COPY

The University of Melbourne
Semester 1 Assessment 2012

Department of Mathematics and Statistics

MAST10007 Linear Algebra

Reading Time: 15 minutes
Writing Time: 3 hours

This paper has: 6 pages

Identical Examination Papers: None
Common Content Papers: MAST10008 Accelerated Mathematics 1

Authorized Materials:

No materials are authorized.

Calculators and mathematical tables are not permitted.

Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:

This examination consists of 13 questions.

The total number of marks is 100.

All questions may be attempted. All answers should be appropriately justified.

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) Consider the following linear system:

$$\begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & - & x_3 & - & 3x_4 & = & 3 \\ 2x_1 & - & 5x_2 & - & 2x_3 & - & 6x_4 & = & 5 \\ x_1 & - & 3x_2 & & & - & x_4 & = & 5 \end{array}$$

(i) Write down the augmented matrix corresponding to the linear system.

(ii) Reduce the matrix in (i) to reduced row-echelon form.

(iii) Use the reduced row-echelon form to give all solutions in $\mathbb{R}^4$ to the linear system.

(b) Determine the values (if any) of k in $\mathbb{R}$ for which the following linear system has:

(i) no real solution,

(ii) a unique real solution,

(iii) infinitely many real solutions.

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 2 \\ x & - & ky & - & 2z & = & -k \\ & & (k+1)y & - & kz & = & k-k^2+3 \end{array}$$

[10 marks]

2. Consider the matrices

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Evaluate, if possible:

(a) AB

(b) $C + C^T$

(c) C^3

(d) ABC

[4 marks]

3. Consider the matrix

$$M = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Use cofactor expansion to calculate its determinant $\det(M)$.

(b) Find the inverse of the matrix M given above or explain why it does not exist.

(c) Suppose that P is a 3×3 invertible matrix. Calculate $\det(PM^{-9}P^{-1})$, where M is the matrix given above.

[6 marks]

4. (a) Let L be the line in $\mathbb{R}^3$ given by the following vector equation:

$$(x, y, z) = (1, 1, 1) + t(1, -2, 2), \qquad t \in \mathbb{R}.$$

(i) Show that the point $(0, 1, 2)$ does not lie on the line L.

(ii) Find the Cartesian equation of the plane that contains the point $(0, 1, 2)$ and contains the line L.

(b) Use vectors to calculate the area of the triangle in $\mathbb{R}^3$ with vertices at $(1, 1, -2)$, $(2, 2, -3)$ and $(1, 5, 2)$.

[8 marks]

5. (a) For each of the following, decide whether or not the given set S is a subspace of the vector space V. Justify your answers by either using appropriate theorems, or providing a counter-example.

(i) $V = \mathcal{P}_3$ (all polynomials of degree at most three) and

$$S = \left\{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 - a_1 + a_2 - a_3 \geq 0\right\}.$$

(ii) $V = M_{2,2}$ (all 2×2 matrices) and

$$S = \left\{A \in M_{2,2} \mid A + A^T = 0\right\}.$$

(b) Let V be a vector space and let H and K be two subspaces of V. Let W denote the subset $H \cap K$ of V. Prove that W is a subspace of V.

[10 marks]

6. Determine whether or not the following set S is linearly independent in the given vector space V. If the set is linearly dependent, express one of its vectors as a linear combination of the other vectors in the set S.

$$S = \left\{ \begin{bmatrix} 1 & 4 \\ -4 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ 5 & -6 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$$

$$V = \left\{ \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

[5 marks]

7. Let

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 2 & -4 & 3 & 3 & 0 \\ 1 & -2 & 1 & 2 & 1 \\ 1 & -2 & 4 & -1 & 1 \\ 0 & 0 & 3 & -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B is the reduced row echelon form of the matrix A. Using this information, or otherwise, answer the following questions, giving reasons for your answers.

(a) Is A invertible? Give a reason.

(b) Write down a basis for the column space of A.

(c) Write down a basis for the row space of A.

(d) Are the vectors $(1, 2, 1, 1, 0), (0, 3, 1, 4, 3), (3, 3, 2, -1, -3)$ linearly independent? If not, write one of these vectors as a linear combination of the others.

