Prac 1. MAT4MDS (2023) (1)

(Solution)

Simplify:

Q1. $\left(\dfrac{(75x^3y^{-2}z)^2}{3z^9y^4x^2}\right) \times \left(\dfrac{5xyz}{15x^2z^{-1}}\right)^{-1} =$

$$= \frac{75^2 x^6 y^{-4} z^2}{3z^9 y^4 x^2} \times \frac{15x^2 z^{-1}}{5xyz}$$

$$= \frac{(3\times 25)^2 x^6 z^2}{3 z^9 y^4 \times y^4 x^2} \times \frac{3\times 5 x^2}{5x y z \times z^1} =$$

$$= \frac{(3\times 5^2)^2 x^{6-2}}{3 z^{9-2} y^{4+4}} \times \frac{3\times 5\ x^{2-1}}{5\times y\ z^{1+1}}$$

$$= \frac{3^2\times 5^4 x^4}{\cancel{3} z^7 y^8} \times \frac{\cancel{3}\times\cancel{5}\times x^1}{\cancel{5}\times y z^2}$$

$$= \frac{3^2\times 5^4 x^{4+1}}{z^{7+2} y^{8+1}} = \frac{3^2\times 5^4 x^5}{z^9 y^9} = \underline{3^2\times 5^4\times x^5 z^{-9} y^{-9}}$$

Prac 1 Mat4MDS (2023)

Q2. Solve for x:

(a) $x^2+4x=21$,

$x^2+4x-21=0$

Is it possible to find a_1, a_2:

$a_1 \times a_2 = -21$

$a_1 + a_2 = 4$.

$7 \times -3 = -21$

$7 + (-3) = 4$

$\Rightarrow$ $x^2+4x-21=0$

Can be written in equivalent form:

$(x+7)(x-3)=0$

$\Rightarrow$ $x+7=0$ or $x-3=0$

$\Rightarrow$ $\boxed{x=-7 \text{ or } x=3}$

Answer

Solving quadratic equations:

1. To solve eq: $x^2+bx+c=0$

(If you can find 2 numbers a_1, a_2 such that:

$$\boxed{\begin{aligned} a_1 \times a_2 &= c \\ a_1 + a_2 &= b \end{aligned}}$$

Then the eq.

$x^2+bx+c=0$

is equivalent:

$(x+a_1)(x+a_2)=0$

$\Rightarrow$ $x+a_1=0$ or $x+a_2=0$

$\Rightarrow$ $\boxed{x=-a_1 \text{ or } x=-a_2}$

2. To solve eq:

$ax^2+bx+c=0$

Use quadr. formula:

$$\boxed{x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

Method 2 works for any quadr. eqation;

Method 1 is easier, but is not alw. working

Q2 (continue)

(b) $e^{2x} - 4e^{x} + 3 = 0$

$\Leftrightarrow (e^{x})^2 - 4e^{x} + 3 = 0$ (1)

must be !

Let $u = e^{x}$ (Note: $u > 0$)

Eq. (1) can be written as:

(2) $u^2 - 4u + 3 = 0$ which is a quadr. eq with respect to u.

Let us solve for u:

$(-3) \times (-1) = 3$

$-3 + (-1) = -4$

So (2) $\Leftrightarrow (u-3)(u-1) = 0$

$u = 3$ or $u = 1$

We have to solve for x, so go back to old variable ($u = e^{x}$)

$\ln e^{x} = \ln 3$ or $\ln e^{x} = \ln 1$

$\boxed{x = \ln 3}$ (is the same as $x = \log_e 3$)

$x = \ln 1 = \log_e 1$

$\boxed{x = 0}$

Answ: $x = 0$ or $x = \ln 3$ ($\log_e 3$)

Q2 (continue)

(c) $6^x - 1 = 6^{1-x}$

Split term 3 into product / quotient using Law 1 or 2

$\Leftrightarrow 6^x - 1 \overset{(\text{Law 2})}{=} \frac{6^1}{6^x}$ ($\times 6^x$ both sides of the equation)

$\Leftrightarrow 6^x \times 6^x - 1 \times 6^x = 6^1$

$\Leftrightarrow (6^x)^2 - 6^x = 6$

$\Leftrightarrow 6^{2x} - 6^x - 6 = 0$

Let $u = 6^x$, $u > 0$

$\Leftrightarrow u^2 - u^x - 6 = 0$

$-3 \times 2 = -6$

$-3 + 2 = -1$

$\Leftrightarrow (u-3)(u+2) = 6$

$u = 3$ or $u = -2 < 0$ (reject).

