MAST10006 problem booklet 2017

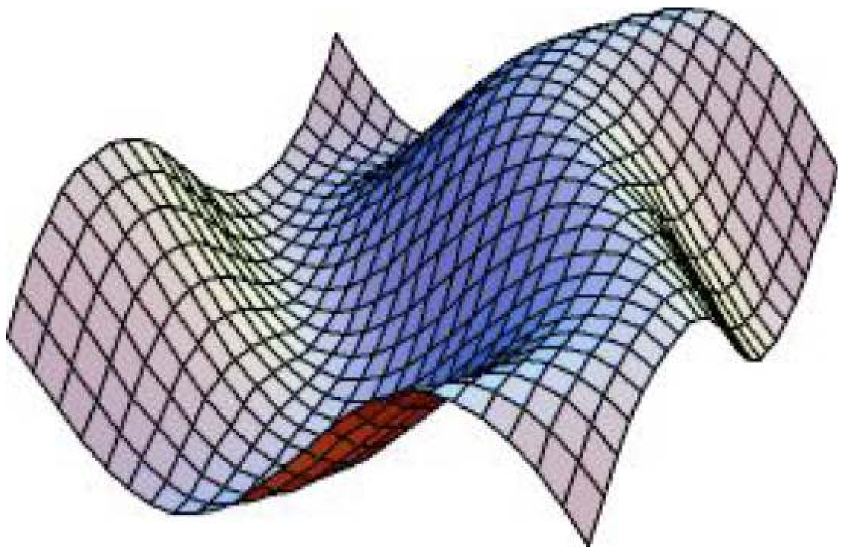
Calculus 2 (University of Melbourne)

The University of Melbourne

School of Mathematics and Statistics

MAST10006
Calculus 2

Semester 1, 2018



STUDENT NAME:	STUDENT NUMBER:

This booklet is for the use of students of the University of Melbourne enrolled in the subject MAST10006 Calculus 2.

MAST10006 Calculus 2
Semester 1, 2018
Subject Information

MAST10006 Calculus 2 is a core mathematics subject that prepares students for further studies in Mathematics and Statistics. Calculus 2 is also a prerequisite for subjects in many other areas, such as the Physical Sciences, Engineering, Actuarial Studies and some branches of Commerce.

This subject will extend knowledge of calculus from school. Students are introduced to hyperbolic functions and their inverses, the complex exponential and functions of two variables. Techniques of differentiation and integration will be extended to these cases. Students will be exposed to a wider class of differential equation models, both first and second order, to describe systems such as population models, electrical circuits and mechanical oscillators. The subject also introduces sequences and series including the concepts of convergence and divergence.

Teaching staff

The subject co-ordinator is Dr. Anthony Morphett, a.morphett@unimelb.edu.au . The lecturers are Prof. Jan de Gier, Dr. Anthony Morphett, Prof. John Sader and Mr. Binzhou Xia.

Expectations

In this subject you are expected to:

- Attend all lectures, and take notes and participate in class activities during lectures.
- Attend all practice classes, participate in groupwork in practice classes, and complete all practice class exercises.
- Work through the problem booklet outside of class in your own time. You should try to keep up-to-date with the problem booklet questions, and aim to have attempted all questions from the problem booklet before the exam.
- Complete all assignments on time.
- Seek help when you need it.

In total, you are expected to dedicate around 170 hours to this subject, including classes. This equates to an average of about 9 hours of additional study, outside of class, per week over 14 weeks.

Classes

Each week there are 3 lectures and 1 practice class. You are expected to attend all classes. Lecture times and locations are listed on the Calculus 2 LMS site. Please attend the lecture stream and practice class in which you are registered. Lectures and practice classes start in week 1.

There are also consultations held throughout the week. These are a chance for you to get individual, one-on-one help with the subject. Attendance at consultations is not required but is highly recommended.

Subject resources

Lecture notes. The MAST10006 Calculus 2 lecture notes are partially complete lecture slides. They contain the theory, diagrams, and statement of the questions to be covered in lectures, and contain spaces for worked examples and notes. Students are expected to bring these partial lecture notes to all lectures, and to take additional notes and fill in the working for examples in the gaps provided.

Problem booklet. This booklet is the MAST10006 Calculus 2 problem booklet. It contains practice exercises for you to work through in your own time. They are not part of the assessment, but are an important part of your study for the subject. You should try to keep up-to-date with the questions in the booklet, and aim to attempt all the problems before the final exam.

There are seven problem sheets with answers in this booklet corresponding to the seven major topics covered in lectures. The questions can be grouped into two types:

- Questions labelled *Revision* cover material that is assumed knowledge from VCE Specialist Mathematics 3/4 and will not be discussed in lectures.
- Questions that do not have a label *Revision* are the core questions for Calculus 2. These questions cover the examinable material for Calculus 2.

Answers to the problem booklet questions are provided at the back of this booklet. You are welcome to seek help with questions from the problem sheets at consultations. Fully worked solutions will not be provided for the problem booklet questions.

The problems you should be able to attempt, based on what we have covered in lectures so far, will be posted on the Calculus 2 LMS site at the end of each week.

Other resources. Other resources, including online videos, applets, and additional notes are available from the Calculus 2 LMS site.

Textbook. There is no assigned textbook for MAST10006 Calculus 2. If you want to use a textbook as a reference, there are many first year level calculus textbooks available for loan in the ERC Library. Two suggested references and some recommended questions from them are listed on the LMS site.

Assessment

Assessment in MAST10006 Calculus 2 consists of:

- 4 assignments, due during semester, contributing 20% of the final MAST10006 grade
- A 3-hour final exam held during the end-of-semester exam period, contributing 80% of the final MAST10006 grade.

Assignments

There are four assignments in MAST10006 Calculus 2, due as follows:

(1) On limits, continuity, sequences, series - due **4pm Monday 19 March**.

(2) On hyperbolic functions, complex numbers, integration - due **4pm Monday 16 April**.

(3) On first order differential equations - due **4pm Monday 30 April**.

(4) On second order differential equations - due **4pm Monday 14 May**.

The assignments will be handed out in lectures and posted on the Calculus 2 LMS site one week before the due date.

Assignment submission. Assignments must be neatly handwritten in blue or black pen. Pencil is not acceptable.

You must attach a completed assignment cover sheet to the front of every assignment. The cover sheet is handed out in lectures along with the assignment questions.

Assignments must be submitted into the appropriate MAST10006 Calculus 2 assignment box on the ground floor of the Peter Hall building. A list of all classes, tutors and the corresponding assignment box will be posted above the assignment boxes and on the LMS early in semester. It is your responsibility to ensure you submit your assignment into the correct assignment box. Assignments submitted into the wrong box may get zero.

Marked assignments will be returned in practice classes in the week after the due date. Full solutions will be posted on the LMS several days after the assignment is due. Marks will be recorded in the LMS Grade Center. If you believe there is an error in the mark recorded in the LMS, please contact the subject co-ordinator.

Every student is required to complete the online Plagiarism Declaration to cover all assignments submitted in this subject this semester. If you do not complete the declaration, you may not receive your assignment marks and may not be able to collect your marked assignments.

Assignment marking criteria. Clear written communication and careful justification is important in mathematics, as well as accuracy of your final answers. In the assignments, marks may be awarded for:

- Correct use of appropriate mathematical techniques
- Accuracy and validity of any calculations or algebraic manipulations
- Clear justification or explanation of techniques and rules used
- Use of correct mathematical notation and terminology

Assignments are marked by tutors according to a marking scheme. A sample marking scheme is available on the Calculus 2 LMS site.

Late assignments. Late assignments which are submitted after the deadline but **within 24 hours of the deadline** will attract a **deduction of 20%** of the total marks.

To submit an assignment late but within 24 hours of the deadline: submit your assignment to the special Calculus 2 late assignment box which will be identified on the signs at the normal assignment boxes.

Do NOT place a LATE assignment into your standard Calculus 2 assignment box - it will not be collected or marked if you do.

Assignments submitted **more than 24 hours** after the deadline will not be accepted and **will receive a mark of 0**, unless prior arrangements for an extension have been made with the subject co-ordinator.

Special consideration for assignments. At times there are factors that affect students capacity to complete a piece of assessment that are unavoidable, unforeseen and outside their control, such as illness or another unexpected occurrence. If this occurs to you, then you may be eligible for consideration in your assessment.

Please note that students are expected to plan around the following circumstances so they would not be grounds for consideration:

- Regular, normal life events such as family life, work, sporting activities, social and other commitments, and:

- Minor interruptions and disruption to routine that might result from minor illness, mishap or other minor adversity.

Consideration for assignments is handled by the subject co-ordinator, following guidelines established by the Faculty of Science.

To apply for special consideration for an assignment, contact the subject co-ordinator Dr. Anthony Morphett as soon as possible and within 4 working days of the assessment due date, and provide suitable supporting documentation (such as a medical certificate) either in person or by email.

Do not submit medical certificates or other documentation into the assignment boxes. You must give the documentation directly to the subject co-ordinator, either in person or by email.

Do not use the student portal to apply for special consideration for MAST10006 assignments. A special consideration application through the student portal is necessary only when applying for special consideration for the final exam, or if instructed to do so by the subject co-ordinator.

Possible outcome if consideration awarded: For an assignment, you may be granted a short extension of up to 3 days or may receive exemption from the assignment. Exemption means that you do not need to complete the assignment, and the weightings of your other assignments are increased to make up for it.

Exemption is usually only given if your circumstances make it impossible to complete the assignment in the allowable timeframe. Exemption will usually be given for at most one assignment for the semester. If you are seeking special consideration for more than one assignment, the subject co-ordinator may ask for additional documentation or require you to submit a formal special consideration application as for examinations.

If you do not contact the subject co-ordinator within 4 working days of the assessment due date, or do not provide supporting documentation, then special consideration will not be awarded. In that case, your assignment will be subject to the normal late policy documented above.

Final exam

The final exam is a written exam worth 80% of the final MAST10006 grade. The time allowed is 3 hours writing time and 15 minutes reading time. No calculators, dictionaries or notes are allowed in the exam. The formula sheet on page viii of this problem booklet is provided in the exam. You can write in pen or pencil. You can attempt questions in any order. The exam covers all subject content including material from lectures, tutorials, problem booklet and assignments. Exam questions are of different lengths. The marks for each question are proportional to the length of the question and the number of marks for each question is indicated on the exam paper. Marking criteria for the exam will be the same as for assignments. The exam timetable is announced via the student portal later in semester. A selection of past exam papers and answers will be provided on the Calculus 2 LMS close to the end of semester.

Information about applying for special consideration for the final exam is available on the Calculus 2 LMS site.

Calculators

There is no formal requirement to possess a calculator for this subject. Calculators are not permitted in the final MAST10006 exam. Assessment in this subject concentrates on the testing of concepts and the ability to conduct procedures in simple cases. Nonetheless, there are some questions on the problem sheets for which calculator usage is appropriate. If you have a calculator, then you will find it useful occasionally.

Intended learning outcomes and generic skills

Students completing this subject should be able to:

- calculate simple limits of a function of one variable;
- determine convergence and divergence of sequences and series;
- sketch and manipulate hyperbolic and inverse hyperbolic functions;
- evaluate integrals using trigonometric and hyperbolic substitutions, partial fractions, integration by parts and the complex exponential;
- find analytical solutions of first and second order ordinary differential equations, and use these equations to model some simple physical and biological systems;
- calculate partial derivatives and gradients for functions of two variables, and use these to find maxima and minima.

In addition to learning specific skills that will assist students in their future careers in science, they will have the opportunity to develop generic skills that will assist them in any future career path. These include:

- problem-solving skills: the ability to engage with unfamiliar problems and identify relevant solution strategies;
- analytical skills: the ability to construct and express logical arguments and to work in abstract or general terms to increase the clarity and efficiency of analysis;
- collaborative skills: the ability to work in a team; and
- time-management skills: the ability to meet regular deadlines while balancing competing commitments.

Prerequisites and required knowledge

The prerequisite for MAST10006 Calculus 2 is a study score of at least 29 in VCE Specialist Mathematics or equivalent, or completion of MAST10005 Calculus 1 or MAST10007 Linear Algebra.

The content of MAST10006 Calculus 2 builds on the content of VCE Specialist Mathematics and MAST10005 Calculus 1. Students who have completed other mathematics qualifications may need to do some additional reading to make up for any topics from VCE Specialist Mathematics/MAST10005 Calculus 1 which are required knowledge for MAST10006 Calculus 2 but were not covered in their previous mathematics studies. The additional reading material can be found on the Calculus 2 LMS.

