Chapter 3: Poisson Process

The Poisson distribution

Reminder: A random variable N taking values in $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ has a *Poisson distribution* with a parameter $\lambda > 0$ (and we write $N \sim \mathrm{Pois}(\lambda)$), if its probability mass function is given by

$$\mathbb{P}(N = n) = \frac{e^{-\lambda}\lambda^n}{n!}, \qquad \text{for } n \in \mathbb{Z}_+.$$

The mean and variance of N are both equal to λ.

The exponential distribution

Reminder: A random variable T has an exponential distribution with parameter $\lambda > 0$ (called the rate), denoted by $T \sim \exp(\lambda)$, if its distribution function is

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t}, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

It follows that the probability density function of T is

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

The mean of T is $1/\lambda$ and the variance of T is $1/\lambda^2$.

The law of rare events

The Poisson distribution arises as the limit of the binomial distribution: Fix $\lambda > 0, k \in \mathbb{Z}_+$.

- If $X_n \sim \mathrm{Bin}(n, \lambda/n)$ and $N \sim \mathrm{Pois}(\lambda)$, then

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \frac{e^{-\lambda}\lambda^k}{k!} = \mathbb{P}(N = k).$$

The exponential distribution arises as the limit of the geometric distribution: Fix $\lambda, t > 0$.

- If $Y_n \sim \mathrm{Geo}(\lambda/n)$ (for $n > \lambda$) and $T \sim \mathrm{Exp}(\lambda)$, then

$$\lim_{n\to\infty} \mathbb{P}(Y_n/n \le t) = 1 - e^{-\lambda t} = \mathbb{P}(T \le t).$$

The Poisson Process

Definition:

A nonnegative integer-valued process $(N_t)_{t \geq 0}$ is a Poisson process with a rate λ if

(i) it has independent increments on disjoint intervals: for $k \geq 2$ and $0 \leq s_1 < t_1 \leq s_2 < \cdots < t_k$,

$$N_{t_1} - N_{s_1}, \ldots, N_{t_k} - N_{s_k}$$

are independent variables, and

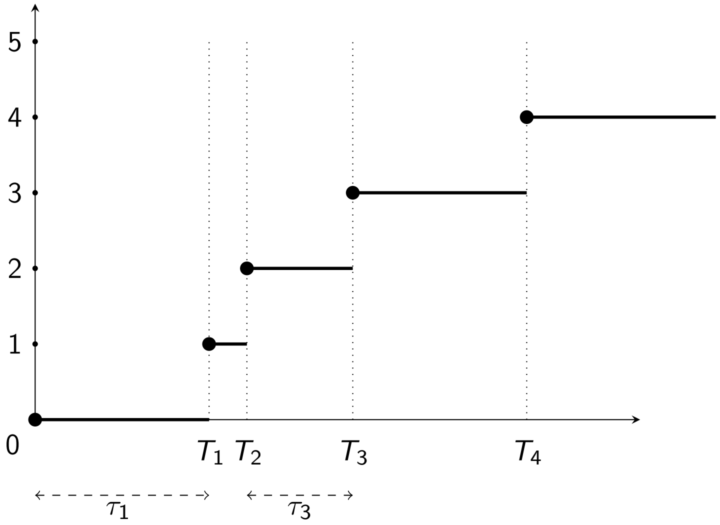
(ii) For each $t > s \geq 0$, $N_t - N_s \sim \mathrm{Pois}(\lambda(t - s))$.

Exercise

If $(N_t)_{t\geq 0}$ is a Poisson process with rate λ, show that for fixed $r > 0$, the process $(N_t^*)_{t\geq 0}$ defined by $N_t^* = N_{t+r} - N_r$, is also Poisson process with rate λ.

[This is true even if r is a stopping time for the process.]

A trajectory



Poisson Process Empirical Data

- Earthquakes
- Grazing animals head raises
- Horse kick deaths

The Poisson Process

Let $T_0 = 0$ and $T_j = \min\{t : N_t = j\}$ (the time of j^{th} jump) and define $\tau_j = T_j - T_{j-1}$ (time between the $(j-1)^{\mathrm{st}}$ and j^{th} jumps).

