PRACTICE EXAM ONLY

Student ID: ☐☐☐☐☐☐☐☐ Seat Number: ☐☐☐

Subject code:	MAT4MDS	Paper number:	1 (of 1)
Reading time:	15	minutes	
Writing time:	120	minutes	
Number of pages:	13 (including 4 page Fact Sheet)	(including cover sheet)	

Campus:

☐ Albury-Wodonga ☐ Bendigo ☒ Bundoora ☐ City ☐ Mildura ☐ Shepparton

Allowable materials:

Number:	Description:

Instructions to candidates:

1. This exam paper consists of 55 marks.
2. Write your answers in the spaces provided using blue or black pen. If you need extra space, continue your answer on an `extra space' page.
3. Attempt all questions. Show all of your working unless instructed otherwise.
4. If you cannot do part of a question, you should still attempt later parts; information given in the question may enable you to answer them correctly.
5. Calculators may NOT be used in the exam.

NOTE: No working pages have been provided in this sample exam – in the actual exam, there are extra blank working pages included.

Question 1. *Total: 7 marks*

Consider the matrix

$$M = \begin{bmatrix} 8 & 9 & 9 \\ 3 & 2 & 3 \\ -9 & -9 & -10 \end{bmatrix}$$

Find the eigenvalues of M.

Question 2. *Total: 12 marks*

(a) Which of the following statements are true, and which are false:

- Any matrix which is self-transpose is invertible.
- Any matrix with non-zero trace is invertible.
- Any matrix with non-zero determinant is invertible.
- Any positive semi-definite matrix is invertible.
- Any full-rank matrix is invertible.
- If none of the eigenvalues of a matrix is zero, it is invertible.
- The all-ones square matrix is invertible.
- The identity matrix is self-transpose.
- The trace of a square matrix of any size is equal to the sum of its eigenvalues.
- The determinant of a square matrix of any size is equal to the product of its eigenvalues.

5 marks

(b) Find the least squares line $y = \alpha x + \beta$ for the following data, with x as the independent variable and y as the dependent variable:

y	-1	0	1	2	3
x	3	2	1	1	0.5

6 marks

(c) If the roles of the variables are interchanged in part (b), the following line is obtained:

$$x = -0.6y + 2.1$$

Explain why re-arranging this line to give y in terms of x does NOT give the same line as found in part (b).

1 mark

Question 3. *Total: 15 marks*

Consider the function

$$h : \mathbb{R} \to \mathbb{R},\ h(x) = \log_e(x^2 + 3)$$

(a) Locate the stationary point of $h(x)$. *2 marks*

(b) Using the second derivative test, classify the point found in (a). *3 marks*

(c) Find any points of inflection of $h(x)$. *2 marks*

(d) State the range of h (that is, the output values of the function). *1 mark*

(e) Sketch the graph of h, marking the points found in (a) and (c). *3 marks*

(f) Find the inverse of the function *4 marks*

$$f : \mathbb{R}^- \to \mathbb{R},\ f(x) = \log_e(x^2 + 3)$$

Question 4. *Total: 7 marks*

(a) Show that

$$\frac{B(x+1,y)}{B(y+1,x)} = \frac{x}{y}$$

2 marks

(b) By an appropriate method, find *5 marks*

$$\int_0^{\sqrt{3}} \log_e(x^2+3)\ dx$$

You may use the fact, which is being given to you, that $\displaystyle\int_0^1 \frac{1}{1+u^2}\ du = \frac{\pi}{4}$

Question 6. *Total: 12 marks*

Consider the function of two variables

$$f(x, y) = e^{-x^2-y^2}.$$

(a) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ *2 marks*

(b) Find $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ *3 marks*

(c) Hence find the second order Taylor polynomial for $f(x, y)$ near $(1, -1)$. *4 marks*

(d) Let $g(x,y) = x^4 e^{-x^2-y^\alpha}$

Find an expression (in terms of y) for

$$\int_0^\infty g(x,y)dx.$$

3 marks

(e) In the function $g(x,y)$ is α a shape parameter or a scale parameter? Why? *2 marks*

$* * * *$ **End of Questions** $* * * *$

Matrices

- For each order $(m \times n)$, the matrix of ones $J_{m\times n}$ is the matrix in which every entry is 1. The square $(n \times n)$ matrix of ones is usually written as J_n.
- The centering matrix $C_n := I_n - \frac{1}{n} J_n$
- When it exists, the inverse of an $n \times n$ matrix A is the unique matrix, denoted by A^{-1}, with the property
$$AA^{-1} = I_n,$$
where I_n is the $n \times n$ identity matrix.
- Multiplicative Property of Determinants: $\det(AB) = \det(A)\det(B)$.
- Associated with each eigenvalue λ of a square matrix A, there is an eigenvector X, which is a non-zero column vector such that $AX = \lambda X$.
- For a 2×2 matrix A, the characteristic equation of A is
$$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0.$$
where trace(A) is the sum of the diagonal entries of A.
- The least squares solution to the system $AX = b$ is given by the solution to $A^T AX = A^T b$.

