MAST20004 Probability

Assignment One [Due Monday 19/08]

Please click on the Plagiarism Declaration link on the LMS and follow the instructions before submitting this assignment. Assignment boxes are located on the ground floor in the Peter Hall Building (near Wilson computer lab). Your solutions to the assignment should be left in the MAST20004 assignment box set up for your tutorial group. Don't forget to staple your solutions and to print your name, student ID, the subject name and code, and your tutor's name on the first page.

Staplers are not available from the School General Office, from individual staff members, or at the Assignment Boxes. Please make sure your assignment is stapled ready for submission before bringing it to Peter Hall.

There are 5 problems in total, of which 3 randomly chosen ones will be marked. You are expected to submit answers to all questions, otherwise a mark penalty will apply. Calculations and reasoning must be given in order to obtain full credit.

Problem 1. A boy has a fever after coming home in the afternoon. His mother thinks that it could be related to the following three possible reasons:

$$\begin{aligned} A &: \text{ He plays football in the rain,} \\ B &: \text{ He takes a cold water shower after playing,} \\ C &: \text{ He eats too many ice creams.} \end{aligned}$$

(i) Show that

$$A \cup B \cup C = A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C),$$

and

$$\mathbb{P}(A \cup B \cup C) \leqslant \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C).$$

(ii) Use the Addition Theorem (cf. Lecture Slide 35, Property (9)) or another method to show that

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) \\ - \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B \cap C).$$

(iii) The mother has 80% confidence that her son's fever is caused by at least one of the three given reasons. She further estimates that the probabilities of the three individual reasons are $0.5, 0.5, 0.2$ respectively, and she believes that they are pairwisely independent. Are the three reasons mutually independent?
(iv) Suppose that the mother is 100% sure that her son's fever is caused by at least one of the three given reasons. Moreover, she believes that they are mutually independent although she doesn't know the exact probabilities of any of the individual reasons. After a moment's thought, she tells her son that one of the three reasons must be certain (that is, one of $\mathbb{P}(A) = 1$ or $\mathbb{P}(B) = 1$ or $\mathbb{P}(C) = 1$ must be true)! Should the boy believe his mother's assertion?

Problem 2. In a standard poker game with 52 playing cards, a *flush* is a hand which contains five cards all of the same suit (diamond, clover, heart or spade). Suppose that 5 cards are randomly selected. Under each of the following two assumptions, describe the sample space for the corresponding random experiment and compute the probability that a flush is obtained.
(i) The 5 cards are selected one by one in order.
(ii) The 5 cards are selected at the same time without orders.

Problem 3. [Hard] A fair coin is tossed for n times independently.
(i) Suppose that $n = 5$. Given the appearance of successive heads, what is the conditional probability that successive tails never appear?
(ii) Let p_n denote the probability that successive heads never appear. Find an explicit formula for p_n.
(iii) Let q_n denote the conditional probability that successive heads appear, given no successive heads are observed in the first $n-1$ tosses. What is $\lim_{n\to\infty} q_n$?

Problem 4. (1) In recent statistics, the success rate of baby delivery is 99%. However, there are 12% of births involving Cesarean (known as C section), and when a C section is performed, the baby survives 96% of the time.
(i) What is the probability that a randomly chosen pregnant woman has a C section?
(ii) If a randomly chosen pregnant woman does not have a C section, what is the probability that her baby survives?
(2) An ectopic pregnancy is three times as likely to develop when the pregnant

woman is a smoker as it is when she is not a smoker. If 25% of pregnant women are smokers, what percentage of women having ectopic pregnancies are smokers?

Problem 5. A man comes home and he wants to open her door. He has 5 keys, two of which are the door keys but he couldn't remember them. Therefore, he tries the 5 keys independently and at random. If a key is unsuccessful, it will be excluded for further selections. Let N be the total number of trials.
(i) Describe the underlying sample space and the set of possible values for N.
(ii) Find the probability mass function and distribution function of N.
(iii) Compute the expectation and variance of N.
(iv) Consider a simpler situation in which there are 3 keys in total (named a, b, c), and only one of them is the correct key, say key a. We would like to compute the probability that the man opens the door at the second trial. Xi has a solution to this problem as follows:

When performing the random experiment, the man keeps trying the keys until he reaches the correct key a. Therefore, the sample space is given by

$$\Omega = \{a, ba, ca, bca, cba\},$$

each outcome recording a possibility of trying the keys until reaching the correct key. Since the problem corresponds to the classical probability model over Ω, we conclude that

$$\mathbb{P}(N = 2) = \mathbb{P}(\{ba, ca\}) = \frac{2}{5}.$$

Do you think Xi's solution is correct? If not, is it possible to fix it without changing his sample space Ω?