Theorem: $(N_t)_{t \geq 0}$ is a Poisson process with rate λ if and only if $(\tau_j)_{j \in \mathbb{N}}$ are independent $\mathrm{Exp}(\lambda)$ random variables.

Proof: The key to the proof is to observe that the event $\{T_j \leq t\}$ is the same as $\{N_t \geq j\}$. That is the waiting time until the jth jump is less than or equal to t if and only if there are j or more jumps up to (and including) time t.

Assume that $(N_t)_{t \geq 0}$ is a Poisson process. Then $\mathbb{P}(T_1 \leq t) = \mathbb{P}(N_t \geq 1) = 1 - \mathbb{P}(N_t = 0) = 1 - e^{-\lambda t}$, so $T_1 \sim \mathrm{Exp}(\lambda)$.

The Poisson Process

Furthermore, we have

$$\begin{aligned}\mathbb{P}(T_j \le t) &= \mathbb{P}(N_t \ge j) \\ &= \sum_{k=j}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!} \\ &= 1 - \sum_{k=0}^{j-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!}.\end{aligned}$$

This final expression is the distribution function for gamma distribution with parameter k and rate λ. You can check this by differentiating to get the density function
$f_{T_j}(t) = e^{-\lambda t}\lambda^j t^{j-1}/(j-1)!$.
So the waiting time until the jth event is the sum of j independent exponentially-distributed inter-event times with parameter λ.

The Poisson Process

This argument also holds in reverse.
Assuming that $\tau_1 \sim$Exp(λ), we know that $\mathbb{P}(T_1 \leq t) = 1 - e^{-\lambda t}$, which tells us that $\mathbb{P}(N_t = 0) = e^{-\lambda t}$.
Furthermore, for $j > 1$, if $\{\tau_1, \ldots, \tau_j\}$ are i.i.d. Exp(λ), then T_j has a Gamma distribution with parameters λ and j. So

$$\mathbb{P}(N_t \geq j) = \mathbb{P}(T_j \leq t) = 1 - \sum_{k=0}^{j-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!},$$

which tells us that N_t has a Poisson distribution with parameter λt.

The Poisson Process

We'd still need to show that $N_{t_i} - N_{s_i}$ are independent $\sim$Pois$(\lambda(t_i - s_i))$ over sets $[s_i, t_i)$ of disjoint intervals.

This follows from the memoryless property of the exponential distribution (so the remaining time from s_i until the next jump doesn't depend on the time $s_i - T_{N_{s_i}}$ since the previous jump) and the independence of the τ_j.

Order statistics

For random variables $\xi_1, \xi_2, \ldots, \xi_k$, denote by $\xi_{(i)}$ the i^{th} smallest of them. Then $\xi_{(1)}, \xi_{(2)}, \ldots, \xi_{(k)}$ are called the order statistics associated with $\xi_1, \xi_2, \ldots, \xi_k$.

For example, if we sample these random variables and find that $\xi_1 = 1.3, \xi_2 = 0.9, \xi_3 = 0.7, \xi_4 = 1.1$ and $\xi_5 = 1.5$, then $\xi_{(1)} = 0.7, \xi_{(2)} = 0.9, \ldots, \xi_{(5)} = 1.5$.

Order statistics play a very important role in applications. For example, the maximum likelihood estimator of θ for a sample $\xi_1, \xi_2, \ldots, \xi_k$ from the uniform $[0, \theta]$ distribution is $\xi_{(k)}$.