Back to x:

$6^x = 3$ or $\underbrace{6^x = -2}$,

$\Leftrightarrow \log_6 6^x = \log_6 3$

No solution, as there is no such x so that 6^x is a negative number.

$\Leftrightarrow$

Answ: $\boxed{x = \log_6 3}$ unique solution.

(If needed use calculator to evaluate this)

Q2 (continue)

(d) $4^{x} - 2^{x+3} + 12 = 0$ (1)

write this term with basis 2

split this term in the product using index Law 1

(the goal is to have all terms with the same basis)

eq. (1) is equivalent to:

$$(2^2)^x - 2^x \times 2^3 + 12 = 0$$

$$\Leftrightarrow \quad 2^{2x} - 8 \times 2^x + 12 = 0 \quad (2)$$

Similar to (b) & (c):

Let: $u = 2^x$, $u > 0$!

Eq (2) is equivalent:

$$u^2 - 8u + 12 = 0 \quad (3)$$

$$\begin{pmatrix} -6 \times -2 = 12 \\ -6 + (-2) = -8 \end{pmatrix}$$

$\Rightarrow$ eq (3) is equivalent to:

$$(u-6)(u-2) = 0$$

$\Rightarrow u - 6 = 0$ or $u - 2 = 0$ (So both u are > 0 $\Rightarrow$ all good)

$u = 6 > 0$ $\quad$ $u = 2 > 0$

Back to x: $2^x = 6$ or $2^x = 2^1$

Answ: $\boxed{x = \log_2 6 \text{ or } x = 1}$

Q2 continue

(e) $9^x = 3^x + 1$

1. Goal: to make all terms with the same Basis.

$$(3^2)^x = 3^x + 1$$

To make it quadratic with respet to 3^x, make the right hand side $= 0$

So: $3^{2x} - 3^x - 1 = 0$

Let: $u = 3^x$, $u > 0$

Then $u^2 - u^x - 1 = 0$ (1)

It is inconvenien to find a_1, a_2:

$$a_1 \times a_2 = -1$$
$$a_1 + a_2 = -1$$

So it is better to solve this eq. usind Quadratic formula:

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For eq. (1): $a = 1$, $b = -1$, $c = -1$

$$\Rightarrow u = \frac{1 \pm \sqrt{1 - 4 \times 1 \times (-1)}}{2 \times 1} = \frac{1 \pm \sqrt{1 + 4}}{2}$$

$\Rightarrow u = \frac{1+\sqrt{5}}{2} > 0$ or $u = \frac{1-\sqrt{5}}{2} < 0$, so reject

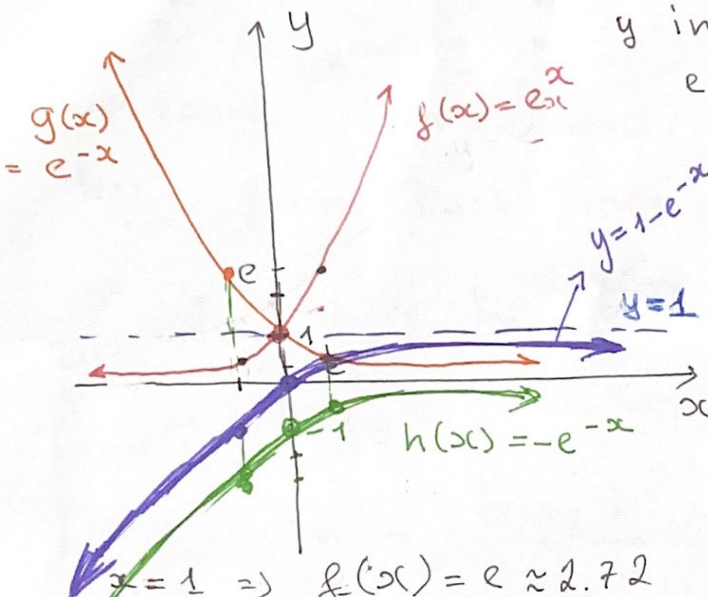
Back to x: $3^x = \frac{1+\sqrt{5}}{2} \Rightarrow \boxed{x = \log_3\left(\frac{1+\sqrt{5}}{2}\right)}$

