Department of Mathematics and Statistics

Discrete Maths and Operations Research

Semester 2, 2013

Exercise Booklet

Discrete Mathematics

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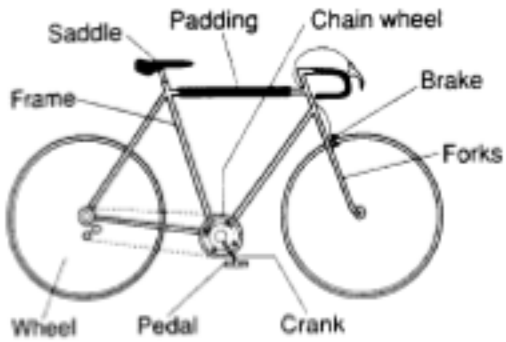
Topic 1: Networks

1. *Constructing an AON activity network.* The bicycle shop at The University of Melbourne is getting its assembly team ready to put together as many bicycles as possible for the O-week rush. The diagram below shows the basic components of a bicycle



Putting together a bicycle is split up into small jobs which can be done by different people. These are

	Activity	Duration (hours)	Precedence
A	Preparation of the frame	9	
B	Mounting and aligning the front wheel	7	A
C	Mounting and aligning the back wheel	7	A
D	Attaching the chain to the wheel crank	2	A
E	Attaching the chain wheel and crank to the frame	2	D
F	Mounting the right pedal	8	D,E
G	Mounting the left pedal	8	D,E
H	Final attachments such as saddle, chain, stickers, etc.	21	B,C,F,G

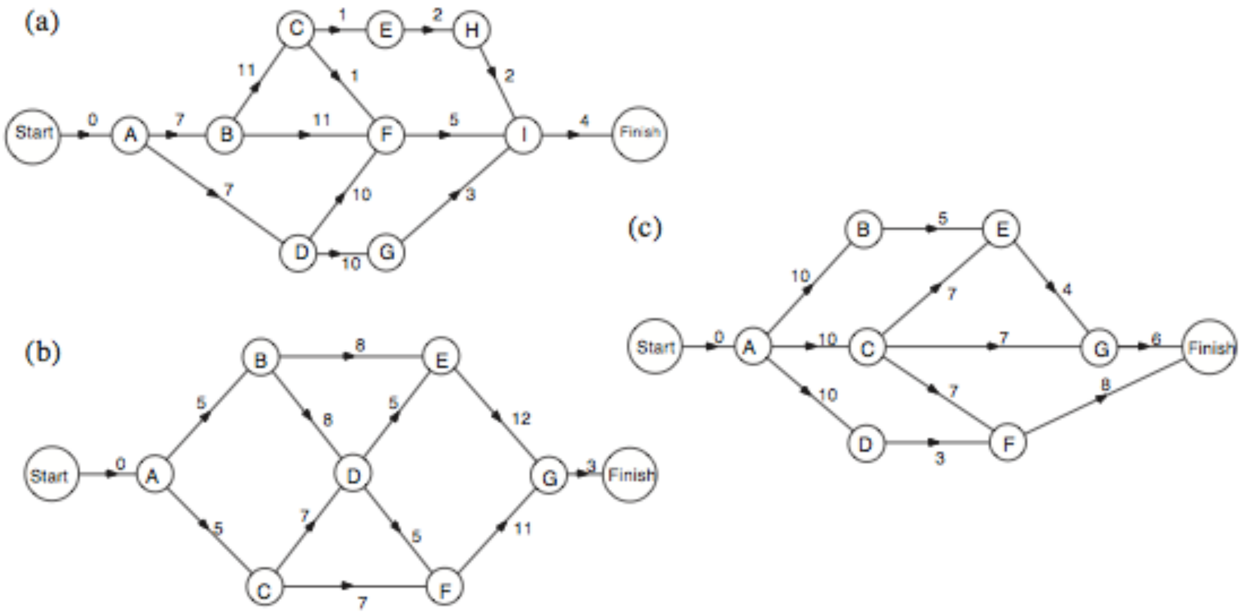
Complete an activity network for this problem.

2. *Constructing an AON activity network.* A furniture maker is going to produce a new wooden framed couch with cloth-covered foam cushions. These are the tasks that have to be done by the furniture maker and assistants, the times they will take and the tasks preceding them

	Activity	Duration (days)	Precedence
A	Make wooden arms and legs	4	B
B	Make wooden back	2	C
C	Make wooden base	3	
D	Cut foam for back and base	2	B
E	Make covers	4	D
F	Fit covers	2	E
G	Put everything together	2	A,F

Complete an activity network for this problem.

3. *Determining a critical path.* Find the critical paths and minimum completion time for each of the activity networks shown below.



4. *Free Float.* Determine the total float and free float of activities B, C and D from part (c) of the above question.

5. *Determining a critical path.* An extension is to be built to a hall. Details of the activities are given below.

	Activity	Duration (days)	Precedence
A	Lay foundations	7	
B	Build walls	10	C
C	Lay drains and floor	15	A
D	Install fittings	8	B
E	Make and fit door frames	2	C
F	Erect roof	5	D,E
G	Plaster ceiling	2	F
H	Fit and paint doors and windows	8	G
I	Fit gutters and pipes	2	F
J	Paint outside	3	H

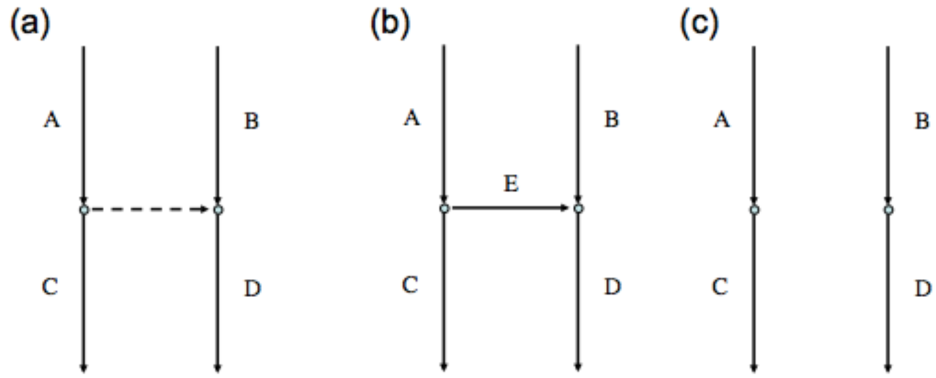
(a) Complete an activity network for this problem.

(b) Find a critical path for the activity network found above.

(c) Fitting the gutters and pipes takes 2 days, if the workman was sick and he started 8 days later would this delay the project?

(d) How many days could the task of making and fitting the doors be delayed before delaying the completion of the hall?

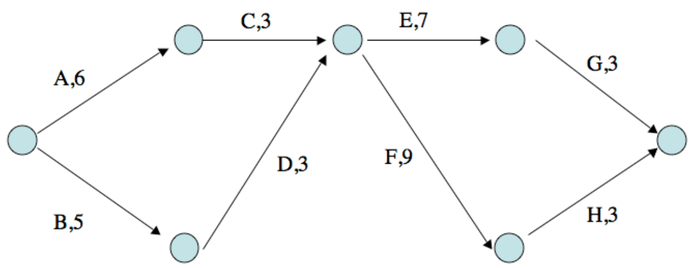
6. *Dummy arrows.* Specify the precedences (if any) of the activities indicated by the following networks



7. *The significance of dummy arrow.* The following table specifies the a project involving 8 activities

Activity	Time (days)	Precedences
A	6	
B	5	
C	3	A
D	3	B
E	7	D, C
F	9	D
G	3	E
H	3	F

A student *incorrectly* draws the AOA network as



(a) What is the critical path and completion time using this *incorrect* network?

(b) What is the error in the AOA network above?

(c) Draw the correct AOA network for the project.

(d) Determine the critical path and correct completion time using you *correct* network.

8. *The* `max` *operation in other settings.* A subsequence is palindromic if it is the same whether read left to right or right to left. We have to devise an algorithm that takes a sequence $x[1,\ldots,n]$ and returns the length of the longest palindromic subsequence. Here $1,\ldots,n$ represent the ordering of the elements in the sequence. Define the maximum length palindrome for the substring $x[i,\ldots,j]$ as $L(i,j)$. For example, for the sequence $arsraara$ we have $L(2,5)=3$, corresponding to the palindrome rsr within the substring $rsra$.

(a) Explain the formulas

$$L(i,j) = \begin{cases} L(i+1,j-1)+2 & \text{if } x[i]=x[j] \\ \max\{L(i+1,j), L(i,j-1)\} & \text{otherwise.} \end{cases}$$
$$L(i,i) = 1, \qquad i = 1,\dots,n$$

(b) Apply the formulas to compute the maximum length palindrome subsequence for *arsraara*.

9. *Matrices and the max-plus algebra.* The following questions relate to the table below.

Activity	Duration	Precedences
A	2	
B	4	A
C	3	B
D	5	A

(a) Construct both an AON and an AOA network.

(b) What is the critical path and minimum completion time of the task?

(c) What is the Q matrix that corresponds to the AON network?

(d) Using the max-plus algebra confirm the minimum time completion and critical path.

(e) What is the Q matrix that corresponds to the AOA network?

