1. The largest negative entry is in column 1, and the associated ratios are $48/8 = 6$, $20/4 = 5$ and $8/2 = 4$. The smallest ratio is 4, in the third row. So we pivot on the entry in row 3, column 1:

$$\left(\begin{array}{cccccc|c} 8 & 6 & 1 & 1 & 0 & 0 & 48 \\ 4 & 2 & 3/2 & 0 & 1 & 0 & 20 \\ \boxed{2} & 3/2 & 1/2 & 0 & 0 & 1 & 8 \\ -60 & -35 & -20 & 0 & 0 & 0 & 0 \end{array}\right).$$

2. (a) There are negative entries in the bottom row, so the optimal solution has not been obtained yet. Applying the ratio test shows that we pivot on row 2, column 1:

$$\left(\begin{array}{ccccc|c} 5 & -2 & 6 & 1 & 0 & 20 \\ \boxed{10} & 4 & -6 & 0 & 1 & 30 \\ -10 & -6 & 8 & 0 & 0 & 0 \end{array}\right).$$

(b) There are no negative entries in the bottom row, so the optimal solution has been found. The basic variables are x_2 and x_3, with $x_2 = 25$ and $x_3 = 35$ (with $x_1 = x_4 = x_5 = 0$).

(c) There are negative entries in the bottom row, so the optimal solution has not been obtained yet. Applying the ratio test shows that we pivot on row 2, column 1:

$$\left(\begin{array}{ccccc|c} -3 & -6 & 0 & 1 & -4 & 35 \\ \boxed{2} & -1 & 1 & 0 & 1 & 20 \\ -2 & 11 & 0 & 0 & 4 & 16 \end{array}\right).$$

3. (a)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 80 \\ \boxed{1} & -4 & 0 & 1 & 20 \\ -8 & -4 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{cccc|c} 0 & \boxed{5} & 1 & -1 & 60 \\ 1 & -4 & 0 & 1 & 20 \\ 0 & -36 & 0 & 8 & 160 \end{array}\right) \begin{array}{l} R_1' = R_1 - R_2 \\ \\ R_3' = R_3 + 8R_2 \end{array}$$

$$\equiv \left(\begin{array}{cccc|c} 0 & 1 & 1/5 & -1/5 & 12 \\ 1 & -4 & 0 & 1 & 20 \\ 0 & -36 & 0 & 8 & 160 \end{array}\right) \begin{array}{l} R_1' = \frac{1}{5}R_1 \\ \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{cccc|c} 0 & 1 & 1/5 & -1/5 & 12 \\ 1 & 0 & 4/5 & 1/5 & 68 \\ 0 & 0 & 36/5 & 4/5 & 592 \end{array}\right) \begin{array}{l} \\ R_2' = R_2 + 4R_1 \\ R_3' = R_3 + 36R_1 \end{array}$$

There are no more negative entries in the bottom row, so the optimal solution has been found. It is attained when $x_1 = 68$ and $x_2 = 12$, with the value of the objective function being $z = 592$.

(b)

$$\left(\begin{array}{ccccc|c} 5 & -2 & 6 & 1 & 0 & 20 \\ \boxed{10} & 4 & -6 & 0 & 1 & 30 \\ -10 & -6 & 8 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{ccccc|c} 5 & -2 & 6 & 1 & 0 & 20 \\ 1 & 2/5 & -3/5 & 0 & 1/10 & 3 \\ -10 & -6 & 8 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ R_2' = \frac{1}{10}R_2 \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{ccccc|c} 0 & -4 & 9 & 1 & -1/2 & 5 \\ 1 & \boxed{2/5} & -3/5 & 0 & 1/10 & 3 \\ 0 & -2 & 2 & 0 & 1 & 30 \end{array}\right) \begin{array}{l} R_1' = R_1 - 5R_2 \\ \\ R_3' = R_3 + 10R_2 \end{array}$$

$$\equiv \left(\begin{array}{ccccc|c} 0 & -4 & 9 & 1 & -1/2 & 5 \\ 5/2 & 1 & -3/2 & 0 & 1/4 & 15/2 \\ 0 & -2 & 2 & 0 & 1 & 30 \end{array}\right) \begin{array}{l} \\ R_2' = \frac{5}{2}R_2 \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{ccccc|c} 10 & 0 & \boxed{3} & 1 & 1/2 & 35 \\ 5/2 & 1 & -3/2 & 0 & 1/4 & 15/2 \\ 5 & 0 & -1 & 0 & 3/2 & 45 \end{array}\right) \begin{array}{l} R_1' = R_1 + 4R_2 \\ \\ R_3' = R_3 + 2R_2 \end{array}$$

$$\equiv \left(\begin{array}{ccccc|c} 10/3 & 0 & 1 & 1/3 & 1/6 & 35/3 \\ 5/2 & 1 & -3/2 & 0 & 1/4 & 15/2 \\ 5 & 0 & -1 & 0 & 3/2 & 45 \end{array}\right) \begin{array}{l} R_1' = \frac{1}{3}R_1 \\ \\ \\ \end{array}$$

$$\equiv \left(\begin{array}{ccccc|c} 10/3 & 0 & 1 & 1/3 & 1/6 & 35/3 \\ 15/2 & 1 & 0 & 1/2 & 1/2 & 25 \\ 25/3 & 0 & 0 & 1/3 & 5/3 & 170/3 \end{array}\right) \begin{array}{l} \\ R_2' = R_2 + \frac{3}{2}R_1 \\ R_3' = R_3 + R_1 \end{array}$$

