Spectral Representation

Let $X(\mathbf{k}), \mathbf{k} \in \mathbb{R}^n$ be a homogenous random field.

Theorem 1 (Multi-dimensional Bochner's Theorem) *A real function $r(\tau)$ on $\mathbb{R}^n$ is positive (semi-)definite if and only if it can be represented in the form*

$$r(\tau) = \int_{\mathbb{R}^n} e^{i\tau k} d^n F(\mathbf{k}),$$

where $F(\,\cdot\,)$ is a non-negative bounded measure.

Bochner's theorem says that all positive definite functions have a unique spectral representation.

Theorem 2 (Weiner-Khintchine's Theorem) *A real function $\rho(\tau)$ on $\mathbb{R}^n$ is a correlation function if and only if it can be represented in the form*

$$\rho(\tau) = \int_{\mathbb{R}^n} e^{i\tau k} d^n F(\mathbf{k}),$$

where $F(\,\mathbf{k}\,)$ on $\mathbb{R}^n$ has the properties of a n–dimensional distribution function.

The n–dimensional distribution function is called the spectral distribution function.

When F is continuous, the spectral density function exists and is defined as

$$f(\mathbf{k}) = \frac{\partial^n F(\mathbf{k})}{\partial k_1 \ldots \partial k_n}.$$

Then

$$\rho(\tau) = \int_{\mathbb{R}^n} e^{i\tau k} d^n F(\mathbf{k}).$$

The spectral density function is obtained from the correlation function by the usual formula for the inversion of an n-dimensional Fourier transform:

$$f(\tau) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\tau k} \rho(\tau) d^n \tau.$$

This gives us the explicit method for verifying the positive definiteness of a correlation function on $\mathbb{R}^n$:

Evaluate the spectral density, $f(\mathbf{k})$, given by the expression above, and check if is non-negative for any $\mathbf{k} \in \mathbb{R}^n$.

For isotropic correlation functions the Weiner–Khintchine theorem takes a simpler form where the $n-$dimensional Fourier integral is replaces by a one–dimensional *Bessel transform:*

Theorem 3 *A real function $\rho(\tau)$ on $\mathbb{R}^n$ is a correlation function if and only if it can be represented in the form*

$$\rho(\tau) = 2^{\frac{n-2}{2}}\Gamma(\tfrac{n}{2}) \int_0^\infty \frac{J_{(n-2)/2}(k\tau)}{(k\tau)^{(n-2)/2}} d\Phi(k),$$

where the function $\Phi(k)$ on $\mathbb{R}$ has the properties of a distribution function and J are Bessel functions of the 1 kind.

A few special cases are of particular interest:

$$\begin{aligned}
\rho &= \int_0^\infty \cos k\tau d\Phi(k) && \text{for } \rho \in \mathcal{D}_1, \\
\rho &= \int_0^\infty J_0 k\tau d\Phi(k) && \text{for } \rho \in \mathcal{D}_2, \\
\rho &= \int_0^\infty \frac{\sin k\tau}{k\tau} d\Phi(k) && \text{for } \rho \in \mathcal{D}_3, \\
\rho &= \int_0^\infty \exp(-k^2\tau^2) d\Phi(k) && \text{for } \rho \in \mathcal{D}_\infty.
\end{aligned}$$

Example 1. $\rho(t) = e^{-t^2}$, $d = 1$, is a correlation function.

$$\begin{aligned}
f(X) =& \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itX} e^{-t^2} dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} exp\left(-\left(t + \frac{iX}{2} \right)^2 + \left(\frac{iX}{2} \right)^2 \right) dt \\
=& \frac{1}{\sqrt{2\pi} \cdot \sqrt{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} \cdot \frac{1}{\sqrt{2}}} e^{-\frac{\left(t + \frac{iX}{2}\right)^2}{2 \cdot \frac{1}{2}}} dt \\
=& \frac{1}{2\sqrt{\pi}} e^{-\frac{X^2}{4}} \geq 0.
\end{aligned}$$

Example 2. $\rho(\tau) = \int_0^\infty \frac{\sin k\tau}{k\tau} d\Phi(k)$.

Let

$$\Phi(k) = \begin{cases} k^2, & \text{if} \quad k \in [0, 1], \\ 1, & \text{if} \quad k > 1. \end{cases}$$

Then

$$\Phi'(k) = \begin{cases} 2k, & \text{if} \quad k \in [0, 1], \\ 1, & \text{if} \quad k > 1. \end{cases}$$

and

$$\begin{aligned}
\rho(\tau) &= \int_0^1 \frac{\sin k\tau}{k\tau} 2k dk = \frac{2}{\tau} \int_0^1 \sin(k\tau) dk \\
&= \frac{2}{\tau^2} (-\cos(k\tau))|_0^1 = \frac{2}{\tau^2}(1 - \cos(\tau)).
\end{aligned}$$

Example 3. $E\left(\frac{\sin K\tau}{K\tau}\right)$, where K is a random variable with cdf $\Phi(\cdot)$.

If $\Phi(a) = 1$, for some $a > 0$, $\Phi(a-) = 0$, then $\rho(\tau) = \frac{\sin a\tau}{a\tau}$.

Similarly we obtain $\rho(\tau) = \cos(a\tau), \rho(\tau) = J_0(a\tau)$ are correlations in $\mathbb{R}^1$ and $\mathbb{R}^1$ respectively.