THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2022

School of Mathematics and Statistics

MAST30001 Stochastic Modelling

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 15 pages (including this page) with 6 questions and 80 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).
- No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.
- You must not write your answers on an iPad or other electronic device.
- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.
- If you are writing answers on the exam or masked exam and you need more space, use blank paper. Note this in the answer box, so the marker knows.
- If you are only writing on blank A4 paper, the first page must contain only your student number, subject code and subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning and Submitting

- **You must not leave Zoom supervision to scan your exam.** Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4.
- Submit your scanned exam as a single PDF file and carefully review the submission in Gradescope. Scan again and resubmit if necessary. Do not leave Zoom supervision until you have confirmed orally with the supervisor that you have received the Gradescope confirmation email.
- **You must not submit or resubmit after having left Zoom supervision.**

Question 1 (20 marks)
A Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{1, 2, 3, 4, 5\}$ has transition matrix

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 \end{pmatrix}.$$

(a) Find the value of p.

(b) Assuming X_0 is uniformly distributed on the set $\{1, 2, 3, 5\}$, find

(i) $\mathbb{P}(X_5 = 3, X_2 = 1 | X_1 = 2)$, and

(ii) $\mathbb{P}(X_3 = 4)$.

(c) Write down the communication classes of the chain. For each class, find the period, determine whether it is essential, and classify it as transient or positive recurrent or null recurrent.

(d) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

(e) Find the expected number of steps to reach state 2 given the chain starts in state 1.

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Question 2 (14 marks)

A group of Galahs perch on a wire with random spacing between them. The wire is attached to a pole on one end and has a mark 10 metres from the pole. We model the positions of Galahs as follows: The Galah closest to the pole perches a random number of centimetres from it, having density

$$f(x) = \frac{3}{4000}(x-10)(30-x), \;\; 10 < x < 30.$$

The next closest Galah perches an independent and random distance from the first one, having a distribution with the same density, and so on.

(a) Estimate the number of Galahs in between the pole and the mark on the wire, and give a symmetric interval around your estimate that has a 95% chance of covering the true number of Galahs there.

(b) Estimate the probability that there are no Galahs perched on the wire within 10 centimetres on either side of the mark.

(c) What is the expected value of the distance between the mark and the Galah closest to the mark?

Question 3 (13 marks)
Customers arrive to a queueing system with two servers (called A and B) according to a Poisson process with rate 2. If an arriving customer finds a server available, they enter service immediately, and they choose a server at random if both are available. If both servers are busy and there is no queue, an arriving customer queues for Server A, waiting for service from A even if the customer getting served by B finishes their service sooner than the customer being served by A. Customers that arrive when there are three customers in the system leave without service. Service times are independent and exponential and Server A works at rate 2 and Server B works at rate 1.

(a) Model this system as a continuous time Markov chain and write down its state space and generator.

(b) Derive the stationary distribution of the chain.

(c) What is the average number of customers in the system in the stationary regime?

(d) What proportion of customers are rejected from the system?

(e) Given that a customer is not immediately rejected from the system, what is the average time they spend in the system?

Question 4 (16 marks)
Emails arrive to your inbox according to a Poisson process with rate 4 per hour. Emails are "important" with probability 1/4, and important emails require a response with probability 2/3, independently of each other and of the arrival process.

(a) What is the chance that at least 2 emails are received in a 15 minute time period?

(b) What is the chance that exactly 2 important emails are received in a 15 minute time interval?

(c) Given that 5 emails were received in a given hour, what is the chance that exactly 3 were received in the first 20 minutes of the hour?

(d) What is the chance that exactly 3 important emails requiring a response and 2 unimportant emails are received in a given hour?

(e) Given that 5 important emails that need a response were received in a given hour, what is the chance that exactly 3 unimportant emails were received in the first 20 minutes of the hour?

(f) Assume now some spammer gets your email address and independently sends you additional emails according to a Poisson process with rate 1 per hour. What is the chance of getting altogether at least 2 emails in a given hour?

Question 5 (12 marks)
Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. For any normal probabilities below, you can write your answer in terms of the standard normal distribution function $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2} dt$.

(a) Compute $\mathbb{P}(B_5 \geq -\sqrt{2} | B_2 = -1/\sqrt{2})$.

(b) Compute $\mathbb{P}(B_2 \geq -\sqrt{2} | B_5 = -1/\sqrt{2})$.

(c) Fix $x > 0$ and let the process $(W_t)_{0 \le t \le 1}$ be distributed as $\big((B_t)_{0 \le t \le 1} | B_1 = x\big)$.

(i) Compute the density of $M := \max_{0 \le t \le 1} W_t$.
Hint: You can understand $\big((B_t)_{0 \le t \le 1} | B_1 = x\big)$ *as* $\lim_{\varepsilon \downarrow 0} \big((B_t)_{0 \le t \le 1} | B_1 \in (x \pm \varepsilon)\big)$.

(ii) Show that $(W_t)_{0 \le t \le 1}$ is a Gaussian process, and identify its parameters.

Question 6 (5 marks)
Below is some R code for simulating the trajectories of a Markov chain with state space $\{1, 2, 3\}$.

```
library(expm)

n<-10000
m<-100

p0<-c(1/3,1/3,1/3)
P<-matrix(c(1/2,1/3,1/6,1/4,1/4,1/2,2/5,0,3/5),nrow=3,byrow=T)

count<-matrix(rep(0,3*n),nrow=3,byrow=T)

for (i in 1:n){
  x<-rep(0,m+1)
  x[1]<-sample(1:3,1,prob = p0)
  for (j in 2:(m+1)){
    x[j]<-sample(1:3,1,prob = P[x[j-1],])
  }
  for(k in 1:3){
    count[k,i]<-sum(x==k)
  }
}

m1<-mean(count[1,]/(m+1))
m2<-mean(count[2,]/(m+1))
m3<-mean(count[3,]/(m+1))
```

(a) What is the transition matrix and the initial distribution of the Markov chain?

(b) Compute the values (up to simulation error) of the outputs `m1`, `m2`, `m3`.

End of Exam — Total Available Marks = 80