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Student Number: ______________

The University of Melbourne

Semester 2 Assessment 2013

Department of Mathematics and Statistics

MAST 10007 Linear Algebra

Reading Time: 15 minutes

Writing Time: 3 hours

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Instructions to Invigilators:

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:

This examination consists of 13 questions.
The total number of marks is 105
All questions may be attempted. All answers should be appropriately justified.

Extra Materials required (please tick & supply)

☐ Graph Paper ☐ Multiple Choice Form ☐ Other (please specify)

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) Consider the following system of linear equations:

$$\begin{array}{rcrcrcrcr} 4x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 2 \\ 4x_1 & & & - & 2x_3 & & & = & 4 \\ 4x_1 & - & 2x_2 & - & 3x_3 & - & x_4 & = & 6 \end{array}$$

   (i) Write down the augmented matrix corresponding to the system of linear equations.

   (ii) Reduce the matrix in (i) to reduced row-echelon form.

   (iii) Use the reduced row-echelon form to give all solutions in $\mathbb{R}^4$ to the system of linear equations.

   (b) Determine the values (if any) of $k \in \mathbb{R}$ for which the following system of linear equations has:

   (i) no real solution,

   (ii) infinitely many real solutions,

   (iii) a unique real solution.

$$\begin{array}{rcrcrcr} x & & & + & z & = & -1 \\ 2x & - & ky & + & 2z & = & -3 \\ 2x & + & ky & + & (k^2-2)z & = & k+1 \end{array}$$

[10 marks]

2. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 6 & 2 \\ 2 & -1 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 & 3 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Evaluate, if possible:

(a) BA

(b) $AC^T + D$

(c) CB^2A

(d) $B^TB - B^{-1}$

[4 marks]

3. Let

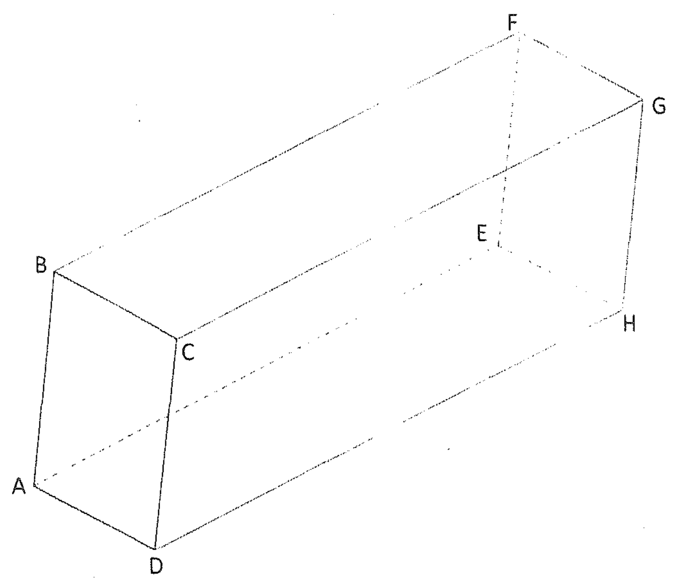
$$M = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 0 & -1 \\ -1 & 4 & -1 \end{bmatrix}$$

(a) Use cofactor expansion along the second row of M to calculate its determinant, $\det(M)$.

(b) Use row-reduction to find the inverse of the matrix M or explain why the inverse does not exist.

[6 marks]

4. Consider the following the parallelepiped:



Some of the corner points of the parallelepiped are $A(2,1,3)$, $B(1,0,3)$, $D(1,1,1)$ and $E(0,4,1)$.

(a) Find the Cartesian equation of the plane that contains the face EFGH of the parallelepiped. Let P denote this plane.

(b) Find a vector equation of the line that passes through the point B and is perpendicular to the plane P. Let L denote this line.

(c) Find the intersection of the line L with the plane P.

[7 marks]

5. Let

$$A = \begin{bmatrix} 1 & -1 & 5 & -2 & -11 \\ 3 & 1 & 7 & 0 & -1 \\ 1 & 1 & 1 & 1 & 5 \\ 3 & 2 & 5 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B is the reduced row-echelon form of the matrix A. Using this information, or otherwise, answer the following questions:

(a) What is the rank of A?

(b) Write down a basis for the row space of A.

(c) Write down a basis for the column space of A.

(d) Is the set of vectors

$$\{(1, 3, 1, 3), (-1, 1, 1, 2), (-2, 0, 1, 0)\}$$

linearly independent? If yes, give a reason. If not, explain why not and write one of these vectors as a linear combination of the others.

(e) Write the vector $(22, 2, -10, -2)$ as a linear combination of vectors in the set

$$\{(1, 3, 1, 3), (-1, 1, 1, 2), (5, 7, 1, 5), (-2, 0, 1, 0)\}$$

(f) Find a basis for the solution space of A.

[8 marks]

6. (a) Let V be a vector space over a field of scalars $\mathbb{F}$ and let $S \subseteq V$ be a subset of V. State the "Subspace Theorem."

(b) For each of the following, decide whether or not the given set S is a subspace of the vector space V. Justify your answers by either using appropriate theorems, or by providing a counter-example.

(i) $V = \mathcal{P}_n$ (all real polynomials of degree at most n) and

$$S = \left\{ p(x) \in \mathcal{P}_n \mid \int_0^1 p(x)dx = 1 \right\}$$

(ii) $V = M_{2,2}$ (all 2×2 matrices with real entries) and

$$S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2} \mid b + c = 0 \right\}$$

[8 marks]

7. (a) Let $\mathcal{P}_3$ be the vector space of all real polynomials of degree at most 3 and let V be the subspace of $\mathcal{P}_3$ given by

$$V = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in \mathcal{P}_3 \mid a_2 = 0\}.$$

Consider the subset S of V given by

$$S = \{1 - 2x,\ 3,\ 1 + x^3,\ x + x^3\}.$$

(i) Determine whether or not the set S is a linearly dependent set of vectors and if it is, express one of its vectors as a linear combination of the other vectors in the set S.

