Semester 2 Assessment, 2014

Department of Mathematics and Statistics

MAST30001 Stochastic Modelling

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam.

This paper consists of 9 pages (including this page)

Authorised materials:

- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You may remove this question paper at the conclusion of the examination.
- This paper has ?? questions. Attempt as many questions, or parts of questions, as you can. The number of marks allocated to each question is shown in the brackets after the question statement. The total number of marks available for this examination is ??. Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination.

1. Let X_n be a Markov chain with transition matrix

$$P = \begin{pmatrix} 1/6 & 1/3 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

Define

$$Y_n = \begin{cases} 1, & X_n \in \{1,2\}, \\ 2, & X_n = 3, \\ 3, & X_n = 4. \end{cases}$$

Is $(Y_n)_{n\geq 0}$ a Markov chain? If so, find its transition matrix and if not, carefully explain why.

[4 marks]

Ans. Y_n is not a Markov chain since

$$P(Y_2 = 2|Y_1 = 1, Y_0 = 2) = \frac{p_{3,1}p_{1,3} + p_{3,2}p_{2,3}}{p_{3,1} + p_{3,2}} = \frac{(1/4)(1/2) + (1/4)(1/4)}{1/4 + 1/4} = 3/8,$$

$$P(Y_2 = 2|Y_1 = 1, Y_0 = 3) = \frac{p_{4,1}p_{1,3}}{p_{4,1}} = \frac{(1/2)(1/2)}{1/2} = 1/2,$$

and these would have to be equal (to $P(Y_2 = 2|Y_1 = 1)$) if Y_n was Markov chain.

2. A Markov chain has transition matrix

$$\begin{pmatrix} 1/3 & 0 & 0 & 2/3 & 0 \\ 1/7 & 2/7 & 2/7 & 2/7 & 0 \\ 0 & 3/7 & 4/7 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 4/9 & 0 & 0 & 5/9 \end{pmatrix}.$$

(a) What is $P(X_4 = 1, X_2 = 2|X_0 = 5)$?

(b) Analyze the state space of the Markov chain: communicating classes, reducibility, periodicity and recurrence.

(c) Describe the long run behavior of the chain.

[8 marks]

Ans.

(a) $P(X_4 = 1, X_2 = 2|X_0 = 5) = (P^2)_{5,2}(P^2)_{2,1} = .08648$.

(b) There are three communicating classes: $\{1,4\}, \{2,3\}, \{5\}$ and only $\{1,4\}$ is essential.

- Reducibility: the chain is reducible since it has more than one communicating class.
- Periodicity: the essential communicating class $\{1,4\}$ has period one because of the "loop" at state 1.
- Recurrence: the essential communicating class is positive recurrent since it's finite and the non-essential communicating classes are transient.

(c) The chain eventually ends up in the essential communicating class $\{1,4\}$ and stays there forever. Since the class is positive recurrent, it has a long run stationary distribution π, the unique solution to $\pi P = \pi$; in this case

$$\pi = (3/7, 4/7).$$

3. Four fair coins are lying on a table. We perform the following procedure: a coin is selected uniformly at random and then tossed and placed back on the table. Let X_n be the number of heads showing among the four coins after performing this procedure n times.

   (a) Model X_n as a Markov chain and write down its transition probabilities.

   (b) If initially there are 2 heads among the four coins lying on the table and then we perform this procedure indefinitely, what is the chance that all four coins show tails before they all show heads?

   (c) If initially all of the coins show tails and then we perform this procedure indefinitely, what is the chance that all four coins show tails again before they all show heads?

   (d) Now assume the coins are biased with chance of heads equal to $p \in (0, 1)$. If initially there are 2 heads among the four coins lying on the table and then we perform this procedure indefinitely, what is the chance that all four coins show tails before they all show heads? (You don't need to simplify your answer past an expression entirely in terms of p.

[8 marks]

Ans.

