MAST30022 Decision Making

Course Guide

School of Mathematics and Statistics
The University of Melbourne

Semester 2, 2021

Course information
Problem sets
Answers/hints to problem sets

Contents

1 Course Information

1.1 General

- **Subject code and title**: MAST30022 Decision Making
- **Coordinator**: Dr Mark Fackrell, Room 148, Peter Hall Building, fackrell@unimelb.edu.au, 8344 6761
- **Contact hours**: 36 one-hour lectures (three per week) and 11 tutorials (one per week starting from the second teaching week)
- **Prerequisites**:
  - MAST20018 Discrete Mathematics and Operations Research

  and one of

  - MAST20004 Probability
  - MAST20006 Probability for Statistics
- **Non-allowed subjects**: None
- **Assessment**: Four written assignments during the semester amounting to a total of up to 50 pages (20%), and a 3-hour written examination in the examination period (80%).
- **Assignments**: There are four assignments due on the following dates:
  - Assignment 1: 4pm Friday 27 August 2021
  - Assignment 2: 4pm Friday 10 September 2021
  - Assignment 3: 4pm Friday 1 October 2021
  - Assignment 4: 4pm Friday 15 October 2021

  The assignments will be uploaded to Canvas approximately two weeks prior to the due date.

  You need to submit all assignments online to Canvas. It must be in a single pdf file. You can use an app on your iPhone or Android phone to take pictures of your handwritten assignment. The app will crop pages and compile into a single pdf. Please make sure all pages are readable. Do not upload more than one file per assignment submission - only the first file will be marked. If you submit early and decide you want to change your answers it is possible to resubmit - only the last assignment submission will be marked.

Do not use the Student Portal to apply for special consideration for assignments; this online application is for the final Decision Making exam only

For the special consideration policies go to `http://ask.unimelb.edu.au`

In each assignment only selected questions, to be chosen *at random*, will be marked. Notwithstanding *you are required to complete all questions in each assignment, otherwise a mark penalty will apply.*

- **Lectures**: The lectures will be face-to-face and Lecture Captured at these times
  - Mondays 12noon–1pm
  - Wednesdays 10am–11am
  - Thursdays 3:15pm–4:15pm
- **Tutorials**: There will be three tutorials every week, beginning in the second teaching week. The tutorials will be held at these times
  - Tuesdays 2:15pm–3:15pm (online)
  - Wednesdays 11am–12noon (face-to-face)
  - Wednesdays 12noon–1pm (face-to-face)
  - Tutorial problems can be found in the tutorial schedule on Canvas
- **Consultation hours**: If you need help please drop into my office (Room 148, Peter Hall), or make a face-to-face appointment via email.

  Alternatively, make a Zoom appointment via email.
- **Problem sets**: There are nine problem sets which are included in this booklet. Answers to selected questions can be found in the booklet as well.
- **Canvas and teaching materials**: Some teaching materials, including skeleton lecture notes, course guide (this booklet), assignments, solutions to assignments, schedule for practical classes, previous exam papers and solutions, etc. will be published on Canvas.

- **Breadth option**: This subject can potentially be taken as a breadth subject for the following courses:
  - Bachelor of Commerce
  - Bachelor of Environments
  - Bachelor of Music

1.2 Description

This subject introduces the essential features of decision-making situations encountered in Operations Research, Management, and Business and Economics. It develops a number of basic mathematical approaches to such situations and the techniques used to solve decision-making problems. The theoretical foundations of these techniques are also discussed. This subject demonstrates the complexity of decision-making situations that may arise from real world applications, the extent and limitations of a number of mathematical techniques used to solve such problems, and the important role that linear programming, calculus, discrete mathematics, and probability theory play in the development of these techniques.

1.3 Objectives

On completion of this subject, students should be able to

- construct mathematical models for practical decision-making problems;
- solve two-person and N-person games, including zero-sum and non-zero-sum games, cooperative and non-cooperative games;
- use decision tree and dynamic programming techniques in solving multiobjective and sequential decision problems;
- solve decision-making problems using utility theory;
- understand the complexity of group decision and social choice problems together with possible approaches;
- solve stochastic decision problems using techniques from probabilistic dynamic programming and Markov decision processes.

1.4 Generic skills

In addition to learning specific skills that will assist students in their future careers in science, they will have the opportunity to develop generic skills that will assist them in any future career path. These include

- problem-solving skills: the ability to engage with unfamiliar problems and identify relevant solution strategies;
- analytical skills: the ability to construct and express logical arguments and to work in abstract or general terms to increase the clarity and efficiency of analysis;
- collaborative skills: the ability to work in a team;
- time-management skills: the ability to meet regular deadlines while balancing competing commitments.

1.5 Course content and timeline

This table gives an overview of the course content and an estimated timeline. It is possible that minor changes will be made to this schedule during the course.

Topics	# Lectures
1. Introduction to game theory	5
2. Two-person zero-sum games	6
3. Two-person non-zero-sum games	4
4. N-person games	5
5. Preference orders and group decision	3
6. Decision making under strict uncertainty	2
7. Utility theory	3
8. Sequential decision making	3
9. Probabilistic DP and MDP	4
Revision	1

1.6 Syllabus

- Introduction to game theory
  - Introduction to game theory: games, aims of game theory
  - Trees: graphs and directed graphs, rooted trees, children, parents, descendants, ancestors, cutting, quotient
  - Game trees
  - Games in extensive form: information sets, choice functions, choice subtrees, game in extensive form, games with perfect information
  - Games with chance moves: choice subtrees for games with chance moves, probability of reaching a leaf, expected payoffs
  - Equilibrium: Nash equilibrium
  - Games in normal form: normal form, equilibrium, constant-sum games, zero-sum games, 2-person games in normal form
- Two-person zero-sum games
  - Saddle points: saddle points, equilibrium v.s. saddle, security levels, lower and upper values
  - Mixed strategies: mixed strategies, expected payoffs, security levels, optimal security levels, equilibrium pair of mixed strategies
  - Fundamental theorem of matrix game theory: fundamental theory, recipe
  - Computational techniques: saddle, dominance elimination, 2×2 formulae, graphical method, linear programming
- Two-person non-zero-sum games
  - Introduction: problem setting, comparison between zero-sum and non-zero-sum games, examples, Prisoner's dilemma
  - Non-cooperative 2-person non-zero-sum games: optimal security levels, equilibrium, Equilibrium theorem of John Nash, finding equilibria for non-cooperative 2-person non-zero-sum games, graphical method
  - Cooperative 2-person games: cooperative payoff set, Pareto boundary, negotiation set
  - Nash's bargaining axioms: Nash's axioms, Nash's theorem

- N-person games
  - Introduction: coalition, characteristic function
  - Games in characteristic function form: essential and inessential games, imputation, dominance of imputations, the core of a game
  - The Shapley value: Shapley's axioms, Shapley's theorem
  - Cost games: Dual games
- Preference orders and group decision
  - Binary relation: binary relation, properties of binary relations, Pareto and lexicographic orders, maximal/minimal elements, greatest/least elements, graphical representation, $P_{\max}(A)$, $P_{\min}(A)$, $L_{\max}(A)$, $L_{\min}(A)$
  - Preference order: strict preference order, indifference relation, weak preference order
  - Group decision and social choice: group decision problem, constitution
  - Arrow's impossibility theorem: Arrow's axioms, Arrow's Impossibility Theorem, discussion
- Decision making under strict uncertainty
  - Introduction: state-of-the-world model, decision table, certainty, risk and strict uncertainty
  - Four decision criteria: Wald's maximin, Hurwicz's maximax, Savage's minimax regret, Laplace
  - An axiomatic approach
- Utility theory
  - Lotteries and utilities: simple lotteries, compound lotteries, von Neumann-Morgenstern Axioms, von Neumann-Morgenstern Theorem, expected utility criterion
  - Decision tables and utility
  - Attitude toward risk: expected value, certainty equivalent, risk premium, attitude toward risk

- Sequential decision making
  - Decision trees: decision trees, incorporating risk aversion into decision tree analysis
  - Shortest path problem for acyclic directed graphs: acyclic directed graphs, proper labelling, shortest path problem, forward dynamic programming (DP) algorithm
  - Multi-objective shortest path problem: Pareto minimal elements, Pareto minimal paths, DP algorithm for finding Pareto minimal paths in acyclic directed graphs
  - Deterministic dynamic programming: Backward DP algorithm, characteristics of dynamic programming, resource allocation problem, knapsack problem
- Probabilistic dynamic programming and Markov decision processes
  - Probabilistic DP: features of probabilistic DP, probabilistic DP equation, examples
  - Review of Markov processes: stochastic processes, Markov chains
  - Markov decision processes
  - Finite horizon MDP: finite horizon MDP, examples
  - Infinite horizon MDP: infinite horizon MDP, stationary policy, optimal policy, value determination equations, Howard's policy iteration, linear programming method, examples

1.7 References

- C. D. Aliprantis and S. K. Chakrabarti, "Game and Decision Making", 2nd edition, Oxford University Press, 2011.
- R. Branzei, D. Dimitrov, and S. Tijs,"Models in Cooperative Game Theory", 2nd edition, Springer, 2008. (electronic resource)
- S. French, "Decision Theory: An Introduction to the Theory of Rationality", Ellis Horwood, 1986.
- M. Maschler, E. Solan, and S. Zamir, "Game Theory", Cambridge University Press, 2013. (electronic resource)
- P. Morris, "Introduction to Game Theory", Springer-Verlag, 1994. (7 day loan and electronic resource)
- M. J. Osborne, "An Introduction to Game Theory", Oxford University Press, 2004.
- M. J. Osborne and A. Rubinstein, "A Course in Game Theory", MIT Press, 1994.
- W. L. Winston, "Operations Research: Applications and Algorithms", 4th edition, Thomson, 2004. (2 hour and 7 day loan)

2 Problem Sets

2.1 Problem Set 1: Introduction to game theory

1. The following payoff matrix is given in the book "Game Theory and the Law" by D. G. Baird, R. H. Gertner and R. C. Picker, Harvard University Press, 1994. There is an interesting discussion all about it which we won't go into here, read it for yourself, it is in the law library! You might also check out the discussion about the movie "The Maltese Falcon".

   Find all pure Nash equilibrium pairs.

		Motorist		
		No care	Some care	Due care
	No care	$(-50, -50)$	$(-99, -2)$	$(-100, -3)$
Pedestrian	Some care	$(-2, -99)$	$(-51, -51)$	$(-101, -3)$
	Due care	$(-3, -100)$	$(-3, -101)$	$(-5, -3)$

2. Prove that in a rooted tree any vertex other than the root has exactly one entering edge.

3. Show that any rooted tree with at least two vertices has at least one terminal vertex.

4. Show that any rooted tree with at least two vertices contains a non-terminal vertex such that all of its children are terminal vertices.

5. The game of Fingers is played as follows: Two players, Alice and Bob, simultaneously hold up either one or two fingers. Alice wins in case of a match (the same number), and Bob wins in case of a non-match. The amount won is the number of fingers held up by the winner. It is paid to the winner by the loser.

   (a) Describe the strategies for both players.

   (b) Write down the normal form (as a pair of payoff matrices).

   (c) Verify that there is no equilibrium pair of strategies.

   (Adapted from "Introduction to Game Theory", P. Morris, 1994)

6. The normal form of a three-person game is given in the following table. Player I has three strategies (1, 2, 3), and Players II and III each have two strategies (1, 2). Find the two equilibrium 3-tuples of strategies. (Adapted from "Introduction to Game Theory", P. Morris, 1994)

Strategy triples	Payoff vectors
(1,1,1)	$(0, -1, 0)$
(1,1,2)	$(0, -2, 0)$
(1,2,1)	$(3, 0, -1)$
(1,2,2)	$(1, -1, -1)$
(2,1,1)	$(0, 0, 0)$
(2,1,2)	$(0, 0, -1)$
(2,2,1)	$(-1, 1, 1)$
(2,2,2)	$(2, 1, -1)$
(3,1,1)	$(0, 0, 2)$
(3,1,2)	$(0, -1, 1)$
(3,2,1)	$(1, -2, 1)$
(3,2,2)	$(1, 1, -1)$

Table 1: Question 6, Problem Set 1

7. Give the normal form of the game shown in Figure 1. (Adapted from "Introduction to Game Theory", P. Morris, 1994)

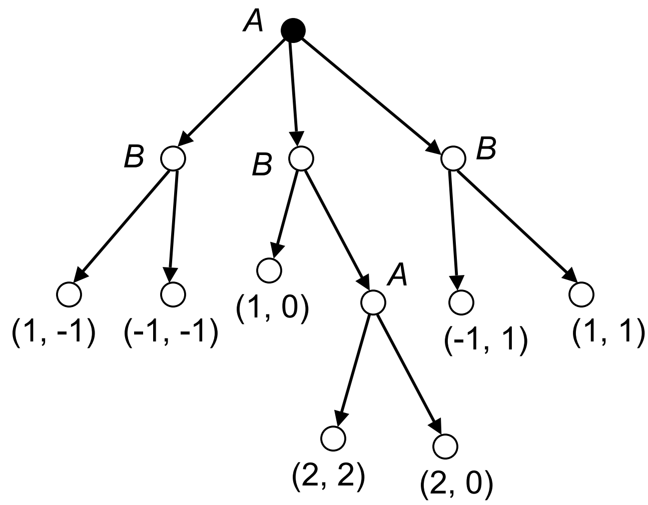

Figure 1: Question 7, Problem Set 1

8. Two nuclear powers are engaged in an arms race in which each power stockpiles nuclear weapons. An issue is the rationality of such a strategy on the part of both powers.