(e) Find a basis for the solution space of A.

[10 marks]

8. Let $M_{2,2}$ denote the vector space of all 2×2 matrices with real entries and let $\mathcal{P}_2$ denote the vector space of all real polynomials of degree at most 2. A linear transformation $T \colon M_{2,2} \longrightarrow \mathcal{P}_2$ is defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + (b + c)x + dx^2.$$

(a) Find the matrix that represents T relative to the bases

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \quad \text{for } M_{2,2}$$

$$(1, x, x^2) \quad \text{for } \mathcal{P}_2.$$

(b) Find bases for the image and kernel of T.

[6 marks]

9. Consider the bases $\mathcal{S} = \{(1,0),(0,1)\}$ and $\mathcal{B} = \{(3,5),(-1,-2)\}$ for $\mathbb{R}^2$.

(a) (i) Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{S}$.

(ii) Find the transition matrix $P_{\mathcal{B},\mathcal{S}}$ from $\mathcal{S}$ to $\mathcal{B}$.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x,y) = (x-2y, x)$.

(i) Find the matrix $[T]_\mathcal{S}$ of the transformation T with respect to the basis $\mathcal{S}$.

(ii) Find the matrix $[T]_\mathcal{B}$ of the transformation T with respect to the basis $\mathcal{B}$.

(iii) If $\boldsymbol{v} = (-4,-7)$ find $[\boldsymbol{v}]_\mathcal{B}$ and $[T\boldsymbol{v}]_\mathcal{B}$.

[8 marks]

10. (a) Consider the vector space $\mathcal{P}_2$ of all polynomials of degree at most 2. Define

$$\langle p, q\rangle = p(0)q(0).$$

Determine whether this formula defines an inner product on $\mathcal{P}_2$. If it is an inner product, prove this by verifying the inner product axioms. If not, give a counterexample to one of the inner product axioms.

(b) (i) Use the Gram-Schmidt procedure to find an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by the vectors

$$(1,-1,1,1), \quad (0,-1,0,1).$$

(Use the dot product on $\mathbb{R}^4$ as the inner product.)

(ii) Find the point in W (from (i)) closest to the vector $\boldsymbol{u} = (1,1,1,1)$.

[10 marks]

11. Let

$$C = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$$

(a) Find all eigenvalues of C.

(b) Find corresponding eigenvectors of C.

(c) Find a matrix P and a diagonal matrix D such that $C = PDP^{-1}$.

(d) Find an orthogonal matrix Q and a diagonal matrix D such that

$$D = Q^T C Q$$

(e) Consider the conic

$$2x^2 + 2\sqrt{2}xy + 3y^2 = 1.$$

Use your previous answers to find a simplified equation for the conic (i.e. an equation in new variables X and Y that does not have an XY term). Hence identify the conic and write down direction vectors for its principal axes.

[10 marks]

12. (a) For each of the following matrices decide whether or not the matrix is diagonalizable over $\mathbb{R}$. You should justify your answers.

$$\begin{bmatrix} -1 & 0 \\ 7 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix} \quad \begin{bmatrix} -5 & -13 \\ 2 & 5 \end{bmatrix}$$

(b) Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(i) Find the eigenvalues of A.

(ii) Find a basis for each eigenspace of A.

(iii) Is the matrix A diagonalizable? Explain your answer.

[8 marks]

13. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

A *generalised eigenvector* of A is a nonzero vector $\boldsymbol{v} \in \mathbb{R}^3$ such that

$$(A - \lambda I)^k \boldsymbol{v} = 0$$

for some scalar $\lambda \in \mathbb{R}$ and some integer $k \geq 1$.

(a) Prove that if $\boldsymbol{v}$ is a generalised eigenvector of A, then the corresponding scalar λ is an eigenvalue of A.

(b) Give an example of a vector $\boldsymbol{v} \in \mathbb{R}^3$ such that $\boldsymbol{v}$ is a generalised eigenvector of A, but $\boldsymbol{v}$ is not an eigenvector of A.

[5 marks]

— END OF EXAMINATION QUESTIONS —