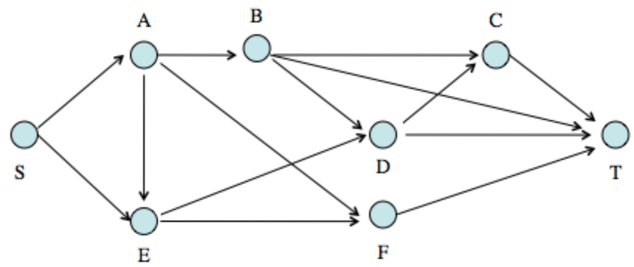
(f) Using the max-plus algebra confirm the minimum time completion and critical path.

(g) Of the methods used in questions (b), (d) and (f) which do you find easiest to determine the critical path?

(h) Which method do you think would be most efficient for a really, really large network?

10. *Shortest Paths.*

(a) Using the tree diagram method, what is the length of the shortest path from S to T in the following graph?



(b) How many shortest paths are there in the above graph?

(c) Determine the incident matrix $[a_{ij}]$ for the above graph. Where

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are connected} \\ 0, & \text{if } i = j \\ \infty, & \text{otherwise} \end{cases}$$

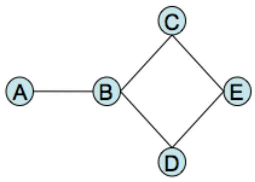
(d) Use the min-plus algebra to confirm the length of the shortest path.

11. *Betweenness.*

    (a) Explain why a vertex of degree 1 (a node with only one edge attached) has betweenness centrality zero.

    (b) Explain why a vertex of degree greater than 1 (a node with only one edge attached) has betweenness centrality can still be zero.

12. *Betweenness.* Compute the betweenness centrality for each node in the network below.



Topic 2: Graph Theory and Scheduling

13. *Brook's Theorem.* Recall that Brook's theorem states that for any non-complete graph G

$$\chi(G) \leq \Delta(G),$$

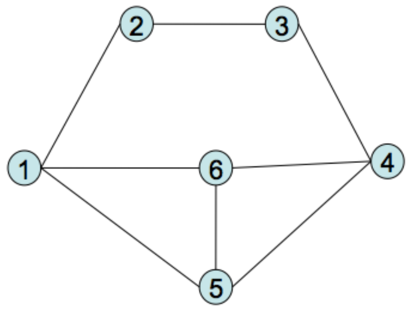
where $\chi(G)$ represents the chromatic number and $\Delta(G)$ is the maximum degree of the vertices in G. A weaker case of Brook's theorem (that is in fact true for any graph)

$$\chi(G) \leq \Delta(G) + 1$$

can be proved by showing that the following algorithm produces a colouring of $\leq \Delta(G) + 1$ colours.

Greedy colouring algorithm. Suppose G has n vertices $\{v_1, v_2, \ldots, v_n\}$ and has maximum degree $\Delta = \Delta(G)$. We will show that the graph can be coloured with the colours $c_0, c_1, \ldots, c_\Delta$. Colour v_1 with c_0. Next consider v_2. If v_2 is connected to v_1 colour it with c_1 otherwise use c_0. Continue with v_3, then v_4, etc, using the colour with the lowest possible subscript. Since any vertex will be connected to at most Δ other vertices, there will always be a colour from $c_0, c_1, \ldots, c_\Delta$ that can be used.

(a) What is $\Delta(G)$ for the below graph?

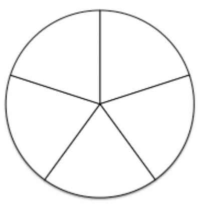


(b) Using the greedy colouring algorithm colour the above graph with $\leq \Delta(G) + 1$ colours.

(c) By Brook's theorem, we know that it is possible to colour the graph with $\leq \Delta(G)$ colours, can you find such a colouring?

14. *Map colouring.* In the lectures we considered vertex colourings. A colouring was an allocation of colours to vertices so that no vertices connected by an edge shared the same colour. This same idea can be used when colouring regions of a graph. We wish to colour a graph so that no regions sharing a common boarder have the same colour. We will call the minimum number of colours used the **region-chromatic number**.

The graph below is basically just a pie chart.



(a) What is the region-chromatic number for the above graph?

(b) Experiment with with finding the region-chromatic number for other pie charts having varying numbers of sectors.

(c) Use your results to predict the region-chromatic number for pie charts with

(i) 37 sectors

(ii) 52 sectors

(d) What can you say, in general, about colouring pie charts?

15. *Assigning fishtanks.* Suppose a pet store has 10 types of fish. Some of the species are prone to attach each other and therefore cannot be stored in the same tank. In the chart below, an X indicates which fish are incompatible with each other.

Species	1	2	3	4	5	6	7	8	9	10
1		X	X	X						
2	X		X	X						
3	X	X		X	X				X	X
4	X	X	X		X					
5			X	X		X				
6					X		X	X	X	
7						X		X	X	
8						X	X		X	X
9			X			X	X	X		X
10			X					X	X	

(a) Draw a graph that represents the information in this table

(b) What are the upper and lower limits of the vertex chromatic number for the graph you've drawn?

(c) What is the chromatic number for the graph you've drawn?

(d) Label the tanks A, B, C, etc and allocate the fish into tanks in a way that each fish is only placed with compatible species.

16. *Scheduling landings.* Suppose a 2 hour time period is broken up into 4 blocks of 30 minutes. Different aircrafts $A, B, \ldots, G$ require use of the landing bays during those time intervals according to the following table.

A	1	2		
B			3	
C		2		4
D	1		3	4
E			3	
F	1			4
G				4

Allocate the aircrafts to bays in a minimal way.

	y_1	y_2	y_3	y_4	y_5
x_1	1	0	0	1	0
x_2	2	1	0	0	1
x_3	0	1	0	1	0
x_4	0	1	1	1	1

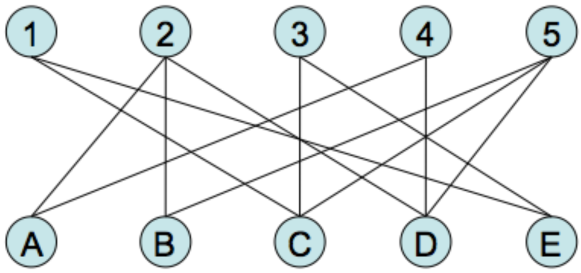
17. *Teacher timetable.* Suppose the teacher / class allocation is given by the following table

    (a) What is the smallest number of time slots required to allocate the teachers and classes so that no teacher is teaching more than one class at the same time and no class is being taught by more than one teacher at a time?

    (b) Adjust the allocations so that there are no more than three allocations per time slot.

    (c) Why might one want to minimize the number of allocations per time slot?

18. *Matchings.* You are given the bipartite graph



    (a) What is the smallest number of colours that are required to paint the edges so that no two edges connected to the same vertex are of the same colour? State the theorem that underlies your answer.

    (b) By decomposing a suitable 0,1 matrix, find a colouring of the above type.

    (c) If your matchings are not of the same size adjust them so they are.

19. *Job assignment.* The matrix below indicates which of the 5 jobs, $\{1, 2, 3, 4, 5\}$, the following 5 people are suited to. Using this information, uniquely allocate one job to each person.

	1	2	3	4	5
a	1	1	0	0	0
b	1	0	1	0	0
c	1	0	0	1	0
d	0	0	0	1	0
e	0	0	0	1	1

20. *Job assignment.* Explain why there is no job allocation for the following situation. State the theorem used.

	1	2	3	4	5
a	1	1	0	0	0
b	1	1	1	1	0
c	0	0	0	1	1
d	0	0	0	1	1
e	0	0	0	1	1

21. *Job assignment.*

    (a) Draw the bipartite graph corresponding to the 0, 1 matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

    with rows labeled $\{A, B, C, D\}$ and columns labeled $\{1, 2, 3, 4\}$.

    (b) Identify an alternating path which involves all the vertices, and give the corresponding matching.

22. *Optimal assignment problem.* Consider the following problem of assigning four jobs to four workers in such a way that the four jobs are completed as quickly as possible.

    The time require for each worker to do each job (in hours) is tabulated below.

	$W1$	$W2$	$W3$	$W4$
$J1$	4	7	4	6
$J2$	8	4	6	9
$J3$	6	3	9	7
$J4$	9	4	7	5

    Find the allocation of workers to jobs so that the time taken to complete all jobs in a minimum.

23. *Optimal assignment problem.* Sarah runs a cleaning business. Her services include, vacuuming, ironing, dusting and general tidying, $\{J1, J2, J3, J4\}$. There are four companies $\{C1, C2, C3, C4\}$ that she can use to hire workers for each job (limit one worker per company). The table below specifies the profit she can make using each company.

	$C1$	$C2$	$C3$	$C4$
$J1$	12	10	3	11
$J2$	17	12	15	14
$J3$	10	8	9	11
$J4$	14	7	11	13

    How can Sarah **maximize** the profit her business makes? Solve this **maximization** problem using the Hungarian algorithm.

Topic 3: Fair Division

Fair Division (Brams and Taylor)

References refer to the textbook: Fair division
S. J. Brams, and A. D. Taylor

24. *Method of markers.* Suppose there are three players and 12 goods, with each player's division given by

$$g_1 \Big|_A g_2 g_3 \Big|_A g_4 g_5 g_6 g_7 g_8 g_9 g_{10} g_{11} g_{12}$$
$$g_1 g_2 g_3 g_4 g_5 \Big|_B g_6 g_7 g_8 g_9 g_{10} \Big|_B g_{11} g_{12}$$
$$g_1 g_2 \Big|_C g_3 g_4 g_5 g_6 g_7 g_8 g_9 \Big|_C g_{10} g_{11} g_{12}$$

Use the method of markers to allocate the goods.