Lecture Outline

This outline is a guide only — material to be covered in each lecture may vary slightly from the following table.

Limits, Continuity, Sequences and Series

1. Intuitive idea of limits. Limit laws. Limits involving infinity.
2. Evaluating limits. Sandwich theorem.
3. Continuity of functions. Limits of compositions of functions.
4. L'Hôpital's rule. Sequences: definition, standard limits, connecting limits of functions and limits of sequences.
5. Limits of sequences examples. Series: definition. Geometric series. Divergence Test.
6. Ratio Test. Comparison test.

Hyperbolic Functions

7. Hyperbolic functions: definition, basic properties, sketching graphs.
8. Hyperbolic functions: identities and applications, differentiation.
9. Inverse hyperbolic functions: definition, sketching graphs, manipulation.

Complex Numbers

10. Inverse hyperbolic functions: differentiation. Cartesian and polar form of a complex number.
11. Complex exponential: definition, properties. De Moivre's theorem.
12. Differentiation and integration using the complex exponential.

Integral Calculus

13. Substitution with derivative present. Trigonometric and hyperbolic substitutions.
14. Hyperbolic substitutions (continued). Products of hyperbolic functions.
15. Partial fractions, including use of polynomial long division.
16. Integration by parts.

First Order Ordinary Differential Equations

17. Definitions: ODE, order, general solution, IVP. Separable ODEs.
18. Linear ODE using integrating factors.
19. Making a substitution to reduce ODE to linear or separable. Equilibrium points. Applications: Doomsday population models without harvesting.
20. Applications: Doomsday population models with harvesting. Logistic population models with and without harvesting.
21. Applications: Mixing problems with constant and variable volume.

Second Order Ordinary Differential Equations

22. Definitions: homogeneous, inhomogeneous, linear/non-linear. General solutions. Solution of homogeneous constant coefficient linear ODEs.
23. Solution of homogeneous/inhomogenous constant coefficient linear ODEs. Particular solutions using method of undetermined coefficients.
24. Particular solutions continued. Superposition of particular solutions.
25. Applications: Free vibrations of hanging spring-mass systems including air resistance.
26. Applications: Forced vibrations of hanging spring-mass systems.

Functions of Two Variables

27. Introduction to functions of two variables. Level curves and cross sections. Sketching surfaces.
28. Sketching surfaces (continued). Limits and continuity.
29. First and second order partial derivatives.
30. Tangent planes. Linear approximations.
31. Chain rule for partial derivatives. Directional derivatives.
32. Directional derivatives (continued). Gradient. Steepest descent.
33. Stationary points. Classification of stationary points using second derivative test.
34. Partial integration. Double integrals over rectangular domains.

Formulae Sheet

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\,dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2 - x^2}\,dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\cot^2 x + 1 = \operatorname{cosec}^2 x \qquad \coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\cos 2x = \cos^2 x - \sin^2 x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cos 2x = 2\cos^2 x - 1 \qquad \cosh 2x = 2\cosh^2 x - 1$$
$$\cos 2x = 1 - 2\sin^2 x \qquad \cosh 2x = 1 + 2\sinh^2 x$$
$$\sin 2x = 2\sin x\cos x \qquad \sinh 2x = 2\sinh x\cosh x$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \qquad \cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$
$$\sin(x+y) = \sin x\cos y + \cos x\sin y \qquad \sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$e^{ix} = \cos x + i\sin x$$
$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$\operatorname{arcsinh} x = \log(x + \sqrt{x^2+1}) \qquad \operatorname{arccosh} x = \log(x + \sqrt{x^2-1})$$

$$\operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \qquad \lim_{n\to\infty} \frac{1}{n^p} = 0, \quad (p > 0)$$

$$\lim_{n\to\infty} r^n = 0, \quad (|r| < 1) \qquad \lim_{n\to\infty} a^{\frac{1}{n}} = 1, \quad (a > 0)$$

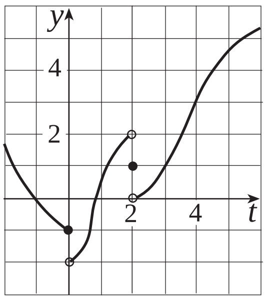
$$\lim_{n\to\infty} \frac{a^n}{n!} = 0, \quad a \in \mathbb{R} \qquad \lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$$

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \mathrm{e}^a, \quad a \in \mathbb{R} \qquad \lim_{n\to\infty} \frac{n^p}{a^n} = 0, \quad (p > 0, a > 1)$$

Sheet 1: Limits, Continuity, Sequences & Series

Limits and Continuity

1.1. *Intuitive Idea of Limit.* For the function $g : [-2, 6] \to \mathbb{R}$ whose graph $y = g(t)$ is



state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t\to 0^-} g(t)$ (b) $\lim_{t\to 0^+} g(t)$ (c) $\lim_{t\to 0} g(t)$

(d) $\lim_{t\to 2^-} g(t)$ (e) $\lim_{t\to 2^+} g(t)$ (f) $\lim_{t\to 2} g(t)$

1.2. *Limit Laws.* Evaluate the limit and justify each step by indicating which limit laws you have used.

(a) $\lim_{x\to -2} (3x^4 + 2x^2 - x + 1)$ (b) $\lim_{x\to 1} \left(\dfrac{1+3x}{1+4x^2+3x^4}\right)^3$

1.3. *New Limit Tricks.* Evaluate the limit, if it exists.

(a) $\lim_{x\to 2} \left(x^2 - 3x + 5\right)$ (b) $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$

(c) $\lim_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$ (d) $\lim_{x\to -4} \dfrac{x^2+7x+12}{x^2+3x-4}$

(e) $\lim_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7}$ (f) $\lim_{x\to -4} \dfrac{\frac{1}{4}+\frac{1}{x}}{4+x}$

1.4. *Off to Infinity.* Evaluate the following limits if they exist.

(a) $\lim_{x\to\infty} \dfrac{3x+5}{x-4}$ (b) $\lim_{x\to\infty} \dfrac{x^3+5x}{2x^3-x^2+4}$

(c) $\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right)$ (d) $\lim_{x\to\infty} \cos x$

1.5. *Sandwich it.* Using the Sandwich theorem, evaluate the following limits:

(a) $\lim_{x\to 0} \left[x^2 \cos\left(\dfrac{20\pi}{x}\right)\right]$ (b) $\lim_{x\to\infty} \left(e^{-2x} \sin x\right)$

(c) $\lim_{x\to 0} \left[x \sin\left(\dfrac{1}{x}\right)\right]$

1.6. *Continuity on Domains.* State the largest possible domain on which the following functions are defined. Explain why the functions are continuous everywhere on their domains.

(a) $F(x) = \dfrac{x}{x^2 + 5x + 6}$ (b) $R(x) = x^2 + \sqrt{2x - 1}$

(c) $G(t) = \log(t^4 - 1)$

1.7. *Continuity and Limits.* Evaluate the following limits if they exist.

(a) $\displaystyle\lim_{x\to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$ (b) $\displaystyle\lim_{x\to \pi} \sin(x + \sin x)$ (c) $\displaystyle\lim_{x\to 1} e^{x^2 - x}$

1.8. *Discontinuities.* Find the values of x at which the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

is discontinuous. Sketch the graph $y = f(x)$.

1.9. *Continuous or Not.* Determine the value of $c \in \mathbb{R}$ such that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

is continuous on $(-\infty, \infty)$. In your answer, include the names of any rules or theorems you have used.

1.10. *L'Hôpital's Rule.* Use L'Hôpital's Rule to evaluate the following limits.

(a) $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h}$ (b) $\displaystyle\lim_{x\to 0} \frac{\tan px}{\tan qx} \quad (q \neq 0)$

(c) $\displaystyle\lim_{x\to \infty} \left(x^3 e^{-x^2}\right)$ (d) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

1.11. *Mixed Limits.* Evaluate the limit, if it exists.

(a) $\displaystyle\lim_{x\to \infty} \left[\log(x^2 - 1) - \log(x + 1)\right]$ (b) $\displaystyle\lim_{x\to \infty} \left(x^2 e^{-x}\right)$

(c) $\displaystyle\lim_{x\to 9} \frac{x - 9}{\sqrt{x} - 3}$ (d) $\displaystyle\lim_{x\to \infty} \frac{3x^{14} - x^{13} - 2}{2x^{15} + x^7 - 3}$

(e) $\displaystyle\lim_{x\to 0} \frac{2x^{10} - 4x^7 + 3x^2 + 4}{7x^{14} + 6x^3 - 7x + 1}$ (f) $\displaystyle\lim_{x\to 0^+} \left(\sqrt{x} \log x\right)$

1.12. *More Mixed Limits.* Evaluate the limit, if it exists.

(a) $\displaystyle\lim_{x\to \infty} \sin(4x + 2)$ (b) $\displaystyle\lim_{x\to -1} \frac{x + 1}{x^2 - x - 2}$

(c) $\displaystyle\lim_{x\to 3} \frac{x^2 + 3x + 4}{x + 3}$ (d) $\displaystyle\lim_{x\to 0} (\tan 7x \operatorname{cosec} 3x)$

(e) $\displaystyle\lim_{x\to 0} \left[x^4 \sin\left(\frac{3}{x}\right)\right]$ (f) $\displaystyle\lim_{x\to 1} \frac{\log x}{\cos\left(\frac{1}{2}\pi x\right)}$

Sequences and Series

1.13. *Sequence Terms.* For each of the sequences whose nth term is given below, find the first four terms. Determine which of the sequences is convergent and which is divergent. Find the limits of the convergent sequences.

(a) $2\sqrt{n}$ (b) $\cos\left(\frac{n\pi}{2}\right)$ (c) $(-1)^n\frac{n}{1+n^2}$

1.14. *Standard Limits.* Use standard limits to decide which sequences, whose nth term is given below, converge. For those that converge, determine their limits.

(a) $\left(\frac{n+3}{n}\right)^n+\frac{1}{4^n}$ (b) $\frac{n!}{10^{6n}}$ (c) $\frac{\log n}{n}+\frac{(2n)^3}{3^n}+\frac{2^n}{(2n)!}$

1.15. *Sequence Limits.* Find the limit (if it exists) of each of the sequences, whose nth term is given below.

(a) $\frac{2n^2+6n+2}{3n^3-n^2-n}$ (b) $\frac{3^n-4^n}{3n^2+4^n+7}$ (c) $\frac{3^n+2n!}{5^n-n!}$

1.16. *Sandwich Rule.* Use the Sandwich Rule to find the limits of the following sequences, whose nth term is given below, if they exist.

(a) $\frac{2+(-1)^n}{n^2}$ (b) $(3^n+1)^{1/n}$ (c) $\frac{n!}{n^n}$

1.17. *Continuity Rule.* Use the Continuity Rule to find the limits of the following sequences, whose nth term is given below, if they exist.

(a) $\exp\left(\frac{n^2-2}{2n^2+1}\right)$ (b) $\cos\left(\frac{n^2+2}{n^3+n+1}\right)$ (c) $\log(2n)-\log(3n+2)$

1.18. *L'Hôpital's Rule.* Use L'Hôpital's Rule to find the limits of the following sequences, whose nth term is given below, if they exist.

(a) $\frac{e^n}{3n+4}$ (b) $\frac{\log(n+1)}{n}$ (c) $n\sin\left(\frac{\pi}{n}\right)$

1.19. *Mixed Sequences.* Find the limits of the following sequences, whose nth term is given below, if they exist.

(a) $\frac{n^3+1}{4n+2n^2}$ (b) $\frac{n^2}{n+1}-\frac{n^2+1}{n}$ (c) $\frac{\log(n+2)}{\log(2n+1)}$

(d) $\frac{\sqrt{n^2+6}}{\sqrt{4n^2-1}}$ (e) $\left(\frac{2n+5}{2n}\right)^n$ (f) $\frac{\cos^2 n}{2^n}$

1.20. *Geometric Series.* Find the sum of each series, if it exists.

(a) $\sum_{n=0}^{\infty}\frac{3^{n+1}}{4^n}$ (b) $\sum_{n=0}^{\infty}\frac{2^{n-1}}{3^n}$ (c) $\sum_{n=0}^{\infty}(2.1)^n$

1.21. *Divergence Test.* Use the divergence test to show that each series is divergent.

