Complementary mathematical topics

These topics will not be tested in STM5001 assignments. They are given for students interested in the mathematical background and justifications of models considered in this and previous weeks lectures.

Spectral Representation.

Let $X(\mathbf{k}), \mathbf{k} \in \mathbb{R}^n$ be a stationary random field. The following result gives necessary and sufficient conditions for covariance functions.

Theorem 1

A function $\rho(\tau)$ on $\mathbb{R}^n$ is a covariance function if and only if it can be represented in the form

$$\boxed{\rho(\tau) = \int_{\mathbb{R}^n} e^{i\tau\mathbf{k}} d^n F(\mathbf{k}),}$$

where $F(\mathbf{k})$ on $\mathbb{R}^n$ has the properties of a n–dimensional cumulative distribution function.

The n–dimensional distribution function $F(\mathbf{k})$ is called the **spectral distribution function.**

When F is smooth, the **spectral density function** is defined as

$$\boxed{f(\mathbf{k}) = \frac{\partial^n F(\mathbf{k})}{\partial k_1 \ldots \partial k_n}.}$$

Then

$$\boxed{\rho(\tau) = \int_{\mathbb{R}^n} e^{i\tau\mathbf{k}} f(\mathbf{k}) d^n\mathbf{k}.}$$

The spectral density function can be obtained from the correlation function by the inversion formula:

$$\boxed{f(\mathbf{k}) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\tau\mathbf{k}} \rho(\tau) d^n\tau.}$$

This gives the explicit method for verifying that a function is a covariance function on $\mathbb{R}^n$:

Evaluate the function $f(\mathbf{k})$, given by the expression above, and check that it is non-negative for all $\mathbf{k} \in \mathbb{R}^n$.

The representation of isotropic correlations has a simpler form with the n-dimensional integral replaced by a one–dimensional **Bessel transform:**

Theorem 2

A real function $\rho(\tau)$ on $\mathbb{R}^n$ is a correlation function if and only if it can be represented in the form

$$\rho(\tau) = 2^{\frac{n-2}{2}}\Gamma\left(\tfrac{n}{2}\right)\int_0^{\infty} \frac{J_{(n-2)/2}(k\tau)}{(k\tau)^{(n-2)/2}}\,d\Phi(k),$$

where the function $\Phi(k)$ on $\mathbb{R}$ is a cumulative distribution function and $J(\cdot)$ are Bessel functions of the 1 kind.

A few special cases are of particular interest:

$$\rho(\tau) = \int_0^\infty \cos(k\tau) d\Phi(k), \qquad \text{for } \mathbb{R}^1,$$

$$\rho(\tau) = \int_0^\infty J_0(k\tau) d\Phi(k), \qquad \text{for } \mathbb{R}^2,$$

$$\rho(\tau) = \int_0^\infty \frac{\sin(k\tau)}{k\tau} d\Phi(k), \qquad \text{for } \mathbb{R}^3,$$

$$\rho(\tau) = \int_0^\infty \exp(-k^2\tau^2) d\Phi(k), \qquad \text{for } \mathbb{R}^\infty.$$

Example 1. For $\mathbb{R}^3$ we know that $\rho(\tau) = \int_0^\infty \frac{\sin k\tau}{k\tau} d\Phi(k)$ is a correlation function.
For example, let us choose

$$\Phi(k) = \begin{cases} 0, & \text{if} \quad k < 0, \\ k^2, & \text{if} \quad k \in [0,1], \\ 1, & \text{if} \quad k > 1. \end{cases}$$

Then

$$\Phi'(k) = \begin{cases} 2k, & \text{if} \quad k \in [0,1], \\ 0, & \text{if} \quad k \notin [0,1]. \end{cases}$$

and the following function is a correlation function:

$$\begin{aligned} \rho(\tau) &= \int_0^1 \frac{\sin k\tau}{k\tau} 2k dk = \frac{2}{\tau} \int_0^1 \sin(k\tau) dk \\ &= \frac{2}{\tau^2}(-\cos(k\tau))|_0^1 = \frac{2}{\tau^2}(1 - \cos(\tau)). \end{aligned}$$

Example 2. We can rewrite $\int_0^\infty \frac{\sin k\tau}{k\tau} d\Phi(k)$ as $E\left(\frac{\sin K\tau}{K\tau}\right)$, where K is a random variable with the cumulative distribution function $\Phi(\cdot)$.

For example, let us $\Phi(a) = 1$, for some $a > 0$, and $\Phi(a-) = 0$. Then $\Phi(\cdot)$ is a cumulative distribution function of a discrete distribution. Moreover, for this distribution $P(K = a) = 1$.

Therefore,

$$\rho(\tau) = E\left(\frac{\sin K\tau}{K\tau}\right) = \frac{\sin(a\tau)}{a\tau}$$

is a correlation functions in $\mathbb{R}^3$.

Similarly we obtain that

$$\rho(\tau) = \cos(a\tau) \quad \text{and} \quad \rho(\tau) = J_0(a\tau)$$

are correlation functions in $\mathbb{R}^1$ and $\mathbb{R}^2$ respectively.