25. *Method of markers.* One drawback of the method of markers lies in the extent to which a particular ordering of the items ($g_1 g_2 g_3$ vs $g_2 g_3 g_1$) favours some players over others. Suppose there are two players and 6 items, two which are particularly highly valued. Let l_1, l_2, l_3, l_4 denote the four low valued items and h_1, h_2 denote the two high valued items.
    (a) Order the items so that both players can get the fairest share.
    (b) Order the items so that one player receives more than the other.

26. *Method of markers.* If there are two players explain why the method of markers always gives an envy free allocation.

27. *Envy free.* Suppose that there are three players, Adam, Barry and Carrie. Adam loves vowels, Barry cares only for consonants and Ted requires at least one vowel and one consonant.
    (a) Using the above information suggest a suitable division of the letters $abcdefghi$ for each player.
    (b) Using your divisions allocate the letters fairly using the method of markers.
    (c) Does your example highlight the fact that the method of markers is not **envy free**? If not, could you come up with a division that does?

28. *Method of sealed bids.* There are four items I_1, I_2, I_3, I_4 to be distributed among three people A, B, C using the method of sealed bids. The bids are given by the following table.

bids	A	B	C
I_1	$10,000	$4,000	$7,000
I_2	$2,000	$1,000	$4,000
I_3	$500	$1,500	$2,000
I_4	$800	$2,000	$1,000

    (a) Extend the table to include the fair share of each player, and their contribution to the pot.
    (b) Compute the excess in the pot.
    (c) Summarize the allocation to each player.
    (d) Show that in this example each person is **envy free**.

29. *Method of sealed bids.* Explain why the method of sealed bids maximises the sum, over all the players, of the monetary value that each player perceives they receive in items and money. This sum is referred to as the **total welfare**.

30. *Method of sealed bids.* A potential drawback of the method of sealed bids is that players can profitably misrepresent their preferences if they know the monetary values that the other player(s) attribute to the items. Show that in the case where there are two players, A, B say, and one item, I_1, that for every extra \$4 that the losing player adds to their bid (without exceeding the winning bid) will result in an extra \$1 they receive in their cash payment.

31. *Adjusted winner method.* A real divorce settlement in the 1980's between two wealthy adults with grown up children saw the division of real estate and liquid assets. The following table indicates the actual value of the items as well as the point allocation given to each item by the husband and wife.

Marital Property	Value	Husband	Wife
Paris apartment	\$642,856	35	55
Paris Studio	\$42,850	6	1
New York City coop	\$103,079	8	1
Farm	\$119,200	8	1
Cash and receivables	\$42,972	5	6
Securities	\$176,705	18	17
Profit-sharing plan	\$120,940	15	15
Life insurance policy	\$24,500	5	4
Total	\$1,273,102	100	100

(a) Using the adjusted winner method determine how the assets will be allocated.

(b) Suppose that the husband knew the wife had her heart set on the Paris apartment and would be allocating the highest proportion of her points to it. And, in knowing this the husband decided to allocate zero points to the apartment and distribute the 35 points evenly over the 7 remaining items. What effect would this have on the allocation of items?

(c) What happens if the husband tries to get as close to the wife's bid on the Paris apartment? Say he allocates 54 points to the apartment and takes the additional 19 points for the other items as evenly as possible.

(d) What if the husband, while trying the get as close to his wife's bid from the Paris apartment as possible, makes a mistake and bids 56 points? (Taking the additional 21 points by subtracting 3 from each of the remaining items)

(e) From the above questions, what conclusions can you make about the adjusted winner method?

(f) Compute the monetary value of the received items using the husband and wife's initial point allocation. Would this effect your confidence in using the adjusted winner method?

32. *Cut and choose method.* Two families, the Thompson's and the Harvey's, decide to buy a beach house together at Cowes. The Thompson's are chosen to do the split and they choose December-February and March-November, as even though it is only three of the 12 months they think Summer is the most valuable.

(a) If the Harvey's value each of the months equally which option would they choose?

(b) What is the value of each family's share in their opinion?

(c) Is the cut and choose method equitable?

(d) Is the cut and choose method envy-free?

33. *Lone divider.* The lone divider method can be extended to work for any number of parties. The method proceeds as follows.

1) The divider divides the item into N pieces, which we will label as $s_1, s_2, \ldots, s_N$.

2) Each of the choosers will separately list which pieces they consider to be a fair share. This is called their declaration or bid.

3) The lists are examined by a referee. There are two possibilities .

- If it is possible to give each party they declared then do so, and the divider gets the remaining piece.

- If two or more parties both want the same piece and no others, then give a non contested piece to the divider and combine the rest of the pieces and repeat the entire procedure with the remaining parties. If there are only two parties left, they can use the cut and choose method.

(a) Suppose that Amie, Bianca, Chris and Dom are dividing a plot of land. Dorian was selected to be the divider, say, through a coin toss. Each person's valuation of each piece of land is shown below.

	Piece 1	Piece 2	Piece 3	Piece 4
Amie	15%	30%	20%	35%
Bianca	30%	35%	10%	25%
Chris	20%	45%	20%	15%
Dom	25%	25%	25%	25%

(i) Based on these values, which pieces of land are acceptable to each person?

(ii) Give a possible allocation according to the lone divider method.

(iii) Does your allocation highlight the fact that the lone divider method is not envy free? If so how?

(iv) Is the lone divider method equitable?

(b) Suppose the valuations of the previous problem were

	Piece 1	Piece 2	Piece 3	Piece 4
Amie	15%	30%	20%	35%
Bianca	20%	35%	10%	35%
Chris	20%	45%	20%	15%
Dom	25%	25%	25%	25%

(i) Based on these values, which pieces of land are acceptable to each person?

(ii) In this case there is no simple settlement. Dom will be given one of the non-disputed pieces. Say we give him piece 3. Suppose Bianca now becomes the divider. She cuts out three new pieces and the valuations are

	Piece 1	Piece 2	Piece 3
Amie	40%	30%	30%
Bianca	33.3%	33.3%	33.3%
Chris	50%	20%	30%

How would you go about dividing up the remaining land between Amie, Bianca and Chris?

34. *Lone divider.* In the lectures we saw that the lone divider method was not envy free. Is it ever envy free? If you think it is, provide an example showing this, if you don't, explain why.

35. *Last diminisher method.* Suppose there are three players A, B, C, and each player values pieces $1, 2, 3, 4$ of a pizza in the ratios $1 : 1 : 1 : 1, 1 : 3 : 2 : 1$ and $3 : 1 : 1 : 2$, respectively, with all parts of each individual piece valued equally by each player. Suppose the players play in order A, B, C.

    (a) What portion of each piece of pizza will each player receive?

    (b) What portion of pizza does each player perceive themselves to have received?

    (c) What portion of pizza does each player perceive the others to have received?

    (d) Is the division envy-free?

36. *Last diminisher method.*

    (a) Is the last diminisher method envy-free?

    (b) Is the last diminisher method Pareto optimal? (Hint: Consider the counterexample from the lecture that was used to show that the lone divider method is not Pareto optimal)

    (c) Is the last diminisher method equitable?

37. *Last diminisher method.* With $n > 2$ will the last diminisher method always end up with one player envying another? Provide details.

38. *Last diminisher method.* In each of the division methods we have investigated each player is always trying to *maximize* their allocation, more money, more cake, a greater inheritance. However, many of these methods could be modified to work for situations where the players wish to *minimize* their allocation, for example in dividing chores between housemates. How could the last diminisher method be adjusted so each player could minimize their allocation?

39. *Envy-free cake division.* Explain why the envy-free cake division is in fact envy free.

Topic 4: Mathematics of Voting

Mathematics of voting and elections: A hands-on approach (Hodge and Klima)

References refer to the textbook: Mathematics of voting and elections
J. Hodge, and R. Klima

40. *Majority and plurality.* Explain why when there are two candidates plurality and majority are the same.

41. *Majority and plurality.* In 2003 Arnold Swarzenegger received 4,206,217 of the 8,657,915 votes cast. The next highest was Bustamante with around 2.7 million votes. Given that he was elected, is it possible that the majority rule was used? How about the plurality method?

42. *Anonymous and neutral.* Suppose three children, Zoe, Emma and Caden, are trying to decide which of their parents, Paul or Karen should choose which board game they should play. To make this decision, they decide to hold an election using a voting system invented by their friend Sally. Table 1 lists three possible combinations of votes by Zoe, Emma and Caden, and the outcome that Sally's voting system would produce for each combination. (Note: In the table, K represents a vote for Karen and P represents a vote for Paul.)

Zoe	Emma	Caden	Winner
P	K	K	P
P	P	K	K
K	K	P	K

Table 1: Results of Sally's Voting System

   (a) Is Sally's voting system anonymous or neutral?
   (b) A **dictatorship** is where one persons vote is the only vote that counts. Is Sally's voting system equivalent to a dictatorship?
   (c) An **imposed rule** is where the outcome is the same no matter how people vote. Is Sally's voting system equivalent to an imposed rule?
   (d) A **minority rule** is where the person with the least number of votes is the winner. Is Sally's voting system equivalent to a minority rule?

