ENGR30002 Fluid Mechanics

Dr Lionel Lam
lionel.lam@unimelb.edu.au

3 Conservation Laws

What we did in the last module

- In the previous module, we discussed dimensional reasoning
  - Deformations in solids and liquids
  - Force that cause/resist flow
  - The use of dimensional analysis to derive dimensionless parameters key to understanding fluid flow
    - Reynolds number, Re
    - Weber number We
    - Bond number, Bo
    - Froude number, Fr

Learning objectives

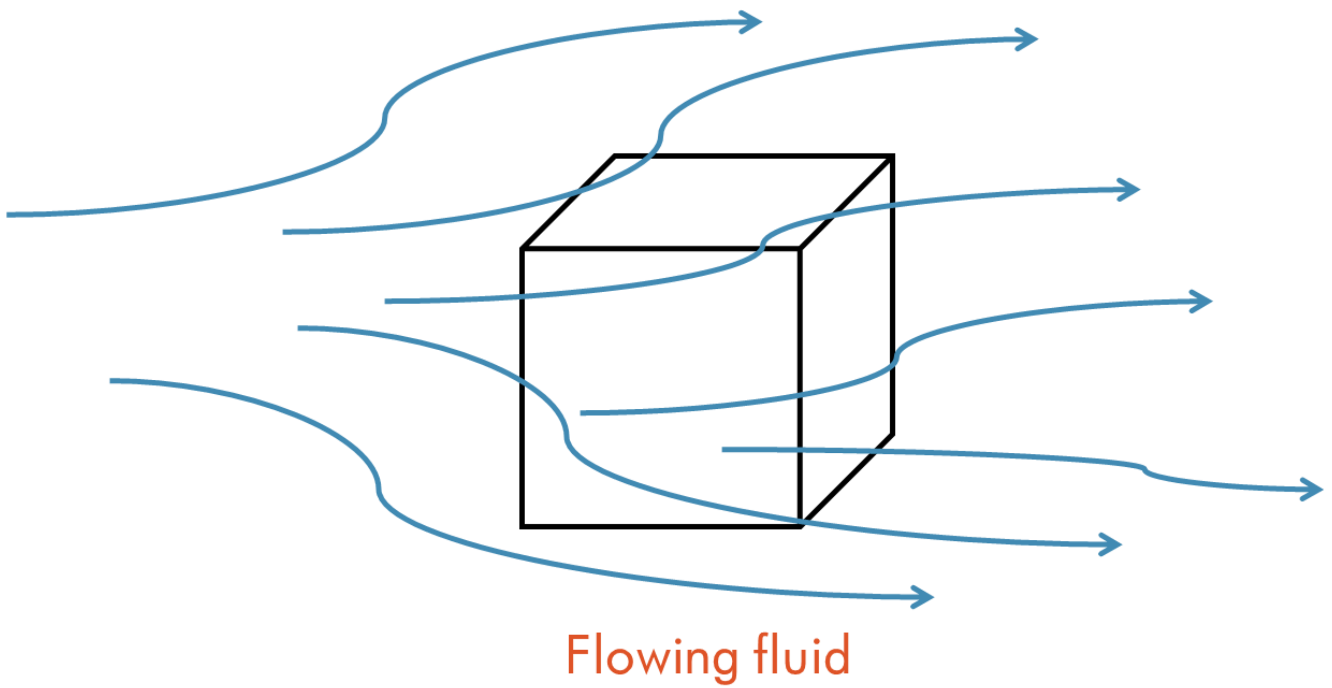
- By the end of this lesson, students should be able to:
  - Understand the derivations of the conservation laws for fluid flow
    - Mass
    - Momentum
    - Energy
  - Describe the assumptions that are built int ach of these equations, and know when they can and cannot be used
  - Use the conservation laws to solve fluid flow problems

The mathematics of fluid flow

- A large portion of this course is dedicated to discovering **mathematical relationships** that will allow you to describe important parameters of fluid flow
  - Pressure
  - Velocity
  - Flow rate (mass flow rate [kg/s] & volumetric flow rate [m^3/s])
- Most of these relationships are based on **conservation laws**
  - Conservation of mass
  - Conservation of momentum
  - Conservation of energy
- More specifically, we will explore conservation of mass, momentum, and energy for **steady state** fluid flow in **closed channels**

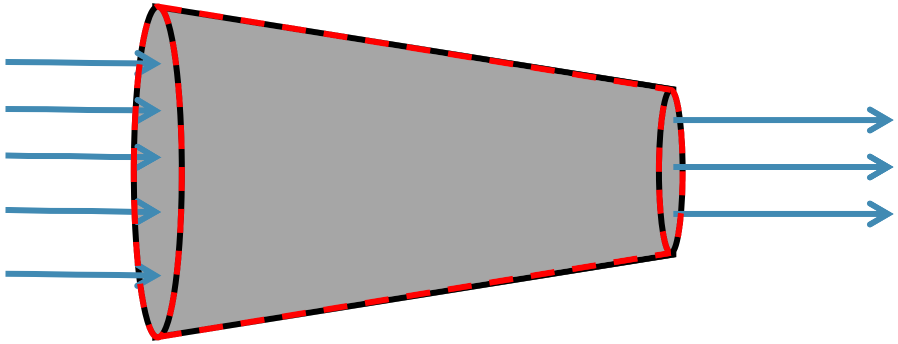
Control volumes

- In order to generate these mathematical relationships, we will define **control volumes** and analyse the movement of mass, energy, and momentum into and out of these volumes

