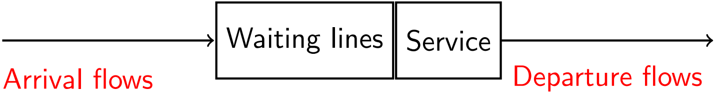
# Queueing systems

# Introduction

Queueing theory is the mathematical study of the operation of stochastic systems describing processing of flows of jobs. Queues occur when current demand for service exceeds the capacity of the service facility.



# Standard setup for arrivals

- We use the terminology 'customers', but they could be telephone calls, computer jobs, information packets, etc.
- Arrival times $T_1,\ T_2,\ T_3,\ \cdots$. The inter-arrival times are $\tau_1 = T_1 - T_0, \tau_2 = T_2 - T_1, \tau_3 = T_3 - T_2 \dots$
- The inter-arrival times are assumed to be i.i.d.
- Alternatively, we could use a counting process $N_t$ giving the number of arrivals in $[0, t]$, $t \geq 0$.

# Standard setup for service

- There is a total of $m$ spaces for both receiving service and waiting for it.
- If there is an idle server, service commences for an arriving customer immediately.
- The service time $S_i^{(j)}$ of the $i$th customer at the $j$th server is a random variable.
- The service times are assumed to be independent (and also identically distributed for each fixed $j$
- When a server is serving a customer, it cannot provide any service to other customers.
- If all servers are busy, then the arriving customers join a queue if there is enough space, otherwise, the customer is rejected.

# Service Disciplines

This could be

- FIFO: First In - First Out (FCFS: First Come - First Served).
- Last Come - First Served (with or without pre-emption).
- Processor Sharing.
- Priority (with or without pre-emption).
- more complicated disciplines?

We will consider only FIFO in this course.
One can use such queueing systems to construct queueing networks by forwarding customers departing from one queue to other queues.

# Quantities of interest

$X_t$ is the number of customers in the system at time $t$ (including those in service and those waiting to begin service).

waiting time: length of time a customer spends in the queue before her/his service commences.

sojourn time: total length of time a customer spends in the system (waiting time plus service time).

# Kendall's notation

This was devised by David Kendall in 1953. It takes the form $A/B/n/m$ where

- $A$ describes the arrival process
  - $A = M$ (Markov) inter-arrival times are exponentially-distributed.
  - $A = GI$ or $(G)$ inter-arrival times have some arbitrary distribution.
  - $A = D$ inter-arrival times are deterministic.

# Kendall's notation

- $B$ describes the service process
  - $B = M$ service times are exponentially-distributed.
  - $B = GI$ or $(G)$ service times have some arbitrary distribution.
  - $B = D$ service times are deterministic.
- $n$ gives the number of servers.
- $m$ gives the capacity of the system. When $m = \infty$, this is usually omitted.

Most common is $M/M/1/\infty$ (or just $M/M/1$).

# Questions

- Does a queueing system have a steady-state regime or does the queue increase unboundedly?
- What is the steady-state queue length distribution if it exists?
- What is the steady-state waiting time distribution if it exists?
- What fraction of time is the server idle?

# $M/M/1$ queue

- Arrival stream: Poisson process with intensity $\lambda$
- Service: $n = 1$ server, service times $\sim \exp(\mu)$
- Infinite space for waiting: $m = \infty$
- The state $X_t$ gives the number of customers at time $t$:
  - If $X_t = 0$ the server is idle.
  - If $X_t = k \geq 1$ one customer is being served and $k - 1$ customers are waiting in the queue.

This is a CTMC (in fact a birth and death process) with non-zero transition rates $q_{i,i+1} = \lambda$ and $q_{i+1,i} = \mu$ for all $i \in \mathbb{Z}_+$.

Exercise: draw the transition diagram for this CTMC

# $M/M/1$ an interpretation

If $X_t = 0$, the process remains at 0 for an $\exp(\lambda)$ time $\tau_+$ until a new customer arrives, so $X_{t+\tau_+} = 1$.
If $X_t = k > 0$, the process remains at $k$ for a time $\tau = \min(\tau_+, \tau_-)$ where

- $\tau_+ \sim \exp(\lambda)$ is the time until the next arrival after $t$
- $\tau_- \sim \exp(\mu)$ is the time until the end of service of the customer in service at $t$.

