MAST30025_2021_SM1 MAST30025 assignment 3

Michael Le

TOTAL POINTS

47 / 48

QUESTION 1

7 pts

1.1 2 / 2

1.2 2 / 2

Why does it equal I?

1.3 3 / 3

QUESTION 2

11 pts

2.1 2 / 2

2.2 2 / 2

2.3 2 / 2

2.4 2 / 2

2.5 3 / 3

QUESTION 3

3 5 / 5

QUESTION 4

18 pts

4.1 3 / 3

4.2 2 / 3

4.3 3 / 3

4.4 3 / 3

4.5 2 / 2

4.6 2 / 2

4.7 2 / 2

QUESTION 5

7 pts

5.1 5 / 5

5.2 2 / 2

MAST30025 Assignment 3 2021 Michael Le LaTex

Michael Le (99811)

May 28, 2021

***Question* 1 :**

Part a:

r(A) = r(AA^CA) $\leq$ r(A^CA) $\leq$ r(A)

From Definition 6.1

Part b:

$(A^C A)^2 =$ (A^CA)(A^CA) $= A^C AA^C A = A^C A$

NOTE: A^CA = I

Part c: Show that this matrix is unique (invariant to the choice of conditional inverse)

A$(A^T A)_1^C$ A^T = A$(A^T A)_2^C$ A^T A$(A^T A)_1^C$ A^T

= A$(A^T A)_2^C$ $[A^T A(A^T A)_1^C A^T]^T$

= A$(A^T A)_2^C A^T$

1.1 **2 / 2**

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Part a:

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From Definition 6.1

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$(A^C A)^2 = (A^C\text{A})(A^C\text{A}) = A^C A A^C A = A^C A$

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Part c: Show that this matrix is unique (invariant to the choice of conditional inverse)

A$(A^T A)_1^C$ A^T = A$(A^T A)_2^C$ A^T A$(A^T A)_1^C$ A^T

= A$(A^T A)_2^C$ $[A^T A (A^T A)_1^C A^T]^T$

= A$(A^T A)_2^C A^T$

1.2 2 / 2

Why does it equal I?

MAST30025 Assignment 3 2021 Michael Le LaTex

Michael Le (99811)

May 28, 2021

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Part a:

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A$(A^T A)_1^C\ A^T$ = A$(A^T A)_2^C\ A^T$ A$(A^T A)_1^C\ A^T$

= A$(A^T A)_2^C\ [A^T A(A^T A)_1^C A^T]^T$

= A$(A^T A)_2^C A^T$

1.3 3 / 3

***Question* 2 :**

Part a:

```
library(MASS)
library(Matrix)
n =12
y = c(22,23,24,22,26,16,18,19,28,27,29,29)
X = matrix(c(rep(1,17),rep(0,12),rep(1,3),rep(0,12),rep(1,4)),12,4)
XtX = t(X)%*%X # From here
#Applying therem 6.2 from the less thank full rank module.
#We first need to compute the rank.
r = rankMatrix(XtX)[1]
#Now we need to find a 3 x 3 minor.
Ac = matrix(0,4,4)
Ac[2:4,2:4] = t(solve(XtX[2:4,2:4]))
Ac = t(Ac)

#Ac is our conditional inverse
Ac
##      [,1] [,2]      [,3] [,4]
## [1,]    0  0.0 0.0000000 0.00
## [2,]    0  0.2 0.0000000 0.00
## [3,]    0  0.0 0.3333333 0.00
## [4,]    0  0.0 0.0000000 0.25
```

2.1 2 / 2

Part b:

```
library(MASS)
library(Matrix)
b = ginv(t(X)%*%X)%*%t(X)%*%y
e = y - X%*%b
SSRes = sum(e^2)
s2 = SSRes/(n-r)
s2
## [1] 2.068519
#[1] 2.068519
```

Different Method. No matter what Conditional Inverse you use, s^2 value does not change.

```
b = Ac%*%t(X)%*%y
c = y - X%*%b
SSRes = sum(c^2)
s2 = SSRes/(n-r)
s2
## [1] 2.068519
#[1] 2.068519
```