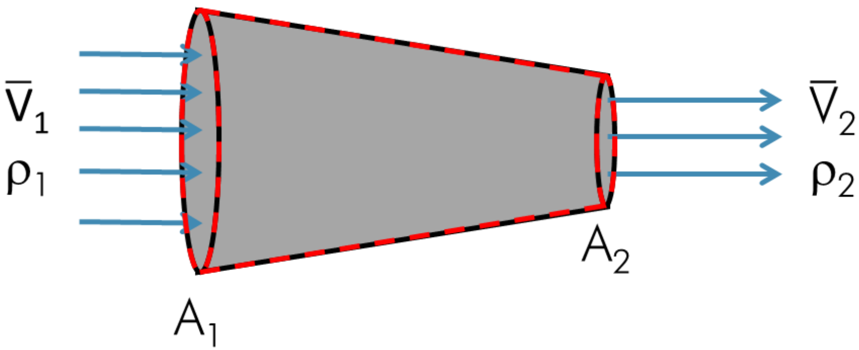


Conservation of mass

- **Conservation of mass** is the simplest of the three relationships:
  - We know that mass cannot be created or destroyed, so all the mass that flows into our control volume will have to flow out
  - Applying a mass balance over a control volume results in the **equation of continuity**
- We have a fluid flowing through a constricting pipe
  - There is no flow of fluid through the walls of the pipe
  - There is no accumulation of mass within the pipe (**steady state)**
  - There entire pipe is the control volume



Equation of continuity



$\bar{V}$ = average velocity
ρ = density
A = cross sectional area

Note: we're working with **average velocity**. This is because the fluid flow across the cross section is not constant, for either laminar or turbulent flow.

- At steady state, the mass of fluid flowing into the pipe through A_1 must be equal to the mass of fluid flowing out of the pipe through A_2

$$G_1 = G_2$$

G is mass flow rate [mass/time]

- But it's easier to measure fluid velocity at a given point in the flow than the mass flow rate, so writing this in terms of velocity:

$$G = \rho V A$$

$$\boxed{\therefore \boldsymbol{\rho_1 V_1 A_1 = \rho_2 V_2 A_2}}$$

Equation of continuity
Incompressible fluids

- If our fluid is a liquid, it is **incompressible**
  - What do we mean when we say a liquid is incompressible?

$$\frac{d\rho}{dP} = 0$$

  - Using this knowledge, we can simplify the equation of continuity:

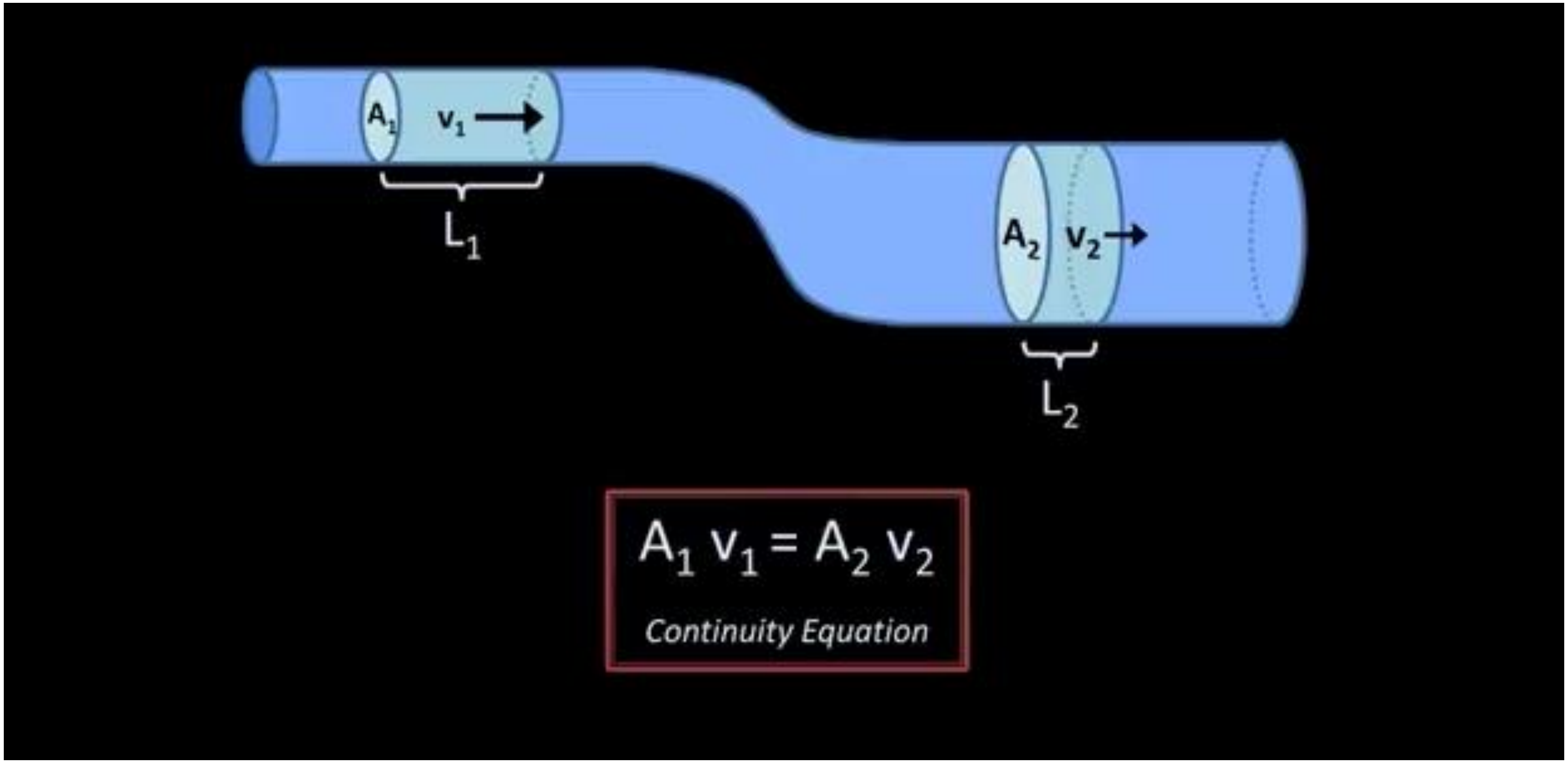
$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$
$$\Rightarrow \rho V_1 A_1 = \rho V_2 A_2$$
$$\Rightarrow V_1 A_1 = V_2 A_2$$
$$\therefore Q_1 = Q_2$$