$X_{t+\tau} = k + 1$ if $\tau_+ < \tau_-$ and $X_{t+\tau} = k - 1$ if $\tau_+ > \tau_-$.

# $M/M/1$ stationary distribution

Using our results from CTMCs, we see that a stationary distribution for $(X_t)_{t\geq 0}$ exists if (and only if) the chain is positive recurrent. This is equivalent to $\rho \equiv \lambda/\mu < 1$, in which case, for $n \in \mathbb{Z}_+$,

$$\pi_n = (\lambda/\mu)^n \pi_0.$$

Using the normalisation condition $\sum_{i=0}^{\infty} \pi_n = 1$, we see that

$$\pi_0 \sum_{i=0}^{\infty} (\lambda/\mu)^i = 1$$

which tells us that

$$\pi_0 = 1 - \rho$$

and, for $n \geq 0$,

$$\pi_n = (1-\rho)\rho^n.$$

So the stationary distribution for the number of customers in the system is geometric*$(1-\rho)$. (Note that this geometric takes values in $\mathbb{Z}_+$).

# $M/M/1$ further questions

- What is the stationary expected number $\ell$ of customers in the system?
- What is the stationary expected number $\ell_q$ of customers in just the queue?
- What is the expected waiting time of a customer in stationarity?
- What is the distribution of the waiting time?

We might guess that $\ell_q = \ell - 1$. However this is not right because the system might be empty.

# $M/M/1$ waiting times in stationarity

- In the stationary regime, a tagged arriving customer will find a random number $N$ of customers where $N \sim (\pi_k)_{k\in\mathbb{Z}_+}$ (PASTA). ((Careful when interpreting this statement, e.g. it's not true for the first customer to arrive after time $t$))
- If $N = 0$, then the customer will go straight into service.
- If $N > 0$, the remaining service time $S_1$ for the customer being served $\sim$ exp($\mu$).
- The service times $S_2, S_3, \ldots, S_N$, for those in the queue are independent exp($\mu$) random variables, also independent of $N$.
- So the waiting time for our tagged customer is $W = \sum_{j=1}^{N} S_j$, where we interpret the empty sum as equal to 0.

# $M/M/1$ waiting times in stationarity

The distribution of a non-negative random variable $Y$ is characterized by its Laplace transform $M_Y(-s) = \mathbb{E}[e^{-sY}]$ for $s > 0$.
We can write

$$\begin{aligned}
\mathbb{E}_\pi[e^{-sW}] &= \mathbb{E}_\pi[\mathbb{E}_\pi[e^{-sW}|N]] = \mathbb{E}_\pi[\mathbb{E}_\pi[e^{-s\sum_{j=1}^{N} S_j}|N]] \\
&= \mathbb{E}_\pi[(\mathbb{E}_\pi[e^{-sS_1}])^N] \\
&= \mathbb{E}_\pi\left[\left(\frac{\mu}{s+\mu}\right)^N\right] = M_N(\log(\mu/(s+\mu))) \\
&= (1-\rho)\sum_{n=0}^{\infty}\rho^n\left(\frac{\mu}{s+\mu}\right)^n \\
&= \frac{(1-\rho)(s+\mu)}{s+\mu-\lambda} \\
&= (1-\rho) + \rho\frac{\mu-\lambda}{s+\mu-\lambda},
\end{aligned}$$

and we see that the distribution of $W$ is a mixture of a 0-random variable and an exponential$(\mu - \lambda)$ random variable. To be precise,

$$\mathbb{P}(W = 0) = 1 - \rho, \qquad \mathbb{P}(W > x) = \rho e^{-x(\mu-\lambda)}, \quad \text{for } x > 0.$$

# $M/M/1$ waiting times in stationarity

It follows that the expected waiting time is

$$\mathbb{E}[W] = \frac{\rho}{\mu - \lambda}.$$

Once we have the expected waiting time, we can calculate the expected total time $d$ in the system via the formula

$$d = \mathbb{E}[W] + \frac{1}{\mu} = \frac{1}{\mu - \lambda}.$$

# Little's law:

Recall that $\ell$ is the expected number of customers in the system, while $\ell_q$ is the expected number of customers waiting for service (both at stationarity).

