# MAST30013 – Techniques in Operations Research

**Semester 1, 2021**

**Tutorial 8 Solutions**

(a) First, we write out the nonlinear program to give us our constraint functions:

$$\begin{array}{lrcl} \min & f(x_1,x_2) & = & -x_1x_2 \\ \text{such that} & g_1(x_1,x_2) & = & 4x_1+x_2-8 \ \le\ 0 \\ & g_2(x_1,x_2) & = & -x_1 \ \le\ 0 \\ & g_3(x_1,x_2) & = & -x_2 \ \le\ 0. \end{array}$$

The Lagrangian is

$$L(x_1,x_2,\lambda_1,\lambda_2,\lambda_3) \;=\; -x_1x_2+\lambda_1(4x_1+x_2-8)-\lambda_2x_1-\lambda_3x_2.$$

The KKT conditions are:

KKTa

$$\begin{aligned} -x_2+4\lambda_1-\lambda_2 &= 0 \\ -x_1+\lambda_1-\lambda_3 &= 0 \end{aligned}$$

KKTb

$$\begin{array}{l} 4x_1+x_2-8 \ \le\ 0, \quad \lambda_1 \ \ge\ 0, \quad \lambda_1(4x_1+x_2-8) \ =\ 0 \\ -\,x_1 \ \le\ 0, \quad \lambda_2 \ \ge\ 0, \quad -\lambda_2x_1 \ =\ 0 \\ -\,x_2 \ \le\ 0, \quad \lambda_3 \ \ge\ 0, \quad -\lambda_3x_2 \ =\ 0 \end{array}$$

KKTc There are no equality constraints.

To solve this, we take all possible combinations of the inequality constraints to be active and see which ones solve the KKT system. Because we have 3 inequality constraints, and any can be active or inactive, this gives us 8 possibilities.

(i) If $\lambda_1,\lambda_2,\lambda_3>0$, then by KKTb, we have $x_1=x_2=0$, but KKTb implies that $-8=0$, which is a contradiction.

(ii) If $\lambda_1=0$, $\lambda_2,\lambda_3>0$, then by KKTb, we have $x_1=x_2=0$, but KKTa implies that $\lambda_2=\lambda_3=0\not>0$, which is a contradiction.

(iii) If $\lambda_2=0$, $\lambda_1,\lambda_3>0$, then by KKTb, $4x_1+x_2-8=0$ and $x_2=0$ which imply that $x_1=2$. Now, from KKTa, we have $\lambda_1=0\not>0$ and $\lambda_3=-2\not>0$, two contradictions.

(iv) If $\lambda_3=0$, $\lambda_1,\lambda_2>0$, then by KKTb, $4x_1+x_2-8=0$ and $x_1=0$ which imply that $x_2=8$. Now, from KKTa, we have $\lambda_1=0\not>0$, a contradiction.

(v) If $\lambda_1=\lambda_2=0$, $\lambda_3>0$, then by KKTa, $x_1=-\lambda_3<0$ and the second constraint is violated.

(vi) If $\lambda_1 = \lambda_3 = 0$, $\lambda_2 > 0$, then by KKTa, $x_2 = -\lambda_2 < 0$ and the third constraint is violated.

(vii) If $\lambda_2 = \lambda_3 = 0$, $\lambda_1 > 0$, then by KKTb, $4x_1 + x_2 - 8 = 0$. From KKTa we have $x_2 = 4\lambda_1$ and $x_1 = \lambda_1$, which imply, using KKTb, that $\lambda_1 = 1$. Then, $x_1 = 1$ and $x_2 = 4$. The KKT point is $(x_1^*, x_2^*) = (1, 4)$ with corresponding Lagrange multipliers $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 0, 0)$.

(viii) If $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then KKTa gives $x_1 = 0$ and $x_2 = 0$. The KKT point is $(x_1^*, x_2^*) = (0, 0)$ with corresponding Lagrange multipliers $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0, 0, 0)$.

Therefore, there are two KKT points, given in (vii) and (viii) above.

(b) The constraint qualifications hold at both KKT points since all three constraints are affine.

(c) We now check the second-order condition at the two KKT points.

(vii) $\boldsymbol{x}^* = (x_1^*, x_2^*) = (1, 4)$, $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 0, 0)$. The active constraint is $4x_1 + x_2 \leq 8$. The critical cone is

$$\begin{aligned} \mathcal{C}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) &= \left\{\boldsymbol{d} \in \mathbb{R}^2 : \boldsymbol{\nabla} g_1(1,4)^T \boldsymbol{d} = 0\right\} \\ &= \left\{\boldsymbol{d} \in \mathbb{R}^2 : \begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0\right\} \\ &= \left\{\boldsymbol{d} \in \mathbb{R}^2 : 4d_1 + d_2 = 0\right\} \\ &= \left\{(d_1, d_2) \in \mathbb{R}^2 : d_2 = -4d_1\right\}. \end{aligned}$$

The Hessian of the Langrangian is

$$\boldsymbol{\nabla}^2_{\boldsymbol{xx}}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Now, for $\boldsymbol{d} \in \mathcal{C}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$,

$$\begin{aligned} \begin{pmatrix} d_1 & -4d_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ -4d_1 \end{pmatrix} &= \begin{pmatrix} d_1 & -4d_1 \end{pmatrix} \begin{pmatrix} 4d_1 \\ -d_1 \end{pmatrix} \\ &= 8d_1^2 \\ &> 0. \end{aligned}$$

Thus, $\boldsymbol{\nabla}^2_{\boldsymbol{xx}}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is positive definite on the critical cone. Therefore, $\boldsymbol{x}^* = (1, 4)$ is a local minimum. However, since the constraint set is closed and bounded, $\boldsymbol{x}^* = (1, 4)$ is a global minimum.