There are no more negative entries in the bottom row, so the optimal solution has been found. It is attained when $x_1 = 0$, $x_2 = 25$ and $x_3 = 35/3$, with the value of the objective function being $z = 170/3$.

4. Problem (b) requires the 2-phase method as it has a contraint of the form $a_1x_1+\ldots+a_nx_n \geqslant b$ with b positive. We introduce slack variables x_3, x_4 and x_5, and an artificial variables x_6 (to the first constraint). The resulting optimisation problem is

$$\begin{aligned} \text{maximise} \quad & z = x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 - x_3 + x_6 = 1 \\ & 2x_1 - x_2 + x_4 = 2 \\ & -x_1 + 2x_2 + x_5 = 3 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

The initial matrix is then

$$\left(\begin{array}{cccccc|c} 1 & 1 & -1 & 0 & 0 & 1 & 1 \\ 2 & -1 & 0 & 1 & 0 & 0 & 2 \\ -1 & 2 & 0 & 0 & 1 & 0 & 3 \\ -1 & -3 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

5. We introduce three slack variables, x_3, x_4 and x_5 and three artificial variables, x_6, x_7 and x_8. We also negate the objective function, so that the problem is

$$\begin{aligned} \text{maximise} \quad & -z = -8x_1 - 9x_2 \\ \text{subject to} \quad & 2x_1 - 3x_2 - x_3 + x_6 = 35 \\ & 3x_1 + 2x_2 - x_4 + x_7 = 100 \\ & 2x_1 + 3x_2 - x_5 + x_8 = 75 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

The phase 1 problem is to maximise

$$\begin{aligned} w &= -x_6 - x_7 - x_8 \\ &= (2x_1 - 3x_2 - x_3 - 35) + (3x_1 + 2x_2 - x_4 - 100) + (2x_1 + 3x_2 - x_5 - 75) \\ &= 7x_1 + 2x_2 - x_3 - x_4 - x_5 - 210. \end{aligned}$$

Applying the simplex algorithm:

$$\begin{aligned} &\left(\begin{array}{cccccccc|c} \boxed{2} & -3 & -1 & 0 & 0 & 1 & 0 & 0 & 35 \\ 3 & 2 & 0 & -1 & 0 & 0 & 1 & 0 & 100 \\ 2 & 3 & 0 & 0 & -1 & 0 & 0 & 1 & 75 \\ -7 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & -210 \end{array}\right) \\ \equiv &\left(\begin{array}{cccccccc|c} 1 & -3/2 & -1/2 & 0 & 0 & 1/2 & 0 & 0 & 35/2 \\ 0 & 13/2 & 3/2 & -1 & 0 & -3/2 & 1 & 0 & 95/2 \\ 0 & \boxed{6} & 1 & 0 & -1 & -1 & 0 & 1 & 40 \\ 0 & -25/2 & -5/2 & 1 & 1 & 7/2 & 0 & 0 & -175/2 \end{array}\right) \\ \equiv &\left(\begin{array}{cccccccc|c} 1 & 0 & -1/4 & 0 & -1/4 & 1/4 & 0 & 1/4 & 55/2 \\ 0 & 0 & 5/12 & -1 & \boxed{13/12} & -5/12 & 1 & -13/12 & 25/6 \\ 0 & 1 & 1/6 & 0 & -1/6 & -1/6 & 0 & 1/6 & 20/3 \\ 0 & 0 & -5/12 & 1 & -13/12 & 17/12 & 0 & 25/12 & -25/6 \end{array}\right) \\ \equiv &\left(\begin{array}{cccccccc|c} 1 & 0 & -2/13 & -3/13 & 0 & 2/13 & 3/13 & 0 & 370/13 \\ 0 & 0 & 5/13 & -12/13 & 1 & -5/13 & 12/13 & -1 & 50/13 \\ 0 & 1 & 3/13 & -2/13 & 0 & -3/13 & 2/13 & 0 & 95/13 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \end{aligned}$$

We have found a corner point of the feasible region, so we proceed to phase 2. We drop the artificial columns, return the original objective function, and then make the matrix canonical:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2/13 & -3/13 & 0 & 370/13 \\ 0 & 0 & 5/13 & -12/13 & 1 & 50/13 \\ 0 & 1 & 3/13 & -2/13 & 0 & 95/13 \\ 8 & 9 & 0 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{ccccc|c} 1 & 0 & -2/13 & -3/13 & 0 & 370/13 \\ 0 & 0 & 5/13 & -12/13 & 1 & 50/13 \\ 0 & 1 & 3/13 & -2/13 & 0 & 95/13 \\ 0 & 0 & -11/13 & 42/13 & 0 & -3815/13 \end{array}\right) \quad R_4' = R_4 - 8R_1 - 9R_2$$