   Let us examine a stylized version. Assume Country 1 moves in the first stage and may choose between nuclear weapons N or non-proliferation NP. Country 2 in stage 2 of the game observes the choice of country 1 and chooses between N and NP. A representative game tree of the situation is shown in Figure 2.

According to this game tree, country 2 likes the option N whether country 1 chooses NP or N. If country 1 chooses NP, then country 2 by choosing N guarantees for itself a very powerful position. If country 1 chooses N, then country 2 would like to choose N as this allows a credible deterrence against a possible nuclear attack by country 1.

Give the normal form of this game. Find all its Nash equilibria or otherwise prove that it has no Nash equilibrium.

(Adapted from "Game and Decision Making", C. D. Aliprantis and S. K. Chakrabarti, 2000)

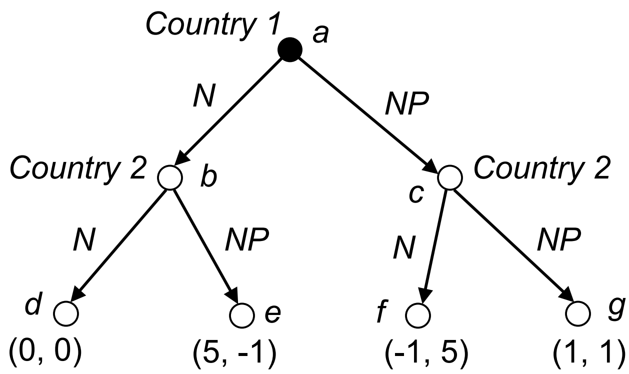

Figure 2: Question 8, Problem Set 1

9. Represent the following game in extensive form, and then give its normal form using a payoff matrix.

   Anne has an Ace and a Queen. Bea has a King and a Joker. The rank of the cards is $A > K > Q$, but the Joker is peculiar as will be seen.

   Each antes (an initial bet before any move is made) a penny to the pot. Each person then selects one of her cards and simultaneously they reveal their selections. If Bea selects the King, the highest card chosen wins the pot and he game ends. If Bea selects the Joker and Ann the Queesn, they split the pot and the game ends. If Bea selects the Joker and Ann the Ace, then Ann may either resign (in which case Bea gets the pot) or demand a replay. If Ann demands a replay they each ante another penny to the pot and play again. This time if Bea selects the Joker and Ann the Ace, Bea gets the pot.

   (Adapted from "Game Theory: Mathematical Models of Conflict", A. J. Jones, 2000)

2.2 Problem Set 2: 2-person zero-sum games

1. (a) Solve the 2-person zero-sum-game with payoff matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 2 \\ -1 & -1 & 3 & 0 \\ 1 & -2 & 5 & 1 \end{bmatrix}$$

(You should find the saddle points, if any, and state the value of the game.)

   (b) In **any** game with the same columns as columns 2 and 3 above, would Player II ever uses the strategy represented by column 3? Explain.

2. Consider the following 2-person zero-sum game:

$$\begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 6 \end{bmatrix}$$

   (a) Find the range of values for p and q that make the entry (a_2, A_2) a saddle point.

   (b) Is it possible to have **both** (a_2, A_2) and (a_3, A_2) saddle points at the same time? What about **both** (a_2, A_2) and (a_2, A_3)? Give reasons for your answers.

3. A football team, The Yarras, wants to travel from A to B by the shortest possible route to give them as much time to recover from the journey as possible. It can only travel over the routes shown in the table below. Unfortunately a rival team, The Crocs, can block one road leading out of C and, at the same time, one road leading out of D. The Yarras will not know which roads have been blocked until they arrive at C or D. Should The Yarras start towards C or D? Which routes should The Crocs block? Set this up as a 2 person constant sum game and solve it. (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

4. Find the values of x for which the following games have a saddle point, and solve the game for these cases.

$$\text{(a)} \begin{bmatrix} x & 2 \\ 1 & x \end{bmatrix}; \qquad \text{(b)} \begin{bmatrix} 1 & 2 & x \\ 7 & 0 & 3 \\ 6 & 5 & 4 \end{bmatrix}$$

Route	Length of route (in kilometers)
A to C	80
A to D	90
D to E	40
D to F	20
C to E	30
C to F	60
E to B	50
F to B	30

5. Let

$$G_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad G_2 = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Show that if no two of a, b, c, d are equal, and if the 2-person zero-sum game with payoff matrix G_1 has a saddle point at (a_1, A_1), then the 2-person zero-sum game with payoff matrix G_2 has a saddle point. (In fact it can be shown that G_2 has a saddle point if and only if G_1 has a saddle point.)

6. Solve the following 2-person constant-sum game:

$$\begin{bmatrix} (3,6) & (4,5) & (8,1) \\ (6,3) & (5,4) & (7,2) \\ (8,1) & (2,7) & (1,8) \end{bmatrix}$$

7. Find the expected payoff to each player if Player I uses mixed strategy $(0.4, 0.6)$ and Player II uses mixed strategy $(0.2, 0.8)$, in the 2-person zero-sum game with payoff matrix:

$$\begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix}$$

8. This question relates to the following known result:

> If one player of a zero-sum two-person game employs a fixed strategy, then the opponent has an optimal counter strategy that is pure. So, if Player I KNOWS II's strategy, Player I can maximize their expected payoff by using a pure strategy, and vice versa.

We explore the above statement in this question for a particular case.

(a) i. Find the expected payoff, $\mathbf{E}(\mathbf{x}, \mathbf{y})$, to Player I if Player I uses mixed strategy $\mathbf{x} = (x_1, 1 - x_1), 0 \leq x_1 \leq 1$, whilst Player II uses mixed strategy $\mathbf{y} = (y_1, 1 - y_1), 0 \le y_1 \le 1$, in the 2-person zero-sum game with payoff matrix:

$$\begin{bmatrix} 5 & 0 \\ -1 & 2 \end{bmatrix}$$

ii. Now think of y_1 as being fixed and find the value(s) of x_1 that maximizes $\mathbf{E}(\mathbf{x}, \mathbf{y})$ for fixed y_1. Hence write down the strategy(ies) for Player I that maximizes $\mathbf{E}(\mathbf{x}, \mathbf{y})$. Deduce that in all cases, for this example, Player I can use a pure strategy to get the maximum $\mathbf{E}(\mathbf{x}, \mathbf{y})$, given that Player I knows Player II's strategy. (If you cannot see what to do, try putting $y_1 = 0.2$, for example, and see what happens. Then try and generalize.)

(b) Suppose that Player I knew that Player II is to use a mixed strategy $(0.2, 0.3, 0.5)$ in the 2-person zero-sum game with matrix below.

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Work out the three expected payoffs if Player I uses a pure strategy and hence deduce the best strategy for Player I.

9. Solve, USING A GRAPHICAL METHOD, the 2-person, zero-sum game whose payoff matrix is

$$\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$$

10. (a) Write down the two linear programming problems induced by the game given in Question 9 above.

    (b) Solve ONE of them BY HAND, and thus find the optimal strategies for both players and the value of the game.

11. Without solving the game, find out whether $\mathbf{x}^* = \frac{1}{27}(18, 7, 2), \mathbf{y}^* = \frac{1}{27}(2, 12, 0, 13)$ is an optimal solution for the 2-person zero-sum game with payoff matrix below. If so, find the value of the game. (Hint: Use an appropriate Theorem!)

$$\begin{bmatrix} -1 & 1 & 2 & 0 \\ 4 & -2 & -3 & 2 \\ 0 & 3 & 1 & -2 \end{bmatrix}$$

12. Solve the 2-person zero-sum game with payoff matrix below in THREE different ways.

$$\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$$

13. The 2-person zero-sum game with payoff matrix

$$V = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$$

may be solved using the linear programming problem

$$\begin{aligned} \max \quad & y_1' + y_2' \\ \text{Subject to} \quad & y_1' + 4y_2' \leq 1 \\ & 2y_1' + y_2' \leq 1 \\ & y_1', y_2' \geq 0 \end{aligned}$$

Starting from $v_2 = \min_{\mathbf{y} \in Y} \max_{i=1,2} V_{i\cdot}\mathbf{y}^T$, clearly explain how this LP problem may be derived. DO NOT SOLVE the linear programming problem.

14. Solve the following problem as a two-person constant-sum problem.

An escaped prisoner can choose two roads, the highway or a forest road. There is only one available police officer to cover these. The prisoner will escape if both take separate roads. If both take the highway, the prisoner will be caught, if both take the forest road the probability of escape is $1 - 1/w$, where $w > 1$ and w depends on the weather. Discuss what happens to the solution as w increases. (Adapted from “Game Theory” by Morton Davis.)

15. (a) In order to use the linear programming method that we developed in lectures, it is necessary that the value of the game be positive (there are other LP methods where this is not the case). Explain why it is true that if one row of the payoff matrix is positive then the value of the game is positive.

(b) Using the Simplex Method, solve the 2-person zero-sum game whose payoff matrix is

$$\begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & -2 \\ -4 & 0 & 1 \end{bmatrix}$$

(c) Check your answer by showing that your solution is an equilibrium point.

(d) Investigate what happens in this example if you do not ensure that the value of the game you solve by the simplex method is positive, i.e. solve it without adding a constant!

16. Solve the 2-person zero-sum game with the following matrix. This time DO NOT use a graphical method and DO NOT use the simplex method. (You

should also show that you get an equilibrium point as a check!)

$$\begin{bmatrix} 3 & 1 & 4 & 3 & -1 \\ 2 & 0 & 2 & 2 & -1 \\ 1 & 5 & 0 & -1 & 1 \\ 2 & 0 & 1 & 0 & -2 \end{bmatrix}$$

17. Solve the 2-person zero-sum game with the following payoff matrix, using, in part, a GRAPHICAL METHOD.

$$\begin{bmatrix} 3 & 4 & 4 & 3 & -1 \\ 2 & -2 & 2 & 2 & -1 \\ 1 & -2 & 0 & -1 & 1 \\ 2 & 0 & 1 & 0 & -2 \end{bmatrix}$$

18. The police have a tip that a weekly consignment of drugs carried by two carriers is coming in to either a certain airport and/or a certain seaport. Two officers are assigned to intercept them. If there are the same number of police as smugglers at a place, nothing will get through. If both of the officers and both carriers are at different places, 100 kilograms of drugs will get through. If one officer is where both carriers are, 70 kilograms will get through and if there is one carrier where there is no officer, 50 kilograms will get through. Set this up as a 2-person zero-sum game and find all optimal strategies for the carriers and officers. ASSUME THAT THE VALUE OF THE GAME IS POSITIVE. How much drugs will get through on average? Notice that there are two different optimal solutions for the smugglers. (Adapted from “Game Theory” by Morton D. Davis.)

19. Consider the 2-person zero-sum game with payoff matrix below. Observe that it has no saddle point.

$$\begin{bmatrix} 0 & -3 \\ -1 & 5 \\ 14 & 2 \end{bmatrix}$$

   (a) Explain how you know that the value of this game is positive. Write down a linear program for determining an equilibrium mixed strategy for Player I. **Do not solve this linear program.**

   (b) Using a graphical method, find the equilibrium mixed strategies for Players I and II and the value of the game.

   (c) Using the solution obtained in (b), deduce the optimal solution of the linear program in (a). Do not just write down the answer. Explain where it comes from.

20. Solve the 2-person zero-sum game with payoff matrix below:

$$\begin{bmatrix} 3 & 2 & 7 \\ 1 & 1 & 1 \\ 1 & a & 8 \end{bmatrix}$$

21. Solve the 2-person zero-sum game with payoff matrix below by means of

    (a) graph and 2×2 formula;

    (b) solving all 2×2 sub-games;

    (c) linear programming.

$$\begin{bmatrix} 2 & 0 & 1 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}$$

22. Is it possible to have a 2×2 matrix A such that A has a saddle but A^T does not have a saddle? Justify your answer.

23. (a) Show that, if (a_i, A_i) and (a_j, A_j) both give saddles in a 2-person zero-sum game, then (a_i, A_j) also gives a saddle, where a_i, a_j are pure strategies for Player I and A_i, A_j are pure strategies for Player II.

    (b) (a) is a special case of the following general result: If both $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ are equilibrium pairs in a two-person zero-sum game, then so are $(\mathbf{x}_1, \mathbf{y}_2)$ and $(\mathbf{x}_2, \mathbf{y}_1)$, where $\mathbf{x}_1, \mathbf{x}_2$ are mixed strategies for Player I and $\mathbf{y}_1, \mathbf{y}_2$ are mixed strategies for Player II. Prove this result. Prove also that $\mathbf{E}(\mathbf{x}_1, \mathbf{y}_1) = \mathbf{E}(\mathbf{x}_1, \mathbf{y}_2)$.

24. Two firms, Hustle (H) and Con (C) are to market washing powder. H can afford to buy 2 blocks of television advertising time and C can buy 3 such blocks. There are morning (M) blocks, afternoon (A) blocks and evening (E) blocks available. The following payoff table is drawn up based on a knowledge of the proportion of viewers watching in each period and the portion of the market each firm gets if they buy more or less or the same as each other in the same period.

    Solve this problem with the aid of dominance and linear programming software. Provide a print out of the final tableau and clear explanations of where your solution comes from. (Adapted from "Game Theory" by Morton D. Davis.)

25. Prove that for a 2-person zero-sum game, $v_2 \geq L$, and hence the value of the game is $\geq L$, where v_2 is the optimal (mixed strategy) security level for Player II and L is the maximum (pure strategy) security level for Player I.