(ii) Determine whether or not the set S is a spanning set for the vector subspace V. If S is a spanning set, find a subset of S that is a basis for V.

(b) Let $M_{2,3}$ be the vector space of all 2×3 matrices with real entries, and let T be the subset of $M_{2,3}$ given by

$$T = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

(i) Determine whether or not the set T is a linearly dependent set of vectors and if it is, express one of its vectors as a linear combination of the other vectors in the set T.

(ii) Determine whether or not the set T is a spanning set for the vector space $M_{2,3}$. If T is a spanning set, find a subset of T that is a basis for $M_{2,3}$.

[8 marks]

8. Let $\mathcal{P}_2$ denote the vector space of all real polynomials of degree at most 2. Consider the subspace

$$V = \left\{ \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in M_{2,2} \mid b_1 = b_4 \right\}$$

of the vector space $M_{2,2}$ of all 2×2 matrices with real entries.

Define $T\colon \mathcal{P}_2 \to V$ by

$$T(a_0 + a_1 x + a_2 x^2) = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_0 \end{bmatrix}$$

(a) Show that T is a linear transformation.

(b) Let $\mathcal{B} = \{1 + x, x, 1 - x^2\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. Find $[T]_{\mathcal{C},\mathcal{B}}$ the matrix that represents T relative to the bases $\mathcal{B}$ for $\mathcal{P}_2$ and $\mathcal{C}$ for V.

(c) Find the kernel of T.

(d) Calculate the nullity of T and the rank of T.

[8 marks]

9. Let U, V and W be vector spaces (over $\mathbb{R}$) and let $S : U \to V$ and $T : V \to W$ be two linear transformations such that $T \circ S = 0$, (i.e. $T(S(\boldsymbol{u})) = \mathbf{0}$ for every $\boldsymbol{u} \in U$).

(a) Show $\text{Im}(S)$ is a subset of $\text{Ker}(T)$.
(That is, show that if $\boldsymbol{v} \in \text{Im}(S)$ then $\boldsymbol{v} \in \text{Ker}(T)$.)

(b) Show that

$$\dim(U) \leq \text{Nullity}(S) + \text{Nullity}(T)$$

[6 marks]

10. Consider the following bases for $\mathbb{R}^2$:

$$\mathcal{S} = \{(1,0),(0,1)\} \quad \mathcal{B} = \{(-1,1),(2,1)\}$$

(a) (i) Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{S}$.

(ii) Find the transition matrix $P_{\mathcal{B},\mathcal{S}}$ from $\mathcal{S}$ to $\mathcal{B}$.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by,

$$T(x,y) = (x+y, 3x-y).$$

(i) Find the matrix $[T]_{\mathcal{S}}$ of the transformation T with respect to the basis $\mathcal{S}$.

(ii) Find the matrix $[T]_{\mathcal{B}}$ of the transformation T with respect to the basis $\mathcal{B}$.

(iii) If $\mathbf{v} = (2,2)$, find $[\mathbf{v}]_{\mathcal{B}}$ and $[T\mathbf{v}]_{\mathcal{B}}$.

[9 marks]

11. Let $\boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Define

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 + x_2y_3 + x_3y_2 + x_3y_3 \qquad (*)$$

You may assume that the formula $(*)$ defines an inner product on $\mathbb{R}^3$.

(a) Find a symmetric 3×3 matrix A such that the above formula $(*)$ can be written in the form

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = [\boldsymbol{x}]^T A [\boldsymbol{y}]$$

for row matrices $[\boldsymbol{x}]^T = [x_1 \ x_2 \ x_3]$ and $[\boldsymbol{y}]^T = [y_1 \ y_2 \ y_3]$.

(c) Let W be the subspace of $\mathbb{R}^3$ that has basis $\mathcal{B} = \{(-1, -1, -1), (1, 2, -1)\}$. Apply the Gram-Schmidt procedure to the basis $\mathcal{B}$ to obtain a basis for W that is orthonormal with respect to the inner product defined above by $(*)$.

(d) Find the point of W that is closest (*with respect to the distance given by the above inner product* $(*)$) to the point $(8, 1, 4)$.

[10 marks]

12. Let

$$M = \begin{bmatrix} 5 & -3 & -6 \\ 6 & -4 & -6 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) Find all the eigenvalues of M.

(b) For each eigenvalue find a basis for the corresponding eigenspace.

(c) Find an invertible matrix P and a diagonal matrix D such that

$$P^{-1}MP = D$$

(d) Use the result of part (c) above to find the determinant of M.

[9 marks]

13. (a) For each of the following three matrices decide whether or not the matrix is diagonalizable over $\mathbb{R}$. You should justify your answers.

(i) $R = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(ii) $S = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -1 & 0 \\ 3 & -2 & 2 \end{bmatrix}$

(iii) $T = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Let $A = \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix}$. You may assume that A has eigenvalues 10 and 15 with corresponding eigenvectors $(2, 1)$ and $(-1, 2)$ respectively.

(i) Find an orthogonal matrix Q and a diagonal matrix D such that

$$D = Q^T A Q$$

(ii) Consider the conic given by the equation

$$11x^2 - 4xy + 14y^2 = 5$$

Use your answer to part (i) to find a simplified equation for the conic. Hence identify the conic and write down direction vectors for its principal axes.

[12 marks]

— END OF EXAMINATION QUESTIONS —