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MAT4MDS **Practice 3**

MATRICES

- Matrices of the same size can be added (or subtracted) element-by-element.
- To multiply a matrix by a scalar, multiply each entry of the matrix by the scalar.
- The transpose of a matrix A is the matrix A^T formed by interchanging the rows and columns of A, e.g. if

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 0 \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 0 \end{bmatrix}$$

- Two matrices A and B can be multiplied to give AB if A is $\ell \times m$ and B is $m \times n$ (where $\ell, m, n \in \mathbb{N}$). The resulting matrix AB is $\ell \times n$.
  We multiply **across-the-rows by down-the-columns**, e.g.

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 0 + 2 \cdot 1 + 1 \cdot (-1) \\ 1 \cdot 0 + 4 \cdot 1 + 0 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

  In the example $(2 \times 3)(3 \times 1) \rightsquigarrow (2 \times 1)$.

Consider the following matrices:

$$C = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & -3 \\ 4 & -2 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}.$$

Question 1. Calculate the following, if it is possible. If it is not possible, give a brief reason why it is not.

(a) $C + D$ (b) $C + E$ (c) $3F$ (d) $G - 4D$ (e) $F + F^T$ (f) $E + E^T$.

Question 2. One application of matrices is in the least-squares method of fitting experimental data to a curve. When p data points are fitted with a polynomial curve of order q, a matrix M of size $p \times q$ is defined. In applying the method, the matrix $M^T M$ must be calculated. What size is this matrix?

Question 3. Calculate the following, if it is possible. If it is not possible, give a brief reason why it is not.

(a) CE (b) FE (c) EC (d) EF (e) CD (f) DC (g) DG.

What do you notice about the order of matrix multiplication? Can you make a general conclusion about the **commutativity** of matrix multiplication?

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Question 4. Using answers from Question 3, calculate the following:

$$C(DG) \qquad \text{and} \qquad (CD)G.$$

What do you notice? Can you make a general conclusion about the **associativity** of matrix multiplication?

Question 5. Properties of the transpose matrix.

(a) Form the matrix E^T, and then transpose again i.e. find $(E^T)^T$. What do you notice? Can you explain why this is true for all matrices?

(b) What do you notice about $F + F^T$ formed in Question 1? (Hint: Try transposing.) Do you think this is true for all matrices?

(c) Which of $C^T D^T$ or $D^T C^T$ do you expect to be $(CD)^T$? Check your answer.

> The **determinant** of a 2×2 matrix is: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Question 6. Find each of the following determinants:

(a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ (b) $\det \begin{pmatrix} 2 & -3 \\ 5 & 7 \end{pmatrix}$ (c) $\begin{vmatrix} 1 & -2 \\ -3 & -4 \end{vmatrix}$ (d) $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T$.

> The **determinant** of a 3×3 matrix is given by:
>
> $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Question 7. Find each of the following determinants by expanding in terms of smaller determinants. (No other method is acceptable.)

(a) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 3 & 4 & 1 \end{vmatrix}$, (b) $\det \begin{bmatrix} 0 & -2 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$, (c) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 0 \end{vmatrix}$, (d) $\begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 10 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix}$.

For each order $(m \times n)$, the **matrix of ones** $J_{m\times n}$ is the matrix in which every entry is 1. The square $(n \times n)$ matrix of ones is usually written as J_n.

Question 8.

(a) Calculate the following:

(i) $\det(J_{2\times 2})$ (ii) $(J_{2\times 2})^2$ (iii) $J_{m\times 1}^T J_{m\times 1}$

(b) Considering your answers in part (a), what will the following be, and why?