Q = volumetric flow rate [vol/time]

- For liquids at constant temperature, both the mass flow rate, G and volumetric flow rate, Q are constant
  - This does not mean that the density is always constant. It can still change with other variables such as temperature. However, it is constant with pressure

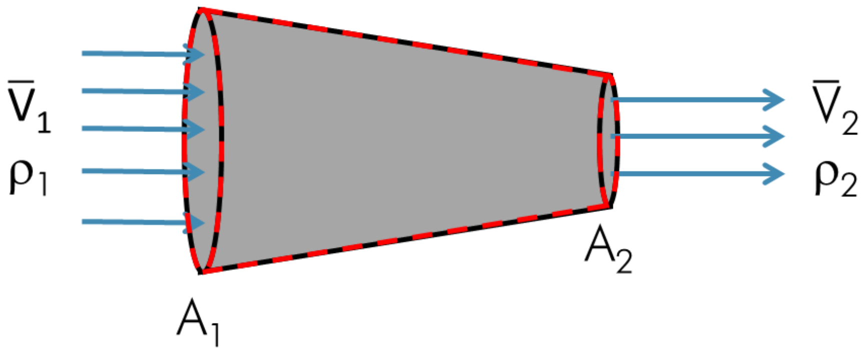
Equation of continuity

Incompressible fluids



Applying the equation of continuity

- An incompressible liquid is flowing through a pipe of variable cross-sectional area. If $\bar{V}_1 = 1$ [m/s], $A_1 = 25$ [cm^2], and $A_2 = 10$ [cm^2], what is $\bar{V}_2$?

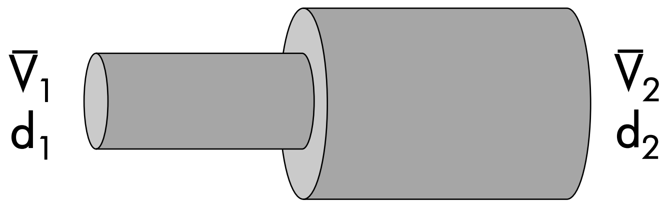


$$\rho_1 \bar{V}_1 A_1 = \rho_2 \bar{V}_2 A_2 \Rightarrow \bar{V}_1 A_1 = \bar{V}_2 A_2$$

$$\Rightarrow \bar{V}_2 = \bar{V}_1 \left(\frac{A_1}{A_2}\right) = (1)\left(\frac{25}{10}\right) = 2.5\ [m/s]$$

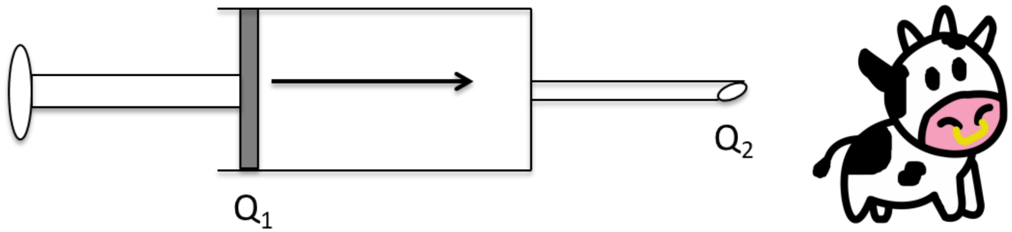
Challenge question 3.1

- An incompressible liquid flows through a rapid expansion in a pipe. The pipe diameter suddenly increases from $d_1=10$ [cm] to $d_2=20$ [cm]. The volumetric flow rate is 10 [L/min]. What are the average velocities in and out?



Example problem 3.1

- A syringe is used to inoculate a cow. The plunger has a face area of 500 [mm^2]. The liquid in the syringe must be injected steadily at a rate of 0.3 [L/min]. At what speed should the plunger advance?



