MAST20004 Probability

Tutorial Set 7

1. Let X be a random variable with distribution function F_X and $Y = \Phi(X)$ where Φ is a strictly decreasing function mapping S_X to the set S_Y.
   (a) Derive an expression for $F_Y(y)$ in terms of $F_X(x)$ that is valid for continuous X, and another that is valid for discrete X.
   (b) Assuming that X is a continuous random variable, and with F_X and Φ both differentiable functions on S_X, derive a formula for the density function of Y in terms of the derivative of Φ^{-1}.
   (c) Give a version of the formula in (b) that doesn't contain the derivative of Φ^{-1}.
   (d) For the random variable $Y = e^{-X}$ where $X \stackrel{d}{=} R(0,1)$, verify that your formulae for (b) and (c) give the same result.
   (e) Use your result from (c) and the corresponding formula for increasing Φ derived on lecture slide 285 to write down a formula valid for either increasing or decreasing Φ.

   Solution: First observe that, because Φ is strictly decreasing, it must be one-to-one and so its inverse Φ^{-1}, which maps S_Y to S_X, must exist. Then for $y \in S_Y$.
   (a)

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(\Phi(X) \le y) \\ &= P(X \ge \Phi^{-1}(y)) \\ &= 1 - P(X < \Phi^{-1}(y)) \\ &= \begin{cases} 1 - F_X(\Phi^{-1}(y)) & \text{if } X \text{ is continuous} \\ 1 - F_X(z) & \text{if } X \text{ is discrete.} \end{cases} \end{aligned}$$

   In the last line z is the largest number less than $\Phi^{-1}(y)$ that X can take with positive probability. A neat way to incorporate both the cases is to write

$$F_Y(y) = \lim_{u \to \Phi^{-1}(y)^-} (1 - F_X(u)).$$

   (b) For $y \in S_Y$,

$$f_Y(y) = -f_X(\Phi^{-1}(y)) \frac{d}{dy} \left(\Phi^{-1}(y) \right).$$

   (c) For $y \in S_Y$,

$$f_Y(y) = -f_X(\Phi^{-1}(y)) \frac{1}{\Phi'(\Phi^{-1}(y))}.$$

(d) For $X \stackrel{d}{=} R(0,1)$,

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Here, $S_X = [0,1]$ and $S_Y = [e^{-1}, 1]$. So, for $y \in S_Y$, $\Phi^{-1}(y) = -\log y$. So the formula from (b) gives

$$f_Y(y) = \begin{cases} \dfrac{1}{y} & \text{if } y \in [e^{-1}, 1] \\ 0 & \text{otherwise.} \end{cases}$$

The formula from (c) gives the same thing.

(e)

$$f_Y(y) = \left| f_X((\Phi^{-1}(y)) \frac{1}{\Phi'(\Phi^{-1}(y))} \right|.$$

2. Let the joint probability mass function of random variables X and Y be given by

$$p(x,y) = \begin{cases} \frac{1}{7}x^2 y & \text{if}(x,y) = (1,1), (1,2), (2,1) \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the probability $P(X \le Y)$.

(b) Are X and Y independent? Why or why not?

Solution:

(a) $P(X \le Y) = P(\{(1,1),(1,2)\}) = \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$.

(b) They are not independent, because, for example, $P((1,1)) = \frac{1}{7}$, $P(X=1) = \frac{3}{7}$, $P(Y=1) = \frac{5}{7}$, and $P((1,1)) \neq P(X=1)P(Y=1)$.

3. Let (X,Y) denote the coordinates of a point chosen at random inside the unit circle with centre at the origin. That is X and Y have joint density function given by

$$f_{(X,Y)}(x,y) = \begin{cases} \dfrac{1}{\pi} & x^2 + y^2 \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density function of X.

(b) Are X and Y independent?

(c) Find the conditional density function of X given that $Y = y$.