Little's law says that

$$\ell = \lambda d,$$

and

$$\ell_q = \lambda \mathbb{E}[W].$$

# Sketch proof of Little's law

Let $N_t$ and $D_t$ denote the number of customers who have *entered* and *departed* from the system in $[0, t]$ respectively. So the number in the system at time $t$ is $X_t = N_t - D_t$. Denoting the area under the function $X_s$ for $s \leq t$ by $A_t$, we calculate $\mathbb{E}[A_t/t]$ in two different ways. First

$$\mathbb{E}\left[\frac{A_t}{t}\right] = \mathbb{E}\left[\frac{1}{t}\int_0^t X_u du\right]$$

which approaches the average number $\ell$ in the system as $t \to \infty$.

# Sketch proof of Little's law

Second, we have,

$$\frac{A_t}{t} = \frac{1}{t}\sum_{n=1}^{N_t} D_n + o(1)$$

where $D_i$ is the time spent in the system by the $i$th customer.

Now

$$\begin{aligned}\mathbb{E}\left[\frac{A_t}{t}\right] &= \frac{1}{t}\mathbb{E}[\sum_{n=1}^{N_t} D_n] + o(1) \\ &= \frac{1}{t}\lambda t d + o(1) \\ &= \lambda d + o(1)\end{aligned}$$

So taking $t$ large we have $\ell = \lambda d$.

# Example

A repairperson is assigned to service a bank of machines in a shop. Assume that failure times occur according to a Poisson process with rate $\lambda = 1/12$ per minute and the repair rate is $\mu = 1/8$ per minute.

# Example

- The traffic intensity is $\rho = 2/3 < 1$, so a stationary distribution exists.
- For $k \geq 0$, the stationary distribution is $\pi_k = (1-\rho)\rho^k$.
- The repairperson is idle with prob $1 - \rho = 1/3$.
- The expected number of machines requiring repair is $\ell = \rho/(1-\rho) = 2$.
- The expected time that a machine spends with the repair person is $d = \ell/\lambda$ $(\text{or } 1/(\mu - \lambda)) = 24$ minutes.
- The expected time waiting for repair is $\mathbb{E}[W] = d - 1/\mu = 16$ minutes.
- Also, e.g. $\mathbb{P}(W > 10) = \rho e^{-(\mu-\lambda)10} \approx 0.44$.

# Example

- Suppose that the failure rate of machines increases (e.g. due to aging) by 16% to $\lambda' = 1/10$, then the new traffic intensity is $\rho' = 4/5$, and $\ell' = 4$ with $d' = 40$ and $\mathbb{E}[W'] = 32$.
- A 16% increase in arrival rate has drastically increased the expected number of failed machines and doubled the time that they have to wait before getting repaired.
- We see that, when $\rho$ is close to 1, the effect of small changes of $\rho$ is profound: if a queueing system has long waiting times and lines, a rather modest increase in the service rate can bring about a dramatic reduction in waiting times.

# Costs example

- Suppose there is a new piece of equipment that will increase the repair rate from $\mu = 1/8$ to $\mu^* = 1/6$, that is, decrease the expected repair time from 8 minutes to 6 minutes.
- The increase in maintenance cost for the new equipment is $c_M = \$6$ per minute.
- The cost of lost production when a machine is out of order is $c_D = \$5$ per minute.
- Should we purchase the new equipment?

# Costs example solution

Without the new equipment

- The expected number of failed machines is $\ell = \rho/(1-\rho) = 2$.
- The expected cost of lost production is $\ell c_D = \$10$ per minute.
- With the new equipment, $\rho^* = 0.5$, and the expected cost is $\ell^* c_D + c_M = \$11$ per minute.
- We should buy the equipment if $\ell^* c_D + c_M < \ell c_D$, so we should not buy the equipment.

