_1 = 0 \text{ vs. } H_1 : \beta_1 \neq 0$$

2. The t-statistic for the test is computed as:

$$t^{act} = \frac{-0.23 - 0}{0.13} = -1.77$$

3b. The corresponding p-value for the test statistic is:

$$\text{p-value} = 2\Phi(-|-1.77|) = 2\Phi(-1.77) = 0.089$$

We fail to reject the null as our p-value is greater than our level of significance: $0.089 > 0.05 = \alpha$

- there is a decent (8.9%) chance of obtaining a value $\hat{\beta}_1 = -0.22$ if the null $\beta_1 = 0$ is true

# Hypothesis Testing with Class Size and Test Score Example

## Noisier Class Size and Test Score Dataset

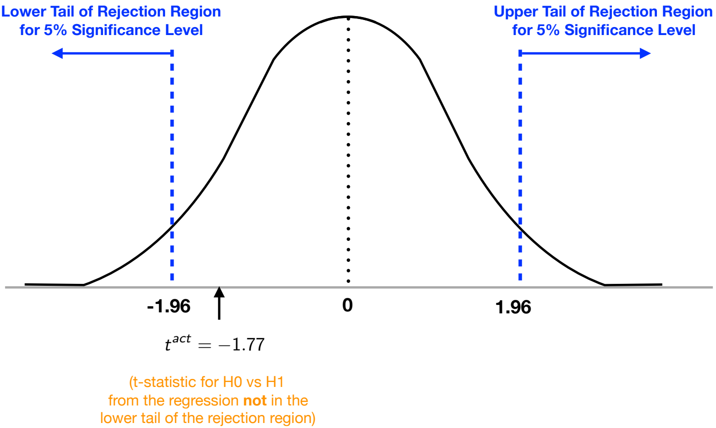
3b. (Alternative test using $t^{act}$ and critical values)
Using $\alpha = 0.05$, we compare $t^{act}$ to the critical value of $t_{crit}^{\alpha=0.05} = 1.96$ and find:

$$|t^{act}| = 1.77 < 1.96$$

which leads us to fail to reject the null hypothesis of $H_0 : \beta_1 = 0$ in favour of the the alternative $H_1 : \beta_1 \neq 0$.

- The evidence in this example is too noisy, and it does support the politician's claim of no test score – class size relationship!

# Hypothesis Testing with Class Size and Test Score Example



# Hypothesis Testing with Class Size and Test Score Example

## Testing Other Types of Null Hypotheses

- Suppose we wanted to run the following hypothesis test with our original dataset:

$$H_0 : \beta_1 = -1 \quad \text{vs.} \quad H_1 : \beta_1 \neq -1$$

1. Obtain regression results:

$$\widehat{TestScore}_i = \underset{(1.55)}{93.99} - \underset{(0.09)}{1.55}\, ClassSize_i, \quad R^2 = 0.63, SER = 7.24$$

2. The t-statistic for the test is computed as:

$$t^{act} = \frac{-1.55 - (-1)}{0.09} = -6.11$$

3b. Using $\alpha = 0.05$, we compare $t^{act}$ to the critical value of $t_{crit}^{\alpha=0.05} = 1.96$ and find:

$$|t^{act}| = 6.11 > 1.96$$

which leads us to reject the null hypothesis of $H_0 : \beta_1 = -1$ in favour of the the alternative $H_1 : \beta_1 \neq -1$.