(a) If there are currently i heads, then in order to increase the number of heads, we need to choose a coin that's currently showing tails and then toss it to heads: this happens with chance $p(4 - i)/4$, $i = 0, 1, 2, 3, 4$ (setting $p = 1/2$). Similarly the chance of decreasing the number of heads by one is $(1-p)i/4$, and otherwise we stay at i. To summarize, for $0 \le i \le 4$:

$$p_{i,i-1} = \frac{i}{4}(1-p), \qquad p_{i,i} = \frac{i}{4}p + \frac{4-i}{4}(1-p), \qquad p_{i,i+1} = \frac{4-i}{4}p.$$

(b) By symmetry in heads and tails, the chance of reaching all heads before all tails is $1/2$.

(c) Let f_j be the chance of reaching all tails before all heads starting from j heads; we want f_0 and part (b) says $f_2 = 1/2$. Then first step analysis implies

$$f_0 = p_{0,1}f_1 + p_{0,0}, \qquad f_1 = p_{1,0} + p_{1,2}f_2 + p_{1,1}f_1,$$

and solving implies $f_1 = 5/8$ and $f_0 = 13/16$.

(d) Use first step analysis to find that we need to solve

$$\begin{aligned}
f_0 &= p_{0,1}f_1 + p_{0,0}, \\
f_1 &= p_{1,0} + p_{1,2}f_2 + p_{1,1}f_1, \\
f_2 &= p_{2,1}f_1 + p_{2,2}f_2 + p_{2,3}f_3, \\
f_3 &= p_{3,2}f_2 + p_{3,3}f_3, \\
f_4 &= p_{4,3}f_3.
\end{aligned}$$

We just solve the middle three equations and this will give f_2.

$$\begin{aligned}
f_1 &= \frac{1-p}{4} + \frac{3p}{4}f_2 + \left(\frac{p}{4} + \frac{3(1-p)}{4}\right) f_1, \\
f_2 &= \frac{1-p}{2}f_1 + \frac{1}{2}f_2 + \frac{p}{2}f_3, \\
f_3 &= \frac{3(1-p)}{4}f_2 + \left(\frac{3p}{4} + \frac{1-p}{4}\right) f_3.
\end{aligned}$$

Some tedious but straightforward calculating give

$$f_2 = \frac{(1-p)^2}{1+2p}\left(1 - \frac{12p(1-p)}{(1+2p)(3-2p)}\right)^{-1},$$

and as a check we see if $p = 1/2$, $f_2 = 1/2$.

4. Let $p \in (0, 1)$. A Markov chain on $\{0, 1, 2, \ldots\}$ has transition probabilities

$$p_{i,i+1} = p, \qquad p_{i,j} = (1-p)/(i+1) \ \text{ for } \ j = 0, \ldots, i.$$

(a) Determine the values of p for which a stationary distribution exists and for these values find the stationary distribution.

(b) Determine the values of p where the chain is transient, null recurrent, positive recurrent.

[6 marks]

Ans.

The chain is irreducible and aperiodic, so by the ergodic theorem for Markov chains, there is a probability vector π that solves $\pi P = \pi$ if and only if P is positive recurrent. $\pi P = \pi$ holds if and only if

$$\pi_0 = (1-p)\pi_0 + (1-p)\sum_{j\geq 1}\frac{\pi_j}{j+1}, \qquad \pi_i = p\pi_{i-1} + (1-p)\sum_{j\geq i}\frac{\pi_j}{j+1}, \qquad i \geq 1.$$

