Decision Making
Part 6: Decision Making Under Strict Uncertainty

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Topics in this part

- Introduction: state-of-the-world model, decision table, certainty, risk, and strict uncertainty

- Four decision criteria: Wald's maximin, Hurwicz's maximax, Savage's minimax regret, Laplace

- An axiomatic approach

Reference:
S. French, Decision theory: an introduction to the mathematics of rationality, Ellis Horwood, 1986, Chapter 2

Introduction

State-of-the-world model

Suppose that for a decision making problem the decision maker has a finite number of possible actions:

$$a_1, a_2, \ldots, a_m.$$

The consequence of any action is determined not only by the action itself but also by a number of external factors. These external factors are beyond the control of the decision maker and are unknown to the decision maker at the time of the decision. We call the complete description of these external factors the state of the world or simply the state.

The state that actually holds will be called the true state.

Assume that there are only a finite number of mutually exclusive states. We label them by

$$\theta_1, \theta_2, \ldots, \theta_n.$$

Decision table

Let x_{ij} denote the consequence of taking action a_i when θ_j is the true state.

Note that x_{ij} may not be a numerical number.

The decision problem can be represented by the following table.

Consequences		States			
		θ_1	θ_2	$\dots$	θ_n
	a_1	x_{11}	x_{12}	$\dots$	x_{1n}
	a_2	x_{21}	x_{22}	$\dots$	x_{2n}
Actions	$\vdots$	$\vdots$			$\vdots$
	a_m	x_{m1}	x_{m2}	$\dots$	x_{mn}

We now assume that the value of x_{ij} can be measured by a real number $v_{ij} = v(x_{ij})$. (It is understood that if the decision maker prefers x_{ij} to x_{kl} then $v_{ij} > v_{kl}$.)

Then we have the following decision table with consequences replaced by their values.

$$
\begin{array}{ll|cccc}
 & & & \text{States} & & \\
\text{Values} & & \theta_1 & \theta_2 & \dots & \theta_n \\
\hline
 & a_1 & v_{11} & v_{12} & \dots & v_{1n} \\
 & a_2 & v_{21} & v_{22} & \dots & v_{2n} \\
\text{Actions} & \vdots & \vdots & & & \vdots \\
 & a_m & v_{m1} & v_{m2} & \dots & v_{mn}
\end{array}
$$

Example 1. (state of the world model, Winston pp737)

A newspaper vendor must determine how many newspapers to order each day. He pays 20c for each paper and sells for 25c each. Unsold papers are worthless at the end of the day. The vendor knows that he can sell between 6 to 10 papers, with each possibility equally likely. This problem fits into the "state-of-the-world" model.

Suppose that he orders i newspapers and j papers are demanded, $i, j \in \{6, \ldots, 10\}$. Then

$$v_{ij} = \begin{cases} 25i - 20i = 5i & \text{if } i \leq j, \\ 25j - 20i & \text{if } i > j. \end{cases}$$

Example. (cont.)

The corresponding decision table is

		States Papers demanded				
		6 (θ_1)	7 (θ_2)	8 (θ_3)	9 (θ_4)	10 (θ_5)
Actions	6 (a_1)	30c	30c	30c	30c	30c
	7 (a_2)	10c	35c	35c	35c	35c
Papers	8 (a_3)	-10c	15c	40c	40c	40c
ordered	9 (a_4)	-30c	-5c	20c	45c	45c
	10 (a_5)	-50c	-25c	0c	25c	50c

Certainty, risk, and strict uncertainty

Decision under certainty: The decision maker knows the true state before he makes his choice. In other words, he can predict the consequences of his action with certainty. Equivalently, there is exactly one column in the decision table.

Decision with risk: The decision maker does not know the true state for certain before he makes his decision. However, he can quantify his uncertainty by a probability distribution on the set of states. That is, he knows the probabilities $\mathbf{Pr}(\theta_1), \mathbf{Pr}(\theta_2), \ldots, \mathbf{Pr}(\theta_n)$.

Decision under strict uncertainty: The decision maker knows nothing at all about the true state at the time of making decision, and he cannot quantify his uncertainty in any way.