Solution:

(a) For $-1 \le x \le 1$,

$$\begin{aligned} f_X(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\ &= \frac{2\sqrt{1-x^2}}{\pi}. \end{aligned}$$

(b) They are not independent. First observe that, by symmetry, for $-1 \le y \le 1$,

$$f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi}.$$

Then, for example, $f_{(X,Y)}\left(\frac{3}{4},\frac{3}{4}\right) = 0$, but $f_X\left(\frac{3}{4}\right) f_Y\left(\frac{3}{4}\right) = 7/\left(4\pi^2\right)$.

(c) For $y \in [-1,1]$ and $x \in \left[-\sqrt{1-y^2}, \sqrt{1-y^2}\right]$,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{(X,Y)}(x,y)}{f_Y(y)} \\ &= \frac{1}{2\sqrt{1-y^2}}. \end{aligned}$$

4. Let X denote the number of DVD players sold during a particular week by a certain store. The pmf of X is

x	0	1	2	3
$p_X(x)$	0.15	0.25	0.35	0.25

It is also known that 60% of all customers who purchase DVD players also buy an extended warranty. Let Y denote the number of purchases during this week who buy an extended warranty.

(a) Calculate $P(X = 3, Y = 2)$. (*Hint*: This probability equals $P(Y = 2|X = 3) \cdot P(X = 3)$; now think of the 3 purchases as 3 trials of a binomial experiment, with success on a trial corresponding to buying an extended warranty.)

(b) Determine the joint pmf of X and Y and then the marginal pmfs of X and Y. (*Note*: It is sufficient to represent a pmf by a table.)

Solution:

(a) $P(X = 3, Y = 2) = P(Y = 2|X = 3)P(X = 3) = 3 \times 0.6^2 \times 0.4 \times 0.25 = 0.108$.

(b)

	0	1	2	3	$p_Y(y)$
0	0.15	0.4×0.25	$(0.4)^2 \times 0.35$	$(0.4)^3 \times 0.25$	0.322
1	0	0.6×0.25	$2 \times 0.6 \times 0.4 \times 0.35$	$3 \times (0.6) \times (0.4)^2 \times 0.25$	0.390
2	0	0	$(0.6)^2 \times 0.35$	$3 \times (0.6)^2 \times (0.4) \times 0.25$	0.234
3	0	0	0	$(0.6)^3 \times 0.25$	0.054
$p_X(x)$	0.15	0.25	0.35	0.25	

5. Let X and Y have joint density function

$$f_{(X,Y)}(x,y) = \begin{cases} 4xy & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density functions of X and Y.

(b) Are X and Y independent?

(c) Find the probability $P(X + Y < 1)$.

Solution:

(a)

$$f_X(x) = \int_0^1 4xydy = 2x,\ 0 \le x \le 1$$

By symmetry, $f_Y(y) = 2y$, $0 \le y \le 1$.

(b) They are independent, since $f_{(X,Y)}(x,y) = f_X(x)f_Y(y)$ for $0 \le x \le 1$, $0 \le y \le 1$.

(c)

$$\begin{aligned} P(X+Y<1) &= \int_0^1 \int_0^{1-x} 4xy\,dy\,dx \\ &= \int_0^1 2x(1-x)^2 dx = \tfrac{1}{6}. \end{aligned}$$

6. Let X and Y have joint density function

$$f_{(X,Y)}(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the marginal density functions of X and Y.

(b) Find $P\left(X \le \frac{1}{2} | Y \ge \frac{1}{4}\right)$.

(c) Find $P\left(X \le \frac{1}{2} | Y = \frac{1}{4}\right)$.

