MAST30001 Stochastic Modelling – 2016

Assignment 2

If you didn't already hand in a completed and signed Plagiarism Declaration Form (available from the LMS or the department's webpage), please do so and attach it to the front of this assignment.

Don't forget to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 21 October by 4pm** in the appropriate assignment box at the north end of Richard Berry Building (near Wilson Lab).

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness and style count.

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1. Assume that rainstorms arrive in Melbourne according to a Poisson process with rate 1/2 per day, the amount of rain that falls during each storm is exponentially distributed with *mean* 4 mm, and the amount of rain that falls in separate storms are independent of each other and the times of arrivals of the storms. Let $(N_t)_{t\geq 0}$ be the number of storms between now and t days from now and $(R_t)_{t\geq 0}$ be the amount of rainfall between now and t days from now.

   (a) What is the expected amount of rainfall in Melbourne over the 5 days?

   (b) What is the probability that over the next 5 days, there are exactly 3 storms, 2 of which have rainfall amounts less than 4 mm, and 1 with a rainfall amount more than 4 mm?

   (c) What is the probability that over the next 5 days, there are exactly 3 storms, 2 of which have rainfall amounts less than 4 mm, and 1 with a rainfall amount more than 6 mm?

   (d) Given that there is exactly 1 storm with a rainfall amount more than 6 mm in the next 5 days, what is the expected number of storms with a rainfall amount less than 7 mm over this same time period?

   Ans.

   Note that $R_t = \sum_{j=1}^{N_t} X_j$, where X_j is the amount of rainfall of the jth storm that occurs in the future. The X_j are i.i.d. and independent of $(N_t)_{t\geq 0}$

   (a) Using the conditional mean formula, $E[R_5] = E\{E[R_5|N_5]\} = E[N_5 E[X_1]] = 10.$

   (b) If M_t denotes the number of storms with rainfall amounts less than 4 mm, then we want to compute

   $$P(M_5 = 2, N_5 - M_5 = 1).$$

   The thinning theorem says that $(M_t)_{t\geq 0}$ and $(N_t - M_t)_{t\geq 0}$ are *independent* Poisson processes with rates

   $$\frac{5}{2}P(X_i < 4) = \frac{5(1-e^{-1})}{2} = 1.5803, \quad \frac{5}{2}P(X_i > 4) = \frac{5e^{-1}}{2} = 0.9197,$$

so that

$$\begin{aligned} P(M_5 = 2, N_5 - M_5 = 1) &= P(M_5 = 2)P(N_5 - M_5 = 1) \\ &= \frac{e^{-1.5803}(1.5803)^2}{2} e^{0.9197}(0.9197) \\ &= 0.0943. \end{aligned}$$

(c) If L_t denotes the number of storms with rainfall amounts more than 6 mm, then we want to compute

$$P(M_5 = 2, L_5 = 1, N_5 - M_5 - L_5 = 0).$$

But $(L_t)_{t\geq 0}$ is a further thinning of $(N_t - M_t)_{t\geq 0}$ (independent of $(M_t)_{t\geq 0}$) and so L_5 is a Poisson process with rate $(5/2)P(X_i > 6) = 5e^{-3/2}/2 = 0.5578$ which is independent of $(M_t)_{t\geq 0}$ and $(N_t - M_t - L_t)_{t\geq 0}$. We can now compute

$$\begin{aligned} P(M_5 = 2, L_5 = 1, &N_5 - M_5 - L_5 = 0) \\ &= P(M_5 = 2)P(L_5 = 1)P(N_5 - M_5 - L_5 = 0) \\ &= \frac{e^{-1.5803}(1.5803)^2}{2} e^{-0.5578}(0.5578)e^{-0.36187} \\ &= 0.0572. \end{aligned}$$

(d) Let $(U_t, V_t, W_t)_{t\geq 0}$ denote the number of storms with rainfall amounts less than 6 mm, between 6 and 7 mm, and greater than 7 mm (to relate to previous notation: $V_t + W_t = L_t$ and $U_t + V_t + W_t = N_t$). In this notation, we want to compute $E[U_5 + V_5|V_5 + W_5 = 1]$. We first compute

$$\begin{aligned} E[V_5|V_5 + W_5 = 1] &= P(V_5 = 1|V_5 + W_5 = 1) \\ &= \frac{P(V_5 = 1, W_5 = 0)}{P(V_5 + W_5 = 1)} \\ &= \frac{e^{-0.1234}(0.1234)e^{-0.4344}}{e^{-0.5578}(0.5578)} \\ &= 0.2212. \end{aligned}$$

These calculations follow since V_5 is Poisson with mean $(5/2)P(6 < X_i < 7) = 0.1234$, independent of W_5 which is distributed Poisson with mean $(5/2)P(X_i > 7) = 0.4344$, and $V_5 + W_5 (= L_t)$ is distributed Poisson with mean 0.5578.