Part c:

```
#Thoerem 6.10
tt = c(1,2,1,0)
tt%*%Ac%*%t(X)%*%X
##      [,1] [,2] [,3] [,4]
## [1,]    3    2    1    0
#It would have been estimable if mew (the intercept) + (any treatment or contast)
#In this case the contasts add up to become non zero, Hence
#It is not estimable
```

2.2 2 / 2

Different Method. No matter what Conditional Inverse you use, s^2 value does not change.

```
b = Ac%*%t(X)%*%y
c = y - X%*%b
SSRes = sum(c^2)
s2 = SSRes/(n-r)
s2
## [1] 2.068519
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```

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## [1,]    3    2    1    0
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#In this case the contasts add up to become non zero, Hence
#It is not estimable
```

Using a different conditional inverse using ginv function.

```
library(MASS)
library(Matrix)
tt = c(1,2,1,0)
XtXc = ginv(t(X)%*%X)
tt%*%XtXc%*%t(X)%*%X
##      [,1] [,2] [,3] [,4]
## [1,]  1.5  1.5  0.5 -0.5
#It would have been estimable if mew (the intercept) + (any treatment or contast)
#In this case the contasts add up to become non zero, Hence
#It is not estimable
```

Part d:

```
#Using R the generalised inverse
tt = c(1,0,1,0)
ta = qt(0.950,n-r)
halfwidth = ta*sqrt(s2*t(tt)%*%XtXc%*%tt)
c(tt%*%b-halfwidth,tt%*%b + halfwidth)
## [1] 16.14451 19.18882
#Using the Conditional Inverse from Theoerm 6.2
tt = c(1,0,1,0)
ta = qt(0.950,n-r)
halfwidth = ta*sqrt(s2*t(tt)%*%Ac%*%tt)
c(tt%*%b-halfwidth,tt%*%b + halfwidth)
## [1] 16.14451 19.18882
```

2.3 2 / 2

Using a different conditional inverse using ginv function.

```
library(MASS)
library(Matrix)
tt = c(1,2,1,0)
XtXc = ginv(t(X)%*%X)
tt%*%XtXc%*%t(X)%*%X
##      [,1] [,2] [,3] [,4]
## [1,]  1.5  1.5  0.5 -0.5
#It would have been estimable if mew (the intercept) + (any treatment or contast)
#In this case the contasts add up to become non zero, Hence
#It is not estimable
```

Part d:

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tt = c(1,0,1,0)
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tt = c(1,0,1,0)
ta = qt(0.950,n-r)
halfwidth = ta*sqrt(s2*t(tt)%*%Ac%*%tt)
c(tt%*%b-halfwidth,tt%*%b + halfwidth)
## [1] 16.14451 19.18882
```

2.4 2 / 2

Part e:

```
#Creating our hypothesis
library(MASS)
library(Matrix)
C = matrix(c(0,1,0,-1),1,4)
SS = t(C%*%b)%*%solve(C%*%Ac%*%t(C))%*%C%*%b

