Question 1

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = -x^3 + 6x^2 + 10$

a) • The function is unimodal in $[-1,1]$

the minimum of the function occurs when $f'(x) = -3x^2 + 12x = 0$

$\Rightarrow 3x(-x+4) = 0$

$x = 0$ $x = 4$

therefore, there's only one minimum ~~in~~ $[-1,1] \Rightarrow$

the function is unimodal

• the derivative is increasing in $[-1,1]$

$f'(x) = -3x^2 + 12x$

$f'' = 12 - 6x \geq 0$ in $[-1,1] \Rightarrow$ f' is increasing in the interval

b)

$\frac{2}{F_n} < 2\varepsilon$

$F_n > \frac{1}{\varepsilon}$

$F_n > 20 \Rightarrow$

1	1	2	3	5	8	13	21
0	1	2	3	4	5	6	7

$n = 7$

c) To find the minimum of f we look for the zero of f'. We saw in a) that f' is increasing in $[-1,1]$, and we also saw that it crosses the x-axis. Therefore, it is possible to find the minimum of f using the method of the false position in f'.

$k=1$	$f'(-1) = -15$ $f'(1) = 9$	$p = -1 + \frac{2\cdot(-15)}{-15-9} = 1/4$	$g(p) = 2.81$, New interval $[-1 \quad 1/4]$

d) In the first iteration we will need f' and f'', and f'' must be $\neq 0$, which is not the case. So it is not possible to start the Newton method with $x_0 = 2$

Q2

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x_1, x_2) = x_1^3 + 6x_1^2 + 2x_2^3 - 6x_2^2 + 30$

a) first order conditons :

$$\nabla f(x_1, x_2) = 0 = \begin{bmatrix} 3x_1^2 + 12x_1 \\ 6x_2^2 - 12x_2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 0 \text{ or } -4 \\ x_2 = 0 \text{ or } 2 \end{matrix}$$

stationary points

$$\underset{P_1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \quad \underset{P_2}{\begin{pmatrix} 0 \\ 2 \end{pmatrix}} \quad \underset{P_3}{\begin{pmatrix} -4 \\ 0 \end{pmatrix}} \text{ and } \underset{P_4}{\begin{pmatrix} -4 \\ 2 \end{pmatrix}}$$

b) $$\nabla^2 f = \begin{bmatrix} 6x_1 + 12 & 0 \\ 0 & 12x_2 - 12 \end{bmatrix}$$

for P_1 $\quad \nabla^2 f = \begin{bmatrix} 12 & 0 \\ 0 & -12 \end{bmatrix}$ $\quad P_1$ saddle

for P_2 $\quad \nabla^2 f = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} > 0$ $\quad P_2$ is <u>min</u>

for P_3 $\quad \nabla^2 f = \begin{bmatrix} -12 & 0 \\ 0 & -12 \end{bmatrix} < 0$ $\quad P_3$ is <u>max</u>

for P_4 $\quad \nabla^2 f = \begin{bmatrix} -12 & 0 \\ 0 & 12 \end{bmatrix}$ $\quad P_4$ is saddle c)

$$P_1 \quad f(x-(0,0)) \cong 30 + \tfrac{1}{2}(x)^T \cdot \begin{bmatrix} 12 & 0 \\ 0 & -12 \end{bmatrix} x$$

$$P_4 \quad f(x-(-4,2)) \cong 54 + \tfrac{1}{2}(x-(-4,2))^T \begin{bmatrix} -12 & 0 \\ 0 & 12 \end{bmatrix} (x-(-4,2))$$

d)

for P_1 (P_2) $d = (1,0)$ increases (decreases) the value

for P_2 (P_1) $d = (0,1)$ increases (decreases) the value

Q3

$$f(x) = 2x_1^2 + 2x_2^2 + 2x_1x_2 - 6x_1 + 2x_3^2$$

a)

$$\nabla f(x) = \begin{bmatrix} 4x_1 + 2x_2 - 6 \\ 4x_2 + 2x_1 \\ 4x_3^2 \end{bmatrix} \quad \text{at } x_0=(1,1,0) \quad \nabla f(x_0) = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

$$X_1 = X_0 - t \cdot \nabla f(x_0)$$

$$X_1 = (1\ 1\ 0) + (0\ -6t\ 0)$$

$$X_1 = (1\ \ 1-6t\ \ 0)$$

$$f(x_1) = 4 + 2(1-6t)^2 + 2(1-6t) - 6$$

it's a single variable function on t

$$f(t) = 2(1-6t)^2 + 2(1-6t) - 2$$

it's stattionary point(s) are given when $f'(t) = 0$

$$f'(t) = -4 \cdot 6 \cdot (1-6t) - 12$$

$$\Rightarrow \quad t = 1/4$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1/4 \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix}$$

$$\nabla f(x_1) = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

b) $90°$.