# Confidence Intervals

- We can compute confidence intervals (CI) for $\beta_1$ in the exact same way that we computed CIs for the population mean $\mu_Y$ in Lecture Note 3
- Remember, the basic idea with CIs: with a random sample, we can never figure out exactly what the true value of $\beta_1$ is
- But we can use our OLS estimate $\hat{\beta}_1$, and its sampling distribution $N(\beta_1, \sigma^2_{\hat{\beta}_1})$, to determine a range in which the true value of $\beta_1$ is, with confidence $1 - \alpha$
- Also remember that if we choose a level of statistical significance $\alpha$ for hypothesis testing, the corresponding level of confidence for constructing CIs is $1 - \alpha$
- Throughout we will focus on CIs for $\beta_1$; CIs for $\beta_0$ are computed the exact same way

# Confidence Intervals

- Let's work with $\alpha = 0.05$ so we have a $1 - 0.05 = 0.95$ or 95% CI
- CIs have two equivalent definitions:
  - Definition 1: the 95% CI is the set of values for the null hypothesis that cannot be rejected using a two-sided hypothesis test with a 5% significant level
  - Definition 2: the 95% CI is an interval that has a 95% probability of containing the true value of $\beta_1$ from the population

# Confidence Intervals

## Constructing a 95% CI from the t-statistic

- What are the null $\beta_1$ values that are not rejected at the 5% level of significance?
  - recall that $t_\alpha = 1.96$ is the critical value for the test corresponding to $\alpha = 5\%$
- Using our rejection rule for a 2-sided hypothesis test. We cannot reject all null values of $\beta_1$ if

$$-1.96 < t^{act} = \frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)} < 1.96$$

- Re-arranging the inequalities, this is equivalent to saying that we cannot reject all null values of $\beta_1$ if

$$\hat{\beta}_1 - 1.96 SE(\hat{\beta}_1) < \beta_1 < \hat{\beta}_1 + 1.96 SE(\hat{\beta}_1)$$

# Confidence Intervals

## Constructing a 95% CI from the t-statistic

- Hence the 95% CI for $\beta_1$ is given by:

$$[\hat{\beta}_1 - 1.96SE(\hat{\beta}_1), \hat{\beta}_1 + 1.96SE(\hat{\beta}_1)]$$

- Interpretations:
  - given our data and our OLS estimates of $\hat{\beta}_1$ and $SE(\hat{\beta}_1)$, any null hypothesis value for $\beta_1$ that falls between $\hat{\beta}_1 - 1.96SE(\hat{\beta}_1)$ and $\hat{\beta}_1 + 1.96SE(\hat{\beta}_1)$ would <u>not</u> be rejected at the 5% level of significance
  - given our data and our OLS estimates of $\hat{\beta}_1$ and $SE(\hat{\beta}_1)$, there is a 95% chance that the true value of $\beta_1$ lies between $\hat{\beta}_1 - 1.96SE(\hat{\beta}_1)$ and $\hat{\beta}_1 + 1.96SE(\hat{\beta}_1)$

# CIs Application to Class Size and Test Scores

- OLS regression results:

$$\widehat{TestScore}_i = \underset{(1.55)}{93.99} - \underset{(0.09)}{1.55}\, ClassSize_i, \quad R^2 = 0.63, SER = 50.46$$

- 95% confidence interval is:

$$[-1.55 - 1.96 \times 0.09, -1.55 + 1.96 \times 0.09]$$

which is

$$[-1.73, -1.37]$$

- Given our data and OLS estimates, there is a 95% chance that the true value of $\beta_1$ lies in the range $[-1.73, -1.37]$
  - with 95% confidence, the true value of $\beta_1$ is as low as -1.73 (one additional student reduces test scores by 1.73 out of 100)
  - with 95% confidence, the true value of $\beta_1$ is as high as -1.37 (one additional student reduces test scores by 1.37 out of 100)

# Class Size and Test Scores - Other Common CIs

- 90% confidence interval is:

$$[-1.55 - 1.65 \times 0.09, -1.55 + 1.65 \times 0.09] = [-1.70, -1.40]$$

- 95% confidence interval is:

$$[-1.55 - 1.96 \times 0.09, -1.55 + 1.96 \times 0.09] = [-1.73, -1.37]$$

- 99% confidence interval is:

$$[-1.55 - 2.58 \times 0.09, -1.55 + 2.58 \times 0.09] = [-1.78, -1.32]$$

- Again, notice how how the confidence interval gets wider as the confidence level goes up
  - You need a wider interval of potential values of $\beta_1$ to be more confident that true value of $\beta_1$ lies in the interval

