TOR – Exam 2019 – Solution

Q1

a) $f(x) = e^x + x\log x$ is cleary continuous on $x > 0$ since e^x is continuous and $x\log x$ is continuous for $x > 0$

Now $f'(x) = e^x + \log x + 1$

$f''(x) = e^x + \frac{1}{x} > 0$ for $x > 0$

$\therefore$ f is strictly convex on $(0;\infty)$

$\therefore$ f is unimodal on $(0;\infty)$

b). $\gamma^n < \frac{2\varepsilon}{b-a}$

$= \frac{2.0.005}{0.1} = 0.1$

$(0.618)^n < 0.1 \Rightarrow n = 5$

$\therefore$ 6 caculations are required

Final interval length is $(0.618)^5(0.2-0.1)$

$= 0.009$

d) Consider $f'(x) = e^{x} + \log x + 1$

$f'(0.1) = -0.1974$

$f'(0.2) = 0.611$

Also, $f''(x) > 0$ for $x > 0$

$\therefore$ $f'(x)$ is increasing on $[0.1 ; 0.2]$

$\therefore$ False position can be used.

e). $f'(0.2) = 0.611$
$f''(0.2) = 6.22$

$$p = 0.2 - \frac{0.611}{6.22} \qquad\qquad p = a - \frac{g(a)}{g'(a)}$$

$= 0.1018.$

c) $\frac{b-a}{F_n} < 2\varepsilon$

$\therefore \quad F_n > \frac{0.1}{0.01} = 10$

$\therefore \quad n = 5$

$p = b - \frac{F_{n-1}}{F_n}(b-a) = 0.2 - \frac{5}{8}(0.2 - 0.1)$

$= 0.1375$

$q = a + \frac{F_{n-1}}{F_n}(b-a) = 0.1 + \frac{5}{8} \cdot 0.1$

$= 0.1625$

$f(p) = e^{0.1375} + 0.1375 \log 0.1375 = 0.8766$
$f(q) = e^{0.1625} + 0.1625 \log 0.1625 = 0.8812$

$f(q) > f(p) \quad \therefore$

$b = q = 0.1625$

New interval is $[0.1 \,;\, 0.1625]$

Q2.

$$f(x) = x_1^3 + x_2^2 + x_3^2 + 6x_1x_2 - 8x_2 - 8x_3$$

a)

$$\nabla f = \begin{bmatrix} 3x_1^2 + 6x_2 \\ 2x_2 + 6x_1 - 8 \\ 2x_3 - 8 \end{bmatrix} = 0$$

$$3x_1^2 + 6x_2 = 0 \quad \ldots \quad (1)$$
$$2x_2 + 6x_1 = 8 \quad \ldots \quad (2)$$
$$2x_3 = 8 \qquad x_3 = 4$$

$(1) - 3\times(2)$: $3x_1^2 - 18x_1 = -24$

$$x_1^2 - 6x_1 + 8 = 0$$
$$(x_1 - 4)(x_1 - 2) = 0$$
$$x_1 = 4 \text{ or } x_1 = 2$$

$$\therefore \quad x_2 = \frac{8 - 6x_1}{2}$$

$$= -8 \text{ or } -2$$

$\therefore$ SPs : $(4, -8, 4)$, $(2, -2, 4)$.

b) $$\nabla^2 f = \begin{bmatrix} 6x_1 & 6 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \therefore \nabla^2 f(4, -8, 4)$$

$$= \begin{bmatrix} 24 & 6 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$(24-\lambda)(2-\lambda)(2-\lambda) - 72 = 0$

$-\lambda^3 + 28\lambda^2 - 100\lambda + 24 = 0$ $\lambda = 0.27,\ 2.0,\ 25.53.$

At $(2, -2, 4)$

$$\nabla^2 f(2,-2,4) = \begin{bmatrix} 12 & 6 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\therefore\ (12-\lambda)(2-\lambda)(2-\lambda) - 72 = 0$

$-\lambda^3 + 16\lambda^2 - 52\lambda - 24 = 0$

$\lambda = -0.81,\ 2,\ 14.81.$

$\therefore\ (4, -8, 4)$ is a local min

$(2, -2, 4)$ is a saddle point.

c) In an unconstrained C^1 problem a global min must be a stationary point. $\therefore$ the local min is a global min. since there is only one and there is no local max.