Conservation of linear momentum

- The next relationship we're going to develop is **conservation of momentum**
- … What's momentum again?
  - Momentum is the product of mass and velocity

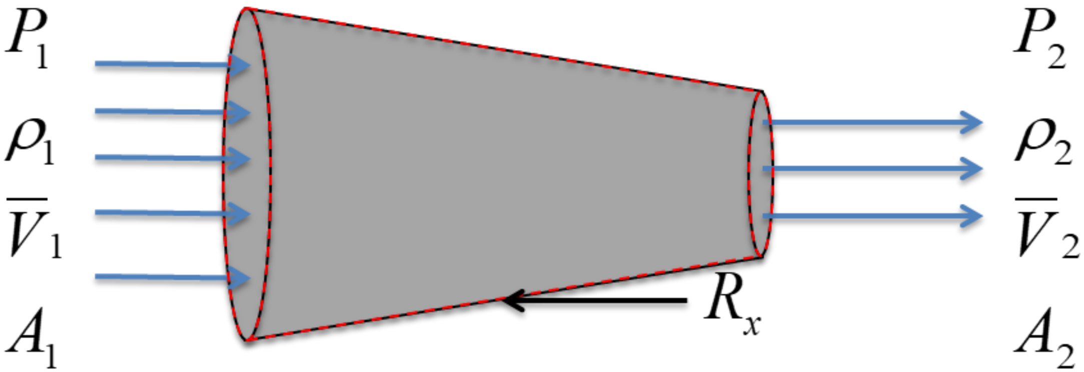
$$p\ [N\ s] = m\ [kg] \times V\ [m/s]$$

- From Newton's Second Law of Motion, we know that force equals the rate of change of momentum

Conservation of linear momentum

- The **rate of change of momentum** as the fluid moves from Point 1 to Point 2 is equal to **momentum out per unit time** minus **momentum in per unit time**

Rate of change of momentum = momentum out per time – momentum in per time



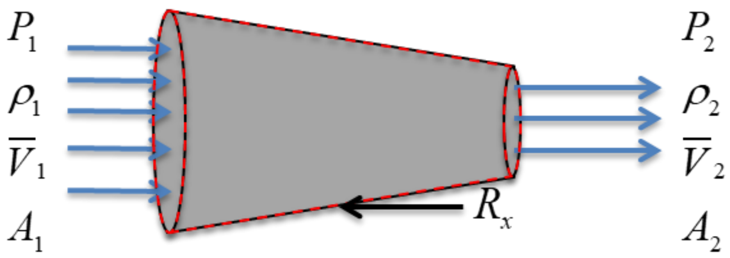
R_x is the force exerted by the duct on the fluid

- We need to recast this relationship in terms of our system variables

Conservation of linear momentum

Right hand side

- Let's determine the RHS of the equation first
  - RHS = momentum out per time – momentum in per time



$$\text{Momentum out per time} = G_2 \times \bar{V}_2 = \rho_2 \bar{V}_2 A_2 \times \bar{V}_2 = \rho_2 A_2 \bar{V}_2^2$$

[N] ≡ [kg m s^{-2}] [kg/s] [m/s]

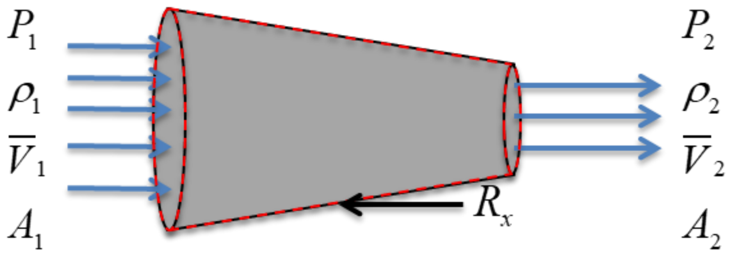
$$\text{Momentum in per time} = \rho_1 A_1 \bar{V}_1^2$$

- RHS $= \rho_1 A_1 \bar{V}_1^2 - \rho_2 A_2 \bar{V}_2^2$

Conservation of linear momentum

Left hand side

- Now for the LHS
  - LHS = Rate of change of momentum
  - We know from Newton's Second Law of Motion, that force = rate of change of momentum
- Therefore, we need to perform a force balance on the fluid to evaluate the LHS of the equation
  - Forces acting in direction of flow = $P_1 A_1$
  - Forces acting in opposite direction of flow = $P_2 A_2 + R_x$
  - LHS = Rate of change of momentum = $P_1 A_1 - P_2 A_2 - R_x$



Conservation of linear momentum

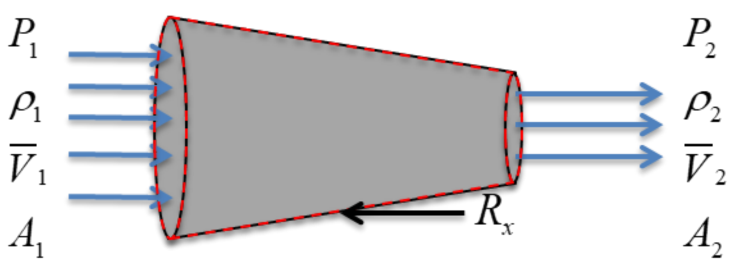
- Putting it all together, our momentum balance becomes:

$$P_1A_1 - P_2A_2 - R_x = \rho_2A_2\bar{V}_2^2 - \rho_1A_1\bar{V}_1^2$$

- But from **conservation of mass**, we know:

$$\rho_2A_2\bar{V}_2 = \rho_1A_1\bar{V}_1$$

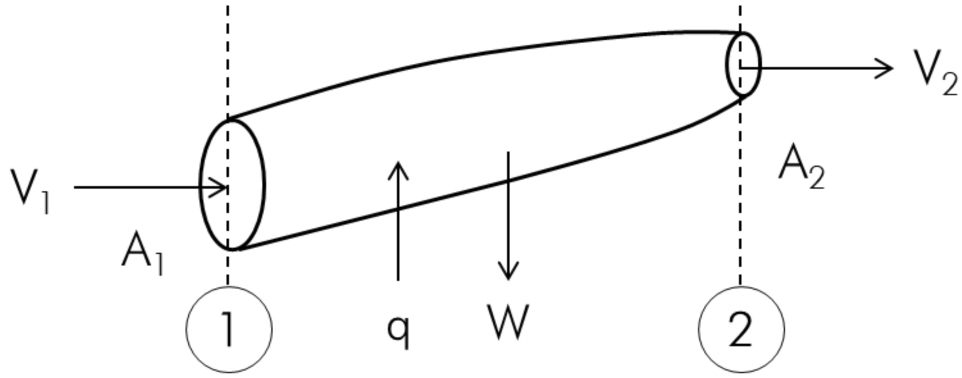
$$\therefore R_x = \rho_1A_1\bar{V}_1^2\left(1 - \frac{\rho_1A_1}{\rho_2A_2}\right) + P_1A_1 - P_2A_2$$



R_x is the force exerted by the duct wall on the fluid. It is also the force required to hold the duct stationary.

Conservation of energy

- **Conservation of energy** will give us the most information about the flow – consider flow through an arbitrarily-shaped volume:



- The most general form of the energy balance has the following form:

$$\underbrace{E_2 - E_1}_{\text{increase in total stored energy per unit mass in the fluid passing from Point 1 to Point 2}} = \underbrace{q}_{\text{net heat per unit mass added to the fluid passing from Point 1 to Point 2}} - \underbrace{W}_{\text{net work done per unit mass by the fluid on the surroundings in passing from Point 1 to Point 2}}$$

increase in total stored energy per unit mass in the fluid passing from Point 1 to Point 2

net heat per unit mass <u>added to the fluid</u> passing from Point 1 to Point 2

net work done per unit mass by the fluid <u>on the surroundings</u> in passing from Point 1 to Point 2

Conservation of energy
Stored energy

$$\underline{E_2 - E_1} = q - W$$

increase in **total stored energy**
per unit mass in the fluid
passing from Point 1 to Point 2

- Energy in the fluid can be stored in **three** forms:
  1. Internal energy, U
     - Energy associated with internal molecular and atomic motions
     - Depends on ρ and T
  2. Kinetic energy, E_K
     - Energy associated with motion
     - Depends on $\frac{1}{2}\bar{V}^2$
  3. Potential energy, E_P
     - Energy associated with position in gravitational field
     - Depends on gz, where z is the height above a reference height

Conservation of energy

Work done by fluid, W

$$E_2 - E_1 = q - \underline{W}$$

net work done per unit mass by the fluid <u>**on the surroundings**</u> in passing from Point 1 to Point 2

- Work done <u>by the fluid</u> can occur in **two** forms:
  1. Flow work
     - Work done on the surroundings as a result of the fluid crossing the control surfaces
     - Here, it is the **work done by pressure** as a fluid enters and leaves the control volume
  2. Shaft work, W_S
     - "Useful work" being done as the fluid flows between Points 1 & 2 e.g. driving a mechanical device via a moving shaft

Conservation of energy

Flow work

- We will not elaborate more on **shaft work, W_S** at this time
- But we can put **flow work** in terms of more useful variables:
  - Flow work is also called **pressure work**
  - It involves work done by the fluid against external pressure at the exit and…
  - Work done on the fluid by pressure at the entry
- Consider a fluid flowing through the following volume over the time duration Δt:
  - During that time, a plug of fluid will move into the control volume
  - Another plug of fluid will move out of the control volume
  - We can quantify the pressure work needed to achieve this fluid motion

Conservation of energy
Flow work

- Work = Force x Displacement
  - The force required to move the plug of fluid into the control volume is P_1A_1
  - The displacement the inlet undergoes during this time is $\bar{V}_1\Delta t$
- The work done on the fluid at Point 1 $= P_1A_1\bar{V}_1\Delta t$
- Similarly, the work done by the fluid at Point 2 $= P_2A_2\bar{V}_2\Delta t$
- So, the net flow work done <u>by the fluid</u> is:

$$P_2A_2\bar{V}_2\Delta t - P_1A_1\bar{V}_1\Delta t$$

  - But we're not quite done yet – our original energy balance was in a per unit mass basis, so this term also needs to be recast in a per unit mass basis…

Conservation of energy

Flow work

- From conservation of mass, the mass flowing into the control volume is equal to the mass following out:
  - Mass flowing in $= \rho_1 A_1 \bar{V}_1 \Delta t$
  - Mass flowing out $= \rho_2 A_2 \bar{V}_2 \Delta t$
- So the net flow work per unit mass becomes:

$$\frac{P_2 A_2 \bar{V}_2 \Delta t}{\rho_2 A_2 \bar{V}_2 \Delta t} - \frac{P_1 A_1 \bar{V}_1 \Delta t}{\rho_1 A_1 \bar{V}_1 \Delta t} = \frac{P_2}{\rho_2} - \frac{P_1}{\rho_1} \equiv P_2 v_2 - P_1 v_1$$

v is the specific volume
(reciprocal of density)
[volume/mass]