(a) $\sum_{n=1}^{\infty}\sqrt[n]{3}$ (b) $\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^n$ (c) $\sum_{n=1}^{\infty}\frac{n+1}{n+3}$

1.22. *Comparison Test.* Use the comparison test to determine if the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}-1}{n^2+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+n-1}$

1.23. *Ratio Test.* Use the ratio test to determine the convergence or divergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

1.24. *Mixed Series.* Determine whether the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{1+n^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{4^n}$ (f) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^n}$

1.25. *Mixed Sequences and Series.* Determine whether the following sequences and series are convergent or divergent.

(a) $\displaystyle\left\{\left(\frac{n}{n+7}\right)^n\right\}$ (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+7}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{9n^4+2n^3+5}{3n^5-n^2}$ (d) $\displaystyle\left\{\frac{9n^4+2n^3+5}{3n^5-n^2}\right\}$

(e) $\displaystyle\left\{\frac{\sin 2n}{n}\right\}$ (f) $\displaystyle\sum_{n=0}^{\infty} \frac{5^{n+2}}{7^n}$

Sheet 2: Hyperbolic Functions

2.1. *Calculating Hyperbolic Functions.* Find the exact numerical value of each expression:
(a) $\sinh(\log 3)$ (b) $\cosh(-\log 2)$ (c) $\tanh(2\log 5)$

2.2. *Hyperbolic Expressions.* Simplify the following expressions:
(a) $\sinh(\log x)$ (b) $\cosh(-3\log x)$ (c) $\tanh(2\log x)$

2.3. *Hyperbolic Functions.*

(a) If $\cosh x = \frac{5}{4}$, what are the possible values of $\sinh x$ and $\tanh x$?

(b) If $\sinh x = -\frac{2}{5}$, compute $\cosh x$, $\tanh x$, $\coth x$, $\operatorname{sech} x$ and $\operatorname{cosech} x$.

2.4. *Standard Hyperbolic Identities.*

Use the definitions of $\cosh x$ and $\sinh x$ to verify the following identities, where $n \in \mathbb{Z}$.

(a) $\cosh x - \sinh x = e^{-x}$

(b) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(c) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

2.5. *Sketching Hyperbolics.* On the same set of axes sketch the following graphs:

$$y = \cosh(2x),\ y = \cosh(2x+3),\ y = \operatorname{sech}(2x+3)$$

2.6. *Manipulating Hyperbolic Functions.* Express

(a) $\sinh^5 x$ in terms of hyperbolic sines of multiples of x.

(b) $\cosh^6 x$ in terms of hyperbolic cosines of multiples of x.

2.7. *Standard Hyperbolic Derivatives.* Using the derivatives of $\sinh x, \cosh x$ and $\tanh x$, show that:

(a) $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x, \quad x \neq 0$

(b) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

(c) $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x, \quad x \neq 0$

2.8. *Hyperbolic Derivatives.* Find the derivatives of the following functions f given by:
(a) $f(x) = \cosh(\sqrt{x})$ (b) $f(x) = \sqrt{\cosh x}$
(c) $f(x) = \tanh(\sin 3x)$ (d) $f(x) = x\sinh\left(\frac{1}{x}\right)$

Where do the derivatives exist?

2.9. *Sketching Inverse Hyperbolics.* On the same set of axes sketch the following graphs:

$$y = \sinh\left(\frac{x}{3}\right),\ y = \operatorname{cosech}\left(\frac{x}{3}\right),\ y = \operatorname{arccosech}\left(\frac{x}{3}\right)$$

2.10. *Functions and Inverses.* Simplify the following expressions:

(a) $\sinh(\operatorname{arccosh} x)$ (b) $\sinh^2(\operatorname{arctanh} x)$ (c) $\tanh(\operatorname{arccosh} x)$

2.11. *Inverse Hyperbolic Functions.*

(a) Prove that, for $x \geq 1$,

$$\operatorname{arccosh} x = \log(x + \sqrt{x^2 - 1})$$

(b) Find the derivative

$$\frac{d}{dx}(\operatorname{arccosh} x)$$

(i) using the formula in part (a) (ii) using implicit differentiation

(c) Using the logarithm formula given in part (a), calculate the following limits:

(i) $\lim_{x\to\infty}(\operatorname{arccosh} x - \log x)$ (ii) $\lim_{x\to 1^+} \operatorname{arccosh} x$

2.12. *Inverse Hyperbolic Derivatives.* Using implicit differentiation, show that:

(a) $\dfrac{d}{dx}(\operatorname{arctanh} x) = \dfrac{1}{1-x^2}, \quad -1 < x < 1$

(b) $\dfrac{d}{dx}(\operatorname{arcsech} x) = \dfrac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$

2.13. *More Inverse Hyperbolic Derivatives.* Find the derivatives of the following functions f given by:

(a) $f(x) = x^3 \operatorname{arcsinh}(e^x)$ (b) $f(x) = \operatorname{arccosh}(\sqrt{x})$

(c) $f(x) = \log(\operatorname{arccosh} 4x)$ (d) $f(x) = \dfrac{1}{\operatorname{arctanh} x}$

Where do the derivatives exist?

Sheet 3: Complex Numbers

Note: In this problem sheet, the principal argument should be used whenever the complex number is written in polar form, that is, $-\pi < \theta \leq \pi$.

3.1. *Cartesian Form (Revision).* Express the following numbers in the Cartesian form $a+ib$ with $a, b \in \mathbb{R}$:

(a) $(1+i)^2 + (1-i)$ (b) $\overline{12+7i}$

(c) $\text{Im}\left(\dfrac{1-5i}{4+i}\right)$ (d) $\text{Re}\left(\dfrac{1-5i}{4+i}\right)$

3.2. *Modulus and Argument (Revision).*

(a) Find the modulus of each of the following complex numbers without multiplying them out into Cartesian form:

(i) $\dfrac{5+2i}{2+5i}$ (ii) $\dfrac{-27i(8+2i)(2+i)}{(4+i)(4-3i)(4-8i)}$

(b) Find an argument θ, $-\pi < \theta \leq \pi$, for the following complex numbers:

(i) $\dfrac{-2}{1+i\sqrt{3}}$ (ii) $\dfrac{i}{-2-2i}$

3.3. *Polar Form.* Express each of the following complex numbers in the polar form $re^{i\theta}$. In each case, choose an angle θ, $-\pi < \theta \leq \pi$.

(a) -1 (b) $-5+5i$

(c) $(1-i)^2$ (d) $2+5i$

3.4. *Powers.* Simplify the following powers of complex numbers:

(a) $(1+i)^{20}$ (b) $(2\sqrt{3}+2i)^5$

(c) $\left(\dfrac{1+i}{\sqrt{3}+i}\right)^{12}$ (d) $\left(\dfrac{1-i}{\sqrt{3}-i}\right)^{7}$

3.5. *Use Complex Exponential.* Using the complex exponential, express:

(a) $\sin^6\theta$ as a sum of $\cos(n\theta)$

(b) $\sin^3\theta\cos^2\theta$ as a sum of $\sin(n\theta)$

3.6. *Trigonometric and Hyperbolic Functions.* If $z = x+iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$, we define

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \quad \text{and} \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

Use these definitions to show that:

(a) $\sin(iz) = i\sinh z$

(b) $\cos(iz) = \cosh z$

(c) $\sin z = \sin x\cosh y + i\cos x\sinh y$

3.7. *Derivatives.* Find the following derivatives with respect to the real variable t:
(a) the third derivative of $e^{(2+3i)t}$ (b) the 18th derivative of $e^{(1-i)t}$

3.8. *Derivatives via Complex Exponential.* Find the following derivatives with respect to the real variable t, using the complex exponential:
(a) the 257th derivative of $e^{-t}\sin t$ (b) the 576th derivative of $e^{4t}\cos 4t$

3.9. *Integrals.* Find the indefinite integrals of the following using the complex exponential:
(a) $e^{x}\cos 3x$ (b) $e^{-2x}\sin 11x$ (c) $e^{5t}\cos 7t$

Sheet 4: Integral Calculus

Techniques of Integration

4.1. *Integration (Revision).* Evaluate the following integrals:

(a) $\displaystyle\int_1^2 (2x-5)^3\,\mathrm{d}x$ (b) $\displaystyle\int \frac{1}{x^2+4x+13}\,\mathrm{d}x$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 7x\,\mathrm{d}x$ (d) $\displaystyle\int \frac{3x}{(x-2)(x+4)}\,\mathrm{d}x$

(e) $\displaystyle\int \frac{1}{x\log x}\,\mathrm{d}x$ (f) $\displaystyle\int \frac{e^x}{e^x+1}\,\mathrm{d}x$

(g) $\displaystyle\int \frac{4x+17}{x^2+10x+25}\,\mathrm{d}x$ (h) $\displaystyle\int \sin^6 x\cos^3 x\,\mathrm{d}x$

4.2. *Derivative Substitutions.* Evaluate the following integrals, where $k \in \mathbb{R}$ is a constant:

(a) $\displaystyle\int \operatorname{cosech}^2 x\,\coth^2 x\,\mathrm{d}x$ (b) $\displaystyle\int \sinh 3x\,\exp(4\cosh 3x)\,\mathrm{d}x$

(c) $\displaystyle\int \frac{2\operatorname{arccosh} x}{\sqrt{x^2-1}}\,\mathrm{d}x$ (d) $\displaystyle\int \frac{\operatorname{sech}^2 kx}{2+\tanh kx}\,\mathrm{d}x$

4.3. *Trigonometric and Hyperbolic Substitutions.* Using an appropriate trigonometric or hyperbolic substitution, find the indefinite integrals of the following functions:

(a) $\sqrt{1+4x^2}$ (b) $\sqrt{4-x^2}$ (c) $\displaystyle\frac{1}{(x^2-1)^{\frac{3}{2}}}$

4.4. *Hyperbolic Powers.* Find the indefinite integrals of the following powers of hyperbolic functions:

(a) $\sinh^6 x\cosh x$ (b) $\cosh^2 3x$ (c) $\sinh^2 x\cosh^3 x$

(d) $\sinh^3 4x$ (e) $\cosh^4 x$

4.5. *Hyperbolic Tangent Integrals.* Evaluate the following integrals:

(a) $\displaystyle\int \tanh x\operatorname{sech}^2 x\,\mathrm{d}x$ (b) $\displaystyle\int \tanh^2 3x\,\mathrm{d}x$

4.6. *Partial Fractions.* Find the indefinite integrals of the following:

(a) $\displaystyle\frac{1}{(x+2)(x^2+1)}$ (b) $\displaystyle\frac{37-11x}{(x+1)(x-2)(x-3)}$

(c) $\displaystyle\frac{4x+3}{(x^2+1)(x^2+2)}$ (d) $\displaystyle\frac{x+13}{x^3+2x^2-5x-6}$

(e) $\displaystyle\frac{x^3+3x-2}{x^2-x}$ (f) $\displaystyle\frac{9}{x(x+3)^2}$

4.7. *By Parts.* Evaluate the following integrals:

(a) $\displaystyle\int x\cos 3x\ \mathrm{d}x$ (b) $\displaystyle\int \arcsin x\ \mathrm{d}x$ (c) $\displaystyle\int_0^1 x\arctan x\ \mathrm{d}x$

(d) $\displaystyle\int x^2\cosh x\ \mathrm{d}x$ (e) $\displaystyle\int \operatorname{arcsinh} x\ \mathrm{d}x$ (f) $\displaystyle\int x^7\log x\ \mathrm{d}x, \quad x>0$

4.8. *By Parts Again.* Using integration by parts twice, evaluate the following integrals:

(a) $\displaystyle\int e^x \cos 3x \; \mathrm{d}x$ (b) $\displaystyle\int e^{-2x} \sin 11x \; \mathrm{d}x$

(c) $\displaystyle\int \cos x \cos 3x \; \mathrm{d}x$ (d) $\displaystyle\int \sin x \sin 4x \; \mathrm{d}x$

4.9. *Mixed Integrals.* Evaluate the following integrals:

(a) $\displaystyle\int \cosh^3 x \sinh^3 x \, \mathrm{d}x$ (b) $\displaystyle\int \sqrt{x} \log x \, \mathrm{d}x, \quad x > 0$

(c) $\displaystyle\int \frac{x^2 + 3x + 4}{x^2 + x} \, \mathrm{d}x$ (d) $\displaystyle\int \frac{x^2 - 2}{x^3 - 6x - 2} \, \mathrm{d}x$