c) $$\nabla^2 f(x) = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 8x_3^2 \end{bmatrix}$$

at $x_0 = (1,1,0)$

$$\nabla f(x_0) = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the matrix is definite positive, so we can compute the Newton direction.

$$d = -\left(\nabla^2 f(x_1)\right)^{-1} \nabla f(x_1) =$$

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

d) i) Same as in a)

$$\begin{bmatrix} 0 \\ -6 \\ 0 \end{bmatrix}$$

ii) Same as in b

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 6 \end{bmatrix}$$

4/

$$f(x_1, x_2) = -x_1 - x_2$$

$$\text{s.t.} \quad x_1^2 + x_2^2 \leq 4$$

$$x_2 \leq 1$$

a/

$$\mathcal{L}(x, \lambda) = -x_1 - x_2 + \lambda_1 (x_1^2 + x_2^2 - 4) + \lambda_2 (x_2 - 1)$$

b/

KKTa:

$$\nabla \mathcal{L}_x(x,\lambda) = 0 = \begin{bmatrix} -1 + 2x_1\lambda_1 \\ -1 + 2x_2\lambda_1 + \lambda_2 \end{bmatrix}$$

KKTb

$$\lambda_1, \lambda_2 \geq 0$$

$$x_1^2 + x_2^2 - 4 \leq 0$$

$$x_2 - 1 \leq 0$$

$$\lambda_1 (x_1^2 + x_2^2 - 4) = 0$$

$$\lambda_2 (x_2 - 1) = 0$$

4 cases

) $\lambda_1 = 0, \ \lambda_2 = 0 \Rightarrow$ KKTa fails $-1 = 0$

) $\lambda_1 = 0, \ \lambda_2 > 0 \Rightarrow$ KKTa fails

) $\lambda_1 > 0, \ \lambda_2 = 0$

$\downarrow$

from KKTa $\rightarrow x_2 = 1/2\lambda_1$

from KKTa $\rightarrow x_1 = 1/2\lambda_1$

$\Rightarrow x_2 = x_1$

from KKTb

$x_1^2 + x_2^2 - 4 = 0$

$2x_1^2 = 4$

$x_1 = \pm\sqrt{2} = x_2$

two candidates

$\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$ $\begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$

violates KKTb

$x_2 < 1$

$\sqrt{2} < 1$ violation

contradiction with hypothesis that $\lambda_1 > 0$

$-1 - 2\sqrt{2}\lambda_1 = 0$

$\lambda_1 = -1/2\sqrt{2}$

4) $\lambda_1 > 0, \ \lambda_2 > 0$

KKTb: $x_2 = 1$

$x_1^2 + 1 = 4$

$x_1 = \pm\sqrt{3}$

$\Longrightarrow$ candidates

$\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ $\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$

from KKTa violates hypothesis that $\lambda_1 > 0$

KKTa $-1 + 2\sqrt{3}\lambda_1 = 0$

$\Rightarrow \lambda_1 = 1/2\sqrt{3}$

KKTb $\lambda_2 = 1 - 2 \cdot 1 \cdot \frac{1}{2\sqrt{3}} = 1 - \frac{1}{\sqrt{3}} = \frac{3-\sqrt{3}}{3}$

$$\boxed{\begin{array}{l} x_1 = \sqrt{3} \\ x_2 = 1 \\ \lambda_1 = 1/2\sqrt{3} \\ \lambda_2 = 3-\sqrt{3}/3 \end{array}}$$

c)

The set of all active gradients are active

$$\begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \text{for} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \Big/ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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we can find $d = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ for which

$$\nabla g_1(x^*)^T \cdot d < 0$$

$$\nabla g_2(x^*)^T \cdot d < 0$$

d)

Critical cone

$$\begin{bmatrix} 2\sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \qquad \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0$$

$$\Rightarrow d_2 = 0$$

$$\Rightarrow d_1 = 0$$

the critical cone is $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ the hessian is definite positive in the critical cone $- \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ = emptyset

the hessian is .

