Fluid Mechanics

Topic 10.2

Conservation Laws and the Navier-Stokes Equations

Goals for this class

Understand conservation laws for mass and momentum

Derive integral expressions for these in three dimensions

Apply these to calculate forces and flow rates

Formulate vector calculus expressions for the Navier-Stokes equations

Conservation laws and the Navier-Stokes equations

These are the Navier-Stokes Equations, which are expressions of the conservation of mass, momentum and energy. For the purposes of this class, we will be examining expressions of the Navier-Stokes equations relating to mass and momentum balances.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{F}$$

These are expressions that state that the amount of mass and momentum in a defined system is conserved.

We will examine the derivation and behavior of these.

Conservation of mass

The conservation of mass is a statement that says that mass cannot be created or destroyed. Though mass can enter or leave a control volume, this means that whatever the change in mass is in the control volume has to be accounted for by the mass flowing into or out of the control volume

Therefore:

Conservation of mass

Here we will examine each of these components (change in mass in control volume V and flow of mass into V) individually and put them into integral form.

<u>Change of mass in V:</u>

Conservation of mass

<u>Flow of mass out of *V*:</u>

Conservation of mass

This gives the conservation of mass in integral form:

This expression states that the sum of the change in mass and flow out of a volume equals zero

Put another way, any change of mass in a volume is equal to the flow out of the volume.

Conservation of mass

Let’s utilize this expression to derive a useful expression for the relationship between velocity and cross-sectional area in a given volume:

nothing new here – we have seen this before, but used a different approach to derive this

Conservation of momentum

The conservation of momentum states that the rate of change in momentum is equal to the sum of forces applied to that system.

Therefore:

Conservation of momentum

Here we will examine each of these components (change in momentum in control volume *V*, flow of momentum into *V* and forces acting on *V*) individually and put them into integral form.

Change of momentum in *V*:

Conservation of momentum

<u>Momentum flow out of *V* (across its surface):</u>

Conservation of momentum

Forces acting on *V* include those acting on and *within* the volume

These are:

1. Gravitational forces (acting within the volume)

2. Pressure forces (acting on the volume's surface)

3. Viscous forces (acting on the volume's surface)

Conservation of momentum

<u>This yields the integral expression for the conservation of momentum:</u>

Conservation of momentum

Let's utilize this expression to derive a useful expression for the conservation of momentum in a control volume.

together these are terms describing the (1) change in momentum, (2) momentum flow and (3) forces acting on the system

Conservation of momentum

We ignore the change in momentum term at steady state.

Let's solve the integral for the momentum flow term:

Conservation of momentum

Let's solve the integral for the forces, starting with Pressure term:

Conservation of momentum

Let’s finally examine the integral <u>for the viscous term:</u>

In summary, this yields the expression:

Relationship between integral expressions and Navier-Stokes equations

You will notice that the expressions we derived share the same physical meaning as the Navier-Stokes equations, but not the same form (what we have are integral forms of these conservation laws). Here we will recast our equations into differential forms.

First, we will examine something called divergence theorem (Gauss's theorem) , which is a statement that the flow of some quantity across a volume's surface (the 'flux' of that quantity) is equal to the sum of sources and sinks within that volume.

We can use this to put the surface integrals in the integral forms of our conservation laws in terms of volumetric integrals.

Vector calculus operators

Spatial derivative in 3D:

gradient ∇

Velocities point outward (positive) or inward (negative) from a point

divergence ∇·v

Navier-Stokes equations: Conservation of Mass

start with our integral form

apply the divergence theorem

substituting this into the conservation of mass gives:

since the volume integral will be equal to zero only if the contents inside are equal to zero, we have the following expression, which is often referred to as the ‘continuity equation’

Navier-Stokes equations: Conservation of Mass

We can also expand these terms, yielding the following

which, if we assume incompressibility, we have:

Since $\nabla \cdot \mathbf{v}$ is the fractional rate of change of the volume of a fluid element, this means that the incompressible form of the conservation of mass requires that the volume of a fluid element remains the same. This does not mean that fluid channels cannot converge/diverge:

Navier-Stokes equations: Conservation of Momentum

We start with our integral form

apply the divergence theorem

The pressure and viscous stress terms can similarly by put in terms of volumetric integrals

Which leads to the differential form of the conservation of momentum

Navier-Stokes equations: Conservation of Momentum

We can convert this into the form given at the start of the module by making use of a vector calculus identity. We start with our derived expression:

Left term – apply product rule:

Right term – apply the divergence of a dyad (the divergence of two multiplied vectors):

combine components

Navier-Stokes equations: Conservation of Momentum

Combine terms with **v**

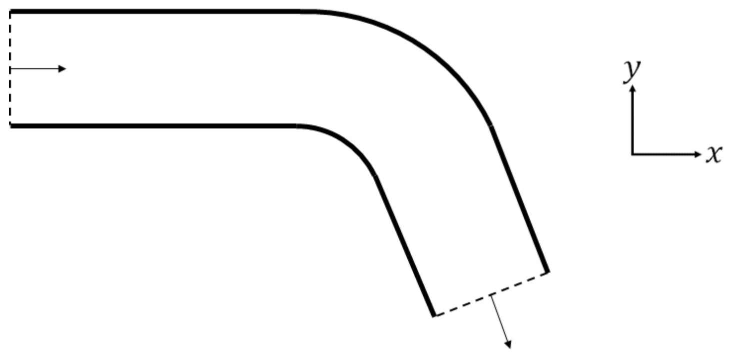
Apply continuity equation

This results in:

We will examine and apply the N.S.E in a separate sub-module. For now let's explore the application of the integral forms of momentum conservation.

Example Problem 9.1

Water with a flow rate of 0.01 m^3/s travels through a 10 cm diameter pipe with a 60° bend. Calculate the force exerted on the pipe – both in the x and y directions and the absolute force magnitude – if it is to remain stationary.



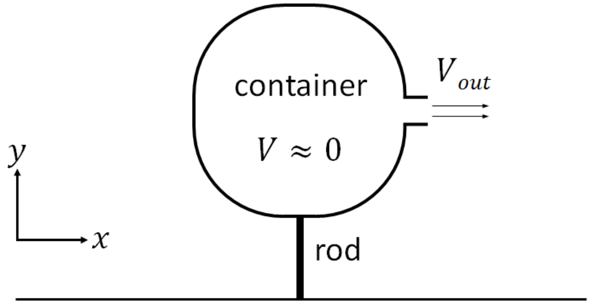
Example Problem 9.2

A large container with pressurized air is being held in place by a rod. This container is emitting air at a velocity of 10 m/s through a small circular orifice with a diameter of 1 cm on the side of the container. The density of the air emitted from the container is 1.3 kg/m^3.

(a) Calculate the mass flux (in kg/m^2s) at the outlet

(b) Calculate the volumetric flow rate (m^3/s) at the outlet

(c) Calculate the force exerted by the rod to keep the container in place.



Example Problem 9.3

A pipe of indeterminant length and constant diameter starts out with a pressure of P_1 = 300 kPa and ends with a pressure of P_2 = 150 kPa. The pipe has a diameter of 10 cm. The flow along the pipe creates viscous drag that pushes it in the direction of travel. Find the counteracting force that must be exerted on the pipe to keep it stationary.