43. *Borda Count.* Suppose Robbo, Shazza, Gazza and Chezza are all running for the coveted office of President of the Cartoon Voice Actor's association of Australia (CVAAA). The preference orders of each of the 27 memebers of the association are given in Table 2.

$$\begin{array}{rl} 12: & R > S > G > C \\ 7: & S > G > C > R \\ 5: & G > C > R > S \\ 3: & C > G > S > R \end{array}$$

Table 2: Preference Schedule for the CVAAA Election

   (a) Who would be the winner under the majority rule?
   (b) Who would be the winner under the plurality method?
   (c) Who would be the winner under the Borda count method?
   (d) Comparing the first and third results, what does this say about the Borda count method?
   (e) Who do you think should win the election?

44. *Borda Count.* Using the Borda count committee scoring system, what would be the total if you added each of the candidates committee Borda count score?

45. *Condorcet Winner.* Is there a Condorcet winner in the CVAAA election (with preferences given by Table 2)?

46. *Condorcet Winner and Majority Rule.* Explain why, whenever the majority rule does not result in a tie, the majority rule winner will be a Condorcet winner.

47. *Independence of irrelevant alternatives.* Another way of looking at the IIA criteria is: If an election is held and a winner is declared, this winning candidate should remain the winner in any recalculation of votes as a result of one or more of the losing candidates dropping out.

    (a) Consider the following contest for best meal on MasterChef between three of the contestants based on preferential voting by 53 taste testers. To respect the privacy of the contestants we'll call them A, B, C.

    $$\begin{aligned} 27 &: A > B > C \\ 24 &: C > B > A \\ 2 &: B > C > A \end{aligned}$$

    Show that C is the loser of this contest if the Borda count method is used.

    (b) C was so embarrassed about losing the contest that they demanded to retroactively withdraw from the contest. Again using the Borda count method, who is the winner now?

    (c) Using the new definition of IIA, what does this say about the IIA criterion and the Borda count method?

    (d) How could some of the details from this question be used as a counter example for our standard definition of the IIA criterion?

48. *Nanson Borda count.*

    (a) Use the Nanson Borda count method to determine the winner of the election with the following voter preferences

    $$\begin{aligned} 6 &: P > R > K \\ 5 &: K > P > R \\ 3 &: R > K > P \\ 2 &: K > R > P \end{aligned}$$

    (b) Is this the same winner as the standard Borda count?

49. *Instant Run-off*

    (a) Show that G wins the election if an instant run-off method is used

    $$\begin{aligned} 6 &: G > M > D > S \\ 5 &: M > G > D > S \\ 4 &: D > S > M > G \\ 2 &: S > D > G > M \end{aligned}$$

    (b) G is also the winner using the plurality method. Does your above work show that a winner determined by the plurality method will always be the winner by instant run-off? Explain.

50. *Instant Run-off.* In an election of 4 candidates what is the smallest and greatest number of rounds it could take to determine a winner (ignore the case where there is no one winner). Explain why there could be no more or no less in each case.

51. *Instant run-off.* Although instant run-off only considers the first place votes in each round, in what way is instant run-off superior to the plurality method?

52. *Smith Set.*

    (a) Explain why the Smith set satisfies the monotonicity criteria.

    (b) Explain why the Smith set satisfies the Condorcet winner criteria.

    (c) Explain why the Smith set does/does not satisfy the majority criteria.

    (d) Explain why the Smith set does/does not satisfy the IIA criteria

    (e) Use this information to make a summary table for the good/bad properties of the Smith set method.

53. *Approval Voting (and more).* The University of Melbourne MathsSquad has been invited to compete in 5 different Mathematics competitions: The Australiasian Problem Solving Mathematical Olympiads (APSMO), Australian Maths Challenge (AMC), Australian Mathematical Olympiad (AMO), Australian Intermediate Mathematical Olympiad (AIMO) and Have Sum Fun Online (HSFO). The club can is only allowed to attend one competition. The club leader decides to use Approval voting to determine which competition to attend. The assistant in charge of the polling the 100 MathsSquad members misunderstood the instructions and asked each member to rank the competitions from 1^{st} to 5^{th} choice and summarized the results in the following preference schedule.

$$\begin{aligned} 39 &: APSMO > AMC > AMO > AIMO > HSFO \\ 16 &: AMC > HSFO > AIMO > AMO > APSMO \\ 45 &: AMO > AMC > HSFO > AIMO > APSMO \end{aligned}$$

    (a) Suppose the club leader decides to change the methods and choose the parade using plurality with elimination. Which competition will the MathsSquad attend?

    (b) Suppose the club leader decides to change methods and choose the competition according to the Condorcet winner. Which competition will the MathsSquad attend?

    (c) Suppose the club leader decides to change methods and choose the competition using Borda count. Which competition will the MathsSquad attend?

    (d) Suppose the club leader decides to try approval voting anyway. Which competition will the MathsSquad attend if, after re-polling, each MathsSquad member approved only his/her first, second and third choices?

    (e) Suppose the club leader decides to try approval voting anyway. Which competition will the MathsSquad attend if, after re-polling, each MathsSquad member approved only his/her first and second choices?

    (f) Suppose the club leader decides to try approval voting anyway. Which competition will the MathsSquad attend if, after re-polling, each MathsSquad member whose first choice was the AMO approved only of the AMO but everyone else approved his/her first three choices?

54. *Weighted Voting Systems.* Two weighted voting systems are said to be **isomorphic** if they have exactly the same winning coalitions. For each of the following weighted voting systems, list all of the winning coalitions. Then decide which of the systems are isomorphic to each other.

    (a) $[4 : 2, 2, 1]$

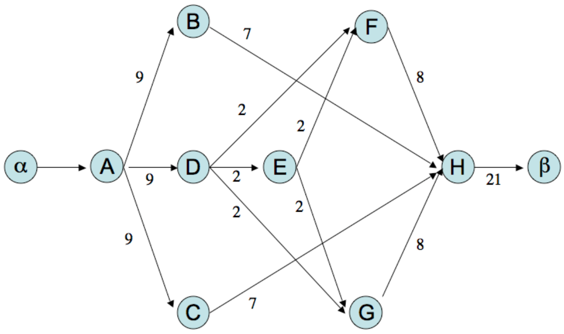
    (b) $[4 : 3, 2, 1]$

    (c) $[5 : 3, 2, 1]$

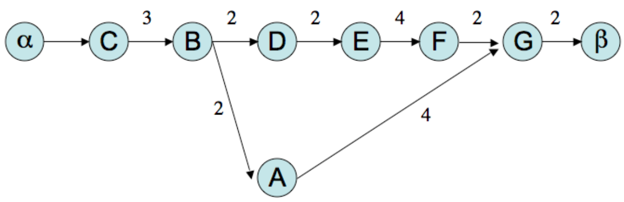
55. *Dictators, Vetos and Dummies* For the weighted voting system $[101 : 101, 97, 2]$ determine which voters are dictators, which are dummies and which have veto power.

Answers for Topic 1: Networks

1. $A : k = 0$, $B, C, D : k = 1$, $E : k = 2$, $F, G : k = 3$, $H : k = 4$, which gives



2. $C : k = 0$, $B : k = 1$, $A, D : k = 2$, $E : k = 3$, $F : k = 4$, $G : k = 5$, which gives

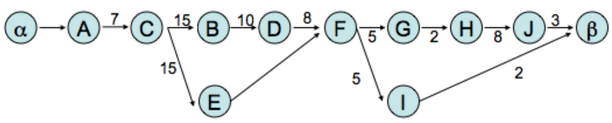


3. (a) $ABCFI$, 28 (b) $ABDEG$, 33 (c) $ACEG$, 27

4.

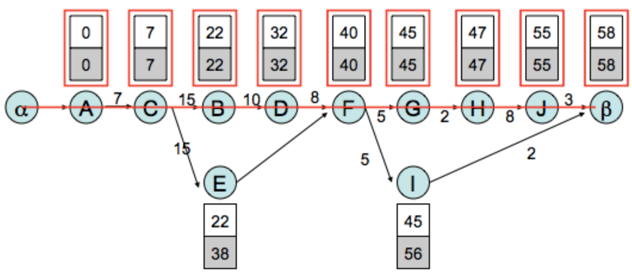
Activity	Total float	Free float
B	2	2
C	0	0
D	6	4

5.



(a)

(b) $ACBDFGHJ$



(c) No. Activity I has a float of 11 and would need to be delayed for longer than 11 days to delay the project.

(d) Activity E has a float of 16, allowing the activity to be delayed by up to and including 16 dayd without effecting the completion date of the hall.

6.

(a)

Activity	Precedence
A	
B	
C	A
D	A, B

(b)

Activity	Precedence
A	
B	
C	A
D	B, E
E	A

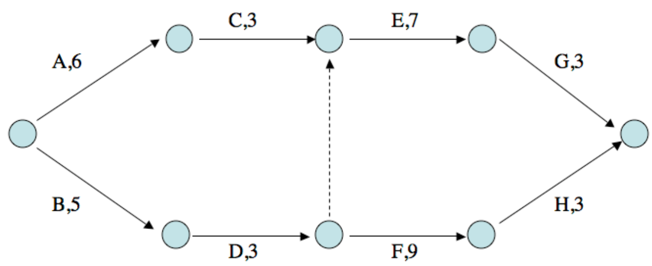
(c)

Activity	Precedence
A	
B	
C	A
D	B

7.

