

MAT4MDS – Practice 11

Gamma and beta functions

The gamma function

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt$$

It has the properties:

$$\Gamma(x+1) = x\Gamma(x) \qquad \Gamma(n+1) = n! \text{ for } n \in \mathbb{N}$$

Special values:

$$\Gamma(1) = 1 \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Question 1.

- (a) Find $\Gamma\left(\frac{5}{2}\right)$.
- (b) Find $\Gamma\left(-\frac{1}{2}\right)$.
- (c) Write $10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$ in terms of the factorial function.
- (d) Write the product $9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$ in terms of the gamma function.
(Hint: there are two different ways to do this. You need to re-write the product so that the factors differ from each other by 1. Then you can use $x\Gamma(x) = \Gamma(x+1)$ repeatedly.)
- (e) Hence write the product $1 \cdot 3 \cdot 5 \dots (2k-1)(2k+1)$ in terms of the gamma function.

Question 2. Using the substitution $t = u^2$, show that another way to represent $\Gamma(x)$ is

$$\Gamma(x) = 2 \int_0^{\infty} e^{-u^2} u^{2x-1} du.$$

Question 3. Using the substitution $u = e^{-t}$, show that another way to represent $\Gamma(x)$ is

$$\Gamma(x) = \int_0^1 \left[\log\left(\frac{1}{u}\right) \right]^{x-1} du.$$

The beta function

$$B(p, q) = \int_0^1 y^{p-1} (1-y)^{q-1} dy = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Question 4. Show, using properties of the gamma function, that

$$B(p, q+1) + B(p+1, q) = B(p, q).$$

The beta distribution is a two-parameter family of distributions that has probability density function

$$f(y) = \begin{cases} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

α and β are both positive shape parameters.

Question 5. In the Week 5 reading on understanding integration by substitution it was shown that the mean of the beta distribution is $\frac{\alpha}{\alpha+\beta}$.

(a) Find $\int_{-\infty}^{\infty} y^2 f(y) dy$.

(b) Hence show that the variance is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

Question 6. The mode for a random variable occurs where its probability density function has a (global) maximum. Use differentiation to find the stationary point of $f(y)$ for the beta distribution. For what values of α and β is this a maximum?

Question 7. Show that with an appropriate substitution (which you must find)

$$B(p, q) = \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} du.$$

The gamma distribution is a two-parameter family of distributions that has probability density function

$$f(y) = \begin{cases} \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

α and β are both positive parameters.

Question 8. Find the mode of the gamma distribution. For what values of the parameters does this apply?

The cumulative distribution function F (CDF) is an antiderivative of the probability density function f (PDF) for continuous data. That is:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$$

Question 9. The following function is to be used as a PDF of a continuous random variable:

$$f(x) = \begin{cases} 2 - 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Plot $f(x)$.

(b) Following the steps below, find the associated CDF.

(i) Find $F(x)$ for, $x < 0$

(ii) Find $F(x)$ for $0 \leq x \leq 1$

(iii) Find $F(x)$ for $x > 1$ (Hint: for $x > 1$ we have $F(x) = F(1) + \int_1^x f(t) dt$).

Question 10. Consider the function $f(x, y) = x^2 e^{2y+x} - \frac{x}{y}$. Integrate $f(x, y)$ with respect to y between $y = 1$ and $y = 8$.
That is, calculate

$$\int_1^8 f(x, y) dy.$$

Two (or more) random variables are jointly continuous when they have a **joint cumulative distribution function** that is continuous in both variables:

$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(s, t) dt ds$$

The function $f(x, y)$ is the **joint probability density function**.

Question 11. Consider the joint probability density function

$$f(x, y) = \begin{cases} 6e^{-2x}e^{-3y} & x \geq 0, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the associated joint cumulative distribution function $F(x, y)$.

Question 12. Check that for the function found in question 7:

- As $x \rightarrow -\infty$ or $y \rightarrow -\infty$, $F(x, y) \rightarrow 0$.
- As $x \rightarrow \infty$ and $y \rightarrow \infty$, $F(x, y) \rightarrow 1$.