MAST30001 Stochastic Modelling – 2020

Assignment 2

If you haven't already, please complete the Plagiarism Declaration Form on the LMS before submitting this assignment.

Submission Instructions:

- Typed submissions (ideally using LaTeX) are preferred. For handwritten solutions:
  - Write your answers on blank paper. Write on one side of the paper only. Start each question on a new page. Write the question number at the top of each page.
  - Scan your solutions to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Upload the PDF file to Gradescope via the LMS. Gradescope will ask you to identify on which of the uploaded pages your answers to each question are located.
- The submission deadline is **5:00pm on Thursday, 29 October, 2020**.

There are 2 questions, and both will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and assume that each arrival has a "type" from the set $\{1,\dots,K\}$, where K is some positive integer. The probability that a given arrival is of type $j \in \{1,\dots,K\}$ is p_j, and the types of different arrivals are independent. For $j=1,\dots,K$ and $t \geq 0$, let $N_t^{(j)}$ be the number of arrivals of type j in the interval $[0,t]$.

   (a) Use the thinning theorem and induction to show that the processes $\big((N_t^{(j)})_{t\geq 0}\big)_{j=1}^K$ are independent Poisson processes, and identify their rates.

   (b) Assuming $K \geq 3$, find the chance that there are exactly two arrivals having type 1 or 2 in the interval $[0,3/4]$, and exactly one arrival of type 3 in the interval $[1/2,1]$.

   (c) Assuming again $K \geq 3$, given that $N_1 = 3$, find the chance that there are exactly two arrivals having type 1 or 2 in the interval $[0,3/4]$, and exactly one arrival of type 3 in the interval $[1/2,1]$.

   (d) For distinct fixed numbers $\alpha_1,\dots,\alpha_K$, show that the process

   $$\left(\sum_{j=1}^{K} \alpha_j N_t^{(j)}\right)_{t\geq 0}$$

   has the same distribution as some compound Poisson process

   $$\left(\sum_{i=1}^{N_t} X_i\right)_{t\geq 0},$$

   and give the distribution of X_1.

   [3 + 2 + 2 + 3 = 10 marks]

Ans.

(a) The induction statements are, for $k \geq 2$, that the result holds with the processes having rates $\lambda \times \mathbb{P}(\text{type } j)$, for any probability distribution with support on $\{1, \dots, k\}$ and any λ. The result is true for $K = 2$ by the thinning theorem. For $K > 2$, assume the statement holds for $K - 1$, and note that, by the usual thinning theorem,

$$(N_t^{(1)})_{t\geq 0} \text{ and } \left(\sum_{j=2}^{K} N_t^{(j)}\right)_{t\geq 0}$$

are independent Poisson processes with rates λp_1 and $\lambda(1 - p_1)$.

Now, to generate $\left((N_t^{(j)})_{t\geq 0}\right)_{j=2}^{K}$ from $\left(\sum_{j=2}^{K} N_t^{(j)}\right)_{t\geq 0}$, we give each arrival type j with probability $p_j/(1 - p_1)$ (the probability of being type j, given not of type 1). Thus the induction hypothesis implies that $\left((N_t^{(j)})_{t\geq 0}\right)_{j=2}^{K}$ are independent Poisson processes with rates $\lambda(1 - p_1)p_j/(1 - p_1) = \lambda p_j$.

(b) Using independence and the previous result, we have

$$\begin{aligned}\mathbb{P}&\left(N_{3/4}^{(1)} + N_{3/4}^{(2)} = 2, N_1^{(3)} - N_{1/2}^{(3)} = 1\right)\\ &= \left[(3\lambda(p_1 + p_2)/4)^2 e^{-3\lambda(p_1+p_2)/4}/2\right]\left[(\lambda p_3/2)\, e^{-\lambda p_3/2}\right].\end{aligned}$$

(c) Given $N_1 = 3$, the times of the three arrivals are i.i.d. distributed as uniform on the interval $(0, 1)$. We need two of these to land in $[0, 3/4]$ and be of type 1 or 2 (together occurring with probability $(3/4)(p_1 + p_2)$) and the other to land in $[1/2, 1]$ and be of type 3 (occurring with probability $(1/2)p_3$, and the number of ways of choosing which arrivals are in which group is $\binom{3}{2}$, so altogether,

$$\mathbb{P}\left(N_{3/4}^{(1)} + N_{3/4}^{(2)} = 2, N_1^{(3)} - N_{1/2}^{(3)} = 1 | N_1 = 3\right) = \binom{3}{2}\left((3/4)(p_1 + p_2)\right)^2 \left((1/2)p_3\right).$$

Alternatively this can be done with a more tedious direct calculation starting from

$$\begin{aligned}\mathbb{P}&\left(N_{3/4}^{(1)} + N_{3/4}^{(2)} = 2, N_1^{(3)} - N_{1/2}^{(3)} = 1 | N_1 = 3\right)\\ &= \frac{\mathbb{P}\left(N_{3/4}^{(1)} + N_{3/4}^{(2)} = 2, N_1^{(3)} - N_{1/2}^{(3)} = 1, N_1 = 3\right)}{\mathbb{P}(N_1 = 3)}.\end{aligned}$$

(d) The simplest way to see this is to note that the process $\left(\sum_{j=1}^{K} \alpha_j N_t^{(j)}\right)_{t\geq 0}$ can be described alternatively as follows. Each arrival of type j contributes α_j to the sum, and each arrival is type j with probability p_j. Thus, each arrival independently contributes a random variable, say distributed as X to the sum, where the random variable has distribution

$$\mathbb{P}(X = \alpha_j) = p_j.$$

Since the types of arrivals are independent, the process has the same distribution as a compound Poisson process as in the problem where X_i is distributed as X.

