Exam 11 June 2012, questions

Stochastic Modelling (University of Melbourne)

The University of Melbourne

Department of Mathematics and Statistics

MAST30001 Stochastic Modelling

Semester 2 Exam — November 19, 2012

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 6 pages

Authorised materials:
Students may bring one double-sided A4 sheet of handwritten notes into the exam room. Hand-held electronic calculators may be used.

Instructions to Invigilators:
Students may take this exam paper with them at the end of the exam.

Instructions to Students:
This paper has seven (7) questions.
Attempt as many questions, or parts of questions, as you can.
The number of marks allocated to each question is shown in the brackets after the question statement.
The total number of marks available for this examination is 100.
Working and/or reasoning must be given to obtain full credit.

1. Consider a random experiment in which two bits of information are transmitted along a cable and the output is observed. Each bit can be 0 or 1, and so the set of possible observations is $\Omega = \{(0,0), (0,1), (1,0), (1,1)\}$.

   (a) Giving your reasons, state whether the sets

   (i)

$$\begin{aligned}\mathcal{F}_1 &= \{\emptyset, \{(0,0)\}, \{(0,1)\}, \{(0,0),(0,1)\}, \{(1,0),(1,1)\},\\ &\quad \{(0,0),(1,0),(1,1)\}, \{(0,1),(1,0),(1,1)\}, \Omega\},\end{aligned}$$

   (ii)

$$\begin{aligned}\mathcal{F}_2 &= \{\emptyset, \{(0,0)\}, \{(0,1)\}, \{(0,0),(0,1)\}, \{(1,0),(1,1)\},\\ &\quad \{(0,0),(0,1),(1,0)\}, \{(0,1),(0,1),(1,1)\}, \Omega\}.\end{aligned}$$

   are σ-algebras of subsets of Ω.

   (b) Let P be a probability measure defined on $\mathcal{F}_1$, with

$$P(\{(0,0),(1,0),(1,1)\}) = P(\{(0,1),(1,0),(1,1)\}) = 2/3.$$

   Justifying your calculations by referring to the probability axioms, give the value of P at each of the other sets in $\mathcal{F}_1$.

   (c) Answer this question with respect to the probability measure that you defined in part (b). For

$$\begin{aligned}A &= \{(0,0),(0,1)\},\\ B &= \{(0,0),(1,0),(1,1)\}, \text{ and}\\ C &= \{(0,1),(1,0),(1,1)\},\end{aligned}$$

   (i) derive the conditional probability $P(A|B)$, and

   (ii) giving your reasons, state whether B and C are independent.

   (d) Define the random variable X such that, for $\omega \in \Omega$, $X(\omega)$ is the number of zeros in the first component of ω.

   (i) Explain why X is measurable with respect to $\mathcal{F}_1$.

   (ii) Give the distribution function and probability mass function of X with respect to the probability measure that you defined in part (b).

[16 marks]

2. (a) Write down the communicating classes for the discrete-time Markov chains (DTMCs) with the following transition matrices. In each case, state whether the communicating class is transient or recurrent, and write down its period.

(i)

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix},$$

(ii)

$$\begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 2/3 & 0 & 0 & 1/3 \end{pmatrix},$$

(iii)

$$\begin{pmatrix} 1/7 & 6/7 & 0 & 0 & 0 & 0 \\ 6/7 & 1/7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1/3 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(b) Consider a DTMC with state space $\mathbb{Z}_+$ with transition matrix given by

$$p_{j,k} = \begin{cases} p/j & k = j+1 \\ 1 - 1/j & k = j \\ (1-p)/j & k = j-1 \end{cases}$$

for $j > 0$, with $p_{0,1} = p, p_{0,0} = 1 - p$ where $p \in (0,1)$. Carefully explaining your reasoning, determine the ranges of p for which the DTMC is

(i) transient,

(ii) null-recurrent, and

(iii) positive-recurrent.

For the positive-recurrent case, determine the stationary distribution π_k in terms of π_0. You do not need to calculate π_0.

[20 marks]

3. A gambler has \$2. She is allowed to play a game of chance three times, and her goal is to maximise the probability that she ends up with at least \$5. On each play of the game, her winning probability is 0.4. If she bets \$$b$ on a play and wins, then her capital increases by \$$b$. Otherwise, she loses and her capital decreases by \$$b$. On any play of the game, the gambler cannot bet any more than she has, and she does not have to bet at all.

Determine the betting strategy that will maximise the gambler's probability of attaining a wealth of at least \$5 by the end of the third game and evaluate this probability.

[8 marks]

4. At a particular location on the boundary of a tectonic plate, earthquakes of magnitude greater than 5 occur according to a Poisson process $\{N_t,\ t \geq 0\}$ with intensity 0.04 per year. With probability 0.75 an earthquake will generate a tsunami.