# CIs for Predicting Effects of Changing $X$

- Our general 95% CI formulas

$$[\hat{\beta}_1 - 1.96 SE(\hat{\beta}_1), \hat{\beta}_1 + 1.96 SE(\hat{\beta}_1)]$$

  are intervals for the average change in $Y$ corresponding to a one-unit change in $X$ of $\Delta X = 1$
- From our example, they correspond to the change in $Y =$ test scores from adding $\Delta X = 1$ student to a class
- The general 95% CI formula for the change in $Y$ for any $\Delta X$ value is given by

$$[(\hat{\beta}_1 - 1.96 SE(\hat{\beta}_1)) \times \Delta X, (\hat{\beta}_1 + 1.96 SE(\hat{\beta}_1)) \times \Delta X]$$

- That is, you multiply the upper and lower limits of the usual CI formula for $\Delta X = 1$ by $\Delta X$ to get the CI for predicted average change in $Y$ corresponding to a given value of $\Delta X$

# CIs for Predicting Effects of Changing $X$

- From our example, the 95% CI for the change in test scores from changing $\Delta X = 2$ (adding 2 additional students to a classroom) is:

  $$[(-1.55 - 1.96 \times 0.09) \times 2, (-1.55 + 1.96 \times 0.09) \times 2]$$

  which equals

  $$[-3.46, -2.74]$$

- Interpretation: Given our data and OLS estimates, there is a 95% chance the true impact on average test scores from adding 2 students to a classroom lies on the interval $[-3.46, -2.74]$

# Dummy Variables

- So far we have focused on a regressor $X$ that is continuous
- Regressions can also incorporate regressors that are binary variables, that is variables that take on the value 0 or 1
- Also called dummy variables and indicator variables, and are denoted by $D_i$ where either $D_i = 0$ or $D_i = 1$
- Examples of binary variables abound:
  - female $D_i = 1$, male $D_i = 0$
  - urban $D_i = 1$, rural $D_i = 0$
  - domestic $D_i = 1$, foreign $D_i = 0$

# Dummy Variables

- Regression model with $D_i$ as the regressor:

$$Y_i = \beta_0 + \beta_1 D_i + u_i$$

- Same regression model as we have been working with, except we have our dummy variable $D_i$ in where $X_i$ was before
- The $\beta_1$ coefficient has a very different interpretation, however: it is <u>not</u> a slope coefficient anymore
- We instead call $\beta_1$ the coefficient on $D_i$

# Dummy Variables - How do They Work?

- Regression model with $D_i$ as the regressor:

$$Y_i = \beta_0 + \beta_1 D_i + u_i$$

- Consider the model when $D_i = 0$ and when $D_i = 1$, that is, when the dummy is "switched off" and "switched on":

$$Y_i = \beta_0 + u_i \quad (D_i = 0)$$

and

$$Y_i = \beta_0 + \beta_1 + u_i \quad (D_i = 1)$$

# Dummy Variables - How do They Work?

- In terms of conditional expectations, we have:

$$E[Y_i|D_i = 0] = E[\beta_0 + \beta_1 \underbrace{D_i}_{0} + u_i|D_i] = \beta_0$$

and

$$E[Y_i|D_i = 1] = E[\beta_0 + \beta_1 \underbrace{D_i}_{1} + u_i|D_i] = \beta_0 + \beta_1$$

- In other words, the population mean is $\beta_0$ for the group where $D_i = 0$, and is $\beta_0 + \beta_1$ when $D_i = 1$
- Therefore, $\beta_1$ is the interpreted as the difference in the population mean between groups for which $D_i = 0$ and $D_i = 1$