(viii) $\boldsymbol{x}^* = (x_1^*, x_2^*) = (0, 0)$, $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0, 0, 0)$. The active constraints are $-x_1 \leq 0$ and $-x_2 \leq 0$ (Note that both corresponding Lagrange multipliers are zero). The critical cone is

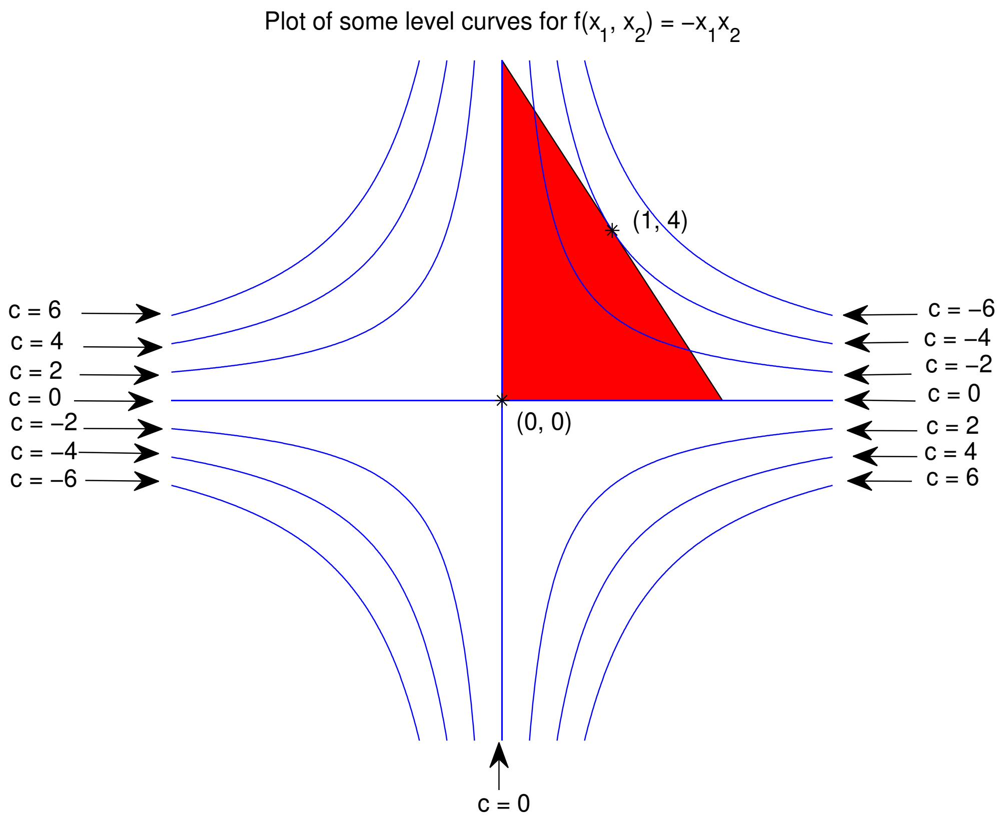
$$\begin{aligned}
\mathcal{C}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) &= \left\{\boldsymbol{d} \in \mathbb{R}^2 : \boldsymbol{\nabla} g_2(0,0)^T \boldsymbol{d} \leq 0, \boldsymbol{\nabla} g_3(0,0)^T \boldsymbol{d} \leq 0, \right\} \\
&= \left\{\boldsymbol{d} \in \mathbb{R}^2 : \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \leq 0, \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \leq 0 \right\} \\
&= \left\{\boldsymbol{d} \in \mathbb{R}^2 : -d_1 \leq 0, -d_2 \leq 0 \right\} \\
&= \left\{(d_1, d_2) \in \mathbb{R}^2 : d_1 \geq 0, d_2 \geq 0 \right\}.
\end{aligned}$$

Now, for $\boldsymbol{d} \in \mathcal{C}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$,

$$\begin{aligned}
\begin{pmatrix} d_1 & d_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= \begin{pmatrix} d_1 & d_2 \end{pmatrix} \begin{pmatrix} -d_2 \\ d_1 \end{pmatrix} \\
&= -2 d_1 d_2 \\
&\leq 0.
\end{aligned}$$

Thus, $\boldsymbol{\nabla}^2_{\boldsymbol{xx}}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is negative semi-definite on the critical cone. Therefore, $\boldsymbol{x}^* = (0,0)$ is a not a local minimum. In fact it is a local maximum but there are points $\boldsymbol{x}'$ in the feasible region such that $f(\boldsymbol{x}') = f(\boldsymbol{x}^*)$. These points are of the form $(x_1, 0)$ with $0 < x_1 \leq 2$ and $(0, x_2)$ with $0 < x_2 \leq 8$. However, since the constraint set is closed and bounded, these points are also global maxima.

(d) Below is a diagram of the feasible region with some level curves for $f(x_1, x_2) = -x_1 x_2$. The minimum is achieved at the KKT point $(1, 4)$. The maximum is achieved on the abovementioned set of points.




It is interesting to note, that at the KKT point $(0, 0)$ both the second and third constraints are active but the corresponding Lagrange multipliers are both zero. If we consider, for example, the second constraint $-x_1 \leq 0$ and decide to increase the right hand side of the inequality by a small amount $\triangle > 0$, we have an enlarged feasible region, $-x_1 \leq \triangle$. Now, the only direction we can go in to achieve a (possibly) lower minimum value for $f(x_1, x_2)$ while maintaining feasibility, is along the negative $x$-axis. Here the function value is 0. Thus, the rate of increase is 0, precisely the Lagrange multiplier corresponding to the second constraint. A similar observation can be made when considering the third constraint $-x_2 \leq 0$..