4.4 The simplex algorithm

The key idea of the simplex algorithm (not to be confused with the downhill simplex method) is to start at one corner of the feasible region, and then move around the corners in such a way that the objective function increases with every step, until finally the optimal solution is reached. The algorithm doesn't necessarily give the shortest route from the origin to the optimal solution; it's not the perfect algorithm. Nevertheless it is very clever; it is a relatively simple algorithm to perform, and it is remarkably efficient.

In this section, we will show how to solve a linear program of the form

$$\begin{aligned} \text{maximise} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}, \end{aligned}$$

with $\mathbf{b} \geqslant \mathbf{0}$. Later we will see what to do with constraints of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$ or $\mathbf{A}\mathbf{x} \geqslant \mathbf{b}$. The main advantage of this form is that, after having introduced slack variables, the corresponding augmented matrix is canonical. Thus we immediately have a basic feasible solution at our disposal. Geometrically, this corresponds to starting the search at the origin.

The idea is then to replace one of the basic variables by another variable. The new basic variable is called the **entering basic variable** and the variable we replace is called the **departing basic variable**. The basic framework for the **simplex algorithm** (which we will refine in the next section) is as follows:

1. Choose the entering basic variable x_e to be the variable with respect to which the partial derivative of z is largest and positive. This corresponds to the variable with the most negative entry in the bottom row.
2. When moving in the x_e-direction, stop as soon as you hit the corner point of the feasible region. This can be checked by applying the **minimum ratio test**. For an entering basic variable in column e, and an augmented matrix
$$\left(\begin{array}{ccccc|c} a_{11} & \dots & a_{1e} & \dots & a_{1n} & b_1 \\ a_{21} & \dots & a_{2e} & \dots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{me} & \dots & a_{mn} & b_m \end{array}\right),$$
compute the ratio b_j/a_{jk} for each $j \leqslant m$ with $a_{je} > 0$. The basic variable x_d in the row with smallest such ratio is chosen as the departing basic variable.
3. Replace the basic variable x_d by basic variable x_e and use row operations to find the canonical form. This process is called **pivoting** on the entry in the e-th column and the d-th row.
4. If there is a variable with a negative entry in the bottom row, repeat all steps. Otherwise, stop.

Let's now apply this to the main example,

$$\left(\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 70 \\ 1 & 1 & 0 & 1 & 0 & 40 \\ 1 & 3 & 0 & 0 & 1 & 90 \\ -40 & -60 & 0 & 0 & 0 & 0 \end{array}\right).$$

The most negative entry occurs in column 2, so x_2 is the entering basic variable. To choose the departing variable, first calculate the ratio of each b_i with the corresponding entries in column 2, provided that the entries in that column are positive:

$$70/1 = 70 \qquad 40/1 = 40 \qquad 90/3 = 30.$$

The smallest value was obtained from from row 3, so the departing basic variable is the basic variable that appears in row 3 (in this case, x_5), and we pivot on row 3, column 2.

To indicate the entry we are pivoting on, we will surround that entry by a box. The first step is:

$$\left(\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 70 \\ 1 & 1 & 0 & 1 & 0 & 40 \\ 1 & \boxed{3} & 0 & 0 & 1 & 90 \\ -40 & -60 & 0 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{ccccc|c} 2 & 1 & 1 & 0 & 0 & 70 \\ 1 & 1 & 0 & 1 & 0 & 40 \\ 1/3 & 1 & 0 & 0 & 1/3 & 30 \\ -40 & -60 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_3' = \tfrac{1}{3}R_3$$

$$\equiv \left(\begin{array}{ccccc|c} 5/3 & 0 & 1 & 0 & -1/3 & 40 \\ 2/3 & 0 & 0 & 1 & -1/3 & 10 \\ 1/3 & 1 & 0 & 0 & 1/3 & 30 \\ -20 & 0 & 0 & 0 & 20 & 1800 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 - R_3 \\ R_2' = R_2 - R_3 \\ \\ R_4' = R_4 + 60R_3 \end{array}$$

We can read the current corner point of the algorithm directly from this matrix. Before introducing slack variables, the variables we had were x_1 and x_2. As x_1 is non-basic, we set it equal to zero, and row 3 indicates that x_2 is 30, which gives $(x_1, x_2) = (0, 30)$, one of the corner points of the feasible region. As there is still a negative entry in the bottom row, we have not found the optimal solution. So we repeat the steps again.

There is only one negative entry, in column 1, so the entering basic variable is x_1. Computing each ratio:

$$40/(5/3) = 24 \qquad 10/(2/3) = 3/2 \qquad 30/(1/3) = 90.$$

The smallest occurs in row 2, so the departing basic variable is x_4 and we pivot on the entry in row 2, column 1:

$$\left(\begin{array}{ccccc|c} 5/3 & 0 & 1 & 0 & -1/3 & 40 \\ \boxed{2/3} & 0 & 0 & 1 & -1/3 & 10 \\ 1/3 & 1 & 0 & 0 & 1/3 & 30 \\ -20 & 0 & 0 & 0 & 20 & 1800 \end{array}\right) \equiv \left(\begin{array}{ccccc|c} 5/3 & 0 & 1 & 0 & -1/3 & 40 \\ 1 & 0 & 0 & 3/2 & -1/2 & 15 \\ 1/3 & 1 & 0 & 0 & 1/3 & 30 \\ -20 & 0 & 0 & 0 & 20 & 1800 \end{array}\right) \quad R_2' = \tfrac{3}{2}R_3$$

