MAST30001 Stochastic Modelling

Tutorial Sheet 7

1. A two state continuous time Markov chain $(X_t)_{t\geq 0}$ has the following generator with transition rates $\lambda, \mu > 0$:

$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$

   (a) Find the time t transition matrix $P^{(t)}$ with $(P^{(t)})_{i,j} = \mathbb{P}(X_t = j | X_0 = i)$.

   (b) Using your answer to part (a) with $\lambda = \mu$, find a simple expression (i.e., not an infinite sum) for the chance that a random variable having the Poisson distribution with mean λ is an even number.

Ans.

(a) We solve the equation

$$\frac{d}{dt}P^{(t)} = P^{(t)}A,$$

where A is the generator in the problem. Thus

$$p'_{11}(t) = -\lambda p_{11}(t) + \mu p_{12}(t)$$

and also use $p_{11}(t) = 1 - p_{12}(t)$. We find

$$p_{11}(t) = \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} \quad \text{and} \quad p_{12}(t) = \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right)$$

and by symmetry

$$p_{22}(t) = \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} \quad \text{and} \quad p_{12}(t) = \frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right).$$

(b) If $\lambda = \mu$, then the exponential waiting time description of the chain with generator in the problem implies that that number of times the chain switches states in $(0, t)$ has a Poisson distribution with mean λt. If the chain starts at state 1, then the chain being at state 1 at time t is the same as there being an even number of jumps. So $p_{11}(1)$ is the chance a Poisson variable with mean λ is even, and according to (a) this is

$$(1/2)(1 + e^{-2\lambda}).$$

Alternatively, you can use Taylor's expansion of e^x to find

$$\frac{e^\lambda + e^{-\lambda}}{2} = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!},$$

which leads to the same conclusion.

2. If $(X_t^{(1)})_{t\geq 0}, \ldots, (X_t^{(k)})_{t\geq 0}$ are i.i.d. continuous time Markov chains on $\{0, 1\}$ each having generator

$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$

then what is the generator for the chain determined by $Y_t = \sum_{i=1}^k X_t^{(i)}$?

Ans. Y_t takes values in $\{0, \ldots, k\}$. $Y_t = i$ means that exactly i of the $X_t^{(j)}$'s are one. So from this point Y_t increases by one at rate $(k-i)\lambda$ (the minimum of the $(k-i)$ exponential clocks where the $X_t^{(j)} = 0$) and decreases by one at rate $i\mu$ (the minimum of the i exponential clocks where the $X_t^{(j)} = 1$); as a check note these formulas are correct for $i = k$ and $i = 0$. So the generator has for $i = 1, \ldots, k-1$

$$a_{ii+1} = (k-i)\lambda, \quad a_{ii-1} = i\mu \;\; a_{ii} = -(k\lambda + i(\mu - \lambda)),$$

$-a_{k,k} = a_{kk-1} = k\mu$, $-a_{00} = a_{01} = k\lambda$, and all other entries 0.

3. A workshop has two machines and one repairperson. Each machine is either functional or broken. If the the ith machine ($i = 1, 2$) is functional, then it fails after an exponential rate λ_i time. If the ith machine is broken, it takes the repairperson an exponential rate μ_i amount of time to fix it and once it is fixed, it's good as new. Assume the repairperson begins work the instant a machine breaks down, that only one machine can be repaired at a time, and all lifetime and repair times are independent.

   (a) Construct an appropriate continuous time Markov chain to describe the system and find the generator.

   (b) If $\lambda_i = \mu_i = i$ for $i = 1, 2$, find the stationary distribution of the process.

   Ans.

   (a) We take the state space to be $\{(1,1), (1,0), (0,1), (0,0,1), (0,0,2)\}$ where the ith coordinate is one if machine i is functional and zero otherwise and $(0,0,i)$ means both machines are broken and the repairperson is working on machine i. Then

$$\begin{aligned}
(1,1) &\to (1,0), \text{ rate } \lambda_2, \\
(1,1) &\to (0,1), \text{ rate } \lambda_1 \\
(1,0) &\to (1,1), \text{ rate } \mu_2, \\
(1,0) &\to (0,0,2), \text{ rate } \lambda_1 \\
(0,1) &\to (1,1), \text{ rate } \mu_1, \\
(0,1) &\to (0,0,1), \text{ rate } \lambda_2 \\
(0,0,1) &\to (1,0), \text{ rate } \mu_1, \\
(0,0,2) &\to (0,1), \text{ rate } \mu_2.
\end{aligned}$$