Q3. $\min f(x_1, x_2) = \frac{4}{3} x_1^3 - x_1 x_2^2 + 3x_2^2 - 8x_2$

a)

$x^0 = (2, 0)$

$$\nabla f = \begin{bmatrix} 4x_1^2 - x_2^2 \\ -2x_1 x_2 + 6x_2 - 8 \end{bmatrix}$$

$$\nabla f(2,0) = \begin{bmatrix} 16 \\ -8 \end{bmatrix}$$

$$\therefore d^0 = \begin{bmatrix} -16 \\ 8 \end{bmatrix} \quad / \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$x^1 = x^0 + t_0 d^0$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 2t \\ t \end{bmatrix}$$

$$\phi(t) = f(x^1) = \frac{4}{3}(2-2t)^3 - (2-2t)(t)^2 + 3t^2 - 8t$$

$$= -\frac{26}{3} t^3 + 33t^2 - 40t + 32/3$$

$$\phi'(t) = -26t^2 + 66t - 40 = 0$$

$$13t^2 - 33t + 20 = 0$$

$$(t-1)(13t - 20) = 0$$

$\phi''(t) = -52t + 66$

$\therefore t = 1$ is min.

$\therefore \quad x^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

b) $\quad \nabla^2 f = \begin{bmatrix} 8x_1 & -2x_2 \\ -2x_2 & -2x_1+6 \end{bmatrix}$

$\nabla^2 f(2,0) = \begin{bmatrix} 16 & 0 \\ 0 & 2 \end{bmatrix}$

pos. def $\quad \lambda = 16, 2$

$d^0 = -\nabla^2 f(2,0)^{-1} \nabla f(2,0)$

$\begin{bmatrix} 16 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -\begin{bmatrix} 16 \\ -8 \end{bmatrix}$

$\therefore \quad 16d_1 = -16 \qquad d_1 = -1$
$\qquad 2d_2 = 8 \qquad d_2 = 4$

$x^1 = x^0 + t d^0 \qquad = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 2-t \\ 4t \end{bmatrix}$

c) $d^0 = -H_0 \nabla f(x^0)$

$$= -\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 16 \\ -8 \end{bmatrix}$$

$$= -\begin{bmatrix} 32 \\ -16 \end{bmatrix} = \begin{bmatrix} -32 \\ 16 \end{bmatrix} \parallel \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

d) Find t such that $\frac{df(x^{k+1})}{dt} = 0$

Now $x^{k+1} = x^k + td^k$

$$\therefore \frac{df(x^{k+1})}{dt} = \nabla f(x^{k+1})^T \frac{d(x^k + td^k)}{dt}$$

$$= \nabla f(x^{k+1})^T d^k$$

$$= 0$$

Now $d^{k+1} = -\nabla f(x^{k+1})$

$$\therefore (d^{k+1})^T . d^k = -\nabla f(x^{k+1})^T d^k = 0$$

$\therefore \perp$

Q4.

$$\min f(x) = 2x_1^3 + 3x_2^2 + 3x_3^2$$

$$\text{st} \quad x_1 + x_2 + x_3 = 0$$

$$x_1 \geq 3.$$

a) $$L(x, \lambda, \mu) = 2x_1^3 + 3x_2^2 + 3x_3^2 + \lambda(3 - x_1) + \mu(x_1 + x_2 + x_3)$$

b) KKTa $$\nabla_x L(x^*, \lambda^*, \mu^*) = 0$$

$$6x_1^{*2} - \lambda^* + \mu^* = 0 \quad \ldots \text{(1)}$$
$$6x_2^* + \mu^* = 0 \quad \ldots \text{(2)}$$
$$6x_3^* + \mu^* = 0 \quad \ldots \text{(3)}$$

KKTb

$$\lambda^* \geq 0$$
$$\lambda^*(3 - x_1^*) = 0 \quad \ldots \text{(4)}$$
$$3 - x_1^* \leq 0$$

KKTc

$$x_1^* + x_2^* + x_3^* = 0 \quad \ldots \text{(5)}$$

case 1 : $\lambda^* > 0$ $\therefore$ $x_1^* = 3$ by (4)

Also, $x_2^* = x_3^*$ by (2), (3)

$\therefore$ By (5), $x_2^* = x_3^* = -\frac{3}{2}$.