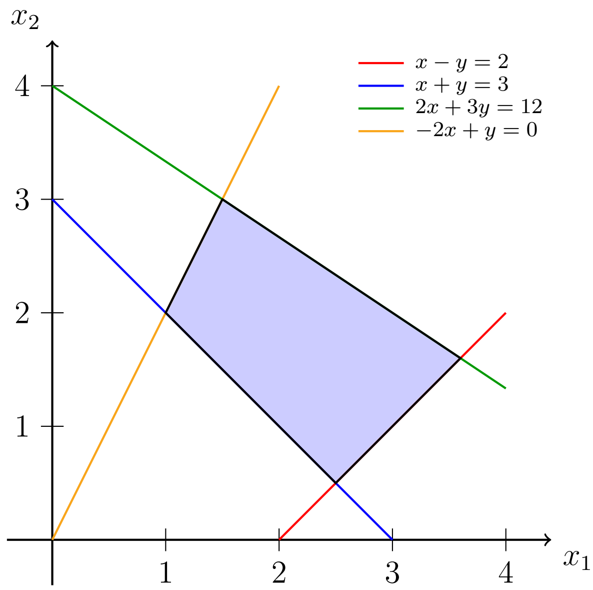
$$\equiv \left(\begin{array}{ccccc|c} 0 & 0 & 1 & 0 & 1/2 & 15 \\ 1 & 0 & 0 & 3/2 & -1/2 & 15 \\ 0 & 1 & 0 & -1/2 & 1/2 & 25 \\ 0 & 0 & 0 & 30 & 10 & 2100 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 - \tfrac{5}{3}R_2 \\ \\ R_3' = R_3 - \tfrac{1}{3}R_2 \\ R_4' = R_4 + 20R_2 \end{array}$$

The entries in the bottom row are now all non-negative, so we have found the optimal solution. The second row indicates that $x_1 = 15$, and the third row indicates that $x_2 = 25$. So the maximiser is $(x_1, x_2) = (15, 25)$, and the value of the objective function is $z = 2100$.

Notice one important caveat of this algorithm: the algorithm can only proceed from one basic solution to another. When the origin is feasible, as is the case in our main example, the algorithm will start there and proceed from one basic solution to the next. However, it need not be the case that the origin is feasible. This occurs when a constraint takes the form $a_1x_1 + \cdots + a_nx_n \geqslant b$ with $b > 0$. For example, consider the following linear program:

$$\begin{array}{rr} \text{maximise} & z = -7x_1 - 5x_2 \\ \text{subject to} & x_1 - x_2 \leqslant 2 \\ & x_1 + x_2 \geqslant 3 \\ & 2x_1 + 3x_2 \leqslant 12 \\ & -2x_1 + x_2 \leqslant 0 \\ & \mathbf{x} \geqslant 0. \end{array}$$

The origin $(x_1, x_2) = (0, 0)$ fails the second constraint, as $0 \not\geqslant 3$. Its feasible region is shown below.



In its current form, the simplex algorithm does not apply to this problem. Before describing how to find an initial basic solution, we will discuss in the next section the concepts of non-uniqueness and degeneracy.

Example: the simplex algorithm I

Consider the following linear program:

$$\begin{aligned} \text{maximise} \quad & z = 4x_1 + 2x_2 + 5x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leqslant 10 \\ & x_1 + 2x_2 + 2x_3 \leqslant 8 \\ & 3x_1 + x_2 + 4x_3 \leqslant 16 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

All of the inequalities are in the required form to apply the methods of this section. We saw in Section 4.3 that the initial matrix for this linear program is

$$\left(\begin{array}{cccccc|c} 2 & 3 & 1 & 1 & 0 & 0 & 10 \\ 1 & 2 & 2 & 0 & 1 & 0 & 8 \\ 3 & 1 & 4 & 0 & 0 & 1 & 16 \\ -4 & -2 & -5 & 0 & 0 & 0 & 0 \end{array}\right).$$

The most negative entry is in column 3, and the ratios are 10/1, 8/2 and 16/4. We could choose row 2 or row 3; here we choose row 2. Pivoting on row 2, column 3 gives:

$$\left(\begin{array}{cccccc|c} 2 & 3 & 1 & 1 & 0 & 0 & 10 \\ 1 & 2 & \boxed{2} & 0 & 1 & 0 & 8 \\ 3 & 1 & 4 & 0 & 0 & 1 & 16 \\ -4 & -2 & -5 & 0 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{cccccc|c} 2 & 3 & 1 & 1 & 0 & 0 & 10 \\ 1/2 & 1 & 1 & 0 & 1/2 & 0 & 4 \\ 3 & 1 & 4 & 0 & 0 & 1 & 16 \\ -4 & -2 & -5 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_2' = \tfrac{1}{2}R_2$$

$$\equiv \left(\begin{array}{cccccc|c} 3/2 & 2 & 0 & 1 & -1/2 & 0 & 6 \\ 1/2 & 1 & 1 & 0 & 1/2 & 0 & 4 \\ 1 & -3 & 0 & 0 & -2 & 1 & 0 \\ -3/2 & 3 & 0 & 0 & 5/2 & 0 & 20 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 - R_2 \\ \\ R_3' = R_3 - 4R_2 \\ R_4' = R_4 + 5R_2 \end{array}$$