Order Statistics: examples

- If X_1, X_2 and X_3 are independent and identically-distributed random variables taking values 1, 2 and 3 each with probability $1/3$, find the joint probability mass function of $(X_{(1)}, X_{(2)}, X_{(3)})$.
- If $Y_1, Y_2, \ldots, Y_k$ are i.i.d. random variables with distribution function F, the distribution function of $Y_{(i)}$ is

$$F_{Y_{(i)}}(x) = \sum_{\ell=i}^{k} \binom{k}{\ell} F(x)^{\ell} (1 - F(x))^{k-\ell}.$$

- So, for the special case where $Y_1, Y_2, \ldots, Y_k$ are i.i.d.$\sim U[0, t]$, for $i \le k$ and $x \le t$, the distribution function of the order statistic $Y_{(i)}$ is given by

$$F_{Y_{(i)}}(x) = \sum_{\ell=i}^{k} \binom{k}{\ell} (x/t)^{\ell} (1 - x/t)^{k-\ell}.$$

Order Statistics

- Above we saw that if $Y_1, Y_2, \ldots, Y_k$ are i.i.d. with distribution function F the distribution function of $Y_{(i)}$ is

$$F_{Y_{(i)}}(x) = \sum_{\ell=i}^{k} \binom{k}{\ell} F(x)^{\ell}(1 - F(x))^{k-\ell}.$$

  If they are also absolutely continuous, with density f, then the density of the order statistic $Y_{(i)}$ is

$$\begin{aligned} f_{Y_{(i)}}(x) &= \binom{k}{i-1}(k-i+1)F(x)^{i-1}f(x)(1-F(x))^{k-i} \\ &= \binom{k}{i} iF(x)^{i-1}f(x)(1-F(x))^{k-i}. \end{aligned}$$

Order Statistics

Similarly the joint densities for $1 \leq r \leq k$ and $x_1 < \cdots < x_r$ are

$$\begin{aligned} & f_{Y_{(i_1)},\ldots,Y_{(i_r)}}(x_1,\ldots,x_r) \\ & \quad = \binom{k}{i_1-1, 1, i_2-i_1-1, 1 \cdots, 1, k-i_r} \\ & \qquad \times \prod_{j=1}^{r} f(x_j) \prod_{j=0}^{r} (F(x_{j+1}) - F(x_j))^{i_{j+1}-i_j-1}, \end{aligned}$$

where $\binom{\ell}{a_1,\cdots,a_j}$ is the number of ways to choose subsets of sizes $a_1,\ldots,a_j$ from a set of size ℓ and for the sake of brevity we set $x_0 = -\infty$ and $x_{r+1} = \infty$ so $F(x_0) = 0$ and $F(x_{r+1}) = 1$.

Order Statistics

In particular for $r = k$, $x_1 < \cdots < x_r$,

$$f_{Y_{(1)},\ldots,Y_{(k)}}(x_1, \ldots, x_k) = k! \prod_{j=1}^{k} f(x_j).$$

The Poisson Process

Theorem: The conditional distribution of $(T_1, \cdots, T_k)$ given that $N_t = k$ is the same as the distribution of order statistics of a sample of k independent and identically-distributed random variables uniformly distributed on $[0, t]$.

I.e.

$$(T_1, \cdots, T_k)|\{N_t = k\} \stackrel{d}{=} (U_{(1)}, \cdots, U_{(k)})$$

where $U_1, \cdots, U_k$ are independent $\sim U(0, t)$.

The same representation holds for the conditional distribution of $(T_1, \cdots, T_k)$ given that $T_{k+1} = t$.

“Proof” of theorem

According to our derivation for order statistics, $(U_{(1)}, \cdots, U_{(k)})$ has density $k!t^{-k}$ for $0 = x_0 < x_1 < \cdots < x_k < t$.

So we show the LHS has the same density:

$$
\begin{aligned}
&\mathbb{P}(T_1 \in dx_1, \cdots, T_k \in dx_k | N_t = k) \\
&\quad = \frac{\mathbb{P}(\tau_1 \in dx_1, \tau_2 \in d(x_2 - x_1), \ldots, \tau_k \in d(x_k - x_{k-1}), \tau_{k+1} > t - x_k)}{\mathbb{P}(N_t = k)} \\
&\quad = \frac{(\prod_{i=1}^{k} \lambda e^{-\lambda(x_i - x_{i-1})}) e^{-\lambda(t - x_k)}}{(\lambda t)^k e^{-\lambda t}/k!} dx_1 \ldots dx_k \\
&\quad = k!t^{-k} dx_1 \ldots dx_k.
\end{aligned}
$$

The proof of the second claim is similar.