		Hustle					
		EE	EA	EM	AA	AM	MM
	EEE	75	60	65	60	50	65
	EEA	65	75	80	60	65	80
	EEM	60	70	75	70	60	65
	EAA	40	65	55	75	80	80
Con	EAM	50	60	65	70	75	80
	EMM	35	45	60	70	70	75
	AAA	40	40	30	65	55	55
	AAM	50	50	40	60	65	55
	AMM	50	35	50	45	60	65
	MMM	35	20	35	45	45	60

26. This problem is taken from "The Compleat Strategyst" by J. D. Williams.

    The firm of Gunning and Kappler manufactures an amplifier whose performance depends critically on a small inaccessible condenser. The condenser costs Gunning and Kappler \$1 but they are set back \$10 on average if the original condenser is defective. There is an alternative superior condenser which costs \$6 and is fully guaranteed, so that there will be no costs to Gunning and Kappler if it is defective. There is a third condenser which is insured in such a way, that if it fails, all costs are covered AND you get your money back for the condenser. This condenser costs \$10. Set this up as a 2-person zero-sum game and solve it.

27. Find the range of values for p and q that make the entry (a_2, A_2) a saddle point in the 2-person zero-sum game with the following payoff matrix.

$$\begin{bmatrix} 3 & 4 & 5 \\ 9 & 7 & q \\ 4 & p & 6 \end{bmatrix}$$

28. Suppose that the payoff matrix of a 2-person zero-sum game is the same as in Question 27 above. Is it possible to have **both** of (a_2, A_2) and (a_3, A_2) as saddle points at the same time? What about **both** of (a_2, A_2) and (a_2, A_3)? Give reasons for your answers.

29. Find the values of x for which the following 2-person zero-sum game has at least one saddle point, and solve the game for these cases.

$$\begin{bmatrix} -1 & 2 & 7 \\ x & 1 & 2 \\ 7 & x & 9 \end{bmatrix}$$

30. Solve the 2-person constant-sum game with the payoff matrix below. Identify an optimal strategy pair.

$$\begin{bmatrix} (16,64) & (32,48) & (39,41) \\ (41,39) & (47,33) & (56,24) \\ (29,51) & (50,30) & (26,54) \end{bmatrix}$$

31. Solve the following 2-person zero-sum game by using the graphical method.

$$\begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$$

32. Consider the following 2-person zero-sum game. Is there an optimal pure strategy pair? If yes, state the optimal pure strategy pair. If not, solve the problem using Linear Programming approach.

$$\begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix}$$

33. Solve the 2-person zero-sum game with the payoff matrix below, using the graphical method, possibly in combination with other methods (e.g. dominance). State explicitly the value of the game and an optimal strategy pair.

$$\begin{bmatrix} 3 & 2 & 1 & 8 & 3 \\ 0 & 2 & 3 & 7 & 1 \end{bmatrix}$$

34. Solve the following 2-person zero-sum game.

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & -1 & 2 \\ 3 & 2 & -3 \end{bmatrix}$$

35. A 2-person zero-sum game is said to be *fair* if its value is zero. Consider the following 2-person zero-sum game:

$$\begin{bmatrix} a & 2 \\ 1 & -1 \end{bmatrix}$$

For which values of the parameter a is the game fair? When does it favor the row player (i.e. positive value)? When does it favor the column player? (Adapted from "Introduction to Game Theory", P. Morris, 1994)

2.3 Problem Set 3: 2-person non-zero-sum games

1. For the 2-person non-zero-sum game with payoff bi-matrix:

		Player II	
		A_1	A_2
Player I	a_1	(2, 10)	(5, 0)
	a_2	(4, 8)	(8, 3)

   (a) clearly explain why (a_2, A_1) is an equilibrium pair;

   (b) clearly explain why (a_2, A_2) is NOT an equilibrium pair;

   (c) classify the other pairs as equilibrium or not equilibrium.

2. Investigate whether the following 2-person non-zero-sum game has equilibrium pairs:

		Player II	
		A_1	A_2
Player I	a_1	(2, 1)	(1, 2)
	a_2	(1, 2)	(2, 1)

Table 2: Question 2, Problem Set 3

3. Use a payoff matrix to describe the following situation as a non-zero-sum game and find all equilibrium pairs.

   Greedy and Greedier are the rulers of two countries that have formed an oil cartel. In order to keep their price up, they have agreed to limit their productions respectively to 1 million and 4 million barrels per day. For each, cheating on this agreement means producing an extra 1 million barrels per day. Thus they would produce in total either 5, 6, or 7 million barrels per day, with corresponding profit margins of \$16, \$12 and \$8 per barrel. (Adapted from "A Gentle Introduction to Game Theory", S. Stahl, 1999)

4. Describe a systematic way of finding equilibrium pairs of **pure strategies** for 2-person non-zero-sum games, and use it to find all equilibrium pairs of pure strategies for the 2-person non-zero-sum game with the following payoff bi-matrix:

		Player II			
		A_1	A_2	A_3	A_4
	a_1	(5, 2)	(3, 0)	(8, 1)	(2, 3)
Player I	a_2	(6, 3)	(5, 4)	(7, 4)	(1, 1)
	a_3	(7, 5)	(4, 6)	(6, 8)	(0, 2)

5. (a) Find security level mixed strategies and associated expected payoffs for the 2-person non-zero-sum game with bi-matrix

$$\begin{bmatrix} (2,1) & (-1,-1) \\ (-1,-1) & (1,2) \end{bmatrix}$$

   (b) Check whether these strategies are in equilibrium.

   (c) Find, by inspection of the matrix, any pure strategy equilibrium pairs.

   (d) i. Use a graphical method to find all equilibrium pairs.

   ii. Check one of these pairs to make sure it is an equilibrium pair.

   iii. Find the expected payoffs associated with each of these equilibrium pairs and compare with the payoff you found for the security level pair.

6. (a) Find all pure strategy equilibria for the 2-person non-zero-sum game with bi-matrix

$$\begin{bmatrix} (3,2) & (2,1) \\ (4,3) & (1,4) \end{bmatrix}$$

   (b) Find security level pure strategies (i.e. if the players just wanted to use pure strategies, what would give best security levels) and associated expected payoffs. What would be the actual payoff if both followed these strategies?

   (c) Find an equilibrium pair for this game by trying to set $\mathbf{x}A\mathbf{y}^{*T} =$ constant for all $\mathbf{x}$ and $\mathbf{x}^*B\mathbf{y}^T =$ constant for all $\mathbf{y}$, as explained in lectures. Find also the expected payoffs for this pair.

   (d) Find all equilibrium pairs for this game using the graphical method. Find also the expected payoffs.

7. (a) For any a and b, find an equilibrium pair $(\mathbf{x}^*, \mathbf{y}^*)$ of mixed strategies for the 2-person non-zero-sum game with payoff matrix below, using the method of setting $\mathbf{x}A\mathbf{y}^{*T} =$ constant for all $\mathbf{x}$ and $\mathbf{x}^*B\mathbf{y}^T =$ constant for

all $\mathbf{y}$.

$$\begin{bmatrix} (2,0) & (0,2) & (4,4) \\ (0,4) & (1,2) & (2,0) \\ (1,a) & (1,b) & (0,0) \end{bmatrix}$$

(b) Show that this method does not work if the entry in the third row and first column of this matrix is changed to give:

$$\begin{bmatrix} (2,0) & (0,2) & (4,4) \\ (0,4) & (1,2) & (2,0) \\ (-1,a) & (1,b) & (0,0) \end{bmatrix}$$

8. In checking whether the pair $\mathbf{x}^* = (1/3, 1/4, 5/12), \mathbf{y}^* = (1/4, 1/4, 1/2)$ is in equilibrium for the 2-person game:

$$\begin{bmatrix} (5,3) & (2,6) & (1,7) \\ (6,2) & (3,5) & (7,1) \\ (4,4) & (2,6) & (4,4) \end{bmatrix},$$

it is worked out that

$$\mathbf{x}^* A \mathbf{y}^{*T} = 175/48, \ \ \mathbf{x} A \mathbf{y}^{*T} = (108x_1 + 276x_2 + 168x_3)/48.$$

(a) CLEARLY explain how we can get that this is not an equilibrium pair from these results.

(b) In going on to show that the pair $\mathbf{x}^* = (0,1,0), \mathbf{y}^* = (0,1,0)$ is an equilibrium pair for this game, it is worked out that

$$\mathbf{x}^* A \mathbf{y}^{*T} = 3, \ \ \mathbf{x} A \mathbf{y}^{*T} = 2x_1 + 3x_2 + 2x_3$$

$$\mathbf{x}^* B \mathbf{y}^{*T} = 5, \ \ \mathbf{x}^* B \mathbf{y}^{T} = 2y_1 + 5y_2 + y_3.$$

CLEARLY explain how we can get that this is an equilibrium pair from these results.

9. (a) The following problem is adapted from an example in "Game Theory and the Law" by D. G. Baird, R. H. Gertner and R. C. Picker, Harvard University Press, 1994.

Suppose that it is in the interest of a landowner to build and maintain a levee to prevent flood damage if and only if an adjacent landowner builds and maintains a levee too. Find all equilibria if the following payoff matrix applies. (Note that we assume here that each makes a decision without knowing what the other will do.)

		Landowner 2	
		Maintain	Don't Maintain
	Maintain	$(-4,-4)$	$(-10,-6)$
Landowner 1	Don't Maintain	$(-6,-10)$	$(-6,-6)$

		Landowner 2	
		Maintain	Don't Maintain
	Maintain	$(-4,-4)$	$(-4,-12)$
Landowner 1	Don't Maintain	$(-12,-4)$	$(-6,-6)$

(b) You should find that the above produces three equilibria. Suppose we now introduce a legal rule such that a landowner pays damages if they fail to maintain it when others do. This changes the payoffs as follows Find all equilibria for the new payoffs.

We see that this gives us a way of analysing the changes brought by the legal rule.

10. Firm I can make either 200 big TV sets a week or 100 big and 100 small sets. Firm II can make either 50 big TV sets a week or 100 small sets each week. There is a weekly demand of 200 big and 100 small TVs. Big TVs sell at \$400 each and small TVs at \$200 each. It costs Firm I \$300 to make each big set and \$120 to make each small set. It costs Firm II \$320 to make each big set and \$170 to make each small set.

    If more televisions are made than there is demand, the firms sell the same proportion of what they made. (E.g. if I makes 200 big TVs and II makes 50 big TVs and demand for big TVs is 200, then I sells $200 \times 200/250 = 160$ and II sells $50 \times 200/250 = 40$.) Sets not sold are worth nothing, but the firms still have to pay the cost of manufacture.

    (a) Set up this problem as a 2×2 non-zero-sum game.

    (b) Find all pure strategy equilibrium pairs for the game and verify that $((11/14, 3/14), (6/13, 7/13))$ is a mixed strategy equilibrium with payoffs $(164/13, 6/7)$.

    (c) Use a zero-sum game method for each player separately to show that for this problem we get payoffs $(164/13, 6/7)$ again.

    (Adapted from "Games, Theory and Applications", L. C. Thomas, 1984)

11. Verify the following for the game with payoff bi-matrix below.

$$\begin{bmatrix} (2,6) & (10,5) \\ (4,8) & (0,0) \end{bmatrix}$$

   (a) Player I's security level is 10/3 and Player II's is 6.

   (b) With status quo point $(10/3, 6)$ the Nash solution is

$$(17/3, 43/6) = (13/18)(4,8) + (5/18)(10,5).$$

12. Find the solution generated by Nash's axioms, for the game with payoff bi-matrix below, using

   (a) the point $(0,0)$ as the status quo point;

   (b) the point $(1,2)$ as the status quo point.

$$\begin{bmatrix} (1,4) & (-1,-4) \\ (-4,-1) & (4,1) \end{bmatrix}$$

13. Determine the solution generated by Nash's axioms, for the game with payoff bi-matrix below, using the security level expected payoff pair as status quo point. (You are expected to draw the cooperative payoff set, show the Pareto optimal set and the negotiation set; find what combination of the given payoffs in the matrix give the Nash solution and what this implies about how the game should be played if obeying Nash's axioms.)

$$\begin{bmatrix} (2,1) & (3,2) & (0,4) \\ (0,1) & (4,0) & (2,1) \end{bmatrix}$$

14. Kalai and Smorodinsky ("Other solutions to Nash's bargaining problem", Econometrica 43 (1975) 513–518) considered the following related arbitration problems, both with status quo points at $(0,0)$.

$$P = \begin{bmatrix} (0,0) & (20,0) \\ (0,20) & (15,15) \end{bmatrix}, \quad Q = \begin{bmatrix} (0,0) & (20,0) \\ (0,20) & (20,14) \end{bmatrix},$$

   (a) Find the Nash solution for both games.

   (b) Notice that the cooperative payoff set for P is completely contained in that for Q. Discuss why this example seems to show that the Nash arbitration scheme is unfair. (Hint: In this example concentrate on why it seems unfair to Player II.)

Kalai and Smorodinsky argued in this fashion and proposed an alternative scheme which does not have this problem. Of course it does not satisfy all of Nash's axioms. (Adapted from "Game Theory and Strategy" by P. D. Straffin.)

15. Show that Nash's solution obeys Nash's (a) symmetry axiom and (b) "invariant under linear transformation" axiom.

16. Consider again the problem of TVs in Question 10 in this problem set. Now regard this as a cooperative game and find the solution which obeys Nash's axioms. Use thousands of dollars for the unit in the payoffs. Use $(18, 3)$ as status quo point.

17. Consider the following 2-person non-cooperative non-zero-sum game.

$$\begin{bmatrix} (1,1) & (0,0) \\ (2,-1) & (-1,2) \end{bmatrix}$$

   (a) Find a security level mixed strategy for each player.