(e) $\displaystyle\int \frac{1}{\sqrt{9 - 4x^2}} \, \mathrm{d}x$ (f) $\displaystyle\int_1^e \log(x^4) \, \mathrm{d}x$

4.10. *More Mixed Integrals.* Evaluate the following integrals:

(a) $\displaystyle\int \sqrt{x^2 - 16} \, dx$ (b) $\displaystyle\int x \sinh 3x \, \mathrm{d}x$

(c) $\displaystyle\int \frac{9}{(x+2)(x-1)^2} \, \mathrm{d}x$ (d) $\displaystyle\int \cosh^5 x \sinh^4 x \, \mathrm{d}x$

(e) $\displaystyle\int \cosh x \sinh x \sqrt[5]{\cosh 2x + 55} \, \mathrm{d}x$ (f) $\displaystyle\int e^{2x} \sin 3x \, \mathrm{d}x$

Applications of Integration

4.11. *Arc Length.* The arc length s of a curve defined by the equation $y = f(x)$ between the points $x = a$ and $x = b$ (the distance along the curve) is given by

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \; dx.$$

Find the length of the arcs of the given curves between the two points indicated:

(a) $y = c \cosh\left(\dfrac{x}{c}\right)$ from $x = 0$ to $x = 1, \quad (c \in \mathbb{R}^+)$

(b) $y = \dfrac{1}{2}x^2 - \dfrac{1}{4}\log x$ from $x = 2$ to $x = 5$

Sheet 5: First Order Differential Equations

Solution of First Order Differential Equations

5.1. *Verification.* Verify (by substituting the function into the differential equation) that the given function is a solution of the differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2x}{t} - 1, \quad x(t) = t + t^2,\ t \neq 0$$

5.2. *Separable Equations.* Obtain the general solutions of the following first order differential equations.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{x-2y}$

(b) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{3t + e^{2t}}{x^2 + e^{-x}}$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 3\sqrt{9 - y^2}\ \sinh^5 4t \cosh 4t$

(d) $(y-1)\dfrac{\mathrm{d}y}{\mathrm{d}x} = (3y^2 + 4y + 1)\cos 3x$

5.3. *Separable Initial Value Problems.* Solve the following initial value problems:

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 5x^2 \cos^2 y$ for $y(1) = 0$

(b) $(x^3 + 3x + 7)\sin^2 y \dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 + 1$ for $y(-1) = 0$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy^3(1 + x^2)^{-\frac{1}{2}}$ for $y(0) = 1$

5.4. *Integrating Factor.* Solve the following linear differential equations:

(a) $xy' + 4y = x^2 - x + 1$

(b) $\dfrac{\mathrm{d}x}{\mathrm{d}t} - 6tx = 5t$

(c) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = x^2 \cosh^5 x$

(d) $t\dfrac{\mathrm{d}x}{\mathrm{d}t} + 3x = t^2 + \log t, \quad t > 0$

5.5. *Integrating Factor Initial Value Problems.* Solve the following initial value problems:

(a) $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \cos x$ for $y\left(\dfrac{\pi}{2}\right) = 0$

(b) $y' - 2y = x^2$ for $y(0) = 1$

(c) $xy' + y = e^{\sin x}\cos x$ for $y(\pi) = 1$

5.6. *Make a Substitution.*

(a) Consider the differential equation:

$$t\frac{\mathrm{d}x}{\mathrm{d}t} = x\left(1 + \log\left[\frac{x}{t}\right]\right), \quad \frac{x}{t} > 0 \tag{1}$$

Make the substitution $u = \dfrac{x}{t}$ and show that the differential equation reduces to

$$t\frac{\mathrm{d}u}{\mathrm{d}t} = u \log u \tag{2}$$

Solve equation (2) for $u(t)$ and hence write down the solution for equation (1).

(b) Consider the differential equation:

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy - y^3 = 0 \tag{3}$$

Make the substitution $u = y^{-2}$ and show that the differential equation reduces to

$$-\frac{1}{2}x^2 \frac{\mathrm{d}u}{\mathrm{d}x} + 2xu - 1 = 0 \tag{4}$$

Solve equation (4) for $u(x)$ and hence write down the solution for equation (3).

(c) Consider the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x+y) \tag{5}$$

Make the substitution $u = x + y$ and show that the differential equation reduces to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos u + 1 \tag{6}$$

Solve equation (6) for $u(x)$ and hence write down the solution for equation (5).

Applications of First Order Differential Equations

5.7. *Malthus Population Growth.* Consider a culture of x_0 bacteria in a petri dish. Assume that the rate of growth of the bacteria is proportional to the population x at time t. After 1 hour there are $\frac{3}{2}x_0$ bacteria.

(a) Find $x(t)$.

(b) When will there be $3x_0$ bacteria?

5.8. *Logistic Population Growth.* The population of a certain animal species is governed by the differential equation

$$1000\frac{\mathrm{d}p}{\mathrm{d}t} = p(100-p)$$

where p is the number of individuals in the colony at time t years. The initial population is known to be 200 individuals.

(a) Find $p(t)$. Sketch the population–time graph.

(b) Will there ever be more than 200 individuals in the colony? Explain.

(c) Will there ever be less than 100 individuals in the colony? Explain.

5.9. *Phase Plane.* Consider

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(2-y)(y-1)$$

(a) Find the equilibrium solutions.

(b) Sketch $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ versus y i.e. draw a phase plane diagram.

(c) For the regions between equilibrium solutions, determine where the solutions are increasing or decreasing.

(d) Hence give a rough sketch of y as a function of t. What is $\lim\limits_{t\to\infty} y(t)$ if:

(i) $y(0) = \frac{1}{2}$ (ii) $y(0) = 3$

(e) For which values of $y(0)$ will $\lim\limits_{t\to\infty} y(t) = 2$?

5.10. *Population with Harvesting.* Consider the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P\left(1 - \frac{P}{100}\right) - h$$

as a model for a fish population, where P is the number of fish at time t months. The constant $h > 0$ represents the harvesting rate.

(a) Determine the equilibrium solution for P as a function of h. Determine the values of h for which there is at least one equilibrium solution.

(b) For which values of h does the fish population die out irrespective of the initial population?

5.11. *Smoking.* Suppose people start smoking in a room of volume 60 m^3, thereby introducing air containing 5% carbon monoxide at a rate of 0.002 m^3/min into the room. Assume that the smoky air mixes immediately and uniformly with the rest of the air, and that this mixture leaves the room at the same rate as the smoky air enters. Assume that there is no carbon monoxide in the room initially.

(a) Prove that the amount x (in m^3) of carbon monoxide in the room at time t minutes satisfies the differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{30000} = \frac{1}{10000}$$

(b) Determine the amount of carbon monoxide in the room at any time. Identify the transient and steady-state parts of your solution.

(c) What happens to the concentration of carbon monoxide in the room in the long term? Sketch a curve of the concentration of carbon monoxide as a function of time.

(d) Medical tests warn that exposure to air containing 0.1% carbon monoxide for some time can lead to coma. How long does it take for the concentration of carbon monoxide in the room to reach this level?

5.12. *The Melting Pot.* A tank containing 20,000 litre of blended fuel oil is stirred continuously to keep the mixture homogeneous. The fuel is blended from a heavy and a light component in equal proportions. The two components are added at a rate of 5 litres/min each, then the blended fuel oil is withdrawn at a rate 10 litres/min to be burnt in a furnace. The blended fuel oil will not burn in the furnace if the heavy component forms more than 80% of the mixture.

At a certain instant the supply of the light component ceases whilst the heavy component continues to be added at the same rate of 5 litres/min and the homogeneous mixture is withdrawn at the same rate of 10 litres/min.

(a) Let x be the amount of heavy fuel in the tank (in litres) at time t minutes after the supply of light oil ceases. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{10x}{20000 - 5t} = 5, \qquad 0 \le t < 4000$$

(b) Solve the differential equation in part (a) for $x(t)$.

(c) Show that it will be 40 hours before the furnace will go out.

5.13. *Factory Pollutant.* A river flows into a lake initially containing 100 $(\mathrm{km})^3$ of pure water at a rate of 48 $(\mathrm{km})^3$/year. Pollutant from a factory is released into the lake at the rate of 2 $(\mathrm{km})^3$/year. Assume that the pollutant mixes immediately and uniformly with the rest of the water in the lake. The lake also flows into another river at the rate of 40 $(\mathrm{km})^3$/year.

(a) If x $(\mathrm{km})^3$ is the amount of pollutant in the lake after t years, show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{4x}{10+t} = 2$$

(b) Solve the differential equation in part (a) to find $x(t)$.

(c) How much pollutant will be in the lake after 10 years?

(d) It has been found that a concentration of over 2% is hazardous for the fish in the lake. Show that it will take $10(2^{\frac{1}{5}} - 1)$ years for the lake to become hazardous to the fish.

Sheet 6: Second Order Differential Equations

Solution of Second Order Differential Equations

6.1. *Verification of Solutions.* Verify (by substituting the functions into the differential equation) that the given functions are solutions of the differential equation:

$$y'' + 2y' - 3y = 0, \quad y_1(x) = e^{-3x}, \quad y_2(x) = e^x$$

6.2. *Homogeneous Solutions/Complementary Functions.* Find the general solutions $y(x)$ to the following second order differential equations:

(a) $y'' - 9y = 0$ (b) $y'' - 12y' + 36y = 0$ (c) $y'' + 49y = 0$

(d) $y'' + 8y' + 17y = 0$ (e) $y'' - 7y' = 0$

6.3. *Different Solutions.* Consider the following second order differential equation

$$y'' + by' + 4y = 0$$

where $b \in \mathbb{R}$. For what values of b does the characteristic equation have:

(a) two real distinct solutions;

(b) one real repeated solution;

(c) two complex conjugate solutions?

6.4. *Second Order Initial Value Problems.* Solve the following initial value problems for $y(x)$:

(a) $y'' + 2y' - 3y = 0, \quad y(0) = 3, \quad y'(0) = -1$

(b) $y'' + 6y' + 9y = 0, \quad y(0) = \frac{1}{3}, \quad y'(0) = 1$

(c) $y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1$

6.5. *Different Right Hand Sides.* Find the general solution of the ordinary differential equation

$$y'' + 4y' + 5y = f(x)$$

in the following cases:

(a) $f(x) = 0$ (b) $f(x) = -5$ (c) $f(x) = 2x$

(d) $f(x) = e^{3x}$ (e) $f(x) = \sin x$

Hence write down the general solution of the differential equation

$$y'' + 4y' + 5y = 10 + 10x + 2e^{3x} + 8\sin x$$

6.6. *General Solutions.* Find the general solutions of the following differential equations:

(a) $y'' - 8y' + 52y = 74e^{5x}$ (b) $y'' + 2y' + y = 3x^2 + 2$

(c) $y'' + y' - 2y = 2t$ (d) $y'' - 2y' - 3y = 2\cos 3x + 5\sin 3x$

6.7. *Look Before you Leap.* Find the general solutions of the following differential equations:

(a) $\ddot{x} + 5\dot{x} + 4x = e^{-4t}$

(b) $y'' + 9y = 12\cos 3x + 72\sin 3x$

(c) $y'' + 6y' + 9y = e^{-3x}$

6.8. *Initial Value Problems.* Solve the following initial value problems:

(a) $y'' - y = 259\cos 6x + 74\sin 6x$ where $y(0) = -6$ and $y'(0) = -11$

(b) $y'' - 7y' + 10y = e^{2x}$ where $y(0) = 4$ and $y'(0) = 50/3$

(c) $y'' + 16y = \cos 4x$ where $y(0) = 6$ and $y'(0) = 4$

Applications of Second Order Differential Equations

6.9. *Submerged Spring.* A mass of 1 kilogram is attached to a spring hanging vertically, stretching it 0.6125 m. The entire system is then submerged in a liquid that imparts a damping force numerically equal to 10 times the instantaneous velocity. Let x be the position of the weight (in metres) below the equilibrium position at time t seconds.

(a) Show that the equation of motion for the mass on the spring is

$$\ddot{x} + 10\dot{x} + 16x = 0$$

(b) Find a general expression for the position of the weight at time t.

(c) Find the position of the weight if the weight is released from rest 1 m below the equilibrium position. In this case, describe the motion as t increases, and sketch the graph of $x(t)$.