- … Why isn't friction included as part of work?
  - W is the work done on the surroundings
  - Frictional forces are considered to occur **within** the system, so work done to overcome such forces is not included in W

Conservation of energy

- Putting it all together, we have:

$$E_2 - E_1 = q - W$$

$$\left(U_2 + \frac{1}{2}\bar{V}_2^2 + gz_2\right) - \left(U_1 + \frac{1}{2}\bar{V}_1^2 + gz_1\right) = q - W_S - \left(\frac{P_2}{\rho_2} - \frac{P_1}{\rho_1}\right)$$

$$\left(U_2 + \frac{P_2}{\rho_2} + \frac{1}{2}\bar{V}_2^2 + gz_2\right) - \left(U_1 + \frac{P_1}{\rho_1} + \frac{1}{2}\bar{V}_1^2 + gz_1\right) = q - W_S$$

$\Delta U = U_2 - U_1$
$\Delta P = P_2 - P_1$
$\Delta z = z_2 - z_1$
$\Delta q =$ heat added per unit mass over infinitesimally short pipe length
$\Delta W_S =$ shaft work per unit mass over infinitesimally short pipe length

$$\Delta U + \Delta\left(\frac{P}{\rho}\right) + \Delta\left(\frac{1}{2}\bar{V}^2\right) + \Delta gz = \Delta q - \Delta W_S$$

Conservation of energy

Mechanical energy balance

- Removing internal energy (U) from our equation:

$$\Delta U + \Delta\left(\frac{P}{\rho}\right) + \Delta\left(\frac{1}{2}\bar{V}^2\right) + \Delta gz = \Delta q - \Delta W_S$$

- From thermodynamics, we have the following relationship:

$$\Delta U = \Delta q - P\Delta v + \Delta F$$

  - Δq: heat added per unit mass
  - $-P\Delta v$: reversible work done on the fluid by compression
  - ΔF: friction, mechanical energy is converted to heat

- Also note that we have the following (chain rule):

$$\Delta\left(\frac{P}{\rho}\right) = (\Delta P)\left(\frac{1}{\rho}\right) + P\Delta\left(\frac{1}{\rho}\right) \equiv \frac{\Delta P}{\rho} + P\Delta v$$

$$\boxed{\therefore \frac{\Delta P}{\rho} + \Delta\left(\frac{1}{2}\bar{V}^2\right) + g\Delta z + \Delta W_S + \Delta F = 0}$$

after 8 slides, we've **finally** arrived at the **mechanical energy balance**!

Conservation of energy
Mechanical energy balance

- The **mechanical energy balance** tells us how energy within the flow is partitioned as the fluid flows between two points:

$$\frac{\Delta P}{\rho} + \Delta\left(\frac{1}{2}\bar{V}^2\right) + g\Delta z + \Delta W_S + \Delta F = 0$$

pressure energy per unit mass

kinetic energy per unit mass

potential energy per unit mass

shaft work per unit mass

friction per unit mass

The Bernoulli equation

- The mechanical energy balance can be written in several forms
- One of the most common alternative forms of the mechanical energy balance is the **Bernoulli equation**, which assumes there is:
  - No shaft work
  - No friction

$$\frac{\Delta P}{\rho} + \Delta\left(\frac{1}{2}\bar{V}^2\right) + g\Delta z + \Delta W_S + \Delta F = 0$$

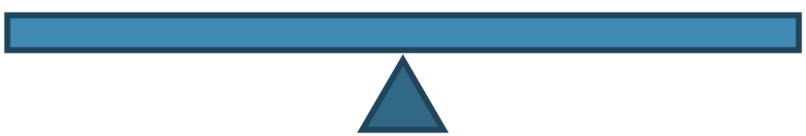
$$\frac{\Delta P}{\rho} + \Delta\left(\frac{1}{2}\bar{V}^2\right) + g\Delta z = 0$$

$$\boxed{\frac{P_1}{\rho} + \frac{1}{2}\bar{V}_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}\bar{V}_2^2 + gz_2}$$