There is one negative entry in the bottom row, in column 1. The ratios are $6/(3/2) = 4$, $4/(1/2) = 8$

and $0/1 = 0$. The smallest is 0, occurring in the third row, so we pivot on row 3 column 1:

$$\left(\begin{array}{cccccc|c} 3/2 & 2 & 0 & 1 & -1/2 & 0 & 6 \\ 1/2 & 1 & 1 & 0 & 1/2 & 0 & 4 \\ \boxed{1} & -3 & 0 & 0 & -2 & 1 & 0 \\ -3/2 & 3 & 0 & 0 & 5/2 & 0 & 20 \end{array}\right) \equiv \left(\begin{array}{cccccc|c} 0 & 13/2 & 0 & 1 & 5/2 & -3/2 & 6 \\ 0 & 5/2 & 1 & 0 & 3/2 & -1/2 & 4 \\ 1 & -3 & 0 & 0 & -2 & 1 & 0 \\ 0 & -3/2 & 0 & 0 & -1/2 & 3/2 & 20 \end{array}\right) \begin{array}{l} R_1' = R_1 - 3/2R_3 \\ R_2' = R_2 - 1/2R_3 \\ \\ R_4' = R_4 + 3/2R_3 \end{array}$$

The most negative entry is $-3/2$, in column 2. Using only the positive entries in column 2, the ratios are $6/(13/2) = 12/13$ and $4/(5/2) = 8/5$. The smallest is in row 1, so we pivot on row 1, column 2:

$$\begin{aligned} &\left(\begin{array}{cccccc|c} 0 & \boxed{13/2} & 0 & 1 & 5/2 & -3/2 & 6 \\ 0 & 5/2 & 1 & 0 & 3/2 & -1/2 & 4 \\ 1 & -3 & 0 & 0 & -2 & 1 & 0 \\ 0 & -3/2 & 0 & 0 & -1/2 & 3/2 & 20 \end{array}\right) \\ \equiv &\left(\begin{array}{cccccc|c} 0 & 1 & 0 & 2/13 & 5/13 & -3/13 & 12/13 \\ 0 & 5/2 & 1 & 0 & 3/2 & -1/2 & 4 \\ 1 & -3 & 0 & 0 & -2 & 1 & 0 \\ 0 & -3/2 & 0 & 0 & -1/2 & 3/2 & 20 \end{array}\right) \begin{array}{l} R_1' = \frac{2}{13}R_1 \\ \\ \\ \\ \end{array} \\ \equiv &\left(\begin{array}{cccccc|c} 0 & 1 & 0 & 2/13 & 5/13 & -3/13 & 12/13 \\ 0 & 0 & 1 & -5/13 & 7/13 & 1/13 & 22/13 \\ 1 & 0 & 0 & 6/13 & -11/13 & 4/13 & 36/13 \\ 0 & 0 & 0 & 3/13 & 1/13 & 15/13 & 278/13 \end{array}\right) \begin{array}{l} \\ R_2' = R_2 - \frac{5}{2}R_1 \\ R_3' = R_3 + 3R_1 \\ R_4' = R_4 + \frac{3}{2}R_1 \end{array} \end{aligned}$$

There are no more negative entries to consider, so the maximiser is $(x_1, x_2, x_3) = \frac{1}{13}(36, 12, 22)$, with $z = \frac{278}{13}$.

Example: the simplex algorithm II

In this example, we will not explain each step; the pivots and ratios are indicated along the way.

$$\begin{aligned} \text{minimise} \quad & z = -30x_1 - 50x_2 + 30x_3 \\ \text{subject to} \quad & 6x_1 + 4x_2 + 2x_3 \leqslant 30 \\ & x_1 + 4x_2 + 2x_3 \leqslant 12 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

This is a minimisation problem, so we negate the objective function. We will need to undo the negation at the end of the algorithm. We have

$$\begin{aligned}
&\left(\begin{array}{ccccc|c} 6 & 4 & 2 & 1 & 0 & 30 \\ 1 & \boxed{4} & 2 & 0 & 1 & 12 \\ -30 & -50 & 30 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} 30/4 = 15/2 \\ 12/4 = 3 \\ \end{array} \\
\equiv &\left(\begin{array}{ccccc|c} 6 & 4 & 2 & 1 & 0 & 30 \\ 1/4 & 1 & 1/2 & 0 & 1/4 & 3 \\ -30 & -50 & 30 & 0 & 0 & 0 \end{array}\right) \quad R_2' = \tfrac{1}{4}R_2 \\
\equiv &\left(\begin{array}{ccccc|c} \boxed{5} & 0 & 0 & 1 & -1 & 18 \\ 1/4 & 1 & 1/2 & 0 & 1/4 & 3 \\ -35/2 & 0 & 55 & 0 & 25/2 & 150 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 - 4R_2 \\ \\ R_3' = R_3 + 50R_2 \end{array} \qquad \begin{array}{l} 18/5 \\ 3/(1/4) = 12 \\ \end{array} \\
\equiv &\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1/5 & -1/5 & 18/5 \\ 1/4 & 1 & 1/2 & 0 & 1/4 & 3 \\ -35/2 & 0 & 55 & 0 & 25/2 & 150 \end{array}\right) \quad \begin{array}{l} R_1' = \tfrac{1}{5}R_1 \\ \\ \end{array} \\
\equiv &\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1/5 & -1/5 & 18/5 \\ 0 & 1 & 1/2 & -1/20 & 3/10 & 21/10 \\ 0 & 0 & 55 & 7/2 & 9 & 213 \end{array}\right) \quad \begin{array}{l} \\ R_2' = R_2 - 1/4R_1 \\ R_2' = R_2 + 35/2R_1 \end{array}
\end{aligned}$$