In this part we will discuss decision making under strict uncertainty.

Four decision criteria

Wald's maximin criterion

This is a conservative and pessimistic approach. It bears similarity with the approach used for 2-person zero-sum games.

Compute the security level of action a_i:

$$s_i = \min_{1 \leq j \leq n} v_{ij}, \ 1 \leq i \leq m.$$

This is the worst consequence if the decision maker chooses a_i.

Wald suggested that the decision maker should choose an action with maximum security level, i.e. an action a_k such that

$$s_k = \max_{1 \leq i \leq m} s_i = \max_{1 \leq i \leq m} \min_{1 \leq j \leq n} v_{ij}.$$

Example 2. (Wald: maximin criterion)

Determine the action of the newspaper vendor if he follows Wald's maximin criterion.

		States					
		Papers demanded					"security
		6 (θ_1)	7 (θ_2)	8 (θ_3)	9 (θ_4)	10 (θ_5)	level"
Actions	6 (a_1)	30c	30c	30c	30c	30c	
	7 (a_2)	10c	35c	35c	35c	35c	
Papers	8 (a_3)	-10c	15c	40c	40c	40c	
ordered	9 (a_4)	-30c	-5c	20c	45c	45c	
	10 (a_5)	-50c	-25c	0c	25c	50c	

Hurwicz's maximax criterion

This is an optimistic approach.

Compute the optimism level of action a_i:

$$o_i = \max_{1 \le j \le n} v_{ij}, \;\; 1 \le i \le m.$$

This is the best consequence if the decision maker chooses a_i.

Hurwicz suggested that the decision maker should choose an action with maximum optimism level, i.e. an action a_k such that

$$o_k = \max_{1 \le i \le m} o_i = \max_{1 \le i \le m} \max_{1 \le j \le n} v_{ij}.$$

This maximax value is the largest entry of the decision table.

Example 3. (Hurwicz: maximax criterion)

Determine the action of the newspaper vendor if he follows Hurwicz's maximax criterion.

		States					
		Papers demanded					"optimism
		6 (θ_1)	7 (θ_2)	8 (θ_3)	9 (θ_4)	10 (θ_5)	level"
Actions	6 (a_1)	30c	30c	30c	30c	30c	
	7 (a_2)	10c	35c	35c	35c	35c	
Papers	8 (a_3)	-10c	15c	40c	40c	40c	
ordered	9 (a_4)	-30c	-5c	20c	45c	45c	
	10 (a_5)	-50c	-25c	0c	25c	50c	

A combination of maximin and maximax

Consider the following weighted average of security and optimism levels:

$$\alpha s_i + (1-\alpha) o_i, \;\; 1 \le i \le m$$

where $0 \le \alpha \le 1$ is specified by the decision maker.

Hurwicz also suggested that the decision maker should choose an action such that $\alpha s_i + (1-\alpha) o_i$ is as large as possible. That is, choose an action a_k such that

$$\alpha s_k + (1-\alpha) o_k = \max_{1 \le i \le m} [\alpha s_i + (1-\alpha) o_i].$$

We call this Hurwicz's α-criterion.

Savage's minimax regret criterion

Savage defined the regret of a consequence v_{ij} as

$$r_{ij} = \max_{1 \leq l \leq m} v_{lj} - v_{ij}.$$

This is the difference between the maximum value given that θ_j is the true state, and the value resulting from a_i under θ_j.

Define the regret table as

$$\begin{array}{ll|cccc} & & & \text{States} & & \\ \text{Regrets} & & \theta_1 & \theta_2 & \dots & \theta_n \\ \hline & a_1 & r_{11} & r_{12} & \dots & r_{1n} \\ & a_2 & r_{21} & r_{22} & \dots & r_{2n} \\ \text{Actions} & \vdots & \vdots & & & \vdots \\ & a_m & r_{m1} & r_{m2} & \dots & r_{mn} \end{array}$$

Savage suggested that the decision maker should choose an action according to the minimax rule applied to the regret matrix.

