School of Mathematics and Statistics
MAST30030
Applied Mathematical Modelling

Assignment 3. Due: 11:59pm Friday June 5th

This assignment counts for 15% of the marks for this subject.

Question 1 [20 marks]

At time $t \geq 0$, a velocity field $\mathbf{u} = u\hat{\mathbf{x}} + v\hat{\mathbf{y}}$ is given by

$$u(x, y, t) = -y; \;\; v(x, y, t) = x + t.$$

(a) Find the streamlines in parametric form.

(b) Are there any requirements for the streamfunction to exist? Calculate the streamfunction under these conditions (if any), and use this result to calculate the streamlines in Cartesian form.

(c) Using the results in (a) and (b), plot the streamline originating at $(x, y) = (1, 0)$ for $t \in \{0, 1, 2\}$.

(d) Find the path of the particle that was located at (X, Y) at $t = 0$.

(e) Using (d), plot the path of the particle originating at $(X, Y) = (1, 0)$. Discuss (briefly) any similarities or differences you observe in comparison to (c).

(f) Verify from the particle path that

$$\frac{\partial \mathbf{u}}{\partial t}\Big|_{\mathbf{R}} = \frac{D\mathbf{u}}{Dt}.$$

Question 2 [20 marks]

(a) The generalised Lorentz reciprocal theorem for two, incompressible, Stokes flows (which may or may not be subject to body forces) in the same domain, with velocity and stress tensor $(\mathbf{u}, \mathbf{T})$ and $(\mathbf{u}', \mathbf{T}')$, respectively, is

$$\int_S \mathbf{n} \cdot (\mathbf{u}' \cdot \mathbf{T} - \mathbf{u} \cdot \mathbf{T}')\, dS = \int_V \mathbf{u} \cdot (\nabla \cdot \mathbf{T}') - \mathbf{u}' \cdot (\nabla \cdot \mathbf{T})\, dV \tag{1}$$

where $\boldsymbol{n}$ is the unit vector into the fluid domain, V, and S is the surface of the domain. Use Cartesian tensor methods to prove this identity. Hint: You may need to use the constitutive equation for an incompressible, viscous fluid and the continuity equation.

(b) Consider two plates located at positions $z = -h/2$ and $z = h/2$ where h is a positive constant; the plates are infinitely long in both the x and y directions. The space between the plates is filled with a fluid of shear viscosity, μ. The top and bottom plates have velocity $\mathbf{u} = (U/2)\hat{\mathbf{x}}$ and $\mathbf{u} = -(U/2)\hat{\mathbf{x}}$ respectively where U

is a positive constant. Solve the Stokes equations (without body force and no applied pressure gradient) for the velocity field. Use your result for the velocity field to calculate the force per unit area exerted on each of the plates by the fluid.

(c) In addition to the problem specification in (b), a body force is acting on the fluid which decelerates fluid particles in close proximity to the walls, this can be modelled as $\mathbf{b} = -A\,(z/h)^2 \sin(2\pi z/h)\hat{\mathbf{x}}$ where A is a positive constant. Using your results from (b) and the Lorentz reciprocal theorem in (a), calculate the force per unit area exerted on the two plates, without explicitly solving for the velocity and stress fields.

Question 3 [10 marks]

A baby seal is swimming in the southern ocean when it spots an orca swimming below it. The baby seal, wanting to avoid becoming the orca's next meal, knows that it can swim fastest if it does not break the surface of the water. However, it also wants to swim as close to the surface as possible to avoid detection by the orca.

Calculate the maximum height the baby seal can swim at before it breaks the surface. You may find a diagram to be helpful (creativity encouraged but not required).

You may make the following assumptions: (i) The flow is steady (this approximation will hold well when the baby seal is gliding). (ii) The flow is inviscid. (iii) The baby seal is 1m long and 40cm thick at its widest point. It swims with its length parallel to the surface of the water. (iv) The gravitational acceleration is 9.8m/s^2. (v) The maximum speed the baby seal can reach while gliding is 5km/h. (vi) It is a calm day and so the surface of the sea, far from the seal, is flat.