Answ: only 1 solution

Q3. Sketch the graph of $y = 1 - e^{-x}$

Follow the steps:

(1) Sketh the graph $f(x) = e^x$



y intercepts: $x = 0$

$e^0 = 1 \Rightarrow$ y-intercep of e^x (and of any exp. function a^x) is $(0, 1)$

if $x \to -\infty$

$e^x = \lim_{x \to -\infty} e^x = 0$

i.e it will never touch or cross the x-axis

if $x \to +\infty$,

$e^x = \lim_{x \to \infty} e^x = \infty$

$x = 1 \Rightarrow f(x) = e \approx 2.72$

$x = -1 \Rightarrow f(x) = e^{-1} = \frac{1}{e} \approx 0.37$

(2) Sketch $g(x) = e^{-x}$ by reflecting graph $f(x) = e^x$ with respect to y-axis (so y-axis is the mirrow)

(3) Sketth $h(x) = -e^{-x}$ by reflecting $g(x) = e^{-x}$ with respect to x-axis

(4) Sketh the graph $y = -e^{-x} + 1 \doteq 1 - e^{-x}$ (the same as) by moving up by 1 unit.

Q4 Consider: $a = b^m$ (1)

take $\log_b$ from both sides of eq (1):

$$\log_b a = \log_b m$$

$$\Leftrightarrow \log_b a = m \quad (2)$$

Now for some (unrelated) let's take $\log_d$ from both sides of eq. (1):

$$\log_d a = \log_d b^m$$

$$\Leftrightarrow \log_d a = m \log_d b \quad \text{(log Law 4)}$$

$$\Leftrightarrow m = \frac{\log_d a}{\log_d b} \quad (3)$$

Combying (2) & (3) we obtain:

$$\boxed{\log_b a = \frac{\log_d a}{\log_d b}}$$

Change of Base rule

Calculators usually have options to calculate $\log_e$ & $\log_{10}$. All other log-s (with different basis can be calculated using Change of base rule, e.g.

$$\log_2 8 = \frac{\log_e 8}{\log_e 2} = \frac{\log_{10} 8}{\log_{10} 2}.$$

Q5 Solve for x

(a) $2\log_3(x) + \log_9(x) = 10$ (1)

The goal is to make all term of same base and bring this equation to the form $\log_a b = \log_a c$.

To do this, first use change of base rule and then Law 3 to get rid of all coefficients:

(1) $\Leftrightarrow$ (change of base) $2\log_3(x) + \dfrac{\log_3 x}{\log_3 9} = 10$

$\Leftrightarrow$ $\log_3 9 = \log_3 3^2 = 2$

$$2\log_3 x + \frac{\log_3 x}{2} = 10 \quad (2)$$

$\Leftrightarrow$ ($\times 2$) eq. (2) to get rid of denominator

$$4\log_3 x + \log_3 x = 20$$

$\Leftrightarrow$ (Law 3)

$$\log_3 x^4 + \log_3 x = \log_3 3^{20}$$

$\Leftrightarrow$ (Law 1)

$$\log_3(x^4 \times x) = \log_3 3^{20}$$

$\Leftrightarrow$

$$\log_3 x^5 = \log_3 3^{20}$$

$\Leftrightarrow$

$$x^5 = 3^{20}$$

$$x = (3^{20})^{\frac{1}{5}}$$

$$x = 3^4 = 81$$

Q5. Solve for x

(b) $\frac{1}{2}\log_e x = \log_e(2x-1)$ $\quad x>0 \quad 2x-1>0 \Rightarrow x>\frac{1}{2}\ (*)$