   (b) Check whether the security level strategies found in (a) give an equilibrium pair.

   (c) Find all equilibrium pairs by the graphical method (Swastika method).

   (d) For each equilibrium pair and the security level pair, give the expected payoff to each player. Compare the payoffs associated with the security level pairs and that associated with the equilibrium pairs, and confirm that the latter is no worse than the former.

18. Consider the following 2-person cooperative non-zero-sum game.

$$\begin{bmatrix} (1,10) & (10,1) \\ (0,-10) & (0,-9) \end{bmatrix}$$

   (a) What are the security levels u^*, v^* for Players I and II?

   (b) Draw a graph to show the cooperative payoff set C, the Pareto optimal boundary and the negotiation set.

   (c) Using the security levels $\mathbf{q} = (u^*, v^*)$ as the status quo point, determine the unique Nash solution $(\underline{u}, \underline{v})$ to the game (that is, the unique point in C which satisfies Nash's Axioms).

19. Consider the 2-person, non-zero-sum, non-cooperative game with bi-matrix

$$\begin{bmatrix} (8,13) & (1,5) \\ (6,4) & (13,8) \end{bmatrix}.$$

(a) Check whether the strategy pairs $((1,0),(0,1))$ and $((1,0),(1,0))$ are equilibrium pairs. Show all your workings.

(b) Find all equilibrium pairs by the graphical method.

2.4 Problem Set 4: N-person games

1. Consider the following 3-person game, taken from "Games, Theory and Applications" by L. C. Thomas, Ellis Horwood, 1984.

   Country 1 has oil which it can use to run its transport system at a profit of \$a per barrel. Country 2 wants to buy the oil to use in its manufacturing industry, where it gives a profit of \$b per barrel. Country 3 wants it for food manufacturing where the profit is \$c per barrel. Assume $a < b \leq c$.

   (a) Justify to yourself that an appropriate characteristic function for this situation is

   $$v(\emptyset) = 0,\ v(\{1\}) = a,\ v(\{2\}) = v(\{3\}) = 0$$

   $$v(\{1,2\}) = b,\ v(\{1,3\}) = c,\ v(\{2,3\}) = 0,\ v(\{1,2,3\}) = c.$$

   (b) Verify that this characteristic function is superadditive.

   In part (c)-(e) let $a = 10$, $b = 16$ and $c = 18$.

   (c) State, with reasons, which of the following allocations are imputations:

   $$\mathbf{x}^1 = (12,2,4),\ \mathbf{x}^2 = (8,6,4),\ \mathbf{x}^3 = (10,3,4),\ \mathbf{x}^4 = (13,3,2),\ \mathbf{x}^5 = (17,0,1).$$

   For each imputation state, with reasons, whether it is a core element or not.

   (d) Find the core of this game. Interpret your result in terms of coalitions formed and who pays whom what!

   (e) Calculate the Shapley value of this game. Is the Shapley value a core element of this game?

2. Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	a	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

   (a) Find the maximum a such that the characteristic function is superadditive.

   (b) With this maximum value of a, show that

   $$\mathbf{p} = (1/4, 1/8, 5/8),\ \mathbf{q} = (1/2, 0, 1/2),\ \mathbf{r} = (5/16, 1/8, 9/16)$$

   are imputations. Which of these imputations are core elements?

(c) Draw the core of the game with a as found in (a) and find its extreme points.

(d) Find the Shapley value with $a = 1/2$.

3. Let $q \in \mathbb{R}$. Consider the game (N, v) with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	4	q	5	8

Determine for which q

(a) $C(v) = \emptyset$,

(b) $C(v)$ consists of one point,

(c) $C(v)$ consists of infinitely many points.

4. Three lawyers have banded together to form a joint practice: the Mighty Trio. The overhead for the practice is \$200,000 per year. Each lawyer brings in annual revenues and incurs annual variable costs as follows: Andrew - \$775,000 in revenue, \$200,000 in variable cost; Liam - \$800,000 in revenue, \$175,000 in variable cost; and Phil - \$600,000 in revenue, \$190,000 in variable cost.

The Mighty Trio wants to use game theory to determine how much each of the lawyers should be paid.

(a) Determine the relevant characteristic function.

(b) Find the core of the game.

(c) Determine the Shapley value of the game in two different ways. Does the Shapley value give a reasonable division of the practices' profits?

(Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

5. (a) Find the core of the 3-person cooperative game with the following characteristic function:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	1

Hint: make a picture to determine the extreme points of the core.

(b) Find all marginal vectors of the game in (a). Compare the marginal vectors with the extreme points of the core.

(Adapted from "Games Theory: Mathematical Models of Conflict", A. J. Jones, 2000)

6. Let $N = \{1, 2, 3\}$ and let $v \in \mathrm{TU}^N$ as described in the table below, with $r, t \in \mathbb{R}$.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	$t+5$	5	7	$2t$	$3t+4$	14	$2t+r$

Furthermore, it is given that $(x_1, x_2, x_3) = (2t, 10, 9) \in C(v)$. Determine t and r.

7. Let (N, v) be a TU-game, with $N = \{1, 2, 3, 4\}$. The following coalitional values of v are given:

$$\begin{aligned} v(\{1,2,3\}) &= 4, \\ v(\{1,2,4\}) &= 7, \\ v(\{1,3,4\}) &= 6, \\ v(\{2,3,4\}) &= 5, \\ v(\{1,2,3,4\}) &= 7. \end{aligned}$$

Show that $C(v) = \emptyset$.

8. Consider the game (N, v), with $N = \{1, 2, 3\}$ and v as described in the table below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	0	0	2	4	5	7

   (a) Write v as a linear combination of unanimity games.

   (b) Use your answer of (a) to compute the Shapley value.

   (c) Compute the Shapley value by calculating the average of all marginal vectors.

9. U.N. Security Council has 15 members, five of whom have vetoes. For a substantive resolution to pass it is necessary to have nine affirmative votes and no vetoes.

   We may treat this as a 15-person game with $N = \{1, 2, 3, 4, 5, 6, \ldots, 15\}$ and suppose that the first five have the veto. Define a coalition S to be winning (that is, $v(S) = 1$) if it can defeat a substantive resolution, otherwise $v(S) = 0$. Thus the characteristic function is:

$$\text{if } i \in S \text{ for some } i,\ 1 \leq i \leq 5, \text{ then } \ v(S) = 1$$

$$\text{otherwise } \; v(S) = \begin{cases} 1, & \text{if } |S| \geq 7 \\ 0, & \text{if } |S| < 7. \end{cases}$$

Compute the Shapley values for this game and give an interpretation of your result. (Adopted from “Games Theory: Mathematical Models of Conflict,” A. J. Jones, 2000)

10. The 3-person game of Couples is played as follows. Each player chooses one of the other two players; these choices are made simultaneously. If a couple forms (e.g. if Player I chooses Player II, and Player II chooses Player I), then each member of that couple receives a payoff of $1/2$, while the person not in the couple receives -1. If no couple forms (e.g. if I chooses II, II chooses III, and III chooses I), then each receives a payoff of zero.

    (a) Using the technique of optimal security levels, determine the corresponding game in characteristic function form.

    (b) Prove that in this game

    $$v(S) + v(N \backslash S) = 0$$

    for all $S \in 2^N$.

    (c) Show that this game is essential (i.e. show that this game is not an additive game).

    (d) Show that this game has an empty core.

    (Adapted from “Games Theory: Mathematical Models of Conflict”, A. J. Jones, 2000)

11. Prove that the core of an n-person game is a convex set (which may be empty). That is, if both $\mathbf{x}$ and $\mathbf{y}$ are in the core, then for any real number λ, $0 \leq \lambda \leq 1$, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is also in the core.

12. Compute the Shapley value for any n-person inessential game.

13. A 4-person game in characteristic function form is given as follows:

$$v(\emptyset) = 0, v(\{1,2,3,4\}) = 2$$

$$v(\{1\}) = -1, v(\{2\}) = 0, v(\{3\}) = -1, v(\{4\}) = 0$$

$$v(\{1,2\}) = 0, v(\{1,3\}) = -1, v(\{1,4\}) = 1$$

$$v(\{2,3\}) = 0, v(\{2,4\}) = 1, v(\{3,4\}) = 0$$

$$v(\{1,2,3\}) = 1, v(\{1,2,4\}) = 2, v(\{1,3,4\}) = 0, v(\{2,3,4\}) = 1$$

(a) Verify that v is superadditive.

(b) Is the core of this game non-empty?

(c) Compute the Shapley value for this game.

(Adapted from “Introduction to Game Theory”, P. Morris, 1994)

14. Let (N, v) be a TU-game. Consider the following one-point solution concepts:

    1. Egalitarion rule, $E(v)$, defined by $E_i(v) = \frac{v(N)}{n}$, for all $i \in N$.
    2. Utopia vector, $U(v)$, defined by $U_i(v) = v(N) - v(N \backslash \{i\})$, for all $i \in N$.

    (a) Indicate for each of the two solution concepts which of the properties efficiency, symmetry, the dummy property and additivity is satisfied by this property. Give a proof if a property is satisfied and provide a counterexample if it is not satisfied.

    (b) Why can none of the above two rules satisfy all four properties?

15. Let v be a 3-person TU-game. Show that if

$$v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) > 2v(N),$$

    then $C(v) = \emptyset$.

16. Let $v \in \mathrm{TU}^N$ and let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two imputations. We say that $\mathbf{y}$ dominates $\mathbf{x}$ if there exists a coalition $S \in 2^N \backslash \{\emptyset\}$ such that

    (i) $y_i > x_i$, for all $i \in S$, and

    (ii) $\sum_{i \in S} y_i \leq v(S)$.

    The *dominance core* $DC(v)$ is defined by

$$DC(v) := \{\mathbf{x} \in I(v) \mid \nexists\ \mathbf{y} \in I(v) \text{ such that } \mathbf{y} \text{ dominates } \mathbf{x}\}.$$

    (a) Let $N = \{1,2,3\}$, $v(N) = 1$, $v(\{1,2\}) = 2$ and $v(S) = 0$ otherwise. Show that $C(v) = \emptyset$ and $DC(v) = \mathrm{conv}\{(1,0,0),(0,1,0)\}$.

    (b) Let $N = \{1,2,3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(N) = 1$. Show that $DC(v) = \emptyset$.

    (c) Prove that $C(v) \subseteq DC(v)$.

(d) Let $v \in \mathrm{TU}^N$ be such that

$$v(N) \geq v(S) + \sum_{i \in N \setminus S} v(\{i\})$$

for all $S \in 2^N$, which is a weaker condition than superadditivity. Show that under this condition

$$C(v) = DC(v).$$

(Adapted from "Game Theory", P. Borm, reader Tilburg University)

17. Using marginal vectors, prove that the Shapley value satisfies efficiency, symmetry, the dummy property and additivity.

18. Let $v \in \mathrm{TU}^N$. Show that

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \big(v(S \cup \{i\}) - v(S)\big).$$

19. In an airport problem there is a set of players N. Each of these players owns an airplane with certain characteristics, which determine the minimal length of a landing strip this plane can use. In this exercise we will use cooperative game theory to answer the question of how to divide the maintenance costs of the total length of the landing strip among all players.

Consider the airport problem as depicted in Figure 3. Here $N = \{1, 2, 3\}$, and each player $i \in N$ owns an airplane of type i. E.g. player 2 has one airplane of type 2 and this airplane needs a longer strip to land than an airplane of type 1. Furthermore, the maintenance costs for the length of the strip needed for this airplane is $50 + 20 = 70$.

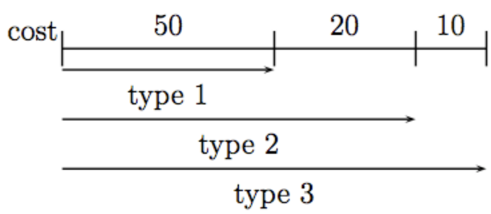

Figure 3: Overview of a landing strip, Question 19, Problem Set 4

(a) Provide a cooperative cost game that resembles this situation.

(b) Determine the Shapley value of this game and interpret its value in the context of this problem.

2.5 Problem Set 5: Preference orders and group decision

1. Determine whether the following satisfy the properties of transitivity, reflexivity, completeness, symmetry, antisymmetry, asymmetry.

   (a) $A = \{$all lines in a given plane$\}$ with binary relation 'is perpendicular to'.

   (b) $A = \{$all lines in a given plane$\}$ with binary relation 'is parallel to'.

   (c) $A = \{$vertices in a network without cycles$\}$ with binary relation 'is a predecessor of' (note that this does not mean 'immediate' predecessor).

2. (a) Given a set A of real numbers and the binary relation θ on A such that

   $$\theta = \{(a, b) : a, b \in A, b = a + 4\},$$

   is θ an order on A?

   (b) If $A = \{(x, y) : x, y \in \mathbb{R}^+\}$ and $\theta = \{(\mathbf{a}, \mathbf{b}) : \mathbf{a} = (a_1, a_2) \in A, \mathbf{b} = (b_1, b_2) \in A, a_1 a_2 \geq b_1 b_2\}$, is θ an order on A?

3. Consider $A = \{(5, 5), (4, 7), (6, 4), (8, 2), (11, -2)\}$ relative to the relation

   $$\theta = \{(\mathbf{a}, \mathbf{b}) : a_1 + a_2 \geq b_1 + b_2\}$$

   where $\mathbf{a} = (a_1, a_2) \in A, \mathbf{b} = (b_1, b_2) \in A$, that is $\mathbf{a}\theta\mathbf{b}$ iff $a_1 + a_2 \geq b_1 + b_2$.

