MAST30022 Decision Making
2021
Tutorial 7

1. **(PS5-7)** Let

$$A = \{1, 2, 3, 4\}$$

and

$$\theta = \{(1, 1), (1, 3), (2, 3), (3, 4), (4, 4)\}.$$

Is θ transitive?

Solution

Since $1\theta3$ and $3\theta4$ but $\neg 1\theta4$, θ is not transitive.

2. **(PS5-8)** Let $\mathbb{Z}$ be the set of integers, and k a positive integer. Let θ be the binary relation on $\mathbb{Z}$ defined by

$$\theta = \{(x, y) : k \text{ is a divisor of } x - y\}.$$

Prove that θ is an equivalence relation on $\mathbb{Z}$, that is, θ is reflexive, transitive, and symmetric.

Solution

- **Reflexivity:** Let $x, y, z \in \mathbb{Z}$. Since k is a divisor of $0 = x - x$, by the definition of θ we have $x\theta x$ and hence θ is reflexive.
- **Transitivity:** If $x\theta y$ and $y\theta z$, then k is a divisor of $x - y$ and a divisor of $y - z$. So $x - y = ak$ and $y - z = bk$ for some integers a and b. Hence $x - z = (x - y) + (y - z) = ak + bk = (a + b)k$, and k is a divisor of $x - z$, that is, $x\theta z$. Thus θ is transitive.
- **Symmety:** Finally, if $x\theta y$, then k is a divisor of $x - y$ and hence is also a divisor of $y - x$, that is, $y\theta x$. Hence θ is symmetric.

3. **(PS5-9)** Let $A_1, \ldots, A_n$ be sets equipped with binary relations $\succeq_1, \ldots, \succeq_n$ respectively. Let

$$A = A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_i \in A_i, 1 \leq i \leq n\}$$

be the Cartesian product of $A_1, \ldots, A_n$. Define a binary relation $\succeq$ on A such that

$$(a_1, \ldots, a_n) \succeq (b_1, \ldots, b_n)$$

if and only if $a_i \succeq_i b_i$ for all $i = 1, 2, \ldots, n$.

(a) Prove that, if all $\succeq_i$ $(i = 1, 2, \ldots, n)$ have one and the same of the following properties:
   – transitivity
   – reflexivity
   – symmetry
   – antisymmetry
   – asymmetry
   then $\succeq$ has the same property.
(b) Prove or disprove the following statement: if all $\succeq_i$ $(i = 1, 2, \ldots, n)$ are weak orders then so is $\succeq$.

Solution

(a) **Transitivity:** Suppose

$$(a_1, \ldots, a_n) \succcurlyeq (b_1, \ldots, b_n)$$

$$(b_1, \ldots, b_n) \succcurlyeq (c_1, \ldots, c_n).$$

By the definition of $\succcurlyeq$,

$$a_i \succcurlyeq_i b_i, b_i \succcurlyeq_i c_i$$

for all $i = 1, 2, \ldots, n$. But each $\succcurlyeq_i$ is transitive by our assumption. So

$$a_i \succcurlyeq_i c_i$$

for all $i = 1, 2, \ldots, n$. This means that

$$(a_1, \ldots, a_n) \succcurlyeq (c_1, \ldots, c_n)$$

and hence $\succcurlyeq$ is transitive.

Reflexivity: Since all $\succcurlyeq_i$ are reflexive by our assumption, we have $a_i \succcurlyeq_i a_i$ for all i. Thus

$$(a_1, \ldots, a_n) \succcurlyeq (a_1, \ldots, a_n)$$

and $\succcurlyeq$ is reflexive.