6.10. *Hanging Spring.* A 10 kilogram mass suspended from the end of a vertical spring stretches the spring $\frac{49}{90}$ metres. At time $t = 0$, the mass is started in motion from the equilibrium position with an initial velocity of 1 m/s in the upward direction. At the same time, a constant downward force of 360 Newtons is applied to the system. Assume that air resistance is equal to 60 times the instantaneous velocity and that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.

(a) Determine the spring constant.

(b) Show that the equation of motion is

$$\ddot{x} + 6\dot{x} + 18x = 36$$

where x is the displacement of the mass below the equilibrium position at time t. In your answer include a diagram of all forces acting on the mass.

(c) Find the position of the mass at any time.

(d) Identify the transient and steady-state parts of your solution for $x(t)$.

6.11. *Spring Resonance.* A mass of 0.5 kg is attached to a spring suspended from the ceiling of a room. The spring constant is 5 N/m. A constant external downward force of α N is applied to the system, where $\alpha \geq 0$. Assume the damping force is β times the velocity of the mass ($\beta \geq 0$) and that the acceleration due to gravity is $9.8\,\mathrm{m/s^2}$. Let y metres be the displacement of the spring below the equilibrium position at time t seconds.

The system is set into motion by pulling the mass down 0.3 m below its equilibrium position and releasing it from rest.

(a) Determine the extension of the spring from its natural length when the system is in equilibrium.

(b) Using Newton's law of motion, derive the equation of motion for the spring-mass system. Include a diagram of all forces acting on the mass.

(c) What are the initial conditions for the system?

(d) For which values of α and β will the motion of the mass be *critically damped*?

(e) Suppose the applied force is changed from α N to $\sin(\omega t)$ N where $\omega \geq 0$.

  (i) Write down the equation of motion for the spring-mass system.

  (ii) For which values of β and ω will the displacement of the spring grow without bound (*resonance*)?

  (iii) For the case of *resonance*, find the position of the mass at any time.

Sheet 7: Functions of Two Variables

Sketching Functions of Two Variables

7.1. *Level Curves.* Draw some level curves for the following functions f given by:

(a) $f(x, y) = xy$ (b) $f(x, y) = \sqrt{x+y}$

(c) $f(x, y) = y - \cos x$ (d) $f(x, y) = x - y^2$

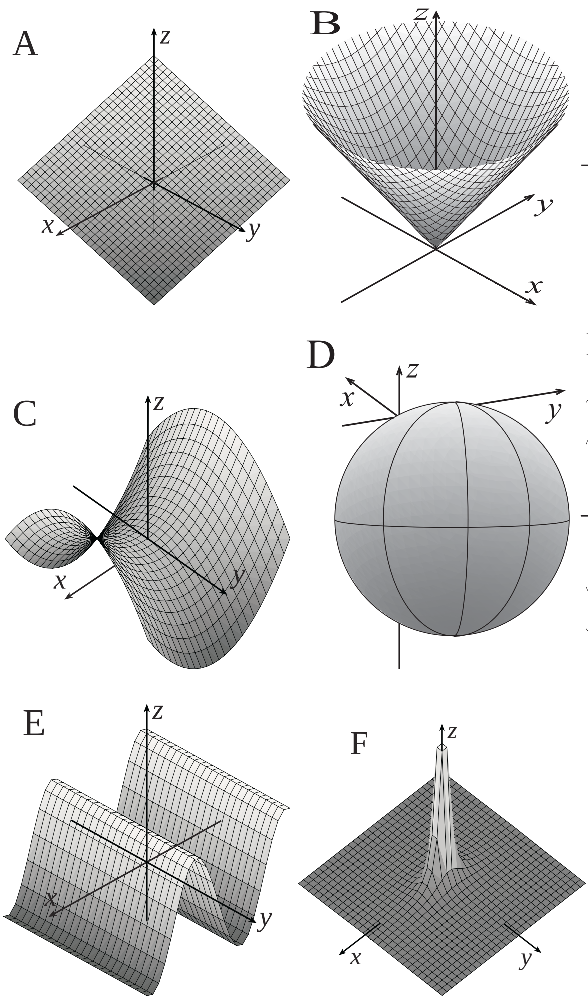
7.2. *Contours and Graphs.* Identify the level sets I - VI and the graph surfaces A - F corresponding to the equations (a) - (f).

(a) $z = \sqrt{x^2 + y^2}$ (b) $z = \sin x$

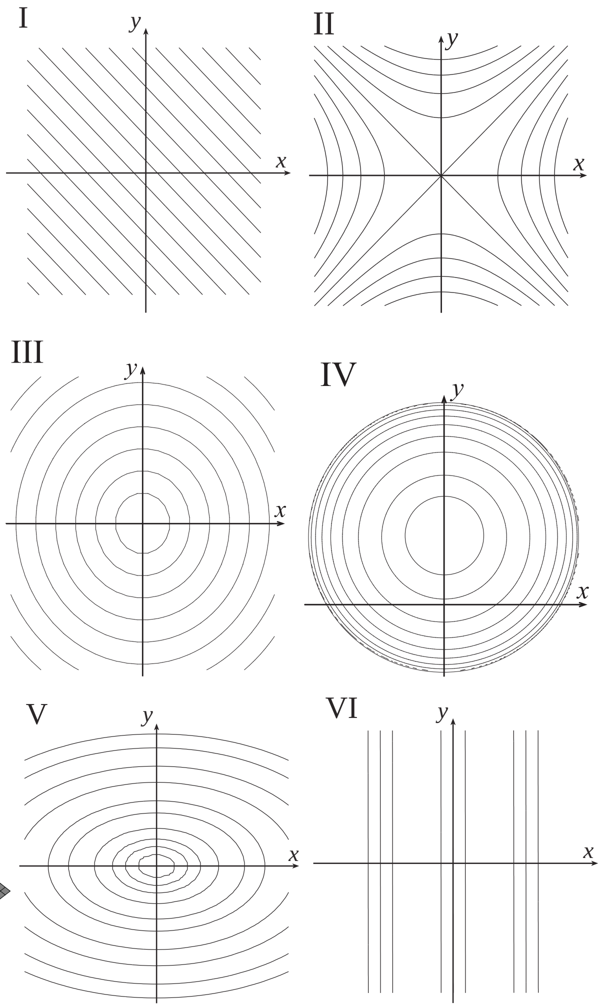
(c) $z = \dfrac{1}{x^2 + 4y^2}$ (d) $z = 1 - 2x - 2y$

(e) $x^2 + (y-1)^2 + (z+2)^2 = 4$ (f) $z = x^2 - y^2$

Graphs:



Level Sets:



7.3. *Surfaces.* By sketching some level curves and cross sections in the x-z and y-z planes, describe and sketch the surface $z = f(x, y)$:

(a) $f(x, y) = \sqrt{1 - x^2 - y^2}$ (b) $f(x, y) = 16y^2 - x^2$

(c) $f(x, y) = 6 - 2x - 3y$ (d) $f(x, y) = 5 - x^2 - y^2$

Limits and Continuity

7.4. *Limits.* Use limit laws and continuity to evaluate the following limits.

(a) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{xy^3}{x+y}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \cos(x^2+2y^2)$

7.5. *Continuity.* State the largest possible domain on which the following functions f given by

(a) $f(x,y) = e^{xy}(xy+8x+y^3)$ (b) $f(x,y) = \sqrt{1-x^2-y^2}$

are defined. Explain why the functions are continuous everywhere on their domains.

Partial Derivatives and Applications

7.6. *Partial Derivatives by First Principal.* Find $f_x(x,y)$ and $f_y(x,y)$ using the limit definition of the partial derivative when

(a) $f(x,y) = 2x^2 - 3y - 4$ (b) $f(x,y) = \dfrac{x}{y}$

7.7. *Partial Derivatives.* Find all first order partial derivatives of the following functions f given by.

(a) $f(x,y) = 3x^2 + 2xy + y^5$ (b) $f(x,y) = \sin x \sin y$

(c) $f(u,v) = \arctan\left(\dfrac{u}{v}\right)$ (d) $f(x,y) = \log\left(x + \sqrt{2+y^2}\right)$

7.8. *Further Partial Derivatives.* Find $f_{xx}(x,y)$, $f_{yy}(x,y)$, $f_{xy}(x,y)$ and $f_{yx}(x,y)$ when

(a) $f(x,y) = e^x \cosh y$ (b) $f(x,y) = \log(4x-5y)$

(c) $f(x,y) = x^2y + x\sqrt{y}$ (d) $f(x,y) = (x^2+y^2)^{\frac{3}{2}}$

7.9. *Solutions of PDEs.*

(a) Verify that $u(x,t) = e^{-\alpha^2k^2t}\sin kx$ is a solution of the heat conduction equation

$$u_t = \alpha^2 u_{xx}$$

(b) If $f(x,y) = xe^{y/x}$, show that

(i) $xf_x + yf_y = f$ (ii) $x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = 0$

7.10. *Tangent Planes.* Find the equation to the tangent plane to the given surface at the point specified:

(a) $z = y^2 - x^2$, $(-4,5,9)$ (b) $z = \log(2x+y)$, $(-1,3,0)$

(c) $z = x\sqrt{y}$, $(1,4,2)$ (d) $z = e^x \cos xy$, $(0,0,1)$

7.11. *Linear Approximations.*

(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = 2x^2 + 3y^2$$

Using a linear approximation, estimate the change in f as (x, y) changes from $(2, 1)$ to $(2.01, 1.02)$. Calculate $f(2.01, 1.02) - f(2, 1)$ exactly. Is your linear estimate for the change in f correct to within one percent?

(b) Write down an approximation for the change in V corresponding to a small change α in x and a small change β in y if $V(x, y) = x^2 y$.

7.12. *Chain Rule.* Use the chain rule to find:

(a) $\dfrac{dz}{dt}$ when $z = x^2 + y^2$, $x = t^3$, $y = 1 + t^2$

(b) $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $z = x^2 - 3x^2y^2$, $x = se^t$, $y = se^{-t}$

(c) $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ at $s = 0$, $t = 1$ when $z = x^2 \tan 2y$, $x = s^2 + t^2$, $y = 2st$

7.13. *Polar Coordinates.* We change variables from Cartesian coordinates (x, y) to polar coordinates (r, θ) by putting $(x, y) = (r\cos\theta, r\sin\theta)$. Let $z = x^2 - y^2 + x^3y$. Find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ at $(r, \theta) = \left(2, \dfrac{\pi}{4}\right)$, using the chain rule.

7.14. *General Chain Rule.* If $z = f(x, y)$, where $x = g(t)$, $y = h(t)$, $g(3) = 2$, $\dot{g}(3) = 5$, $h(3) = 7$, $\dot{h}(3) = -4$, $f_x(2, 7) = 6$, and $f_y(2, 7) = -8$, find $\dfrac{dz}{dt}$ when $t = 3$.

7.15. *Gradient Vectors.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = 5xy^2 - 4x^3y$$

(a) Find the gradient of f.

(b) Evaluate the gradient at the point $(1, 2)$.

(c) Find the rate of change of f at the point $(1, 2)$ in the direction of the vector $\left(\dfrac{5}{13}, \dfrac{12}{13}\right)$.

7.16. *Directional Derivative.* Find the directional derivatives of the following functions at the indicated point in the direction specified.

(a) $f(x, y) = x^3y^2$ at $(-1, 2)$ in the direction parallel to a vector $\dfrac{\pi}{3}$ clockwise from the positive x-axis

(b) $g(x, y) = \sin(xy)$ at $\left(\dfrac{1}{6}, \pi\right)$ in the direction of the unit vector $\left(\dfrac{3}{5}, \dfrac{-4}{5}\right)$

(c) $f(x, y) = \arcsin\left(\dfrac{x}{y}\right)$ at the point $(1, 2)$ towards the point $(3, 0)$

7.17. *Steepest Descent.*

(a) Find the direction in which the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^3 + y^2 - 6xy$$

increases most rapidly at the point $(3, 3)$.

(b) Find the direction in the xy-plane one should travel, starting from the point $(1, 1)$, to obtain the most rapid rate of decrease of

$$f(x, y) = (x + y - 2)^2 + (3x - y - 6)^2$$

(c) In which direction in the xy-plane is the directional derivative of the function $f : \mathbb{R}^2\backslash\{(0, 0)\} \to \mathbb{R}$ given by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

at the point $(1, 1)$ equal to zero?