# Dummy Variables Application to Class Size and Test Scores

THE AGE

NATIONAL VICTORIA

**Country students falling behind at school**

By Jewel Topsfield Education Editor
27 August 2014 — 3:44pm

- Politician says: "That is fake news! There is no gap between the academic performance of urban and regional students"
- Let's use our class size, test scores, and (new) urban/regional status dataset to rigorously shed some light on this debate
- The dummy variable of interest is $Urban_i$
  - $Urban_i = 1$ if class $i$ is in a location with $> 100,000$ people
  - $Urban_i = 0$ if class $i$ is in a location with $< 100,000$ people

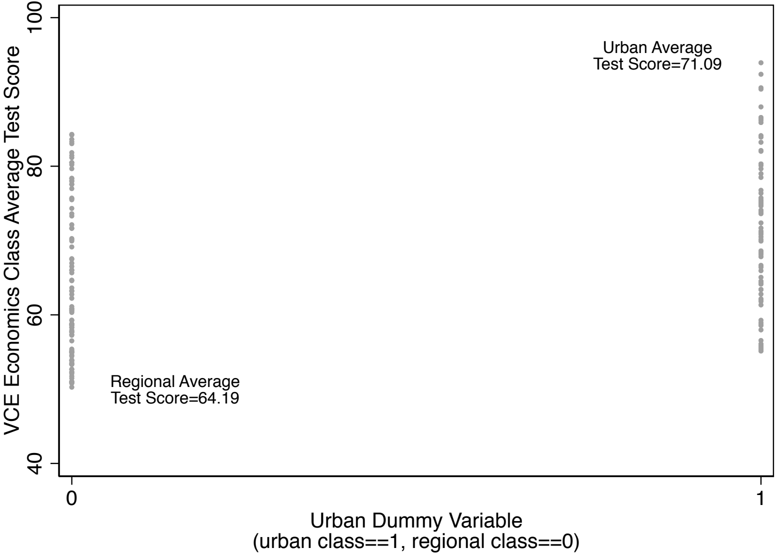
# Dummy Variables Application to Class Size and Test Scores

## Dummy Variables Data

| classid | class_size | grade | urban |
|---|---|---|---|
| 1 | 12 | 75.23 | 1 |
| 2 | 28 | 53.41 | 0 |
| 3 | 23 | 67.82 | 1 |
| 4 | 19 | 73.31 | 0 |
| 5 | 10 | 65.94 | 1 |
| 6 | 24 | 55.27 | 1 |
| 7 | 7 | 80.31 | 1 |
| 8 | 24 | 52.10 | 0 |
| 9 | 17 | 71.68 | 1 |
| 10 | 14 | 61.98 | 1 |
| 11 | 24 | 59.29 | 0 |
| 12 | 10 | 74.85 | 1 |
| 13 | 10 | 92.38 | 1 |
| 14 | 16 | 67.48 | 0 |
| 15 | 17 | 68.33 | 1 |
| 16 | 16 | 70.13 | 0 |
| 17 | 12 | 82.12 | 1 |
| 18 | 12 | 83.29 | 0 |
| 19 | 15 | 82.01 | 1 |
| 20 | 16 | 64.67 | 0 |
| 21 | 19 | 64.13 | 1 |
| 22 | 19 | 69.93 | 1 |
| 23 | 15 | 75.65 | 0 |
| 24 | 24 | 55.17 | 1 |
| 25 | 13 | 64.53 | 1 |

# Dummy Variables Application to Class Size and Test Scores

## Dummy Variables Graphically



# Dummy Variables Application to Class Size and Test Scores

## Dummy Variables Regression

- We estimate OLS regressions, conduct hypothesis tests, and construct confidence intervals with dummy variables $D_i$ *exactly* as we do with continuous regressors $X_i$
- Regression results based on the urban/rural dummy variable:

$$\widehat{TestScore}_i = \underset{(1.06)}{64.19} + \underset{(1.58)}{6.90}\, Urban_i, \quad R^2 = 0.10, SER = 10.37$$

