MAST30025: Week 2 Tutorial / Lab.

Q1) Show that X^TX is a symmetric matrix.

let, $X = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ for example.

Attempt 1

$(X^TX)^T = XX^T$ ✗

$X^T(X^T)^T$

$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $X^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$X^TX = 0$

$(X^TX)^T = X^T(X^T)^T = X^TX$ ✓ (Commutative)? No.

$X^TX = XX^T$?

commutative law of multiplication.

$XX^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Q2a). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

be a non singular 2x2 matrix, ~~Show~~ Show ~~that~~ by direct multiplication that,

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$A = \cancel{(A^{-1})}$

$\cancel{A \cdot A^{-1} = I}$

$\cancel{A^{-1} = I}$

$AA^{-1} = I$

$\cancel{A \cdot A^{-1} A^{-1} = IA^{-1}}$

$AA^{-1} = I$

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} A^{-1} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$ ✓

$AA^{-1} = I$

$AA^{-1}A = IA$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ c/d & 1 & 0 & 1/d \end{array}\right]$$

if ~~and~~ $a \neq 0$, $d \neq 0$.

$$\begin{bmatrix} 1 & b/a \\ \frac{c}{d} & 1 \end{bmatrix}$$

Soln

$AA^{-1} = I$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

substitute A & A^{-1}

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -cb+da \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 .$$

b) Find the inverse of, $\begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix} = A$,

$$A^{-1} = \frac{1}{-6-4} \begin{bmatrix} -3 & -4 \\ -1 & 2 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} -3 & -4 \\ -1 & 2 \end{bmatrix}$$

$c = \dfrac{1}{\sqrt{1^2+1^2+1^2+1^2}} = \dfrac{1}{2}$

$Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Q3) Attempt 2.

$X = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Sl. 221 linear Algebra slides. Orthonormal sets.

No, As $\{(1,1,1,1), (1,-1,-1,1), (-1,1,-1,1), (-1,-1,1,1)\}$ does not form a ~~orthonormal~~ orthonormal set.

$X \Rightarrow$ ~~$\det(X) = \ldots$~~

norm > 1
Hence $c = 1/2$ ~~for each~~ vectors

$X^{-1} \Rightarrow$ ~~Xa~~

$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Leftarrow$ Ignore this!

$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$

$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$

To rephrase it –
The matrix is not orthogonal, as the columns do not form an orthonormal set (i.e. first column has norm > 1). However they do form an orthogonal set, so we can just normalise each vector to produce an orthogonal matrix.
This gives $c = 1/2$.

QED. ✓

$X^T X = 4I$

$\neq$

Q4) a. Find the eigen values, and an associated eigenvector for each eigen value, of the matrix,

$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

$(A - \lambda I)x = 0$

where x is an eigenvector.

$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = 0$$

$$\cancel{4} - 4\lambda + \lambda^2 - \cancel{4} = 0$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0$$

$$\lambda = 0, 4. \quad \text{eigenvalues.}$$

Case 1 $\lambda = 0$.

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

let $y = t$

$x + t = 0$

$x = -t$

$t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ let $t = -1$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ eigenvector

Case 2: $\lambda = 4$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$y = s$

$-x + s = 0$

$x = s$

$s = 1$

$s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigen vector.

eigen vectors ← $P^TAP = D$ ← eigenvalues

Q4b) find an orthogonal matrix P such that P^TAP is diagonal. Soln

$P^TAP = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$

$P = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$

definition of diagonalisation

$P \rightarrow$ orthonormal eigenvectors

$\therefore P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$D = \begin{bmatrix} 4 = \lambda_1 & 0 \\ 0 & 0 = \lambda_2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$D = P^TAP$ $\quad P =$

Trail & Error.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2 \end{bmatrix}$

wait.

Attempt 2

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$P^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$P^TAP = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\cancel{\lambda} - 4\lambda + \lambda^2 - \cancel{4}$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0$$

$$\lambda = 0, 4.$$

$$P^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} 1 & +1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{0}4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = \left(\frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)\right)$$

Q4b)

where $u_1 = \frac{1}{\sqrt{2}}(1,1)$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

where $u_2 = \frac{1}{\sqrt{2}}(1,-1)$

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\sqrt{1^2+1^2} = \sqrt{2}$$

$$\sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

take out the factor $\frac{1}{\sqrt{2}}$!!

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

* magnitute

$$P^TAP = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$= \begin{bmatrix}4 & 0\\ 0 & 0\end{bmatrix}$$ consequential

(c) Write down P^TAP for the P given in part b

Q5: let $A = \begin{bmatrix}1 & 4 & 3\\ -2 & 0 & 2\\ 4 & 4 & 0\end{bmatrix} \rightarrow$

(a) Write down the trace of A.

