SCHOOL OF MATHEMATICS AND STATISTICS

MAST30022 Decision Making **Semester 2, 2021**

Assignment 3 Solutions

1. (a)(i)

   - **Transitivity**: Assume $\boldsymbol{a}\theta\boldsymbol{b}$ and $\boldsymbol{b}\theta\boldsymbol{c}$. We need to check if we then have $\boldsymbol{a}\theta\boldsymbol{c}$. By the definition of θ,
   $$\boldsymbol{a}\theta\boldsymbol{b} \iff a_1a_2 - b_1b_2 = 2k+1$$
   and
   $$\boldsymbol{b}\theta\boldsymbol{c} \iff b_1b_2 - c_1c_2 = 2l+1$$
   for some integers k, l.
   Therefore,
   $$(a_1a_2 - b_1b_2) + (b_1b_2 - c_1c_2) \ = \ a_1a_2 - c_1c_2 = 2\,(k+l+1),$$
   (the sum of two odd numbers is an even number), and we have $\neg\boldsymbol{a}\theta\boldsymbol{c}$. Hence, θ is not transitive.
   - **Reflexivity**: We need to check if $\boldsymbol{a}\theta\boldsymbol{a}$ for any $\boldsymbol{a} \in \mathbb{Z}^2$. By the definition of θ, $\boldsymbol{a}\theta\boldsymbol{a} \iff a_1a_2 - a_1a_2 = 0$ is odd, which is not true since 0 is even. So, θ is not reflexive.
   - **Comparability**: We need to check if for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}\times\mathbb{Z}$ we have $\boldsymbol{a}\theta\boldsymbol{b}$ or $\boldsymbol{b}\theta\boldsymbol{a}$ or both. It is not the case here since, for example, if $\boldsymbol{a} = (2,1)$ and $\boldsymbol{b} = (1,2)$, then $a_1a_2 - b_1b_2 = 0$ and $b_1b_2 - a_1a_2 = 0$, neither of them being odd. Hence, θ is not comparable.
   - **Symmetry**: We need to check if for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}\times\mathbb{Z}$ we have $\boldsymbol{a}\theta\boldsymbol{b} \iff \boldsymbol{b}\theta\boldsymbol{a}$. This is indeed the case since $a_1a_2 - b_1b_2$ is odd if and only if $b_1b_2 - a_1a_2$ is odd. Hence, θ is symmetric.
   - **Asymmetry**: We need to check if $\boldsymbol{a}\theta\boldsymbol{b} \implies \neg\boldsymbol{b}\theta\boldsymbol{a}$. It is not the case since we have shown that θ is symmetric.
   - **Antisymmetry**: Assume $\boldsymbol{a}\theta\boldsymbol{b}$ and $\boldsymbol{b}\theta\boldsymbol{a}$. We need to check if we then have $\boldsymbol{a} = \boldsymbol{b}$. From the symmetry property, $\boldsymbol{a}\theta\boldsymbol{b} \iff \boldsymbol{b}\theta\boldsymbol{a}$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}\times\mathbb{Z}$, but it is clear that $\boldsymbol{a}$ is not necessarily equal to $\boldsymbol{b}$. Hence, θ is not antisymmetric.

   (a)(ii) If we replace "odd" by "even" in the definition of θ, we gain transitivity because the sum of two even numbers is an even number.

   We gain reflexivity because 0 is an even number.

   We still do not gain comparability because, for example, if $\boldsymbol{a} = (1,1)$ and $\boldsymbol{b} = (2,2)$, then $a_1a_2 - b_1b_2 = -3$ and $b_1b_2 - a_1a_2 = 3$, neither of which is even.

   We keep symmetry because $a_1a_2 - b_1b_2$ is even if and only if $b_1b_2 - a_1a_2$ is even.

   We still do not gain asymmetry because the new θ is still symmetric.

   We still do not gain antisymmetry because from the symmetry property, $\boldsymbol{a}\theta\boldsymbol{b} \iff \boldsymbol{b}\theta\boldsymbol{a}$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}\times\mathbb{Z}$, but it is clear that $\boldsymbol{a}$ is not necessarily equal to $\boldsymbol{b}$.