Then complete the simplex algorithm:

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2/13 & -3/13 & 0 & 370/13 \\ 0 & 0 & \boxed{5/13} & -12/13 & 1 & 50/13 \\ 0 & 1 & 3/13 & -2/13 & 0 & 95/13 \\ 0 & 0 & -11/13 & 42/13 & 0 & -3815/13 \end{array}\right) \equiv \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -3/5 & 2/5 & 30 \\ 0 & 0 & 1 & -12/5 & 13/5 & 10 \\ 0 & 1 & 0 & 2/5 & -3/5 & 5 \\ 0 & 0 & 0 & 6/5 & 11/5 & -285 \end{array}\right)$$

There are no more negative entries in the bottom row, so the optimal solution has been found. The minimial solution is then $z = 285$ when $x_1 = 30$ and $x_2 = 5$.

6.
```
function [r,c] = minimumRatio(M)
% It is assumed that the matrix M is canonical and has at least one
%   negative entry in the bottom row.
% The outputs r and c indicate the pivot element according to the minimum
%   ratio test.

% Extract the last row, but ignore the value of the objective function
lastRow = M(end, 1:end-1);
% Find the column with the most negative entry
[~, c] = min(lastRow);
% Compute the relevant ratios
ratios = M(1:end-1,end) ./ M(1:end-1, c);
% If the ratio should be ignored, replace it with Inf
ratios(M(1:end-1, c) <= 0) = Inf;
% Find the index of the minimum ratio
[~, r] = min(ratios);
end
```

7. First define the initial matrix:

```
M = [1/2, -11/2, -5/2, 9, 1, 0, 0, 0;
     1/2, -3/2, -1/2, 1, 0, 1, 0, 0;
     1, 0, 0, 0, 0, 0, 1, 1;
     -10, 57, 9, 24, 0, 0, 0, 0];
```

Then by running the following line of code multiple times, you will find that the initial matrix `M` is obtained again after the 6th iteration:

```
[r, c] = minimumRatio(M); M = pivot(M, r, c)
```

If you wanted to automate this you could instead write:

```
M_original = [1/2, -11/2, -5/2, 9, 1, 0, 0, 0;
              1/2, -3/2, -1/2, 1, 0, 1, 0, 0;
              1, 0, 0, 0, 0, 0, 1, 1;
              -10, 57, 9, 24, 0, 0, 0, 0];

[r,c] = minimumRatio(M_original);
M = pivot(M_original, r, c);
count = 1;

while ~isequal(M, M_original)
    [r,c] = minimumRatio(M);
    M = pivot(M, r, c);
    count = count+1;
end
fprintf('The matrix repeated after %d iterations.\n', count)
```

To solve it using Bland's rule, we will determine the pivots manually (because it requires some very careful bookkeeping to correctly code the condition "if there is a tie, choose the row with the leftmost basic variable").

The first few iterations will proceed identically to the simplex algorithm:

```
M = [1/2, -11/2, -5/2, 9, 1, 0, 0, 0;
     1/2, -3/2, -1/2, 1, 0, 1, 0, 0;
     1, 0, 0, 0, 0, 0, 1, 1;
     -10, 57, 9, 24, 0, 0, 0, 0];
M = pivot(M, 1, 1)
M = pivot(M, 2, 2)
M = pivot(M, 3, 1)
M = pivot(M, 1, 3)
M = pivot(M, 2, 4)
M = pivot(M, 1, 5)
```

At this step, the matrix is

$$\left(\begin{array}{ccccccc|c} -4 & 8 & 2 & 0 & 1 & -9 & 0 & 0 \\ 1/2 & -3/2 & -1/2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -22 & 93 & 21 & 0 & 0 & -24 & 0 & 0 \end{array}\right)$$

Applying the simplex algorithm would result in pivoting on column 6, as it is the most negative entry. This is the step that would return the algorithm to the initial matrix.

Applying Bland's rule instead results in pivoting on column 1, as it is the leftmost negative entry. The smallest ratio in column 1 is $0/(1/2) = 0$ in row 2, so we pivot on row 2, column 1:

```
M = pivot(M, 2, 1)
M = pivot(M, 3, 3)
```

The resulting matrix is

$$\left(\begin{array}{ccccccc|c} 0 & 2 & 0 & 4 & 1 & -5 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 3 & 1 & -2 & 0 & -2 & 1 & 1 \\ 0 & 30 & 0 & 42 & 0 & 18 & 1 & 1 \end{array}\right),$$

showing that the maximum is 1 at $(x_1, x_2, x_3, x_4) = (1, 0, 1, 0)$.