$tr(A) = 1 + 0 + 0 = 1.$

(b) Are the columns of A linearly independent? Justify your answer.

$X_1 = \begin{bmatrix}1\\ -2\\ 4\end{bmatrix}, \quad X_2 = \begin{bmatrix}4\\ 0\\ 4\end{bmatrix}, \quad X_3 = \begin{bmatrix}3\\ 2\\ 0\end{bmatrix}$

$X_1 - X_2 + X_3 = 0$

$\Rightarrow$ linearly dependent; **NO**,

X_1 is not a constant of X_2

$\Rightarrow \{X_1, X_2\}$ is linearly independent, Q5C) find the rank A. doing row operations?

no. A has 2 linearly indep. vectors, $\therefore$ rank(A) = 2

$$\cancel{\begin{bmatrix} 1 & 4 & 3 \\ -2 & 0 & 2 \\ 4 & 4 & 0 \end{bmatrix} \xrightarrow[R_2 = \frac{R_2}{2}]{R_3 = \frac{R_3}{4}} \begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 0 & 2 \\ 4 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 \\ -2 & 0 & 2 \\ 4 & 4 & 0 \end{bmatrix} \xrightarrow[R_3 = R_1 - R_3]{} \begin{bmatrix} 1 & 4 & 3 \\ -2 & 0 & 2 \\ -3 & 0 & 3 \end{bmatrix} \xrightarrow[R_2 = \frac{R_2}{2} - \frac{R_3}{-3}]{} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \begin{matrix} R_2 \leftrightarrow R_3 \\ R_3 = R_3 + R_1 \end{matrix}$$

$$\xrightarrow[\substack{R_3 \leftrightarrow R_2 \\ R_2 = \frac{R_2}{4}}]{} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 = R_1 - 3R_2]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank(A) = 2.

Soln There is a shorter method ~~on~~ to this. OR since colns 1 & 3 from A are not multiples of each other, hence R(A) = 2

QED.

Q6) Show that if X is of full rank, then

$I - X(X^T X)^{-1} X^T$ is an idempotent matrix.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

use definition.

$A = A^2$

$$\left(I - X(X^T X)^{-1} X^T\right)\left(I - X(X^T X)^{-1} X^T\right)$$

$$I^2 - X(X^T X)^{-1} X^T - X(X^T X)^{-1} X^T + X\cancel{(X^T X)^{-1}} \cancel{X^T X}(X^T X)^{-1} X^T$$

A is nonsingular.
then A^{-1} exists.

$$AA^{-1} = I$$

$$AI = A$$

$$I^2 = II = I$$

idempotent

$$\cancel{(X^T X)^{-1} X^T} X = I$$

$$\left(I - X(X^T X)^{-1} X^T\right)\left(I - X(X^T X)^{-1} X^T\right)$$

$$= I^2 - X(X^T X)^{-1} X^T - X(X^T X)^{-1} X^T + X\underline{\cancel{(X^T X)^{-1} X^T X}}(X^T X)^{-1} X^T$$

$$= I - 2X(X^T X)^{-1} X^T + X(X^T X)^{-1} X^T$$

$$= I - X(X^T X)^{-1} X^T$$

∴ you see in the next page.

let, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$X^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad X^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow \quad X^T X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+dc & b^2+d^2 \end{bmatrix}$$

$$\cancel{(X^T X)^{-1} = \frac{1}{(a^2+c^2)(b^2+d^2) - (ab+dc)(ab+cd)}\begin{bmatrix} & \\ & \end{bmatrix}}$$

$$(X^T X)^{-1} = \frac{1}{(a^2+c^2)(b^2+d^2) - (ab+dc)(ab+cd)}\begin{bmatrix} b^2+d^2 & -(ab+cd) \\ -(ab+dc) & a^2+c^2 \end{bmatrix}$$

$$X(X^T X)^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{(a^2+c^2)(b^2+d^2) - (ab+dc)(ab+cd)}\begin{bmatrix} b^2+d^2 & -(ab+cd) \\ -(ab+dc) & a^2+c^2 \end{bmatrix}$$

$$= \frac{1}{(a^2+c^2)(b^2+d^2) - (ab+dc)(ab+cd)}\begin{bmatrix} a(b^2+d^2) - b(ab+dc) & -a(ab+cd)+b(a^2+c^2) \\ c(b^2+d^2) - d(ab+dc) & -c(ab+cd)+d(a^2+c^2) \end{bmatrix}$$

