MAST10006 lecture slides 2019 s1 print version

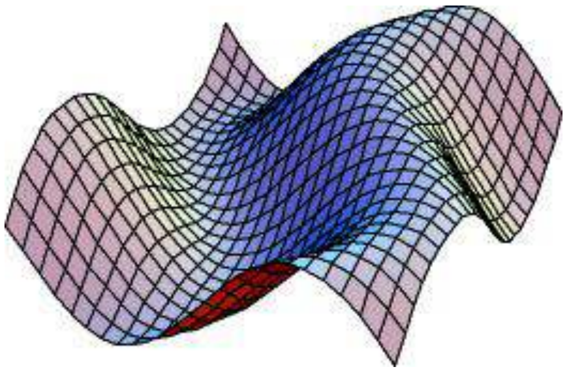
Calculus 2 (University of Melbourne)

THE UNIVERSITY OF MELBOURNE

SCHOOL OF MATHEMATICS AND STATISTICS

MAST10006 Calculus 2

Lecture Notes



STUDENT NAME:	**STUDENT NUMBER:**

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Section 0 - Notation used in MAST10006 Calculus 2

Standard Abbreviations

1. such that or given that: $|$
2. therefore: $\therefore$
3. for all: $\forall$
4. there exists: $\exists$
5. equivalent to: $\equiv$
6. that is: $i.e$
7. approximate: $\approx$
8. much smaller than: $\ll$

Standard Notation for Sets of Numbers

1. natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$
2. integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$
3. rational numbers: $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$
4. real numbers: $\mathbb{R}$ (rational numbers plus irrational numbers)
5. complex numbers: $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R},\ i^2 = -1\}$
6. $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ (xy plane)
7. $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ (3 dimensional space)

Standard Notation for Intervals

1. element of: $\in$
   so $a \in X$ means "a is an element of the set X "
2. open interval: (a, b)
   so $x \in (0, 1)$ means "$0 < x < 1$"
3. closed interval: $[a, b]$
   so $x \in [0, 1]$ means "$0 \leq x \leq 1$"
4. partial open and closed interval: $(a, b]$ or $[a, b)$
   so $x \in [0, 1)$ means "$0 \leq x < 1$"
5. not including: $\setminus$
   so $x \in \mathbb{R} \setminus \{0\}$ means "x is any real number excluding 0".
   Alternatively, we could write $(-\infty, 0) \cup (0, \infty)$ where $\cup$
   means the "union of the two intervals".

More Standard Notation

1. natural logarithm: $\log x$
   base 10 logarithm: $\log_{10} x$

   Alternative notations for natural logarithms used in textbooks: $\log_e x, \ln x$

2. inverse trigonometric functions: $\arcsin x$, $\arctan x$ etc

   Alternative notations used in textbooks: $\sin^{-1} x$, $\tan^{-1} x$ etc

3. implies: $\Rightarrow$
   so $p \Rightarrow q$ means "p implies q"

4. if and only if (iff): $\Leftrightarrow$ (means both $\Leftarrow$ and $\Rightarrow$)
   so $p \Leftrightarrow q$ means "p implies q" AND "q implies p"

5. approaches: $\rightarrow$
   so $f(x) \rightarrow 1$ as $x \rightarrow 0$ means "$f(x)$ approaches 1 as x approaches 0"

Greek Alphabet

α	alpha	ν	nu
β	beta	ξ	xi
γ	gamma	o	omicron
δ	delta	π	pi
ϵ or ε	epsilon	ρ	rho
ζ	zeta	σ	sigma
η	eta	τ	tau
θ	theta	υ	upsilon
ι	iota	ϕ	phi
κ	kappa	χ	chi
λ	lambda	ψ	psi
μ	mu	ω	omega

Section 1: Limits, Continuity, Sequences, Series

Limits

Let f be a real-valued function.

We say that f has the limit L as x approaches a,

$$\lim_{x \to a} f(x) = L,$$

if $f(x)$ gets arbitrarily close to L whenever x is close enough to a but $x \neq a$.

Note:

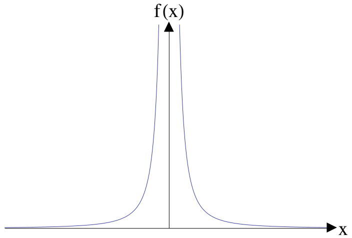
1. The formal definition of limits can be found in more advanced subjects such as MAST20026 Real Analysis.
2. If exists, the limit L must be a unique finite real number.

Example 1.1: If $f(x) = 2x$, evaluate $\lim_{x \to 1} f(x)$.

Solution:

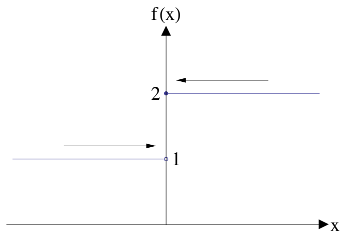
Note:
We can easily evaluate this limit by limit laws in the next few slides.

Example 1.2: If $f(x) = \dfrac{1}{x^2}$, evaluate $\lim_{x \to 0} f(x)$.



Solution:

Example 1.3: If $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$, evaluate $\lim_{x \to 0} f(x)$.



Solution:

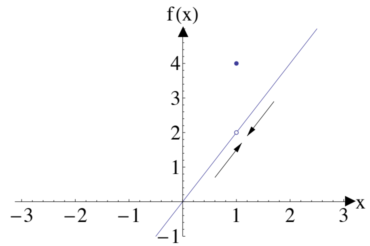
We can describe this behaviour in terms of one-sided limits. We write

Theorem:

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

Thus the limit exists if and only if the left and right hand limits exist and are equal.

Example 1.4: If $f(x) = \begin{cases} 2x & x \neq 1 \\ 4 & x = 1 \end{cases}$, evaluate $\lim_{x\to 1} f(x)$.



Solution:

Note:
The limit of f as x approaches a does not depend on $f(a)$. The limit can exist even if f is undefined at $x = a$.

Limit Laws

Let f and g be real-valued functions and let $c \in \mathbb{R}$ be a constant. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$.
3. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
4. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$.
5. $\lim_{x\to a} c = c$.
6. $\lim_{x\to a} x = a$.

The limit laws can be proved using the definition of limits.

We give the idea of the proof of Limit Law 1 as an example: (A rigorous proof will need the formal definition of limits.)

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

For an arbitrary positive real number ε, to make

$$|f(x) + g(x) - (L + M)| < \varepsilon$$

we only need to make $|f(x) - L| < \frac{\varepsilon}{2}$ and $|f(x) - M| < \frac{\varepsilon}{2}$.

These will be satisfied whenever x is close enough to a but $x \neq a$ since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

Hence $f(x) + g(x)$ can be arbitrarily close to $L + M$ whenever x is close enough to a but $x \neq a$, which means that

$$\lim_{x\to a} [f(x) + g(x)] = L + M = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Example 1.5: Use the limit laws to evaluate $\lim_{x \to 2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

Solution:

Limits as x Approaches Infinity

We say that f has the limit L as x approaches positive infinity,

$$\lim_{x \to \infty} f(x) = L,$$

if $f(x)$ gets arbitrarily close to L whenever x is sufficiently large and positive.

We say that f has the limit M as x approaches negative infinity:

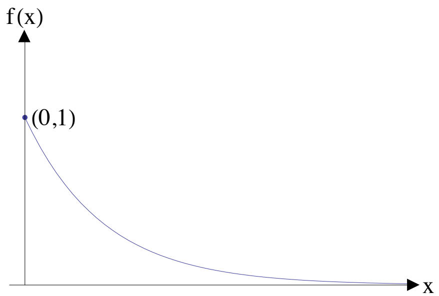
$$\lim_{x \to -\infty} f(x) = M$$

if $f(x)$ gets arbitrarily close to M whenever x is sufficiently large and negative.

Note:

1. L and M must be finite.
2. Limit laws (1)-(5) apply.

Example 1.6: If $f(x) = e^{-x}$, evaluate $\lim_{x \to \infty} f(x)$.



Solution:

Evaluating Limits with Indeterminate Forms

We say a function $\frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ as $x \to a$ if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

Example 1.7: Evaluate $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution:

We say a function $\dfrac{f(x)}{g(x)}$ has indeterminate form $\dfrac{\infty}{\infty}$ as $x \to a$ if $f(x) \to \infty$ and $g(x) \to \infty$.

Example 1.8: Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 2x + 3}{x^2 + 4x + 4}$.

Solution:

We say a function $f(x) - g(x)$ has indeterminate form $\infty - \infty$ as $x \to a$ if $f(x) \to \infty$ and $g(x) \to \infty$.

Example 1.9: Evaluate $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right)$.

Solution:

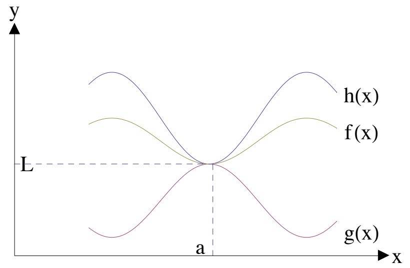
Sandwich Theorem:

If $g(x) \le f(x) \le h(x)$ when x is near a but $x \ne a$, and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} f(x) = L.$$



Note:

1. "x is near a but $x \ne a$" means that x lies in $(b, a) \cup (a, c)$ for some $b < a < c$.
2. The validity of Sandwich Theorem is based on the fact that
$$g(x) \le f(x) \le h(x)$$
$$\Rightarrow |f(x) - L| \le |g(x) - L| + |h(x) - L| \text{ for all } L.$$
Can you prove this inequality or even the stronger conclusion that $|f(x) - L| \le \max\{|g(x) - L|, |h(x) - L|\}$?
3. Sandwich Theorem works for limits as x approaches infinity. For example, if $g(x) \le f(x) \le h(x)$ when $x \in (c, \infty)$ for some real number c, and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$, then
$$\lim_{x\to\infty} f(x) = L.$$
The similar theorem holds when x approaches $-\infty$.

Example 1.10: Evaluate $\lim_{x \to 0} \left[x^2 \sin\left(\frac{1}{x}\right) \right]$.

Solution:

Example 1.11: Evaluate $\lim_{x \to 0} \left[x \sin\left(\frac{1}{x}\right) \right]$.

Solution:

Continuity

Let f be a real-valued function.

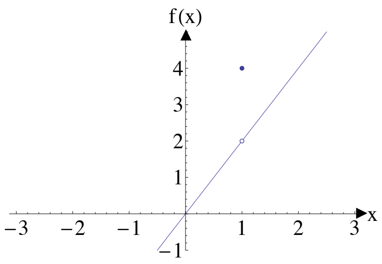
The function f is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Example 1.12: Let

$$f(x) = \begin{cases} 2x & x \neq 1 \\ 4 & x = 1. \end{cases}$$

Is f continuous at $x = 1$?



Solution:

Example 1.13: Let $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2. \end{cases}$

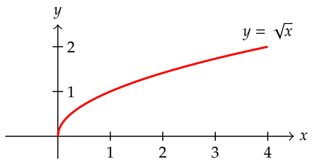
Is f continuous at $x = 2$?

Solution:

At the endpoints of a domain, we cannot take both left and right hand limits, so we use the appropriate limit to test continuity.

1. A function f is left continuous (continuous from the left) at $x = a$ if $\lim\limits_{x\to a^-} f(x) = f(a)$.
2. A function f is right continuous (continuous from the right) at $x = a$ if $\lim\limits_{x\to a^+} f(x) = f(a)$.

Example 1.14: Is $f(x) = \sqrt{x}$ continuous in its domain?



Solution:

Let f and g be real-valued functions and let $c \in \mathbb{R}$ be a constant.

Continuity Theorem 1:

If the functions f and g are continuous at $x = a$, then the following functions are continuous at $x = a$:

1. $f + g$,
2. cf,
3. fg,
4. $\dfrac{f}{g}$ if $g(a) \neq 0$.

Note:
The theorem follows from limit laws.

Continuity Theorem 2:

If f is continuous at $x = a$ and g is continuous at $x = f(a)$, then $g \circ f$ is continuous at $x = a$.

[Recall that $(g \circ f)(x) = g(f(x))$.]

Continuity Theorem 3:

The following function types are continuous at every point in their domains: polynomials, trigonometric functions, exponentials, logarithms, nth root functions, hyperbolic functions.

Example 1.15: Let $f(x) = \dfrac{\log x + \sin x}{\sqrt{x^2 - 1}}$.

For which values of x is f continuous?

Solution:

Example 1.16: $f(x) = \begin{cases} x^3 - cx + 8, & x \leq 1 \\ x^2 + 2cx + 2, & x > 1. \end{cases}$

For which values of c is f continuous? Justify your answer.

Solution:

Theorem:

If f is continuous at b and $\lim_{x \to a} g(x) = b$ then

$$\lim_{x \to a} f[g(x)] = f\left[\lim_{x \to a} g(x)\right] = f(b).$$

Note:
This theorem also holds for limits as $x \to \infty$, as long as $b \in \mathbb{R}$ is finite.

Example 1.17: Evaluate $\lim_{x \to \infty} \sin\left(e^{-x}\right)$.

Solution:

Differentiability

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. The derivative of f at $x = a$ is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

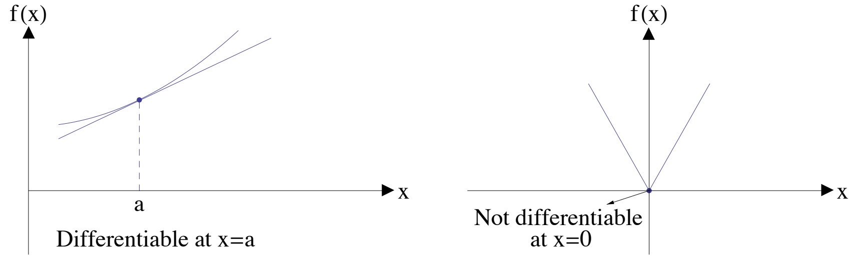
The function f is differentiable at $x = a$ if this limit exists.

Geometrically, f is differentiable at $x = a$ if the graph $y = f(x)$ has a *tangent line* at $x = a$ given by

$$y - f(a) = f'(a)(x - a)$$

which gives a good approximation to the graph near $x = a$.

Note:
We can also define left differentiable and right differentiable.



If f is differentiable at $x = a$, the linear approximation of f near $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x - a)$$

Theorem:
If f is differentiable at $x = a$, then f is continuous at $x = a$.

L'Hôpital's Rule

Let f and g be differentiable functions near $x = a$, and $g'(x) \neq 0$ at all points x near a with $x \neq a$. If

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

if the limit involving the derivatives exists.

Note:
L'Hôpital's Rule also holds when x approache

Example 1.18: Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$. $\left(\frac{0}{0}\right)$

Solution:

Example 1.19: Evaluate $\lim_{x\to\infty} \dfrac{3x^2 - 2x + 3}{x^2 + 4x + 4}$. $\left(\dfrac{\infty}{\infty}\right)$

Solution:

Example 1.20: Evaluate $\lim_{x\to\infty} \left(x^{-\frac{1}{3}} \log x\right)$. $(0 \cdot \infty)$

Solution:

Sequences

A sequence is a function $f : \mathbb{N} \to \mathbb{R}$.
It can be thought of as an ordered list of real numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n \ldots$$

Thus, $f(n) = a_n$.
The sequence is denoted by $\{a_n\}$, where a_n is the n^{th} term.

Example

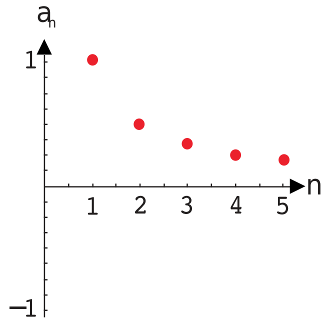
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \quad \Longrightarrow \quad a_n = \frac{1}{n}$$

Example

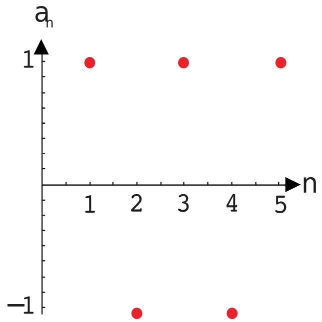
$$1, -1, 1, -1, 1, -1, \ldots \quad \Longrightarrow \quad a_n = (-1)^{n-1}$$

The graph of a sequence $\{a_n\}$ can be plotted on a set of axes with n on the x-axis and a_n on the y-axis.