- Hypothesis testing: the 6.90 coefficient on $Urban_i$ has:
  - t-statistic of $t^{act} = 4.38$
  - p-value of 0.000001
  - 95% CI of [3.80,9.99]
  - Strongly reject the null at the $\alpha = 0.05$ level of significance
- Evidence here is very worrisome for policy: students in urban markets tend to have 6.90 more points out of 100 on their test scores

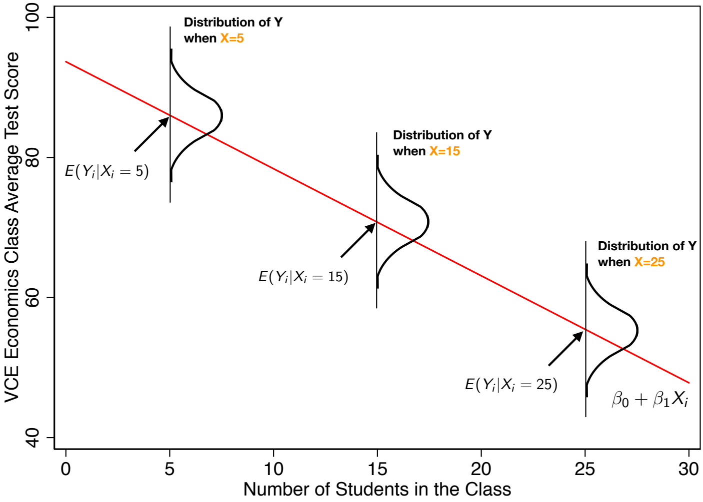
# Heteroskedasticity and Homoskedasticity

- Let's return to our standard regression model

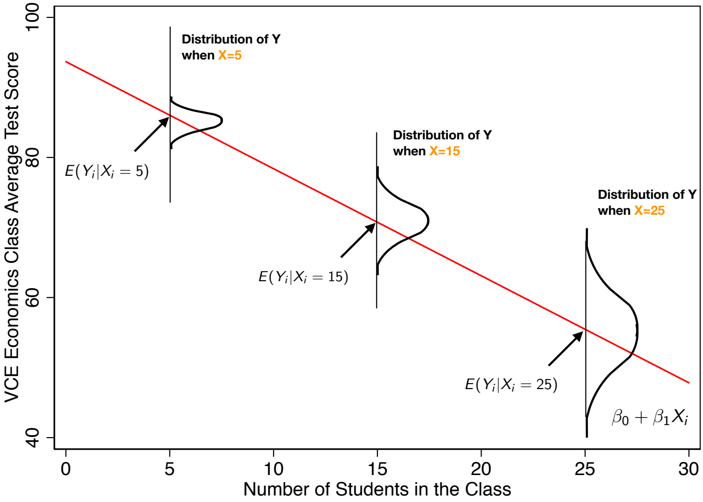
$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- Recall that throughout we have maintained 3 Least Squares Assumptions in estimating and testing our regression model:
  1. Independence of $u_i$ and $X_i$
  2. IID Sampling of $(X_i, Y_i)$
  3. No Outliers
- We can also characterise the variance of the errors $u_i$ given $X_i$, $\sigma^2_{u|X}$. We say the errors are ...
  - homoskedastic if $\sigma^2_{u_i|X_i}$ is <u>the same</u> no matter what $X_i$ is
  - heteroskedastic if $\sigma^2_{u_i|X_i}$ <u>varies</u> with $X_i$

# Homoskedasticity Graphically



# Heteroskedasticity Graphically



# Heteroskedasticity vs Homoskedasticity Example

- A good example from the text that highlights heteroskedasicity is the relation between earnings and gender
- Suppose we had a random sample on $Earnings_i$ for males and females, with a dummy variable $Male_i$ that equals 1 if person $i$ is male and 0 otherwise
- We could run the following regression:

$$Earnings_i = \beta_0 + \beta_1 Male_i + u_i$$

  where $\beta_1$ is the difference in mean earnings between males and females
- We could also assume homoscedastic errors, then $u_i$ does not depend the regressor, which in this example is $Male_i$.
- This is equivalent to assuming that the variance of earnings is the same for men as it is for women
- Problem: we know that this is just plain wrong! Males tend to have more dispersed earnings than females.