Symmetry: Suppose

$$(a_1, \ldots, a_n) \succcurlyeq (b_1, \ldots, b_n).$$

Then

$$a_i \succcurlyeq_i b_i$$

for all $i = 1, 2, \ldots, n$. Since all $\succcurlyeq_i$ are symmetric by our assumption, we have

$$b_i \succcurlyeq_i a_i$$

for all $i = 1, 2, \ldots, n$. Thus,

$$(b_1, \ldots, b_n) \succcurlyeq (a_1, \ldots, a_n).$$

So we have proved that

$$(a_1, \ldots, a_n) \succcurlyeq (b_1, \ldots, b_n) \Longrightarrow (b_1, \ldots, b_n) \succcurlyeq (a_1, \ldots, a_n)$$

and hence $\succcurlyeq$ is symmetric.

Antisymmetry: Suppose

$$(a_1, \ldots, a_n) \succcurlyeq (b_1, \ldots, b_n)$$

$$(b_1, \ldots, b_n) \succcurlyeq (a_1, \ldots, a_n).$$

Then

$$a_i \succcurlyeq_i b_i, b_i \succcurlyeq_i a_i$$

for all $i = 1, 2, \ldots, n$. But each $\succcurlyeq_i$ is antisymmetric. So we must have

$$a_i = b_i$$

for each i. In other words,

$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$$

and hence $\succcurlyeq$ is antisymmetric.

Asymmetry: Suppose

$$(a_1, \ldots, a_n) \succcurlyeq (b_1, \ldots, b_n).$$

Then

$$a_i \succcurlyeq_i b_i$$

for all $i = 1, 2, \ldots, n$. But each $\succcurlyeq_i$ is asymmetric. So we must have

$$b_i \not\succcurlyeq_i a_i$$

for each i. This implies that

$$(b_1, \ldots, b_n) \not\succcurlyeq (a_1, \ldots, a_n)$$

and hence $\succcurlyeq$ is asymmetric. (Note that $(b_1, \ldots, b_n) \not\succcurlyeq (a_1, \ldots, a_n)$ as long as $b_i \not\succcurlyeq_i a_i$ for at least one i.)

(b) The statement is false since $\succcurlyeq$ does not inherit comparability from $\succcurlyeq_i$'s. For example, if we take all $A_i = \mathbb{R}$ and all $\succcurlyeq_i$'s to be $\geq$ in the usual sense, then $\succcurlyeq_i$'s are comparable but $\succcurlyeq$ is not comparable. For instance, in the case where $n = 2$ we have

$$(1, 4) \not\succcurlyeq (3, 2) \text{ (as } 1 \not\geq 3)$$

and

$$(3, 2) \not\succcurlyeq (1, 4) \text{ (as } 2 \not\geq 4).$$

4. **(PS5-10)** Let

$$A = \{(-2,3,1),(0,1,-1),(1/2,4,2),(3,2,1),(5,-1,0)\}.$$

(a) List the lexicographic order of A, and find the greatest and least elements of A.

(b) For the Pareto order on A draw the corresponding directed graph such that for each directed edge the tail is "better than" the head. Find the sets $P_{\min}(A)$ and $P_{\max}(A)$, and the Pareto greatest and least elements (if any) of A.

Solution

(a) The lexicographic order is:

$$(5,-1,0),(3,2,1),(1/2,4,2),(0,1,-1),(-2,3,1)$$

Under this order the greatest element is $(5,-1,0)$ and the least element is $(-2,3,1)$.

(b) Denote

$$a = (-2,3,1), b = (0,1,-1), c = (1/2,4,2)$$
$$d = (3,2,1), e = (5,-1,0).$$

The graphical representation of the Pareto order on A is shown in Figure 1.

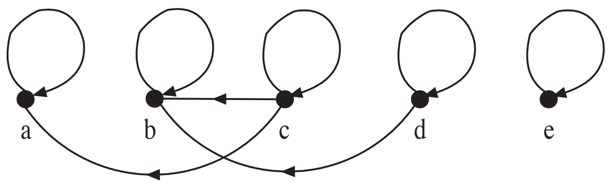

Figure 1: PS5-10

From the graph we see that there is no greatest element or least element in A under the Pareto order.

$P_{\min}(A)$ is the set of vertices without leaving edges (apart from loops).

