1. (a) The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is independent if the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{0}$$

has just one solution, namely $a_1 = a_2 = 0$. The corresponding system of equations is

$$\begin{aligned} -3a_1 + 7a_2 &= 0, \\ a_1 - 5a_2 &= 0, \\ 7a_1 + 7a_2 &= 0. \end{aligned}$$

Running `rref([-3 7; 1 -5; 7 7])` in MATLAB shows that the only solution is $a_1 = a_2 = 0$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is independent.

(b) $\mathrm{Sp}(\mathbf{v}_1)$ is the line in $\mathbb{R}^3$ passing through the origin and the point $(-3, 1, 7)$.
Two examples of vectors in $\mathrm{Sp}(\mathbf{v}_1)$ are $2\mathbf{v}_1 = (-6, 2, 14)$ and $-\mathbf{v}_1 = (3, -1, -7)$.

(c) $\mathrm{Sp}(\mathbf{v}_2)$ is the line in $\mathbb{R}^3$ passing through the origin and the point $(7, -5, 7)$.
Two examples of vectors in $\mathrm{Sp}(\mathbf{v}_2)$ are $2\mathbf{v}_2 = (14, -10, 14)$ and $-\mathbf{v}_2 = (-7, 5, -7)$.

(d) $\mathrm{Sp}(\mathbf{v}_1, \mathbf{v}_2)$ is the plane in $\mathbb{R}^3$ passing through the origin and the points $(-3, 1, 7)$ and $(7, -5, 7)$.
Two examples of vectors in $\mathrm{Sp}(\mathbf{v}_1, \mathbf{v}_2)$ are $\mathbf{v}_1 + \mathbf{v}_2 = (4, -4, 14)$ and $\mathbf{v}_1 - \mathbf{v}_2 = (-10, 6, 0)$.

2. (a) Running `rref([1 2 3 1; 0 1 2 3; 2 1 0 0])` shows that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) No; from the matrix above, we can observe that $\mathbf{a}_3 = -\mathbf{a}_1 + 2\mathbf{a}_2$.

(c) From (a), we have

$$\begin{aligned} \mathrm{Ker}(\mathbf{A}) &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3, x_2 = -2x_3, x_4 = 0\} \\ &= \{(x_3, -2x_3, x_3, 0) \in \mathbb{R}^4 : x_3 \in \mathbb{R}\} \\ &= \mathrm{Sp}((1, -2, 1, 0)), \end{aligned}$$

so the basis for $\mathrm{Ker}(\mathbf{A})$ is $\{(1, -2, 1, 0)\}$.
Running `rref([1 2 3 1; 0 1 2 3; 2 1 0 0]')` shows that

$$\mathbf{A}^T \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\mathrm{Im}(\mathbf{A}) = \mathrm{Sp}((1, 0, 0), (0, 1, 0), (0, 0, 1)) = \mathbb{R}^3$ (which has $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ as a basis).

3. By running the following,

```
B = [4 3 1; -1 -4 3; 1 -2 3];
rref(B)
rref(B')
```

we find that

$$\mathbf{B} \equiv \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B}^T \equiv \begin{pmatrix} 1 & 0 & 6/13 \\ 0 & 1 & 11/13 \\ 0 & 0 & 0 \end{pmatrix}.$$

From this, we deduce

$$\begin{aligned} \mathrm{Ker}(\mathbf{B}) &= \{(-x_3, x_3, x_3) \in \mathbb{R}^3 : x_3 \in \mathbb{R}\} = \mathrm{Sp}((-1, 1, 1)), \\ \mathrm{Im}(\mathbf{B}) &= \mathrm{Sp}((1, 0, \tfrac{6}{13}), (0, 1, \tfrac{11}{13})). \end{aligned}$$

A basis for $\mathrm{Ker}(\mathbf{B})$ is $\{(-1, 1, 1)\}$ and a basis for $\mathrm{Im}(\mathbf{B})$ is $\{(1, 0, \tfrac{6}{13}), (0, 1, \tfrac{11}{13})\}$.

4. We run the following code:

```
C = [3 1 5; 4 -4 -4; -4 -2 -8; 5 1 7];
rref(C)
rref(C')
```

This gives

$$\mathbf{C} \equiv \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{C}^T \equiv \begin{pmatrix} 1 & 0 & -3/2 & 3/2 \\ 0 & 1 & 1/8 & 1/8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This gives

$$\begin{aligned} \operatorname{Ker}(\mathbf{C}) &= \{(-x_3, -2x_3, x_3) \in \mathbb{R}^3 : x_3 \in \mathbb{R}\} = \operatorname{Sp}((-1,-2,1)), \\ \operatorname{Im}(\mathbf{C}) &= \operatorname{Sp}((1,0,-\tfrac{3}{2},\tfrac{3}{2}),(0,1,\tfrac{1}{8},\tfrac{1}{8})) \end{aligned}$$

A basis for $\operatorname{Ker}(\mathbf{C})$ is $\{(-1,-2,1)\}$ and a basis for $\operatorname{Im}(\mathbf{C})$ is $\{(1,0,-\frac{3}{2},\frac{3}{2}),(0,1,\frac{1}{8},\frac{1}{8})\}$.