7.18. *Classification.* Find all the stationary points of the following functions f, and classify these points as maxima, minima or saddle points.

(a) $f(x, y) = x^2 + y^2 + 4x - 6y$

(b) $f(x, y) = x^2 + y^2 + x^2y + 4$

(c) $f(x, y) = -4x^2 + xy - y^2$

(d) $f(x, y) = xye^{-x-y}$

(e) $f(x, y) = x^3 + y^3 - 6xy$

(f) $f(x, y) = \cos x \cos y$

7.19. *Surfboards.* A company produces two types of surfboard, x thousand of type A and y thousand of type B, per year. If the sales revenue R and production/distribution cost C for the year (in millions of dollars) are given by

$$\begin{aligned} R(x, y) &= 2x + 3y \\ C(x, y) &= x^2 - 2xy + 2y^2 + 6x - 9y + 5 \end{aligned}$$

determine how many of each type of surfboard should be made per year in order to maximise the profit. What is the maximum profit?

7.20. *Least Squares.* A satellite television repeater station is to be located at a point P with coordinates (x, y) so that the sum of the squares of the distances from the three towns A, B and C it serves is a minimum. The three towns are located at the positions with coordinates $(0, 0)$, $(2, 6)$ and $(10, 0)$ respectively. Find the coordinates of the repeater station.

Double Integrals

7.21. *Partial Integrals.* Find the partial integrals of

$$f(x, y) = 3x^2 + 2xy + y^5$$

(a) with respect to x

(b) with respect to y

7.22. *Double Integrals.* Evaluate the following double integrals where $R = [0,1] \times [0,1]$.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 (y\cos x + 2)\,\mathrm{d}y\mathrm{d}x$ (b) $\displaystyle\int_{-1}^{0} \int_1^2 (-xe^y)\,\mathrm{d}y\mathrm{d}x$

(c) $\displaystyle\iint_R ye^{xy}\,\mathrm{d}A$ (d) $\displaystyle\iint_R x^2y^2\cos(x^3)\,\mathrm{d}A$

7.23. *Mixed Differentiation and Integration.* Evaluate the following derivatives and integrals

(a) $\displaystyle\int_0^2 \int_0^1 (2x^2 + y^4 - 4xy + 2)\,\mathrm{d}y\mathrm{d}x$ (b) $\displaystyle\frac{\partial}{\partial x}\left(x^2y^2 + \sinh(x^2 + y)\right)$

(c) $\displaystyle\frac{\partial^2}{\partial x \partial y}\left(ye^{xy} + x^2e^y\right)$ (d) $\displaystyle\int_0^{\pi} (x^2y^2 + \cos y)\,\mathrm{d}y$

(e) $\displaystyle\int (2x^4y^3 - 4x + y^2)\,\mathrm{d}x$ (f) $\displaystyle\frac{\partial^2}{\partial y^2}\left(2x^4y^3 - 4x + y^2\right)$

7.24. *Volume.* Using double integrals, find the volume of the solid bounded by the graph of $f(x,y) = 1 + 2x + 3y$, the rectangle $R = [1,2] \times [0,1]$ and the vertical sides of R.

7.25. *Average Value.* The average value of a continuous function f over a rectangle R is defined as

$$f_{average} = \frac{1}{\text{Area of } R} \iint_R f(x,y)\mathrm{d}A$$

Find the average value of $f(x,y) = y\sin(xy)$ over the rectangle $[0,1] \times [0, \frac{\pi}{2}]$.

7.26. *Centre of Mass.* The coordinates $(\bar{x}, \bar{y})$ of the centre of mass of a lamina occupying the region D, and having density $\rho(x,y)$ where $\rho : \mathbb{R}^2 \to [0,\infty)$ is a continuous function, are

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\mathrm{d}A \qquad \bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\mathrm{d}A,$$

where the mass is given by

$$m = \iint_D \rho(x,y)\mathrm{d}A$$

Find the centre of mass of a rectangular lamina with vertices at $(0,0)$, $(1,0)$, $(1,3)$ and $(0,3)$ if the density is $\rho(x,y) = x^2y$.

Answers for sheet 1: Limits, Continuity, Sequences & Series

1.1. (a) -1 (b) -2 (c) Does not exist (d) 2 (e) 0 (f) Does not exist

1.2. (a) 59 (b) $\frac{1}{8}$

1.3. (a) 3 (b) 5 (c) $2x$ (d) $\frac{1}{5}$ (e) $\frac{1}{6}$ (f) $-\frac{1}{16}$

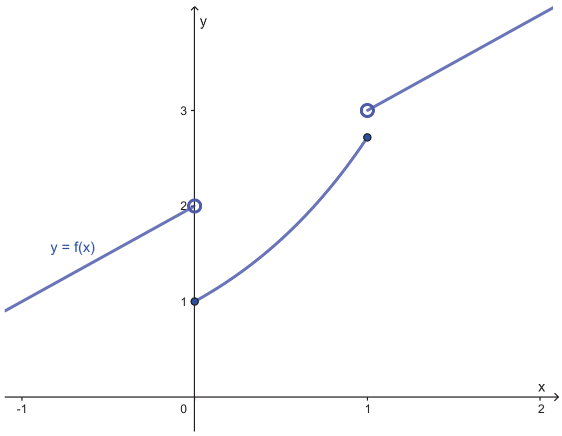
1.4. (a) 3 (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) Does not exist

1.5. (a) 0 (b) 0 (c) 0

1.6. (a) $\{x \mid x \neq -3, -2\}$ (b) $\left[\frac{1}{2}, \infty\right)$ (c) $(-\infty, -1) \cup (1, \infty)$

1.7. (a) $\frac{7}{3}$ (b) 0 (c) 1

1.8. 0, 1



1.9. $\frac{2}{3}$

1.10. (a) 1 (b) $\frac{p}{q}$ (c) 0 (d) $-\frac{1}{2}$

1.11. (a) Does not exist but approaches ∞ (b) 0 (c) 6 (d) 0 (e) 4 (f) 0

1.12. (a) Does not exist (b) $-\frac{1}{3}$ (c) $\frac{11}{3}$ (d) $\frac{7}{3}$ (e) 0 (f) $-\frac{2}{\pi}$

1.13.	(a) $2,\ 2\sqrt{2},\ 2\sqrt{3},\ 4$; divergent to ∞.	(b) $0, -1, 0, 1$; divergent as oscillating.	
	(c) $-\frac{1}{2}, \frac{2}{5}, -\frac{3}{10}, \frac{4}{17}$; convergent to 0.		
1.14.	(a) e^3	(b) divergent to ∞	(c) 0
1.15.	(a) 0	(b) -1	(c) -2
1.16.	(a) 0	(b) 3	(c) 0
1.17.	(a) $\sqrt{e}$	(b) 1	(c) $\log\left(\frac{2}{3}\right)$
1.18.	(a) divergent to ∞	(b) 0	(c) π
1.19.	(a) divergent to ∞	(b) -1	(c) 1
	(d) $\frac{1}{2}$	(e) $e^{5/2}$	(f) 0
1.20.	(a) 12	(b) $\frac{3}{2}$	(c) diverges to ∞
1.21.	(a) Proof required.	(b) Proof required.	(c) Proof required.
1.22.	(a) divergent	(b) convergent	(c) divergent
1.23.	(a) convergent	(b) divergent	(c) convergent
1.24.	(a) convergent	(b) divergent	(c) convergent
	(d) divergent	(e) convergent	(f) divergent
1.25.	(a) convergent	(b) divergent	(c) divergent
	(d) convergent	(e) convergent	(f) convergent

Answers for sheet 2: Hyperbolic Functions

2.1. (a) $\frac{4}{3}$ (b) $\frac{5}{4}$ (c) $\frac{312}{313}$

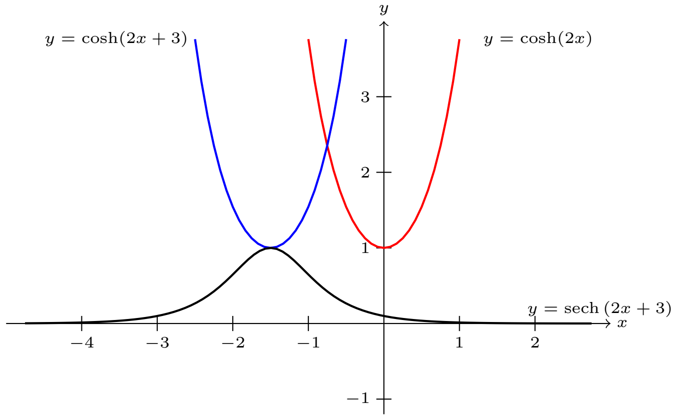
2.2. (a) $\frac{1}{2}\left(x - \frac{1}{x}\right)$ (b) $\frac{1}{2}\left(x^{-3} + x^3\right)$ (c) $\frac{x^2 - x^{-2}}{x^2 + x^{-2}}$

2.3. (a) $\sinh x = \pm\frac{3}{4}$, $\tanh x = \pm\frac{3}{5}$

(b) $\cosh x = \sqrt{29}/5$, $\tanh x = -2/\sqrt{29}$, $\coth x = -\sqrt{29}/2$
$\operatorname{sech} x = 5/\sqrt{29}$, $\operatorname{cosech} x = -5/2$

2.4. Requires proof.

2.5.



2.6. (a) $\frac{1}{16}\left(\sinh 5x - 5\sinh 3x + 10\sinh x\right)$

(b) $\frac{1}{32}\left(\cosh 6x + 6\cosh 4x + 15\cosh 2x + 10\right)$

2.7. Proof required.

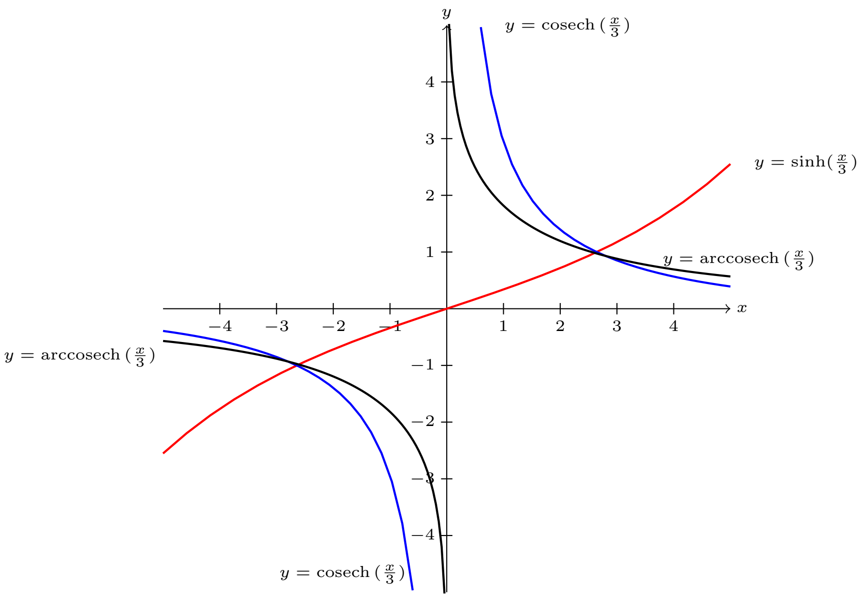
2.8. (a) $\dfrac{\sinh(\sqrt{x})}{2\sqrt{x}}$, for $x > 0$

(b) $\dfrac{\sinh x}{2\sqrt{\cosh x}}$, for all x

(c) $3\cos 3x \operatorname{sech}^2(\sin 3x)$, for all x

(d) $\sinh(1/x) - (1/x)\cosh(1/x)$, for $x \neq 0$

2.9.



2.10. (a) $\sqrt{x^2 - 1}$ (b) $x^2/(1 - x^2)$ (c) $\sqrt{x^2 - 1}$

2.11. (a) Proof required.