$\therefore$ By ②, $\mu^* = -6x_2^* = -6\left(-\frac{3}{2}\right)$

$= 9$

$\therefore$ By ①, $\lambda^* = 6x_1^{*2} + \mu^*$

$= 6(3)^2 + 9$

$= 63$

KKT point : $(x^*, \lambda^*, \mu^*) = \left(\left(3, -\frac{3}{2}, -\frac{3}{2}\right), 63, 9\right)$

Case 2: $x_1^* > 3$

$\therefore \lambda^* = 0$

By ①, ②, ③

$x_1^{*2} = x_2^* = x_3^* = -\frac{\mu^*}{6}$

By ⑤,

$x_1^* + 2x_1^{*2} = 0$

$\therefore x_1^* = 0$ or $x_1^* = -2$

Contradiction $x_1^* > 3$

c) $C(x^*, \lambda^*) = \{ d \in \mathbb{R}^3 ; \langle \nabla g(x^*), d \rangle = 0,$
$\langle \nabla h(x^*), d \rangle = 0 \}$

where $g(x) = x_3 - x_1$
$h(x) = x_1 + x_2 + x_3$

$\nabla g(x) = (-1, 0, 0)^T$

$\nabla h(x) = (1, 1, 1)^T$

$\therefore \quad (-1, 0, 0) \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = 0 \Rightarrow d_1 = 0$

$(1, 1, 1) \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = 0 \Rightarrow d_2 + d_3 = 0$

$\therefore \quad C(x^*, \lambda^*) = \{ (0, d, -d) : d \in \mathbb{R} \}$

d) Now $\nabla^2_{xx} L(x, \lambda, \mu)$

$$= \begin{bmatrix} 12x_1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\therefore \quad \nabla^2_{xx} L(x^*, \lambda^*, \mu^*) = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Eigenvalues of $\nabla^2_{xx} L(x^*, \lambda^*, \mu^*)$ are $36, 6, 6$.

$\therefore$ L is pos def everywhere and $\therefore$ pos def on the critical cone.

$\therefore$ (x^*, λ^*, μ^*) is a local minimum

e) f changes by $-\mu^* \Delta = -(9)\cdot(1) = -9$

Q5.

$$\min f(x_1, x_2) = x_1^4 - 2x_1^2 + x_2^2$$

$$x_1 \geq 0 \implies -x_1 \leq 0$$
$$x_2 \geq 3 \implies 3 - x_2 \leq 0$$

a) $$P_k(x_1, x_2) = x_1^4 - 2x_1^2 + x_2^2 + \frac{k}{2}\left((-x_1)_+\right)^2 + \frac{k}{2}\left((3-x_2)_+\right)^2$$

b) $$\nabla P_k(x) = \begin{bmatrix} 4x_1^3 - 4x_1 - k(-x_1)_+ \\ 2x_2 - k(3-x_2)_+ \end{bmatrix} = 0$$

If $x_1 < 0$ then $4x_1^3 - 4x_1 + kx_1 = 0$

$\therefore x_1\left(4x_1^2 - 4 + k\right) = 0$

$\implies x_1 = 0$ or $x_1 = \pm\sqrt{\frac{4-k}{4}}$

contradiction since $x_1 < 0$ and $k \to \infty$

$\therefore x_1 \geq 0$.