Example: $a_n = \frac{1}{n}$



Example: $a_n = (-1)^{n-1}$



Limits of Sequences

A sequence $\{a_n\}$ has the limit L if a_n approaches L as n approaches infinity. Note, that L must be finite.

We write

$$\lim_{n\to\infty} a_n = L$$

or $a_n \to L$ as $n \to \infty$.

If the limit exists we say that the sequence converges. Otherwise, we say that the sequence diverges.

Example 1.21: Determine whether the following sequences converge or diverge: (a) $\left\{\frac{1}{n}\right\}$ (b) $\left\{(-1)^{n-1}\right\}$ (c) $\{n\}$

Solution:

The only difference between $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{x\to\infty} f(x) = L$ is that n is a natural number whereas x is a real number.

Theorem:

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $\{a_n\}$ be a sequence of real numbers such that $a_n = f(n)$. If

$$\lim_{x\to\infty} f(x) = L \quad \text{then} \quad \lim_{n\to\infty} a_n = L.$$

This means that we can use the techniques for evaluating limits of functions to evaluate limits of sequences.

Note:

$$\lim_{n\to\infty} a_n = L \quad \not\Rightarrow \quad \lim_{x\to\infty} f(x) = L.$$

eg. $a_n = \sin(2\pi n)$, $f(x) = \sin(2\pi x)$.

Theorem:

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and $c \in \mathbb{R}$ a constant.

If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist, then

1. $\lim\limits_{n\to\infty} [a_n + b_n] = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$.
2. $\lim\limits_{n\to\infty} [ca_n] = c \lim\limits_{n\to\infty} a_n$.
3. $\lim\limits_{n\to\infty} [a_n b_n] = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$.
4. $\lim\limits_{n\to\infty} \left[\frac{a_n}{b_n}\right] = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ provided $\lim\limits_{n\to\infty} b_n \neq 0$.

Sandwich Theorem:

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers.

If $a_n \le c_n \le b_n$ for all $n > N$ for some N, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L.$$

The Factorial Function

The factorial function $n!$ $(n = 0, 1, 2, ...)$ is defined by

$$n! = n(n-1)!\ , \quad 0! = 1$$

or

$$n! = n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1$$

Therefore

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \times 1 = 2 \\ 3! &= 3 \times 2 \times 1 = 6 \\ 4! &= 4 \times 3 \times 2 \times 1 = 24 \end{aligned}$$

Example

$$(2n+2)! = (2n+2) \times (2n+1) \times (2n) \times (2n-1) \times ... \times 3 \times 2 \times 1$$

or

$$(2n+2)! = (2n+2) \times (2n+1) \times (2n)!$$

Standard Limits

(1) $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0 \quad (p > 0)$

(2) $\lim\limits_{n\to\infty} r^n = 0 \quad (|r| < 1)$

(3) $\lim\limits_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0)$

(4) $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

(5) $\lim\limits_{n\to\infty} \dfrac{a^n}{n!} = 0 \quad (a \in \mathbb{R})$

(6) $\lim\limits_{n\to\infty} \dfrac{\log n}{n^p} = 0 \quad (p > 0)$

(7) $\lim\limits_{n\to\infty} \left(1 + \dfrac{a}{n}\right)^n = \mathrm{e}^a \quad (a \in \mathbb{R})$

(8) $\lim\limits_{n\to\infty} \dfrac{n^p}{a^n} = 0 \quad (p \in \mathbb{R}, a > 1)$

Note:

Standard limits (1), (3), (4), (6), (7), (8) also hold for limits of real-valued functions as $x \to \infty$. Standard limit (2) also holds for $x \to \infty$ when $0 \le r < 1$.

Example 1.22: Evaluate $\lim\limits_{n\to\infty} \left[\left(\dfrac{n-2}{n} \right)^n + \dfrac{4n^2}{3^n} \right]$.

Solution:

Example 1.23: Find the limit of the sequence

$$a_n = \frac{3^n + 2}{4^n + 2^n}, \quad n \geq 1.$$

Solution:

Note:
The order hierarchy can be used to help identify the largest term in an expression:

$$\log n \ll n^p \ll a^n \ll n!$$

Example 1.24: Prove Standard Limit 6:

$$\lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$$

Solution:

Note:
We must change to a continuous variable $x \in \mathbb{R}$ before applying L'Hôpital's rule.

Example 1.25: Evaluate $\lim_{n\to\infty} [\log(3n-2) - \log n]$.

Solution:

Note:
… a continuous variable $x \in \mathbb{R}$ before applying …rem.

Example 1.26: Evaluate $\lim_{n\to\infty} \dfrac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$.

Solution:

Adding Infinitely Many Numbers

Starting with any sequence $\{a_n\}$, adding the a_n's together in order gives a sequence $\{s_n\}$:

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ \vdots\ & \quad \vdots \quad\ \vdots \quad\ \vdots \end{aligned}$$

The sequence of partial sums $\{s_n\}$ may or may not converge. If it does converge, we call

$$S = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(a_1 + a_2 + \ldots + a_n)$$

the sum of the a_n's.

Example 1.27: Find the sum S of $a_n = \left(\frac{1}{2}\right)^n, n \geq 1$.

Solution:

Series

The series with terms a_n is denoted by the sum

$$\sum_{n=1}^{\infty} a_n.$$

If $\lim_{n\to\infty} s_n$ exists, we say that the series converges. Otherwise we say that the series diverges.

Example

The sequence $\{n\} = 1, 2, 3, 4, \ldots$

The series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \ldots$

The sequence and series both diverge to infinity, so the sum does not exist.

Application: Decimals

The decimal representation of a number is actually a series.

Example

The sequence $\left\{\frac{1}{10^n}\right\} = 0.1, 0.01, 0.001, \ldots.$

The series $\sum_{n=1}^{\infty} \frac{1}{10^n} = 0.1 + 0.01 + 0.001 + \ldots = 0.11111111\ldots$

The sequence converges to 0 while the series converges to $\frac{1}{9}$.

In General

For a number $x \in (0, 1)$ with decimal digits $d_1,\ d_2,\ d_3,\ d_4,\ \ldots$

$$x = 0.d_1d_2d_3d_4\ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

Properties of Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series, and $c \in \mathbb{R}$ a constant.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then

1. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ converges.
2. $\sum_{n=1}^{\infty}(ca_n) = c\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty}(ca_n)$ diverges.

Note:

These properties follow from the properties c sequences.

Geometric Series

A geometric series has the form

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

where $a \in \mathbb{R}$ and $r \in \mathbb{R}$.

The series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

If $|r| < 1$, we have

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Note:

This follows from the fact that $\sum_{k=0}^{n} ar^k = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$.

Example 1.28: What does the series

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$$

converge to?

Solution:

Harmonic p Series

A harmonic p series has the form

$$\sum_{n=1}^{\infty}\frac{1}{n^p}.$$

The series converges if $p > 1$ and diverges if $p \leq 1$.

Example

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges BUT $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Note:

If $\lim_{n\to\infty} a_n = 0$ then

1. $\displaystyle\sum_{n=1}^{\infty} a_n$ may converge or diverge.
2. The Divergence Test is not relevant, so we need to use another test to determine if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges or diverges.

Example 1.29: Does the series $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n}$ converge?

Solution:

Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series.

1. If $a_n \leq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $a_n \geq b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

To apply the comparison test we compare a given series to a harmonic p series or geometric series.

Example 1.30: Does $\sum_{n=1}^{\infty} \frac{7}{2n^2 + 4n + 3}$ converge or diverge?

Solution:

Example 1.31: Does $\sum_{n=1}^{\infty} \frac{n^2 + 4}{n^3 + 1}$ converge or diverge?

Solution:

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series and

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

1. If $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the ratio test is inconclusive.

The ratio test is useful if a_n contains an exponential or factorial function of n.

Example 1.32: Does $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converge or diverge?

Solution:

Example 1.33: Does $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\, n!}$ converge or diverge?

Solution:

Section 2: Hyperbolic Functions

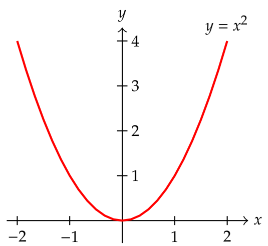
Even Functions

A function f is an even function if

$$f(x) = f(-x)$$

Example

$f(x) = \cos x$ and $f(x) = x^2$



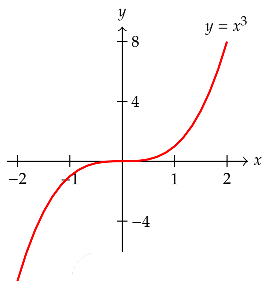
Odd Functions

A function f is an odd function if

$$f(x) = -f(-x)$$

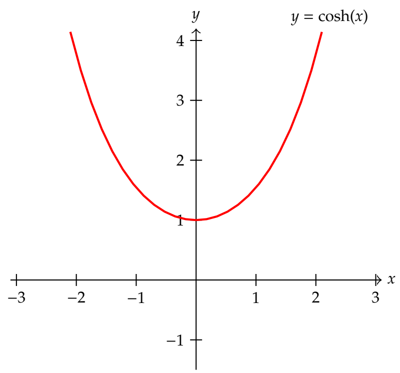
Example

$f(x) = \sin x$ and $f(x) = x^3$



We define the hyperbolic cosine function:

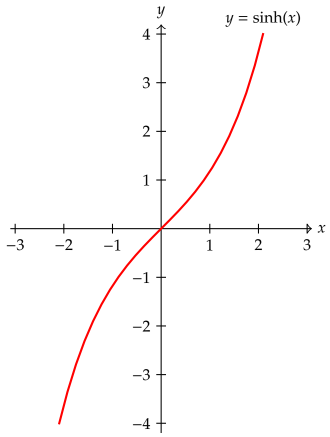
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right), \quad x \in \mathbb{R}$$



Properties

We define the hyperbolic sine function:

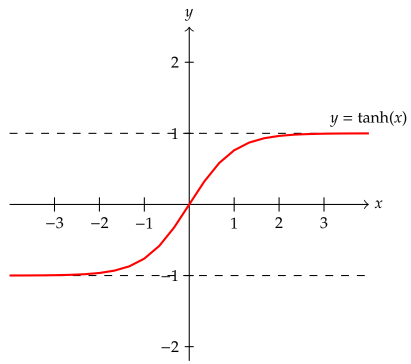
$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right), \quad x \in \mathbb{R}$$



Properties

We define the hyperbolic tangent function:

$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x} \\ &= \frac{\frac{1}{2}\left(e^x - e^{-x}\right)}{\frac{1}{2}\left(e^x + e^{-x}\right)} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbb{R}.\end{aligned}$$

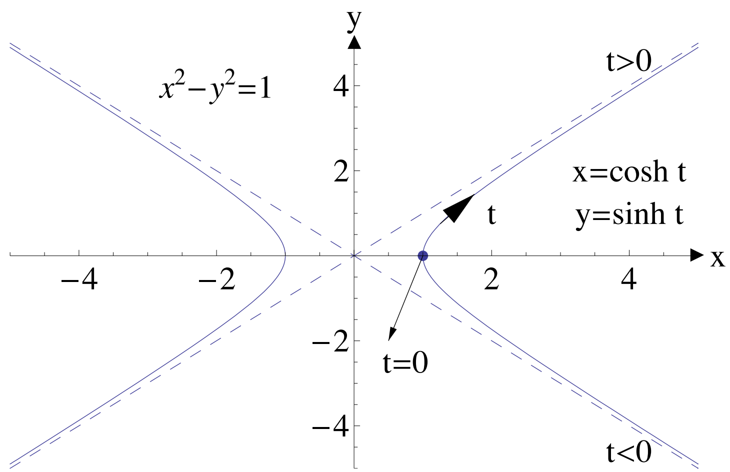


Properties

Why call them hyperbolic functions?

Let $x = \cosh t$ and $y = \sinh t$ then

So $(x, y) = (\cosh t, \sinh t)$ denotes a point on the hyperbola $x^2 - y^2 = 1$. Since $x \geq 1$, the right hand branch of the hyperbola can be parametrised by $x = \cosh t, y = \sinh t, t \in \mathbb{R}$.

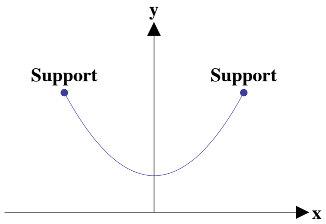


Application: Catenary

A flexible, heavy cable of uniform mass per length ρ and tension T at its lowest point has shape

$$y = \frac{T}{\rho g} \cosh\left(\frac{\rho g x}{T}\right)$$

where g is the acceleration due to gravity.



Example 2.1: Simplify $\sinh(2 \log x)$.

Solution:

Example 2.2: If $\cosh x = \frac{13}{12}$ and $x < 0$ find $\sinh x$ and $\tanh x$.

Solution:

Example 2.3: Write $\cosh^3 x$ in terms of the functions $\cosh(nx)$ for integers n.

Solution:

Addition Formulae

$$\begin{aligned} \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \\ \sinh(x-y) &= \sinh x \cosh y - \cosh x \sinh y \\ \cosh(x-y) &= \cosh x \cosh y - \sinh x \sinh y \end{aligned}$$

Example 2.4: Prove the $\sinh(x+y)$ addition formula.

Solution:

Double Angle Formulae

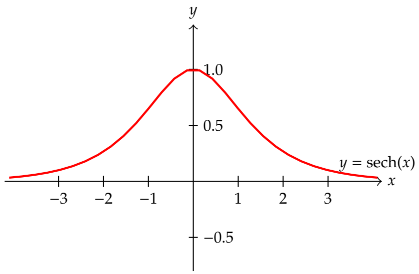
$$\begin{aligned} \sinh(2x) &= 2 \sinh x \cosh x \\ \cosh(2x) &= \cosh^2 x + \sinh^2 x \\ \cosh(2x) &= 2\cosh^2 x - 1 \\ \cosh(2x) &= 2\sinh^2 x + 1 \end{aligned}$$

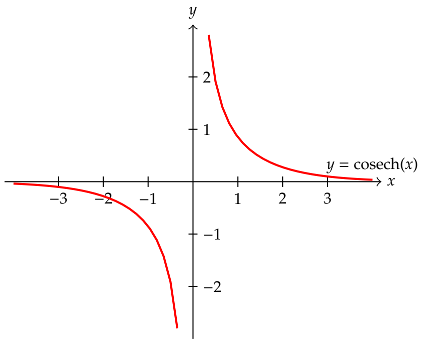
These can be proved using the addition formulae.

Reciprocal Hyperbolic Functions

We define the three reciprocal hyperbolic functions:

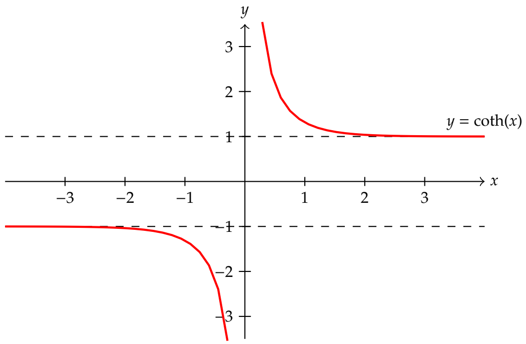
$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad x \in \mathbb{R} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}, x \in \mathbb{R} \setminus \{0\}$$





Reciprocal Hyperbolic Functions

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}, \quad x \in \mathbb{R} \setminus \{0\}$$



Basic Identities

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \coth^2 x - 1 &= \operatorname{cosech}^2 x \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \end{aligned}$$

Derivatives of Hyperbolic Functions

$$\begin{aligned} \frac{d}{dx}(\cosh x) &= \sinh x, \quad x \in \mathbb{R} \\ \frac{d}{dx}(\sinh x) &= \cosh x, \quad x \in \mathbb{R} \\ \frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x, \quad x \in \mathbb{R} \\ \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x, \quad x \in \mathbb{R} \\ \frac{d}{dx}(\operatorname{cosech} x) &= -\operatorname{cosech} x \coth x, \quad x \in \mathbb{R} \setminus \{0\} \\ \frac{d}{dx}(\coth x) &= -\operatorname{cosech}^2 x, \quad x \in \mathbb{R} \setminus \{0\} \end{aligned}$$

Example 2.5: Prove that $\dfrac{d(\cosh x)}{dx} = \sinh x$.

Solution:

Example 2.6: Let $y = \sqrt{\sinh(6x)}$, $x > 0$. Find $\dfrac{dy}{dx}$.