(b)(i)

- **Transitivity**: Assume $\boldsymbol{a}\bar{\theta}\boldsymbol{b}$ and $\boldsymbol{b}\bar{\theta}\boldsymbol{c}$. We need to check if we then have $\boldsymbol{a}\bar{\theta}\boldsymbol{c}$. This does not always hold, take for example $\boldsymbol{a} = (0, 2)$, $\boldsymbol{b} = (2, 1)$, and $\boldsymbol{c} = (1, 3)$. We have $\boldsymbol{a}\bar{\theta}\boldsymbol{b}$ since $a_2 \geq b_2$ and $\boldsymbol{b}\bar{\theta}\boldsymbol{c}$ since $b_1 \geq c_1$. But we don't have $\boldsymbol{a}\bar{\theta}\boldsymbol{c}$ since $a_1 < c_1$ and $a_2 < c_2$. Therefore, $\bar{\theta}$ is not transitive.
- **Reflexivity**: We need to check if $\boldsymbol{a}\bar{\theta}\boldsymbol{a}$ for any $\boldsymbol{a} \in \mathbb{Z} \times \mathbb{Z}$. By the definition of $\bar{\theta}$, $\boldsymbol{a}\bar{\theta}\boldsymbol{a} \iff a_1 \geq a_1$ or $a_2 \geq a_2$. Since this holds with equality, $\bar{\theta}$ is reflexive.
- **Comparability**: We need to check if for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z} \times \mathbb{Z}$ we have $\boldsymbol{a}\theta\boldsymbol{b}$ or $\boldsymbol{b}\theta\boldsymbol{a}$ or both. If $\neg\boldsymbol{a}\bar{\theta}\boldsymbol{b}$ we have $a_1 < b_1$ and $a_2 < b_2$, in which case $b_1 \geq a_1$ and $b_2 \geq a_2$, hence $\boldsymbol{b}\bar{\theta}\boldsymbol{a}$. Therefore, $\bar{\theta}$ is comparable.
- **Symmetry**: We need to check if for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z} \times \mathbb{Z}$ we have $\boldsymbol{a}\bar{\theta}\boldsymbol{b} \iff \boldsymbol{b}\bar{\theta}\boldsymbol{a}$. This is not the case, take for instance $\boldsymbol{a} = (2, 3)$ and $\boldsymbol{b} = (1, 2)$. Then since $a_1 \geq b_1$ (and $a_2 \geq b_2$), we have $\boldsymbol{a}\bar{\theta}\boldsymbol{b}$, but we do not have $\boldsymbol{b}\bar{\theta}\boldsymbol{a}$. So, $\bar{\theta}$ is not symmetric.
- **Asymmetry**: We need to check if $\boldsymbol{a}\bar{\theta}\boldsymbol{b} \implies \neg\boldsymbol{b}\bar{\theta}\boldsymbol{a}$. This is not the case as we can have both $\boldsymbol{a}\bar{\theta}\boldsymbol{b}$ and $\boldsymbol{b}\bar{\theta}\boldsymbol{a}$ for some $\boldsymbol{a}$ and $\boldsymbol{b}$ (but not for all $\boldsymbol{a}$ and $\boldsymbol{b}$ since $\bar{\theta}$ is not symmetric). Take for example $\boldsymbol{a} = (3, 1)$ and $\boldsymbol{b} = (2, 3)$. So $\bar{\theta}$ is not asymmetric.
- **Antisymmetry**: Assume $\boldsymbol{a}\bar{\theta}\boldsymbol{b}$ and $\boldsymbol{b}\bar{\theta}\boldsymbol{a}$. We need to check if we then have $\boldsymbol{a} = \boldsymbol{b}$. This is not the case since for $\boldsymbol{a} = (3, 1)$ and $\boldsymbol{b} = (2, 3)$, as shown above we have $\boldsymbol{a}\bar{\theta}\boldsymbol{b}$ and $\boldsymbol{b}\bar{\theta}\boldsymbol{a}$ but $\boldsymbol{a} \neq \boldsymbol{b}$. So $\bar{\theta}$ is not antisymmetric.

(b)(ii) By replacing "$a_1 \geq b_1$ or $a_2 \geq b_2$" by "$a_1 > b_1$ or $a_2 > b_2$" in $\bar{\theta}$, we still do not have transitivity. Take the above counterexample replacing "$\geq$" with "$>$".

We lose reflexivity since now $\neg\boldsymbol{a}\bar{\theta}\boldsymbol{a}$ for all $\boldsymbol{a} \in \mathbb{Z} \times \mathbb{Z}$ since $a_1 > a_1$ and $a_2 > a_2$ do not hold.

We lose comparability because $\bar{\theta}$ is now not reflexive and we cannot compare an element with itself.

We still do not have symmetry. Take the above counterexample replacing "$\geq$" with "$>$".

We still do not have asymmetry. Take the above counterexample replacing "$\geq$" with "$>$".

We still do not have antisymmetry. Take the above counterexample replacing "$\geq$" with "$>$".