Subtracting the equation for π_{i+1} from that of π_i, we have that for $i \geq 0$ (setting $\pi_{-1} = 0$),

$$\pi_{i+1} - \pi_i = p\pi_i - p\pi_{i-1} - \frac{(1-p)}{i+1}\pi_i,$$

or better

$$\pi_{i+1} = \pi_i(1 + p - \frac{1-p}{i+1}) - p\pi_{i-1}.$$

Building these up from the bottom we find $\pi_1 = 2p\pi_0$, $\pi_2 = 3p^2\pi_0$, $\pi_3 = 4p^3\pi_0$ and we guess the solution is

$$\pi_i = (i+1)p^i\pi_0,$$

which is easily check to satisfy the equations above. We only need to determine for which p this can be used to define a probability distribution, that is when

$$\pi_0 = \left(\sum_{i\geq 0}(i+1)p^i\right)^{-1} > 0.$$

But this sum is always finite and we see that π is the negative binomial distribution with parameters 2 and $1-p$. Thus the chain is positive recurrent for all values of $p < 1$.

5. A shop has two machines that operate independently and occasionally break. At the beginning of the day, each machine is in perfect working order and the times until failure of each machine are independent and exponentially distributed with rate μ. When a machine breaks, service is immediately started on it and it is repaired in an exponential rate λ amount of time; service times are independent of each other and failure times.

   (a) Model the number of working machines as a continuous time Markov chain and write down its generator.

   (b) Argue the Markov chain is ergodic and find its steady state distribution.

   (c) Assume $\lambda = 1$. What is the maximum rate μ so that the chance at least one machine is working is at least 95%?

   (d) Assume now that $\lambda = \mu$. Derive the transition probability $p_{0,1}(t)$ and verify that the limiting probability as $t \to \infty$ matches that of part (b). [9 marks]

Ans.

(a) When both machines are broken (the chain is in state 0) the chain moves to state 1 at rate 2λ. When one machine is broken, the chain moves to state 2 at rate λ and state 0 at rate μ. When both machines are working the chain moves to state 1 at rate 2μ. The generator is

$$A = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}.$$

(b) The Markov chain is irreducible on a finite state space and so is ergodic. Its long run steady state distribution is the unique solution to $\pi A = 0$ which in this case is given by

$$\pi = (\lambda+\mu)^{-2}(\mu^2, 2\mu\lambda, \lambda^2).$$

(c) If $\lambda = 1$ then from (b), the steady state distribution is

$$\pi = (1+\mu)^{-2}(\mu^2, 2\mu 1, 1),$$

so we want the maximum μ so that the first component is $< .05$ which is given by

$$\mu < \frac{1}{1-\sqrt{.05}} - 1 \approx 0.288.$$

(d) The forward equation says

$$p'_{0,1}(t) = 2\lambda p_{0,0}(t) - 2\lambda p_{0,1}(t) + 2\lambda p_{0,2}(t).$$

But $p_{0,2}(t) = 1 - p_{0,0}(t) - p_{0,1}(t)$ and so we find

$$p'_{0,1}(t) = -4\lambda p_{0,1}(t) + 2\lambda$$

and solving this (using an integrating factor or otherwise) with the boundary condition $p_{0,1}(t) = 0$ we have

$$p_{0,1}(t) = \frac{1}{2}(1 - e^{-4\lambda t}).$$

Another way to understand this result is that the chain moves from state 1 to states 0 or 2 at rate 2λ and then returns in an exponential rate 2λ time regardless of where the chain jumped. So we can view this chain as a two state Markov chain with generator

$$\begin{pmatrix} -2\lambda & 2\lambda \\ 2\lambda & -2\lambda \end{pmatrix},$$

and we've shown in tutorial that the chance you're at a different state than you started from after t time units in such a chain is given by the formula for $p_{0,1}(t)$ above.

6. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

   (a) If the price of a stock at time t hours into a trading day is given by

   $$S_t = 25\exp\{2B_t - t\},$$

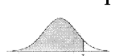
   (i) What is the chance that the stock price is higher than it started 4 hours into the day given that it was equal to $25e^2$ dollars two hours into the day?

   (ii) Given that the stock's price four hours into the day was 25 dollars, what is the chance the price was less than $25e^2$ dollars 2 hours into the day?