Define

$$\rho_i = \max_{1 \leq j \leq n} r_{ij}, \;\; 1 \leq i \leq n.$$

This is the worst regret resulting from choosing a_i.

Savage's minimax regret criterion chooses an action with minimum ρ_i, that is, an action a_k such that

$$\rho_k = \min_{1 \leq i \leq m} \rho_i = \min_{1 \leq i \leq m} \; \max_{1 \leq j \leq n} r_{ij}.$$

Example 4. (Savage: minimax regret criterion)

Determine the action of the newspaper vendor if he follows Savage's minimax regret criterion.

		States				
		Papers demanded				
		6 (θ_1)	7 (θ_2)	8 (θ_3)	9 (θ_4)	10 (θ_5)
Actions	6 (a_1)	30c	30c	30c	30c	30c
	7 (a_2)	10c	35c	35c	35c	35c
Papers	8 (a_3)	-10c	15c	40c	40c	40c
ordered	9 (a_4)	-30c	-5c	20c	45c	45c
	10 (a_5)	-50c	-25c	0c	25c	50c

Laplace's criterion

Laplace argued that "knowing nothing about the true state" is equivalent to "all states having equal probability".

Define

$$\bar{v}_i = \sum_{j=1}^{n} \frac{1}{n} \cdot v_{ij}, \;\; 1 \leq i \leq m.$$

Since all states are assumed to happen equally probably, this can be interpreted as the expected value of a_i.

If you agree with this, you may choose an action with maximum expected value, i.e. an action a_k such that

$$\bar{v}_k = \max_{1 \leq i \leq m} \bar{v}_i = \max_{1 \leq i \leq m} \sum_{j=1}^{n} \frac{1}{n} \cdot v_{ij}.$$

Example 5. (Laplace's criterion)

Determine the action of the newspaper vendor if he follows Laplace's criterion.

		States					
		Papers demanded					"expected
		6 (θ_1)	7 (θ_2)	8 (θ_3)	9 (θ_4)	10 (θ_5)	reward"
Actions	6 (a_1)	30c	30c	30c	30c	30c	
	7 (a_2)	10c	35c	35c	35c	35c	
Papers	8 (a_3)	-10c	15c	40c	40c	40c	
ordered	9 (a_4)	-30c	-5c	20c	45c	45c	
	10 (a_5)	-50c	-25c	0c	25c	50c	
	Prob.	1/5	1/5	1/5	1/5	1/5	

Example 6. (Milnor 1954)

Consider the following decision table

	θ_1	θ_2	θ_3	θ_4
a_1	2	2	0	1
a_2	1	1	1	1
a_3	0	4	0	0
a_4	1	3	0	0

Develop a course of action based on the above described criteria.

Example (cont.)

An axiomatic approach

Some properties of decision rules

Axiom 1. (Complete ranking) A decision rule should provide a complete ranking (i.e. linear order) on the set of possible actions.

This means that

- any two actions are "comparable" (for any two actions a_i, a_j, either a_i is no worse than a_j, or a_j is no worse than a_i, or both);
- if a_i is no worse than a_j, and a_j is no worse than a_k, then a_i is no worse than a_k.
- if a_i is no worse than a_j, and a_j is no worse than a_i, then $a_i = a_j$.

A consequence of this axiom is that a decision rule should be able to assign a real-valued index V_i to every a_i in such a way that

$$\text{“}a_i \text{ is better than } a_j\text{”} \iff V_i > V_j.$$

Axiom 2. (Independence of labelling) A decision rule's choice should be independent of the labels of the actions and also independent of the labels of the states.

If π is a permutation of actions and τ is a permutation of states, let (v'_{ij}) be the decision table whose $\pi(i)$-th row is the i-th row of (v_{ij}) and whose $\tau(j)$-th column is the j-th column of (v_{ij}).

The axiom above says that a decision rule should assign the values V, V' respectively to the two tables in such a way that, for any $1 \leq i, k \leq m$,

$$V_i > V_k \Longleftrightarrow V'_{\pi(i)} > V'_{\pi(k)}.$$

Axiom 3. (Independence of value scale) A decision rule's choice should be invariant under linear transformations.