(b) $\dfrac{1}{\sqrt{x^2-1}}$

(c) (i) $\log 2$ (ii) 0

2.12. Requires proof.

2.13. (a) $3x^2 \operatorname{arcsinh}(e^x) + \dfrac{x^3 e^x}{\sqrt{1+e^{2x}}},\ x \in \mathbb{R}$

(b) $\dfrac{1}{2\sqrt{x}\sqrt{x-1}}$, $x > 1$

(c) $\dfrac{4}{(\operatorname{arccosh} 4x)\sqrt{16x^2-1}}$, $x > \frac{1}{4}$

(d) $\dfrac{-1}{(1-x^2)\operatorname{arctanh}^2 x}$, $-1 < x < 0 \cup 0 < x < 1$

Answers for sheet 3: Complex Numbers

3.1. (a) $1+i$ (b) $12-7i$
(c) $-21/17$ (d) $-1/17$

3.2. (a) (i) 1 (ii) 27/10
(b) (i) $2\pi/3$ (ii) $-3\pi/4$

3.3. (a) $-1 = e^{i\pi}$ (b) $5\sqrt{2}e^{3\pi i/4}$
(c) $2e^{-i\pi/2}$ (d) $\sqrt{29}e^{i\alpha}$, where $\alpha = \arctan(5/2)$

3.4. (a) -1024 (b) $-512\sqrt{3}+512i$
(c) $-1/64$ (d) $\frac{1}{2^{7/2}}e^{-7\pi i/12}$

3.5. (a) $\frac{1}{32}(10-15\cos 2\theta+6\cos 4\theta-\cos 6\theta)$ (b) $\frac{1}{16}(2\sin\theta+\sin 3\theta-\sin 5\theta)$

3.6. Requires proof.

3.7. (a) $(9i-46)e^{(2+3i)t}$ (b) $-512ie^{(1-i)t}$

3.8. (a) $2^{128}e^{-t}(\cos t-\sin t)$ (b) $32^{288}e^{4t}\cos 4t$

3.9. (a) $\frac{e^x}{10}(\cos 3x+3\sin 3x)+C$ (b) $\frac{e^{-2x}}{125}(-11\cos 11x-2\sin 11x)+C$
(c) $\frac{e^{5t}}{74}(5\cos 7t+7\sin 7t)+C$

Answers for sheet 4: Integral Calculus

4.1.
(a) -10
(b) $\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)+C$
(c) $\frac{\pi}{4}$
(d) $\log|x-2|+2\log|x+4|+C$
(e) $\log|\log x|+C$
(f) $\log(e^x+1)+C$
(g) $4\log|x+5|+\frac{3}{x+5}+C$
(h) $\frac{1}{7}\sin^7 x-\frac{1}{9}\sin^9 x+C$

4.2.
(a) $-\frac{1}{3}\coth^3 x+C$
(b) $\frac{1}{12}\exp(4\cosh 3x)+C$
(c) $\operatorname{arccosh}^2 x+C$
(d) $\frac{1}{k}\log|2+\tanh kx|+C$

4.3.
(a) $\frac{1}{4}\left(\operatorname{arcsinh} 2x+2x\sqrt{1+4x^2}\right)+C$
(b) $\frac{x}{2}\sqrt{4-x^2}+2\arcsin\left(\frac{x}{2}\right)+C$
(c) $\dfrac{-x}{\sqrt{x^2-1}}+C$

4.4.
(a) $\frac{1}{7}\sinh^7 x+C$
(b) $\frac{1}{2}x+\frac{1}{12}\sinh 6x+C$
(c) $\frac{1}{3}\sinh^3 x+\frac{1}{5}\sinh^5 x+C$
(d) $\frac{1}{12}\cosh^3 4x-\frac{1}{4}\cosh 4x+C$
(e) $\frac{3}{8}x+\frac{1}{4}\sinh 2x+\frac{1}{32}\sinh 4x+C$

4.5.
(a) $\frac{1}{2}\tanh^2 x+C$
(b) $x-\frac{1}{3}\tanh 3x+C$

4.6.
(a) $\frac{1}{5}\log|x+2|-\frac{1}{10}\log(x^2+1)+\frac{2}{5}\arctan x+C$
(b) $4\log|x+1|-5\log|x-2|+\log|x-3|+C$
(c) $2\log(x^2+1)+3\arctan x-2\log(x^2+2)-\frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+C$
(d) $-2\log|x+1|+\log|x+3|+\log|x-2|+C$
(e) $\frac{1}{2}x^2+x+2\log|x|+2\log|x-1|+C$
(f) $\frac{3}{x+3}+\log|x|-\log|x+3|+C$

4.7.
(a) $\frac{1}{3}x\sin 3x+\frac{1}{9}\cos 3x+C$
(b) $x\arcsin x+\sqrt{1-x^2}+C$
(c) $\frac{\pi}{4}-\frac{1}{2}$
(d) $x^2\sinh x-2x\cosh x+2\sinh x+C$
(e) $x\operatorname{arcsinh} x-\sqrt{x^2+1}+C$
(f) $\frac{1}{8}x^8\log x-\frac{1}{64}x^8+C$

4.8.
(a) $\frac{e^x}{10}(\cos 3x+3\sin 3x)+C$
(b) $\frac{e^{-2x}}{125}(-11\cos 11x-2\sin 11x)+C$
(c) $\frac{3}{8}\cos x\sin 3x-\frac{1}{8}\sin x\cos 3x+C$
(d) $\frac{1}{15}\cos x\sin 4x-\frac{4}{15}\sin x\cos 4x+C$

4.9.
(a) $\dfrac{1}{4}\sinh^4 x+\dfrac{1}{6}\sinh^6 x+C=\dfrac{1}{6}\cosh^6 x-\dfrac{1}{4}\cosh^4 x+D=\dfrac{1}{48}\cosh^3 2x-\dfrac{1}{16}\cosh 2x+E$
(b) $x^{3/2}\left(\dfrac{2}{3}\log x-\dfrac{4}{9}\right)+C$
(c) $x+4\log|x|-2\log|x+1|+C$
(d) $\dfrac{1}{3}\log|x^3-6x-2|+C$
(e) $\dfrac{1}{2}\arcsin\left(\dfrac{2x}{3}\right)+C$
(f) 4

4.10.
(a) $\dfrac{x}{2}\sqrt{x^2-16}-8\operatorname{arccosh}\left(\dfrac{x}{4}\right)+C$
(b) $\dfrac{x}{3}\cosh(3x)-\dfrac{\sinh(3x)}{9}+C$
(c) $-\log|x-1|+\log|x+2|-\dfrac{3}{x-1}+C$
(d) $\dfrac{1}{9}\sinh^9 x+\dfrac{2}{7}\sinh^7 x+\dfrac{1}{5}\sinh^5 x+C$
(e) $\frac{5}{24}(\cosh 2x+55)^{6/5}+C$
(f) $\dfrac{e^{2x}}{13}(-3\cos 3x+2\sin 3x)+C$

4.11.
(a) $c\sinh(1/c)$
(b) $\frac{21}{2}+\frac{1}{4}\log 5-\frac{1}{4}\log 2$

Answers for sheet 5: First Order Differential Equations

5.1. Requires verification.

5.2. (a) $y(x) = \frac{1}{2}\log(2e^x + C)$ (b) $\frac{1}{3}x^3 - e^{-x} = \frac{3}{2}t^2 + \frac{1}{2}e^{2t} + C$

(c) $y(t) = 3\sin\left(\frac{1}{8}\sinh^6 4t + C\right)$ (d) $\log|y+1| - \frac{2}{3}\log|3y+1| = \frac{1}{3}\sin 3x + C$

5.3. (a) $y(x) = \arctan\left(\frac{5}{3}(x^3 - 1)\right)$ (b) $\frac{1}{2}y - \frac{1}{4}\sin 2y = \frac{1}{3}\log|x^3+3x+7| - \frac{1}{3}\log 3$

(c) $y(x) = \left(3 - 2(1+x^2)^{\frac{1}{2}}\right)^{-\frac{1}{2}}$

5.4. (a) $y(x) = \frac{1}{6}x^2 - \frac{1}{5}x + \frac{1}{4} + Cx^{-4}$

(b) $x(t) = -\frac{5}{6} + Ce^{3t^2}$

(c) $y(x) = x\sinh x + \frac{2}{3}x\sinh^3 x + \frac{1}{5}x\sinh^5 x + Cx$

(d) $x(t) = \frac{1}{5}t^2 + \frac{1}{3}\log t - \frac{1}{9} + \frac{C}{t^3}$

5.5. (a) $y(x) = \frac{1}{x^2}(\sin x - 1)$ (b) $y(x) = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{2x}$

(c) $y(x) = \dfrac{e^{\sin x} + \pi - 1}{x}$

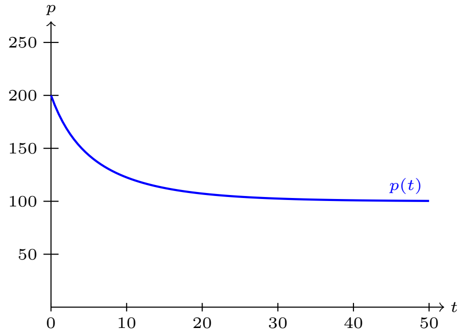
5.6. (a) $x(t) = te^{At}$ (b) $y^2 = \dfrac{5x}{2 + Ax^5}$

(c) $y(x) = 2\arctan(x + C) - x$

5.7. (a) $x(t) = x_0\left(\frac{3}{2}\right)^t$ (b) 2.71 hours

5.8.

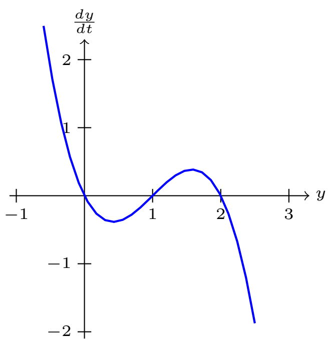
(a) $p(t) = \dfrac{200}{2 - e^{-t/10}}$ (b) No (c) No



5.9.

(a) $y = 0,\ y = 2,\ y = 1$

(b)



(c) Increasing for $y < 0$ and $1 < y < 2$, decreasing for $0 < y < 1$ and $y > 2$

(d) (i) 0 (ii) 2

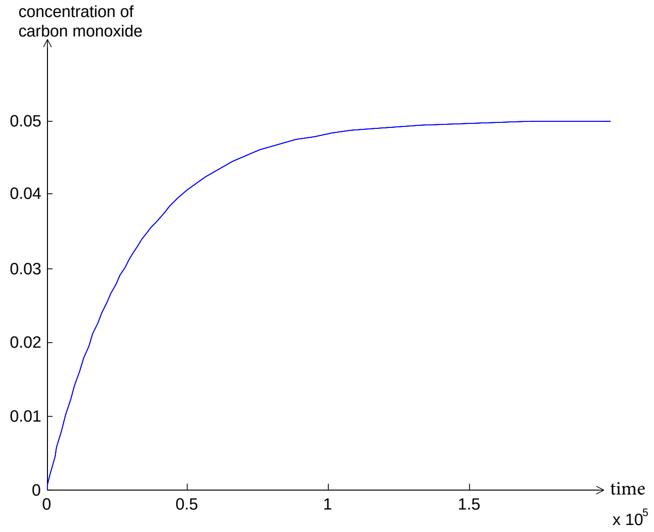
(e) $y(0) > 1$

5.10. (a) $P = 50\left(1 \pm \sqrt{1 - \frac{h}{25}}\right)$, $0 < h \le 25$ (b) $h > 25$

5.11. (a) Proof required

(b) $x(t) = 3(1 - e^{-t/30000})$, Transient solution: $-3e^{-t/30000}$, Steady state solution: 3

(c) concentration goes to $1/20$ or 5%



(d) 10 hours 6 minutes

5.12. (b) $x(t) = 5(4000 - t) - \dfrac{1}{1600}(4000 - t)^2$

5.13. (b) $x(t) = \dfrac{2}{5}(10 + t) - \dfrac{40000}{(10 + t)^4}$ (c) 7.75 km^3

Answers for sheet 6: Second Order Differential Equations

6.1. Requires verification.

6.2. (a) $y(x) = \mathrm{A}e^{3x} + \mathrm{B}e^{-3x}$ (b) $y(x) = (\mathrm{A} + \mathrm{B}x)e^{6x}$
(c) $y(x) = \mathrm{A}\cos 7x + \mathrm{B}\sin 7x$ (d) $y(x) = e^{-4x}(\mathrm{A}\cos x + \mathrm{B}\sin x)$
(e) $y(x) = \mathrm{A} + \mathrm{B}e^{7x}$