   (b) Let $0 < t_1 < t_2$. Find the expected value of $M = \max\{B_s : t_1 \leq s \leq t_2\}$. (You may want to use the fact that $\max\{B_s : 0 \leq s \leq t\} \stackrel{d}{=} |B_t|$.)

[7 marks]

Tables of the Normal Distribution



Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015

Ans.

(a) For (i), we want

$$P(S_4 > 25|S_2 = 25e^2) = P(B_4 > 2|B_2 = 2) = 1/2,$$

since conditional on B_s, $(B_{t+s} - B_s) \stackrel{d}{=} B_t$. For (ii), we want

$$P(S_2 > 25|S_4 = 25e^2) = P(B_2 > 1|B_4 = 3).$$

Since $(B_2/\sqrt{2}, B_4/2)$ are standard bivariate normal with correlation $1/\sqrt{2}$, we can write

$$B_2 = B_4/2 + Z,$$

where Z is standard normal independent of B_4 and so

$$P(B_2 < 2|B_4 = 0) = P(Z < 1) = 0.8413$$

according to the normal table.

(b) Since $(B_{t_1+s} - B_{t_1})_{s\geq 0}$ is a standard Brownian motion independent of $(B_s)_{s<t_1}$ (by the independent increments property), we can write

$$M \overset{d}{=} \max\{B_s : 0 \leq s \leq t_2 - t_1\} + B_{t_1},$$

and so

$$E[M] = E[\max\{B_s : 0 \leq s \leq t_2 - t_1\}] + E[B_{t_1}].$$

The second expectation on the right hand side is zero and using the hint in the problem, if Z is a standard normal variable then

$$E[\max\{B_s : 0 \leq s \leq t_2 - t_1\}] = \sqrt{t_2 - t_1}E|Z| = \sqrt{\frac{2(t_2 - t_1)}{\pi}}.$$

7. A boutique dress shop has two entrances, one on A street and the other on B street. Customers enter the shop from A street according to a rate one Poisson process and from B street according to a rate two Poisson process independent of the flow of customers from A street (units are in hours). Customers who enter the shop buy something with probability $1/3$, independent of other customers' behavior.

   (a) What is the chance that exactly 3 people enter the shop from A street between noon and 1pm?
   (b) What is the distribution of time until the first person enters the shop after it opens?
   (c) What is the chance that exactly 4 people enter the shop (total from A and B street) between noon and 1pm?
   (d) Given 2 people entered the shop between noon and 1pm, what is the chance that exactly one person entered the shop from A street between noon and 12:30pm?
   (e) What is the chance that exactly 3 people buy something between noon and 1pm?
   (f) What is the chance that between 1pm and 2pm, exactly 2 people who entered from A street buy something and exactly 1 person who entered from B street buys something?

   Customers can purchase a dress in advance and then drop in at the shop to have it fitted. Assume that these customers arrive according to a rate one Poisson process and that there are two tailors to fit dresses. The service times for each tailor are independent and both distributed as exponential with rate $3/4$.

   (g) Model the number of customers in the shop who purchased a dress in advance to be fitted as a queuing system and determine its steady state distribution.
   (h) What is the average amount of time a customer who purchased a dress in advance to be fitted spends in the store?
   (i) What is the average amount of time a customer who purchased a dress in advance waits to be served?

   [12 marks]

Ans.

A_t, B_t are the Poisson processes governing the customers entering from A and B streets, rates 1 and 2, with $t = 0$ corresponding to noon.

(a) $P(A_1 = 3) = e^{-1}/3!$.

(b) Exponential rate three.

(c) By superposition, $A_t + B_t$ is a Poisson process rate 3 and so $P(A_1 + B_1 = 4) = 3^4 e^{-3}/4!$.

