School of Mathematics and Statistics
MAST30030
Applied Mathematical Modelling

Problem Sheet VECTOR - Practice Class.
Revision of Vector Analysis

Question 1

Vector Properties. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Prove using the standard identities of vector analysis that

(a) $\nabla^2 \left(\dfrac{1}{r}\right) = 0, \quad r \neq 0;$ (b) $\nabla^2 \left(r^n\right) = n(n+1)r^{n-2};$ (c) $\nabla \cdot \left(\dfrac{\mathbf{r}}{r^3}\right) = 0;$

(d) $\nabla \cdot (r^n \mathbf{r}) = (n+3)r^n;$ (e) $\nabla \times \left(\dfrac{\mathbf{r}}{r}\right) = \mathbf{0};$ (f) $\nabla \times (r^n \mathbf{r}) = \mathbf{0}.$

where n is any real number.

Question 2

Cylindrical Coordinates. Define curvilinear coordinates (σ, ϕ, z) by

$$x = \sigma \cos\phi, \qquad y = \sigma \sin\phi, \qquad z = z.$$

where $\sigma \geq 0, 0 \leq \phi \leq 2\pi$. If n is an integer, evaluate the following quantities:

(a) $\nabla\phi$ (b) $\nabla\sigma^n$ (c) $\nabla^2(\sigma^2 \cos\phi)$.

Note that in MAST30030, we denote cylindrical coordinates by (σ, ϕ, z) since we reserve ρ for the density.

Question 3

Spherical Coordinates. Define curvilinear coordinates (r, θ, ϕ) by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

where $r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$. Evaluate the following quantities:

(a) $\nabla\phi$ (b) $\nabla\theta$ (c) $\nabla \cdot (\hat{\mathbf{r}} \cot\phi - 2\hat{\phi})$.

Question 4

Coordinate Systems. Using spherical coordinates, express each of the orthonormal vectors $\hat{\mathbf{r}}$, $\hat{\theta}$ and $\hat{\phi}$ in terms of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ and (x, y, z).

Some Applications of Vector Analysis

Question 5

Consider the vector $\mathbf{q} = q(\theta)\mathbf{k}$ where $\mathbf{k}$ is the unit vector in the z direction and θ is the spherical polar angle,

i. express the integral

$$\int_S d\mathbf{S} \cdot \mathbf{q}$$

in terms of the r and θ components of $\mathbf{q}$ if S is the surface of a sphere of radius a

ii. hence evaluate the integral if $q(\theta) = \cos\theta$

Question 6

The torque (about the origin) acting on a particle is defined by

$$\Lambda = \mathbf{r} \times \mathbf{F}$$

where $\mathbf{r}$ is the position vector of the point of application of the force $\mathbf{F}$, relative to the origin. The total torque on a rigid body is found by integrating the torque acting on each element of the body. In fluid mechanics, the torques usually act only on the surface of the body.

i. Suppose a cylinder of radius a is rotating about its axis so that the surrounding fluid produces a force per unit area acting on its surface:

$$\mathbf{f} = -D\mathbf{e}_\phi$$

Find the total torque acting on the cylinder, per unit length.

ii. Suppose a sphere of radius a is rotating about the z-axis so that the surrounding fluid produces a force per unit area acting on its surface:

$$\mathbf{f} = -D\mathbf{e}_\phi$$

Find the total torque acting on the sphere.