The Poisson Process

The theorem implies that if $\tau_1, \ldots, \tau_n$ are i.i.d. exponential variables, then

$$\left(\frac{\tau_1}{\sum_{j=1}^{n+1} \tau_j}, \frac{\tau_1 + \tau_2}{\sum_{j=1}^{n+1} \tau_j}, \ldots, \frac{\sum_{j=1}^{n} \tau_j}{\sum_{j=1}^{n+1} \tau_j} \right)$$

have the same distribution as $U(0, 1)$ order statistics.

Superposition of Poisson processes

Theorem: Let $(N_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$ be two independent Poisson processes with rates λ_1 and λ_2 respectively and $L_t = N_t + M_t$. Then $(L_t)_{t\geq 0}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

Proof:

- By independence, $L_t - L_s \sim \text{Pois}(\lambda_1(t-s) + \lambda_2(t-s))$.
- For disjoint $[s_1, t_1]$ and $[s_2, t_2]$,

$$\begin{aligned} L_{t_1} - L_{s_1} &= (N_{t_1} - N_{s_1}) + (M_{t_1} - M_{s_1}) \\ L_{t_2} - L_{s_2} &= (N_{t_2} - N_{s_2}) + (M_{t_2} - M_{s_2}) \end{aligned}$$

which are independent because of the same property of $(N_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$. This argument extends to all finite collections of disjoint intervals.

Superposition example

A shop has two entrances, one from East St, the other from West St. Flows of customers through the two entrances are independent Poisson processes with rates 0.5 and 1.5 per minute, respectively.

- What is the probability that no new customers enter the shop in a fixed three minute time interval?
- What is the mean time between arrivals of new customers?
- What is the probability that a given customer entered from West St?

Thinning of a Poisson process

Suppose in a Poisson process $(N_t)_{t \geq 0}$ each 'customer' is 'marked' independently with probability p. Let M_t count the number of 'marked customers' that arrive on $[0, t]$.

Theorem. The processes $(M_t)_{t \geq 0}$ and $(N_t - M_t)_{t \geq 0}$ are independent Poisson processes with rates λp and $\lambda(1 - p)$ respectively.

Thinning - "proof"

"Proof".

$$\begin{aligned}
\mathbb{P}(M_t = j, N_t - M_t = k) &= \mathbb{P}(M_t = j, N_t = k + j) \\
&= \mathbb{P}(M_t = j | N_t = k + j)\mathbb{P}(N_t = k + j) \\
&= \binom{k+j}{j} p^j (1-p)^k \frac{e^{-\lambda t}(\lambda t)^{k+j}}{(k+j)!} \\
&= \frac{e^{-p\lambda t}(p\lambda t)^j}{j!} \frac{e^{-(1-p)\lambda t}((1-p)\lambda t)^k}{k!}.
\end{aligned}$$

Thinning - example

The flow of customers to a shop is a Poisson process with rate 25 customers per hour. Each of the customers independently has a probability $p = 0.8$ of making a purchase.

- What is the probability that all customers who enter the shop during the time interval from 11.00 am to 11.15 am make a purchase?
- What is the probability that, conditional on there being two customers that made a purchase during that period, all customers who enter the shop during the time interval make a purchase?

The Compound Poisson Process

Suppose that $(N_t)_{t \geq 0}$ is a Poisson process and $(X_i)_{i \in \mathbb{N}}$ are independent and identically-distributed random variables, which are also independent of $(N_t)_{t \geq 0}$.

For $t \geq 0$, define $Y_t = \sum_{j \leq N_t} X_j$. Then $(Y_t)_{t \geq 0}$ is called a compound Poisson process.

It can be shown that $(Y_t)_{t \geq 0}$ has independent increments and it is possible to compute the distribution of Y_t by conditioning on N_t.

Compound Poisson example

Suppose claims made to an insurance company arrive according to a Poisson process with rate λ, and each policy holder carries a policy for an amount X_k. Assume $X_1, X_2, \ldots$ are independent and identically-distributed, and the number of claims and the size of claims are independent.

Calculate the mean and variance of the total amount of claims on the company up to time t.