So the minimiser is $(x_1, x_2, x_3) = (\frac{18}{5}, \frac{21}{10})$, with $-z = 213$, so the minimum is $z = -213$.

4.5 Non-uniqueness and degeneracy

There are some edge cases of the simplex algorithm that require some careful consideration.

The optimal solution may not be unique, which occurs when there is an edge of the feasible region which is parallel to the level sets of the objective function. When this happens, the optimal solution is attained at two (or more) corners, and at every point on the line segment that joins them (or, if more than two, the polytope with them as vertices). From a practical point of view, if the optimal solution isn't unique, then it is sensible to find all the corners at which the optimum occurs. This knowledge can be useful, since there might be a preference for one solution over another for reasons that haven't been encoded in the constraint relations.

Non-uniqueness of the solution is detected by inspecting the final matrix. In the final matrix of the simplex algorithm, all the entries in the bottom row will be non-negative (because if there was a negative entry the algorithm would continue); if there is a zero entry in a column of a non-basic variable, then the solution won't be unique. Using this variable as a new entering basic variable, and performing another step in the algorithm, will lead to another optimal solution (which, of course, has the same z value).

In addition, it is possible that there is no optimal solution. This can occur when the feasible region is unbounded, and is detected when the minimum ratio test cannot be applied; i.e., when none of the entries in the column of the entering basic variable are positive. In this case, the objective function can be made arbitrarily large without violating the constraints, so there is no maximum.

The completed **simplex algorithm** is then:

1. Determine the entering basic variable by choosing the biggest negative entry in the bottom row.
2. If none of the entries in the column of the entering basic variable are positive, then there are arbitrarily large solutions. STOP.
3. Determine the departing basic variable rule by applying the minimum ratio test.
4. Apply row operations to determine the new basic feasible solution.
5. If there is a variable with a negative entry in the bottom row, this is not an optimal solution; return to step 1.
6. If there are no negative entries in the bottom row, and there is a zero entry in the column of a *non-basic* variable, and this solution has not yet been recorded, then record this solution as optimal and return to step 3 using this variable as the entering basic variable.
7. If there are no negative entries in the bottom row, the optimal solution has been found. STOP.

Example: non-uniqueness

Consider the following:

$$\begin{aligned} \text{maximise} \quad & z = 60x_1 + 35x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 \leqslant 48 \\ & 4x_1 + 2x_2 + \tfrac{3}{2}x_3 \leqslant 20 \\ & 2x_1 + \tfrac{3}{2}x_2 + \tfrac{1}{2}x_3 \leqslant 8 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

The initial matrix is

$$\left(\begin{array}{cccccc|c} 8 & 6 & 1 & 1 & 0 & 0 & 48 \\ 4 & 2 & 3/2 & 0 & 1 & 0 & 20 \\ 2 & 3/2 & 1/2 & 0 & 0 & 1 & 8 \\ -60 & -35 & -20 & 0 & 0 & 0 & 0 \end{array}\right)$$

We leave out the details, but after pivoting twice, the matrix becomes

$$\left(\begin{array}{cccccc|c} 0 & -2 & 0 & 1 & 2 & -8 & 24 \\ 0 & -2 & 1 & 0 & 2 & -4 & 8 \\ 1 & 5/4 & 0 & 0 & -1/2 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 10 & 10 & 280 \end{array}\right).$$

This gives the maximum of 280 attained when $(x_1, x_2, x_3) = (2, 0, 8)$. However, observe that the second column has a zero entry in the bottom row, but x_2 is not a basic variable. We can choose x_2 as an entering basic variable and pivot once more:

$$\left(\begin{array}{cccccc|c} 0 & -2 & 0 & 1 & 2 & -8 & 24 \\ 0 & -2 & 1 & 0 & 2 & -4 & 8 \\ 1 & \boxed{5/4} & 0 & 0 & -1/2 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 10 & 10 & 280 \end{array}\right) \equiv \left(\begin{array}{cccccc|c} 0 & -2 & 0 & 1 & 2 & -8 & 24 \\ 0 & -2 & 1 & 0 & 2 & -4 & 8 \\ 4/5 & 1 & 0 & 0 & -2/5 & 6/5 & 8/5 \\ 0 & 0 & 0 & 0 & 10 & 10 & 280 \end{array}\right) \quad R_3' = \tfrac{4}{5}R_3$$

$$\equiv \left(\begin{array}{cccccc|c} 8/5 & 0 & 0 & 1 & 6/5 & -28/5 & 136/5 \\ 8/5 & 0 & 1 & 0 & 6/5 & -8/5 & 56/5 \\ 4/5 & 1 & 0 & 0 & -2/5 & 6/5 & 8/5 \\ 0 & 0 & 0 & 0 & 10 & 10 & 280 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 + 2R_3 \\ R_2' = R_2 + 2R_3 \end{array}$$

