THE UNIVERSITY OF MELBOURNE

Student number

Semester 2 Assessment, 2020

School of Mathematics and Statistics

MAST30001 Stochastic Modelling

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 25 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier, blank loose-leaf paper and a Casio FX-82 calculator.
- One double sided A4 page of notes (handwritten or printed).

Instructions to Students

- There are 7 questions with marks as shown. The total number of marks available is 80.
- Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.
- During writing time you may only interact with the device running the Zoom session with supervisor permission. The screen of any other device must be visible in Zoom from the start of the session.
- If you have a printer, print the exam one-sided. If you cannot print, download the exam to a second device, which must then be disconnected from the internet.
- Write your answers in the boxes provided on the exam that you have printed or the masked exam template that has been previously made available. If you are unable to answer the whole question in the answer space provided then you can append additional handwritten solutions to the end after the 25 numbered pages. If you do this you MUST make a note in the correct answer space or page for the question, warning the marker that you have appended additional remarks at the end.
- If you have been unable to print the exam and do not have the masked template write your answers on A4 paper. The first page should contain only your student number, the subject code and the subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.
- Assemble all exam pages (or masked template pages) in correct page number order and the correct way up. Add any extra pages with additional working at the end. Use a mobile phone scanning application to scan all pages to a single PDF file. Scan from directly above to reduce keystone effects. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Submit your PDF file to the Canvas Assignment corresponding to this exam using the Gradescope window. Before leaving Zoom supervision, confirm with your Zoom supervisor that you have Gradescope confirmation of submission.

Question 1 (16 marks)

A Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{1, 2, 3, 4, 5\}$ has transition matrix

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/5 & 4/5 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/3 & 0 & 0 & 0 & 2/3 \end{pmatrix}.$$

(a) Assuming X_0 is uniformly distributed on the set $\{2, 3, 5\}$, find

(i) $\mathbb{P}(X_4 = 3, X_2 = 3 | X_1 = 2)$, and

(ii) $\mathbb{P}(X_4 = 3, X_2 = 3)$.

(b) Write down the communication classes of the chain. For each class, find the period, determine whether it is essential, and classify it as transient or positive recurrent or null recurrent.

(c) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

(d) Find the probability of reaching state 1 before state 2 given the chain starts in state 4.

Question 2 (15 marks)

A train line operates from 5am to midnight, and the times between trains that stop at a certain station are independent with distribution uniform between 10 and 20 minutes.

(a) Estimate how many trains stop at the station between 5am and 10am, and give a symmetric interval around your estimate that has a 95% chance of covering the true number of trains that stop.

(b) You arrive at the station at 7pm. What is a good estimate for the mean of the time until the next train stops at the station?

(c) Now assume that trains still arrive at the station in the same way, but each train doesn't stop at the station with probability 1/10, independently between trains. What is a good estimate of the number of trains that stop at the station between 5am and 10am?

Question 3 (11 marks)

Customers arrive to an outback auto repair shop according to a Poisson process with rate 3 per day. The shop has one mechanic who takes an exponential with rate 3 per day amount of time to repair a car. In addition, if there are no cars in the shop, the mechanic will wait an exponential rate 1 per day time, and if no car has arrived in that time, the mechanic will leave the shop and take a nap for an exponential rate 10 per day time. If a car arrives while the mechanic is waiting to leave to take a nap, they will begin work on the car. Cars that arrive when the mechanic is working form a queue. When the queue has three cars in it, or the mechanic is taking a nap, cars arriving for repair will move on to the next repair shop.

(a) Model this system as a continuous time Markov chain and write down its state space and generator.

(b) Determine the stationary distribution of the chain.

(c) What is the stationary average number of customers at the auto repair shop (both those waiting for service and in service)?

(d) What proportion of customers are rejected from the system?

(e) Given that a customer is not immediately rejected from the system, what is the average time they spend waiting to be served?

Question 4 (17 marks)

Phone calls arrive to a large phone bank according to a Poisson process with rate 2 per minute. Received calls are answered immediately, and the time a call lasts (rounded up to the nearest minute), is distributed as a geometric random variable X satisfying

$$\mathbb{P}(X = k) = (2/3)^{k-1}(1/3), \;\; k = 1, 2, \dots,$$

and the times calls last are independent.

(a) What is the chance that at least 2 calls are received in a 2 minute time interval?

(b) What is the chance that at least 2 calls lasting exactly 1 minute are received in a 2 minute time interval?

(c) Given that 10 calls were received in a given 5 minute time interval, what is the chance that exactly 3 were received in the first minute of the interval?

(d) Given that 10 calls were received in a given 5 minute time interval, what is the chance that exactly 3 calls lasting exactly 1 minute were received in the first minute of the interval?

(e) Assuming there are no queued calls when the phone bank opens, for each $n = 1, 2, \ldots,$ find the distribution of X_n equal to the number of calls currently being handled n minutes after the phone bank opens.

(f) Now assume the phone bank also receives calls from an additional independent source according to a Poisson process with rate 5 per minute, and having the same service discipline. Now what is the chance that at least 2 calls lasting exactly 1 minute are received (from either source) in a 2 minute time interval?

Question 5 (11 marks)

Let $(B_t)_{t\geq 0}$ be a Brownian motion. For any normal probabilities below, you can write your answer in terms of the standard normal distribution function $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}dt$.

(a) Find $\mathbb{P}(B_{10} \geq -2 | B_4 = -1)$.

(b) Find $\mathbb{P}(B_{10} \geq -2 | B_4 = -1, B_2 = 1)$.

(c) Find $\mathbb{P}(B_4 \geq -2 | B_{10} = -1)$.

(d) Let Z be standard normal random variable that is independent of $(B_t)_{t\geq 0}$, and for $0 \leq t \leq 1$ set

$$X_t = (B_t - tB_1) + tZ.$$

Show that $(X_t)_{0\leq t\leq 1}$ is a standard Brownian motion (restricted to the interval $[0, 1]$).

Question 6 (6 marks)

Let K be a positive integer. A Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{0, 1, \ldots, K\}$ has transition probabilities

$$p_{i,i+1} = p_{i,i-1} = 1/2, \;\; 1 \leq i \leq K-1,$$

and

$$p_{0,1} = 1 - p_{0,0} = p_{K,K-1} = 1 - p_{K,K} = \alpha.$$

Let $T = \inf\big\{n \geq 1 : X_n \in \{0, K\}\big\}$. For each $i \in \{0, 1, \ldots, K\}$ find a simple expression in terms of α, K, and i for

$$\mathbb{E}[T | X_0 = i].$$

Question 7 (4 marks)

Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on $S = \{0, 1, 2, \ldots\}$ with generator $A = (a_{ij})_{i,j\in S}$ satisfying

$$a_{i,i+1} = 2^{i+1/2}, \ \ i \geq 0, \quad a_{i,i-1} = 2^{i}, \ \ i \geq 1,$$

and with all other off-diagonal entries zero. Show that $(X_t)_{t\geq 0}$ is an explosive continuous time Markov chain.

End of Exam — Total Available Marks = 80