   (a) Is θ an order on A?

   (b) Find all greatest, least, maximal and minimal elements, if they exist for A.

4. (a) Let θ be an order on some set A. Show that if the order is antisymmetric then there is at most one least element of A.

   (b) Show that, if A is a finite non-empty set and θ is an antisymmetric, complete order on A, then A has at most one maximal element.

5. Consider the set $A = \{(1, 2), (2, 1), (2, 3), (2, 4), (2, 5), (3, 2), (3, 4), (4, 2), (4, 3)\}$.

   Represent A graphically and hence find the greatest, least, maximal and minimal elements, if they exist, for

   (a) the lexicographic order;

   (b) the Pareto order.

Show clearly on separate graphs or with different colours where the maximal and minimal elements come from.

(We can represent any weak order θ on a finite set A by a directed graph whose vertices are the elements of A such that there is an directed edge from a to b iff $a\theta b$.)

6. (a) Prove that the transitivity property holds for the lexicographic order.
   (b) Prove that the lexicographic order is complete.
   (c) Prove that the lexicographic order is antisymmetric.

   These and a result in lectures imply that, for a finite subset A of $\mathbb{R}^n$, with respect to the lexicographic order there is exactly one greatest element and exactly one least element, and these are also the unique maximal and minimal elements of A.

7. Let
$$A = \{1, 2, 3, 4\}$$
and
$$\theta = \{(1, 1), (1, 3), (2, 3), (3, 4), (4, 4)\}.$$
Is θ transitive ?

8. Let $\mathbb{Z}$ be the set of integers, and k a positive integer. Let θ be the binary relation on $\mathbb{Z}$ defined by
$$\theta = \{(x, y) : k \text{ is a divisor of } x - y\}.$$
Prove that θ is an equivalence relation on $\mathbb{Z}$, that is, θ is reflexive, transitive and symmetric.

9. Let $A_1, \ldots, A_n$ be sets equipped with binary relations $\succeq_1, \ldots, \succeq_n$ respectively. Let
$$A := A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_i \in A_i, 1 \le i \le n\}$$
be the Cartesian product of $A_1, \ldots, A_n$. Define a binary relation $\succeq$ on A such that
$$(a_1, \ldots, a_n) \succeq (b_1, \ldots, b_n)$$
if and only if $a_i \succeq_i b_i$ for all $i = 1, 2, \ldots, n$.

   (a) Prove that, if all $\succeq_i$ $(i = 1, 2, \ldots, n)$ have one and the same of the following properties:
   – transitivity;

– reflexivity;

– symmetry;

– antisymmetry

– asymmetry

then $\succeq$ has the same property.

(b) Prove or disprove the following statement: if all $\succeq_i$ $(i = 1, 2, \ldots, n)$ are weak orders then so is $\succeq$.

10. Let

$$A = \{(-2, 3, 1), (0, 1, -1), (1/2, 4, 2), (3, 2, 1), (5, -1, 0)\}.$$

(a) List the lexicographic order of A, and find the greatest and least elements of A.

(b) For the Pareto order on A draw the corresponding directed graph such that for each directed edge the tail is "better than" the head. Find the Pareto-minimal and Pareto-maximal element sets P-min(A) and P-max(A), and the Pareto greatest and least elements (if any) of A.

11. Let

$$A = \{(0, 2), (0, 0), (1, 3), (1, 1), (2, 4), (2, 1), (3, 3)\}.$$

Define a binary relation θ on A by

$$(x_1, x_2)\theta(y_1, y_2) \text{ if and only if } x_1 > y_1 \text{ or } x_2 > y_2.$$

Is θ an order? Give reasons.

12. Let A be the same as in Question 11, and let θ be the binary relation on A with the Boolean matrix below (e.g. $(0, 2)\theta(0, 2), (0, 2)\theta(0, 0), (0, 2)\theta(1, 3)$, etc.)

	(0, 2)	(0, 0)	(1, 3)	(1, 1)	(2, 4)	(2, 1)	(3, 3)
(0,2)	×	×	×	×		×	
(0,0)							
(1,3)	×	×	×	×		×	
(1,1)	×	×	×	×		×	
(2,4)	×	×	×	×	×	×	×
(2,1)	×	×	×	×		×	
(3,3)	×	×	×	×	×	×	×

Find all least, greatest, minimal and maximal elements (if any) of θ.

13. You are decorating a room for your indecisive grandmother, who will not tell you what colour paint to use, but will give you a strict and consistent (transitive) preference between any pair of colours. The choice of paint is restricted to: Lilac (L), Blue (B), Brown (Br), Mushroom (M), White (W), Green (G), Pink (P).

    You have asked her seven questions and learnt that she holds:

    $$M > P, M > G, P > W, B > M, G > W, Br > L, L > M$$

    where "$x > y$" means that she prefers x over y. What is the next and last question you ask her in order to determine her choice of colour for the room? In all you have asked her eight questions. What is the least number of questions you could have asked her, and is there any strategy for questioning her that ensures you only ask the least number possible? Give reasons for your answer. (Adapted from "Decision Theory", S. French, 1986)

14. Let $A = \{(2,0,3), (1,2,3), (-4,-2,1), (8,1,1), (-5,0,2), (2,1,1), (-6,-2,0)\}$.

    (a) List the lexicographic order of A, and find the greatest and least elements of A under this order.

    (b) Give the Pareto order of A and represent it by drawing a directed graph such that for each arc the tail is "better than" the head. Find $P_{min}(A)$ and $P_{max}(A)$. Also, find the greatest and least elements (if any) of A under the Pareto order.

15. Seven clerks share an office. Each has an ideal working temperature τ_i ($i = 1, 2, \dots, 7$) with

    $$\tau_1 < \tau_2 < \tau_3 < \tau_4 < \tau_5 < \tau_6 < \tau_7.$$

    Their individual preferences between any two room temperatures t and t' depends only on the magnitude of the departure of t and t' from their ideal, i.e.

    $$t \succeq_i t' \Leftrightarrow |t - \tau_i| \leq |t' - \tau_i|$$

    for $i = 1, 2, \dots, 7$.

    They decide as a group to adopt the fourth clerk's preference and set the room temperature to τ_4 because three would prefer a cooler room and three would prefer a warmer room. Show that in doing so they are implicitly adopting the simple majority rule; i.e. show that, when $t \succ_4 t'$, at least three others hold $t \succ_i t'$, and that when $t \sim_4 t'$, the number who hold $t \succ_i t'$ equal the number who hold $t' \succ_i t$. (Adapted from "Decision Theory", S. French, 1986)

16. The individual preference orders are said to satisfy the *single-peakedness condition* if there exists some underlying descriptive ordering of the alternatives such that in passing through the alternatives in this order, each individual's preference strictly increases until his most preferred alternative is reached, after which his preference strictly decreases. The underlying descriptive order is common to all individuals, but their preference orders may differ.

    Now consider a group of $n = 2r$ individuals who have to choose between three alternatives $\{a, b, c\}$. Suppose that

    $$a \succ_i b \succ_i c \text{ for } i = 1, 2, \ldots, r$$

    and

    $$b \succ_i c \succ_i a \text{ for } i = r+1, r+2, \ldots, 2r.$$

    Show that the single-peakedness condition holds. Show also that the simple majority rule gives

    $$a \sim_g b,\ b \succ_g c \text{ and } c \sim_g a.$$

    (Adapted from "Decision Theory", S. French, 1986)

2.6 Problem Set 6: Decision making under strict uncertainty

1. Find the rankings generated by Wald's Maximin, Hurwicz's Maximax, Savage's Minimax Regret and Laplace's Criteria for the decision problem under strict uncertainty as shown in Table 3.

		Sates		
		θ_1	θ_2	θ_3
	a_1	3	4	7
	a_2	2	3	3
Actions	a_3	5	3	2
	a_4	4	2	5
	a_5	2	6	4

Table 3: Question 1, Problem Set 6

2. Find what each of the decision methods: Wald's Maximin, Hurwicz's Maximax, Savage's Minimax Regret and Laplace's Criterion, would tell a company manager to do in the following decision situation. The manager has no information about what the economy will be like 3 years from now when the payoff will come, and so he/she may suppose that each state may occur equally likely. The figures in the table are profit to company in \$ million. (Adapted from P. D. Straffin, "Game Theory and Strategy".)

		Economy			
		Way up	Slightly up	Slightly down	Way down
	Hold steady	3	2	2	0
Manager	Expand slightly	4	2	0	0
	Expand greatly	6	2	0	-2
	Diversity	1	1	2	2

Table 4: Question 2, Problem Set 6

3. Pizza King and Nobel Greek are two competing restaurants. Each must determine the price they will charge for each pizza sold. Pizza King believes that Nobel Greek's price is a random variable D having the following probability distribution: $\mathbf{Pr}(D = \$6) = 1/4$, $\mathbf{Pr}(D = \$8) = 1/2$, $\mathbf{Pr}(D = \$10) = 1/4$. If Pizza King charges a price p_{PK} and Noble Greek charges a price p_{NG}, Pizza King will sell $100 + 25(p_{NG} - p_{PK})$ pizzas. It costs \$4 to make a pizza. Pizza

King is considering charging \$5, \$6, \$7, \$8, or \$9 for a pizza. Use each of the four decision criteria (Wald's Maximin, Hurwicz's Maximax, Savage's Minimax Regret, and Laplace) to determine the price that Pizza King should charge. (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

4. Consider decision making with risk in which probabilities $\mathbf{Pr}(\theta_j)$, $j = 1, 2, \ldots, n$, are associated with the states. Consider the *expected utility rule* which chooses a_k to maximize

$$V_k = \sum_{j=1}^{n} \mathbf{Pr}(\theta_j) v_{kj}.$$

Show that this rule satisfies the following six Axioms: complete ranking, independence of labelling, independence of value scale, strong domination, independence of irrelevant alternatives, independence of addition of a constant to a column. (Adapted from "Decision Theory", S. French, 1986)

5. Consider the decision table of Milnor (as discussed in the lecture):

		States			
		θ_1	θ_2	θ_3	θ_4
	a_1	2	2	0	1
Actions	a_2	1	1	1	1
	a_3	0	4	0	0
	a_4	1	3	0	0

Suppose that the situation is one under risk. Show that there are values of the probabilities $\mathbf{Pr}(\theta_1), \mathbf{Pr}(\theta_2), \mathbf{Pr}(\theta_3)$ and $\mathbf{Pr}(\theta_4)$ such that a_1 is optimal under the expected utility rule (see Question 4 above). Similarly, show that there are other sets of probabilities such that a_2 and a_3 are optimal under the expected utility rule. However, show that there is no set of probabilities such that a_4 is optimal under this rule. (Adapted from "Decision Theory", S. French, 1986)

6. Consider the decision table below (where x is a real number).

   (a) Find which decision will be taken, as a function of x, according to: (i) Wald's Maximin criterion; (ii) Hurwicz's criterion (take $\alpha = 1/2$); (iii) Laplace's criterion; or (iv) Savage's Minimax Regret criterion.

   (b) Find the range(s) of x for which all four criteria uniquely lead to the same choice.

(Adapted from "Decision Theory", S. French, 1986)

		States			
		θ_1	θ_2	θ_3	θ_4
	a_1	x	3	4	6
Actions	a_2	2	2	2	4
	a_3	3	2	1	9
	a_4	6	6	1	3

2.7 Problem Set 7: Utility theory

The first four questions below are adapted from “Operations Research: Appl. & Alg.”, W. L. Winston, 4th ed., 2004, and the rest questions from “Decision Theory”, S. French, 1986.

1. Suppose that Bob’s utility function for asset position x is given by $u(x) = \ln x$.

   (a) Is Bob risk-averse, risk-neutral, or risk-seeking?

   (b) Assume that Bob has \$20,000 and he is considering the following two lotteries:

   L_1: With probability 1, one will lose \$1,000

   L_2: With probability 0.9 one will gain \$0, and with probability 0.1 one will lose \$10,000.

   Determine which lottery Bob prefers and the risk premium of L_2.

2. Answer Question 1 above for a utility function $u(x) = x^2$.

3. Answer Question 1 above for a utility function $u(x) = 2x + 1$.

4. Alice is trying to determine which of two courses to take. If she takes the decision making course, she believes that she has a 10% chance of receiving an A, a 40% chance for a B, and a 50% for a C. If Alice takes a statistics course, she has a 70% chance for a B, a 25% chance for a C, and a 5% chance for a D. Alice is indifferent between $\langle 1, C\rangle$ and $\langle 0.25, A; 0.75, D\rangle$. She is also indifferent between $\langle 1, B\rangle$ and $\langle 0.7, A; 0.3, D\rangle$. If Alice wants to take the course that maximizes the expected utility of her final grade, which course should she take?

5. Consider the following decision table under risk, in which the consequences are monetary payoffs. Convert this problem into one of choosing between lotteries.

Consequences	x_{ij}	States θ_1	θ_2	θ_3
	a_1	\$100	\$110	\$120
Actions	a_2	\$90	\$100	\$120
	a_3	\$100	\$110	\$100
		Probabilities 1/4	1/2	1/4

The decision maker holds the following indifferences with reference lotteries:

$$\$100 \sim \langle 0.5, \$120; 0.5, \$90 \rangle$$

$$\$110 \sim \langle 0.8, \$120; 0.2, \$90 \rangle$$

Which action should the decision maker choose?