# Another costs example

At a service station the rate of service is $\mu$ cars per hour, and the rate of arrivals of cars is $\lambda$ per hour. The cost incurred by the service station due to delaying cars is \$$c_1$ per car per hour and the operating and service costs are \$$\mu c_2$ per hour. The rate of service $\mu$ is a control parameter. Determine the value of $\mu$ so that the least expected cost is achieved and find the value of the latter.

# Another costs example solution

- If there are $Y$ cars in the service station, the cost is \$$c_1 Y + \mu c_2$ per hour.
- In the stationary regime, $\mathbb{E}[Y] = \rho/(1-\rho)$.
- The expected total cost per hour is \$$c(\mu) = c_1\rho/(1-\rho) + \mu c_2$
- To find the minimum, we find $\mu$ such that $c'(\mu) = 0$ and since $\mu > \lambda$, we have a solution $\mu_0 = \lambda + \sqrt{\lambda c_1/c_2}$.
- We can check that $\mu_0$ achieves the minimum and $c(\mu_0) = \lambda c_2 + 2\sqrt{\lambda c_1 c_2}$.

# $M/M/a$ Queue

This system has the following properties

- $a \geq 1$ servers,
- a Poisson arrival process with rate $\lambda$,
- a FIFO service discipline,
- independent $\exp(\mu)$ service times,
- when an arrival finds more than one idle server, it chooses one at random,
- when $k$ servers are working, the total service rate is $k\mu$.

# $M/M/a$ Queue

The transition rates are $q_{i,i+1} = \lambda$, for $i \geq 0$ and $q_{i,i-1} = \mu \times \min(a, i)$ for $i \geq 1$.

Exercise: draw the transition diagram

This is a birth-and-death process with $\nu_i = \lambda$ for $i = 0, 1, 2, \ldots$ and $\mu_i = i\mu$ for $i = 1, 2, \ldots, a$ and $\mu_i = a\mu$ for $i > a$.

# $M/M/a$ ergodicity

$$\kappa_j \equiv \frac{\nu_0 \cdots \nu_{j-1}}{\mu_1 \cdots \mu_j} = \begin{cases} (\lambda/\mu)^j/j! & \text{if } j \leq a \\ \frac{(\lambda/\mu)^j}{a! a^{j-a}} & \text{if } j > a. \end{cases}$$

We know that

$$\sum_{j=0}^{\infty} \kappa_j < \infty \Longleftrightarrow \sum_{j=a}^{\infty} \kappa_j < \infty.$$

This occurs if $\lambda < a\mu$, in which case

$$\sum_{j=0}^{\infty} \kappa_j = \sum_{k=0}^{a-1} \frac{\lambda^k}{k! \mu^k} + \frac{\lambda^a}{a! \mu^a} \frac{a\mu}{a\mu - \lambda}.$$

# $M/M/a$ ergodicity

So the $M/M/a$ queue is ergodic if and only if arrival rate $\lambda$ is less than the maximum service rate $a\mu$.

In this case, the stationary distribution is given by

$$\pi_k = \begin{cases} \pi_0(\lambda/\mu)^k/k! & \text{if } k < a \\ \pi_0(\lambda/\mu)^k/(a!a^{k-a}) & \text{if } k \geq a, \end{cases}$$

where

$$\pi_0 = \left( \sum_{j=0}^{\infty} \kappa_j \right)^{-1}.$$

# $M/M/a$ busy servers

For what proportion of time $\delta_q$ are all the servers busy? This is the same as the probability that an arriving customer will have to wait (recall the PASTA principle).
We have

$$\delta_q = \sum_{k=a}^{\infty} \pi_k = \pi_0 \frac{\lambda^a}{\mu^a a!} \frac{a\mu}{a\mu - \lambda}.$$

The expected queue length is

$$\ell_q = \mathbb{E}[\max(X_t - a, 0)] = \frac{\lambda}{a\mu - \lambda} \delta_q.$$