Finally, U_5 is independent of $V_5 + W_5$, so we have $E[U_5|V_5 + W_5 = 1] = E[U_5] = (5/2)(1 - e^{-3/2}) = 1.942$, and

$$E[U_5 + V_5|V_5 + W_5 = 1] = 1.9422 + 0.2212 = 2.1634.$$

2. Customers arrive to a queuing system according to a Poisson process with rate 4 per hour. If there are fewer than 3 people in the queue, then an arriving customer will join the queue, and otherwise will leave the system. At exponential rate 2 (per hour) times, a server arrives and instantaneously serves all customers in the queue (if there is no one in the system, the server does nothing and exits the system).

(a) What is the long run proportion of time there is no one in the queue?

(b) What is the average number of customers in the system?

(c) What is the expected amount of time that customers that enter the system have to wait for service?

(d) What is the long run proportion of arriving customers that enter the system?

(e) Given an arriving server finds customers in the system, what is the expected number of customers served?

(f) If X_t denotes the number of customers in the system at time $t \geq 0$, find $P(X_t = 0 | X_0 = 0)$.

Ans.

We can view this system as a CTMC with states $\{0, 1, 2, 3\}$ with generator matrix For this CTMC the A is given by, for $\lambda = 4, \mu = 2$,

$$\begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 \\ \mu & 0 & -(\mu+\lambda) & \lambda \\ \mu & 0 & 0 & -\mu \end{pmatrix},$$

and it is ergodic since it's irreducible with a finite state space. The steady state regime is given by the stationary distribution π satisfying $\pi A = 0$, which is solved by (now putting in numbers)

$$\pi = \frac{1}{27}(9, 6, 4, 8).$$

(a) $\pi_0 = 1/3$.

(b) The average number of customers in the system is

$$\pi_1 + 2\pi_2 + 3\pi_3 = 38/27 = 1.4074.$$

(c) When a customer enters the system, they only need to wait until the next server arrives, which is distributed exponential rate 2 (memoryless property), which has mean 1/2.

Note that from (b), the expected length of the queue $L_q = 38/27$ and so using Little's law naively gives for the average waiting time $L_q/\lambda = 17/57 \neq 1/2$. However, using PASTA, the rate of cars *entering* the system is $\lambda(1 - \pi_3) = 4 \cdot 19/27 = 2 \cdot 38/27$ and with this correct rate, Little's law gives $L_q/(\lambda(1 - \pi_3)) = 38/(2 \cdot 38) = 1/2$.

(d) As noted in the second part of the solution to (b), the long run proportion of customers entering the system are those not finding 3 customers in the queue when they arrive, which, according to PASTA, is $1 - \pi_3 = 19/27$.

(e) If there are customers in the system, either a server arrives before a new customer, this happens with probability $\mu/(\lambda + \mu) = 1/3$ (competing exponential clocks), or a new customer arrives with the complementary probability. Starting from the system being empty, a new customer eventually arrives. The probability the next service then serves 1 customer is 1/3 (the chance of a service before an arrival). The probability the next service serves 2 customers is $(2/3)(1/3)$ since we need a customer to arrive (with probability 2/3) and then for a service to occur (with probability 1/3). Finally, the probability the next service serves 3 customers is $(2/3)^2 = 1 - 1/3 - (2/3)(1/3)$ (since once there are 3 customers in the system the next service will always have 3 customers). Therefore the expected number of customers served by a non-empty service is

$$\frac{1}{3} + 2\frac{2}{9} + 3\frac{4}{9} = \frac{19}{9} = 2.111.$$

Another way to do the problem is to note that according to PASTA, an arriving server finds the system in stationary. Thus the distribution of the number of customers in the queue found by a non-empty service is the stationary distribution conditioned to be positive, i.e.,

$$(\pi'_1, \pi'_2, \pi'_3) = (1/3, 2/9, 4/9),$$

and taking expectation against this yields 19/9.

(f) If $p_{i,j}(t) := P(X_t = j | X_0 = i)$, then the Kolmogorov forward equations applied to the generator matrix of part (a) imply

$$\begin{aligned}\frac{d}{dt}p_{0,0}(t) &= -\lambda p_{0,0}(t) + \mu(p_{0,1}(t) + p_{0,2}(t) + p_{0,3}(t)) \\ &= -\lambda p_{0,0}(t) + \mu(1 - p_{0,0}(t)),\end{aligned}$$

where we use that $p_{0,0}(t) + p_{0,1}(t) + p_{0,2}(t) + p_{0,3}(t) = 1$. Solving this equation with the initial condition $p_{0,0}(0) = 1$ implies

$$p_{0,0}(t) = \frac{\mu + \lambda e^{-(\lambda+\mu)t}}{\lambda + \mu} = \frac{1 + 2e^{-6t}}{3}.$$