6. Suppose you wish to assess a decision maker's utility function for money, $u(x)$, for r different values $x = x_1, x_2, \ldots, x_r$. Suppose further that the decision maker has given you $(r-2)$ indifferences of the form

$$\langle p_1^{(i)}, x_1; p_2^{(i)}, x_2; \ldots; p_r^{(i)}, x_r \rangle \sim \langle q_1^{(i)}, x_1; q_2^{(i)}, x_2; \ldots; q_r^{(i)}, x_r \rangle, \;\; i = 1, \ldots, r-2.$$

Give a necessary and sufficient condition on the probabilities $p_j^{(i)}$ and $q_j^{(i)}$ which ensures that this is sufficient information to determine an agreeing utility function, unique up to positive linear transformation. (A linear transformation $u' = au + b$ is called *positive* if $a > 0$.)

7. An investor has \$1000 to invest in two types of share. If he invests $\$m$ in share A he will invest $\$(1000 - m)$ to share B. An investment in share A has a 0.7 chance of doubling in value and a 0.3 chance of being lost altogether. An investment in share B has a 0.6 chance of doubling in value and a 0.4 chance of being lost altogether. The chances associated with share A are independent of those associated with share B. Determine the optimal value of m, if the decision maker's utility function for a gain or loss x is $u(x) = \ln(x + 3000)$. What would be the optimal value of m, if his utility fuction were instead $u(x) = (x + 3000)^2$?

8. Two individuals have the same attitude to risk modelled by $u(x) = x^{1/3}$ for a change in assets of x. One of them is given a lottery ticket that gives him a chance p of winning $\$R$ and a chance $1 - p$ of winning nothing. Show that there is a price $\$b$ acceptable to both men such that the one who receives the lottery ticket can sell it to the other for $\$b$. Find further the range of b such that this sale is acceptable to both men.

9. You have already been given one play in the gamble described in the St. Petersburg paradox (or the French paradox as called in Russia:). A coin is to be tossed and you will receive \$$2^n$ if it first lands "heads" on the nth toss. However, before the coin is tossed, someone offers to buy your unknown winnings from you. He will give you \$$x$ and then take whatever prize results from the tossing of the coin. If your utility function is $u(y) = \log_2 y$, what is the least value of x that you will accept?

   You may need to know the following summation:

$$\sum_{n=1}^{\infty} \frac{n}{a^n} = \frac{1}{a(1-\frac{1}{a})^2} \text{ for } a > 1.$$

10. Let $\mathcal{L}$ be a set of simple and finitely compounded lotteries and X the set of all possible rewards. Denote the maximum value of X by x_{max} and the minimum value of X by x_{min}. Let A be the union of X, $\mathcal{L}$ and all lotteries of the type $\langle p, x_{max}; 1-p, x_{min}\rangle$. Assume that a decision maker's preferences over A satisfy all the seven axioms of von Neumann-Morgenstern.

    a) By continuity (Axiom 7), for each $x_i \in X$, there exists u_i with $0 \le u_i \le 1$ such that

$$x_i \sim \langle u_i, x_{max}; 1-u_i, x_{min}\rangle.$$

    Using Axiom 1 (weak order) and Axiom 6 (monotonicity), show that u_i is unique.

    b) Using Axiom 1 (weak order), Axiom 3 (reduction of compound lotteries), Axiom 4 (substitutability) and Axiom 7 (continuity), show that for each $x_i \in X$

$$x_i \sim \langle 0, x_1; \ldots; 1, x_i; \ldots; 0, x_r\rangle.$$

2.8 Problem Set 8: Sequential decision making

1. Consider the distance matrix below, representing a network where the arc labels are (cost, time).

		Vertices								
		1	2	3	4	5	6	7	8	9
	1	-	-	(0,8)	-	(2,5)	-	-	-	-
	2	-	-	-	(8,2)	-	-	-	-	-
	3	-	-	-	(11,2)	-	-	-	-	-
	4	-	-	-	-	-	-	-	-	-
Vertices	5	-	-	-	(4,4)	-	-	-	-	-
	6	-	-	-	-	-	-	(2,5)	(4,3)	-
	7	(4,6)	-	-	-	-	-	-	-	(1,2)
	8	-	(7,2)	-	-	-	-	-	-	(1,5)
	9	-	(6,3)	-	-	-	-	-	-	-

   (a) Explain how you can tell from this matrix that the vertices are not 'properly' labelled.

   (b) Find the Pareto and lexicographic shortest distances from vertex 1 to vertex 9 (these refer to 'properly labelled' vertices). You are expected to:

      i. compute proper labels for the vertices,

      ii. draw the network with the new labeling,

      iii. solve the functional equations on the network,

      iv. recover all the non-dominated paths from the origin to the terminal vertex and give final answer in terms of original vertex labels.

   (c) If cost was our only concern what would the shortest path be?

   (d) If time was our only concern what would the shortest path be?

2. A group of students is planning a car trip from 1 = Melbourne to 7 = Sunshine Coast, in the summer vacation. The various routes they can follow are given by the network in Figure 4.

   Three things are important to them when planning which route to take: time, cost and "pleasure". The pleasure factor is based on scenery, nice places to stay, not too much traffic, etc. and is the most important to them. Cost is the second most important factor. Naturally, they want to maximize pleasure and minimize cost and time. They estimate these as follows:

   Find a shortest path through the network, based on this information.

Road	Time	Cost	Pleasure
1–2	5	1	3
1–3	8	2	7
1–4	4	2	6
1–6	9	2	4
2–3	6	1	4
3–7	7	1	7
4–5			
4–6			
5–6			
5–7			
6–7			

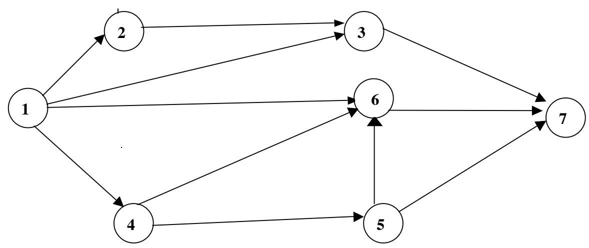

Figure 4: Question 2, Problem Set 9

3. Suppose that a new car costs \$10,000 and that the annual operating cost and resale value of the car are as shown below (Table 5). If I have a new car now, determine a replacement policy that minimizes the net cost of owning and operating a car for the next six years. (This is a deterministic dynamic programming problem.)

Average of Car (years)	Resale Value (\$)	Operation Cost (\$)
1	7,000	300 (year 1)
2	6,000	500 (year 2)
3	4,000	800 (year 3)
4	3,000	1,200 (year 4)
5	2,000	1,600 (year 5)
6	1,000	2,200 (year 6)

Table 5: Question 3, Problem Set 9

4. Consider the network shown in Figure 5 below.

    (a) Is the labelling given on the network a proper labelling? If it is not proper, find a proper labelling by using the method of counting the number of predecessors of each vertex.

    (b) Based on the proper labelling you found in (a), find **all** Pareto minimal paths from vertex u to vertex v.

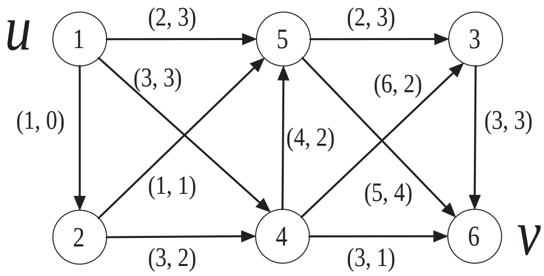

Figure 5: Question 4, Problem Set 9

5. Oilco must determine whether or not to drill for oil in the South China Sea. It costs \$100,000, and if oil is found the value is estimated to be \$600,000. At present, Oilco believes that there is a 45% chance that the field contains oil. Before drilling, Oilco can hire (for \$10,000) a geologist to obtain more information about the likelihood that the field will contain oil. There is a 50% chance that the geologist will issue a favorable report and a 50% chance of an unfavorable report. Given a favorable report, there is an 80% chance that the field contains oil. Given an unfavorable report, there is a 10% chance that the field contains oil. Assume that Oilco is risk neutral towards risk. Determine Oilco's optimal course of action. (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

6. Consider the network shown in Figure 6 below.

   (a) Convince yourself that this graph is acyclic. Also convince yourself that the given labelling is proper, that is, for each arc of the network the label of the tail is smaller than the label of the head.

   (b) Based on the given proper labelling, find **all** Pareto minimal paths from vertex 1 to vertex 8.

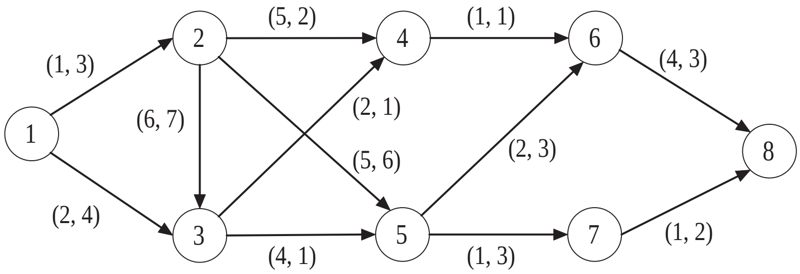

Figure 6: Question 6, Problem Set 9

7. A pharmaceutical firm X faces a decision concerning the introduction of a new drug. This means there is an initial decision about how much to spend on research and development (R & D), the possibility that the drug may fail to be developed on schedule, and the fact that the drug may not be quite successful in the market. At each stage of this decision making process, we notice the presence of uncertainty.

   At the initial stage firm X has to decide whether to spend a large amount (Hi) or a small amount (Lo) on R & D. The result of this investment could either lead to success (S) or failure (F), with the probability p of success being higher in the case of Hi expenditure on R & D. Even when the drug is successfully produced, the firm may decide not to market it. A decision tree for this problem is shown in Figure 7, with M and DM standing for "market" and "do not market" respectively.

   Suppose the decision maker is risk-neutral. Determine the firm's optimal decision path. (Adapted from "Game and Decision Making", C. D. Aliprantis and S. K. Chakrabarti, 2000)

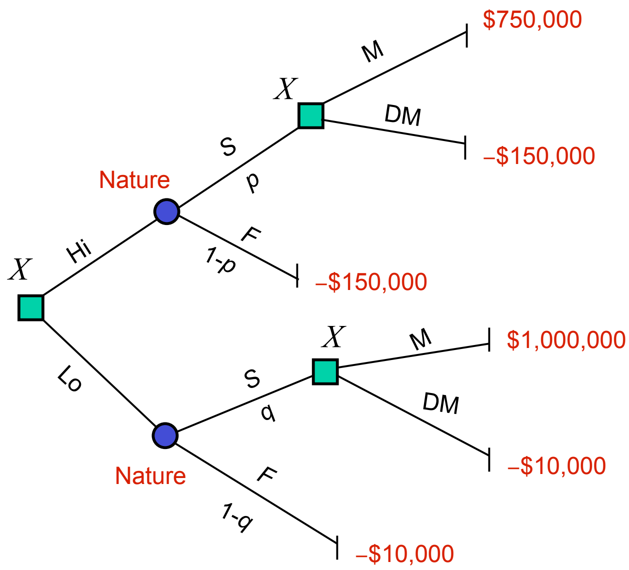

Figure 7: Question 7, Problem Set 9

8. J. R. Carrington has \$4 million to invest in three oil well sites. The amount of revenue earned from site i ($i = 1, 2, 3$) depends on he amount of money invested in site i as shown in the table below. Assuming that the amount invested in a site must be an exact multiple of \$1 million, use dynamic programming to determine an investment policy that will maximize the revenue that J. R. Carrington will earn from his three oil wells.

		Revenue (\$ Million)		
		Site 1	Site 2	Site 3
Amount Invested (\$ Million)	0	4	3	3
	1	7	6	7
	2	8	10	8
	3	9	12	13
	4	11	14	15

(Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

9. **The knapsack problem.** You have k types of items, and you want to bring some of them to an aircraft. Each item of type i has value v_i and weight w_i. The airline's weight limit is w. How should you pack your belongings such that the total value of the packed items is maximized subject to the weight constraint?

Assume you pack y_i items of type i. Then the problem is

$$\max \sum_{i=1}^{k} y_i v_i$$

$$s.t. \sum_{i=1}^{k} y_i w_i \leq w$$

$$x_1, \ldots, y_k \geq 0 \text{ are integers.}$$

This is a special case of the resource allocation problem for which

$$g_i(y_i) = y_i w_i, \;\; r_i(y_i) = y_i v_i, \;\; i = 1, \ldots, k.$$

Let

$$f_i(x_i) = \text{max. total value with weight limit } x_i \text{ and types } i, i+1, \ldots, k$$

Then the DP equation is

$$f_i(x_i) = \max\{y_i v_i + f_{i+1}(x_i - y_i w_i) : y_i \geq 0 \text{ an integer s.t. } y_i w_i \leq x_i\}.$$

Beginning with $f_k(\cdot)$ and $y_k(\cdot)$ and working backward, compute

$$f_{k-1}(\cdot), y_{k-1}(\cdot), \ldots, f_1(\cdot), y_1(\cdot)$$

sequentially.

Solve the knapsack problem with 3 types of items such that

$$w_1 = 4, \; w_2 = 3, \; w_3 = 5,$$

$$v_1 = 11, \; v_2 = 7, \; v_3 = 12,$$

and $w = 10$. (Winston, Section 18.4, knapsack problem)

10. Use dynamic programming to solve a knapsack problem in which the knapsack can hold up to 13 kg (see the table below).

Item	Weight (kg)	Value
1	3	12
2	5	25
3	7	50

(Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

11. A company needs the following number of workers during each of the next five years: year 1, 15; year 2, 30; year 3, 10; year 4, 30; year 5, 20. At present, the company has 20 workers. Each worker is paid \$30,000 per year. At the beginning of each year, workers may be hired or fired. It costs \$10,000 to hire a worker and \$20,000 to fire a worker. A newly hired worker can be used to meet the current year's worker requirement. During each year, 10% of all workers quit (workers who quit do not incur any firing cost).