The Bernoulli equation

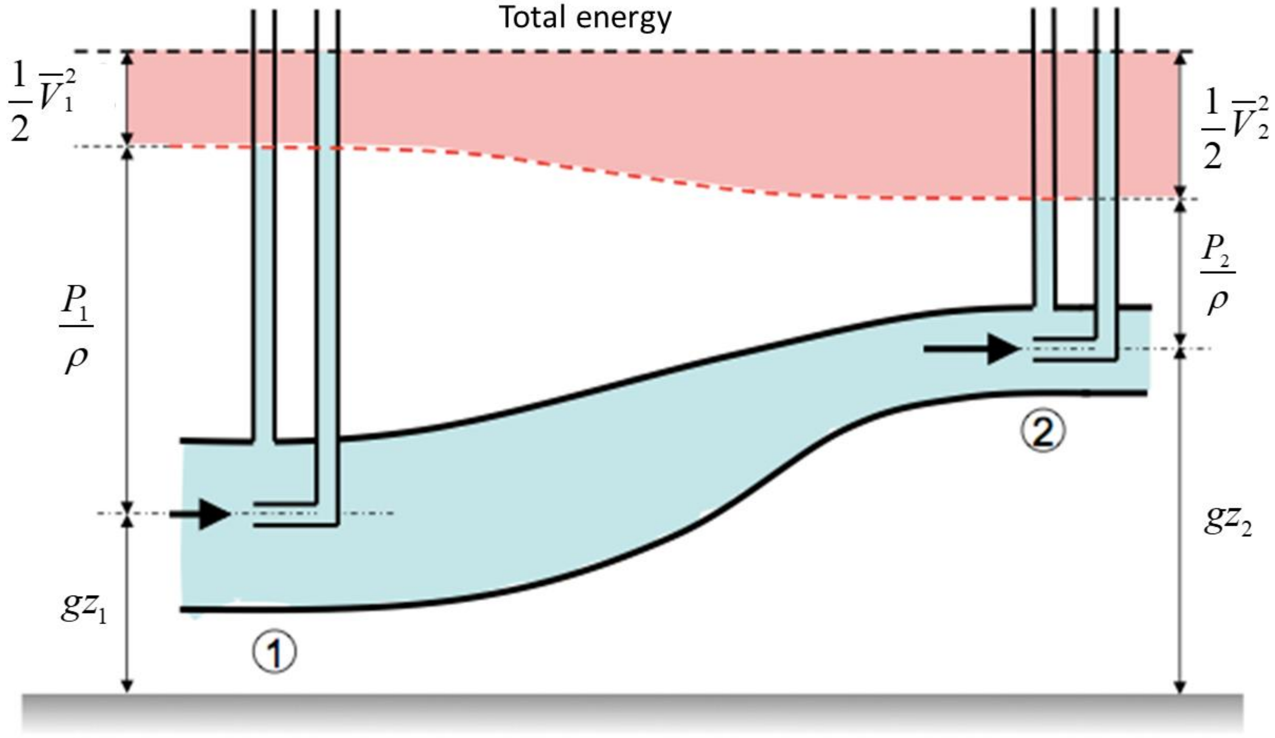
- The Bernoulli equation provides us with three key pieces of information:
    1. A stationary fluid only has potential energy ($\rho g h$), while a flowing fluid has potential energy, kinetic energy, and pressure energy
    2. The mechanical energy at Point 1 equals the mechanical energy at Point 2 – therefore, the total mechanical energy of the fluid is conserved
    3. However, the partitioning of the fluid's energy can change as a fluid flows

$$\frac{P_1}{\rho} + \frac{1}{2}\bar{V}_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}\bar{V}_2^2 + gz_2$$



The Bernoulli equation

- When friction is neglected and there is no shaft work, the energy at any point in the flow is constant
- However, the partitioning of the energy changes



The Bernoulli equation

- The Bernoulli equation can also be written in terms of **head** by dividing through by gravity

$$\frac{P_1}{\rho} + \frac{1}{2}\bar{V}_1^2 + g z_1 = \frac{P_2}{\rho} + \frac{1}{2}\bar{V}_2^2 + g z_2$$

Bernoulli equation written in terms of energy

- Pressure energy
- Kinetic energy
- Potential energy

$$\frac{P_1}{\rho \cancel{g}} + \frac{1}{2\cancel{g}}\bar{V}_1^2 + z_1 = \frac{P_2}{\rho \cancel{g}} + \frac{1}{2\cancel{g}}\bar{V}_2^2 + z_2$$

Bernoulli equation written in terms of head

- Pressure head
- Velocity head
- Gravitational head

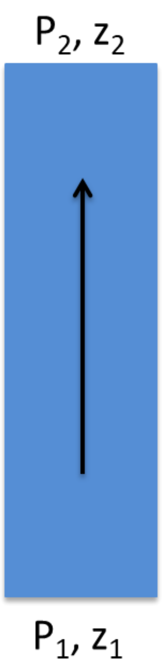
- These equations represent the same thing – they are just different notations. Notice the units of head is metres [m].

Who's Bernoulli?

- **Daniel Bernoulli** was a Swiss mathematician and physicist. He lived from 1700-1782. A large portion of his work focused on applying mathematics to mechanics, specifically fluid mechanics.
- His name is commemorated in **Bernoulli's principle**, an example of conservation of energy in fluid mechanics which states that an increase in speed of a fluid occurs simultaneously with a decrease in pressure or a decrease in potential energy.

Examples of Bernoulli's principle

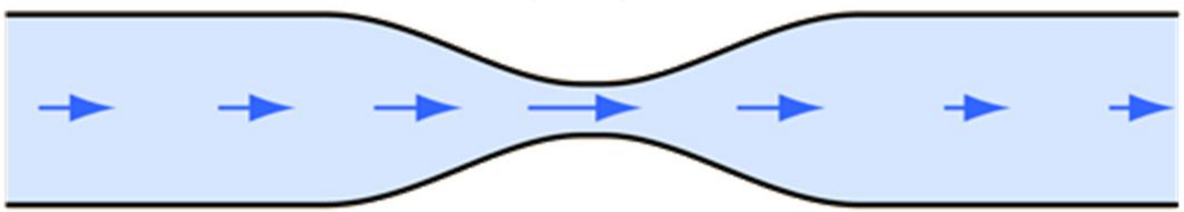
- Example: you are pumping a fluid vertically against gravity
  - In this scenario, the fluid will flow from high to low pressure, so as the fluid flows from Point 1 to Point 2, its pressure energy decreases
  - However, due to its increase in height in the gravitational field, that pressure energy is converted into potential energy



$$\frac{P_1}{\rho} + \frac{1}{2}\bar{V}_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}\bar{V}_2^2 + gz_2$$