$P_{\max}(A)$ is the set of vertices without entering edges (apart from loops).

Hence

$$P_{\min}(A) = \{a,b,e\}, \quad P_{\max}(A) = \{c,d,e\}.$$

5. **(PS5-14)** Let $A = \{(2,0,3), (1,2,3), (-4,-2,1), (8,1,1), (-5,0,2), (2,1,1), (-6,-2,0)\}$.

(a) List the lexicographic order of A, and find the greatest and least elements of A under this order.

(b) Give the Pareto order of A and represent it using a Boolean matrix. Find $P_{\min}(A)$ and $P_{\max}(A)$. Also, find the greatest and least elements (if any) of A under the Pareto order.

Solution

(a) The lexicographic order of A is:

$$(8,1,1), (2,1,1), (2,0,3), (1,2,3), (-4,-2,1), (-5,0,2), (-6,-2,0).$$

So $(8,1,0)$ is the greatest element and $(-6,-2,0)$ is the least element of A w.r.t. the lexicographic order.

(b) Let

$$a = (2,0,3),\ b = (1,2,3),\ c = (-4,-2,1),\ d = (8,1,1),$$
$$e = (-5,0,2),\ f = (2,1,1),\ g = (-6,-2,0).$$

The Boolean matrix representation is

$\succeq_P$	a	b	c	d	e	f	g
a	$\times$		$\times$		$\times$		$\times$
b		$\times$	$\times$		$\times$		$\times$
c			$\times$				$\times$
d			$\times$	$\times$		$\times$	$\times$
e					$\times$		$\times$
f			$\times$			$\times$	$\times$
g							$\times$

Minimal elements with respect to $\succeq_P$ have, for every $\times$ in a row of the Boolean matrix, a symmetric $\times$. Thus

$$P_{\min}(A) = \{g\}.$$

Maximal elements with respect to $\succeq_P$ have, for every $\times$ in a column of the Boolean matrix, a symmetric $\times$. Thus

$$P_{\max}(A) = \{a, b, d\}.$$

Least elements have the entire column populated with $\times$s. Thus g is a least element (the only one). Greatest elements have the entire row populated with $\times$s. Thus there are no greatest elements.

6. **(PS5-15)** Seven clerks share an office. Each has an ideal working temperature τ_i ($i = 1, 2, \ldots, 7$) with

$$\tau_1 < \tau_2 < \tau_3 < \tau_4 < \tau_5 < \tau_6 < \tau_7.$$

Their individual preferences between any two room temperatures t and t' depends only on the magnitude of the departure of t and t' from their ideal, that is,

$$t \succeq_i t' \iff |t - \tau_i| \leq |t' - \tau_i|$$

for $i = 1, 2, \ldots, 7$.

They decide as a group to adopt the fourth clerk's preference and set the room temperature to τ_4 because three would prefer a cooler room and three would prefer a warmer room. Show that in doing so they are implicitly adopting the simple majority rule; that is, show that, when $t \succ_4 t'$, at least three others hold $t \succ_i t'$, and that when $t \sim_4 t'$, the number who hold $t \succ_i t'$ equals the number who hold $t' \succ_i t$. (Adapted from "Decision Theory", S. French, 1986)

Solution

One can show that, if t is nearer τ_4 than t', then t is nearer at least three other ideal points than t'. (There are two possibilities to consider: t and t' are on different and on the same sides of τ_4.) In other words, if $t \succ_4 t'$, then there are at least three other $i \neq 4$ such that $t \succ_i t'$.

If $t \sim_4 t'$, then $|t - \tau_4| = |t' - \tau_4|$ and t, t' are on different sides of τ_4. So the number who hold $t \succ_i t'$ equal the number who hold $t' \succ_i t$.

In particular, since $\tau_4 \succ_4 t'$ for any $t' \neq \tau_4$, most clerks prefer τ_4 than any other choice. Thus by choosing τ_4 the clerks are using the simple majority rule.