If

$$v'_{ij} = av_{ij} + b,\ \ 1 \leq i \leq m, 1 \leq j \leq n$$

for some $a > 0$ and b, then a decision rule should assign the values V, V' to $(v_{ij}), (v'_{ij})$ respectively such that, for all $1 \leq i, k \leq m$,

$$V_i > V_k \Longleftrightarrow V'_i > V'_k.$$

Axiom 4. (Strong dominance) Suppose there are two actions a_i and a_k such that

$$v_{ij} > v_{kj} \quad \text{for all } j.$$

Then a decision rule should assign values to the actions such that

$$V_i > V_k.$$

Axiom 5. (Independence of irrelevant alternatives) Let (v_{ij}) be a decision table with actions a_i and states θ_j. Let a second table (v'_{ij}) be constructed from (v_{ij}) by adding an extra row to (v_{ij}).

Then a decision rule should assign values V, V' respectively to the actions in the two tables such that, for all $1 \leq i, k \leq m$,

$$V_i > V_k \Longleftrightarrow V'_i > V'_k.$$

Axiom 6. (Independence of addition of a constant to a column) Let (v'_{ij}) be constructed from (v_{ij}) by adding a constant to every entry in one column and keeping all other entries unchanged.

Then a decision rule should assign values V, V' respectively to the actions in $(v_{ij}), (v'_{ij})$ such that, for all $1 \leq i, k \leq m$,

$$V_i > V_k \iff V'_i > V'_k.$$

Axiom 7. (Independence of row permutation) Suppose there exist two actions a_i and a_k and a permutation τ of $\{1, 2, \ldots, n\}$ such that

$$v_{ij} = v_{k\tau(j)}.$$

Then a decision rule should assign values V to the actions such that

$$V_i = V_k.$$

Rationale underlying this axiom: Since the true state is strictly uncertain, if $v_{ij} = v_{k\tau(j)}$ for some permutation τ, then a_i and a_k should be viewed as indifferent and hence a decision rule should assign the same value to them.

Axiom 8. (Independence of column duplication) Let (v'_{ij}) be constructed from (v_{ij}) by duplicating the last column of (v_{ij}).

Then a decision rule should assign values V, V' respectively to the actions in $(v_{ij}), (v'_{ij})$ such that, for all $1 \leq i, k \leq m$,

$$V_i > V_k \Longleftrightarrow V'_i > V'_k.$$

This axiom allows one to duplicate the last column more than once by repeated application.

Theorem 1. The decision rules of Wald, Hurwicz, Savage, and Laplace satisfy certain axioms above as shown in the following table.

	Four decision rules			
	Wald	Hurwicz	Savage	Laplace
1. Complete ranking	×	×	×	×
2. Independence of labeling	×	×	×	×
3. Independence of value scale	×	×	×	×
4. Strong dominance	×	×	×	×
5. Independence of irrelevant alternatives	×	×		×
6. Ind. add. const. to a column			×	×
7. Independence of row permutation	×	×		×
8. Independence of column duplication	×	×	×	

Proof. See [S. French, Decision theory: an introduction to the mathematics of rationality, Ellis Horwood, 1986, pp.46–49]

Theorem 2. Suppose a decision rule satisfies the Axioms of “complete ranking”, “strong dominance”, “independence of irrelevant alternatives”, “independence of addition of a constant to a column”, and “independence of row permutation”. Then the rule assigns index V_i to each action a_i such that

$$V_i \geq V_k \iff \sum_{j=1}^{n} \frac{1}{n} \cdot v_{ij} \geq \sum_{j=1}^{n} \frac{1}{n} \cdot v_{kj}.$$

In other words, the rule satisfies the Laplace criterion.

Proof. See [S. French, Decision theory: an introduction to the mathematics of rationality, Ellis Horwood, 1986, pp.49–52]

Since the Laplace criterion does not satisfy the axiom of “independence of column duplication”, the theorem above implies:

Corollary 1. No decision rule can satisfy all of the above-mentioned eight Axioms.