#Fstat
Fstat = (SS/1)/s2
Fstat
##          [,1]
## [1,] 25.27037
#p value
pf(Fstat,1,n-r,lower=F)
##              [,1]
## [1,] 0.0007123037
```

Reject the null hypothesis, p value $= 0.0007123037 < \alpha = 0.05$

2.5 3 / 3

Question 3 :

$$X^TX = \left[\begin{array}{c|c} X_1^TX_1 & X_1^TX_2 \\ \hline X_2^TX_1 & X_2^TX_2 \end{array}\right]$$

Show that $X^TXz=t$. where z has a solution.,
we need to also show,

$$r([X^TX|t]) = r(X^TX) = r(X) = r(X_1) + r(X_2).$$

We first observe that since $t_1^T\beta_1$ is estimable in the first model, z_1 has a solution to $X_1^TX_1z=t_1$, Similary for z_2 to the system $X_2^TX_2z_2=t_2$, Assume X_2 is at full rank. Thus,

$$r([X^TX|t]) \geq r(X^TX).$$

Show the reverse inequality,

$$r([X^TX|t]) = r\left(\left[\begin{array}{cc|c} X_1^TX_1 & X_1^TX_2 & t_1 \\ X_2^TX_1 & X_2^TX_2 & t_2 \end{array}\right]\right)$$

$$= r\left(\left[\begin{array}{cc|c} X_1^TX_1 & X_1^TX_2 & X_1^TX_1z_1 \\ X_2^TX_1 & X_2^TX_2 & X_2^TX_2z_2 \end{array}\right]\right)$$

$$= r\left(\begin{bmatrix} X_1^T & 0 \\ 0 & X_2^T \end{bmatrix}\left[\begin{array}{cc|c} X_1 & X_2 & X_1z_1 \\ X_1 & X_2 & X_2z_2 \end{array}\right]\right)$$

$$\leq r\left(\begin{bmatrix} X_1^T & 0 \\ 0 & X_2^T \end{bmatrix}\right) = r(X_1) + r(X_2).$$

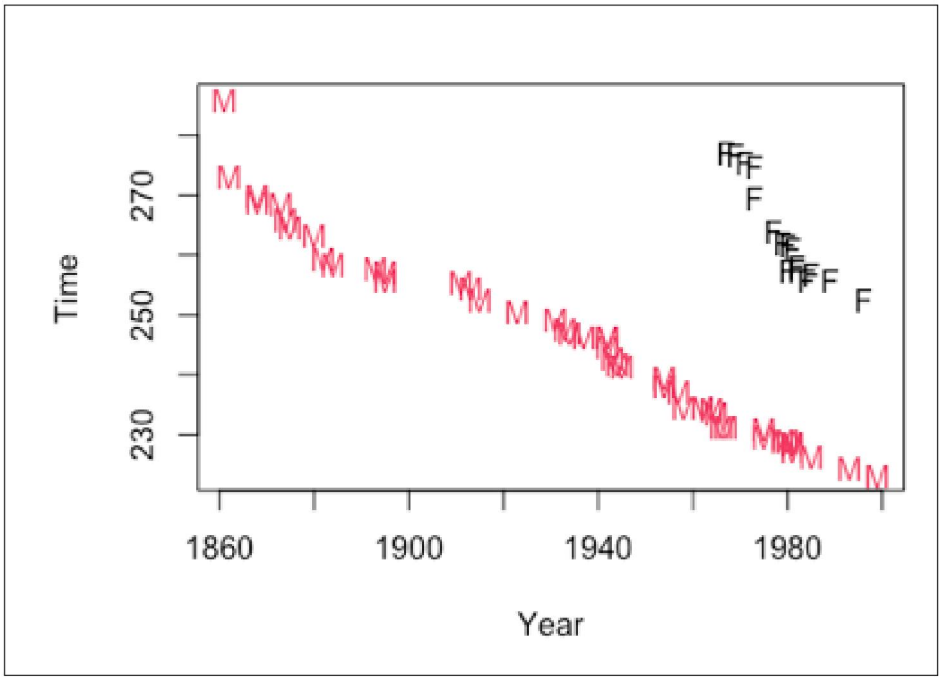
$\rightarrow$ Equality is proved, $t^T\beta$ is estimable in the full model.

3 5 / 5

***Question* 4 :**

Part a:

```
setwd("~/Desktop/UNIMELB 2021 Material/UNIMELB S1 2021
(Currently)/MAST30025/Tutorials /Tutorials/Rfile/data")
mile = read.csv("mile.csv")
str(mile)
## 'data.frame':    62 obs. of  6 variables:
##  $ Year   : int  1861 1862 1868 1868 1873 1874 1875 1880 1882 1884 ...
##  $ Time   : num  286 273 270 269 269 ...
##  $ Name   : chr  "N.S.,Greene" "George,Farran" "Walter,Chinnery" "William,Gibbs" ...
##  $ Country: chr  "IRL" "IRL" "GBR" "GBR" ...
##  $ Place  : chr  NA NA NA NA ...
##  $ Gender : chr  "Male" "Male" "Male" "Male" ...
#Ensure you convert gender into your Factor Type.
genderfactor = factor(mile$Gender)
plot(Time ~ Year, pch=as.character(genderfactor), col=genderfactor, data=mile)
```



The data may look linear, but its decreasing as the years gone by. There are not in dependant and does not satisfy the linear model assumptions.

4.1 3 / 3

Part b:

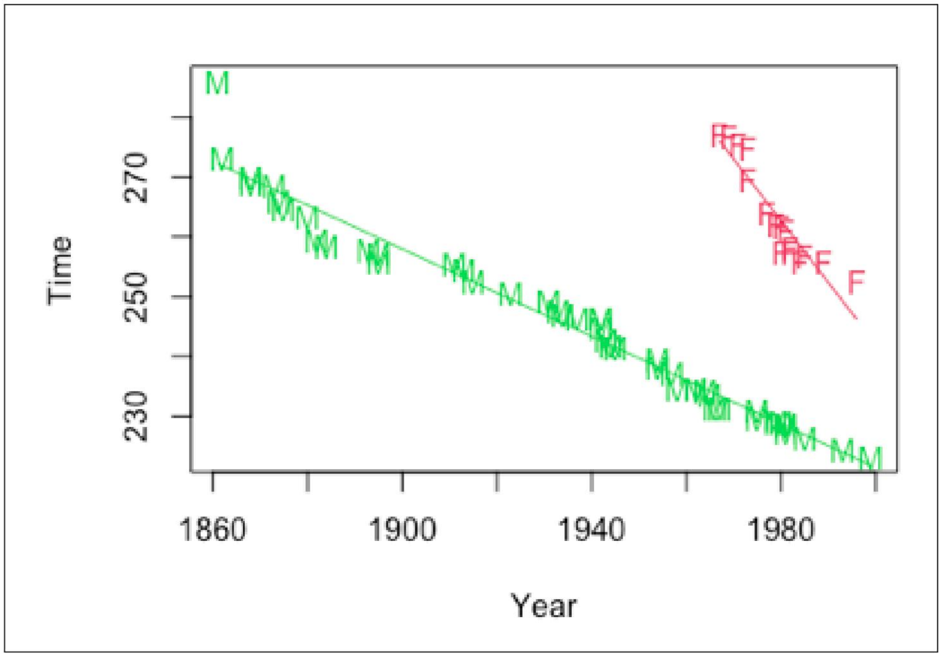
```
imodel = lm(Time~Year*Gender, data = mile)
amodel = lm(Time~Year+Gender,data = mile)
anova(imodel,amodel)
## Analysis of Variance Table
##
## Model 1: Time ~ Year * Gender
## Model 2: Time ~ Year + Gender
##   Res.Df    RSS Df Sum of Sq      F    Pr(>F)
## 1     58 518.03
## 2     59 895.62 -1   -377.59 42.276 2.001e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Since the p value is 2.001e-0.8 < 0.05, we reject the null. Conclude that here is significant evidence that there is no interaction difference between the rate of improvements in the male and female records.

4.2 **2 / 3**

Part c:

```
imodel$coef[c(1,2)] + imodel$coef[c(3,4)]
## (Intercept)        Year
## 953.7469611  -0.3661867
plot(Time ~ Year, data = mile, col=as.numeric(genderfactor)+1,pch=as.character(genderfactor))
for (i in 1:2) { with(mile, lines(Year[as.numeric(genderfactor)==i],fitted(imodel)
[as.numeric(genderfactor)==i], col=i+1))}
```



Case 1 for females

Time = 2309 - 1.034*Year

Case 2 for males

Time = 954 - 0.366*Year

4.3 **3 / 3**

Part d:

```
-imodel$coef[3]/imodel$coef[4]
## GenderMale
##    2030.95
```

We expect that the world records will be equal around the year 2031. However this is unlikely to be an accurate estimate, as we are extrapolating well beyond the range of the data.

Part e:

This is not an estimable quantity; it is not expressible as a linear function of the parameters. This is consistent with part (d), because "estimable" really means linearly estimable. So we can estimate a value for this quantity even though it is not estimable.

Part f:

```
confint(imodel)[4,]
##      2.5 %    97.5 %
## 0.4620087 0.8730100
```

4.4 3 / 3

Part d:

```
-imodel$coef[3]/imodel$coef[4]
## GenderMale
##    2030.95
```

We expect that the world records will be equal around the year 2031. However this is unlikely to be an accurate estimate, as we are extrapolating well beyond the range of the data.

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Part f:

```
confint(imodel)[4,]
##      2.5 %    97.5 %
## 0.4620087 0.8730100
```

4.5 2 / 2

Part d:

```
-imodel$coef[3]/imodel$coef[4]
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We expect that the world records will be equal around the year 2031. However this is unlikely to be an accurate estimate, as we are extrapolating well beyond the range of the data.

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Part f:

```
confint(imodel)[4,]
##     2.5 %    97.5 %
## 0.4620087 0.8730100
```

4.6 **2 / 2**

Part g:

```
library(car)
## Loading required package: carData
linearHypothesis(imodel, c(0,1,0,1), -0.4)
## Linear hypothesis test
##
## Hypothesis:
## Year  + Year:GenderMale = - 0.4
##
## Model 1: restricted model
## Model 2: Time ~ Year * Gender
##
##   Res.Df    RSS Df Sum of Sq      F   Pr(>F)
## 1     59 604.84
## 2     58 518.03  1    86.806 9.7191 0.002837 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We should reject the null hypothesis p value = 0.002837 $< \alpha = 5$ per cent.

4.7 2 / 2

***Question* 5 :**

Part a:

$\mathrm{Var}\ \underbrace{\widehat{\tau_i - \tau_3}}$

comparing all contrasts from treatment 3.

$= \sigma^2 \left(\frac{1}{n_i} + \frac{1}{n_3} \right)$ where, $i = 1, 2, 3$.

we want to minimize,

$$f(n_1, n_2, n_3, \lambda) = \sigma^2 \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_3} \right) + \lambda (5n_1 + 2n_2 + n_3 - 100)$$

$$= \sigma^2 \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{2}{n_3} \right) + \lambda (5n_1 + 2n_2 + n_3 - 100)$$

This gives, $\frac{\partial f}{\partial n_1} = \frac{-\sigma^2}{n_1^2} + 5\lambda = 0$ — (1)

$\frac{\partial f}{\partial n_2} = \frac{-\sigma^2}{n_2^2} + 2\lambda = 0$ — (2)

$\frac{\partial f}{\partial n_3} = \frac{-2\sigma^2}{n_3^2} + \lambda = 0$ — (3)

$\Rightarrow n_1^2 = \frac{\sigma^2}{5\lambda} = \frac{2}{5} n_2^2 = \frac{1}{10} n_3^2$

$\Rightarrow n_1 = \frac{1}{\sqrt{10}} n_3, \quad n_2 = \frac{1}{2} n_3$

Then, $\frac{5}{\sqrt{10}} n_3 + \frac{2}{2} n_3 + n_3 = 100$

Thus, $n_3 = 27.92\ (\approx 27)$, $n_2 = 13.5 \approx 14$, $n_1 = 9$

Part b:

5.1 5 / 5

```
x = sample(50,50)
x[1:9]
## [1] 43  3 30  8  4 15 22 36 44
x[10:23]
##  [1] 42 21  2  5 17 49 19 27 35 31 47 14  9 12
x[24:50]
##  [1] 45 26 10 11 18 37  6 28 13 32 41 50 38 40 48 23 46 39 16 24 29 25 20 34  7
## [26]  1 33
```

END OF ASSIGNMENT.

5.2 2 / 2