   (a) What is the distribution of the time T until the first earthquake?

   (b) During a planning horizon of 100 years, what is the probability that there is no earthquake?

   (c) During the same time, what is the probability that there is no tsunami?

   (d) Calculate the expected number of tsunamis during a 100 year period.

   (e) Given that there are six earthquakes in 100 years, what is the probability that two or less generate tsunamis?

   (f) Given that there are two tsunamis in a 100-year period, what is the expected number of earthquakes that occurred in the 100-year period?

[12 marks]

5. Consider a continuous-time Markov chain (CTMC) $X(t)$ that is modelling a piece of equipment which can either be operating (state 1) or under repair (state 2). The equipment stays in the operating state for an exponentially-distributed amount of time with parameter 1 before breaking down. Then it takes an exponentially-distributed amount of time with parameter 12 before it is repaired.

   (a) Derive the stationary distribution $\pi_k, k = 1, 2$ for $X(t)$.

   (b) Find $P(X(t) = k | X(0) = 1)$ for $k = 1, 2$.

   (c) Verify that $\lim_{t\to\infty} P(X(t) = k | X(0) = 1)$ is the same as the stationary distribution that you calculated in part (a).

   (d) Find $P(X(t) = k)$ for $k = 1, 2$ when $P(X(0) = k) = \pi_k$.

   (e) Management has decreed that it would like the equipment to be operating for 95% of the time. Carefully justifying your reasoning, give the minimal rate of repair that will achieve this.

[16 marks]

6. (a) **Hint:** In this question you may use the fact that the Laplace-Stieltjes transform of an exponential random variable with parameter γ is

$$\frac{\gamma}{s+\gamma}.$$

Consider an $M/M/1$ queue with service rate 5 and arrival rate 3.

(i) Quoting any results that you need from lectures about the stationary distribution of a continuous-time birth-and-death process, write down the stationary distribution $\pi_k, k = 0, 1, \ldots$ of the $M/M/1$ queue.

(ii) Explain why a typical customer arriving to the queue in a stationary regime will see k customers with probability π_k.

(iii) Assume that a tagged customer arrives to find $k \geq 1$ customers already in the system (that is the queue plus the server). What is the Laplace-Stieltjes transform of the time until the customer in service completes that service?

(iv) What is the Laplace-Stieltjes transform of the total time that it takes the $k-1$ customers that are in the queue ahead of the tagged customer in (c) to complete service?

(v) Using your results from parts (i), (ii), (iii) and (iv), write down the Laplace-Stieltjes transform of the tagged customer's stationary waiting time in the queue, conditional on it arriving to find $k \geq 1$ customers already in the system.

(vi) Hence give the stationary waiting time distribution.

(b) Consider a tandem network of two queues. Customers arrive to the first queue in a Poisson process with parameter 6. Customers in the first queue are served at rate 10 and customers in the second queue are served at rate 8.

(i) The tandem network can be described by a continuous-time Markov chain with state space $\mathbb{Z}^2$. Write down the transition rates between

- state (n_1, n_2) and state (n_1+1, n_2),
- state (n_1, n_2) and state (n_1-1, n_2+1),
- state (n_1, n_2) and state (n_1, n_2-1),

You may find it useful to use the indicator functions $I(n_1 > 0)$ and $I(n_2 > 0)$ to take account of the possibilities that n_1 or n_2 might be zero.

(ii) Hence write down the equations $\pi A = 0$ for the stationary distribution of the continuous-time Markov chain.

(iii) Verify that the solution of these equations has the form

$$\pi(n_1, n_2) = K\left(\frac{3}{5}\right)^{n_1}\left(\frac{3}{4}\right)^{n_2}.$$

(iv) Does there exist a stationary distribution for the continuous-time Markov chain? If so, calculate the value of the constant K. If not, explain why not.

(v) Now assume that 30% of customers are routed from the second queue back to the first queue. Giving your reasons, explain whether the network has a stationary distribution and give its form if one exists.

[18 marks]

7. Consider a renewal process N_t, with inter-renewal times τ_i that have distribution $F(x) = x^2$ for $x \in [0, 1]$.

   (a) Calculate the mean and variance of the τ_i.

   (b) Let $Y_t = T_{N_t+1} - t$ be the residual lifetime at time t. Derive an expression for $\lim_{t\to\infty} P(Y_t \leq y)$.

   (c) Give an (approximate) interval to which the random variable N_{300} belongs with probability 95%. You may use the fact that, for a standard normal random variable Z, $P(-1.96 < Z \leq 1.96) \approx 0.95$.

[10 marks]

End of the exam