6.3.
(a) $|b| > 4$ (b) $b = \pm 4$ (c) $|b| < 4$

6.4.
(a) $y(x) = e^{-3x} + 2e^{x}$ (b) $y(x) = \dfrac{1}{3}e^{-3x} + 2xe^{-3x}$

(c) $y(x) = \cos 2x + \dfrac{1}{2}\sin 2x$

6.5. (a) $y(x) = e^{-2x}(A\cos x + B\sin x)$
(b) $y(x) = e^{-2x}(A\cos x + B\sin x) - 1$
(c) $y(x) = e^{-2x}(A\cos x + B\sin x) + \dfrac{2}{5}x - \dfrac{8}{25}$
(d) $y(x) = e^{-2x}(A\cos x + B\sin x) + \dfrac{1}{26}e^{3x}$
(e) $y(x) = e^{-2x}(A\cos x + B\sin x) + \dfrac{1}{8}(\sin x - \cos x)$

The general solution is $y(x) = e^{-2x}(A\cos x + B\sin x) + 2x + \frac{2}{5} + \frac{1}{13}e^{3x} + \sin x - \cos x$

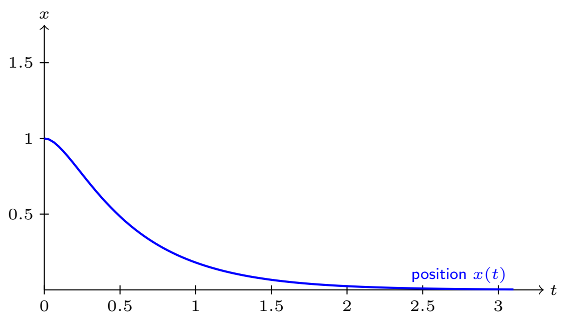
6.6. (a) $y(x) = e^{4x}(\mathrm{A}\cos 6x + B\sin 6x) + 2e^{5x}$ (b) $y(x) = (A + Bx)e^{-x} + 3x^2 - 12x + 20$
(c) $y(x) = Ae^{t} + Be^{-2t} - t - \frac{1}{2}$ (d) $y(x) = Ae^{3x} + Be^{-x} + \frac{1}{30}\cos 3x - \frac{2}{5}\sin 3x$

6.7. (a) $x(t) = Ae^{-t} + Be^{-4t} - \dfrac{1}{3}te^{-4t}$ (b) $y(x) = (A - 12x)\cos 3x + (B + 2x)\sin 3x$
(c) $y(x) = (A + Bx + \dfrac{1}{2}x^2)e^{-3x}$

6.8. (a) $y(x) = e^{x} - 7\cos 6x - 2\sin 6x$ (b) $y(x) = (1 - \frac{1}{3}x)e^{2x} + 3e^{5x}$
(c) $y(x) = 6\cos 4x + (1 + \frac{1}{8}x)\sin 4x$

6.9. (b) $x(t) = Ae^{-2t} + Be^{-8t}$
(c) $x(t) = \frac{4}{3}e^{-2t} - \frac{1}{3}e^{-8t}$

motion is strongly- or over-damped, smooth and non-oscillatory. It dies away to zero with time, and the derivative is zero at the origin and then negative, so that the decay is monotone.



6.10. (a) 180 N/m

(c) $x(t) = -2e^{-3t}\cos 3t - \dfrac{7}{3}e^{-3t}\sin 3t + 2$

(d) Transient solution: $-2e^{-3t}\cos 3t - \dfrac{7}{3}e^{-3t}\sin 3t$. Steady state solution: 2

6.11. (a) 0.98 metres

(b) $\dfrac{d^2y}{dt^2} + 2\beta\dfrac{dy}{dt} + 10y = 2\alpha$

(c) $y(0) = 0.3,\ \dfrac{dy}{dt}(0) = 0$

(d) $\beta = \sqrt{10}$ and $\alpha = 0$

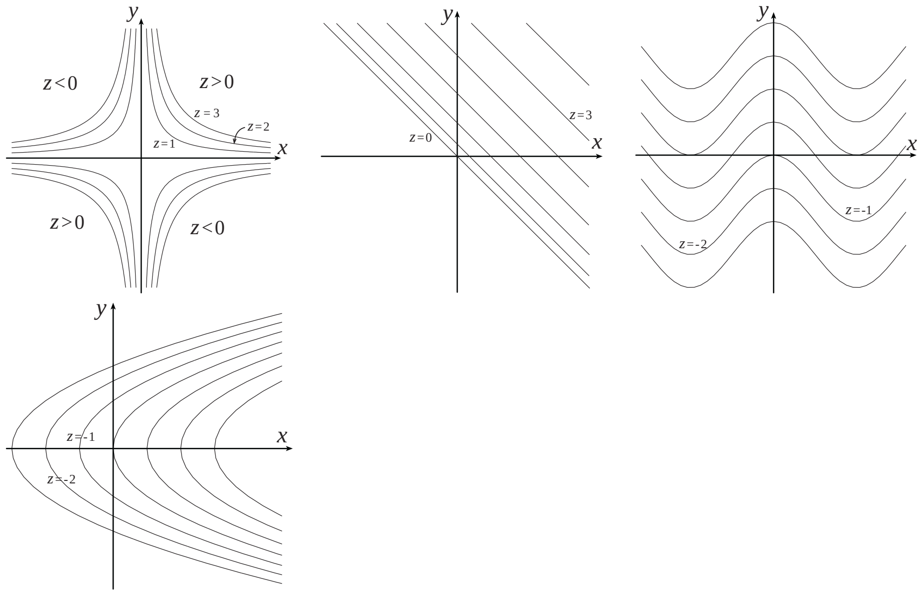
(e) (i) $\dfrac{d^2y}{dt^2} + 2\beta\dfrac{dy}{dt} + 10y = 2\sin(\omega t)$

(ii) $\beta = 0$ and $\omega = \sqrt{10}$

(iii) $\dfrac{3}{10}\cos(\sqrt{10}t) + \dfrac{1}{10}\sin(\sqrt{10}t) - \dfrac{t\sqrt{10}}{10}\cos(\sqrt{10}t)$

Answers for sheet 7: Functions of Two Variables

7.1.



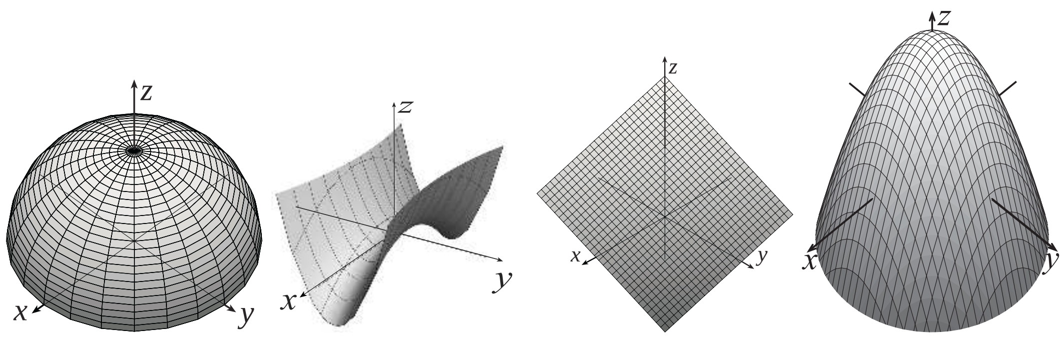
7.2. (a)-B-III (b)-E-VI (c)-F-V
(d)-A-I (e)-D-IV (f)-C-II

7.3. (a) upper hemisphere radius 1, centre at (0, 0, 0)

(b) saddle

(c) plane with intercepts (3, 0, 0), (0, 2, 0), (0, 0, 6)

(d) upside down parabolic bowl with intercept at (0, 0, 5)



7.4. (a) $\frac{8}{3}$ (b) 1

7.5. (a) all of the xy plane (b) the closed unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$

7.6. (a) $f_x(x, y) = 4x,\ f_y(x, y) = -3$

(b) $f_x(x, y) = \dfrac{1}{y},\ f_y(x, y) = -\dfrac{x}{y^2}$

7.7. (a) $f_x(x, y) = 6x + 2y,\ f_y(x, y) = 2x + 5y^4$

(b) $f_x(x, y) = \cos x \sin y,\ f_y(x, y) = \sin x \cos y$

(c) $f_u(u, v) = v/(u^2 + v^2),\ f_v(u, v) = -u/(u^2 + v^2)$

(d) $f_x(x, y) = 1/(x + \sqrt{x^2 + y^2}),\ f_y(x, y) = y/(x^2 + y^2 + x\sqrt{x^2 + y^2})$

7.8. (a) $f_{xx}(x,y) = e^x \cosh y,\ f_{yy}(x,y) = e^x \cosh y,\ f_{xy}(x,y) = f_{yx}(x,y) = e^x \sinh y$

(b) $f_{xx}(x,y) = -\dfrac{16}{(4x-5y)^2},\ f_{yy}(x,y) = -\dfrac{25}{(4x-5y)^2}$
$f_{xy}(x,y) = f_{yx}(x,y) = \dfrac{20}{(4x-5y)^2}$

(c) $f_{xx}(x,y) = 2y,\ f_{yy}(x,y) = -\frac{x}{4}y^{-\frac{3}{2}},\ f_{xy}(x,y) = f_{yx}(x,y) = 2x + 1/(2\sqrt{y})$

(d) $f_{xx}(x,y) = \dfrac{3(2x^2+y^2)}{\sqrt{x^2+y^2}},\ f_{yy}(x,y) = \dfrac{3(x^2+2y^2)}{\sqrt{x^2+y^2}},\ f_{xy}(x,y) = f_{yx}(x,y) = \dfrac{3xy}{\sqrt{x^2+y^2}}$

7.9. Verification required.

7.10. (a) $z = 8x + 10y - 9$ (b) $z = 2x + y - 1$

(c) $z = 2x + \frac{1}{4}y - 1$ (d) $z = x + 1$

7.11. (a) 0.2, 0.2014, YES (b) $2xy\alpha + x^2\beta$

7.12. (a) $dz/dt = 6t^5 + 4t^3 + 4t$

(b) $\partial z/\partial s = 2se^{2t} - 12s^3$, $\partial z/\partial t = 2s^2e^{2t}$

(c) $\partial z/\partial s = 4$, $\partial z/\partial t = 0$

7.13. $8, -16$

7.14. 62

7.15. (a) $(5y^2 - 12x^2y, 10xy - 4x^3)$ (b) $(-4, 16)$

(c) $172/13$

7.16. (a) $6 + 2\sqrt{3}$ (b) $\pi 3\sqrt{3}/10 - \sqrt{3}/15$ (c) $\frac{3}{2\sqrt{6}} = \frac{\sqrt{6}}{4}$

7.17. (a) $\frac{1}{5}(3,-4)$ (b) $\frac{1}{\sqrt{10}}(3,-1)$

(c) $\frac{1}{\sqrt{2}}(1,1),\ \frac{1}{\sqrt{2}}(-1,-1)$

7.18. (a) local minimum at $(-2,3)$

(b) local minimum at $(0,0)$ and two saddle points at $(\pm\sqrt{2},-1)$

(c) $(0,0)$ local maximum

(d) $(1,1)$ local maximum, $(0,0)$ saddle

(e) $(0,0)$ saddle; $(2,2)$ local minimum

(f) $(n\pi + \frac{\pi}{2}, m\pi + \frac{\pi}{2})$ are saddles $(n\pi, m\pi)$ are local minima if $n+m$ is odd, local maxima if even; n, m integers

7.19. 2000 of type A and 4000 of type B for maximum profit of 15 million dollars.

7.20. P with coordinates$(4,2)$ gives a minimum

7.21. (a) $x^3 + x^2y + xy^5 + g(y)$ where g is an arbitrary function

(b) $3x^2y + xy^2 + \frac{1}{6}y^6 + h(x)$ where h is an arbitrary function

7.22. (a) $\frac{1}{2} + \pi$ (b) $\frac{1}{2}(e^2 - e)$
(c) $e - 2$ (d) $\frac{1}{9}\sin 1$

7.23. (a) $\frac{86}{15}$ (b) $2xy^2 + 2x\cosh(x^2+y)$
(c) $2ye^{xy} + 2xe^y + xy^2e^{xy}$ (d) $\frac{\pi^3}{3}x^2$
(e) $\frac{2}{5}x^5y^3 - 2x^2 + xy^2 + c(y)$ (f) $12x^4y + 2$

7.24. $\frac{11}{2}$

7.25. $1 - \frac{2}{\pi}$

7.26. $(\bar{x}, \bar{y}) = \left(\frac{3}{4}, 2\right)$ [Note $m = 1.5$]