We find a second corner of the feasible region that attains the optimal solution: $(x_1, x_2, x_3) = (0, \frac{8}{5}, \frac{56}{5})$. The same observation as earlier applies; in this case, the first column has a zero entry but is non-basic, so x_1 would enter. However, this would result in basic variables x_1, x_3 and x_4, which are the same as earlier, and hence a repeat solution. So we have found all corners were the optimal solution is attained.

Thus, all optimal solutions to this problem lie on the line segment joining the points $(2, 0, 8)$ and $(0, \frac{8}{5}, \frac{56}{5})$, so that the maximisers are given by

$$\begin{aligned} (x_1, x_2, x_3) &= \alpha(2, 0, 8) + (1-\alpha)(0, \tfrac{8}{5}, \tfrac{56}{5}) \\ &= (2\alpha, \tfrac{8-8\alpha}{5}, \tfrac{56-16\alpha}{5}) \end{aligned}$$

with $\alpha \in [0, 1]$.

Example: unbounded solutions

Consider the following:

$$\begin{aligned} \text{maximise} \quad & z = x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 - x_2 - x_3 \leqslant 3 \\ & 2x_1 + x_2 - x_3 \leqslant 2 \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

Applying the simplex algorithm:

$$\left(\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 0 & 3 \\ \boxed{2} & 1 & -1 & 0 & 1 & 2 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 0 & 3 \\ 1 & 1/2 & -1/2 & 0 & 1/2 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{array}\right) \quad R_2' = \tfrac{1}{2}R_2$$

$$\equiv \left(\begin{array}{ccccc|c} 0 & -3/2 & -1/2 & 1 & -1/2 & 2 \\ 1 & 1/2 & -1/2 & 0 & 1/2 & 1 \\ 0 & -1/2 & -3/2 & 0 & 1/2 & 1 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 - R_2 \\ \\ R_3' = R_3 + R_2 \end{array}$$

The most negative entry in the bottom row is in column 3, but there are no positive entries in column 3. We can immediately deduce that there is no optimal solution; the objective function can be made arbitrarily large.

In $\mathbb{R}^2$ the corner points of the feasible region are usually given by the intersection of two constraint lines. Similarly, in $\mathbb{R}^n$ corners would generically be the intersection of n constraint hyperplanes. The term **degeneracy** refers to the situation where a corner point lies on the intersection of an excessive number of hyperplanes. Degeneracy can lead to **stalling** in the simplex algorithm, which occurs when the simplex algorithm repeats the same non-optimal corner point across several consecutive iterations, but eventually arrives at the optimal solution. In the augmented matrix, this situation is detected by zeros in the resource column. One of the examples in Section 4.4 exhibits stalling; see if you can find it.

Worse than stalling is **cycling**, which occurs when the simplex algorithm returns to a non-optimal solution that it has already left. In that case, the algorithm will never terminate and never find the optimal solution. The first example of cycling in the simplex algorithm was described by Hoffman in 1951 [5]. A simpler example, adopted from Chvátal [4], is as follows:

$$\begin{aligned} \text{maximise} \quad & z = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} \quad & \tfrac{1}{2}x_1 - \tfrac{11}{2}x_2 - \tfrac{5}{2}x_3 + 9x_4 \leqslant 0 \\ & \tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 \leqslant 0 \\ & x_1 \leqslant 1 \\ & x_1, x_2, x_3, x_4 \geqslant 0. \end{aligned}$$

If you apply the simplex algorithm to this linear program, you will find that it cycles through the same 6 pivots before returning to the initial matrix. Consequently, the simplex algorithm will never find the solution; it cycles endlessly at the origin without finding the optimal solution at the corner point $(x_1, x_2, x_3, x_4) = (1, 0, 1, 0)$.

One way to resolve cycling is due to Bland [1], and called **Bland's rule**, which modifies the selection of entering and departing basic variables. All steps remain the same, except for the following:

- Determine the entering basic variable by choosing the *leftmost* negative entry in the bottom row.
- Determine the departing basic variable using the minimum ratio test, but if there is a tie, choose the row with the leftmost basic variable.

Bland proved that this method will always terminate, but it should be noted that in practice, cycling is rare and Bland's rule is not as efficient.

In the final section of this chapter, we explain how to deal with linear programs for which we don't known a basic feasible solution.

4.6 The 2-phase method

The simplex algorithm that we have described only deals with constraints of the form

$$a_1x_1 + \cdots + a_nx_n \leqslant b,$$

where the resource value b is non-negative. The simplex algorithm requires an initial corner of the feasible region, and under this condition, the origin is a corner.