Solution:

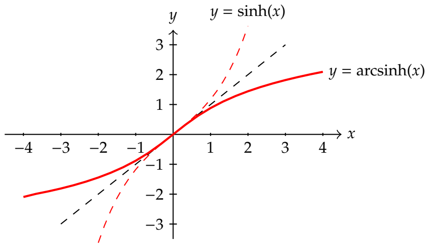
Inverses of Hyperbolic Functions

We define three inverse hyperbolic functions.

1. Inverse hyperbolic sine function: $\operatorname{arcsinh} x$

Since $\sinh x$ is a 1-1 function

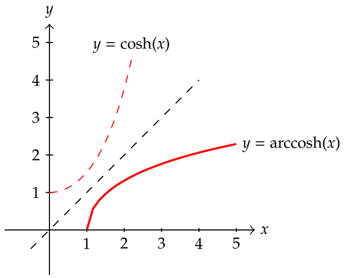
$$\begin{aligned}
\text{domain } \operatorname{arcsinh} x &= \text{range } \sinh x = \mathbb{R}. \\
\text{range } \operatorname{arcsinh} x &= \text{domain } \sinh x = \mathbb{R}. \\
\operatorname{arcsinh}(\sinh x) &= x, \quad x \in \mathbb{R}. \\
\sinh(\operatorname{arcsinh} x) &= x, \quad x \in \mathbb{R}.
\end{aligned}$$



2. Inverse hyperbolic cosine function: $\operatorname{arccosh} x$

Restrict domain of $\cosh x$ to be $[0, \infty)$ to give a 1-1 function. Then

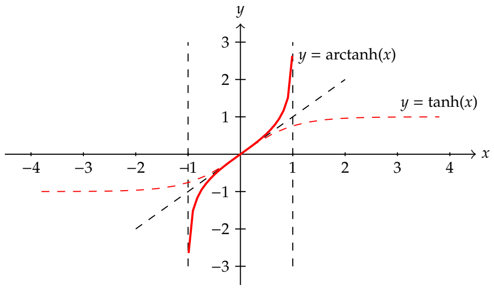
$$\begin{aligned}
\text{domain } \operatorname{arccosh} x &= \text{range } \cosh x = [1, \infty). \\
\text{range } \operatorname{arccosh} x &= \text{restricted domain } \cosh x = [0, \infty). \\
\cosh(\operatorname{arccosh} x) &= x, \quad x \geq 1. \\
\operatorname{arccosh}(\cosh x) &= x, \quad x \geq 0.
\end{aligned}$$



3. Inverse hyperbolic tangent function: $\operatorname{arctanh} x$

Since $\tanh x$ is a 1-1 function

$$\begin{aligned}
\text{domain } \operatorname{arctanh} x &= \text{range } \tanh x = (-1, 1).\\
\text{range } \operatorname{arctanh} x &= \text{domain } \tanh x = \mathbb{R}.\\
\tanh(\operatorname{arctanh} x) &= x, \quad -1 < x < 1.\\
\operatorname{arctanh}(\tanh x) &= x, \quad x \in \mathbb{R}.
\end{aligned}$$



The inverse hyperbolic functions can be expressed in terms of natural logarithms.

$$\boxed{\begin{aligned}
\operatorname{arcsinh} x &= \log\left(x + \sqrt{x^2+1}\right), & x \in \mathbb{R}\\
\operatorname{arccosh} x &= \log\left(x + \sqrt{x^2-1}\right), & x \geq 1\\
\operatorname{arctanh} x &= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), & -1 < x < 1
\end{aligned}}$$

We can also define inverse reciprocal hyperbolic functions:

- $\operatorname{arcsech} x \quad (0 < x \leq 1)$
- $\operatorname{arccosech} x \quad (x \neq 0)$
- $\operatorname{arccoth} x \quad (x < -1$ or $x > 1)$

Example 2.7: Proof of $\operatorname{arcsinh} x$ relation.

Solution:

Example 2.8: Find the exact value of $\sinh[\operatorname{arccosh}(3)]$.

Solution:

Example 2.9: Simplify $\cosh(\operatorname{arctanh} x)$ for $-1 < x < 1$.

Solution:

Derivatives

$$\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}} \quad (x \in \mathbb{R})$$

$$\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$$

$$\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2} \quad (-1 < x < 1)$$

Each formula is derived using implicit differentiation or by differentiating the logarithm definition of each function.

Example 2.10: Prove that $\dfrac{d}{dx}(\operatorname{arcsinh} x) = \dfrac{1}{\sqrt{x^2+1}}$.

Solution:

Example 2.11: Find $\dfrac{d}{dx}(\operatorname{arctanh}(2x)\cosh(3x))$.

Solution:

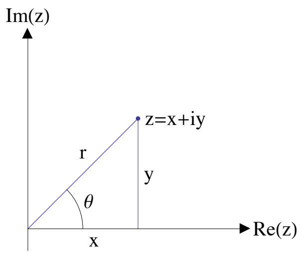
Section 3: Complex Numbers

The Cartesian form of a complex number $z \in \mathbb{C}$ is

$$\boxed{z = x + iy} \quad \text{where} \quad x, y \in \mathbb{R}$$

and

- $x = \mathrm{Re}(z)$ is the real part of z,
- $y = \mathrm{Im}(z)$ is the imaginary part of z,
- $i^2 = -1$.



The complex number can be written as

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where

- $r = |z| = \sqrt{x^2 + y^2}$
- $\tan\theta = \dfrac{y}{x}$

Note:

The angle θ is not unique – only defined up to multiples of 2π.
We choose θ such that $-\pi < \theta \le \pi$ and call this angle the principal argument of z.

The Complex Exponential

We define the complex exponential using Euler's formula

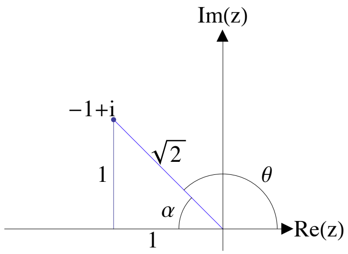
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

for $\theta \in \mathbb{R}$.

We can then write the polar form of a complex number as

$$\boxed{z = re^{i\theta}}$$

Example 3.1: Write $z = -1 + i$ in polar form.



Solution:

Properties of the Complex Exponential

1. $\boxed{e^{i0} = 1}$

Proof:
$e^{i0} = \cos 0 + i\sin 0 = 1.$

2. $\boxed{e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}}$

Proof:

$$\begin{aligned}
e^{i\theta}e^{i\phi} &= (\cos\theta + i\sin\theta)\big(\cos\phi + i\sin\phi\big) \\
&= \cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi - \sin\theta\sin\phi \\
&= \big(\cos\theta\cos\phi - \sin\theta\sin\phi\big) + i\big(\cos\theta\sin\phi + \sin\theta\cos\phi\big) \\
&= \cos\big(\theta+\phi\big) + i\sin(\theta+\phi) \\
&= e^{i(\theta+\phi)}.
\end{aligned}$$

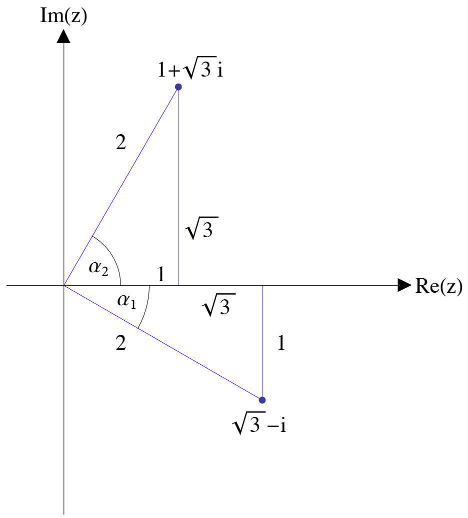
Products and Division in Polar Form

If $z = r_1e^{i\theta}$ and $w = r_2e^{i\phi}$ then

$$\boxed{\begin{aligned}
zw &= r_1r_2e^{i(\theta+\phi)} \\
\frac{z}{w} &= \frac{r_1}{r_2}e^{i(\theta-\phi)}
\end{aligned}}$$

Example 3.2: Using the complex exponential, simplify

$(\sqrt{3}-i)(1+\sqrt{3}i)$ and $\dfrac{\sqrt{3}-i}{1+\sqrt{3}i}$.



Solution:

De Moivre's Theorem:

If $z = re^{i\theta}$ and n is a positive integer then

$$\boxed{z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}}$$

Example 3.3: Evaluate $\left(1 + \sqrt{3}i\right)^{15}$.

Solution:

Exponential Form of $\sin\theta$ and $\cos\theta$

$$\begin{aligned} \text{Now } e^{i\theta} &= \cos\theta + i\sin\theta && (1) \\ \Rightarrow e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ \Rightarrow e^{-i\theta} &= \cos\theta - i\sin\theta && (2) \end{aligned}$$

Equation $(1) + (2)$ gives

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

Equation $(1) - (2)$ gives

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\Rightarrow \boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

Note:

These formulae give a connection between the hyperbolic and trigonometric functions.

$$\begin{aligned} \cosh(i\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta \\ \sinh(i\theta) &= \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = i\sin\theta \end{aligned}$$

Example 3.4: Express $\sin^5\theta$ in terms of the functions $\sin(n\theta)$ for integers n.

Solution:

Differentiation via the Complex Exponential

If $z = x + yi$ where $x, y \in \mathbb{R}$ then we define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Derivatives of functions from $\mathbb{R}$ to $\mathbb{C}$ are defined similarly as those from $\mathbb{R}$ to $\mathbb{R}$.

Differentiation to functions from $\mathbb{R}$ to $\mathbb{C}$ is also linear and follows the product law.

Show that $\dfrac{d}{dt}\left(e^{kt}\right) = ke^{kt}$ when $k = a + bi \in \mathbb{C}$.

$$\frac{d}{dt}\left[e^{(a+bi)t}\right] = \frac{d}{dt}\left[e^{at}e^{ibt}\right]$$

$$\begin{aligned}
&= \frac{d}{dt}\left[e^{at}\left(\cos(bt) + i\sin(bt)\right)\right] \\
&= ae^{at}\left[\cos(bt) + i\sin(bt)\right] + e^{at}\left[-b\sin(bt) + bi\cos(bt)\right] \\
&= ae^{at}\left[\cos(bt) + i\sin(bt)\right] + e^{at}\left[bi^2\sin(bt) + bi\cos(bt)\right] \\
&= ae^{at}\left[\cos(bt) + i\sin(bt)\right] + bie^{at}\left[\cos(bt) + i\sin(bt)\right] \\
&= (a + bi)e^{at}\left[\cos(bt) + i\sin(bt)\right] \\
&= (a + bi)e^{at}e^{ibt} \\
&= (a + bi)e^{(a+ib)t}.
\end{aligned}$$

Example 3.5: Find $\dfrac{d^{56}}{dt^{56}}\left(e^{-t}\cos t\right)$.

Solution:

Note:

Example 3.5 also gives the answer to $\dfrac{d^{56}}{dt^{56}}\left(e^{-t}\sin t\right)$.

Integration via the Complex Exponential

Since $\dfrac{d}{dx}\left(e^{kx}\right) = k\,e^{kx}$ if $k = a + bi \ (a, b \in \mathbb{R})$, then

$$\int k\,e^{kx}\,dx = e^{kx} + C$$

$$\Rightarrow \int e^{kx}\,dx = \frac{1}{k}e^{kx} + D$$

Example 3.6: Evaluate $\displaystyle\int e^{3x}\sin(2x)\,dx$.

Solution:

Note:

Example 3.6 also gives the answer to $\int e^{3x}\cos(2x)\,dx$.

Section 4: Integral Calculus

Derivative Substitutions

To evaluate

$$\int f[g(x)]g'(x)dx$$

put $u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$.

Then

$$\begin{aligned}\int f[g(x)]g'(x)dx &= \int f(u)\frac{du}{dx}\,dx \\ &= \int f(u)\,du\end{aligned}$$

Example 4.1: Evaluate $\int (6x^2 + 10)\sinh(x^3 + 5x - 2)dx$.

Solution:

Example 4.2: Evaluate $\int \frac{\operatorname{cosech}^2(3x)}{10 - 2\coth(3x)} dx$.

Solution:

Trigonometric and Hyperbolic Substitutions

We can use trigonometric and hyperbolic substitutions to integrate expressions containing

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2},$$

where a is a positive real number.

Method:

Put $x = g(\theta)$. Then

$$\int f(x)\, dx = \int f[g(\theta)]g'(\theta)\, d\theta$$

Integrand	Substitution
$\sqrt{a^2 - x^2}$, $\frac{1}{\sqrt{a^2 - x^2}}$, $\left(a^2 - x^2\right)^{\frac{3}{2}}$ etc.	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$, $\frac{1}{\sqrt{a^2 + x^2}}$, $\left(a^2 + x^2\right)^{-\frac{3}{2}}$ etc.	$x = a\sinh\theta$
$\sqrt{x^2 - a^2}$, $\frac{1}{\sqrt{x^2 - a^2}}$, $\left(x^2 - a^2\right)^{\frac{5}{2}}$ etc.	$x = a\cosh\theta$
$\frac{1}{a^2 + x^2}$	$x = a\tan\theta$

Example 4.3: Evaluate $\int \frac{1}{\sqrt{x^2 + 25}} dx$ using a substitution.

Solution:

Example 4.4: Evaluate $\int \frac{1}{x^2+2}\,dx$ using a substitution.

Solution:

Example 4.5: Evaluate $\int \sqrt{9-4x^2}\,dx$ if $|x| \le \frac{3}{2}$.

Solution:

Example 4.6: Evaluate $\int \left(x^2 - 1\right)^{\frac{3}{2}}\, dx$ if $x \geq 1$.

Solution:

Powers of Hyperbolic Functions

Consider the integral:

$$\int \sinh^m x \cosh^n x \, dx$$

where m, n are integers (≥ 0).

- If m or n is odd, create a "derivative" substitution by rewriting one of the odd power terms using identities.

- If m and n are even, use double angle formulae.

Example 4.7: Evaluate $\int \sinh^4 \theta \, d\theta$.

Solution:

Finish Example 4.6:

Example 4.8: Evaluate $\int \sinh^5 x \cosh^6 x \, dx$.

Solution:

Example 4.9: Evaluate $I = \int \sinh^5 x \cosh^7 x \, dx$.

Solution:

Partial Fractions

Let $f(x)$ and $g(x)$ be polynomials, then

$$\frac{f(x)}{g(x)} \quad \begin{array}{l} \longrightarrow \text{ degree } n \\ \longrightarrow \text{ degree } d \end{array}$$

can be written as the sum of partial fractions.

Case 1: $n < d$

1. Factorise g over the real numbers.
2. Write down partial fraction expansion.
3. Find unknown coefficients

$$A, A_1, A_2, \ldots, A_r, B, B_1, B_2, \ldots$$

Denominator Factor	Partial Fraction Expansion
$(x-a)$	$\frac{A}{x-a}$
$(x-a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$
(x^2+bx+c)	$\frac{Ax+B}{x^2+bx+c}$
$(x^2+bx+c)^r$	$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(x^2+bx+c)^r}$

Example 4.10: Evaluate $\int \frac{4}{x^2(x+2)}\,dx \quad (x \neq 0, -2)$.

Solution:

Example 4.11: Evaluate $\int \frac{4x}{(x^2+4)(x-2)}\,dx \quad (x \neq 2)$.

Solution:

Example 4.12: Evaluate $\int \frac{2x^4 + 3x^2}{(x^2+1)^2(x^2+2)}\,dx.$

Solution:

Note:
In general, for a positive integer n if we put $x = \tan\theta$ then
$$\int \frac{1}{(x^2+1)^n}\,dx = \int \cos^{2n-2}\theta\,d\theta.$$

Case 2: $n \geq d$

Use long division, then apply case 1.

Example 4.13: Find
$$\int \frac{5x^4+13x^3+6x^2+4}{x^3+2x^2}\,dx \quad (x \neq 0, -2).$$

Solution:

Integration by Parts

The product rule for differentiation is

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Integrate

$$\int \frac{d}{dx}(uv)\,dx = \int \left(\frac{du}{dx}v + u\frac{dv}{dx}\right) dx$$

$$\Rightarrow uv = \int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx$$

$$\Rightarrow \quad \boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

Example 4.14: Evaluate $\int x^2 \log x\,dx \qquad (x > 0)$.

Solution:

Example 4.15: Evaluate $\int xe^{5x}\,dx$.

Solution:

Example 4.16: Evaluate $\int \log x\,dx \qquad (x > 0)$.

Solution:

Note:
This technique can also be used to integrate inverse trigonometric functions and inverse hyperbolic functions.