(a) $ACFH$, 21 days.

(b) The AOA shown has C as a precedence of F.



(c)

(d) $BDFH$, 20 days.

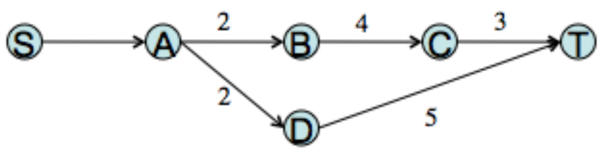
8.

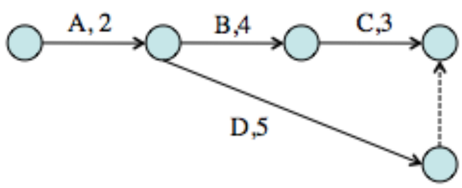
(a) If in $x[i, \ldots, j]$ the first and the last letters match, then the maximum length palindromic subsequence is two more than that for substring with the first and last letters removed. Thus $L(i, j) = L(i+1, j-1)+2$ for $x[i] = x[j]$. If $x[i] \neq x[j]$ then both these letters cannot be part of a palindromic subsequence. Removing either one gives $L(i, j) = \max\{L(i+1, j), L(i, j-1)\}$ in this circumstance. Furthermore, when there is only one letter, the maximum length palindromic subsequence is unity.

(b) We can construct the following table

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
$j=1$	1							
$j=2$	1	1						
$j=3$	1	1	1					
$j=4$	3	2	1	1				
$j=5$	4	2	1	1	1			
$j=6$	4	4	2	2	2	1		
$j=7$	4	4	2	2	2	1	1	
$j=8$	6	4	4	3	3	2	1	1

9.





(a)

(b) ABC, 9

(c)

$$\begin{bmatrix} 0 & 2 & -\infty & 2 & -\infty \\ -\infty & 0 & 4 & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty & 3 \\ -\infty & -\infty & -\infty & 0 & 5 \\ -\infty & -\infty & -\infty & -\infty & 0 \end{bmatrix}$$

(d)

$$[\ 0 \quad -\infty \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q = [\ 0 \quad 2 \quad -\infty \quad 2 \quad -\infty\ ]$$

$$\begin{aligned} [\ 0 \quad -\infty \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q^2 &= [\ 0 \quad 2 \quad -\infty \quad 2 \quad -\infty\ ] \otimes Q \\ &= [\ 0 \quad 2 \quad 6 \quad 2 \quad 7\ ] \end{aligned}$$

$$\begin{aligned} [\ 0 \quad -\infty \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q^3 &= [\ 0 \quad 2 \quad 6 \quad 2 \quad 7\ ] \otimes Q \\ &= [\ 0 \quad 2 \quad 6 \quad 2 \quad 9\ ] \end{aligned}$$

$$\begin{aligned} [\ 0 \quad -\infty \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q^4 &= [\ 0 \quad 2 \quad 6 \quad 2 \quad 9\ ] \otimes Q \\ &= [\ 0 \quad 2 \quad 6 \quad 2 \quad 9\ ] \end{aligned}$$

The minimum completion time is 9. Working backwards through the network we see that the critical path is ABC.

(e)

$$\begin{bmatrix} 0 & 2 & -\infty & -\infty & -\infty \\ -\infty & 0 & 4 & 5 & -\infty \\ -\infty & -\infty & 0 & -\infty & 3 \\ -\infty & -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty & 0 & 0 \end{bmatrix}$$

(f)

$$[\ 0 \quad -\infty \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q = [\ 0 \quad 2 \quad -\infty \quad -\infty \quad -\infty\ ]$$

$$\begin{aligned} [\ 0 \quad -\infty \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q^2 &= [\ 0 \quad 2 \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q \\ &= [\ 0 \quad 2 \quad 6 \quad 7 \quad -\infty\ ] \end{aligned}$$

$$\begin{aligned} [\ 0 \quad -\infty \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q^3 &= [\ 0 \quad 2 \quad 6 \quad 7 \quad -\infty\ ] \otimes Q \\ &= [\ 0 \quad 2 \quad 6 \quad 7 \quad 9\ ] \end{aligned}$$

$$\begin{aligned} [\ 0 \quad -\infty \quad -\infty \quad -\infty \quad -\infty\ ] \otimes Q^4 &= [\ 0 \quad 2 \quad 6 \quad 7 \quad 9\ ] \otimes Q \\ &= [\ 0 \quad 2 \quad 6 \quad 7 \quad 9\ ] \end{aligned}$$

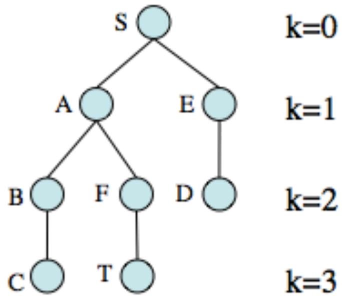
The minimum completion time is 9. Working backwards through the network we see that the critical path is ABC.

(g) (b), where either observation or the ES/LS approach was used. Both methods involve mucg less work than the Q matrix.

(h) I would assume that method (f) would be simplest for a larger project as the matrix arithmetic could be computed using technology.

10.

(a) We have



From the above tree diagram we see that the shortest path is 4.

(b) We have

$$\begin{aligned}
\mu(T|T) &= 1\\
\mu(C|T) &= 0\\
\\
\mu(B|T) &= \mu(C|T) + \mu(T|T)\\
&= 0 + 1 = 1\\
\mu(F|T) &= \mu(T|T)\\
&= 1\\
\mu(D|T) &= \mu(C|T) + \mu(T|T)\\
&= 0 + 1 = 1\\
\\
\mu(A|T) &= \mu(B|T) + \mu(F|T)\\
&= 1 + 1 = 2\\
\mu(E|T) &= \mu(D|T) + \mu(F|T)\\
&= 1 + 1\\
\\
\mu(S|T) &= \mu(A|T) + \mu(E|T)\\
&= 2 + 2 = 4
\end{aligned}$$

There are four shortest paths (SAFT, SABT, SEDT and SEFT).

(c)

$$A = \begin{bmatrix}
0 & 1 & \infty & \infty & \infty & 1 & \infty & \infty\\
1 & 0 & 1 & \infty & \infty & 1 & 1 & \infty\\
\infty & 1 & 0 & 1 & 1 & \infty & \infty & 1\\
\infty & \infty & 1 & 0 & 1 & \infty & \infty & 1\\
\infty & \infty & 1 & 1 & 0 & 1 & \infty & 1\\
1 & 1 & \infty & \infty & 1 & 0 & 1 & \infty\\
\infty & 1 & \infty & \infty & \infty & 1 & 0 & 1\\
\infty & \infty & 1 & 1 & 1 & \infty & 1 & 0
\end{bmatrix}$$

(d)

$$\begin{aligned}
[\ 0 \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty\ ] \otimes_{\min} A &= [\ 0 \quad 1 \quad \infty \quad \infty \quad \infty \quad 1 \quad \infty \quad \infty]\\
[\ 0 \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty\ ] \otimes_{\min} A^2 &= [\ 0 \quad 1 \quad \infty \quad \infty \quad \infty \quad 1 \quad \infty \quad \infty\ ] \otimes_{\min} A\\
&= [\ 0 \quad 1 \quad 2 \quad \infty \quad 2 \quad 1 \quad 2 \quad \infty\ ]\\
[\ 0 \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty \quad \infty\ ] \otimes_{\min} A^3 &= [\ 0 \quad 1 \quad 2 \quad \infty \quad 2 \quad 1 \quad 2 \quad \infty\ ] \otimes_{\min} A\\
&= [\ 0 \quad 1 \quad 2 \quad 3 \quad 2 \quad 1 \quad 2 \quad 3\ ]
\end{aligned}$$

Which confirms that the shortest path from S to T is 3

11.

(a) No shortest paths will contain this vertex unless it is an endpoint and therefore won't ever contribute to the summation.

(b) Consider the complete graph of three vertices. The shortest path between any two vertices will never contain any vertices other than the initial vertices.

12. The betweenness centrality for node A is zero, since no shortest paths pass through A without having it as an endpoint. To calculate the betweenness centrality for node B we compute $\sigma_{st}(v)/\sigma_{st}$ for $st = AD, AE, AC, CD, CE$ and DE to obtain $1/1, 2/2, 1/1, 1/2, 0/1$, and $0/1$ respectively. Therefore the betweenness centrality for node B is 3.5. Similar calculations show that the betweenness centrality for nodes C, D and E are $1, 1$ and 0.5 respectively.

Answers for Topic 2: Graph Theory and Scheduling

13.

(a) 3

(b) $c_0 = \{v_1, v_3\}, c_1 = \{v_2, v_4\}, c_2 = \{v_5\}, c_3 = \{v_6\}$

(c) $c_0 = \{v_1, v_4\}, c_1 = \{v_2, v_5\}, c_2 = \{v_3, v_6\}$

14.