    With dynamic programming, formulate a recursion that can be used to minimize the total cost incurred in meeting the worker requirements of next five years. (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

12. (a) The dynamic programming for the shortest path problem as discussed in lectures is based on the so-called Bellman's Principle, which asserts that a segment of a shortest path is also a shortest path. More explictly, if $G = (V, E)$ is a weighted acyclic directed graph whose vertices are properly labelled, and if $P : 1, \ldots, v, j, i$ is a shortest path in G from vertex 1 to vertex i with j the second last vertex, then the segment of P from 1 to j (i.e. the path $P[1, j] : 1, \ldots, v, j$) is a shortest path in G from 1 to j. Why is this true? Convince yourself that indeed we need this principle in forming the recursive equation:

    $$f(i) = \min\{f(j) + c_{ji} : j \in P(i)\}, \ \ i = 1, 2, \ldots, n.$$

    (b) Our dynamic programming method for the Pareto minimal path problem is based on a similar principle. Using the notation in (a) above, it asserts that if $P : 1, \ldots, v, j, i$ is a Pareto minimal path in G from 1 to i, then $P[1, j]$ is a Pareto minimal path in G from 1 to j. Prove this statement and convince yourself that we need this in forming the recursive equation:

    $$f(i) = P_{min}\left(\cup_{j \in P(i)}(f(j) + (c^1_{ji}, c^2_{ji}))\right), \ \ i = 1, 2, \ldots, n.$$

    (c) State the counterpart principle underlying our dynamic programming method for the lexicographic minimal path problem, and justify it by yourself.

2.9 Problem Set 9: Probabilistic dynamic programming and Markov decision processes

1. Suppose that \$4 million is available for investment in three projects. The probability distribution of the net profit earned from each project depends on how much is invested in each project. Let X_t be the random variable denoting the net profit earned by project t. The distribution of X_t depends on the amount of money invested in project t, as shown in the following table (a zero investment in a project always earns a zero net profit), where $\mathbf{Pr}(X_t = a) = p$ means that the probability of earning a millions by project t is equal to p. Use probabilistic dynamic programming to determine an investment allocation that maximises the expect net profit obtained from the three investments. (Adapted from "Operations Research: Applications and Algorithms", W. L. Winston, 4th ed., 2004)

Project	Investment (million)	Probability
1	\$1	$\mathbf{Pr}(X_1 = 2) = 0.6, \mathbf{Pr}(X_1 = 4) = 0.3, \mathbf{Pr}(X_1 = 5) = 0.1$
	\$2	$\mathbf{Pr}(X_1 = 4) = 0.5, \mathbf{Pr}(X_1 = 6) = 0.3, \mathbf{Pr}(X_1 = 8) = 0.2$
	\$3	$\mathbf{Pr}(X_1 = 6) = 0.4, \mathbf{Pr}(X_1 = 7) = 0.5, \mathbf{Pr}(X_1 = 10) = 0.1$
	\$4	$\mathbf{Pr}(X_1 = 7) = 0.2, \mathbf{Pr}(X_1 = 9) = 0.4, \mathbf{Pr}(X_1 = 10) = 0.4$
2	\$1	$\mathbf{Pr}(X_2 = 1) = 0.5, \mathbf{Pr}(X_2 = 2) = 0.4, \mathbf{Pr}(X_2 = 4) = 0.1$
	\$2	$\mathbf{Pr}(X_2 = 3) = 0.4, \mathbf{Pr}(X_2 = 5) = 0.4, \mathbf{Pr}(X_2 = 6) = 0.2$
	\$3	$\mathbf{Pr}(X_2 = 4) = 0.3, \mathbf{Pr}(X_2 = 6) = 0.3, \mathbf{Pr}(X_2 = 8) = 0.4$
	\$4	$\mathbf{Pr}(X_2 = 3) = 0.4, \mathbf{Pr}(X_2 = 8) = 0.3, \mathbf{Pr}(X_2 = 9) = 0.3$
3	\$1	$\mathbf{Pr}(X_3 = 0) = 0.2, \mathbf{Pr}(X_3 = 4) = 0.6, \mathbf{Pr}(X_3 = 5) = 0.2$
	\$2	$\mathbf{Pr}(X_3 = 4) = 0.4, \mathbf{Pr}(X_3 = 6) = 0.4, \mathbf{Pr}(X_3 = 7) = 0.2$
	\$3	$\mathbf{Pr}(X_3 = 5) = 0.3, \mathbf{Pr}(X_3 = 7) = 0.4, \mathbf{Pr}(X_3 = 8) = 0.3$
	\$4	$\mathbf{Pr}(X_3 = 6) = 0.1, \mathbf{Pr}(X_3 = 8) = 0.5, \mathbf{Pr}(X_3 = 9) = 0.4$

2. An advertising firm has D dollars to spend on reaching customers in T separate markets. Market t consists of k_t people. If x dollars are spent on advertising in market t, the probability that a given person in market t will be reached is $p_t(x)$. Each person in market t who is reached will buy c_t units of the product. A person who is not reached will not buy any of the product. Formulate a dynamic programming recursion that could be used to maximize the expected number of units sold in T markets. (Adapted from "Operations Research: Applications and Algorithms", W. L. Winston, 4th ed., 2004)

3. Sunco Oil has D dollars to allocate for drilling at sites $1, 2, \ldots, T$. If x dollars are allocated to site t, with probability $q_t(x)$ oil will be found on site t. Sunco

estimates that if site t has any oil, it is worth r_t dollars. Formulate a probabilistic dynamic model that could be used to enable Sunco to maximise the expected value of all oil found on sites $1, 2, \ldots, T$. (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

4. When Sally Mutton arrives at the bank, 30 minutes remains on her lunch break. If Sally makes it to the head of the line and enters service before the end of her lunch break, she earns reward r. However, Sally does not enjoy waiting in lines, so to reflect her dislike for waiting in line, she incurs a cost c for each minute she waits. During a minute in which n people are ahead of Sally, there is a probability $p(x|n)$ that x people will complete their transactions. Suppose that when Sally arrives, 20 people are ahead of her in line. Use dynamic programming to determine a strategy for Sally that will maximises her expected net revenue (reward minus waiting costs). (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

5. The owner of a winery can use advertisement through one of three media: radio, TV, or newspaper. The weekly costs of advertisement in the three media are 1, 2 and 1.5 thousands respectively. The winery classifies its sales volume during each week as (1) fair, (2) good, or (3) excellent. The transition probabilities associated with each choice of advertisement medium are as follows.

   Transition matrix for Radio:

	fair	good	excellent
fair	0.4	0.5	0.1
good	0.1	0.7	0.2
excellent	0.1	0.2	0.7

   (For example, if sales volume is good and the company chooses to advertise in radio, then with probability 0.2 sales will be excellent in the next period.)

   Transition matrix for TV:

	fair	good	excellent
fair	0.7	0.2	0.1
good	0.3	0.6	0.1
excellent	0.1	0.7	0.2

   Transition matrix for Newspaper:

	fair	good	excellent
fair	0.2	0.5	0.3
good	0	0.7	0.3
excellent	0	0.2	0.8

A return value has been determined for each pair of states and each choice of media, as shown in the following matrices.

Weekly returns (in thousands) when advertising with Radio:

	fair	good	excellent
fair	4	5.2	6
good	3	4	7
excellent	2	2.5	5

(For example, the entry in the 1st row and 2nd column of the return matrix is 5.2, so if in a fair state the company chooses to advertise in radio, and the subsequent state is good, the return to the company will be $5,200.)

Weekly returns (in thousands) when advertising with TV:

	fair	good	excellent
fair	10	13	16
good	8	10	17
excellent	6	7	11

Weekly returns (in thousands) when advertising with Newspaper:

	fair	good	excellent
fair	4	5.3	7.1
good	3.5	4.5	8
excellent	2.5	4	6.5

The winery aims at the maximum expected net return over N weeks.

(a) Formulate this problem as a Markov decision problem.

(b) Find the optimal decision for the last week (i.e. for week $t = N$).

(c) Help the winery to make an optimal advertising policy so as to maximise the net return over 2 weeks. What is the maximum expected net return for two weeks?

6. A company has an end-of-period capacity of 3 units. During a period in which production takes place, a setup cost of $4 is incurred. A $1 holding cost is assessed against each unit of a period's ending inventory. Also, a variable production cost of $1 per unit is incurred. During each period, demand is equally likely to be 1 or 2 units. All demand must be met on time, and the discounted factor is $\alpha = 0.8$. The goal is to minimize expected discounted costs over an infinite horizon.

(a) Use the policy iteration method to determine an optimal stationary policy.

(b) Use linear programming to determine an optimal stationary policy.

(Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

7. A machine in excellent condition earns \$100 profit per week; a machine in good condition earns \$70 per week; and a machine in bad condition earns \$20 per week. At the beginning of any week, a machine may be sent out for repairs at a cost of \$90. A machine that is sent out for repairs returns in excellent condition at the beginning of the next week. If a machine is not repaired, the condition of the machine evolves in accordance with the Markov chain shown in the following table. The company wants to maximize its expected discounted profit over an infinite horizon ($\alpha = 0.9$).

	Next Week		
This Week	Excellent	Good	Bad
Excellent	0.7	0.2	0.1
Good	0	0.7	0.3
Bad	0	0.1	0.9

(a) Use the policy iteration method to determine an optimal stationary policy.

(b) Use linear programming to determine an optimal stationary policy.

(Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

8. For a price of \$1/gallon, the Safeco Supermarket chain has purchased 6 gallons of milk from a local dairy. Each gallon of milk is sold in the chain's three stores for \$2/gallon. The dairy must buy back for \$0.5/gallon any milk that is left at the end of the day. Demand for each of the three stores is uncertain. Past data indicate that the daily demand for each store is as shown in Table 6.

How should Safeco allocate the 6 gallons of milk to the three stores in order to maximise the expected net daily profit (revenues less costs) earned from milk? (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

Store	Daily demand (in gallons)	Probability
1	1	0.6
	2	0
	3	0.4
2	1	0.5
	2	0.1
	3	0.4
3	1	0.4
	2	0.3
	3	0.3

Table 6: Question 8, Problem Set 9

3 Answers and Hints to Selected Questions in Problem Sets

3.1 Answers/Hints to Problem Set 1

1. (a_3, A_3)

5. (b) The payoff matrices for Alice and Bob are $\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$, respectively.

6. (1, 2, 1), (3, 1, 1)

3.2 Answers/Hints to Problem Set 2

1. (a) (a_1, A_2) and (a_2, A_2) are saddles, value $= -1$ (b) No

2. (a) $p \leq 7$ and $q \geq 7$ (b) No. Yes if $q = 7$.

3. Yarra starts towards C, Crocs block CE and DF

4. (a) Saddle iff $1 \leq x \leq 2$; $1 < x < 2$ saddle (a_1, A_1); $x = 1$ saddles (a_1, A_1) and (a_2, A_1); $x = 2$ saddles (a_1, A_1) and (a_1, A_2)

6. Saddle (a_2, A_2), payoffs $(5, 4)$

7. 1.24 to Player I, -1.24 to Player II

8. (a)(i) $\mathbf{E}(\mathbf{x}, \mathbf{y}) = 8x_1y_1 - 2x_1 - 3y_1 + 2$ (b) $(0, 1, 0)$

9. $\mathbf{x}^* = (2/3, 1/3), \mathbf{y}^* = (1/2, 1/2)$, value $= 3$

10. Same as Question 9

11. Theorem: equilibrium implies optimum, so show equilibrium. Value $= \mathbf{x}^* V \mathbf{y}^{*T} = 10/27$.

13. (b) Player I: $(1/4, 3/4)$, Player II: $(3/4, 1/4)$

14. Prisoner will escape with probability $w/(1+w)$. Optimal strategies for both: $(1/(1+w), w/(1+w))$

16. $\mathbf{x}^* = (1/3, 0, 2/3, 0), \mathbf{y}^* = (0, 0, 0, 1/3, 2/3), v = 1/3$

17. $\mathbf{x}^* = (3/8, 0, 5/8, 0), \mathbf{y}^* = (0, 1/4, 0, 0, 3/4), v = 1/4$

18. One solution $\mathbf{x}^* = (1/2, 0, 1/2) = \mathbf{y}^*$. Average 50 kg of drugs gets through.

19. (b) $\mathbf{x}^* = (0, 2/3, 1/3), \mathbf{y}^* = (1/6, 5/6), v = 4$ (c) $x_1' = 0, x_2' = 1/6, x_3' = 1/12$; optimal function value $= 1/4$

21. $\mathbf{x}^* = (1/2, 1/2), \mathbf{y}^* = (3/4, 1/4, 0, 0)$ or $\mathbf{y}^* = (1/2, 0, 1/2, 0)$

22. No

24. One solution (there are more than one solutions): $\mathbf{x}^* = (0, 2/3, 1/3, 0, 0, 0, 0, 0, 0, 0)$, $\mathbf{y}^* = (2/5, 0, 0, 1/3, 4/15, 0)$; value $= 190/3$

26. Buy cheap or insured with odds slightly in favour of cheap: $(10/19, 0, 9/19)$. Average will pay for condenser is \$5.26, (100/19).

27. (a_2, A_2) is a saddle point of the 2-person zero-sum game with the given payoff matrix if and only if $p \leq 7$ and $q \geq 7$.