Example 4.17: Evaluate $\int e^{3x} \sin(2x)\, dx$.

Solution:

Section 5: First Order Differential Equations

Ordinary Differential Equations

(1) An equation of the form

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

is an ordinary differential equation (o.d.e) of order n.

Example 5.1: What order is $3\frac{d^4y}{dx^4} = \left(\frac{dy}{dx}\right)^2 + 2x^2y$?

Solution:

(2) A solution of an o.d.e is a function y that satisfies the o.d.e for all x in some interval.

Example 5.2: Verify that $y(x) = x^2 + \dfrac{2}{x}$ is a solution of

$\dfrac{dy}{dx} + \dfrac{y}{x} = 3x$ for all $x \in \mathbb{R} \setminus \{0\}$.

Solution:

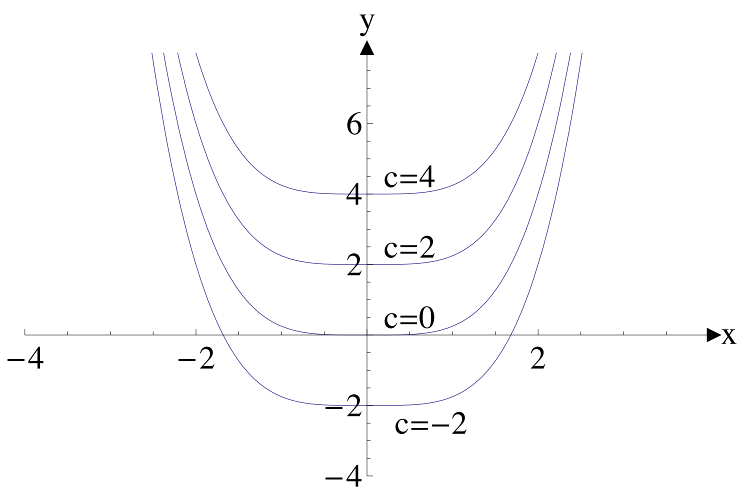
First Order O.D.E's

The general form of a first order o.d.e is $\dfrac{dy}{dx} = f(x, y)$.

Example 5.3: Solve $\dfrac{dy}{dx} = x^3$.

Solution:

This is the general solution where $c \in \mathbb{R}$ is an arbitrary constant.
Each value of c corresponds to a different solution of the o.d.e.



Initial value problem for a first order o.d.e

Solve $\dfrac{dy}{dx} = f(x, y)$ subject to the condition $y(x_0) = y_0$.

Example 5.4: Solve $\dfrac{dy}{dx} = x^3$ given $y(0) = 2$.

Solution:

Separable O.D.E'S

A separable first order o.d.e has the form:

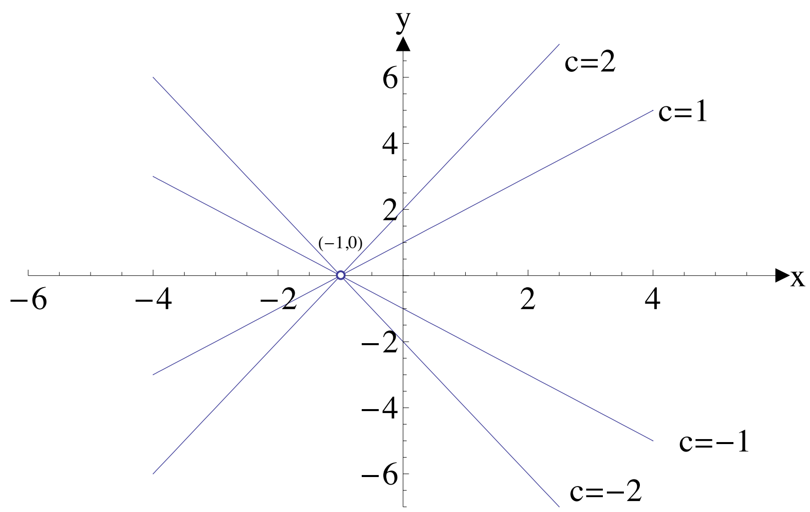
$$\frac{dy}{dx} = \mathcal{M}(x)\mathcal{N}(y), \quad (\mathcal{M}(x) \neq 0, \ \ \mathcal{N}(y) \neq 0)$$

To solve use *separation of variables*

$$\begin{aligned} \frac{dy}{dx} &= \mathcal{M}(x)\mathcal{N}(y) \\ \Rightarrow \quad \frac{1}{\mathcal{N}(y)}\frac{dy}{dx} &= \mathcal{M}(x) \\ \Rightarrow \quad \int \frac{1}{\mathcal{N}(y)}\frac{dy}{dx}\,dx &= \int \mathcal{M}(x)dx \\ \Rightarrow \quad \int \frac{1}{\mathcal{N}(y)}dy &= \int \mathcal{M}(x)\,dx \end{aligned}$$

Example 5.5: Solve $\frac{dy}{dx} = \frac{y}{1+x} \qquad (x \neq -1).$

Solution:

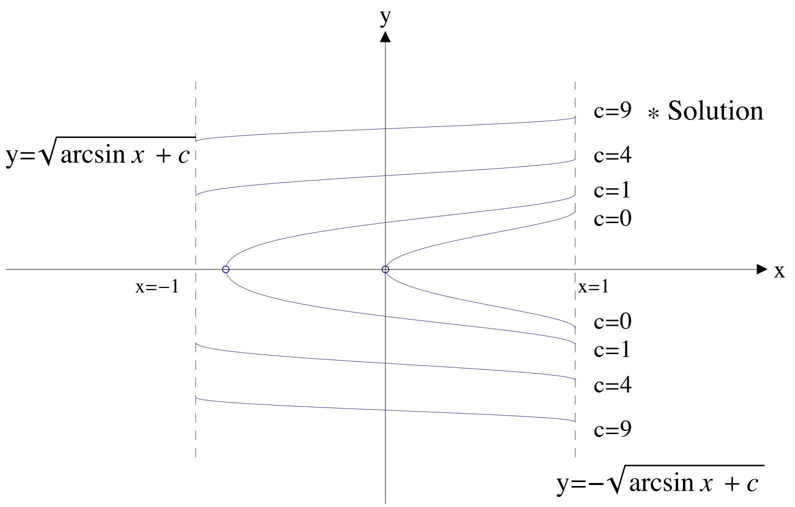


Example 5.6: Solve

$$\frac{dy}{dx} = \frac{1}{2y\sqrt{1-x^2}} \qquad (-1 < x < 1,\ y \neq 0)$$

if $y(0) = 3$.

Solution:



Linear First Order O.D.E's

Example 5.7: Solve $x\dfrac{dy}{dx} + y = e^x$.

Solution:

A linear first order o.d.e has the form:

$$\frac{dy}{dx} + \mathcal{P}(x)y = \mathcal{Q}(x)$$

To solve:

Multiply o.d.e by $\mathcal{I}(x)$

$$\mathcal{I}(x)\frac{dy}{dx} + \mathcal{P}(x)\mathcal{I}(x)y = \mathcal{Q}(x)\mathcal{I}(x)$$

If the left side can be written as the derivative of $y(x)\mathcal{I}(x)$, then

$$\frac{d}{dx}\left[y(x)\mathcal{I}(x)\right] = \mathcal{Q}(x)\mathcal{I}(x)$$

which can be solved by integrating with respect to x.

Aim:

Find an integrating factor $\mathcal{I}$ so the left side will be the derivative of $y\mathcal{I}$. Then

$$\frac{d}{dx}(y\mathcal{I}) \equiv \mathcal{I}\frac{dy}{dx} + \mathcal{P}\mathcal{I}y$$

$$\Rightarrow \frac{dy}{dx}\mathcal{I} + y\frac{d\mathcal{I}}{dx} = \mathcal{I}\frac{dy}{dx} + \mathcal{P}\mathcal{I}y$$

$$\Rightarrow y\frac{d\mathcal{I}}{dx} = \mathcal{P}\mathcal{I}y$$

To solve for all y

$$\Rightarrow \boxed{\frac{d\mathcal{I}}{dx} = \mathcal{P}\mathcal{I}} \qquad \text{(separable)}$$

$$\begin{aligned}
\Rightarrow \frac{1}{\mathcal{I}}\frac{d\mathcal{I}}{dx} &= \mathcal{P} \\
\Rightarrow \int \frac{1}{\mathcal{I}} d\mathcal{I} &= \int \mathcal{P} dx \\
\Rightarrow \log|\mathcal{I}| &= \int \mathcal{P}\, dx + c \\
\Rightarrow |\mathcal{I}| &= e^{\int \mathcal{P}dx + c} \\
&= e^{\int \mathcal{P}dx} \cdot e^{c} \\
\Rightarrow \mathcal{I} &= \underbrace{\pm e^{c}}_{\text{constant}} \cdot e^{\int \mathcal{P}dx}
\end{aligned}$$

So one integrating factor is

$$\boxed{\mathcal{I}(x) = e^{\int \mathcal{P}dx}}$$

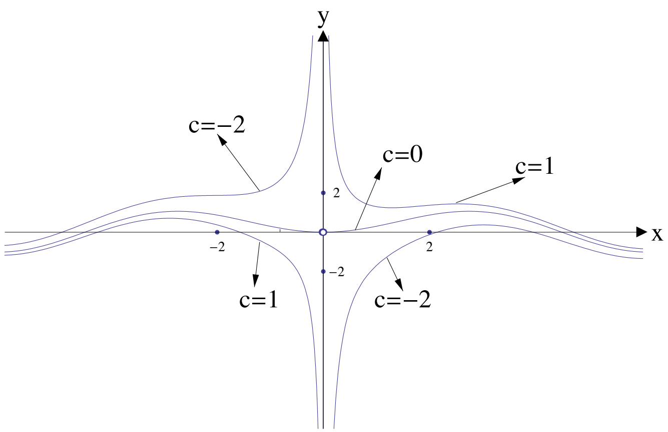
Note:

Since we only need one integrating factor $\mathcal{I}$, we can neglect the '$+c$' and modulus signs when calculating $\mathcal{I}$.

Example 5.8: Find the general solution of

$$\frac{dy}{dx} + \frac{y}{x} = \sin x \qquad (x \neq 0).$$

Solution:

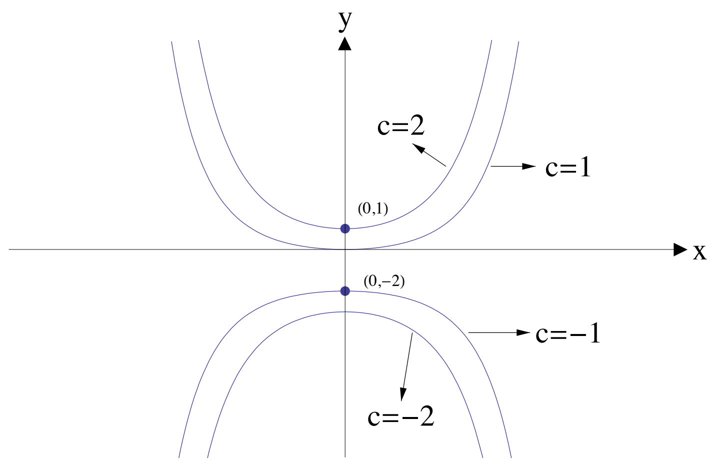


$$y(x) = -\cos x + \frac{1}{x}\sin x + \frac{c}{x}$$

Example 5.9: Solve $\frac{1}{2}\frac{dy}{dx} - xy = x$ if $y(0) = -3$.

Solution:

Note:



$y = -1 + ce^{x^2}$

Other First Order O.D.E's

Sometimes it is possible to make a substitution to reduce a general first order o.d.e to a separable or linear o.d.e.

- A homogeneous type o.d.e has the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Substituting $u = \frac{y}{x}$ reduces the o.d.e to a separable o.d.e.

- Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Substituting $u = y^{1-n}$ reduces the o.d.e to a linear o.d.e.

Example 5.10: Solve the homogeneous type differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right) \qquad \left(-\frac{\pi}{2} < \frac{y}{x} < \frac{\pi}{2}\right)$$

by substituting $u = \dfrac{y}{x}$.

Solution:

Example 5.11: Solve the Bernoulli equation

$$\frac{dy}{dx} + y = e^{3x}y^4 \qquad (y \neq 0)$$

by substituting $u = y^{-3}$.

Solution:

Population Models

Malthus (Doomsday) model

Rate of growth is proportional to the population p at time t.

$$\begin{aligned} \frac{dp}{dt} &\propto p \\ \Rightarrow \frac{dp}{dt} &= kp \qquad \text{(separable/linear)} \end{aligned}$$

where k is a constant of proportionality representing net births per unit population per unit time.

If the initial population is $p(0) = p_0$, then the solution is

$$p(t) = p_0 e^{kt}$$

Note:

The Doomsday model is unrealistic since if

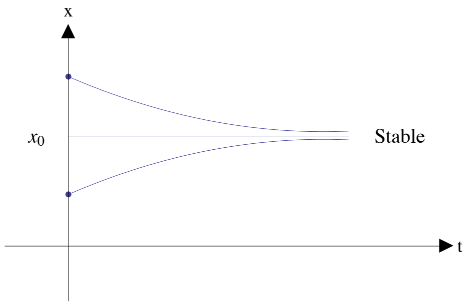
- $k > 0$ – unbounded exponential growth
- $k < 0$ – population dies out
- $k = 0$ – population stays constant

Equilibrium Solutions

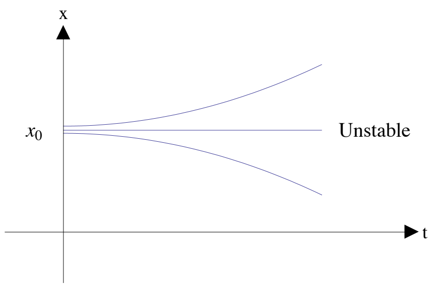
1. An equilibrium solution is a solution that does not change with time.

$$\text{i.e. } \frac{dx}{dt} = 0 \quad \Rightarrow \quad x(t) = x_0$$

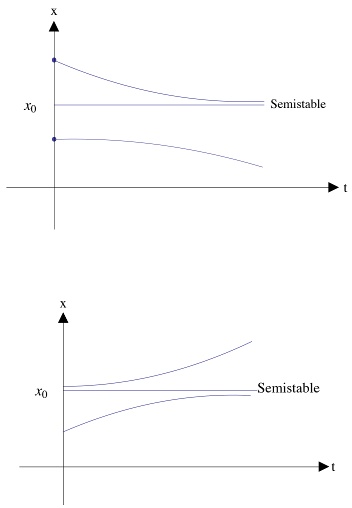
2. Stable equilibrium – solutions that start nearby move closer as t increases.



3. Unstable equilibrium – solutions that start nearby move further away as t increases.



4. Semistable equilibrium – on one side of x_0 solutions that start nearby move closer as t increases whereas on the other side of x_0 solutions move further away as t increases.



5. Phase plots:

If $\frac{dx}{dt} = f(x)$, a plot of $\frac{dx}{dt}$ as a function of x will give the equilibrium solutions and the behaviour of solutions close by.

Doomsday model with harvesting.

Remove some of the population at a constant rate.

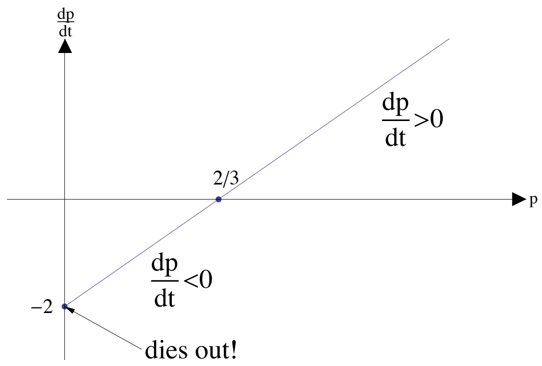
$$\frac{dp}{dt} = kp - h, \quad h > 0.$$

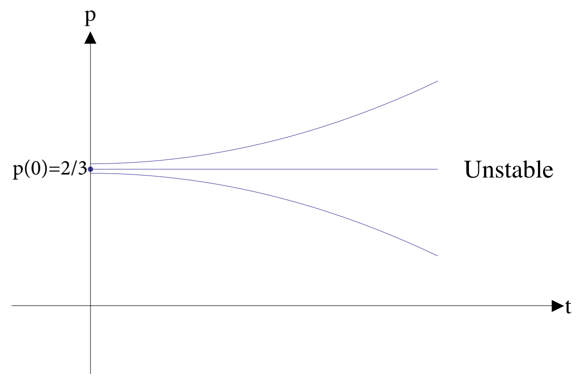
Example 5.12: $\frac{dp}{dt} = 3p - 2 \qquad (k = 3, h = 2)$

Solution:

- Equilibrium solutions

- Phase plot





Note:

Solving $\frac{dp}{dt} = 3p - 2$ with $p(0) = p_0$ gives $p(t) = \frac{2}{3} + \left(p_0 - \frac{2}{3}\right)e^{3t}$ which agrees with predicted behaviour.

