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MAT4MDS ASSIGNMENT 2, Semester 2, 2021

Submit by 11:59 pm on Thursday August 19th.

Scan your answers to the assignment, and submit it as a single pdf file through the LMS. The printers around campus are also scanners. By submitting your work electronically, you are affirming that it is all your own work, and you will be asked to confirm this as you submit it.

You may use facts from the Readings, or derived in the Practice Classes, to answer these questions. In marking your assignment, as well as accuracy, the clarity of your communication will be considered. Use the practice class solutions for a model of the standard expected. Working must be shown to support your answers. Answer in sentences.

Question 1 Consider the function $f : (0,1] \to \mathbb{R}$, given by $f(x) = x^{-1}\sqrt[n]{\exp(mx)}$, $(*)$
where m and n the last two non-zero digits of your student number ordered so that $n \leq m$.

(a) Write down your student number, your m and your n.

(b) Find the first two derivatives of $f(x)$, i.e. find $f'(x)$, $f''(x)$.

(c) Are there any stationary point in the domain $(0,1]$?? If so, identify them and check whether they are minima, maxima, or neither. If not, determine whether f is increasing or decreasing on the domain?

(d) Are there any points of inflection in the domain $(0,1]$? If so, identify them and state which intervals are concave up and which ones are concave down. If not, determine whether the function is concave up or down on the domain.

(e) Redo your calculations for (b) leaving the variables m and n in place (without replacing them with digits from your student number). The easiest way to calculate $f''(x)$ is to write $f''(x) = (f'(x))' = (g(x)f(x))'$ and then use the product rule and the fact that you already know $f'(x)$.

(f) Let

$$f^{(k)}(x) = (-1)^k \frac{f(x)}{(nx)^k}\left(k!n^k + \sum_{i=1}^{k}(-1)^i \frac{k!}{i!} n^{k-i}(mx)^i\right),$$

where $f(x)$ is given by $(*)$. Verify that your answers to (e) satisfy $f'(x) = f^{(1)}(x)$ and $f''(x) = f^{(2)}(x)$.

(g) (Challenging, 2 extra marks!) Show that the derivative of $f^{(k)}(x)$ is equal to $f^{(k+1)}(x)$.

(h) What can you conclude from (f) and (g)? You can still draw a conclusion if you did not do (g).

Question 2 The shape of the function $y = f(x) = x\ln(x^4 - 2x^2 + 2)$ is complicated enough that it is hard to draw using hand calculations. You will use Taylor polynomials to approximate different parts of the function to estimate the shape of its graph, and to approximate some stationary points. If you wish you can use a graphing calculator or an app or website to see the computed shape of the graph, however this is not necessary and your answer should contain only information you have calculated below, not information from another source. It is fine to use a normal calculator if you need to find the decimal expression of a value to plot it.

(a) Calculate the first and second derivatives $f'(x)$ and $f''(x)$ (no need to simplify the expressions).

(b) Calculate the three different second order Taylor polynomials $(T_2 f)(x)$ centred around three x-values: $x = -1$, $x = 0$ and $x = +1$. Show all working.

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(c) Draw the graphs of the three Taylor polynomials found in (b) in one figure, where $-1.5 \leq x \leq 1.5$ and $-1.5 \leq y \leq 1.5$. Make this graph roughly square in shape and at least the area of one quarter of the sheet of paper.

(d) The function f has 4 stationary points. You may already know two of them. Where are they? Explain your answer.

(e) We will try to find approximations for the other two, using Taylor polynomials. Calculate the two second order Taylor polynomials $(T_2 f)(x)$ centred around the x-values: $x = -0.5$ and $x = 0.5$.

(f) Determine the stationary point of each Taylor polynomial found in (e). Verify that these points are better approximations to the stationary points of f than the points $x = -0.5$ and $x = 0.5$.

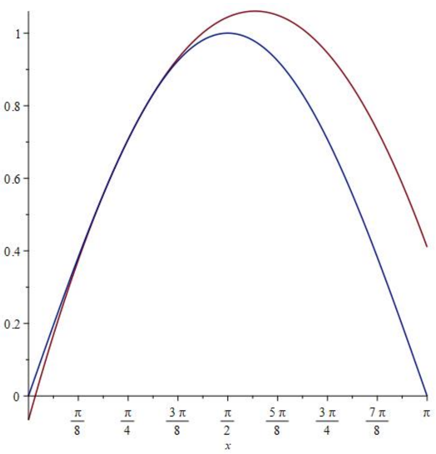
(g) Recall that a Taylor polynomial is a good approximation close to the point it is centred around, but often a bad approximation away from that point. Use this fact and the five graphs of the Taylor functions to deduce the approximate shape of the graph of $f(x)$ between $x = -1.5$ and $x = 1.5$. Mark it on your graph in a different colour or different style (e.g. dashed) so it looks different from the Taylor polynomials you have drawn.

Question 3

Consider the function $f(x) = \sin(x)$ and its Taylor polynomial about $\pi/4$:

$$(T_2 f)(x) = \frac{\sqrt{2}}{2}\left(1 + \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2\right).$$

Their graphs, on the interval $x \in [0, \pi]$, are given on the right. As one can see, the Taylor approximation is reasonably ok, with the error being maximal at π.



(a) Use Taylor's Theorem (restated below), to estimate the error at π. Discuss whether this is a good estimate of the error.

(b) Same question for the error at $\pi/2$.

TAYLOR'S THEOREM *Let f be a function which has an $(n+1)$th derivative defined at least throughout some interval I containing d. If there is a number M such that $|f^{(n+1)}(x)| \leq M$ for all $x \in I$ then*

$$|(E_n f)(x)| \leq \frac{M|(x-d)^{n+1}|}{(n+1)!} \text{ for all } x \in I.$$

Question 4 - video recording

Make a recording explaining the reasoning behind your answer to either 1(g), 1(h) or 3 (your choice).

(a) Introduce yourself. Point the camera at yourself so you are recognisable and say hello and say your name.

(b) Point the camera / share screen at your solution, or whatever you choose to use to explain your reasoning.

(c) Make your description clear but keep your video to less than 1 minute in length. Any extra will not be watched or marked.