Topic 2: Covariance functions and variograms modelling

In this topic we continue to study covariance functions and variograms. In particular, we consider

- **Properties** of covariance functions.
- Methods to obtain **basic covariance functions.**

Covariance functions and positive definiteness

In spatial analysis covariance functions are used to describe dependencies between different spatial locations.

Unlike spatial trends that can be arbitrary functions covariance functions belong to the class of positive definite functions. We use $\mathcal{P}_T$ to denote the family of positive definite functions on the set T.

Recall that by the definition, $B(t, s) \in \mathcal{P}_T$ if and only if

$$\boxed{\sum_{k=1}^{n} \sum_{l=1}^{n} B(t_k, t_l) c_k \bar{c}_l \geq 0}$$

for all integer n, elements $t_k, k = 1, ..., n$, from the set T, and complex numbers $c_k, k = 1, ..., n$.

Properties of elements in $\mathcal{P}_T$

We start with some properties of positive definite functions. Let $B(t,s) \in \mathcal{P}_T$. Then:

(1) $B(t,t) \geq 0$ for all $t \in T$.
(2) $B(t,s) = \overline{B(s,t)}$.
(3) $|B(t,s)|^2 \leq B(t,t)B(s,s)$.

If a function does not satisfy these properties, then it is not a covariance functions. It can be used to eliminate not valid models for dependencies.

Example 3. Let $B(t,s) = t + s - 1,\ T = [0,1]$. Selecting $s = t = 0$ we get

$$B(0,0) = 0 + 0 - 1 = -1 < 0,$$

which contradicts property (1). Hence, $B(t,s) = t + s - 1$, is not a covariance function on $T = [0,1]$.

Methods to obtain covariance functions.

If a function satisfies the mentioned properties it is not necessary from $\mathcal{P}_T$ as there may be other properties not listed here that fail. Thus, not all functions that satisfy properties (1)-(3) above are covariance functions.

How can one then to check that a function is a positive definite? One can start with the following method to obtain basic covariance functions.

Consider any function $f(t) : T \to \mathbb{C}$. Then

$$\boxed{B(t,s) = f(t)\overline{f(s)} \in \mathcal{P}_T.}$$

Example 4. Let $f(t) = t,\ T = [0, 1]$. Then, using the above method we obtain that $B(t, s) = f(t)\overline{f(s)} = ts$ is a covariance function on $T = [0, 1]$.

As it was discussed in the previous weeks, we can use basic covariance models to construct more complex ones. If one can represent a function by using "simple" covariance functions and valid transformations, then the result is a covariance function. For example,

Example 5.

If one knows that $B_1(t, s) = ts$ is a covariance function on $T = [0, 1]$, then

$$B(t, s) = B_1(t, s) + 3B_1^2(t, s) = ts + 3t^2s^2$$

is also a covariance function on $T = [0, 1]$.