Logistic model.

Include “competition” term in Malthus’ model since overcrowding, disease, lack of food and natural resources will cause more deaths.

$$\frac{dp}{dt} = \underbrace{kp}_{\text{net birth rate}} - \underbrace{\frac{k}{a}p^2}_{\text{competition term}} = kp\left(1 - \frac{p}{a}\right)$$

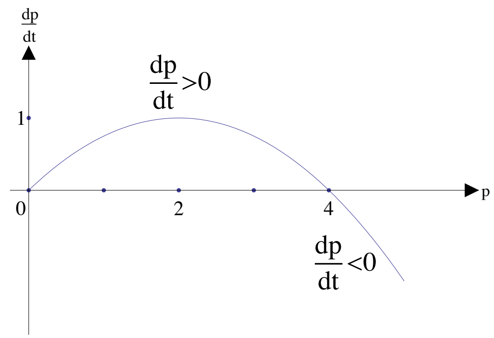
where $a > 0$ is the carrying capacity.

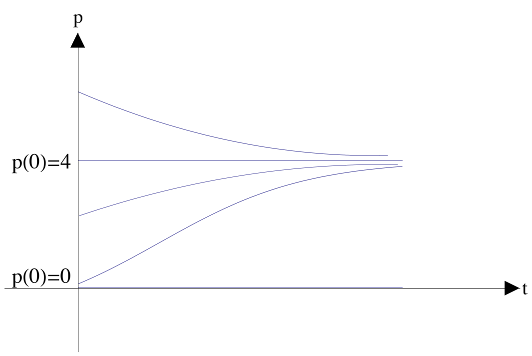
Example 5.13: $\frac{dp}{dt} = p\left(1 - \frac{p}{4}\right)$ $(k = 1, a = 4)$

Solution:

- Equilibrium solutions

- Phase plot

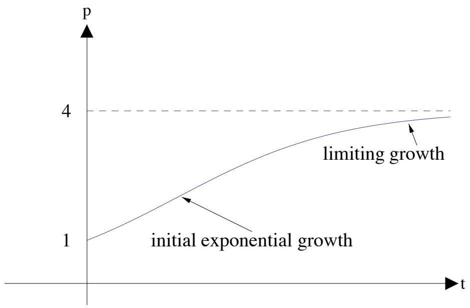




- Exact solution

$$\frac{dp}{dt} = \frac{p}{4}(4 - p) \qquad \text{(separable)}$$

Suppose $p(0) = 1$



Note:

Logistic model accurately predicts

- population in a limited space (e.g. bacteria culture).
- population of USA from 1790-1950.

Logistic model with harvesting.

Remove some of the population at constant rate:

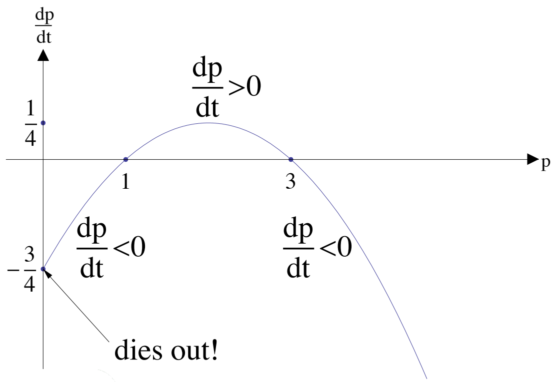
$$\frac{dp}{dt} = kp\left(1 - \frac{p}{a}\right) - h, \quad h > 0, \ a > 0$$

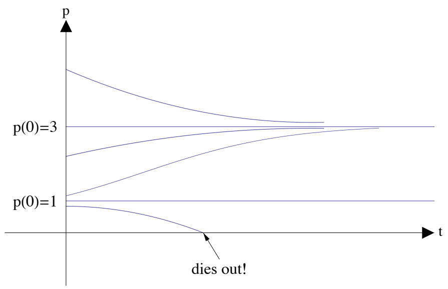
Example 5.14:

$$\frac{dp}{dt} = p\left(1 - \frac{p}{4}\right) - \frac{3}{4} \qquad \left(a = 4, \ k = 1, \ h = \frac{3}{4}\right)$$

Solution:

- Phase plot





Find the time taken until the population dies out if $p(0) = \frac{1}{2}$.

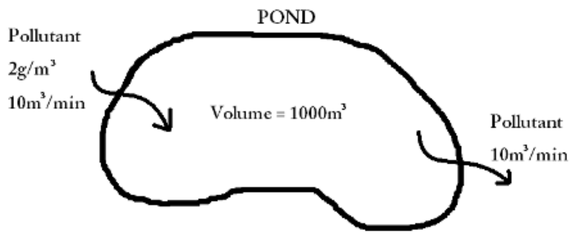
$$\frac{dp}{dt} = -\frac{1}{4}(p-3)(p-1) \quad \text{(separable)}$$

Mixing Problems

Example 5.15: Effluent (pollutant concentration $2g/m^3$) flows into a pond (volume $1000m^3$, initially 100g pollutant) at a rate of $10m^3/min$. The pollutant mixes quickly and uniformly with pond water and flows out of pond at a rate of $10m^3/min$.

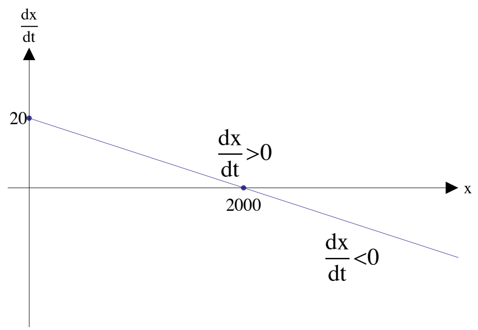
Find the concentration of pollutant in the pond at any time.

Solution:

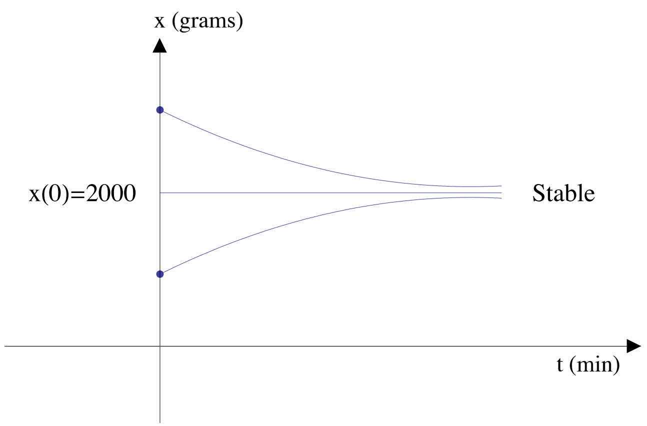


Let x be the amount (grams) of pollutant in pond at time t minutes. Then $C = \frac{x}{V}$ is the concentration of pollutant in pond $(grams/m^3)$, where V is the volume of the pond (m^3) at time t.

$$\boxed{\frac{dx}{dt} = \text{rate pollutant flows in - rate pollutant flows out}}$$

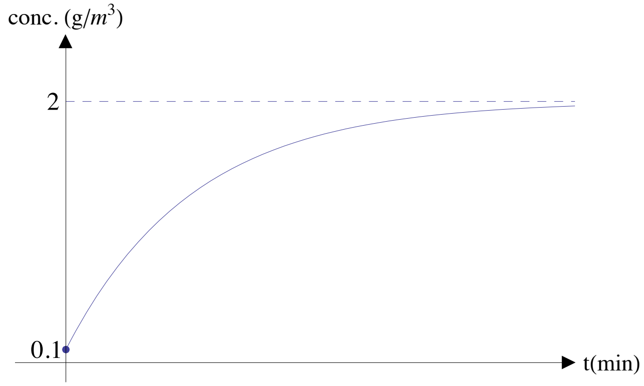


Expect solution (for all initial conditions) to look like



- Exact solution

$$\frac{dx}{dt} + \frac{x}{100} = 20 \qquad \text{(Linear/Separable)}$$



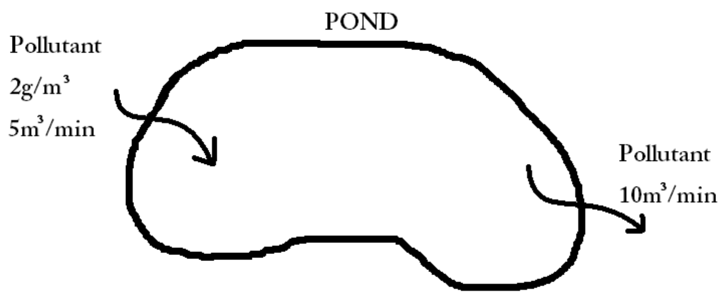
Definitions

1. Transient terms: terms decaying to 0 as $t \to \infty$.
2. Steady state terms: terms NOT decaying to 0 as $t \to \infty$.

The solution for the concentration can be classified as follows.

Example 5.16: Find the concentration of pollutant in pond if input flow rate is decreased to $5m^3/min$.

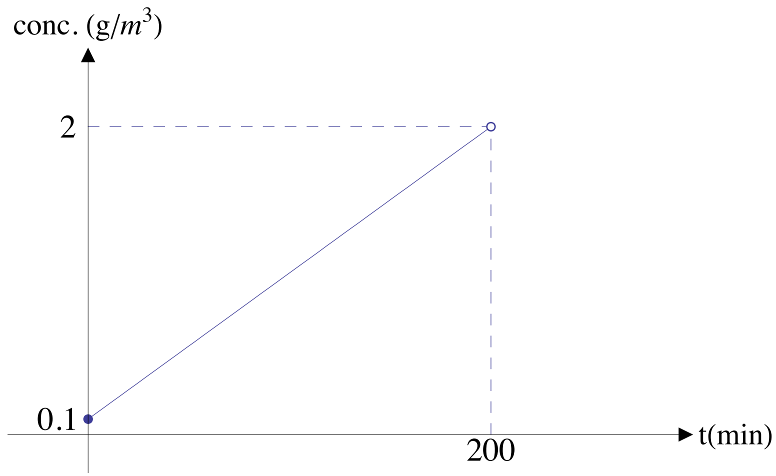
Solution:



Let V be volume in pond (m^3) at time t minutes.

Note:

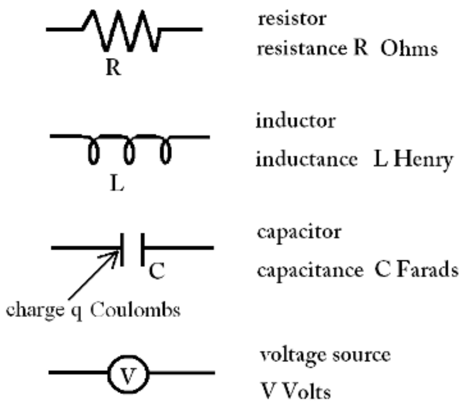
There are no equilibrium solutions for x.



Electric Circuits

An electric circuit is a path (eg. wire) for electrons (charge) to move along.

Circuit elements



Aim:

To find the current, i Amp, in the circuit and the charge, q Coulomb, on the capacitor plates.

$$\boxed{i = \frac{dq}{dt}}$$

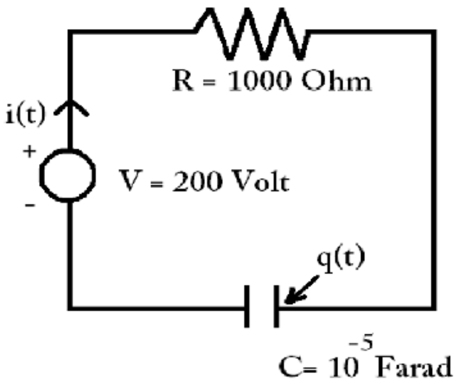
Kirchhoff's Voltage Law:

The sum of the voltages around a closed circuit is zero.

Voltage drop across:

- resistor $= iR$
- inductor $= L\dfrac{di}{dt}$
- capacitor $= \dfrac{q}{C}$

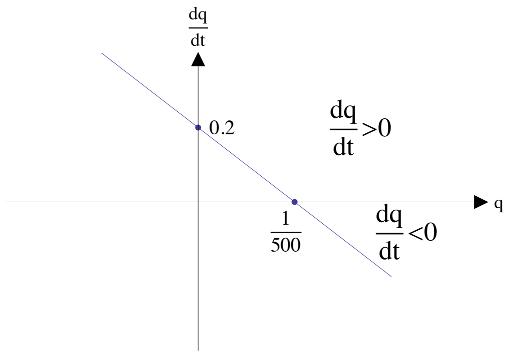
Example 5.17: R-C Series Circuit

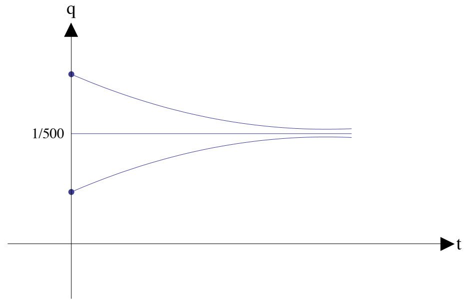


Find the charge on the capacitor at any time, if the current is initially 0.4 Amp.

Solution:

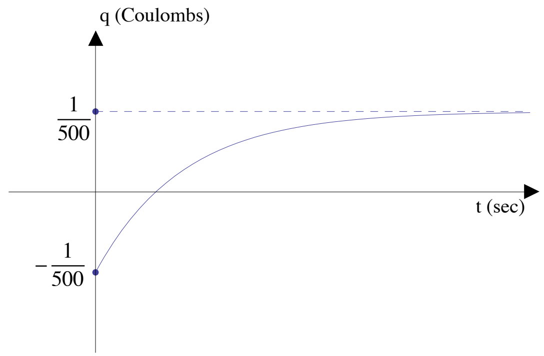
- Equilibrium solutions + phase plot



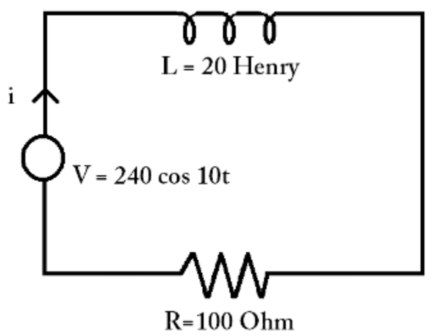


- Exact solution

Solve $\dfrac{dq}{dt} + 100q = 0.2$ to give $q(t) = \dfrac{1}{500} + ce^{-100t}$



Example 5.18: L-R Series Circuit



Find the current in the circuit at any time, if initially there is no current.

Solution:

Note:

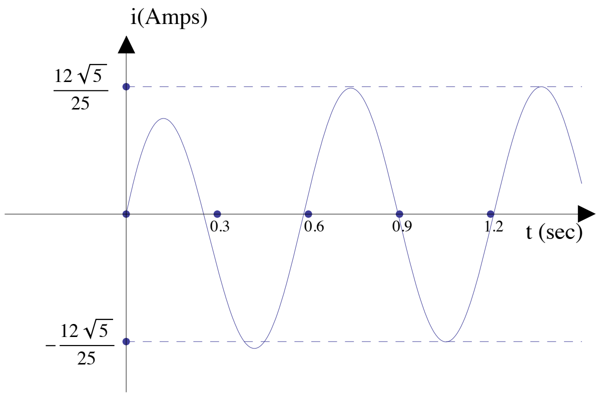
There are no equilibrium solutions for i.

Note:

Can also write solution as

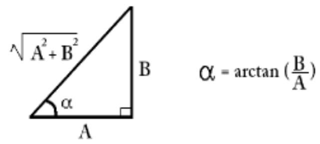
$$i(t) = \frac{12\sqrt{5}}{25}\cos(10t - \alpha) - \frac{12}{25}e^{-5t}$$

where $\alpha = \arctan 2 \approx 63.4°$



In general,

$$A\cos\theta + B\sin\theta = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\theta + \frac{B}{\sqrt{A^2+B^2}}\sin\theta\right)$$



$$= \sqrt{A^2+B^2}\,(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$$

$$= \sqrt{A^2+B^2}\,\cos(\theta-\alpha).$$

Section 6: Second Order Differential Equations

A second order o.d.e has the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

The general form of a linear second order o.d.e is

$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$

- If $\mathcal{R}(x) = 0$, the o.d.e is homogeneous (H).
- If $\mathcal{R}(x) \neq 0$, the o.d.e is inhomogeneous (IH).