(a) 3

(b)

Number of sectors	Region chromatic number
1	1
2	2
3	3
4	2
5	3

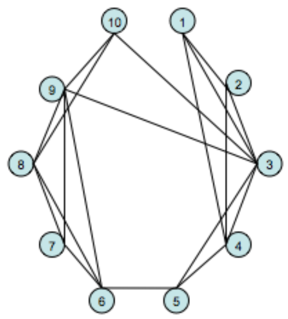
(c) (i) 3

(ii) 2

(d) If there is one sector the region-chromatic number is 1. If the number of sectors is even the region-chromatic number is 2, if the number of sectors is odd (and $\neq 1$) the region chromatic number is 3.

15.

(a) The graph that represents the information is

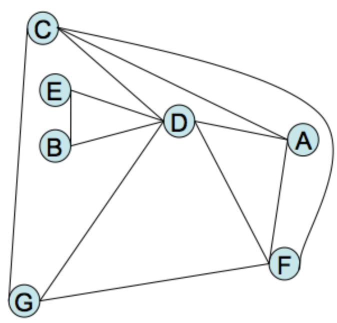


(b) By Brook's theorem the upper limit is 6, and by the size of the largest clique the lower limit is 4.

(c) The chromatic number is 4

(d) Tank A contains species $1, 6, 10$, tank B contains species $2, 5, 7$, tank C contains species $3, 8$ and tank D contains species $4, 9$.

16.



We have $c_0 = \{C, E\}, c_1 = \{D\}, c_2 = \{B, F\}, c_3 = \{A, G\}$. The smallest number of gates is therefore 4.

17.

(a) The highest row/column sum is 4 so we need 4 time slots. We decompose the matrix ensuring that at each stage we reduce the row/column with the greatest sum. We have

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

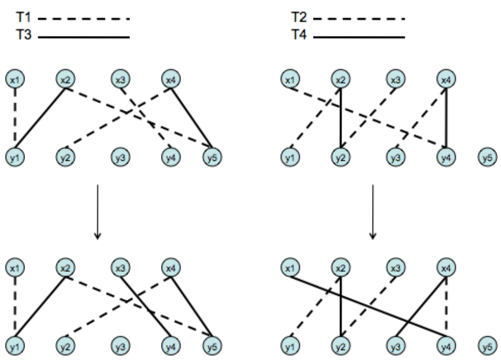
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This teacher timetable problem requires four time slots.

$$T_1 = \{x_1y_1, x_2y_5, x_3y_4, x_4y_2\}, T_2 = \{x_1y_4, x_2y_1, x_3y_2, x_4y_3\}, T_3 = \{x_2y_1, x_4y_5\}, T_4 = \{x_2y_2, x_4y_4\}.$$

(b) We make the changes



We read off the new allocations to be

$$T_1 = \{x_1y_1, x_2y_5, x_4y_2\}, T_2 = \{x_2y_1, x_3y_2, x_4y_4\}, T_3 = \{x_2y_1, x_3y_4, x_4y_5\}, T_4 = \{x_1y_4, x_2y_2, x_4y_3\}.$$

(c) By minimizing the number of allocations per time slot we minimize the number of rooms required at any one time.

18.

(a) 3. A bipartite graph with maximum degree of the vertices equal to Λ can always be edge-coloured using Λ different colours, which is the smallest possible.

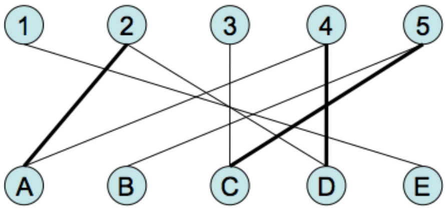
(b)

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The colouring can therefore be given by $c_0 = \{A4, B5, C3, D2, E1\}, c_1 = \{B2, C1, D5, E3\}, c_2 = \{A2, C5, D4\}$

(c) The matchings will be of the same size if one edge from c_0 is moved to c_2. We can determine which one to change by considering the following graph (with the edges of c_2 in bold)

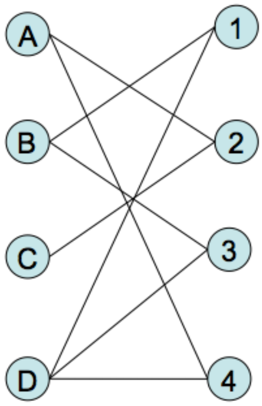


We see that there is an alternating path between $B5C3$ by changing the matching so $C3$ and $B5$ are in c_2 and $C5$ is in c_0 we get the desired result.

19. $(a, 2), (b, 3), (c, 1), (d, 4), (e, 5)$

20. Hall's theorem states that a necessary and sufficient condition for it to be possible to choose a 1 in each row, no two of which are in the same column, is that for each subset of k rows, there are 1's in at least k different columns. If we select rows $3, 4, 5$ there are not three different columns that contain 1's (the only columns containing $1's$ are columns 4 and 5), and therefore no allocation can be found.

21.



(a)

(b) (C2A4D3B1), $\{C2\}, \{A4\}, \{D3\}, \{B1\}$.

22. We use the Hungarian algorithm. First subtract the smallest number from each row, then subtract the smallest number from each column

$$\begin{bmatrix} 4 & 7 & 4 & 6 \\ 8 & 4 & 6 & 9 \\ 6 & 3 & 9 & 7 \\ 9 & 4 & 7 & 5 \end{bmatrix} \to \begin{bmatrix} 0 & 3 & 0 & 2 \\ 4 & 0 & 2 & 5 \\ 3 & 0 & 6 & 4 \\ 5 & 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 3 & 0 & 1 \\ 4 & 0 & 2 & 4 \\ 3 & 0 & 6 & 3 \\ 5 & 0 & 3 & 0 \end{bmatrix}$$

Secondly, cover the zeros by lines using the minimum number of lines. The zeros in this matrix can be covered by three lines, rows 1 and 4 and column 2. Since we do not require 4 lines we locate the smallest uncovered number, here 2 in position $(2, 3)$. We subtract 2 from every uncovered entry and add 2 to all the entries that are crossed twice. Giving us

$$\begin{bmatrix} 0 & 5 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 1 & 0 & 4 & 1 \\ 5 & 2 & 3 & 0 \end{bmatrix}$$

We now have 4 independent zeros and the minimal allocation is $\{J1, W1\}$,$\{J2, W3\}$,$\{J3, W2\}$,$\{J4, W4\}$.

23. To be able to use the Hungarian algorithm we must first turn this maximization problem into a minimization problem. We do this be computing $\max_{i,j}\{c_{ij}\} - c_{ij} = 17 - c_{ij}$ for each entry of the matrix. This gives us

$$\begin{bmatrix} 5 & 7 & 14 & 6 \\ 0 & 5 & 2 & 3 \\ 7 & 9 & 8 & 6 \\ 3 & 10 & 6 & 4 \end{bmatrix}$$

We can now proceed with the Hungarian algorithm. We begin with row and column reduction

$$\begin{bmatrix} 5 & 7 & 14 & 6 \\ 0 & 5 & 2 & 3 \\ 7 & 9 & 8 & 6 \\ 3 & 10 & 6 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 9 & 1 \\ 0 & 5 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 0 & 7 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & \mathbf{0} & 7 & 1 \\ 0 & 3 & \mathbf{0} & 3 \\ 1 & 1 & 0 & \mathbf{0} \\ \mathbf{0} & 5 & 1 & 1 \end{bmatrix}$$

We have four independent zeros (highlighted in bold). We therefore allocate the companies to the jobs as $\{J1, C2\}, \{J2, C3\}, \{J3, C4\}, \{J4, C1\}$.

Answers for Topic 3: Fair Division

24. A receives g_1, B receives g_{11}, g_{12} and C receives $g_3, g_4, g_5, g_6, g_7, g_8, g_9$.

25.
   (a) $l_1 l_2 h_1 l_3 l_4 h_2$
   (b) $l_1 l_2 l_3 l_4 h_1 h_2$

26. If there are two players, each player must, in order to create nonempty consecutive segments, place their marker somewhere along the line. If both markers are in the same place, give one player the items to the left of the marker and one player the items to the right of their marker. Both players will value each half equally and therefore won't envy each other. If the markers are in different places (without loss of generality assume the first players marker is before the second) player one receives the goods to the left of the first line and player two receives goods to the right of the second. Neither player will be envious of the other as in their eyes their apponent receives less than half. (The only time the allocation will not be envy-free is if the allocation of the remaining goods is done in an eviable way.)

27.
   (a) One possible set of divisions

$$a\Big|_A bcde\Big|_A fghi$$
$$abc\Big|_B def\Big|_B ghi$$
$$ab\Big|_C cde\Big|_C fghi$$

   (b) Adam gets a, Barry gets ghi and Carrie gets cde
   (c) Adam could envy Barry who even though gets the same number of vowels as him, has three letters rather than one.

28.
   (a) The extended table reads

bids	A	B	C
I_1	$10,000	$4,000	$7,000
I_2	$2,000	$1,000	$4,000
I_3	$500	$1,500	$2,000
I_4	$800	$2,000	$1,000
Total	$13,300	$8,500	$14,000
Fair Share	$4,433	$2,833	$4,667
Items Won	I_1	I_4	I_2, I_3
Pot	$5,567	−$833	$1,333

   (b) The pot contains an excess of $6,067 (so each player gets $2,022)
   (c) A gets I_1, and pays $-5567 + 2,022 = \$3,545$ cash. B gets I_4, and receives $833 + 2,022 = \$2,855$ cash. C gets I_2, I_3, and receives $-1333 + 2,022 = \$689$ cash. Note that it is the rounding that causes the error in total deficits and total excess of cash.
   (d)
   - In the eyes of person A, they receive $6,455 in monetary value. They value B's allocation as $\$800 + \$2,855 = \$3,655$ and C's allocation as $\$2,500 + \$689 = \$3,139$. So A is envy-free.
   - In the eyes of person B, they receive $4,855 in monetary value. They value A's allocation as $\$4,000 - \$3,545 = \$455$ and C's allocation as $\$2,500 + \$689 = \$3,189$. So B is envy-free.
   - In the eyes of person C, they receive $6,689 in monetary value. They value A's allocation as $\$7,000 - \$3,545 = \$3,455$ and B's allocation as $\$1,000 + \$2,855 = \$3,855$. So B is envy-free.

