**Question 4 Solution:**

Part a:

Given information:

Let,

$x_1$,$x_2$,$x_3$$\sim$ $(N(\mu$,$\sigma^2))$ be a sequence of independent normal random variables,

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$\boldsymbol{x^T} = (x_1, x_2, x_3)^T$

Supposed to be $x^T$ as noted!

$\boldsymbol{y}$ = (x$_1$ - ,$x_2$- ,$x_3$- $)^T$

To solve A from:

$$\boldsymbol{y} = A\boldsymbol{x}$$

$$\begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A $= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Where A is symmetric and idempotent!

$A^2 = \frac{1}{9}\begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} =$ A

Part b: Finding the rank of A

Proof: that there is a linear combination for any columns?

```{r}
A = matrix(c(2,-1,-1,-1,2,-1,-1,-1,2)/3,3,3)

```

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           [,1]       [,2]       [,3]
[1,]  0.6666667 -0.3333333 -0.3333333
[2,] -0.3333333  0.6666667 -0.3333333
[3,] -0.3333333 -0.3333333  0.6666667
```

# Finding rank of A

```{r}
rankMatrix(A)[1]
```

```
[1] 2
```

Each column all added up together gives us 0!.

$x_1 + x_2 + x_3 = 0$

Can be written as,

$x_1 = -x_2 - x_3$

That are linearly dependant and similar for $x_2 and x_3$

Hence r(A) = 2

Part c: Computing $E[y^T y]$

Using Theorem 3.5:

$E[y^T Ay] = tr(AV) + \mu^T A\mu$

since A = I,

$= tr(V) + \mu^T \mu$

V = vary = varAx = AvarxA$^T$

since A is symmetric and idempotent!!

$$var(x_i) = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$

$$V = \frac{1}{3} \begin{bmatrix} 2\sigma^2 & -1 & -1 \\ -1 & 2\sigma^2 & -1 \\ -1 & -1 & 2\sigma^2 \end{bmatrix}$$

$\mu = E[y] = E[Ax] = AE[x]$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} = 0$$

$$E[y^T y] = tr(\frac{1}{3} \begin{bmatrix} 2\sigma^2 & -1 & -1 \\ -1 & 2\sigma^2 & -1 \\ -1 & -1 & 2\sigma^2 \end{bmatrix}) + 0$$

$= 2\sigma^2$

Part d:

Using Theorem 3.5:

Proof:

Assuming that A is idempotent and has rank k. Because it is symmetric, it can be diagonalised. Let the (orthogonal) diagonalising matrix be P.

$$D = P^T AP = \begin{bmatrix} \lambda_1 & ...... & 0 \\ ........ & \lambda_2 & ...... \\ 0 & ....... & \lambda_k \end{bmatrix}$$

since A is symmetric and idempotent, all eigenvalues are either 0 or 1. We know from definition:

$$tr(A) = r(A) = k$$

A $=\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$A^2 =$ A $= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

from Part 4b, we find out the rank and trace of matrix A we found in Part 4a. Is also is the same number of degrees of freedom for the chi squared distribution.

$$tr(A) = r(A) = 2$$

Therefore, A must have two eigenvalues of 1 and one eigenvalue of 0.

Using Theorem 3.5 and Corollary 3.7:

with our non central parameter $\lambda$!

$\lambda = \frac{1}{2}\mu^T A\mu$

$= \frac{1}{2}\begin{bmatrix} \mu & \mu & \mu \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix}$

$= 0$

$\iff$ : *if and only if*

E[y] = E[ $\begin{bmatrix} x_1 - \mu \\ x_2 - \mu \\ x_3 - \mu \end{bmatrix}$ ]

Since $x_1, x_2$ *and* $x_3$ *is identically independently distributed*! *and taking* the expectation of the expectation is the expectation itself!

E[y] = E[ $\begin{bmatrix} \mu - \mu \\ \mu - \mu \\ \mu - \mu \end{bmatrix}$ ] = 0

NOTE: $\mu = \bar{x}$

In which case,

$\frac{y^T y}{\sigma^2}$

is just the sum of two independent standard normal's. This is just an ordinary (central) chi squared distribution $\chi^2_2$.

with expectation of 2 and variance of 4 with 2 degrees of freedom. In which A is symmetric and idempotent!