   So the generator can be written

$$A = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_2 & \lambda_1 & 0 & 0 \\ \mu_2 & -(\lambda_1+\mu_2) & 0 & 0 & \lambda_1 \\ \mu_1 & 0 & -(\lambda_2+\mu_1) & \lambda_2 & 0 \\ 0 & \mu_1 & 0 & -\mu_1 & 0 \\ 0 & 0 & \mu_2 & 0 & -\mu_2 \end{pmatrix}.$$

   (b) Solving $\pi A = 0$ yields

$$\pi = (7/34, 8/34, 5/34, 10/34, 4/34).$$

4. (CTMCs as limits of DTMCs) Let P be a one step transition matrix for a discrete time Markov chain on $0, 1 \dots$ such that $p_{ii} = 0$ for all i. Also let $0 < \lambda_0, \lambda_1, \dots$ be such that $\max_{i \geq 0} \lambda_i < N$, with N an integer. Define the discrete time Markov chain $Y_0, Y_1, \dots$ by

$$\mathbb{P}(Y_{n+1}^{(N)} = i | Y_n^{(N)} = i) = \left(1 - \frac{\lambda_i}{N}\right),$$

and for $i \neq j$

$$\mathbb{P}(Y_{n+1}^{(N)} = j | Y_n^{(N)} = i) = \frac{\lambda_i}{N} p_{ij}.$$

We can think of the discrete jumps of $Y^{(N)}$ occurring at times on the lattice $\{0, 1/N, 2/N \dots\}$ and make a continuous time process by defining

$$X_t^{(N)} = Y_{\lfloor Nt \rfloor}^{(N)},$$

where $\lfloor a \rfloor$ is the greatest integer not bigger than a.

(a) What does a typical trajectory of $X^{(N)}$ look like? Does it have jumps? At what times? How do jumps correspond to $Y^{(N)}$?

(b) Given $X_0^{(N)} = i$, what is the distribution of the random time

$$T^{(N)}(i) = \min\{t \geq 0 : X_t^{(N)} \neq i\}$$

(c) As $N \to \infty$, to what distribution does that of $(T^{(N)}(i) | X_0^{(N)} = i)$ converge?

(d) Based on the previous two items and comparing to the previous problem, do you think that $X^{(N)}$ converges as $N \to \infty$ to a continuous time Markov chain (not worrying about what exactly convergence means)? What is its generator?

Ans.

(a) The chain only has jumps at times k/N for k an integer. Given the chain is in state i, it stays there for a geometric λ_i/N (> 0) number of $1/N$ time units and then jumps according to the one step transition matrix P. The number of these time units are the number of integer time units between jumps in the $Y^{(N)}$ chain.

(b) As mentioned in the previous problem, the $Y^{(N)}$ chain stays at state i for a geometric (λ_i/N) number of time units before jumping. Then considering the time change to get from $Y^{(N)}$ to $X^{(N)}$, the variable $NT^{(N)}(i)$ is geometric λ_i/N (> 0); that is for $k = 1, 2, \dots$

$$\mathbb{P}(T^{(N)}(i) = k/N) = \frac{\lambda_i}{N}\left(1 - \frac{\lambda_i}{N}\right)^{k-1}.$$

(c) A standard calculation (do it!) shows that if Z_p is geometric p, then pZ_p converges in distribution to an exponential distribution with mean 1 as $p \to 0$. Since $NT^{(N)}(i)$ is geometric λ_i/N, $T^{(N)}(i)$ converges to an exponential variable with rate λ_i (or what's the same an exponential rate one variable divided by λ_i).

(d) Since the holding times converge to exponential variables as N goes to infinity, the description of the limiting chain is as follows. Given $X_0 = i$ the chain waits an exponential with rate λ_i time and then jumps to state j with probability p_{ij} (the state jumped to is independent of the time of the jump). Then the chain stays in state j and exponential λ_j amount of time and jumps to state k with probability p_{jk}, and so on. According to our interpretation of the entries of the generator from lecture, the (i, j)th entry of the generator is $\lambda_i p_{ij}$ for $i \neq j$ and $p_{ii} = -\lambda_i$.