$X(X^T X)^{-1} X^T$

$$= \frac{1}{(a^2+c^2)(b^2+d^2) - (ab+dc)(ab+cd)}\begin{bmatrix} a(b^2+d^2) - b(ab+dc) & -a(ab+cd)+b(a^2+c^2) \\ c(b^2+d^2) - d(ab+dc & -c(ab+cd)+d(a^2+c^2) \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$= \cancel{a^2b^2} + a^2d^2 + c^2b^2 + \cancel{c^2d^2} - \cancel{a^2b^2} - abcd + abcd + \cancel{c^2d^2}$$

$$= \frac{1}{a^2b^2 + a^2d^2 + c^2b^2 + \cancel{c^2d^2} - \cancel{a^2b^2} - 2abcd - \cancel{c^2d^2}} \begin{bmatrix} \cancel{ab^2} + ad^2 - \cancel{ab^2} - bdc & -\cancel{a^2b} - acd + \cancel{a^2b} + bc^2 \\ cb^2 + \cancel{cd^2} - abd - \cancel{dc} & -abc - \cancel{c^2d} + a^2d + \cancel{c^2d} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$= \frac{1}{(ad-cb)^2} \begin{bmatrix} ad^2 - bdc & bc^2 - acd \\ cb^2 - abd & a^2d - abc \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$= \frac{1}{(ad-cb)^2} \begin{bmatrix} a^2d^2 - abdc + b^2c^2 - abcd & \underbrace{\cancel{acd^2} - \cancel{bdc^2} + \cancel{bc^2d} - \cancel{acd^2}}_{0} \\ \underbrace{\cancel{acb^2} - \cancel{a^2bd} + \cancel{a^2bd} - \cancel{ab^2c}}_{0} & c^2b^2 - abcd + a^2d^2 - abcd \end{bmatrix}$$

$$= \frac{1}{(ad-cb)^2} \begin{bmatrix} (ad-bc)^2 & 0 \\ 0 & (bc-ad)^2 \end{bmatrix}$$

ref

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I. \leftarrow \text{just to verify.}$$

$X(X^TX)^{-1}X^T$

since, A ∴ exists and is idempotent.

<u>soln</u> refer to slide 43/70. (Linear Algebra section)

$A^2 = A$

$$, \; \left(X(X^TX)^{-1}X^T\right)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = X(X^TX)^{-1}X^T$$

Slide 61/76

Q7) Prove that a (real) symmetric matrix A is positive semidefinite if and only if and only if all of its eigen values are non-negative, and positive definite if and only if all of its eigenvalues are strictly positive

$y^T A y \geq 0$

$\parallel$

D $\qquad$ y

$\Rightarrow y$ is a $k \times 1$ vector;

Soln

$P^T A P = D$, $\qquad$ where P is orthogonal matrix,

($\Leftarrow$) let $\lambda_1, \ldots, \lambda_n \geq 0$ be eigenvalues of A.
for any x we have, for $z = P^T x = (z_1, \ldots, z_n)^T$,

$x^T A x = x^T P \cdot D P^T x = z^T D z = \sum_{i=1}^{n} z_i^2 \lambda_i \geq 0$,

thus. A is positive semidefinite as required.

($\Rightarrow$) Suppose A is positive semidefinite. let x_i be its normalised i-th eigenvector, then,

$0 \leq x_i^T A x_i = \lambda_i x_i^T x_i = \lambda_i$

So the eigenvalues of A are non-negative as required.

$\Rightarrow$ Could possibly be an assignment question

Any matrix A,

Q8) $r(A) = r(A^T) = r(A^T A)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$r(A) = k$

n columns

$$A = \begin{bmatrix} | & | & & | & & | \\ a_1 & a_2 & \cdots & a_k & \cdots & a_n \\ | & | & & | & & | \end{bmatrix}$$

k indept. columns.

can write as linear combinations of other columns

Basis.

let $\underbrace{x_1, x_2 \ldots x_k}_{\text{vectors}}$ be the basis for the column space of A.

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_k \\ | & | & & | \end{bmatrix}$$

$r(X) = k = r(A)$
X has k columns.

Definition of basis: every column vector of A is a linear combination of the column vectors of X

$a_1 = b_1 x_1 + b_2 x_2 \ldots b_k x_k$ (Def. linear comb.)

b is a scalar

$$B = \begin{bmatrix} - & b_1 & - \\ - & b_2 & - \\ & \vdots & \\ - & b_k & - \end{bmatrix}$$

$$\overset{A}{\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & b_1 & - \\ - & b_2 & - \\ & \vdots & \\ - & b_k & - \end{bmatrix}$$

$r(X) = k$

$A = XB$

$A^T = (XB)^T = B^T X^T$

$r(A^T) \le r(A)$

$r(A) \le r(A^T)$

$r(A) = r(A^T)$