# Heteroskedasticity vs Homoskedasticity: Who Cares?

- The variance of $u_i$ conditional on $X_i$ underlies the variance of the residuals $\hat{u}_i$ from an OLS regression
- The variance of the residuals in turn is critical for the calculation of the standard error of $\hat{\beta}_1$, $SE(\hat{\beta}_1)$
- Finally, $SE(\hat{\beta}_1)$ directly enters into our calculations of t-statistics and confidence intervals
- So ultimately the variance of $u_i$ is a critical component to hypothesis testing
- Bottom line: if we get the variance of $u_i$ conditional on $X_i$ wrong, our hypothesis tests are wrong!
- Note: regardless of whether we assume heteroskedasticity vs homoskedasticity, OLS is unbiased $(E(\hat{\beta}_1) = \beta_1)$

# A Key Result Under Homoskedasticity

- Homoskedasticity underlies an important theoretical property of OLS estimators:
- Gauss-Markov Theorem: Under the 3 Least Squares assumptions, and if $u_i$ is homoskedastic, then the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ is the Best Linear Unbiased Estimator of the linear regression true parameters $\beta_0$ and $\beta_1$
- That is, the OLS is BLUE

# A Key Result Under Homoskedasticity

- Let's break down what "OLS is BLUE" means
- Best: the OLS estimator is the most efficient estimator (i.e., smallest sampling variance, so smallest standard errors) among all the estimators that are linear in $Y_1, \ldots, Y_n$.
- Linear: OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are linear functions of $Y_1, \ldots, Y_n$, which they are:
  - $\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})}$
  - $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$
- Unbiased: expected values of the OLS estimators are equal to their true values in the population:

$$E(\hat{\beta}_1) = \beta_1 \text{ and } E(\hat{\beta}_0) = \beta_0$$

- Estimator: OLS values $\hat{\beta}_1$ and $\hat{\beta}_1$ are estimators of population true values $\beta_1$ and $\beta_0$

# BUT.....

- While OLS is BLUE is a key result in theory, it is virtually never relevant in practice
- In general, the variance of $u_i$ does vary with $X_i$, like the earnings and male/female example highlighted
- With continuous variables like income or age, the variance of $u_i$ almost always varies with $X_i$
- Because heteroskedasticity is a prominent practical phenomenon in data, we always do econometrics assuming heteroskedasticity in the errors
- Therefore, in conducting hypothesis tests and constructing confidence intervals, we compute $SE(\hat{\beta}_0)$ and $SE(\hat{\beta}_1)$ using heteroskedasticity-robust standard errors, which are often called robust standard errors or White standard errors
    - in honour of the late Prof. Halbert White who would have won the Nobel Prize before he passed away in 2012

# Heteroskedastic Standard Errors

- The complex formulas for heteroskedasticity robust standard errors are in the textbook
- In practice statistical programs like R readily compute robust standard errors for you
- In practice, if you incorrectly assume homoskedasticity, you will compute incorrect $SE(\hat{\beta}_0)$ and $SE(\hat{\beta}_1)$!

# Heteroskedastic Standard Errors

- In contrast, if you use robust standard errors when the errors are in fact homoskedastic, you will still obtain correct standard errors by using robust standard errors $\rightarrow$ it's the safe choice!
- Therefore, for the remainder of the subject, we will assume heteroskedasticity robust standard errors in conducting all hypothesis tests and in computing confidence intervals (and will use R to compute them)

Note: in the text, we stop at the end of Section 5.4; ignore Section 5.5 (Theoretical Foundations of OLS) and Section 5.6 (t-Statistics in Regression When Sample Size is Small)