Alternatively, for fixed t, we can use independence and the formula for the moment generating function of a Poisson variable to compute the moment generating function

$$\begin{aligned}\mathbb{E}\Big[\exp\Big\{\theta\sum_{j=1}^{K}\alpha_j N_t^{(j)}\Big\}\Big] &= \exp\Big\{-\lambda t\Big(1-\sum_{j=1}^{K}p_j e^{\theta\alpha_j}\Big)\Big\} \\ &= \exp\big\{-\lambda t\big(1-\mathbb{E}[e^{\theta X}]\big)\big\},\end{aligned}$$

which is the moment generating function of the marginal distribution of a compound Poisson process as above. Since both processes have independent increments and the same marginal at time t, they have the same distribution.

2. Customers arrive to an outback auto repair shop according to a Poisson process with rate λ per day. The shop has one mechanic who takes an exponential with rate μ per day amount of time to repair a car. In addition, if there are no cars in the shop, the mechanic will wait an exponential rate ν per day time, and if no car has arrived in that time, the mechanic will leave the shop and take a nap for an exponential rate ν time. If a car arrives while the mechanic is waiting to leave to take a nap, they will begin work on the car. During a nap, cars arriving for repair will move on to the next repair shop. Cars that arrive when the mechanic is working form a queue.

   (a) Model this system as a continuous time Markov chain and write down its state space and generator.

   (b) Determine values of λ, μ, and ν such that the stationary distribution of the Markov chain exists. For these values, determine the stationary distribution.

   (c) What is the stationary average number of customers waiting for service?

   (d) Given that a customer is not immediately rejected from the system, what is the average time they spend in the system?

   (e) What proportion of customers are rejected from the system?

   [2 + 3 + 2 + 2 + 1 = 10 marks]

Ans.

(a) We view the system as a CTMC with states $\{(0,0),(0,1),1,2,3,\ldots,\}$, where $(0,0)$ ($(0,1)$) means there are no cars in the shop and the mechanic is napping (not napping), and otherwise the states represent the number of cars in the shop (including those in service. When there are cars in the shop, the system behaves as a birth-death process with birth rates λ and death rates μ. When there are no cars in the shop, the process moves from $(0,0)\mapsto(0,1)$ at rate ν, and moves from $(0,1)\mapsto 1$ at rate λ and $(0,1)\mapsto(0,0)$ at rate ν. Therefore the generator satisfies

$$\begin{aligned}a_{(0,0),(0,1)} &= a_{(0,1),(0,0)} = \nu \\ a_{(0,1),1} &= a_{i,i+1} = \lambda \\ a_{1,(0,1)} &= a_{i+1,i} = \mu\end{aligned}$$

where $i \geq 1$, and the diagonals are chosen to make the row sums zero.

(b) The system has a stationary distribution if $\pi A = 0$ has a probability solution. These equations are, for $i \geq 2$:

$$(\mu + \lambda)\pi_i = \mu\pi_{i+1} + \lambda\pi_{i-1},$$

which has solution for $i \geq 1$,

$$\pi_i = c(\lambda/\mu)^i,$$

where c is some constant. Plugging into the equations

$$\begin{aligned}(\lambda + \mu)\pi_1 &= \mu\pi_2 + \lambda\pi_{(0,1)}, \\ (\lambda + \nu)\pi_{(0,1)} &= \nu\pi_{(0,0)} + \mu\pi_1,\end{aligned}$$

implies that

$$\pi_{(0,1)} = \pi_{(0,0)} = c.$$

We need to choose c so that the $\pi.$'s sum to one, which can be done if and only if $\lambda < \mu$, in which case

$$c = \frac{\mu - \lambda}{2\mu - \lambda}.$$

(c) The average number of customers waiting for service is

$$\sum_{i=2}^{\infty}(i-1)\pi_i = c\sum_{i=2}^{\infty}(i-1)(\lambda/\mu)^i = \frac{c\lambda^2}{(\mu-\lambda)^2} = \frac{\lambda^2}{(2\mu-\lambda)(\mu-\lambda)}. \tag{1}$$

(d) Using PASTA, an arriving customer finds the system in stationary. The chance they enter the system is $(1 - \pi_{(0,0)})$ and if they enter the system, the time they wait for service is the sum of the exponential service times of all customers already in the system, which is zero if there are no customers, and, if there are $i \geq 1$ customers, is distributed gamma with parameters i and μ. Thus the expected waiting time is

$$\frac{\sum_{i\geq 1}\pi_i(i/\mu)}{1 - \pi_{(0,0)}} = \frac{\lambda}{(2\mu-\lambda)(\mu-\lambda)(1-\pi_{0,0})} = \frac{\lambda}{\mu(\mu-\lambda)}, \tag{2}$$

and the time in the system is this plus their expected service time, which is $1/\mu$, and the final answer is

$$\frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu-\lambda}.$$

Alternatively, to derive (2), Little's law says that it is equal (1) divided by the rate of arrival, which, with rejections, is $\lambda(1 - \pi_{(0,0)})$.

(e) Again using PASTA, an arriving customer finds the system in stationary, and is rejected if the system is in state $(0,0)$, which occurs with probability $\pi_{(0,0)}$. This is also the long run proportion of customers rejected.