(i) $\det(J_n)$ (ii) $(J_n)^2$

Question 9. Consider the following data set, arranged in a matrix D. (This is part of a well-known data set used by the statistician Fisher, and collected by the biologist Anderson.) Each row in this matrix represents an iris (flower) specimen, and the columns correspond to sepal length in cm, sepal width in cm, petal length in cm, and petal width in cm.

$$D = \begin{bmatrix} 5.0 & 3.4 & 1.5 & 0.2 \\ 4.4 & 2.9 & 1.4 & 0.2 \\ 4.9 & 3.1 & 1.5 & 0.1 \\ 5.4 & 3.7 & 1.5 & 0.2 \\ 4.8 & 3.4 & 1.6 & 0.2 \\ 4.8 & 3.0 & 1.4 & 0.1 \\ 4.3 & 3.0 & 1.1 & 0.1 \\ 5.8 & 4.0 & 1.2 & 0.2 \\ 5.7 & 4.4 & 1.5 & 0.4 \\ 5.4 & 3.9 & 1.3 & 0.4 \\ 5.1 & 3.5 & 1.4 & 0.3 \\ 5.7 & 3.8 & 1.7 & 0.3 \\ 4.7 & 3.2 & 1.6 & 0.2 \end{bmatrix}$$

(a) D has 13 rows. What is the meaning of the entries in the matrix product $J_{13}D$? (Note: you are not being asked to calculate this product, just interpret it.)

(b) What is the meaning of the entries in the matrix $\frac{1}{13}J_{13}D$?

(c) Finally, what is the result of forming the product $C_{13}D$, where $C_{13} := I_{13} - \frac{1}{13}J_{13}$?

Question 10. The matrix $C_n := I_n - \frac{1}{n}J_n$ is called the centering matrix.

(a) Why is C_n called the centering matrix?

(b) Form and simplify the product C_n^2 (using properties of I_n and J_n). Explain this result in terms of the action of C_n.

When it exists, the **inverse** of an $n \times n$ matrix A is the unique matrix, denoted by A^{-1}, with the property

$$AA^{-1} = I_n,$$

where I_n is the $n \times n$ identity matrix. Thus,

$$AB = I_n \implies A^{-1} = B.$$

We also have $BA = A^{-1}A = I_n$.

Question 11. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(a) Find AB. and BA. (b) Is B the inverse of A? (c) Does A have an inverse?

A Key Property of Determinants:
A square matrix has an inverse if and only if its determinant is non-zero.

2×2 inverse: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has non-zero determinant, then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Question 12. Consider the following matrices:

$$C = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & -3 \\ 4 & -2 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}.$$

(a) Use determinants to decide which of these matrices are invertible.

(b) For each of the 2×2 invertible matrices identified in (a), write down the inverse.

Consider the matrix equation $AX = B$, where A and B are known matrices and X is unknown. If A is an invertible $n \times n$ matrix, then we can solve for X by **multiplying on the left** by A^{-1}:

$$\begin{aligned} AX = B &\implies A^{-1}AX = A^{-1}B \\ &\implies \quad I_nX = A^{-1}B \implies X = A^{-1}B. \end{aligned}$$

Note that we have used $A^{-1}A = I_n$ and $I_nX = X$.

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Question 13. In each of the following cases, use the method in the box above to solve the equation $AX = B$ for X. (Check your answers by verifying that $AX = B$.)

(a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, and $X = \begin{pmatrix} x \\ y \end{pmatrix}$. (b) $A = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$, and $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Multiplicative Property of Determinants: $\det(AB) = \det(A)\det(B)$.

Question 14. Let A be an invertible $n \times n$ matrix.

(a) Use the fact in the box above to write $\det(A^{-1})$ in terms of $\det(A)$. (You may assume that $\det(I_n) = 1$.)

(b) Assume that A is an invertible 2×2 matrix. Use the formula for A^{-1} given in the second box on page 1 to verify that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

(c) For the matrices D and G of Question 2 (and the inverses that you found for them):

(i) Check that the property in (a) is satisfied.

(ii) Calculate DG and GD and $|DG|$ and $|GD|$. Check that the multiplicative property of determinants is satisfied for these matrices.

(iii) Which of $D^{-1}G^{-1}$ or $G^{-1}D^{-1}$ do you expect to be $(DG)^{-1}$? Check your answer.