# $M/M/a$ busy servers

In stationarity, the expected number $b_s$ of busy servers is

$$\mathbb{E}_\pi[\min(X_t, a)] = \frac{\lambda}{\mu}.$$

Note that, provided that $\lambda < a\mu$, this does not depend on $a$. The expected number of customers in the system is

$$\ell = b_s + \ell_q = \frac{\lambda}{\mu} + \frac{\lambda}{a\mu - \lambda}\delta_q$$

# $M/M/a$ waiting times

By Little's Law, the expected waiting time is

$$\mathbb{E}_\pi[W] = \frac{\ell_q}{\lambda} = \frac{\delta_q}{a\mu - \lambda}.$$

(can also get the above directly from $\pi$)
The expected delay is

$$d = \mathbb{E}_\pi[W] + \frac{1}{\mu} = \frac{\delta_q}{a\mu - \lambda} + \frac{1}{\mu}.$$

# $M/M/a$ Example

An insurance company has 3 claim adjusters in its branch office. Claims against the company arrive according to a Poisson process at an average rate of 20 per 8 hour day. The amount of time an adjuster spends with a claimant is exponentially-distributed with mean service time 40 minutes.

- How many hours a week can an adjuster expect to spend with claimants?
- How much time, on average, does a claimant spend in the branch office?

# $M/M/a$ example solution

- The arrival rate is $\lambda = 20/8 = 2.5$ per hour.
- The service rate is $\mu = 1.5$ per hour.
- $\lambda/(a\mu) = 5/9 < 1$, so a stationary distribution exists.
- We get $\mathbb{P}_\pi$(adjuster is busy) by noticing that

$$\mathbb{E}_\pi[\text{number of busy adjusters}] = \sum_{i=1}^{3} \mathbb{E}_\pi[\mathbb{1}_{\{i\text{th adjuster is busy}\}}]$$

So, by symmetry,

$$\begin{aligned} \mathbb{E}_\pi[\text{number of busy adjusters}] &= 3\mathbb{E}_\pi[\mathbb{1}_{\{\text{a given adjuster is busy}\}}] \\ &= 3\mathbb{P}_\pi(\text{a given adjuster is busy}). \end{aligned}$$

# $M/M/a$ example solution

Substituting the parameter values, we calculate that each adjuster spends 22.2 hours a week on claims and that $\pi_0 = 24/139$, $\delta_q = 125/417$ and $d = 0.817$ hours (which corresponds to 49 minutes).

If there were only two adjusters, we could similarly calculate

- An adjuster will spend on average 33.3 hours with claimants.
- We can calculate that $\pi_0 = 1/11$, $\delta_q = 25/33$ and $d = 2.18$ hours.

We can use this information to quantify the trade-off between the cost of an extra adjuster and the extra level of service that is produced.

# Single or multiple servers?

Which is better? A single fast server or several smaller ones with the same "cumulative service rate"? Assume that the arrival process is Poisson with rate $\lambda$, and compare:

- A single server with service rate $a\mu$, and
- $a$ servers with service rate $\mu$ each.

A heuristic argument tells us that

- if $X_t \geq a$, both systems work with the same rate, but
- if $X_t = k < a$ the rate for the $a$ server queue is $k\mu$, which is less than the rate $a\mu$ for the single server.

So we might conclude that the single server is better. This is "easy" to prove via the technique of *coupling*.

# Single or multiple servers?

We saw that, for the $M/M/a$ queue, the expected number in the system is

$$= \frac{\lambda}{\mu} + \frac{\lambda}{a\mu - \lambda}\delta_q$$

and the expected time in the system is

$$= \frac{1}{\mu} + \frac{1}{a\mu - \lambda}\delta_q.$$

For the $M/M/1$ queue with service rate $a\mu$, the expected number in the system is

$$= \frac{\lambda}{a\mu - \lambda}$$

and the expected time in the system is

$$= \frac{1}{a\mu - \lambda}.$$

# Single or multiple servers?

With some work, we can show that $\delta_q + (a\mu - \lambda)/\mu > 1$, so both the expected number in the system and the expected time in the system are smaller for the $M/M/1$ queue, which proves our conjecture.

As an exercise, think about the waiting time, rather than the time in the system, for each of the systems.

# What if our queue is not Markovian??

Then we need renewal theory!