The University of Melbourne
Semester 2 Assessment 2008

Department of Mathematics and Statistics
620-156 Linear Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.

This paper has: 5 pages.

Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
No materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 13 questions. The total number of marks is 90. All questions may be attempted. All answers should be appropriately justified.

This paper may be held by the Baillieu Library.

— BEGINNING OF EXAMINATION QUESTIONS —

1. Let

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}.$$

Evaluate, if possible:

(a) AB
(b) BA
(c) BA^T
(d) $B + A^T$

[6 marks]

2. (a) Let

$$G = \begin{bmatrix} -1 & 3 & 2 \\ 0 & 2 & 2 \\ -2 & 4 & 8 \end{bmatrix}$$

Calculate the determinant of G using either row operations or cofactor expansion.

(b) Let K and H be two 4×4 matrices such that $\det(K) = -3$ and $\det(H) = 2$. Calculate:

(i) $\det(KH^2)$
(ii) $\det(K^{-1})$
(iii) $\det(2K)$

[6 marks]

3. (a) Find the inverse, if it exists, of the square matrix

$$\begin{bmatrix} 3 & 3 & 3 \\ 1 & 3 & -1 \\ 6 & -2 & 9 \end{bmatrix}$$

using row operations. At each step you should clearly indicate the row operations that you have performed.

(b) Using your answer to part (a), solve the following linear system:

$$\begin{aligned} 6x + 6y + 6z &= 12 \\ 2x + 6y - 2z &= 20 \\ 12x - 4y + 18z &= -10 \end{aligned}$$

[6 marks]

4. (a) We are given a point $A(1,1,0)$ and the line

$$l : \frac{x-1}{2} = y = \frac{2-z}{3}$$

   (i) Find a vector, $\boldsymbol{v}$, parallel to l.

   (ii) Find the cartesian equation of the plane π that contains A and l.

   (b) Two vectors are given by $\boldsymbol{a} = (0,1,-1)$ and $\boldsymbol{b} = (1,-1,0)$. Determine the area of the parallelogram with edges $\boldsymbol{a}$ and $\boldsymbol{b}$.

[6 marks]

5. (a) Let $C = \{(x,y) \ : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ be the set of 2-dimensional real vectors on or inside the unit circle. Determine whether or not C is a subspace of the vector space $\mathbb{R}^2$.

   (b) Let W be the subset of $\mathcal{P}_3$ given by $W = \{a + bx + cx^3 | a + b + c = 0\}$. Determine whether or not W is a subspace of $\mathcal{P}_3$.

   Justify your answers by either appealing to appropriate theorems, or providing a counter-example.

[6 marks]

6. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 & 24 \\ 2 & 5 & 7 & -1 & 0 \\ -1 & -2 & -8 & 1 & 24 \\ 3 & -2 & 96 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 26 & 0 & -6 \\ 0 & 1 & -9 & 0 & 10 \\ 0 & 0 & 0 & 1 & 38 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   In this question you may assume the fact that the matrix B is obtained from the matrix A by applying elementary row operations. Using this information, or otherwise, answer the following:

   (a) What is the rank of A?

   (b) Are the rows of A linearly independent? Explain your answer.

   (c) Write down a basis for the row space of A.

   (d) Write down a basis for the column space of A.

   (e) Do the vectors $(1,2,-1,3), (3,5,-2,-2), (-1,7,-8,96), (0,-1,1,1), (24,0,24,0)$ span $\mathbb{R}^4$? Give a reason.

   (f) Write $(-1,7,-8,96)$ as a linear combination of $(1,2,-1,3)$ and $(3,5,-2,-2)$.

   (g) Find a basis for the solution space of A.

[8 marks]

7. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the y-axis and let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\frac{\pi}{2}$ anticlockwise about the origin.

   (a) Write down the standard matrix representations for S and R.

   (b) Use part (a) to find the standard matrix representation for $S \circ R$ (R followed by S).

[4 marks]

8. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10x_1 + 8x_2 + 3x_3 - 16x_4 \\ 5x_1 + 6x_2 + x_3 - 7x_4 \\ 6x_1 + 8x_2 + x_3 - 8x_4 \end{bmatrix}$$

   (a) Find a basis $\mathcal{B}$ for $\mathrm{Ker}(T)$, the kernel of T.

   (b) Find a basis $\mathcal{C}$ for $\mathrm{Im}(T)$, the image of T.

   (c) State the Rank-Nullity Theorem for linear transformations, and verify that it holds for T.

   (d) Show that $(6, 7, 10)$ is in $\mathrm{Im}(T)$ and find its co-ordinates with respect to $\mathcal{C}$.

[8 marks]

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by:

$$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_2 - x_3, 3x_1 + x_2 + 3x_3)$$

   (a) Write down the matrix $[T]_{\mathcal{S}}$ representing T with respect to the standard basis

$$\mathcal{S} = \{(1,0,0), (0,1,0), (0,0,1)\}.$$

   (b) Find the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{S}$, where

$$\mathcal{B} = \{(-1,1,1), (1,-1,2), (1,0,0)\}.$$

   (c) Use you answer to the previous part to find the transition matrix $P_{\mathcal{B},\mathcal{S}}$.

   (d) Calculate the matrix $[T]_{\mathcal{B}}$ representing T with respect to the basis $\mathcal{B}$.

   (e) Using your answer to the previous part, explain why T is *not* diagonalisable.

[10 marks]

10. Consider the matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

(a) Calculate the characteristic polynomial of A.

(b) Find the eigenvalues and eigenvectors of the matrix

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

[8 marks]

11. (a) Show that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_1 + 4x_2y_2 + 4x_3y_3 + x_2y_3 + x_3y_2$$

defines an inner product on $\mathbb{R}^3$.

(b) Starting with the basis $\{(1,0,0),(0,1,0),(0,0,1)\}$, use the Gram-Schmidt procedure to find an orthonormal basis for $\mathbb{R}^3$ with respect to the inner product of part (a).

[8 marks]

12. (a) Find the line of best fit $y = a + bx$ to the data:

x	-3	-1	0	1	2	3
y	4	4	3	1	-1	-1

(b) Sketch the line of best fit and include the data points on your graph.

(c) With this line of best fit, calculate the size of the error at $x = 0$.

[6 marks]

13. Consider the matrix

$$A = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(a) Find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$.

(b) Use your answer to part (a) to calculate $\lim_{n\to\infty} A^n$.

[8 marks]

— END OF EXAMINATION QUESTIONS —