28. It is impossible to have both (a_2, A_2) and (a_3, A_2) as saddle points. Both (a_2, A_2) and (a_2, A_3) are saddle points if and only if $p \leq 7$ and $q = 7$.

29. Saddle iff $2 \leq x \leq 7$; $x = 2$: saddle (a_3, A_2), value $= 2$; $2 < x < 7$: saddle (a_3, A_2), value $= x$; $x = 7$: saddle (a_3, A_1) and (a_3, A_2), value $= 7$

30. equilibrium: (a_2, A_1); value $= 41$

31. equilibrium: $\mathbf{x} = (1/2, 1/2), \mathbf{y} = (3/4, 1/4)$, value $= 3$

32. No optimal pair of pure strategies; optimal strategy for Player I: $(x_1, x_2, x_3) = (4/9, 11/45, 14/45)$; optimal strategy for Player II: $(y_1, y_2, y_3) = (14/45, 11/45, 4/9)$; value $= -29/45$

33. $v = 9/5$, $\mathbf{x}^* = (3/5, 2/5)$, $\mathbf{y}^* = (2/5, 0, 3/5, 0)$

34. $v = 0$, $\mathbf{x}* = (3/4, 0, 1/4)$, $\mathbf{y}* = (1/2, 0, 1/2)$

3.3 Answers/Hints to Problem Set 3

1. (c) (a_1, A_1) and (a_1, A_2) are not equilibrium pairs.

2. No equilibrium pair

6. (a) None; (b) $(1, 0), (1, 0)$, actual payoffs $(3, 2)$

7. (a) $(\mathbf{x}^*, \mathbf{y}^*) = ((1/2, 1/2, 0), (2/9, 2/3, 1/9))$

9. (a) $((0, 1), (0, 1)), ((2/3, 1/3), (2/3, 1/3)), ((1, 0), (1, 0))$; (b) $((1, 0), (1, 0))$

10. (a) Entries represent thousands of dollars:

		Firm 2	
		50 big	100 small
Firm 1	200 big	$(4, 0)$	$(20, 3)$
	100/100	$(18, 4)$	$(8, -7)$

(b) $(20, 3)$ and $(18, 4)$

13. Nash solution $(11/4, 13/6)$; use (a_1, A_3) with probability 1/12 and (a_1, A_2) with probability 11/12

14. $(15, 15), (20, 14)$

16. Nash solution $(19, 13/6)$; use (a_2, A_1) with probability 1/2 and (a_1, A_2) with probability 1/2

17. (a) a_{12} is a saddle point for Player I, and the security level strategy of Player I is $\mathbf{x}^* = (1, 0)$; optimal security level mixed strategy for Player II is $\mathbf{y}^* = (1/2, 1/2)$; $u^* = 0$, $v^* = 1/2$ (b) No (c) $(\mathbf{x}^*, \mathbf{y}^*) = ((3/4, 1/4), (1/2, 1/2))$ (d) $\mathbf{x}^* A\mathbf{y}^* = 1/2$, $\mathbf{x}^* B\mathbf{y}^* = 1/2$

18. (a) $u^* = 1$, $v^* = -9$ (c) Nash solution $(\underline{u}, \underline{v}) = (10, 1)$; play (a_1, A_2) all the time *if both players follow Nash's Axioms*

19. (a) $((1,0),(0,1))$: no; $((1,0),(1,0))$: yes

3.4 Answers/Hints to Problem Set 4

1. (c) Imputations are $\mathbf{x}^1$, $\mathbf{x}^4$ and $\mathbf{x}^5$ (d) $C(v) = \{(x_1, 0, 18 - x_1) : 16 \leq x_1 \leq 18\} = \text{conv}\big\{(16,0,2),(18,0,0)\big\}$ (e) $(15,1,2)$ and since $v(\{1,3\}) = 18 > \Phi_1(v) + \Phi_3(v)$, $\Phi(v) \notin C(v)$.

2. (a) $a = \frac{1}{2}$. (b) $\mathbf{p}$ is a core element. (c) $C(v) = \text{conv}\big\{(\frac{1}{4},0,\frac{3}{4}),(\frac{1}{4},\frac{1}{4},\frac{1}{2}),(0,\frac{1}{2},\frac{1}{2}),(0,\frac{1}{4},\frac{3}{4})\big\}$. (d) $\Phi(v) = (\frac{1}{8},\frac{1}{4},\frac{5}{8})$).

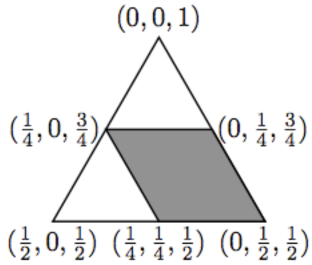

Figure 8: Core of the game of Exercise 2

3. (a) $q > 7$, (b) $q = 7$, (c) $q < 7$.

4. (a) $v(\emptyset) = 0, v(\{A\}) = 375,000, v(\{L\}) = 425,000, v(\{P\}) = 210,000, v(\{A,L\})$ $= 1000,000, v(\{A,P\}) = 785,000, v(\{L,P\}) = 835,000, v(\{A,L,P\}) = 1410,000$

(b) $$C = \left\{ (x_A, x_L, 1410,000 - x_A - x_L) : \begin{array}{l} 375,000 \leq x_A \leq 575,000 \\ 425,000 \leq x_L \leq 625,000 \\ 1200,000 \geq x_A + x_L \geq 1000,000 \end{array} \right\}$$

(c) $(508,333\frac{1}{3}, 558,333\frac{1}{3}, 343,333\frac{1}{3})$

5. (a) $C(v) = \{(x_1,x_2,x_3) : 0 \leq x_1 \leq 3/4, 0 \leq x_2 \leq 1/2, 0 \leq x_3 \leq 2/3, x_1+x_2+x_3 = 1\} = \text{conv}\big\{(\frac{1}{3},0,\frac{2}{3}),(0,\frac{1}{3},\frac{2}{3}),(0,\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},0),(\frac{3}{4},\frac{1}{4},0),(\frac{3}{4},0,\frac{1}{4})\big\}$. (b) The marginal vectors are equal to the extreme points of the core. In general: games for which all marginal vectors are core elements are called *convex* games (in fact in that case the marginal vectors are the extreme points of the core). Convex games have some special properties, which will not be discussed here.

6. $r = 19$, $t = 5$.

8. (a) $v = u_{\{1\}} + u_{\{1,2\}} + 3u_{\{1,3\}} + 5u_{\{2,3\}} - 3u_{\{1,2,3\}}$. (b) $\Phi(v) = (2, 2, 3)$.

9. $x_1 = \cdots = x_5 = 0.19627...$, $x_6 = \cdots = x_{15} = 0.00186...$; 98.1% of the power is in the hands of the members who have a veto.

10.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	-1	-1	-1	1	1	1	0

12. $\phi(v) = (v(\{1\}, v(\{2\}, \ldots, v(\{n\})$

13. (b) Yes, e.g. $(1, 0, 0, 1), (0, 1, 0, 1) \in C(v)$ (c) $(1/4, 13/12, -1/4, 11/12)$

14. (a) The egalitarian rule satisfies efficiency, symmetry and additivity. The utopia vector satisfies symmetry, the dummy property and additivity.

16. (b) Hint: if $\mathbf{x} \in I(v)$, the there exists $i, j \in N$ such that $x_i + x_j < v(\{i, j\})$. Use this observation to find an imputation $\mathbf{y} \in I(v)$ that dominates $\mathbf{x}$.

18. Rearrange expressions in the formula of the Shapley value that uses marginal vectors.

19. (a)

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	50	70	80	70	80	80	80

(b) $\Phi(c) = (\frac{50}{3}, \frac{80}{3}, \frac{110}{3})$. The Shapley value divides the costs of each part of the strip equally among all its users.

20. (a)

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$c^T(S)$	7	3	6	4	13	5	6

(b) $C(c^T) = \Big\{(x_1 + x_2 + x_3) \mid x_1 + x_2 + x_3 = 6,\ 1 \leq x_1 \leq 7,\ -7 \leq x_2 \leq 3,\ 2 \leq x_3 \leq 6\Big\}$
(c) $\beta = (1, 3, 2)$.

3.5 Answers/Hints to Problem Set 5

2. (a) No

3. (a) Yes (b) least and minimal element $(11, -2)$, greatest and maximal element $(4, 7)$

5. (a) Least and minimal $(1, 2)$; greatest and maximal $(4, 3)$ (b) No greatest, no least; minimal: $(1, 2), (2, 1)$; maximal: $(2, 5), (3, 4), (4, 3)$

7. No

10. (a) Lexicographic order: $(5, -1, 0), (3, 2, 1), (1/2, 4, 2), (0, 1, -1), (-2, 3, 1)$; greatest: $(5, -1, 0)$; least $(-2, 3, 1)$ (b) $P_{min} = \{a, b, e\},\ P_{max} = \{c, d, e\}$.

11. No

12. Least: none; greatest: $(2, 4), (3, 3)$; minimal: $(0, 0)$; maximal: $(2, 4), (3, 3)$

13. Least number: 6

14. (a) Lexicographic: $(8, 1, 1)$, $(2, 1, 1)$, $(2, 0, 3)$, $(1, 2, 3)$, $(-4, -2, 1)$, $(-5, 0, 2)$, $(-6, -2, 0)$; greatest: $(8, 1, 0)$; least: $(-6, -2, 0)$ (b) no greatest; least: $(-6, -2, 0)$; $P_{max} = \{(8, 1, 1), (2, 0, 3), (1, 2, 3)\}$; $P_{min} = \{(-6, -2, 0)\}$

15. Hint: When t is nearer τ_4 than t', then t is nearer at least three other ideal points than t'.

16. Hint: Underlying descriptive order: a, b, c

3.6 Answers/Hints to Problem Set 6

1. Maximin: $a_1 \succ a_2 \sim a_3 \sim a_4 \sim a_5$; maximax: $a_1 \succ a_5 \succ a_4 \sim a_3 \succ a_2$; Minimax Regret: $a_1 \succ a_5 \succ a_2 \sim a_4 \succ a_3$; Expected Value: $a_1 \succ a_5 \succ a_4 \succ a_3 \succ a_2$

2. Maximin: diversify; Maximax: Expand greatly; Minimax Regret: Expand slightly; Expected Value: Hold steady

3. Maximin: charge \$7, profit $\geq$ \$225; Maximax: charge \$9, profit $\leq$ \$625; Minimax

regret: \$8; Laplace: \$8, expected profit \$400

5. Hint: Try different probability distributions to make a_1, a_2 or a_3 optimal. Proof by contradiction for a_4

6. (a)(i) a_1 if $x > 2$, a_1 or a_2 if $x = 2$, a_2 if $x < 2$; (ii) a_1 if $x > 7$, a_1 or a_3 if $x = 7$, a_3 if $x < 7$; (iii) a_1 if $x > 3$, a_1 or a_4 if $x = 3$, a_4 if $x < 3$; (iv) a_1 if $x \geq 6$, a_1 if $x > 2$, a_1 or a_3 if $x = 2$, a_3 if $x < 2$ (b) a_1 if $x > 7$

3.7 Answers/Hints to Problem Set 7

1. (a) Risk-averse (b) Prefer L_1; risk premium for $L_2 = \$339$

2. (a) Risk-seeking (b) Prefer L_2; risk premium for $L_2 = -\$235$

4. Statistics course

5. a_1

6. Hint: Simultaneous linear equations

7. Invest \$826 in share A and \$174 in share B; invest \$1000 in share A

8. $p^3R < b < p^3R/(p^3 + (1-p)^3)$

9. \$4

3.8 Answers/Hints to Problem Set 8

1. In terms of original vertex labels: Lexico: $6 \to 7 \to 1 \to 5 \to 4$ [12, 20]; Pareto: $6 \to 7 \to 1 \to 5 \to 4$ [12, 20], $6 \to 8 \to 2 \to 4$ [19, 7], $6 \to 7 \to 9 \to 2 \to 4$ [17, 12]

3. Trade in car whenever it is two years old (at times 2, 4 and 6). Net cost is \$14,400.

4. Pareto minimal paths: $1-3-6$ with distance $(6,4)$, $1-2-3-6$ with distance $(7,3)$

5. Optimal strategy: Hire a geologist to obtain more information. If the report is favorable, drill for oil; otherwise, do not drill.

6. Pareto minimal path for $(9, 9)$: $1-3-4-6-8$; Pareto minimal path for $(8, 10)$: $1-3-5-7-8$

8. Site 1: \$1 million; site 2: \$2 million; site 3: \$1 million. Total revenue is \$24 million.

9. One type 2 item and one type 3 item yield a total value of 75.

3.9 Answers/Hints to Problem Set 9

1. The maximum expected net profit is 10.8 millions. Initially, invest $y_1^* = 2$ millions to the first project 1. Then at stage 2 invest $y_2^* = 0$ to the second project. The remaining 2 millions are invested to the third project.

2. Define $f_t(d)$ to be the maximum expected number of units sold in markets $t, t+1, \ldots, T$, given that d dollars are available to spend on these markets. Then $f_T(d) = c_T k_T p_T(d)$ and for $t < T$, $f_t(d) = \max_{0 \le x \le d}(c_t k_t p_t(x) + f_{t+1}(d - x))$. Work backward until $f_1(D)$ is computed.

5. (c) Optimal policy for two weeks period ($N = 2$): Advertise with TV in each week, regardless sales volumes (states). Maximum expected return 17.83 if the initial sales volume is fair, 16.29 if the initial sales volume is good, and 13.43 if the initial sales volume is excellent.

6. (a) (Period's beginning inventory, period's production level): (0, 4), (1, 3), (2, 0), (3, 0)