However, as mentioned in Section 4.4, if a constraint takes the form $a_1x_1 + \cdots + a_nx_n \geqslant b$ with $b > 0$, then the origin is not a corner of the feasible region. To see how this is a problem when implementing the simplex algorithm, consider the following linear program:

$$\begin{aligned} \text{maximise} \quad & z = -7x_1 - 5x_2 \\ \text{subject to} \quad & x_1 - x_2 \leqslant 2 \\ & x_1 + x_2 \geqslant 3 \\ & 2x_1 + 3x_2 \leqslant 12 \\ & -2x_1 + x_2 \leqslant 0 \\ & \mathbf{x} \geqslant 0. \end{aligned}$$

The origin $(x_1, x_2) = (0, 0)$ fails the second constraint, as $0 \ngeqslant 3$.

Recall from Section 4.3 that for inequalities of the form $a_1x_1 + \cdots + a_nx_n \geqslant b$, introducing a slack variable results in a negative coefficient. So, by introducing slack variables we get the constraints:

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_1 + x_2 - x_4 &= 3 \\ 2x_1 + 3x_2 + x_5 &= 12 \\ -2x_1 + x_2 + x_6 &= 0 \\ \mathbf{x} &\geqslant 0. \end{aligned}$$

Since the slack variable x_4 has coefficient -1, the resulting augmented coefficient matrix is not in canonical form. We can't just multiply both sides of the equation by -1, either. Although that would bring the system into canonical form, the corresponding basic solution would not be feasible. This is why the simplex algorithm cannot be directly applied. So, the first phase is to find a corner of the feasible region.

The clever idea behind the 2-phase method is that the first phase can itself be regarded as a linear programming problem. After writing the problem in standard form, for each equation that does not have a basic variable, we introduce an **artificial variable** a by replacing $a_1x_1 + \ldots + a_nx_n = b$ with $a_1x_1 + \ldots + a_nx_n + a = b$ and the constraint $a \geqslant 0$. For our example, we introduce a single artificial variable x_7:

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_1 + x_2 - x_4 + x_7 &= 3 \\ 2x_1 + 3x_2 + x_5 &= 12 \\ -2x_1 + x_2 + x_6 &= 0 \\ \mathbf{x} &\geqslant 0. \end{aligned}$$

Comparing to the original problem, if $x_1 + x_2 - x_4 = 3$, then $x_7 > 0$ will violate the original constraint; on the other hand, finding values that make $x_7 = 0$ will be consistent with the original formulation.

So, we use linear programming to find a solution that minimizes the artificial variable x_7, or in other words, maximises the objective function $w = -x_7$. However, we want this function to be expressed in terms of non-basic variables. The initial basic variables are x_3, x_5, x_6, x_7, so we have to make a substitution to eliminate the dependence on x_7. The second constraint gives $x_7 = 3 - x_1 - x_2 + x_4$, and so

$$w = -x_7 = x_1 + x_2 - x_4 - 3.$$

Thus, the phase 1 linear program is:

$$\begin{aligned} \text{maximise} \quad & z = -7x_1 - 5x_2 \\ \text{subject to} \quad & x_1 - x_2 + x_3 = 2 \\ & x_1 + x_2 - x_4 + x_7 = 3 \\ & 2x_1 + 3x_2 + x_5 = 12 \\ & -2x_1 + x_2 + x_6 = 0 \\ & \mathbf{x} \geqslant 0, \end{aligned}$$

which is in canonical form. By adding the equation

$$w - x_1 - x_2 + x_4 = -3$$

to the constraints, the augmented matrix for the system (without the first w-column) is:

$$\left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & 3 \\ 2 & 3 & 0 & 0 & 1 & 0 & 0 & 12 \\ -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & -3 \end{array}\right)$$

and the initial basic feasible solution is given by $\mathbf{x} = (0, 0, 2, 0, 12, 0, 3)$. We now apply the simplex algorithm to find an initial corner.

As entering variable we can choose either x_1 or x_2, as their coefficients in the bottom row are both the most negative. We choose x_1. In the first column, the entries in the first three rows are positive. The ratios are $2/1 = 2$, $3/1 = 3$ and $12/2 = 6$, with the smallest in row 1, so we pivot on row 1, column 1.

$$\left(\begin{array}{ccccccc|c} \boxed{1} & -1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & 3 \\ 2 & 3 & 0 & 0 & 1 & 0 & 0 & 12 \\ -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & -3 \end{array}\right) \equiv \left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 5 & -2 & 0 & 1 & 0 & 0 & 8 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 4 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \begin{array}{l} \\ R_2' = R_2 - R_1 \\ R_3' = R_3 - 2R_1 \\ R_4' = R_4 + 2R_1 \\ R_5' = R_5 + R_1 \end{array}$$

There is a negative entry in column 2, with two positive entries in the column above. The ratios are $1/2$ and $8/5$, and as $1/2 < 8/5$ we pivot on the entry in row 2:

$$\left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & \boxed{2} & -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 5 & -2 & 0 & 1 & 0 & 0 & 8 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 4 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \equiv \left(\begin{array}{ccccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 & 1/2 & 1 \\ 0 & 5 & -2 & 0 & 1 & 0 & 0 & 8 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 4 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right) \begin{array}{l} \\ R_2' = R_2/2 \\ \\ \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{ccccccc|c} 1 & 0 & 1/2 & -1/2 & 0 & 0 & 1/2 & 5/2 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 5/2 & 1 & 0 & -5/2 & 11/2 \\ 0 & 0 & 3/2 & -1/2 & 0 & 1 & 1/2 & 9/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} R_1' = R_1 + R_2 \\ \\ R_3' = R_3 - 5R_2 \\ R_4' = R_4 + R_2 \\ R_5' = R_5 + 2R_1. \end{array}$$