5. (a) $D\mathbf{h}(x) = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 \\ 2x_1 & 2x_3 & 2x_2 \end{pmatrix}$

   (b) We have

$$\mathbf{h}(\mathbf{p}) = \begin{pmatrix} \frac{1}{4}((\sqrt{3}-1)^2 + (\sqrt{3}+1)^2) \\ \frac{1}{2}(\sqrt{3}-1)(\sqrt{3}+1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

   as required.

   (c) We have

$$D\mathbf{h}(\mathbf{p}) = \begin{pmatrix} 0 & \sqrt{3}-1 & \sqrt{3}+1 \\ 0 & \sqrt{3}+1 & \sqrt{3}-1 \end{pmatrix}.$$

   As $N\mathcal{H}(\mathbf{p})$ equals $\operatorname{Im}(D\mathbf{h}^T(\mathbf{p}))$, a basis is found by row reducing $(D\mathbf{h}^T(\mathbf{p}))^T = D\mathbf{h}(\mathbf{p})$ and reading the rows as vectors. We have

$$\begin{pmatrix} 0 & \sqrt{3}-1 & \sqrt{3}+1 \\ 0 & \sqrt{3}+1 & \sqrt{3}-1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   so $N\mathcal{H}(\mathbf{p}) = \operatorname{Sp}((0,1,0),(0,0,1))$.
   The rows of $D\mathbf{h}(\mathbf{p})$ are independent, so $\mathbf{p}$ is regular.

   (d) From (c), we have

$$D\mathbf{h}(\mathbf{p}) = \begin{pmatrix} 0 & \sqrt{3}-1 & \sqrt{3}+1 \\ 0 & \sqrt{3}+1 & \sqrt{3}-1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   As $T\mathcal{H}(\mathbf{p})$ equals $\operatorname{Ker}(D\mathbf{h}(\mathbf{p}))$, we obtain

$$T\mathcal{H}(\mathbf{p}) = \{(x_1, 0, 0) \in \mathbb{R}^3 : x_1 \in \mathbb{R}\} = \operatorname{Sp}((1,0,0)).$$

6. (a) $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x_1 & 1 & 1 \\ x_2x_3 & x_1x_3 & x_1x_2 \end{pmatrix}$

   (b) At the point $\mathbf{p}_1 = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}^T$, we have

$$D\mathbf{f}(\mathbf{p}_1) = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \end{pmatrix}$$

   The second row is twice the first row, so they are not independent. Hence $\mathbf{p}_1$ is not regular.
   At the point $\mathbf{p}_2 = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}^T$, we have

$$D\mathbf{f}(\mathbf{p}_2) = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}.$$

   As the rows of this matrix are independent, the point $\mathbf{p}_2$ is regular.

(c) The tangent space is

$$T\mathcal{F}_1(\mathbf{p}_1) = \operatorname{Ker}\begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \end{pmatrix}.$$

Applying the Gaussian algorithm we find

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned} T\mathcal{F}_1(\mathbf{p}_1) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = 0\} \\ &= \{(-\tfrac{1}{2}x_2 - \tfrac{1}{2}x_3, x_2, x_3) : x_2, x_3 \in \mathbb{R}\} \\ &= \{(x_2 + x_3, -2x_2, -2x_3) : x_2, x_3 \in \mathbb{R}\} \\ &= \operatorname{Sp}((1, -2, 0), (1, 0, -2)). \end{aligned}$$

The set $\{(1, -2, 0), (1, 0, -2)\}$ is a basis.

(d) A basis for the normal space $N\mathcal{F}_2(\mathbf{p}_2) = \operatorname{Im}\begin{pmatrix} 4 & 2 \\ 1 & 2 \\ 1 & 4 \end{pmatrix}$ is $\{(4, 1, 1), (1, 1, 2)\}$.

(e) Here are two strategies:

- With $Q = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$ and $x_3 = -2x_1 - x_2$, we get

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = -3(5x_1^2 + 4x_1x_2 + 2x_2^2) = -3\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

  The matrix $\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$ is positive definite as it has positive leading principal minors 5 and 6, so with the negative coefficient -3 we find $\mathbf{x}^T\mathbf{Q}\mathbf{x} < 0$ when $\mathbf{x} \neq \mathbf{0}$.
- With $\mathbf{v} = a(1, -2, 0) + b(1, 0, -2)$ we find $\mathbf{v}^T\mathbf{Q}\mathbf{v} = -3(5a^2 + 2ab + 5b^2)$ which is negative definite as the matrix $\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$ has positive leading principal minors 5 and 24.