SCHOOL OF MATHEMATICS AND STATISTICS

MAST30013 Techniques in Operations Research **Semester 1, 2021**

Assignment 1 SOLUTION

1. Consider the problem:
$$\min f(x) := e^{-x} - \cos x$$
where $x \in [0, 1]$.

   (a) Prove that f is a unimodal function and there is a unique global minimum in the interior of $[0, 1]$.

   SOLUTION:

   Note that $e^{-x} - \cos x$ is continuous on $[0, 1]$. Note also that $f'(x) = \sin x - e^{-x}$ is continuous on $[0, 1]$ and that $f''(x) = \cos x + e^{-x} > 0$ on $[0, 1]$. Therefore $f'(x)$ is continuous and strictly increasing on $[0, 1]$. We can therefore confirm that $f(x)$ has a unique stationary point on $(0, 1)$ by confirming that $f'(0) \times f'(1) < 0$. But $f'(0) = -1$ and $f'(1) = \sin(1) - e^{-1} > 0$, and the result is confirmed. Now, since $f'(0) < 0$ and $f'(1) > 0$ it follows that neither 0 nor 1 are local minimums of f on $[0, 1]$. Therefore it only remains to show that the unique stationary point of f in $(0, 1)$ is a local minimum. But this follows from the fact that $f''(x) > 0$ for all $x \in [0, 1]$. Finally, note that any global minimum is also a local minimum, and therefore the unique stationary point of f in $(0, 1)$ is a unique global min of f in $(0, 1)$.

   (b) Reduce the size of the interval containing the global minimum to less or equal to 0.5 using Golden section method.

   SOLUTION:

   We want $\gamma^n(1 - 0) \leq 0.5$. Therefore we need $n + 1 = 3$ function evaluations.

   $a = 0, b = 1,$
   $p = b - \gamma(b - a) = 0.382,$
   $q = a + \gamma(b - a) = 0.618$
   $f(p) = -0.245$ (evaluation 1)
   $f(q) = -0.276$ (evaluation 2).

   Next step:
   $f(p) > f(q)$, therefore
   $a = p = 0.382$

$b = b = 1$
$p = q = 0.618$
$q = a + \gamma(b - a) = 0.764$
$f(q) = -0.256$ (evaluation 3).

Note that all three evaluations of f have now been performed.
Next step:
$f(p) < f(q)$, therefore
$a = a = 0.382$
$b = q = 0.764$.

Final interval is $[a, b] = [0.382, 0.764]$ and $b - a = 0.382 \leq 0.5$.

2. Use the Fibonacci algorithm to find the minimum of

$$f(x) = -\frac{1}{(x-1)^2}\left(\log x - \frac{2(x-1)}{x+1}\right),$$

which is known to lie in $[1.5, 4.5]$. Reduce the interval to $1/21$ of the original.

SOLUTION:

We need $(b - a)/F_n \leq (b - a)/21$. Therefore $F_n \geq 21$ and therefore $n \geq 7$. Hence we choose $n = 7$.
We now implement the algorithm step by step.
$k = 7$
$a = 1.5$
$b = 4.5$
$p = 2.643$
$q = 3.357$
$f(p) = -0.0259$
$f(q) = -0.0232$

$k = 6$
$a = 1.5$
$b = 3.357$
$p = 2.214$
$q = 2.643$
$f(p) = -0.0267$
$f(q) = -0.0259$

$k = 5$
$a = 1.5$
$b = 2.643$

$p = 1.929$
$q = 2.214$
$f(p) = -0.0262$
$f(q) = -0.0267$

$k = 4$
$a = 1.929$
$b = 2.643$
$p = 2.214$
$q = 2.357$
$f(p) = -0.0267$
$f(q) = -0.0266$

$k = 3$
$a = 1.929$
$b = 2.357$
$p = 2.071$
$q = 2.214$
$f(p) = -0.0266$
$f(q) = -0.0267$

Now we use the standard method to try to avoid duplicating a point:
$k = 2$
$a = 2.071$
$b = 2.357$
$p = 2.214$
$q = a + 2\epsilon = 2.071 + 0.143 = 2.214$
$f(p) = f(q) = -0.0267$

Note, we got duplication anyway, but this is because 21 is a Fibonacci number. We now try points around $p = 2.214$, eg. $p + \delta$ where $|\delta|$ is small.
We find that the optimal lies in the interval: $[2.071, 2.214 + \delta]$.

3. Suppose that $f : [0, \infty) \to \mathbb{R}$ is a continuous unimodal function with $f'(0) < 0$. Show that, for $\sigma \in (0, 1)$ and $\mu \in [\sigma, 1)$, there exists a stepsize $t > 0$ that satisfies both the Armiji-Goldstein condition,

$$f(t) \;\leq\; f(0) + t\sigma f'(0),$$

and the Wolfe condition,

$$f'(t) \;\geq\; \mu f'(0).$$

In other words, if $\sigma \in (0,1)$ and $\mu \in [\sigma, 1)$, the two regions defined by the Armijo-Goldstein and Wolfe conditions overlap.

SOLUTION:

Suppose $\tilde{t}$ is the largest possible value that satisfies the Armijo-Goldstein condition. Therefore, $f(\tilde{t}) = f(0) + \tilde{t}\sigma f'(0)$ and, for $t > \tilde{t}$, $f(t) > f(0) + t\sigma f'(0)$ (that is, the Armijo-Goldstein condition is not satisfied when $t > \tilde{t}$). Now, since the line $y = f(0) + t\sigma f'(0)$ intersects the curve $y = f(t)$ when $t = \tilde{t}$ and never again when $t > \tilde{t}$, we have

$$\begin{aligned} f'(\tilde{t}) &\geq \sigma f'(0) \\ &\geq \mu f'(0) \end{aligned}$$

since $\mu \in [\sigma, 1)$ and $f'(0) < 0$. Thus, $\tilde{t}$ also satisfies the Wolfe condition.