(d) By the thinning theorem, each of the two arrivals has chance $1/3$ of being from street A and the chance that an arrival in $(0, 1)$ is in $(0, 1/2)$ is $1/2$, since the time of an arrival in $(0, 1)$ is uniformly distributed in that interval. Thus the number of arrivals from A in $(0, 1/2)$ is binomial with parameters 2 and $1/6$ and so the chance this is one is

$$2(1/6)(5/6).$$

(e) The total number of customers is a Poisson process rate 3 and the number that buy something is thinned process with rate 1. The chance that there are three arrivals from this process in $(0, 1)$ is $e^{-1}/3!$.

(f) The number that buy something from A, respectively B, street are independent Poisson processes rates $1/3$ and $2/3$, so the probability is

$$\frac{(1/3)^2 e^{-1/3}}{2!}(2/3)e^{-2/3}.$$

(g) This is an $M/M/2$ queue with arrival rate $\lambda = 1$ and $\mu = 3/4$. From lecture, since $\lambda < 2\mu$, the steady state distribution exists and equals

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{1}{2^{k-1}},$$

for $k \geq 1$, where $\pi_0 = (2\mu - \lambda)/(2\mu + \lambda)$.

(h) Little's law says this is equal to the expected number of customers in the stationary distribution divided by λ, which is equal to, using formulas from lecture,

$$L = \frac{1}{\mu}\left(1 + \frac{\lambda^2}{4\mu^2 - \lambda^2}\right).$$

(i) The total time in the store is the waiting time plus the service time: $L = W + S$ and so on average, $W = L - 1/\mu$, where L is given in part (f).

8. A rat has a cage with a tunnel loop that he runs through periodically. The rat stays in the cage for a random amount of time and then runs through the loop, and then starts the process anew. Assume that both the amount of time it takes the rat to run through the loop and the amount of time the rat stays in the cage between times when it runs through the loop are independent and (continuously) uniformly distributed between zero and five minutes.

   (a) Model the number N_t of times the rat returns to the cage from the loop up to t minutes into its day as a renewal process and determine the density, mean, and variance of the random times between renewals.

   (b) On average, about how many times does the rat return to its cage from the loop in the first 4 hours of its day?

   (c) Give an interval around your estimate from (b) that will have a 95% chance of covering the true number of times the rat returned to its cage from the loop in the first 4 hours of its day.

(d) If you go visit the rat's cage at exactly 4 hours into its day, about what is the mean and variance of the time since the rat last returned from the loop?

[12 marks]

Ans.

(a) The inter-renewal time τ is distributed as a sum of two independent uniform $(0,5)$ variables. The density is the triangular density

$$f(t) = \begin{cases} t/25, & 0 < t < 5, \\ (10-t)/25, & 5 < t < 10. \end{cases}$$

The mean and variance are obtained by summing the mean and variance of the uniform variables:

$$E[\tau] = 5, \quad Var(\tau) = 2 * 5^2/12 = 5^2/6.$$

(b) $N_t/t \to 1/E[\tau] = 1/5$ as $t \to \infty$, and so in the first 240 minutes of the day we expect the rat to return about $240/5 = 48$ times.

(c) The renewal CLT says that $N_t \approx \mathrm{N}(t/\mu, t\sigma^2/\mu^3)$ and so for $t = 240$, we expect with there is a 95% chance the rat has returned to the cage a number of times in the interval

$$\left(240/5 - (1.96)\sqrt{240 \cdot 5^2/(6 * 5^3)}, 240/5 + (1.96)\sqrt{240 \cdot 5^2/(6 * 5^3)}\right) \approx 48 \pm 5.54.$$

(d) Defining the renewal times $T_k = \sum_{i=1}^k \tau_i$, we know that for large t and $A_t := t - T_{N_t}$ roughly has density $(1 - F(t))/\mu$ for $0 < t < 10$. In this case this density on this support is

$$\begin{array}{ll} 1/5 - t^2/250, & 0 < t < 5, \\ (10-t)^2/250, & 5 < t < 10. \end{array}$$

and now its only a matter of computing the first and second moments to find $E[A_t] \approx 35/12$ and $Var(A_t) \approx 12.2569$.

End of Exam