Note:
A homogeneous linear o.d.e is different to a homogeneous type

Initial value problem for a second order o.d.e

Solve
$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$
subject to the conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$.

Boundary value problem for a second order o.d.e

Solve
$$\frac{d^2y}{dx^2} + \mathcal{P}(x)\frac{dy}{dx} + \mathcal{Q}(x)y = \mathcal{R}(x)$$
subject to the conditions $y(a) = y_0$ and $y(b) = y_1$.

Homogeneous 2^{nd} Order Linear O.D.E's

Theorem:
The general solution of
$$y'' + \mathcal{P}(x)y' + \mathcal{Q}(x)y = 0$$
is the function y given by
$$y(x) = c_1y_1(x) + c_2y_2(x)$$
where

- y_1, y_2 are two linearly independent solutions of the homogeneous o.d.e,
- $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Definition:
Two functions y_1 and y_2 are linearly independent if
$$c_1y_1(x) + c_2y_2(x) = 0 \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Example 6.1: Are $y_1(x) = x^2,\ y_2(x) = 2x^2$ linearly independent?

Solution:

Example 6.2: Are $y_1(x) = e^{2x},\ y_2(x) = xe^{2x}$ linearly independent?

Solution:

Homogeneous 2^{nd} Order Linear O.D.E's with Constant Coefficients

General form:
$$ay'' + by' + cy = 0$$
where a, b, c are constants.

To solve for $y(x)$:

$$\begin{aligned} \text{Try } y(x) &= e^{\lambda x} \\ \Rightarrow y'(x) &= \lambda e^{\lambda x}, \quad y''(x) = \lambda^2 e^{\lambda x} \end{aligned}$$

$$\text{so } \left(a\lambda^2 + b\lambda + c\right)\underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\Rightarrow \boxed{a\lambda^2 + b\lambda + c = 0} \qquad \text{Characteristic Equation}$$

$$\Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1: $b^2 - 4ac > 0$

- 2 distinct real values λ_1, λ_2
- 2 linearly independent solutions

$$e^{\lambda_1 x}, \quad e^{\lambda_2 x}$$

- General Solution:

$$\boxed{y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}}$$

Example 6.3: Solve $y'' + 7y' + 12y = 0$ for $y(x)$.

Solution:

Case 2: $b^2 - 4ac = 0$

- 1 real value $\lambda = \dfrac{-b}{2a}$
- 1 solution is $e^{\lambda x}$
- 2^{nd} linearly independent solution is $xe^{\lambda x}$ (found using variation of parameters — not in syllabus).
- General Solution:

$$\boxed{y(x) = Ae^{\lambda x} + Bxe^{\lambda x}}$$

We now verify that $xe^{\lambda x}$ is a solution:

If $y(x) = xe^{\lambda x}$, then

$$y'(x) = (\lambda x + 1)\, e^{\lambda x},$$

$$y''(x) = \left(\lambda^2 x + 2\lambda\right) e^{\lambda x}.$$

So $ay'' + by' + cy$

$$\begin{aligned} &= a\left(\lambda^2 x + 2\lambda\right) e^{\lambda x} + b\,(\lambda x + 1)\, e^{\lambda x} + cxe^{\lambda x} \\ &= xe^{\lambda x}\underbrace{\left(a\lambda^2 + b\lambda + c\right)}_{=0} + \underbrace{(2\lambda a + b)}_{=0} e^{\lambda x} \\ &= 0 \end{aligned}$$

So $y(x) = xe^{\lambda x}$ is a solution.

Example 6.4: Solve $y'' + 2y' + y = 0$ for $y(x)$.

Solution:

Case 3: $b^2 - 4ac < 0$

- 2 complex conjugate values

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$

- 2 complex linearly independent solutions

$$e^{(\alpha+i\beta)x}, \quad e^{(\alpha-i\beta)x}$$

- General Solution:

$$\begin{aligned} y(x) &= C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} \qquad \text{where } C_1, C_2 \in \mathbb{C} \\ &= C_1 e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right) + C_2 e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right) \\ &= \underbrace{(C_1 + C_2)}_{A} e^{\alpha x}\cos(\beta x) + \underbrace{(C_1 i - C_2 i)}_{B} e^{\alpha x}\sin(\beta x) \end{aligned}$$

Put $A = C_1 + C_2$ and $B = (C_1 - C_2)i$. If $C_1 = \overline{C_2}$, then $A, B \in \mathbb{R}$.

Note:
Imposing real conditions on the o.d.e will always lead to real coefficients A and B.

• 2 real linearly independent solutions

$$e^{\alpha x}\cos(\beta x), \quad e^{\alpha x}\sin(\beta x)$$

Real General Solution:

$$\boxed{y(x) = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)}$$

Example 6.5: Solve $y'' - 4y' + 13y = 0$ for $y(x)$ if $y(0) = -1$ and $y'(0) = 2$.

Solution:

Inhomogeneous 2^{nd} Order Linear O.D.E's

Theorem:

The general solution of

$$y'' + \mathcal{P}(x)y' + \mathcal{Q}(x)y = \mathcal{R}(x)$$

is the function y given by

$$y(x) = y_{\mathcal{H}}(x) + y_{\mathcal{P}}(x)$$

where

• $y_{\mathcal{H}}(x) = c_1y_1(x) + c_2y_2(x)$ is the general solution of the homogeneous o.d.e (called the homogeneous solution, GS(H)),

• $y_{\mathcal{P}}(x)$ is a solution of the inhomogeneous o.d.e (called a particular solution, PS(IH)),

Inhomogeneous 2^{nd} Order Linear O.D.E's with Constant Coefficients

General form:

$$ay'' + by' + cy = \mathcal{R}(x)$$

where a, b, c are constants.

Example 6.6: Solve $y'' + 2y' - 8y = \mathcal{R}(x)$ where

(a) $\mathcal{R}(x) = 1 - 8x^2$

(b) $\mathcal{R}(x) = e^{3x}$

(c) $\mathcal{R}(x) = 85\cos x$

(d) $\mathcal{R}(x) = 3 - 24x^2 + 7e^{3x}$.

Solution:

Step 1: Find the general solution of $y'' + 2y' - 8y = 0$.

Step 2: Find a particular solution of $y'' + 2y' - 8y = \mathcal{R}(x)$.

(a) $\mathcal{R}(x) = 1 - 8x^2$: $\quad y'' + 2y' - 8y = 1 - 8x^2$

(b) $\mathcal{R}(x) = e^{3x}$: $\quad y'' + 2y' - 8y = e^{3x}$

$\left(e^{3x}\right.$ is NOT part of $\left.GS(H)\right)$

(c) $\mathcal{R}(x) = 85\cos x$: $\quad y'' + 2y' - 8y = 85\cos x$

Superposition of Particular Solutions

Theorem:

A particular solution of

$$ay'' + by' + cy = c_1\mathcal{R}_1(x) + c_2\mathcal{R}_2(x)$$

is $y_{\mathcal{P}}(x) = c_1y_1(x) + c_2y_2(x)$ where

- $y_1(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_1(x)$,
- $y_2(x)$ is a particular solution of $ay'' + by' + cy = \mathcal{R}_2(x)$,
- a, b, c, c_1, c_2 are constants.

Example 6.6 (d): $\mathcal{R}(x) = 3 - 24x^2 + 7e^{3x}$.

Solution:

Example 6.7: Solve $y'' - y = e^x$.

Solution:

$GS(H): y_{\mathcal{H}}(x) = Ae^x + Be^{-x}$

Example 6.8: Solve $y'' + 2y' + y = e^{-x}$.

Solution:

$GS(H): \; y_{\mathcal{H}}(x) = (A + Bx)\, e^{-x}$

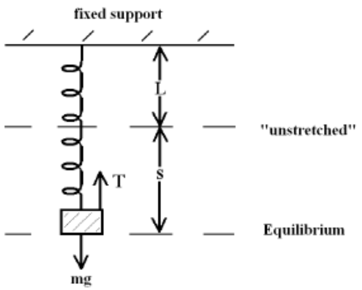
Example 6.9: Solve $y'' + 49y = 28\sin(7t)$.

Solution:

$GS(H) : y_{\mathcal{H}}(t) = A\cos(7t) + B\sin(7t)$

Springs - Free Vibrations

An object (mass m kg) stretches a spring (natural length L m) hanging from a fixed support by s m.



The forces are:

• gravitational force $= mg \qquad (g = 9.8 \text{ m/s}^2)$

• restoring force in spring (from Hooke's Law)

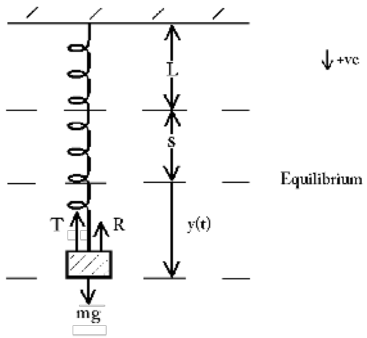
$$T = \underset{\substack{\nwarrow \\ \text{spring constant}}}{k} \cdot \text{extension} \qquad (k > 0)$$

At equilibrium, forces balance so:

Suppose the mass is set in motion. Let y be the displacement of the object from the equilibrium position ($y = 0$) at any time t.

Assume

- downward direction is positive
- spring is stretched below equilibrium
- mass is moving down (so damping is upwards)



Extra force:
• damping force is proportional to velocity

$$R = \underset{\substack{\nwarrow \\ \text{damping constant}}}{\beta}\dot{y} \qquad (\beta \geq 0)$$

Using Newton's Law $\quad (F = ma)$

To solve, try $y(t) = e^{\lambda t}$

$$\Rightarrow \quad m\lambda^2 + \beta\lambda + k = 0$$

$$\Rightarrow \quad \lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

• If $\beta = 0$: $\quad \lambda = \pm ib \qquad$ simple harmonic motion

• If $0 < \beta < 2\sqrt{mk}$: $\quad \lambda = a \pm ib \qquad$ underdamped, weak damping

• If $\beta = 2\sqrt{mk}$: $\quad \lambda = a, a \qquad$ critical damping

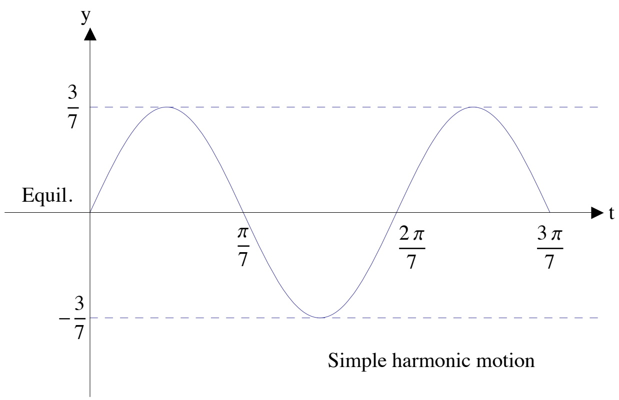
• If $\beta > 2\sqrt{mk}$: $\quad \lambda = a, b \qquad$ overdamped, strong damping

Example 6.10: A $\frac{40}{49}$ kg mass stretches a spring hanging from a fixed support by 0.2m. The mass is released from the equilibrium position with a downward velocity of 3m/s. Find the position of the mass y below equilibrium at any time t, if the damping constant β is:

(a) 0 (b) $\frac{160}{49}$ (c) $\frac{80}{7}$ (d) $\frac{2000}{49}$

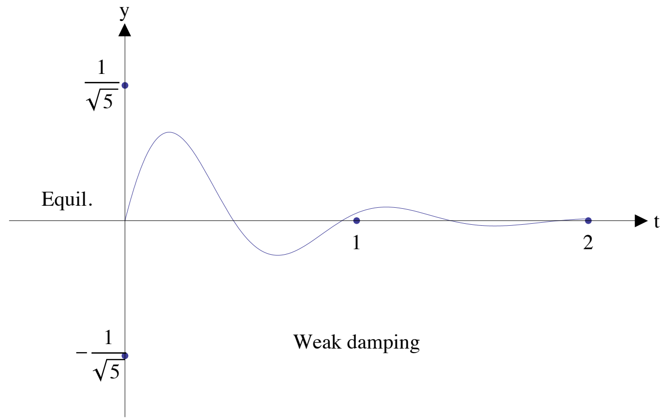
Solution:

(a) $\beta = 0: \quad \ddot{y} + 49y = 0$

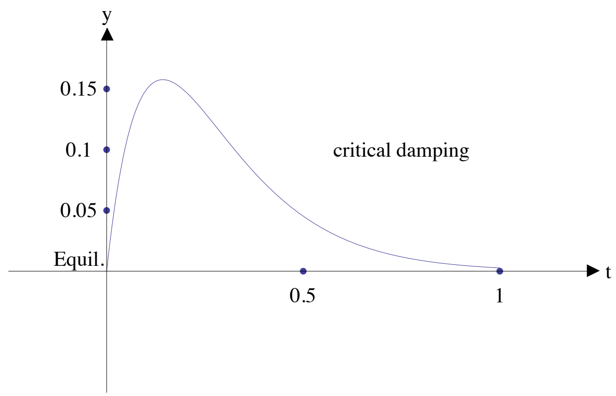


Simple harmonic motion

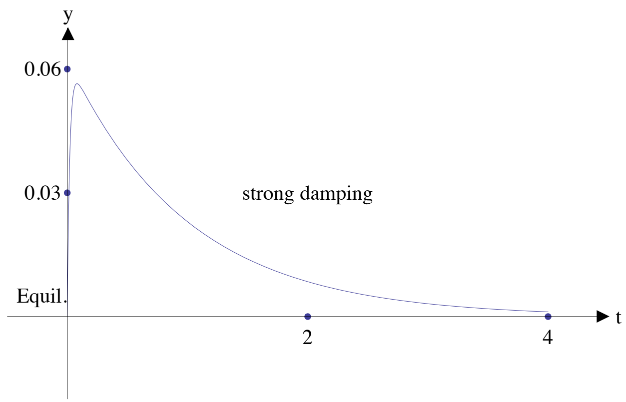
(b) $\beta = \frac{160}{49}$: $\quad \ddot{y} + 4\dot{y} + 49y = 0$



(c) $\beta = \frac{80}{7}$: $\quad \ddot{y} + 14\dot{y} + 49y = 0$

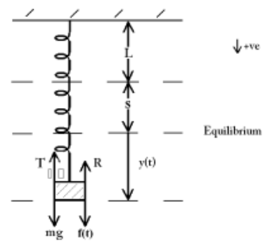


(d) $\beta = \frac{2000}{49}$: $\quad \ddot{y} + 50\dot{y} + 49y = 0$



Springs - Forced Vibrations

If an external downwards force f is applied to the spring-mass system at time t, the forces acting on the mass are:

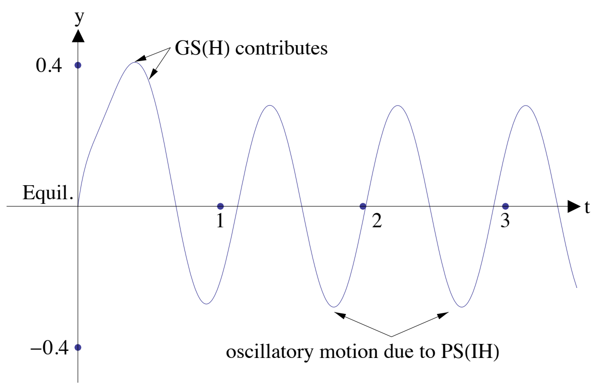


Example 6.11: Apply an external downwards force $f(t) = \frac{160}{7}\sin(7t)$ in Example 6.10.