29. Each player is receiving the items that they value more than any other players (the compensation paid by some players to other players does not effect the total welfare).

bids	A	B
I_1	$\$x$	$\$y$
Total	$\$x$	$\$y$
Fair Share	$\$x/2$	$\$y/2$
Items Won	I_1	
Pot	$\$x/2$	- $\$y/2$

30. Suppose without loss of generality that player A wins the item. (see table above)

The pot contains an excess of $\$x/2 - \$y/2$ (positive because $x > y$ since A is the winner of the item). So each player gets $\$(x - y)/4$. Therefore A receives the item and gives up $\$x/2 - \$(x - y)/4 = \$(x + y)/4$ while B receives $-\$y/2 + \$(x - y)/4 = \$(x + y)/4$. Therefore any \$4 increase in B's bid (ensuring that it does not exceed A's bid) results in \$1 extra in their cash payout.

31.

(a) The initial situation gives The above allocation (indicated by bold-type) indicates that after the winning

Marital Property	Value	Husband	Wife
Paris apartment	$\$642,856$	35	**55**
Paris Studio	$\$42,850$	**6**	1
New York City coop	$\$103,079$	**8**	1
Farm	$\$119,200$	**8**	1
Cash and receivables	$\$42,972$	5	**6**
Securities	$\$176,705$	**18**	17
Profit-sharing plan	$\$120,940$	**15**	15
Life insurance policy	$\$24,500$	**5**	4
Total	$\$1,273,102$	100	100

phase the husband receives the Paris studio, New York City Coop, Farm, Securities, Profit sharing plan and life insurance policy (60 points) and the wife receives the Paris apartment and the cash and receivables (61 points). This means that the wife must transfer all or part of the most similarly rated item. This is the cash and receivables, (ratio 1.2). To find the proportion of this asset that the wife must give up we solve

$$60 + 5x = 55 + 6(1 - x)$$

The solution is $x = 1/11$. Therefore the final division is: the husband receives the Paris studio, New York City Coop, Farm, Securities, Profit sharing plan, life insurance policy and $\$3,906$ of the cash and receivables (60.5 points) and the wife receives the Paris apartment and $\$39,066$ of the cash and receivables (60.5 points)

(b) Alternatively, the points would be allocated as shown below. This outcome puts the husband on 100

Marital Property	Value	Husband	Wife
Paris apartment	$\$642,856$	0	**55**
Paris Studio	$\$42,850$	**11**	1
New York City coop	$\$103,079$	**13**	1
Farm	$\$119,200$	**13**	1
Cash and receivables	$\$42,972$	**10**	6
Securities	$\$176,705$	**23**	17
Profit-sharing plan	$\$120,940$	**20**	15
Life insurance policy	$\$24,500$	**10**	4
Total	$\$1,273,102$	100	100

points and the wife on 55. The first item that gets transferred is the profit sharing plan. Still the husband has more points than his wife (80 vs 70) and the next item to transfer is the securities, though only part of. We solve

$$57 + 23x = 70 + 17(1 - x)$$

and see that $x = 3/4$. So in this case the wife receives more (the Paris apartment, the profit share plan, plus \$132,529 of the cash and securities).

(c) Alternatively, the points would be allocated as shown in the table. The above allocation (indicated by

Marital Property	Value	Husband	Wife
Paris apartment	\$642,856	54	**55**
Paris Studio	\$42,850	**3**	1
New York City coop	\$103,079	**5**	1
Farm	\$119,200	**5**	1
Cash and receivables	\$42,972	2	**6**
Securities	\$176,705	15	**17**
Profit-sharing plan	\$120,940	13	**15**
Life insurance policy	\$24,500	3	**4**
Total	\$1,273,102	100	100

bold-type) indicates that after the winning phase the husband receives the Paris studio, New York City Coop and Farm (13 points) and the wife receives the Paris apartment, the cash and receivables, Securities, Profit sharing plan and life insurance policy (97 points). The closest ratio is the Paris apartment so we proceed to solve

$$42 + 52x = 13 + 54(x - 1)$$

The solution is $x = 25/109$. This means that the wife will receive the cash and receivables, the securities, the profit-sharing plan, the life insurance policy and \$76,897 of the Paris apartment and the husband will receive the Paris studio, the New York coop, the farm and \$565,959 of the Paris apartment.

(d) Alternatively, the points would be allocated as

Marital Property	Value	Husband	Wife
Paris apartment	\$642,856	**56**	55
Paris Studio	\$42,850	**3**	1
New York City coop	\$103,079	**5**	1
Farm	\$119,200	**5**	1
Cash and receivables	\$42,972	2	**6**
Securities	\$176,705	15	**17**
Profit-sharing plan	\$120,940	12	**15**
Life insurance policy	\$24,500	2	**4**
Total	\$1,273,102	100	100

The above allocation (indicated by bold-type) indicates that after the winning phase the husband receives the Paris apartment, Paris studio, New York City Coop and Farm (69 points) and the wife receives the the cash and receivables, Securities, Profit sharing plan and life insurance policy (42 points). This means that the husband must transfer all or part of the most similarly rated item. This is the Paris apartment. To find the proportion of the Paris apartment that needs to be transferred is then

$$13 + 56x = 42 + (1 - x)55$$

The solution is $x = 28/37$. Therefore the final division is: the husband receives the Paris studio, New York City Coop, Farm and \$486,486 of the Paris apartment (55.4 points) and the wife receives the cash and receivables, securities, profit sharing plan, life insurance policy and \$156,370 of the Paris apartment (55.4 points).

(e) Attempts to manipulate the adjusted winner method can badly hurt the players in practice, unless, of course, their intelligence about the party's allocation is perfect. Such intelligence on the other party's allocations will in most cases be difficult to exploit. Hence, adjusted winner is likely to induce honest point allocations.

(f) The wife receives \$681,922 and the husband receives \$591,180. (Possible response) Although it appears that the wife comes out better than her husband (by \$90,742, none the less!) according to the value they place on the items the allocation is fair and therefore the adjusted winner method is a procedure I believe to be fair and worth using.

32.

(a) March-November

(b) Thompson's 50% (as they did the cut at what they perceived to be $50-50$), Harvey's 75% (they value each month equally and they get 3/4 of the months in their share).

(c) No, the Thompson's get less than the Harvey's.

(d) Yes, neither families would prefer the others' months over their own.

33.

(a) (i) Amie will accept pieces 2 and 4, Bianca will accept pieces 1,2 and 4 and Chris will accept piece 2.
(ii) Give Chris piece 2, Amie piece 4, Bianca piece 1 and Dom piece 3.
(iii) Bianca will envy Chris who gets a piece of land she values more than the piece she received.
(iv) This method is not equitable as Chris receives a piece he values more than the value each other person places on the piece they receive.

(b) (i) Amie will accept pieces 2 and 4, Bianca will accept pieces 2 and 4 and Chris will accept piece 2.
(ii) Here the only piece that is accepted is piece 1. In this case we can give Bianca either of pieces 2 and 3 and then let Amie and Chris use the cut and choose method with the remaining land.

34. Yes. For example, if there was a piece of land that when divided by the divider resulted in sections that each person valued equally then the allocations will be envy-free.

35.

(a) A makes the first cut 1/3 of the way through the second piece, (since piece 1 and 1/3 of piece 2 is 1/3 of the total pizza in A's eyes). In B's opinion this is only $(1/7)+(1/3)*(3/7)=(2/7)<(1/3)$ of the pizza so B leaves it. C believes 1/3 of the value of the pizza is 7/9 of the first piece (since $(7/9)*(3/7)=(1/3)$). C takes this portion of the pizza and leaves the game.

Player B now cuts. There remains 2/9 of piece 1 and all of pieces $2,3,4$. These have values to player B $(2/9):3:2:1=2:27:18:9$. One half of this consists of the remains of piece 1 and 26/27 of piece 2. For A, who has values $(2/9):1:1:1=2:9:9:9$, half of the remaining pizza is all that remains of piece 1, the whole of piece 2 and 7/18 of piece 3. So A leaves the piece to B and takes the remaining 1/27 of piece 2 and all of pieces 3 and 4.

In summary, Player C receives the first 7/9 of piece 1, player B receives the last 2/9 of piece 1 and 26/27 of piece 2, and player A receives the last 1/27 of piece 2 and all of pieces 3 and 4.

(b) C believes they received exactly 1/3, B believes they received $(2/9)*(1/7)+(26/27)*(3/7)=(4/9)\approx 0.44$ and A believes they received $(1/27)*(1/4)+(1/4)+(1/4)=(15/29)\approx 0.51$.