If $x_2 \geq 3$ then

$2x_2 - k(0) = 0 \implies x_2 = 0$ contradiction

$\therefore x_2 < 3$.

c)

$\nabla P_k(x) = 0 \quad , \quad x_1 \geq 0 \quad , \quad x_2 \leq 3$

$\therefore \quad 4x_1^3 - 4x_1 = 0 \qquad x_1(x_1^2 - 1) = 0$

$\Rightarrow x_1^k = 0 \text{ or } x_1^k = 1 \qquad x_1^k \neq -1$

$2x_2 - K(3 - x_2) = 0$

$\Rightarrow x_2^k = \dfrac{3K}{2+K}$

$\therefore \quad x^k = \left(0, \dfrac{3k}{2+k}\right) \quad \text{or} \quad x^k = \left(1, \dfrac{3k}{2+k}\right)$

d) $x^* = \lim\limits_{k \to \infty} \left(0, \dfrac{3k}{2+k}\right) = (0, 3)$

$x^* = \lim\limits_{k \to \infty} \left(1, \dfrac{3k}{2+k}\right) = (1, 3)$

Q6

a) $\min f(x_1, x_2) = x_1^2 - 2x_1x_2 + 2x_2^2 - 14x_2$

$$x_1^2 + x_2^2 \leq 5$$
$$x_1 - x_2 \geq 1$$

$$\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 - 2x_1x_2 + 2x_2^2$$

$$\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{bmatrix}$$

$$\frac{1}{2}ax_1^2 + bx_2x_1 + \frac{1}{2}cx_2^2$$

$\frac{1}{2}a = 1 \qquad b = -2 \qquad \frac{1}{2}c = 2$

$a = 2 \qquad\qquad\qquad\quad c = 4$

$$f(x_1, x_2) = \frac{1}{2}x^T A x - \begin{bmatrix} 0 \\ 14 \end{bmatrix}^T x$$

$$\nabla^2 f = A = \begin{bmatrix} 2 & -2 \\ -2 & +4 \end{bmatrix}$$

$$(2-\lambda)(4-\lambda) - 4 = 0$$

$\lambda = 0.76\ldots, \quad \lambda = 5.23$

$\therefore$ pos. def.

$\therefore f$ is convex

b) $L(x,\lambda) = x_1^2 - 2x_1x_2 + 2x_2^2 - 14x_2$

$+ \lambda_1(x_1^2 + x_2^2 - 5) + \lambda_2(1 - x_1 + x_2)$

$x^* = (2,1)$

$\therefore \; L(x^*,\lambda) = 2^2 - 2(2)(1) + 2(1)^2 - 14(1)$

$+ \lambda_1((2)^2 + (1)^2 - 5) + \lambda_2(1 - (2) + (1))$

$= 4 - 4 + 2 - 14 + 0$

$= -12$

$\lambda \geq 0: \; L(x^*,\lambda^*) = -12$

$L(x^*,\lambda) \leq L(x^*,\lambda^*)$

$\lambda^* = (2, 10)$

$\therefore \; L(x,\lambda^*) = x_1^2 - 2x_1x_2 + 2x_2^2 - 14x_2$

$+ 2(x_1^2 + x_2^2 - 5) + 10(1 - x_1 + x_2)$

$= 3x_1^2 + 4x_2^2 - 2x_1x_2 - 4x_2 - 10x_1$

$\therefore \; \nabla_x L(x,\lambda^*) = \begin{bmatrix} 6x_1 - 2x_2 - 10 \\ 8x_2 - 2x_1 - 6 \end{bmatrix} = 0$

$\Rightarrow x_1 = 2 \quad x_2 = 1$

$\therefore$ $(2,1)$ minimises $L(x,\lambda^*)$ since

$$\nabla^2 L((2,1),\lambda^*) = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$$

is pos def

$\therefore$ $L(x^*,\lambda^*) \leq L(x,\lambda^*)$ for all x

$\therefore$ saddle inequality holds

c) Since $\nabla^2 L(x^*,\lambda^*)$ is pos def, the program is <u>strictly</u> convex $\therefore$ unique minimum and all local mins are global mins.

d) Wolfe Dual

$$\max_{x,\lambda} L(x,\lambda) \quad \lambda \geq 0 \quad \nabla_x L(x,\lambda) = 0 \equiv \max_{x,\lambda} x_1^2 - 2x_1x_2 + 2x_2^2 - 14x_2 + \lambda_1(x_1^2+x_2^2-5) + \lambda_2(1-x_1+x_2)$$

s.t.
$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0$$
$$2x_1 - 2x_2 + 2\lambda_1 x_1 - \lambda_2 = 0$$
$$-2x_1 + 4x_2 - 14 + 2\lambda_1 x_2 + \lambda_2 = 0$$