Solution:

(a)

$$f_X(x) = \int_0^x 8xydy = 4x^3,\ 0 \le x \le 1,$$

$$f_Y(y) = \int_y^1 8xydx = 4y - 4y^3,\ 0 \le y \le 1.$$

(b)

$$\begin{aligned} P\left(X \le \tfrac{1}{2} | Y \ge \tfrac{1}{4}\right) &= \frac{\int_{1/4}^{1/2} \int_{1/4}^{x} 8xydydx}{\int_{1/4}^{1} \int_{1/4}^{x} 8xydydx} \\ &= \frac{9}{256} / \frac{225}{256} \\ &= \frac{9}{225} \\ &= 0.04. \end{aligned}$$

(c) Since

$$f_{X|Y}(x|\tfrac{1}{4}) = \frac{2x}{1-(1/4)^2} = \frac{32x}{15},\ \tfrac{1}{4} \le x \le 1,$$

then

$$P\left(X \le \tfrac{1}{2} | Y = \tfrac{1}{4}\right) = \int_{1/4}^{1/2} \frac{32x}{15} dx = \frac{1}{5} = 0.2.$$

MAST20004 Probability

Computer Lab 7

In this lab you

- investigate how the parameters for a bivariate normal affect the shape of the pdf (using a Matlab m-file which allows you to view a 3D plot of the pdf surface from various directions).
- generate sets of observations on various bivariate distributions and view the pattern of points and the empirical marginal pdfs for X and Y.

Exercise A - Bivariate Normal pdf

The Matlab m-file **Lab7ExA.m** displays a 3D plot of a bivariate normal pdf and allows you to view it from various directions simply by clicking and dragging the plot. You can change the surface type using the 'Type' drop down list and use the sliders on the bottom to change any of the five parameters of the bivariate normal, including the correlation coefficient rho (ρ).

Experiment with various parameter values and views to get a feel for the shape of the distribution and how it varies. Remember that the probability that the random bivariate normal point (X, Y) lies in any region R on the plane is given by the volume under the surface above the region R and that the total volume is always 1.

Exercise B - Simulating various bivariate distributions

1. This exercise uses the Matlab m-file **Lab7ExB.m** available on the server.

   Lab7ExB.m generates observations on a bivariate random variable (X, Y). The default distribution is standard bivariate normal, so $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $V(X) = V(Y) = 1$. The program prompts the user to input the single parameter of the standard bivariate normal - the correlation 'rho' (ρ). The program plots 'npts' observations of the bivariate random variable and also plots empirical marginal pdfs for both X and Y. You can change the value of 'npts' in the program itself. You can see another set of observations simply by hitting any key. You must hold down the 'Control' key and hit 'C' to terminate the program.

   You can change to an alternative bivariate distribution (Methods 2 and 3) by commenting out the Method 1 code and un-commenting the desired option as explained in the program.

2. Copy the program and open it in the m-file editor. Study the code which generates the bivariate normal (Method 1). This code uses the fact that you can simulate a bivariate normal (X, Y) by setting

$$X = Z_1, \quad Y = \rho Z_1 + \sqrt{1 - \rho^2} Z_2,$$

where Z_1 and Z_2 are independent standard normal random variables. Using known properties of expectations and variances, confirm that $\mathbb{E}(X) = \mathbb{E}(Y) = 0$ and $V(X) = V(Y) = 1$ in this case. As Y is a so called 'mixture' of Z_1 and Z_2 its dependence on X varies as ρ changes over its range $[-1, 1]$. We will learn more about this 'correlation' later in the course.

3. Run the program using Method 1 for a variety of 'rho' values and 'npts' values and note the resulting scatter of observations and the empirical marginal distributions. What should the theoretical marginal distributions be in this case? Do they change with the value of 'rho'? What happens when 'rho' takes the values 1 or -1?

4. Before switching to Method 2 study lecture slides 354 and try to predict what the output of the program will look like for the bivariate distribution given in the example. Then run the program and check your answer. Consider what the output would be if the expression 'if $\text{rand}(1, 1) < 0.5$' was replaced by 'if $x(i) > 0$'. Again check your answer. [Challenge task: Derive the theoretical marginal distribution for Y in this case].

5. Before switching to Method 3 first try to predict the program output and the shape of the marginal distribution for Y. Check your result by running the program. What is the theoretical marginal distribution for Y in this case?