Solution:

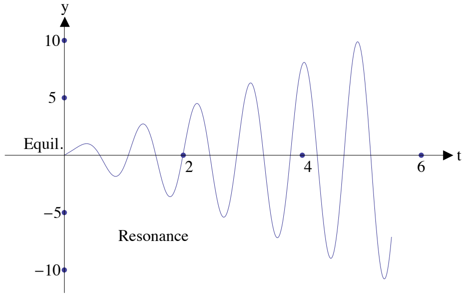
$$(a) \quad \beta = \frac{80}{7}: \quad \ddot{y} + 14\dot{y} + 49y = 28\sin(7t)$$

$$GS(IH): \quad y(t) = (A + Bt)\,e^{-7t} - \frac{2}{7}\cos(7t)$$



$$(b) \quad \beta = 0: \quad \ddot{y} + 49y = 28\sin(7t)$$

$$GS(IH): \quad y(t) = A\cos(7t) + B\sin(7t) - 2t\cos(7t)$$



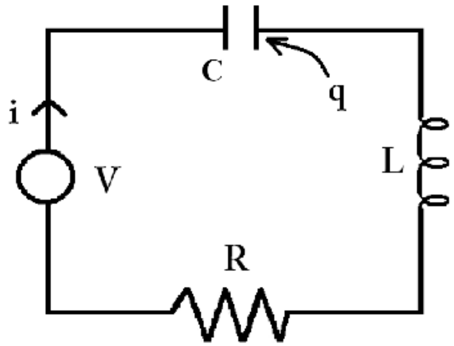
Definition

Resonance: Resonance occurs when the external force f has the same form as one of the terms in the $GS(H)$.

If $\beta = 0$, then the $PS(IH)$ will grow without bound as $t \to \infty$.

Example 6.12: LRC Series Circuit.

Consider a series circuit containing an inductor (inductance L Henry), a capacitor (capacitance C Farad), a resistor (resistance R Ohm) and a voltage source (V Volt).

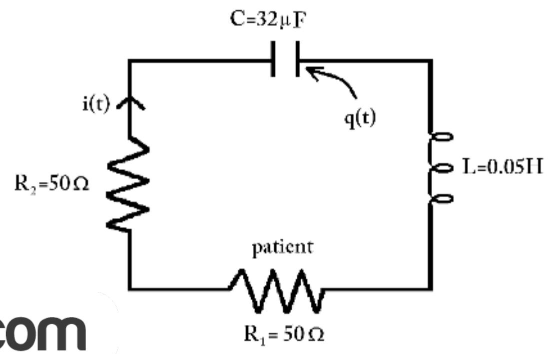


Solution:

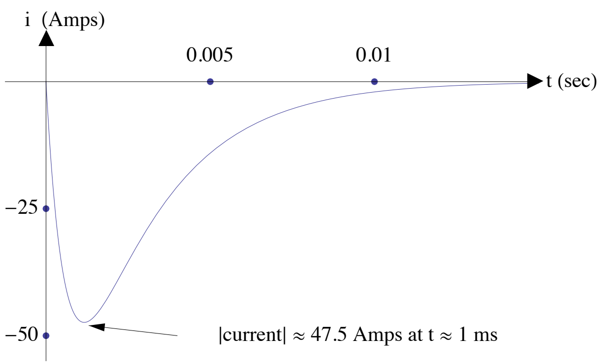
Note:
The o.d.e for the electric circuit gives the full range of solutions obtained in vibrations of springs.

Example 6.13: Defibrillator

A defibrillator discharges a current through the patient in an attempt to restart the patient's heart. It consists of an open circuit containing a capacitor of 32μF, an inductor of 0.05H and a resistor of 50Ω. The patient has a resistance of 50Ω when the device is discharged through them. Find the current during discharge, if the capacitor is initially charged to 6000V.



Solution:



Section 7: Functions of Two Variables

Example

The temperature T at a point on the Earth's surface at a given time depends on the latitude x and the longitude y. We think of T being a function of the variables x, y and write $T = f(x, y)$.

In general

A function of two variables is a mapping f that assigns a unique real number $z = f(x, y)$ to each pair of real numbers (x, y) in some subset D of the xy plane $\mathbb{R}^2$. We also write

$$f : D \to \mathbb{R}$$

where D is called the domain of f.

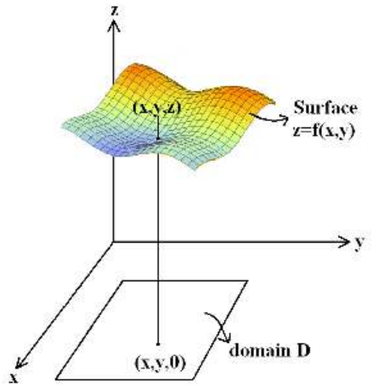
Example

If $f(x, y) = x^2 + y^3$ then $f(2, 1) = 4 + 1 = 5$.

We can represent the function f by its graph in $\mathbb{R}^3$. The graph of f is:

$$\{(x,y,z) : (x,y) \in D \text{ and } z = f(x,y)\}.$$

This is a surface lying directly above the domain D. The x and y axes lie in the horizontal plane and the z axis is vertical.



Equations of a Plane

The Cartesian equation of a plane has the form

$$ax + by + cz = d$$

where a, b, c, d are real constants.

In fact, the plane passing through a point (x_0, y_0, z_0) with a normal vector (a, b, c) consists of the points (x, y, z) such that (a, b, c) is perpendicular to $(x - x_0, y - y_0, z - z_0)$ and thus has equation

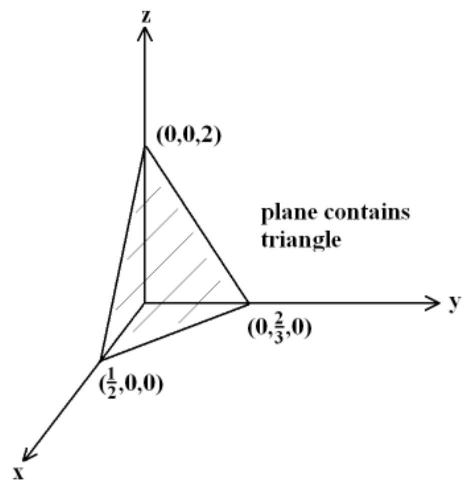
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

that is,

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

Example 7.1: The plane $4x + 3y + z = 2$ can be written as $z = 2 - 4x - 3y$, so is the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2 - 4x - 3y$. Sketch the plane.

Solution:



Level Curves

A curve on the surface $z = f(x, y)$ for which z is a constant is a contour.

The same curve drawn in the xy plane is a level curve.

So a level curve of f has the form

$$\{(x, y) : f(x, y) = c\}$$

where $c \in \mathbb{R}$ is a constant.

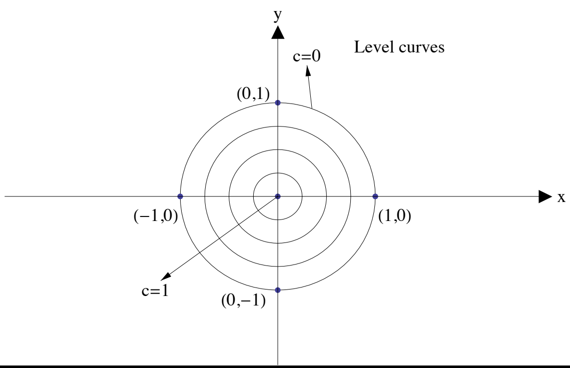
Sketching Functions of Two Variables

The key steps in drawing a graph of a function of two variables $z = f(x, y)$ are:

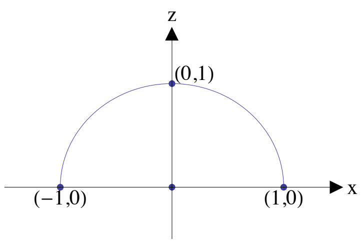
1. Draw the x, y, z axes.
   For right handed axes: the positive x axis is towards you, the positive y axis points to the right, and the positive z axis points upward.
2. Draw the $y - z$ cross section.
3. Draw some level curves and their contours.
4. Draw the $x - z$ cross section.
5. Label any x, y, z intercepts and key points.

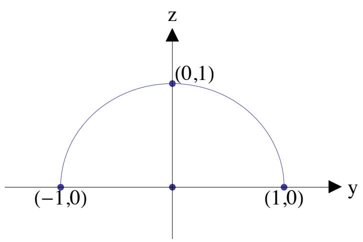
Example 7.2: Find the level curves of $z = \sqrt{1 - x^2 - y^2}$.
Hence identify the surface and sketch it.

Solution:

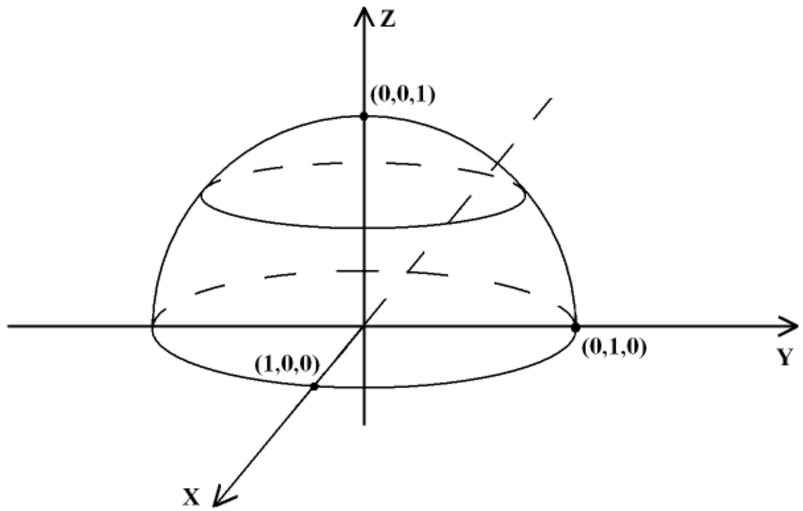


Consider cross sections (slices) to help sketch graph.



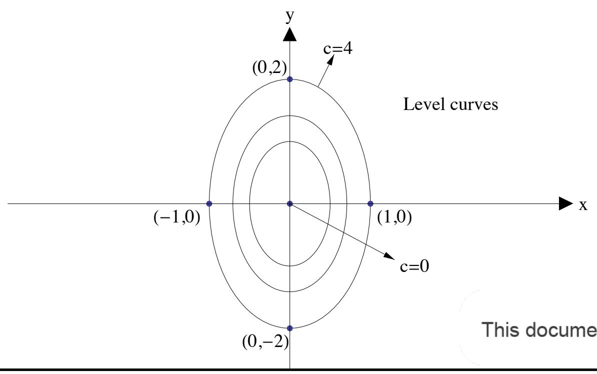


Surface is a hemisphere radius 1, centre at $(0, 0, 0)$ for $z \geq 0$.

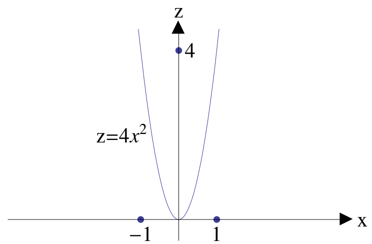


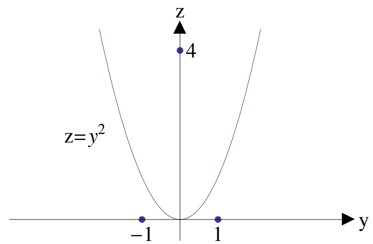
Example 7.3: Sketch the graph of $z = 4x^2 + y^2$.

Solution:

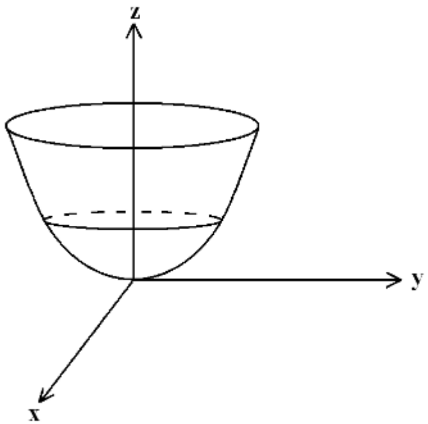


Consider cross sections (slices) to help sketch graph.



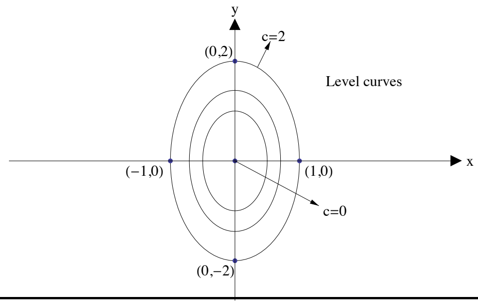


The surface is an elliptic paraboloid (parabolic bowl).

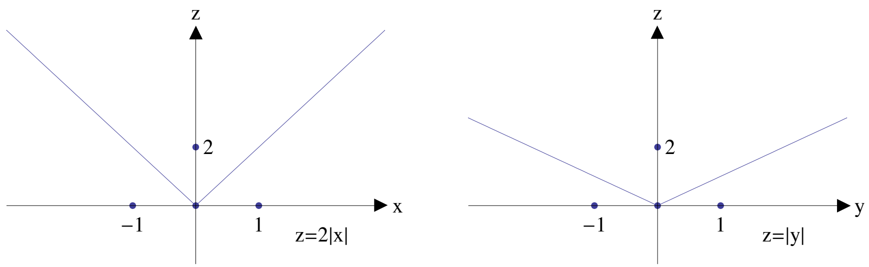


Example 7.4: Sketch the graph of $z = \sqrt{4x^2 + y^2}$.

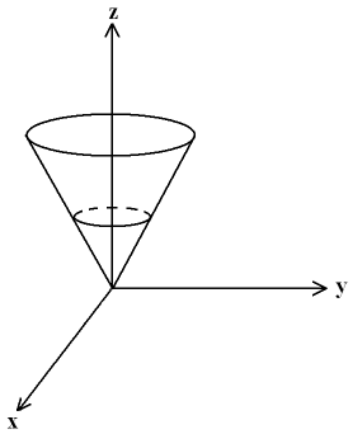
Solution:



Cross sections



The surface is an elliptic cone.



Limits

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

We say f has the limit L as (x, y) approaches (x_0, y_0)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if when (x, y) approaches (x_0, y_0) along ANY path in the domain, $f(x, y)$ gets arbitrarily close to L.

Note:

1. L must be finite.
2. The limit can exist if f is undefined at (x_0, y_0).
3. The usual limit laws apply.

Continuity

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function.

f is continuous at $(x, y) = (x_0, y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

Note:
The continuity theorems for functions of one variable can be generalised to functions of two variables.

Example 7.5: Let $f(x, y) = x^2 + y^2$. For which values of x and y is f continuous?

Solution:

Example 7.6: Evaluate $\lim_{(x,y)\to(2,1)} \log(1 + 2x^2 + 3y^2)$.

Solution:

First Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. The first order partial derivatives of f with respect to the variables x and y are defined by the limits:

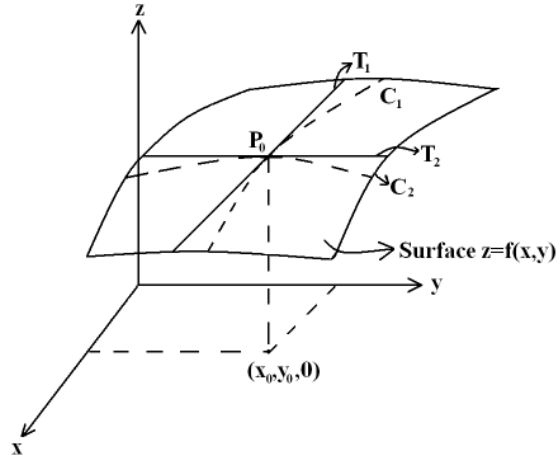
$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Note:

- $\frac{\partial f}{\partial x}$ measures the rate of change of f with respect to x when y is held constant.
- $\frac{\partial f}{\partial y}$ measures the rate of change of f with respect to y when x is held constant.

Geometric Interpretation of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$



Let C_1 be the curve where the vertical plane $y = y_0$ intersects the surface. Then $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ gives the slope of the tangent to C_1 at (x_0, y_0, z_0).

Let C_2 be the curve where the vertical plane $x = x_0$ intersects the surface. The $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ gives the slope of the tangent to C_2 at (x_0, y_0, z_0).

- T_1 and T_2 are the tangent lines to C_1 and C_2.

Example 7.7: Let $f(x,y) = xy^2$. Find $\frac{\partial f}{\partial y}$ from first principles.

Solution:

Example 7.8: Let $f(x, y) = 3x^3y^2 + 3xy^4$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution:

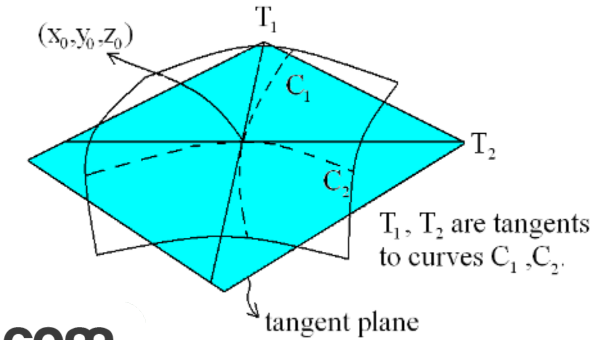
Example 7.9: Let $f(x, y) = y \log x + x \tanh(3y)$. Find f_x, f_y at $(1, 0)$.