Now the bottom row is non-negative. There are zero entries in non-basic columns (all the non-artificial ones). However, in phase 1, we are not interested in *all* solutions; as soon as we have found one solution with $w = 0$,

$$\mathbf{x} = (5/2, 1/2, 0, 0, 11/2, 9/2, 0)$$

we proceed to phase 2.

For phase 2 we drop the artificial columns, insert the original objective function ($z + 7x_1 + 5x_2 = 0$) in the bottom row (omitting the z-column), and make it canonical:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 1/2 & -1/2 & 0 & 0 & 5/2 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 5/2 & 1 & 0 & 11/2 \\ 0 & 0 & 3/2 & -1/2 & 0 & 1 & 9/2 \\ 7 & 5 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{cccccc|c} 1 & 0 & 1/2 & -1/2 & 0 & 0 & 5/2 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 5/2 & 1 & 0 & 11/2 \\ 0 & 0 & 3/2 & -1/2 & 0 & 1 & 9/2 \\ 0 & 0 & -1 & 6 & 0 & 0 & -20 \end{array}\right) \quad R_5' = R_5 - 7R_1 - 5R_2$$

Now we apply the simplex algorithm; in this case, the solution is obtained in one step

$$\left(\begin{array}{cccccc|c} 1 & 0 & 1/2 & -1/2 & 0 & 0 & 5/2 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 5/2 & 1 & 0 & 11/2 \\ 0 & 0 & \boxed{3/2} & -1/2 & 0 & 1 & 9/2 \\ 0 & 0 & -1 & 6 & 0 & 0 & -20 \end{array}\right) \equiv \left(\begin{array}{cccccc|c} 1 & 0 & 1/2 & -1/2 & 0 & 0 & 5/2 \\ 0 & 1 & -1/2 & -1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 5/2 & 1 & 0 & 11/2 \\ 0 & 0 & 1 & -1/3 & 0 & 2/3 & 3 \\ 0 & 0 & -1 & 6 & 0 & 0 & -20 \end{array}\right) \quad R_4' = \tfrac{2}{3}R_4$$

$$\equiv \left(\begin{array}{cccccc|c} 1 & 0 & 0 & -1/3 & 0 & -1/3 & 1 \\ 0 & 1 & 0 & -2/3 & 0 & 1/3 & 2 \\ 0 & 0 & 0 & 8/3 & 1 & -1/3 & 4 \\ 0 & 0 & 1 & -1/3 & 0 & 2/3 & 3 \\ 0 & 0 & 0 & 17/3 & 0 & 2/3 & -17 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 - \frac{1}{2}R_4 \\ R_2' = R_2 + \frac{1}{2}R_4 \\ R_3' = R_3 - \frac{1}{2}R_4 \\ \\ R_5' = R_5 + R_4 \end{array}$$

This gives the optimal solution $z = -17$ at the corner $(1, 2)$ of the feasible region.

This method can also be applied if there are equality constraints in the original problem. For a general linear program

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}_1\mathbf{x} \leqslant \mathbf{b}_1 \\ & \mathbf{A}_2\mathbf{x} = \mathbf{b}_2 \qquad (4.1) \\ & \mathbf{A}_3\mathbf{x} \geqslant \mathbf{b}_3 \\ & \mathbf{x} \geqslant \mathbf{0}, \qquad (4.2) \end{aligned}$$

with $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ all non-negative, the **2-phase method** is as follows:

Phase 1

1. Introduce slack variables to represent the problem in standard form.
2. For each equality constraint in the standard form that does not have a basic variable, introduce an artificial variable to that constraint.
3. Use the simplex method to minimise the sum of artificial variables.

Phase 2

4. Remove the artificial columns and insert the original objective function.
5. Apply the simplex algorithm to find the optimal solution.

For reference, we repeat the full version of the simplex algorithm here:

1. Determine the entering basic variable by choosing the biggest negative entry in the bottom row.
2. If none of the entries in the column of the entering basic variable are positive, then there are arbitrarily large solutions. STOP.
3. Determine the departing basic variable rule by applying the minimum ratio test.
4. Apply row operations to determine the new basic feasible solution.
5. If there is a variable with a negative entry in the bottom row, this is not an optimal solution; return to step 1.
6. If there are no negative entries in the bottom row, and there is a zero entry in the column of a *non-basic* variable, and this solution has not yet been recorded, then record this solution as optimal and return to step 3 using this variable as the entering basic variable.
7. If there are no negative entries in the bottom row, the optimal solution has been found. STOP.