Calculus

function	$f(x)$	x^r $(r \neq 0)$	constant	e^x	$\log_e(x)$
derivative	$f'(x)$	rx^{r-1}	0	e^x	$\dfrac{1}{x}$

- **The constant rule**
  If $y = cu = cg(x)$ where $c \in \mathbb{R}$ then $\quad \dfrac{dy}{dx} = c\dfrac{du}{dx} = cg'(x)$.
- **The sum rule**
  If $y = u + v = g(x) + h(x)$ then $\quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} = g'(x) + h'(x)$.
- **The product rule**
  If $y = u{\cdot}v = g(x){\cdot}h(x)$ then $\quad \dfrac{dy}{dx} = \dfrac{du}{dx}\,v + u\,\dfrac{dv}{dx} = g'(x)\,h(x) + g(x)\,h'(x)$
- **The extended product rule**
  If $y = f(x){\cdot}g(x){\cdot}h(x)$ then $\quad \dfrac{dy}{dx} = f'(x)g(x)h(x) + f(x)g'(x)\,h(x) + f(x)g(x)\,h'(x)$
- **The chain rule**
  If $y = f(u) = f(g(x))$ and $u = g(x)$ then $\quad \dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx} = f'(g(x))\,g'(x)$.
- **The quotient rule**
  If $y = \dfrac{u}{v} = \dfrac{g(x)}{h(x)}$ where $h(x) \neq 0$ then $\dfrac{dy}{dx} = \dfrac{\frac{du}{dx}\,v - u\,\frac{dv}{dx}}{v^2} = \dfrac{g'(x)\,h(x) - g(x)\,h'(x)}{[h(x)]^2}$.
- **The Second Derivative test** Suppose that $f'(x_0) = 0$.
  - If $f''(x_0) > 0$, then x_0 is a local minimum point.
  - If $f''(x_0) < 0$, then x_0 is a local maximum point.
  - If $f''(x_0) = 0$, the test is inconclusive. (This point may be a local maximum, a local minimum or a point of inflection.)
- The nth **Taylor polynomial** to f about a is the function $T_n f : \mathbb{R} \to \mathbb{R}$ where

$$(T_n f)(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

  where $f^{(n)}(x) = \dfrac{d^n}{dx^n}(f(x))$ is the n^{th} derivative of f. $T_1 f$ is a linear approximation to f.
- **Taylor's Theorem.** Let f be a function which has an $(n+1)^{th}$ derivative defined on an interval I containing 0. If there is a positive number M such that $-M \leqslant f^{(n+1)}(x) \leqslant M$ for all $x \in I$ then

$$\big|(E_n f)(x)\big| \;\leqslant\; \frac{M|x|^{n+1}}{(n+1)!} \qquad \text{for } x \in I.$$

  Here $(E_n f)(x) = f(x) - (T_n f)(x)$ is the error which arises when $T_n f$ is used as an approximation to f.

Table of Common Antiderivatives.

$f(x)$	Anti-derivative $F(x)$	Comments
x^k	$\dfrac{1}{k+1}x^{k+1}$	$k \neq -1,\ x > 0.$
e^{ax}	$\frac{1}{a}e^{ax}$	
$\dfrac{1}{x}$	$\log_e(x)$	$x > 0$
$\log_e(x)$	$x\log_e(x) - x$	$x > 0$

- **Sum/difference property:** $\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$
- **Constant multiple property:** $\alpha f(x)\ dx = \alpha \int f(x)\ dx$ for all $\alpha \in \mathbb{R}$.
- For all $a, b \in \mathbb{R}$ with $a \neq 0$: If $\int f(x)\ dx = F(x)$ then $\int f(ax+b)\ dx = \frac{1}{a}F(ax+b)$
- **Substitution rule:** For suitable functions f and g we have

$$\int_a^b f(u)\frac{du}{dx}\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

  where $u = g(x)$.

- **Integration by Parts**

$$\int_a^b u\frac{dv}{dx}\ dx = uv|_a^b - \int_a^b v\frac{du}{dx}\ dx$$

- The **cumulative distribution function** F is an anti-derivative of the **probability density function** f for continuous data. That is:

$$P(X \leqslant x) = F(x) = \int_{-\infty}^{x} f(t)dt$$

  The **mean** value is given by

$$\int_{-\infty}^{\infty} xf(x)dx$$

- **The trapezoidal rule:** The integral on $[a, b]$ of the function f can be approximated by

$$\frac{(b-a)}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

  where $x_k = a + k\frac{(b-a)}{n}, \quad k = 0, 1, \ldots n$

Special functions and functions of two variables:

- **The Gamma Function:**
$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$$
It has the properties:
$$\Gamma(x+1) = x\Gamma(x) \qquad\qquad \Gamma(n+1) = n! \text{ for } n \in \mathbb{N}$$
Special values:
$$\Gamma(1) = 1 \qquad\qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

- **The Beta Function**
$$B(p,q) = \int_0^1 y^{p-1}(1-y)^{q-1}dy = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

- **The second order Taylor polynomial** of a function of two variables $f(x,y)$ about (a,b):
$$\begin{aligned} T^2_{(a,b)}f(x,y) =& f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &+ \frac{1}{2}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right\}. \end{aligned}$$