Complementary mathematical topics

These topics will not be tested in STM5001 assignments. They are given for students interested in the mathematical background and justifications of models considered in this week lectures.

Finite-dimensional distribution.

A random field is completely described by its **finite-dimensional (cumulative) distributions** defined by

$$F_{\mathbf{t}_1,\ldots,\mathbf{t}_n}(x_1,\ldots,x_n) = \mathrm{P}\{X_{\mathbf{t}_1} \le x_1,\ldots,X_{\mathbf{t}_n} \le x_n\},$$

where $n \in \mathbb{N}$, $\{\mathbf{t}_1,\ldots,\mathbf{t}_n\} \subset T$, and $\{x_1,\ldots,x_n\} \subset \mathbb{R}$.

These finite-dimensional distributions can be used to compute all numerical characteristics of random fields. In particular,

Expectation and Covariance

The **expectation** of a random field equals

$$m(\mathbf{t}) = E\{X_{\mathbf{t}}\} = \int_{\mathbb{R}} x dF_{\mathbf{t}}(x).$$

The **(auto-) covariance function** is defined by

$$\begin{aligned} C(\mathbf{t}, \mathbf{s}) &= Cov\{X_{\mathbf{t}}, X_{\mathbf{s}}\} = E\{X_{\mathbf{t}} X_{\mathbf{s}}\} - m(\mathbf{t}) m(\mathbf{s}), \\ &= \int\int_{\mathbb{R}^2} xy d^2 F_{\mathbf{t},\mathbf{s}}(x, y) - m(\mathbf{t}) m(\mathbf{s}), \end{aligned}$$

whereas the *variance* is

$$\sigma^2(\mathbf{t}) = C(\mathbf{t}, \mathbf{t}).$$

Positive definiteness

Definition 1

Let n be a positive integer, and let $\mathbf{t}_k \in T$ and $c_k \in \mathbb{C}$ (or $\mathbb{R}$) for $k = 1, \ldots, n$. Then a function $B(\cdot, \cdot)$ is positive definite on T if

$$\boxed{\sum_{k=1}^{n} \sum_{l=1}^{n} c_k \bar{c}_l B(\mathbf{t}_k, \mathbf{t}_l) \geq 0}$$

for any $n, \{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$, and $\{c_1, \ldots, c_n\}$ ($\bar{c}_k$ is a complex conjugate of c_k).

Properties of $\mathcal{P}_T$

Let $\mathcal{P}_T$ be the class of positive functions on T.

(1) $B(t, s) \in \mathcal{P}_T, \alpha \geq 0 \Rightarrow \alpha \cdot B(t, s) \in \mathcal{P}_T$.

(2) $B_1(t, s) \in \mathcal{P}_T, B_2(t, s) \in \mathcal{P}_T \Rightarrow B_1(t, s) + B_2(t, s) \in \mathcal{P}_T$.

Proofs of the properties.

(1) If $B(t,s) \in \mathcal{P}_T$, then for any $n, \{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$, and $\{c_1, \ldots, c_n\}$ by Definition 1

$$\sum_{k=1}^{n} \sum_{l=1}^{n} c_k \bar{c}_l B(\mathbf{t}_k, \mathbf{t}_l) \geq 0.$$

Therefore, for any $\alpha \geq 0$ it holds

$$\sum_{k=1}^{n} \sum_{l=1}^{n} c_k \bar{c}_l \alpha B(\mathbf{t}_k, \mathbf{t}_l) = \alpha \sum_{k=1}^{n} \sum_{l=1}^{n} c_k \bar{c}_l B(\mathbf{t}_k, \mathbf{t}_l) \geq 0.$$

Thus, by Definition 1 we obtain $\alpha \cdot B(t,s) \in \mathcal{P}_T$.

(2) If $B_1(t,s) \in \mathcal{P}_T$ and $B_2(t,s) \in \mathcal{P}_T$, then by Definition 1

$$\sum_{k=1}^{n}\sum_{l=1}^{n} c_k \bar{c}_l B_1(\mathbf{t}_k, \mathbf{t}_l) \geq 0 \quad \text{and} \quad \sum_{k=1}^{n}\sum_{l=1}^{n} c_k \bar{c}_l B_2(\mathbf{t}_k, \mathbf{t}_l) \geq 0.$$

Therefore, for for any $n, \{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$, and $\{c_1, \ldots, c_n\}$ it holds

$$\sum_{k=1}^{n}\sum_{l=1}^{n} c_k \bar{c}_l \left(B_1(\mathbf{t}_k, \mathbf{t}_l) + B_2(\mathbf{t}_k, \mathbf{t}_l)\right) = \sum_{k=1}^{n}\sum_{l=1}^{n} c_k \bar{c}_l B_1(\mathbf{t}_k, \mathbf{t}_l)$$

$$+ \sum_{k=1}^{n}\sum_{l=1}^{n} c_k \bar{c}_l B_2(\mathbf{t}_k, \mathbf{t}_l) \geq 0.$$

Thus, by Definition 1 we obtain $B_1(t,s) + B_2(t,s) \in \mathcal{P}_T$.