(c) A believes B received $(2/9)*(1/4)+(26/27)*(1/4)=(301/1044)\approx 0.30$ and believes C receives $(7/9)*(1/4)=(7/36)\approx 0.19$ (Check: A+B+C=0.51+0.30+0.19=1). B believes A receives $(1/27)*(3/7)+(2/7)+(1/7)\approx 0.44$ and believes C receives $(7/9)*(1/7)=(1/9)\approx 0.11$ (Check: $A+B+C=0.44+0.44+0.11=0.99\approx 1$). C believes A receives $(1/27)*(1/7)+(1/7)+(2/7)=(82/189)\approx 0.43$ and believes B receives $(2/9)*(3/7)+(26/27)*(1/7)=(44/189)\approx 0.23$ (Check: $A+B+C=0.43+0.23+0.33=0.99\approx 1$).

(d) This example is envy-free as no player perceives another player to receive more than themselves.

36.

(a) No. Although each player can ensure they receive a piece at least $1/n$ by not diminishing it unless they view it greater than $1/n$, all but the last two players may envy those who receive pieces later. More specifically, because the earlier players have exited the game they cannot prevent a piece they regard greater than $1/n$ from going to one of the other players.

(b) No. Suppose we have a cake, 1/3 chocolate, 1/3 vanilla and 1/3 strawberry. Assume Adam only likes chocolate, Bette only likes vanilla and Con only likes strawberry. Suppose Adam is the cutter, he will cut the cake to what he considers 1/3 of its total value. Thus he will cut a piece consisting of 1/3 chocolate and maybe some vanilla and strawberry as well. If it contains less than 1/3 of both the vanilla and strawberry, then Adam will end up with this piece. On the other hand, if it is trimmed by one of the other players to what they consider to be 1/3 of the cake's value, then it still cannot end up being one of the three pieces in the unique Pareto optimal allocation.

(c) No. As in the first explanation.

37. The only case that will result with no players being envious is when every player has the same preferences. If this does not happen the player who receives the first piece will always be envious of one of the later player. This is because the first player leaves what they believe to be exactly $(N-1)/N$ of the object left and since there is at least one player with a preference that differs from that of the first player there exists some i for which the amount of cake left after the i^{th} cutter is greater than $(N-i)/N$, leaving more than $1/N$ for each of the $N-i$ remaining players.

38. A piece would be cut by a player and passed along with each player increasing the piece until it equaled $1/n$, thereby ensuring the remaining chores (or part of chores) are not subsequently divided in such a way - say equally 0 that they may be forced to take more than $1/n$. If a player thought a piece was at least $1/n$, they would not add to it, because this would mean that they might end up with more than their proportional share. Similarly, the aim of a player using this procedure would be to take minimal rather than maximal choices at each stage so as to ensure they don't obtain more than their fair share.

39. After B has done the trimming and the players have chosen their pieces (in order C, B, A) the state will be envy free. C is envy-free because they picked first. B is envy free because they choose one of two of their equally-favourite pieces (one of which is T). A is envy-free since they were the one who divided the piece equally in the first place, they only piece they may not want is T, but the protocol guarantees they won't be left with it.

The situation remains envy free after the trimmings have been divided. It is clearly envy-free for the first person who chooses. It's also envy-free for A who never envies the person who originally received the trimmed piece and it is envy free for the last person, the person who trimmed, because they believe each portion of the trimmed section is the same.

Note: That we require this process to be additive, that is, if you don't envy someone's slice at stage one and you don't envy their slice at stage two, then you don't envy their combination.

Answers for Topic 4: Mathematics of Voting

40. If there are two candidates, A and B. With A being the winner using the plurality method, this means A gets more votes than B, since there are only two candidates this must be more than half the votes, making A the winner under the majority rule also. Alternatively, if A is the winner by the majority rule, A gets more than half the votes. Therefore A must receive more votes than B, making A the winner by the plurality method also. We can therefore conclude that when there are only two candidates that plurality and majority are the same.

41. To be the winner under the majority rule Swarzenegger needed over 4,328,958 votes, which he did not. However, the plurality method could have been used as Swarzenegger did get more votes than any other candidate.

42.

   (a) The first and third rows show that the voting system isn't anonymous and the last two rows show that it is not neutral.

   (b) The voting system is not equivalent to a dictatorship since neither Zoe, Emma or Caden always agree with the winning outcome.

   (c) The voting system is not equivalent to an imposed rule as the winner is not the same for each combination of votes.

   (d) The voting system is not equivalent to a minority rule as in the third row the winner Karen even though Paul receives the fewest votes.

43.

   (a) There would be no winner since no one receives 14 votes or more (the majority of the 27 votes).

   (b) Robbo would be the winner under the plurality method since he receives more votes than any other candidate.

   (c) Using the committee Borda count score we have $R = 4\times 12 + 1\times 7 + 2\times 5 + 1\times 3 = 68$, $S = 3\times 12 + 4\times 7 + 1\times 5 + 2\times 3 = 75$, $G = 2\times 12 + 3\times 7 + 4\times 5 + 3\times 3 = 74$, $C = 1\times 12 + 2\times 7 + 3\times 5 + 4\times 3 = 53$. Making Shazza the winner.

   (d) The Borda count method does not satisfy the majority criterion.

   (e) There is no right answer. However, there is a wrong answer, that is Chezza, unless you thought the minority rule was a good way to go!

44.

Let c be the number of candidates and N be the number of voters. The total of all candidates committee Borda score is $(1+2+\ldots+c)N = \frac{c(c+1)}{2}N$. If we give the candidate in the last preference place a score of 1 and the second best place 2 etc then every person contributes $1+2+\ldots+c$ votes, making it a total of $(1+2+\ldots+c)N$ for all voters votes. Note: The identity $1+2+\ldots+c = \frac{c(c+1)}{2}$ can be seen by considering the following table

$$\begin{array}{cccc} 1 & 2 & \ldots & c \\ c & c-1 & \ldots & 1 \end{array}$$

Each column adds up to $c+1$ and there are c of them therefore $2(1+2+\ldots+c) = c(c+1)$. The identity follows by dividing both sides by 2.

45. No candidate beats all other candidates. Robbo loses to Chezza and Gazza, Shazza loses to Robbo, Gazza loses to Shazza and Chezza loses to Shazza and Gazza.

46. If majority rule does not result in a tie, then there must be one candidate in the election who is ranked first by more than half of the voters. Even without the voters of any of the other votes in the election this candidate would win a head-to-head contest against any of the other candidates.

47.

   (a) Using the committee Borda count scores we have A with a score of 107, B a score of 108 and C a score of 103. Making C the loser.

   (b) The new situation is represented by

$$\begin{array}{r} 27: A > B \\ 24: B > A \\ 2: B > A \end{array}$$

   The new scores are $A = 80$ and $B = 79$, making A the new winner.

(c) This example is a counter example showing that the Borda count does not satisfy the IIA criterion.

(d) Suppose the contest was just between A and B, (A the winner in this case), then C entered later, making B the new winner.

48.

(a) Borda scores are as follows $P = 1$, $K = 1$, $R = -2$ (Check: these add to zero.) Eliminating R gives

$$\begin{array}{l} 6 : P > K \\ 5 : K > P \\ 3 : K > P \\ 2 : K > P \end{array}$$

New Borda scores are $P = -4$, $K = 4$. K is the winner using the Nanson Borda count method.

(b) No, there would have been a tie between K and P had the standard Borda count method been used.

49.

(a) S is eliminated first leaving

$$\begin{array}{l} 6 : G > M > D \\ 5 : M > G > D \\ 4 : D > M > G \\ 2 : D > G > M \end{array}$$

M is then eliminated leaving

$$\begin{array}{l} 6 : G > D \\ 5 : G > D \\ 4 : D > G \\ 2 : D > G \end{array}$$

G is the final winner.

(b) No. Although you only need one counter example to show something is not true, to show something is true one example will rarely be enough. For example, if G was placed after D in the last two rows we would have had D the run-off winner, even though G would remain the winner by the plurality method. To show something is true you need to consider the general case.

50. The smallest number would be 1, for example suppose one candidate has 2 votes while the remaining 3 candidates have 1 vote each. These candidates would be eliminated in the first round leaving only one candidate, the winner, left. The largest number is 3 since at least one candidate must be eliminated each round. After one round there will be at most three candidates, after two rounds at most two candidates, after three at most one candidate remains.

51. When using the plurality method, all of the information in the preference schedule not related to the first place is ignored. While, in instant run-off the lower order preferences (2^{nd}, 3^{rd}, etc) can make it into the first preference position in later rounds.

52.

(a) Ranking a candidate higher will only increase their head to head wins making the voting system monotone.

(b) If a candidate one all head to heads they would immediately become the winner using the Smith set.

(c) If one candidate receives the majority of the votes they immediately win all head to heads and become the winner.

(d) The Smith set will satisfy the IIA criterion as the additional candidate can not make a person who originally lost the election a winner as this additional candidate will not effect the head to heads between the winning candidate and the losing candidates.

53.

(a) AMO

(b) AMC

(c) AMC

(d) AMC

(e) AMC

(f) AMO

54.

(a) The winning coalitions are $\{v_1, v_2\}$ and $\{v_1, v_2, v_3\}$

(b) The winning coalitions are $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_1, v_2, v_3\}$

(c) This system is isomorphic to (will work in the same way as) the one in part (a)

55. v_1 is a dictator and has veto power. v_2 and v_3 are dummies.