Examples of Bernoulli's principle

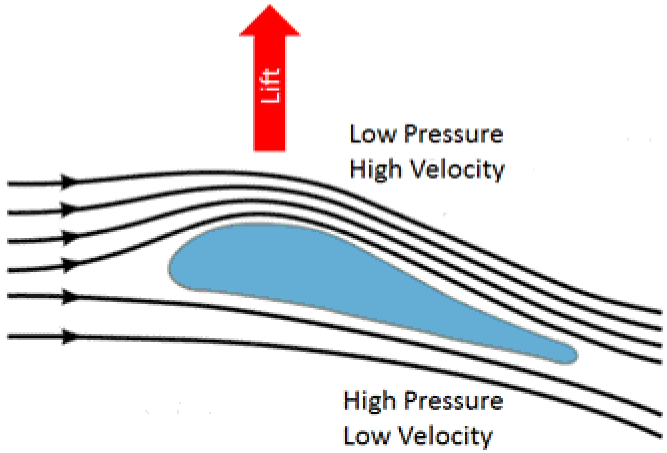
- A liquid flows horizontally through a pipe with a constriction:

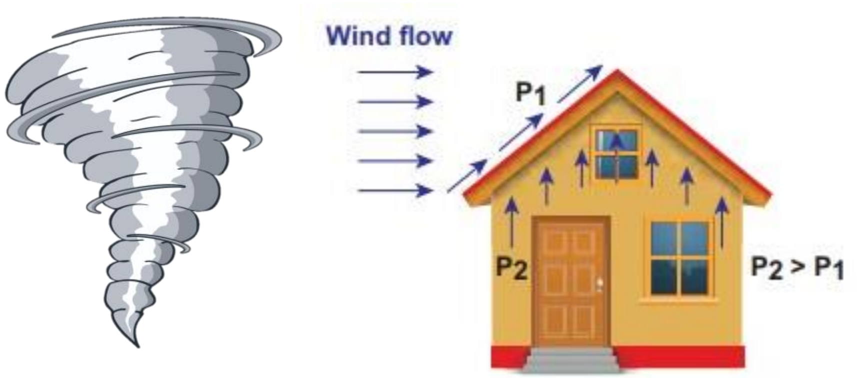


$$\frac{P_1}{\rho} + \frac{1}{2}\bar{V}_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}\bar{V}_2^2 + gz_2$$

  - Does the potential energy of the fluid change in this system?
  - What will happen to kinetic energy as the fluid passes through the constriction?
  - How will that affect the pressure energy of the system?

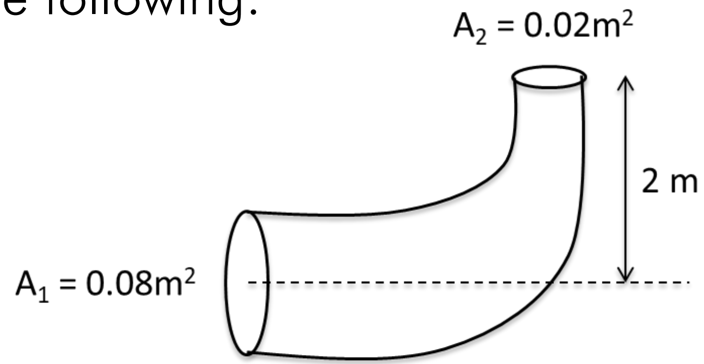
Examples of Bernoulli's principle





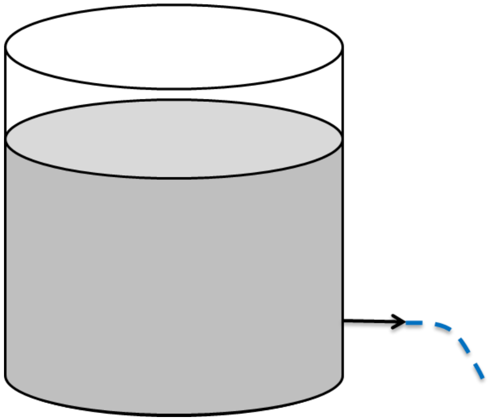
Example problem 3.2

- Water is flowing through a pipe at the rate of 0.08 [m^3/s]. The pressure at Point 1 is 180 [kPa]. Find the following:
  - The velocity of the fluid at Point 1
  - The velocity of the fluid at Point 2
  - The pressure at Point 2



Example problem 2.4

- A nearsighted sheriff fires his gun at a cattle thief. Fortunately for the thief, the bullet misses him and penetrates the town's water tank instead and cases a leak. The top of the tank is open to the atmosphere. Determine the speed at which the water leaves the hole when the water level is 0.5 m above the hole.



Summary

- Several conservation laws can be used to quantify fluid flow
  - Conservation of mass
    - For steady state flow, the mass flow in is equal to the mass flow out
    - For incompressible flow, volumetric flow in is equal to volumetric flow out
  - Conservation of momentum
    - Can determine the amount of force the pipe is experiencing due to the flow
  - Conservation of energy
    - Energy leaves/enters the system as heat or shaft work
    - Energy within the flow can be in multiple forms:
      - Pressure energy
      - Kinetic energy
      - Potential energy
    - The Bernoulli equation is an idealised form of the energy balance where shaft work and friction are assumed to be zero
    - When using the Bernoulli equation, total energy is constant but can change from one from to another