Get rid of all coefficients by log:

$$\log_e x = 2\log_e(2x-1)$$

$\Leftrightarrow$ (Law 3)

$$\log_e x = \log_e(2x-1)^2$$

$$\Leftrightarrow \quad x = (2x-1)^2$$

$$\Leftrightarrow \quad x = 4x^2 - 4x + 1$$

Perfect square rule: $(a\pm b)^2 = a^2 \pm 2ab + b^2$

$$\Leftrightarrow \quad 4x^2 - 5x + 1 = 0$$

$\Leftrightarrow$ quadr. formula $\quad x = \frac{5 \pm \sqrt{25-16}}{8} = \frac{5\pm\sqrt{9}}{8}$

$$x = \frac{5+3}{8} = 1$$

or $\quad x = \frac{5-3}{8} = \frac{2}{8} = \frac{1}{4}$

$\Rightarrow \quad x = 1$ or $x = \frac{1}{4}$

but x must be $> \frac{1}{2}$, so we reject $x = \frac{1}{2}$

as $\log(2\times\frac{1}{4}-1) = \log(\frac{1}{2}-\cancel{1})$

$= \log(-\frac{1}{2})$ does NOT exist

Answer: $\boxed{x = 1}$

Q 6 To get rid of $\log_{10}$ we have to apply exponent. funtion, e.g.

$$\log_{10} x = 0$$
$$\Leftrightarrow \quad x = 10^0$$
$$\Leftrightarrow \quad x = 1$$

Or

$$\log_{10} x = -1$$
$$\Leftrightarrow \quad x = 10^{-1}$$
$$\Leftrightarrow \quad \boxed{x = \frac{1}{10}}$$

Or

$$\log_{10} x = 1$$
$$x = 10^1$$
$$\boxed{x = 10}$$

Q7. Similarly to q. 6 :

$$\log_2 x = 0$$
$$x = 2^0$$
$$x = 1$$

Or $\log_2 x = 1$

$$x = 2^1 = 2$$

Or $\log_2 x = -1$

$$x = 2^{-1} = \frac{1}{2}$$

Or $\log_2 x = 2 \Rightarrow x = 2^2 = 4.$

Q8. It is log-lin graph

(a) x-axis has lin scale

(distance between all units (marks) on the x-axis is the same)

y-axis has log scale

(marks) units are not equaly distanced; every mark is power of 2

2 consequent marks differ in multiplication factor 2.

(b) Let us consider

fatalities on Jun 29

and # ~~deads~~ infections on Jun 29:

fatalities $\approx$ 256 $= 2^8$

infenctions $\approx 2^{16} = 65536$

$$\text{Ratio } \frac{\#\text{ fatalities}}{\#\text{ infections}} = \frac{2^8}{2^{16}} = \frac{1}{2^8} = \frac{1}{256} << \frac{1}{2}$$

$\approx 0.004 = 0.4\%$

So it is ~~two~~ much smaller than half.

(c) Green (dark) indicates # of infections in USA.

Blue = # of infections total

On Jun 29 for example

inf. in USA $= 2^{15}$

inf tota $= 2^{16}$

$$\Rightarrow \frac{\#\text{ inf USA}}{\#\text{ inf tot}} = \frac{2^{15}}{2^{16}} = \frac{1}{2}$$

So Half of all infection accured in USA

Q9 (continue)

(c) Fig 3 is a barchart ; lin-log barchart

- Conclusion: barchart of log datas follows approx. normal distribution

(d) Fig 5, graph is log-log

That means it would be correct to say: log of frequensies against log of daily rainfall is linear.