Example: the 2-phase method

Consider the following linear program:

$$\begin{aligned} \text{maximise} \quad & z = -x_1 + 3x_2 - x_3 + 4x_4 \\ \text{subject to} \quad & 5x_1 + 2x_2 - x_3 + 3x_4 \leqslant 4 \\ & 3x_1 - 5x_2 + 5x_3 + 2x_4 \geqslant 4 \\ & 3x_1 - 3x_3 + x_4 = 0 \\ & x_1, x_2, x_3, x_4 \geqslant 0. \end{aligned}$$

We first express the problem in standard form. The first constraint is a $\leqslant$-inequality, so its slack variable gets a positive coefficient. The second constraint is a $\geqslant$-inequality, so its slack variable gets a negative coefficient. The third constraint is an equation, so it does not need a slack variable. In standard form, the problem is:

$$\begin{aligned} \text{maximise} \quad & z = -x_1 + 3x_2 - x_3 + 4x_4 \\ \text{subject to} \quad & 5x_1 + 2x_2 - x_3 + 3x_4 + x_5 = 4 \\ & 3x_1 - 5x_2 + 5x_3 + 2x_4 - x_6 = 4 \\ & 3x_1 - 3x_3 + x_4 = 0 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geqslant 0. \end{aligned}$$

The second and third equations do not have basic variables, so we introduce two artificial variables, x_7 and x_8. The equations become:

$$\begin{aligned} 5x_1 + 2x_2 - x_3 + 3x_4 + x_5 &= 4 \\ 3x_1 - 5x_2 + 5x_3 + 2x_4 - x_6 + x_7 &= 4 \\ 3x_1 - 3x_3 + x_4 + x_8 &= 0. \end{aligned}$$

Use the latter two equations to express the artificial variables in terms of non-basic variables:

$$\begin{aligned} x_7 &= 4 - 3x_1 + 5x_2 - 5x_3 - 2x_4 + x_6 \\ x_8 &= -3x_1 + 3x_3 - x_4. \end{aligned}$$

We wish to minimise $x_7 + x_8$; i.e., maximise

$$\begin{aligned} w &= -x_7 - x_8 \\ &= -4 + 3x_1 - 5x_2 + 5x_3 + 2x_4 - x_6 + 3x_1 - 3x_3 + x_4 \\ &= -4 + 6x_1 - 5x_2 + 2x_3 + 3x_4 - x_6, \end{aligned}$$

which we write as

$$w - 6x_1 + 5x_2 - 2x_3 - 3x_4 + x_6 = -4.$$

Adding this to the constraints gives the initial matrix for phase 1,

$$\left(\begin{array}{cccccccc|c} 5 & 2 & -1 & 3 & 1 & 1 & 0 & 0 & 4 \\ 3 & -5 & 5 & 2 & 0 & -1 & 1 & 0 & 4 \\ 3 & 0 & -3 & 1 & 0 & 0 & 0 & 1 & 0 \\ -6 & 5 & -2 & -3 & 0 & 1 & 0 & 0 & -4 \end{array}\right)$$

We leave the steps as an exercise, but the final matrix for phase 1 is

$$\left(\begin{array}{cccccccc|c} 0 & 9/2 & 0 & 5/6 & 1 & 1/2 & -1/2 & -7/6 & 2 \\ 0 & -5/8 & 1 & 1/8 & 0 & -1/8 & 1/8 & -1/8 & 1/2 \\ 1 & -5/8 & 0 & 11/24 & 0 & -1/8 & 1/8 & 5/24 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

The solution we find has $x_1 = 2$, $x_2 = 1/2$ and $x_3 = 1/2$. For phase 2, we now drop the artificial columns and the bottom row, return the objective function $z = -x_1 + 3x_2 - x_3 + 4x_4$, and make it canonical:

$$\begin{aligned} &\left(\begin{array}{cccccc|c} 0 & 9/2 & 0 & 5/6 & 1 & 1/2 & 2 \\ 0 & -5/8 & 1 & 1/8 & 0 & -1/8 & 1/2 \\ 1 & -5/8 & 0 & 11/24 & 0 & -1/8 & 1/2 \\ 1 & -3 & 1 & -4 & 0 & 0 & 0 \end{array}\right) \\ \equiv &\left(\begin{array}{cccccc|c} 0 & 9/2 & 0 & 5/6 & 1 & 1/2 & 2 \\ 0 & -5/8 & 1 & 1/8 & 0 & -1/8 & 1/2 \\ 1 & -5/8 & 0 & 11/24 & 0 & -1/8 & 1/2 \\ 0 & -7/4 & 0 & -55/12 & 0 & 1/4 & -1 \end{array}\right) \quad R_4' = R_4 - R_2 - R_3 \end{aligned}$$

Once again, we omit the details of the simplex algorithm, but continuing from here will reach the final matrix

$$\left(\begin{array}{cccccc|c} -10/31 & 1 & 0 & 0 & 11/62 & 4/31 & 6/31 \\ -13/31 & 0 & 1 & 0 & 5/62 & -1/31 & 14/31 \\ 54/31 & 0 & 0 & 1 & 15/62 & -3/31 & 42/31 \\ 230/31 & 0 & 0 & 0 & 44/31 & 1/31 & 172/31 \end{array}\right),$$

resulting in the optimal solution $z = \frac{172}{31}$ when $(x_1, x_2, x_3, x_4) = \frac{1}{31}(0, 6, 14, 42)$.