2. (a) Suppose $a\theta^* b$ and $b\theta^* c$. Then there exists a sequence $a_1, a_2, \ldots, a_k \in A$ such that $a = a_1$, $b = a_k$ and $a_i \theta a_{i+1}$ for all $i = 1, \ldots, k-1$, and a sequence $b_1, b_2, \ldots, b_\ell \in A$ such that $b = b_1$, $c = b_\ell$ and $b_i \theta b_{i+1}$ for all $i = 1, \ldots, \ell - 1$.

Now note that $a_k = b = b_1$. Rename $b_2 = a_{k+1}$, $b_3 = a_{k+2}, \ldots, b_\ell = a_{k+\ell-1}$.

Then for the sequence $a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_{k+\ell-1} \in A$, we have

- $a_i \theta a_{i+1}$ for all $i = 1, \ldots, k + \ell - 1$
- $a_1 = a$
- $a_{k+\ell-1} = c$

So $a\theta^* c$ by the definition of θ^*, and therefore θ^* is transitive.

(b) (i) Suppose $a, b, c \in V$ and $a\theta b$, $b\theta c$, that is, a is a child of b and b is a child of c. Then a cannot be a child of c, since otherwise the underlying graph would contain a cycle which contradicts the fact that (T, r) is a rooted tree. Specific counter-example: $r = c \to b \to a$, then $a\theta b$, $b\theta c$, but $\neg a\theta c$.

(ii) If $a\theta^* b$, then there exists a path from b to a in (T, r). Or, in other words, a is a descendent of b.

3. (a) Let

$$\boldsymbol{a} = (1, 2, -1),\ \boldsymbol{b} = (2, 1, -1),\ \boldsymbol{c} = (-1, 2, 1),\ \boldsymbol{d} = (2, -1, 1),$$
$$\boldsymbol{e} = (2, -1, -1),\ \boldsymbol{f} = (2, -2, 1),\ \boldsymbol{g} = (-1, 1, 1).$$

Boolean matrix:

$\geq_L$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$
$\boldsymbol{a}$	×		×				×
$\boldsymbol{b}$	×	×	×	×	×	×	×
$\boldsymbol{c}$			×				×
$\boldsymbol{d}$	×		×	×	×	×	×
$\boldsymbol{e}$	×		×		×	×	×
$\boldsymbol{f}$	×		×			×	×
$\boldsymbol{g}$							×

$L_{\max}(A) = \{\boldsymbol{b}\}$.

$L_{\min}(A) = \{\boldsymbol{g}\}$.

$\boldsymbol{b}$ is the greatest element and $\boldsymbol{g}$ is the least element.

(b) (i)
- **Reflexivity**: we need to check if $\boldsymbol{x}\theta^P\boldsymbol{x}$ for any $\boldsymbol{x} \in A$. By definition of θ^P, we have $\boldsymbol{x}\theta^P\boldsymbol{x}$ for any $\boldsymbol{x} \in A$ if and only if $f(\boldsymbol{x})Pf(\boldsymbol{x})$ for any $\boldsymbol{x} \in A$, which holds by reflexivity of P. Hence, θ^P is reflexive.
- **Transitivity**: Assume $\boldsymbol{x}\theta^P\boldsymbol{y}$ and $\boldsymbol{y}\theta^P\boldsymbol{z}$. We need to check if we then have $\boldsymbol{x}\theta^P\boldsymbol{z}$.
  By the definition of θ^P,
  $$\boldsymbol{x}\theta^P\boldsymbol{y} \Longleftrightarrow f(\boldsymbol{x})Pf(\boldsymbol{y})$$
  and
  $$\boldsymbol{y}\theta^P\boldsymbol{z} \Longleftrightarrow f(\boldsymbol{y})Pf(\boldsymbol{z}).$$
  By the transitivity of P, we then have $f(\boldsymbol{x})Pf(\boldsymbol{z})$, which means that $\boldsymbol{x}\theta^P\boldsymbol{z}$. Hence, θ^P is transitive.

- **Antisymmetry**: Assume $\boldsymbol{x}\theta^P\boldsymbol{y}$ and $\boldsymbol{y}\theta^P\boldsymbol{x}$. We need to check if we then have $\boldsymbol{x}=\boldsymbol{y}$. By the definition of θ^P

$$\boldsymbol{x}\theta^P\boldsymbol{y} \Longleftrightarrow f(\boldsymbol{x})Pf(\boldsymbol{y}) \quad \text{and} \quad \boldsymbol{y}\theta^P\boldsymbol{x} \Longleftrightarrow f(\boldsymbol{y})Pf(\boldsymbol{x}),$$

then by antisymmetry of P, we obtain $f(\boldsymbol{x}) = f(\boldsymbol{y})$. But that does not necessarily mean that $\boldsymbol{x}=\boldsymbol{y}$, take for example $\boldsymbol{x}=(1,2,-1)$ and $\boldsymbol{y}=(2,1,-1)$. Hence, θ^P is not antisymmetric.

(ii) We have

$$f(\boldsymbol{a}) = (3,-1),\ f(\boldsymbol{b}) = (3,-1),\ f(\boldsymbol{c}) = (1,1),\ f(\boldsymbol{d}) = (1,1),$$
$$f(\boldsymbol{e}) = (1,-1),\ f(\boldsymbol{f}) = (0,1),\ f(\boldsymbol{g}) = (0,1).$$

Boolean matrix:

θ^P	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$
$\boldsymbol{a}$	×	×			×		
$\boldsymbol{b}$	×	×			×		
$\boldsymbol{c}$			×	×	×	×	×
$\boldsymbol{d}$			×	×	×	×	×
$\boldsymbol{e}$					×		
$\boldsymbol{f}$						×	×
$\boldsymbol{g}$						×	×

$\theta^P_{\max}(A) = \{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\}$.
$\theta^P_{\min}(A) = \{\boldsymbol{e},\boldsymbol{f},\boldsymbol{g}\}$.
There is no greatest element and no least element.

4. (a) The decision table is

	θ_1	θ_2	θ_3	θ_4	s_i	o_i	$(s_i+o_i)/2$	$\bar{v}_i$
a_1	16	1	7	16	1	16	17/2	40/4
a_2	4	1	25	7	1	25	26/2	37/4
a_3	4	4	10	4	4	10	14/2	22/4
a_4	7	10	16	x	$\min(7,x)$	$\max(16,x)$	$(\min(7,x)+\max(16,x))/2$	$(33+x)/4$

The regret matrix is

	θ_1	θ_2	θ_3	θ_4	ρ_i
a_1	0	9	18	$\max(0,x-16)$	$\max(18,x-16)$
a_2	12	9	0	$\max(9,x-7)$	$\max(12,x-7)$
a_3	12	6	15	$\max(12,x-4)$	$\max(15,x-4)$
a_4	9	0	9	$\max(16-x,0)$	$\max(9,16-x)$

(i) **Wald's maximin criterion**: the decision maker chooses a_3 if $x<4$, a_4 if $x>4$, and is indifferent between a_3 and a_4 if $x=4$.

(ii) **Hurwicz's α-criterion**: the decision maker chooses a_2 if $x<19$, a_4 if $x>19$, and is indifferent between a_2 and a_4 if $x=19$.

Indeed, there are two critical values of x: $x=7$ and $x=16$. If $x\le 7$, then $\min(7,x)+\max(16,x) = x+16 \le 23 < 26$; if $7<x\le 16$, then $\min(7,x)+\max(16,x) = 23 < 26$; and if $x>16$, then $\min(7,x)+\max(16,x) = 7+x > 23$ and $7+x>26 \Longleftrightarrow x>19$.

(iii) **Laplace's criterion**: the decision maker chooses a_1 if $x < 7$, a_4 if $x > 7$, and is indifferent between a_1 and a_4 if $x = 7$.

(iv) **Savage's minimax regret criterion**: the decision maker chooses a_2 if $x < 4$, a_4 if $x > 4$, and is indifferent between a_2 and a_4 if $x = 4$.

Indeed, there are two critical values of x: $x = 4$ and $x = 7$. We can check that if $x \geq 7$ then the optimal action is a_4. If $x < 7$, then the decision maker chooses a_2 if $16 - x > 12$, that is, if $x < 4$, and a_4 if $16 - x < 12$, that is, if $x > 4$. He is indifferent between a_2 and a_4 if $x = 4$.

(b) All criteria lead to the same choice of action a_4 if $x > 19$. If $x = 19$, then Wald's, Laplace's and Savage's criteria all lead to choose a_4, while Hurwicz's α-criterion leads to indifference between actions a_4 and a_2.

(c) Choose $x = 19$. The decision table is

	θ_1	θ_2	θ_3	θ_4	$(s_i + o_i)/2$
a_1	16	1	7	16	17/2
a_2	4	1	25	7	26/2
a_3	4	4	10	4	14/2
a_4	7	10	16	19	26/2

Hurwicz's α-criterion leads to $a_2 \sim a_4 \succ a_1 \succ a_3$.

If we add $c = 10$ to the third column of the decision matrix we get

	θ_1	θ_2	θ_3	θ_4	$(s_i + o_i)/2$
a_1	16	1	17	16	18/2
a_2	4	1	35	7	36/2
a_3	4	4	20	4	24/2
a_4	7	10	26	19	31/2

Hurwicz's α-criterion leads to $a_2 \succ a_4 \succ a_3 \succ a_1$, which is a different preference order. Therefore, Hurwicz's α-criterion does not satisfy the axiom of *independence of addition of a constant to a column.*