Moreover, the hessian is always > 0 for $\lambda_1, \lambda_2 \geq 0$

Q5

a) $$P_\alpha(x) = x_1^2 - x_2 + \frac{\alpha}{2}\left((1-x_1)_+^2 + (x_2-3)_+^2 \right)$$

b)

$$\nabla_x P_\alpha(x) = 0 = \begin{bmatrix} 2x_1 - \alpha \cdot (1-x_1)_+ \\ -1 + \alpha \cdot (x_2-3)_+ \end{bmatrix}$$

c)
we can treat x_1 and x_2 independently in this

case :

* if $x_1 \geq 1 \Rightarrow (1-x_1)_+ = 0 \Rightarrow 2x_1 = 0$ (contradiction)

$x_1 < 1 \Rightarrow (1-x_1)_+ = (1-x_1) \Rightarrow 2x_1 - \alpha(1-x_1) = 0$

$(2+\alpha)x_1 = \alpha$

$\boxed{x_1 = \alpha/2+\alpha}$

<1 ✓

* if $x_2 \leq 3 \Rightarrow (x_2-3)_+ = 0 \Rightarrow -1 = 0$ (contradiction)

$x_2 > 3 \Rightarrow (x_2-3)_+ = (x_2-3) \Rightarrow \alpha(x_2-3) = 1$

$\boxed{x_2 = \frac{1+3\alpha}{\alpha}}$

>3 ✓

d)

$$x_1^* = \lim_{\alpha \to \infty} \frac{\alpha}{2+\alpha} = \boxed{1 = x_1^*}$$

$$x_2^* = \lim_{\alpha \to \infty} \frac{1+3\alpha}{\alpha} = \boxed{3 = x_2^*}$$

$$\lambda_1^* = \lim_{\alpha \to \infty} \alpha\left(1 - \frac{\alpha}{2+\alpha}\right)$$

$$= \lim_{\alpha \to \infty} \alpha\left(\frac{2}{2+\alpha}\right)$$

$$= \lim_{\alpha \to \infty} \left(\frac{2\alpha}{2+\alpha}\right) = \boxed{2 = \lambda_2^*}$$

$$\lambda_2^* = \lim_{\alpha \to \infty} \alpha\left(\frac{1+3\alpha}{\alpha} - 3\right)$$

$$= \lim_{\alpha \to \infty} \alpha\left(\frac{1+3\alpha-3\alpha}{\alpha}\right) = \boxed{1 = \lambda_2^*}$$

Q6

$$f(x) = x_1^2 + x_2^2 + 2x_2 - 3$$

s.t.

$$x_1^2 + x_2^2 \leq 1$$

$$x_1 + x_2 = 1$$

a) the objective function is convex:

$$\nabla f(x) = \begin{bmatrix} 2x_1 \\ 2x_2 + 2 \end{bmatrix} \Rightarrow \nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} > 0$$

the inequality constraint is convex:

$$\nabla g(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \Rightarrow \nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} > 0$$

the equality constraint is clearly affine.

b) $$f_0(x,\lambda) = x_1^2 + x_2^2 + 2x_2 - 3 + \lambda \cdot (x_1^2 + x_2^2 - 1) + \eta(x_1 + x_2 - 1)$$

c) Saddle inequalities:

$$\mathcal{L}(x^*, \lambda, \eta) \leq \mathcal{L}(x^*, \lambda^*, \eta^*) \leq \mathcal{L}(x, \lambda^*, \eta^*)$$

(1) (2) (3)

(1) $\mathcal{L}(x^*, \lambda, \eta) = 1 - 3 + \lambda(0) + \eta \cdot (0)$

$\mathcal{L}(x^*, \lambda, \eta) = -2$

which is independent of η and λ, thus

$\mathcal{L}(x^*, \lambda, \eta) = -2 = \mathcal{L}(x^*, \lambda^*, \eta^*)$ and

therefore $\boxed{\mathcal{L}(x^*, \lambda, \eta) \leq \mathcal{L}(x^*, \lambda^*, \eta^*)}$

(3) $\mathcal{L}(x, \lambda^*, \eta^*) = x_1^2 + x_2^2 + 2x_2 - 3 - 2(x_1 + x_2 - 1)$

$= x_1^2 + x_2^2 - 2x_1 - 1$

which is minimised at:

$$\nabla f(x) = \begin{bmatrix} 2x_1 - 2 \\ 2x_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} x_1^* = 1 \\ x_2^* = 0 \end{matrix}$$

thus $\mathcal{L}(x^*, \lambda^*, \eta^*) \leq \mathcal{L}(x, \lambda^*, \eta^*)$

d) the problem is convex

e)

$$\max_{x,\lambda,\eta} \quad x_1^2 + x_2^2 + 2x_2 - 3 + \lambda(x_1^2 + x_2^2 - 1) + \eta(x_1 + x_2 - 1)$$

s.t.

$$\lambda \geq 0$$

$$2x_1 + 2x_1\lambda + \eta = 0$$

$$2x_2 + 2 + 2x_2\lambda + \eta = 0$$