MAST20004 Probability

Assignment 3

- Assignment boxes are located on the ground floor in the Peter Hall Building (near Wilson computer lab) and your tutor's name and box number for submission are on the signs above the assignment boxes.
- Your solutions to the assignment should be left in the MAST20004 assignment box set up for your tutorial group.
- **Don't forget** to staple your solutions and to print your name, student ID, the subject name and code, and your tutor's name on the first page.
- The submission deadline is **3 pm on Friday 10 May**.
- There are 4 questions, of which 2 randomly chosen questions will be marked. Note you are expected to submit answers to all questions, otherwise a mark penalty will apply. Working and reasoning **must** be given to obtain full credit. Give clear and concise explanations. Clarity, neatness, and style count.

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1. Let X be a random variable with probability density function f_X given by

$$f_X(x) = \begin{cases} \dfrac{\gamma\alpha^\gamma}{(x+\alpha)^{\gamma+1}}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

where $\alpha > 0$ and $\gamma > 0$.

   (a) Find the cumulative distribution function (cdf) F_X of X.

   (b) Let $Y = \log\left(\frac{X+\alpha}{\alpha}\right)$. Find the cdf of Y and identify the distribution.

   (c) How could a realisation of X be generated from an $R(0,1)$ random number generator?

   (d) Let $Z = \min(X, M)$, where $M > 0$ is a fixed constant. Derive the cdf F_Z of Z and compute its mean.

2. Roll a fair tetrahedral (4-sided) die once and let X be the face value. Then toss a fair coin X times and let Y be the number of heads.

   (a) Derive the joint probability mass function (pmf) $p_{(X,Y)}$.

   (b) Calculate the marginal probability mass function of Y.

   (c) Calculate the conditional distribution of X given $Y = 3$.

   (d) Are X and Y independent? Justify your answer.

[Please Turn Over!]

3. The joint probability density function (pdf) of (X, Y) is given by

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{12}{7}x(x+y), & \text{for } 0 \le y \le 1,\ 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the cumulative distribution function of (X, Y).

Make sure you derive expressions for the cdf in the regions

- $x < 0$ or $y < 0$;
- $0 \le x \le 1,\ 0 \le y \le 1$;
- $x > 1,\ 0 \le y \le 1$;
- $0 \le x \le 1,\ y \ge 1$;
- $x > 1, y > 1$.

(b) Evaluate

(i) $\mathbb{P}(X > Y)$;

(ii) $\mathbb{P}(Y \le X^2)$.

(c) Derive the pdf of X and then compute the mean and variance of X.

(d) Find the pdf of Y and compute the mean and variance of Y.

(e) Calculate the conditional pdf of Y given $X = x$.

(f) Compute the correlation coefficient $\rho(X, Y)$.

(g) Are X and Y independent? Explain.

4. An urn contains $n > 1$ balls numbered $1, 2, \ldots, n$, of which k $(1 \le k < n)$ are chosen at random *without* replacement. Let T be the sum of the values showing on the k balls.

(a) Download the Matlab file **Assn_3_Qu_4_Matlab.m** from the LMS. Let $n = 10$ and $k = 1, 2, \ldots, 9$. Estimate the mean and variance of T for each case. You will need to add the appropriate lines of code. Report your results in a table and comment on them.

(b) Make a copy of the Matlab file and change one line so that now the balls are sampled *with* replacement. Call the sum S. For $n = 10$ and $k = 1, 2, \ldots, 9$, estimate the mean and variance of S for each case. Report your results in a table. Compare the results with those of Part (a) and comment on your findings.

Please print out your code for Part (b) and include it with your assignment. A mark penalty will apply to assignments with no code or incorrect code.