Solution:

Tangent Planes and Differentiability

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. We say that f is differentiable at (x_0, y_0) if the tangent lines to all curves on the surface $z = f(x, y)$ passing through (x_0, y_0, z_0) form a plane, called the tangent plane.

This holds if f_x and f_y exist and are continuous near (x_0, y_0).



The tangent line T_1 has equation ($y = y_0$ fixed):

$$z - z_0 = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0)$$

The tangent line T_2 has equation ($x = x_0$ fixed):

$$z - z_0 = \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0)$$

Since a plane passing through (x_0, y_0, z_0) has the form

$$z - z_0 = \alpha(x - x_0) + \beta(y - y_0)$$

the tangent plane has equation

$$\boxed{z - z_0 = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0).}$$

Example 7.10: Find the equation of the tangent plane to the surface $z = f(x, y) = 2x^2 + y^2$ at $(1, 1, 3)$.

Solution:

Linear Approximations

If f is differentiable at (x_0, y_0), we can approximate $z = f(x, y)$ by its tangent plane at (x_0, y_0, z_0).

This linear approximation of f near (x_0, y_0) is:

$$\boxed{f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0)}$$

Let $\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta f = z - z_0 = f(x, y) - f(x_0, y_0)$.

Then the approximate change in f near (x_0, y_0), for given small changes in x and y, is:

$$\boxed{\Delta f \approx \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}\Delta x + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\Delta y}$$

Example 7.11: Let $z = f(x, y) = x^2 + 3xy - y^2$. If x changes from 2 to 2.05 and y changes from 3 to 2.96, estimate the change in z.

Solution:

Note:
The actual change in f is

$$\begin{aligned} \Delta f &= f(2.05, 2.96) - f(2, 3) \\ &= 13.6449 - 13 \\ &= 0.6449 \end{aligned}$$

Example 7.12: Find the linear approximation of $f(x, y) = xe^{xy}$ at $(1, 0)$. Hence, approximate $f(1.1, -0.1)$.

Solution:

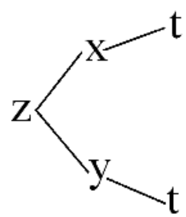
Note:
The actual value is

$$(1.1)e^{-0.11} \approx 0.98542$$

Chain Rule

1. If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$ are differentiable functions, then $z = f(g(t), h(t))$ is a function of t, and

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$



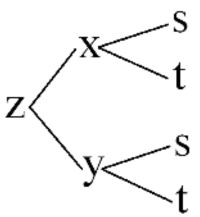
Example 7.13: If $z = x^2 - y^2$, $x = \sin t$, $y = \cos t$. Find $\frac{dz}{dt}$ at $t = \frac{\pi}{6}$.

Solution:

2. If $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$ are differentiable functions, then z is a function of s and t with

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



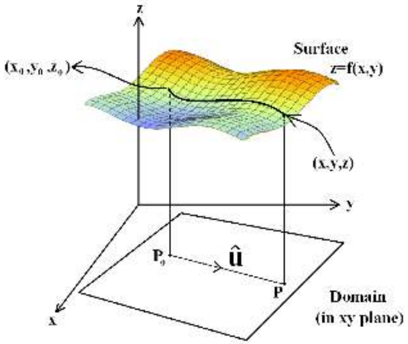
Example 7.14: If $z = e^x \sinh y$, $x = st^2$, $y = s^2 t$.
Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution:

Directional Derivatives

Let $\hat{\mathbf{u}} = (u_1, u_2)$ be a unit vector in the xy-plane (so $u_1^2 + u_2^2 = 1$). The rate of change of f at $P_0 = (x_0, y_0)$ in the direction $\hat{\mathbf{u}}$ is the directional derivative $D_{\hat{\mathbf{u}}} f\big|_{P_0}$.

Geometrically this represents the slope of the surface $z = f(x, y)$ above the point P_0 in the direction $\hat{\mathbf{u}}$.



The straight line starting at $P_0 = (x_0, y_0)$ with velocity $\hat{\mathbf{u}} = (u_1, u_2)$ has parametric equations:

$$x = x_0 + tu_1, \quad y = y_0 + tu_2.$$

Hence,

$$\begin{aligned}
D_{\hat{\mathbf{u}}} f\big|_{P_0} &= \text{rate of change of } f \text{ along the straight line at } t = 0 \\
&= \text{value of } \frac{d}{dt} f(x_0 + tu_1, y_0 + tu_2) \text{ at } t = 0 \\
&= f_x(x_0, y_0)x'(0) + f_y(x_0, y_0)y'(0) \qquad \text{by the chain rule} \\
&= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2.
\end{aligned}$$

We can also write this as a dot product

$$D_{\hat{\mathbf{u}}} f\big|_{P_0} = \left(\frac{\partial f}{\partial x}\bigg|_{P_0}, \frac{\partial f}{\partial y}\bigg|_{P_0} \right) \cdot (u_1, u_2).$$

Gradient Vectors

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, we can define the gradient of f to be the vector

$$\boxed{\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)}$$

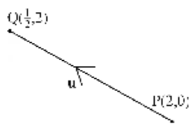
Then the directional derivative of f at the point P_0 in the direction $\hat{\mathbf{u}}$ is the dot product

$$\boxed{D_{\hat{\mathbf{u}}} f\big|_{P_0} = \nabla f\big|_{P_0} \cdot \hat{\mathbf{u}}}$$

Example 7.15: Find the directional derivative of $f(x, y) = xe^y$ at $(2, 0)$ in the direction from $(2, 0)$ towards $\left(\frac{1}{2}, 2\right)$.

Solution:

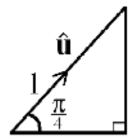
- direction $\hat{\mathbf{u}}$



Example 7.16: Find the directional derivative of $f(x,y) = \arcsin\left(\frac{x}{y}\right)$ at $(1,2)$ in the direction $\frac{\pi}{4}$ anticlockwise from the positive x axis.

Solution:

- direction $\hat{\mathbf{u}}$



Properties of ∇f and $D_{\hat{\mathbf{u}}} f$

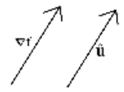
The directional derivative of f is

$$
\begin{aligned}
D_{\hat{\mathbf{u}}} f &= \nabla f \cdot \hat{\mathbf{u}} \\
&= |\nabla f|\,|\hat{\mathbf{u}}| \cos\theta \\
&= |\nabla f| \cos\theta
\end{aligned}
$$

where θ is the angle between ∇f and $\hat{\mathbf{u}}$, and $|\mathbf{v}|$ denotes the length of a vector $\mathbf{v}$.

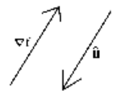
So for fixed ∇f:

- $D_{\hat{\mathbf{u}}} f$ is maximum when $\cos\theta = 1$ so $\theta = 0$



$\Rightarrow f$ increases most rapidly along ∇f.

- $D_{\hat{\mathbf{u}}} f$ is minimum when $\cos\theta = -1$ so $\theta = \pi$



$\Rightarrow f$ decreases most rapidly along $-\nabla f$.

- $D_{\hat{\mathbf{u}}} f = 0$ when $\cos\theta = 0$ so $\theta = \dfrac{\pi}{2}$ and $\nabla f \perp \hat{\mathbf{u}}$.

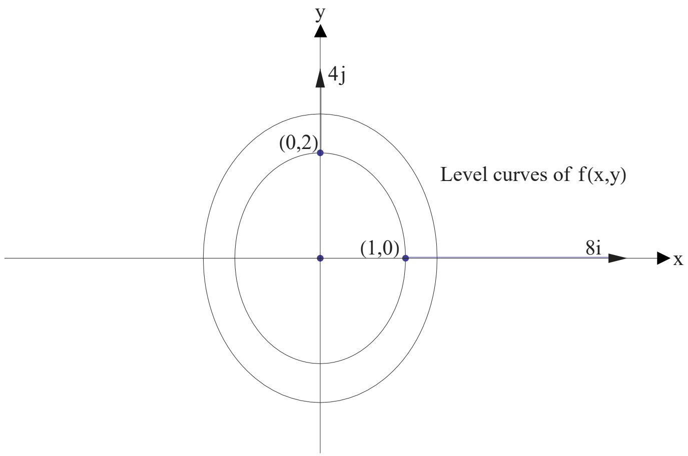
But $D_{\hat{\mathbf{u}}} f = 0$, whenever $\hat{\mathbf{u}}$ is tangent to a level curve of f (where f = constant).

$$\Rightarrow \nabla f \perp \text{ level curves of } f$$

Example 7.17: Let $f(x, y) = 4x^2 + y^2$.
(a) Find ∇f at $(1, 0)$ and $(0, 2)$.
(b) Show that ∇f is perpendicular to the level curves, by sketching ∇f at these points and the level curves of f.

Solution:

(b)



Example 7.18: In what direction does $f(x, y) = xe^y$

(a) increase (b) decrease

most rapidly at $(2, 0)$? Express direction as a unit vector.

Solution:

From Example 7.15

$$\nabla f(2, 0) = \mathbf{i} + 2\mathbf{j}$$

Second Order Partial Derivatives

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. The second order partial derivatives of f with respect to x and y are defined by:

- $f_{xx} = \left(f_x\right)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$
- $f_{yy} = \left(f_y\right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$
- $f_{xy} = \left(f_x\right)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$
- $f_{yx} = \left(f_y\right)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$

Theorem:

If the second order partial derivatives of f exist and are continuous then $f_{xy} = f_{yx}$.

Example 7.19: Find the second order partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x\sin(x + 2y)$.

Solution:

Note:
$f_{xy} = f_{yx}$ as expected since trigonometric functions and polynomials are continuous for all $(x, y) \in \mathbb{R}^2$.

Stationary Points

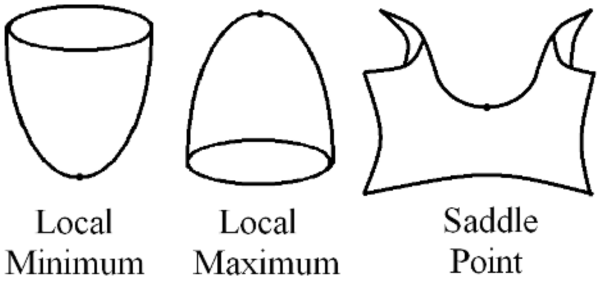
A stationary point of f is a point (x_0, y_0) at which

$$\nabla f = \mathbf{0}$$

So $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously at (x_0, y_0).

Geometrically, this means that the tangent plane to the graph $z = f(x, y)$ at (x_0, y_0) is horizontal, i.e. parallel to the xy-plane.

Three important types of stationary points are

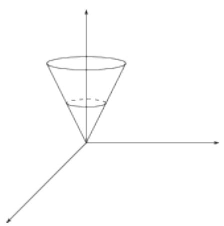


A function f has a

1. local maximum at (x_0, y_0) if $f(x, y) \le f(x_0, y_0)$ for all (x, y) in an open disk centred at (x_0, y_0),
2. local minimum at (x_0, y_0) if $f(x, y) \ge f(x_0, y_0)$ for all (x, y) in an open disk centred at (x_0, y_0),
3. saddle point at (x_0, y_0) if (x_0, y_0) is a stationary point, and there are points near (x_0, y_0) with $f(x, y) > f(x_0, y_0)$ and other points near (x_0, y_0) with $f(x, y) < f(x_0, y_0)$.

Any local maximum or minimum of f will occur at a critical point (x_0, y_0) such that

1. $\nabla f(x_0, y_0) = \mathbf{0}$ <u>or</u>
2. $\dfrac{\partial f}{\partial x}$ and/or $\dfrac{\partial f}{\partial y}$ do not exist at (x_0, y_0).



$z = \sqrt{x^2 + y^2}$. Minimum at $(0, 0)$ BUT ∇f does not exist at $(0, 0)$.

Second Derivative Test

If $\nabla f(x_0, y_0) = \mathbf{0}$ and the second partial derivatives of f are continuous on an open disk centred at (x_0, y_0), consider the Hessian function

$$H(x, y) = f_{xx} f_{yy} - (f_{xy})^2$$

evaluated at (x_0, y_0).

Then (x_0, y_0) is a

1. local minimum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$.
2. local maximum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$.
3. saddle point if $H(x_0, y_0) < 0$.

Note: Test is inconclusive if $H(x_0, y_0) = 0$.

Example 7.20: Find and classify the stationary points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

Solution:

Example 7.21: Find and classify the stationary points of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y \sin x$.

Solution:

Partial Integration

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

The partial indefinite integrals of f with respect to the first and second variables (say x and y) are denoted by:

$\int f(x, y)\, dx$ and $\int f(x, y)\, dy$.

- $\int f(x, y)\, dx$ is evaluated by holding y fixed and integrating with respect to x.
- $\int f(x, y)\, dy$ is evaluated by holding x fixed and integrating with respect to y.

Example 7.22: Evaluate $\int (3x^2y + 12y^2x^3)\, dx$.

Solution:

Note:

Example 7.23: Evaluate $\int_0^1 (3x^2y + 12y^2x^3)\, dy$.

Solution:

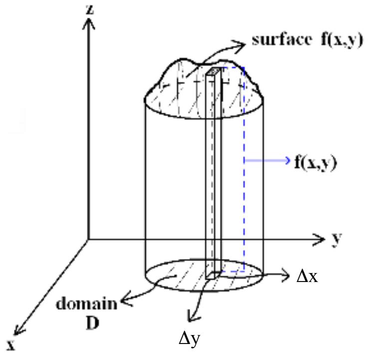
Double Integrals

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over a domain D in $\mathbb{R}^2$.

We can evaluate the double integral:

$$\boxed{\iint_D f(x,y)\, dA = \iint_D f(x,y)\, dx\, dy}$$

$\iint_D f(x,y)\, dA$ is the volume under the surface $z = f(x,y)$ that lies above the domain D in the xy plane, if $f(x,y) \geq 0$ in D.



$$\text{Volume of thin rod} = \underbrace{(\text{Area base})}_{\substack{\| \\ \Delta x \Delta y}} \cdot \underbrace{(\text{height})}_{\substack{\| \\ f(x,y)}}$$

The double integral is defined as the limit of sums of the volumes of the rods:

$$\begin{aligned}\iint_D f(x,y)\, dA &= \iint_D f(x,y)\, dx\, dy \\ &= \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{i=1}^{n} [f(x,y)\Delta x \Delta y]_i\end{aligned}$$

Note:

If $f(x,y) = 1$ then

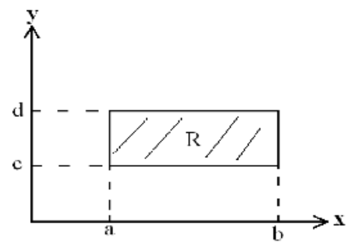
$$\iint_D dA = \iint_D dx\, dy$$

gives the area of the domain D.

Double Integrals Over Rectangular Domains

Definitions

1. $R = [a,b] \times [c,d]$ is a rectangular domain defined by $a \leq x \leq b, \quad c \leq y \leq d.$



2. $\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy$ means integrate ... to x first and then integrate with respect to y.

Fubini's Theorem:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over the domain $R = [a, b] \times [c, d]$. Then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

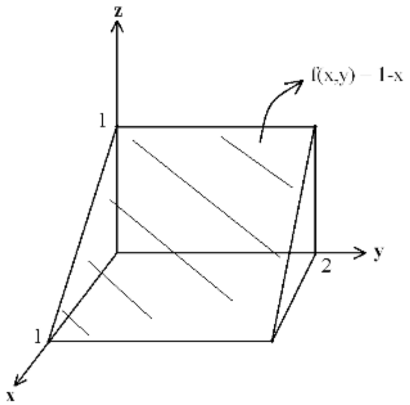
So order of integration is not important.

Example 7.24: Evaluate $\iint_R (x^2 + y^2)\, dx\, dy$ if $R = [-1, 1] \times [0, 1]$.

Solution:

Note:
As expected, the order of integration is not important since polynomials are continuous for all $(x, y) \in \mathbb{R}^2$.

Example 7.25: Using double integrals, find the volume of the wedge shown below.



Solution: