MDS -La Trobe MAT5OPT Assignment 1
Semester 1, 2024

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1 Wordly scenario

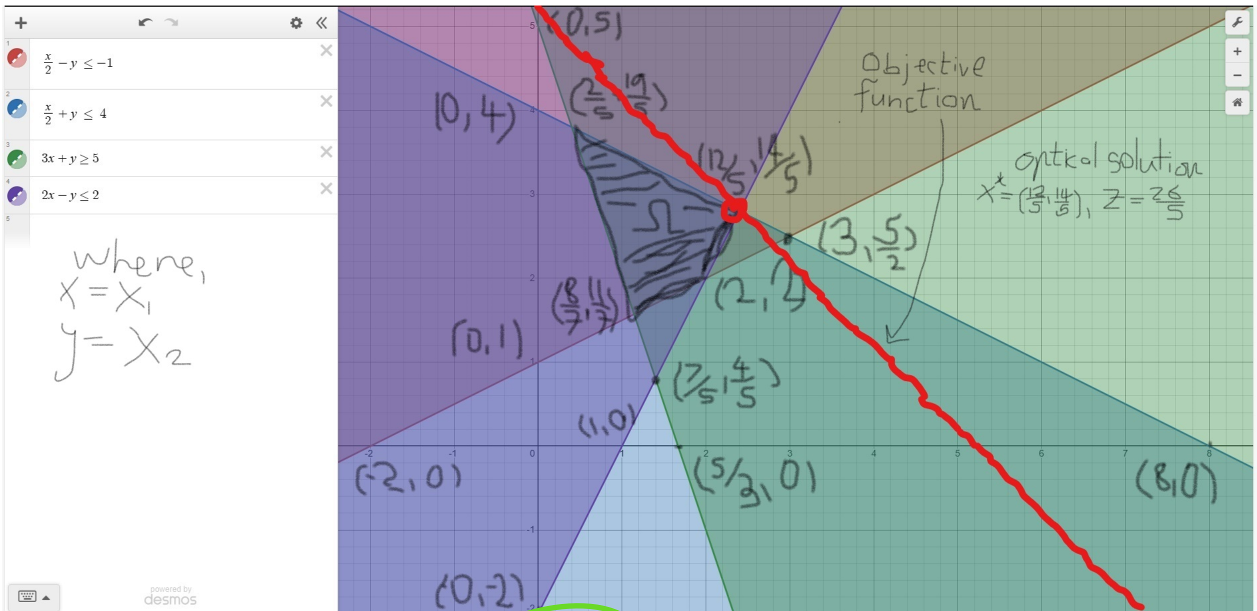
We wanted to maximise the optimisation problem to keep up with the demand.

Let's denote:

x_1*: Own production*
x_2*: Cheap Wine*
x_3*: Expensive Wine*
~~x_2: *Cheap Wine*~~
$max\ z = 500x_1 + 400x_2 + 900x_3$ ✓
$1250x_1 + 2000x_2 + 800x_3 \geq 2750$ ✗
$80x_1 + 200x_2 + 60x_3 \leq 100(x_1+x_2+x_3)$ ✗
$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$ ✓

0.8

2 Visual method.



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3 MATLAB functions manipulating arrays.

```
%Part a.
% This produces an error which has incorrect dimensions for raising a
% matrix. Since the operation is reserved for matrix powers.
%[1 2 3]^2 ;

%. This allows to square on each element of an array indidually.
%[1 2 3].^2;

%. Adding one as a constant to a vector adds one for each element of the
%. vector. %
%[1 2 3] + 1;


%Part b.
X = zeros(1, 9);  % Preallocate array X
for j = 1:9
    X(j) = 5 + 3*j;
end

%Part c.
%See F.m file
function output_array= F(input_array)
odd = input_array(1:2:end);
even = input_array(2:2:end).^2 + 1;

%This works for input_array lengths are even.
if mod(length(input_array),2) == 0
     new_array = horzcat(odd,even);
     new_array = reshape(new_array,length(input_array)/2,2);
     new_array = transpose(new_array);
     output_array = reshape(new_array,1,length(input_array));


%This works for input_array lengths are odd.
else
    new_array = horzcat(odd,even,NaN);
    new_array = reshape(new_array,(length(input_array)+1)/2,2);
    new_array = transpose(new_array);
    output_array = reshape(new_array,1,length(input_array)+1);
    output_array =  rmmissing(output_array);

end
```

```
%Part d.
%See D.m file
function y = D(x)
    n = length(x);  % Get the length of array x
    y = zeros(1, n-1);  % Preallocate array y with length n-1

    for i = 1:n-1
        y(i) = x(i+1) - x(i);  % Compute the first difference
    end
end

%x = [1, 3, 6, 10, 15];  % Example array
%y = D(x);  % Compute first difference
%disp(y);   % Display the result

%Part e.
F = @(x) x.^2; % Define function F, for example, squaring each element
D = @(x) [0, diff(x)]; % Define function D to compute first difference
A = @(x) sum(x(mod(x, 12) == 0));
% Anonymous function A to sum
% elements divisible by 12

X = [1, 4, 6, 12, 15, 18]; % Example array X

result = A(D(F(X))); % Apply A to D(F(X))

disp(result); % Display the result
```

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...

4 Definiteness

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Part a:

$$A(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 + x_3 & x_1 + x_3 & x_1 + x_2 + x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + x_1x_2 + x_1x_3 + x_1x_2 + x_2x_3 + x_1x_3 + x_2x_3 + x_3^2$$

$$= x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + x_3^2$$

Part b:

If $x^T = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$

Then, $A(x) = 2(1)(1) + (1)^2 = 3$

and if, $x^T = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$

Then, $A(x) = 2(1)(-1) + (-1)^2 = -1$

As the quadratic form $A(x)$ *can take values of opposite signs, the matrix* A *is indefinite.*

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Part c:

As $x_3 - x_1 + 2x_2 = 0$

We have, $x_3 = x_1 - 2x_2$ *Then,*

$$A(x) = A(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_1 - 2x_2 \end{pmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1 - 2x_2 \end{pmatrix}$$

$$A(x) = A(x_1, x_2, x_3) = \begin{pmatrix} x_1 + x_2 + x_1 - 2x_2 & x_1 + x_2 - 2x_2 & x_1 + x_2 + x_1 - 2x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1 - 2x_2 \end{pmatrix}$$

$$A(x) = A(x_1, x_2, x_3) = \begin{pmatrix} 2x_1 - x_2 & 2x_1 - 2x_2 & 2x_1 - x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_1 - 2x_2 \end{pmatrix}$$

$= 2x_1^2 - x_1x_2 + 2x_1x_2 - 2x_2^2 + (2x_1 - x_2)(x_1 - 2x_2)$

$= 2x_1^2 - x_1x_2 + 2x_1x_2 - 2x_2^2 + 2x_1^2 - 4x_1x_2 - x_2x_1 + 2x_2^2$

After simplifying we get:

$= 4x_1^2 - 4x_1x_2$

$= x^T \begin{bmatrix} 4 & -2 \\ -2 & 0 \end{bmatrix} x$

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The leading principal minors of B $:= \begin{bmatrix} 4 & -2 \\ -2 & 0 \end{bmatrix}$ **are,**

$\Delta_1 = 4$ **and** $\Delta_2 = det(B) = 4(0) - (-2)(-2) = -4$

To check definiteness. Using, Sylvester's criterion.

$\Delta_1 = 4 > 0$ *and* $\Delta_2 = det(B) = 4(0) - (-2)(-2) = -4 < 0$

Conclude it is not positive definite nor negative definite. Hence B, is indefinite and FONC implies x is neither a local minimiser nor a local maximiser.

(examples preferred)

5 Conditions for minimisers, an unconstrained problem.

Part a:

$\nabla f = \begin{bmatrix} \nabla f_{x_1} \\ \nabla f_{x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$

$= \begin{bmatrix} 3x_1^2 + 2x_2 \\ 2x_1 + 2x_2 + 1 \end{bmatrix}$

$D^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 6x_1 & 2 \\ 2 & 2 \end{bmatrix}$

Part b:

We require,

$3x_1 + 2x_2 = 0$ *(Equation 1)* 2

$2x_1 + 2x_2 + 1 = 0$ *(Equation 2)*

Equate Equation 1 and Equation 2 together.

$3x_1^2 = -2x_2$. *(Equation 3)*

Substitute Equation 3 into Equation 2:

$2x_1 - 3x_2^2 + 1 = 0$ *(Equation 4)*

Rearrange Equation 4 to form a quadratic equation.

$3x_1^2 - 2x_1 - 1 = 0$

$(3x_1 + 1)(x_1 - 1) = 0$

Solving x_1 gives us

$x_1 = \frac{-1}{3}, x_1 = 1$

Substitute $x_1 = \frac{-1}{3}$ into Equation 3 gives us.

$3(\frac{-1}{3})^2 = -2x_2$

$3\frac{1}{9} = -2x_2$

$\frac{1}{3} = -2x_2$

$x_2 = \frac{-1}{6}$

Substitute $x_1 = 1$ into Equation 3 gives us.

$3(1)^2 = -2x_2$

$3 = -2x_2$

$x_2 = \frac{-3}{2}$

Hence our solutions are:

$x^* = \begin{pmatrix} 1 & \frac{-3}{2} \end{pmatrix}^T$

and

$x^* = \begin{pmatrix} \frac{-1}{3} & \frac{-1}{6} \end{pmatrix}^T$

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Part c:

For $p^T = \begin{pmatrix} p_1 & p_2 \end{pmatrix}$, the Hessian is,

For $D^2 f(x) = \begin{pmatrix} 6p_1 & 2 \\ 2 & 2 \end{pmatrix}$, and the leading principal minors are, $\Delta_1 = 6p_1$ and $\Delta_2 = det(D^2 f(x)) = 6p_1(2) - (2)(2) = 12p_1 - 4$.

By Sylvester's criterion

-If $\Delta_1, \Delta_2 > 0$, then $D^2 f(x)$ is positive definite, and the SOSC implies that x is a local minimiser.

-If $\Delta_1 < 0, \Delta_2 > 0$, then $D^2 f(x)$ is negative definite, and the SOSC implies that x is a local maximiser.

-If $\Delta_1, \Delta_2 > 0$, then $D^2 f(x)$ is indefinite, and the SONC implies that x is neither a local minimiser nor local maximiser.

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-*For* $p^T = \begin{pmatrix} \frac{-1}{3} & \frac{-1}{6} \end{pmatrix}$, *we have* $\Delta_1 = -2$ *and* $\Delta_2 = -8$

If $p^T = \begin{pmatrix} \frac{-1}{3} & \frac{-1}{6} \end{pmatrix}$ *is not an extremiser. Applying SONC if* $\Delta_1 = -2 < 0$ *and* $\Delta_2 = -8 < 0$. *This point is a SADDLE POINT.*

If $p^T = \begin{pmatrix} 1 & \frac{-3}{2} \end{pmatrix}$ *is a local minimiser. Applying SOSC if* $\Delta_1 = 6 > 0$ *and* $\Delta_2 = 8 > 0$.

Part d:

From this statement. If x^* *is a global minimiser* ($x^* = \begin{pmatrix} 1 & \frac{-3}{2} \end{pmatrix}^T$ *of* f *over* Ω, *we can write.*

$f(x^*) = min_{x\in\Omega} f(x)$ and $x^* \in \ argmin_{x\in\Omega} f(x)$

Unconstrained optimisation

min f(x) $= x_1^3 + (2x_1 + x_2 + 1)x_2$

subject to $x \in \Omega$ *Where* Ω *is a feasible set.*

Suppose, A global minimiser of f *over* Ω *is a feasible vector* x^* *for which the value of the function is the smallest possible, i.e.*

$(\forall \in \Omega)$ f(x) $\geq$ f(x^*))

for all x (x vectors is an element of a feasible set)

From answers from 5b and 5c,

$\begin{pmatrix} 1 & \frac{-3}{2} \end{pmatrix}^T$ *and*

$\begin{pmatrix} \frac{-1}{3} & \frac{-1}{6} \end{pmatrix}^T$

-$\begin{pmatrix} \frac{-1}{3} & \frac{-1}{6} \end{pmatrix}^T$ *is a local minimiser*

-$\begin{pmatrix} \frac{-1}{3} & \frac{-1}{6} \end{pmatrix}^T$ *is not an extermisier*

Then, $f(x^*) = f((1, \frac{-3}{2}))$

$= 1^3 + (2(1) - \frac{3}{2} + 1)(\frac{-3}{2})$

$= 1^3 + (3 - \frac{3}{2})(\frac{-3}{2})$

$= 1 + (\frac{3}{2})(\frac{-3}{2})$

$= 1 + \frac{-9}{4}$

$= 1 - \frac{9}{4}$

$= \frac{4}{4} - \frac{9}{4}$

$= \frac{4-9}{4}$

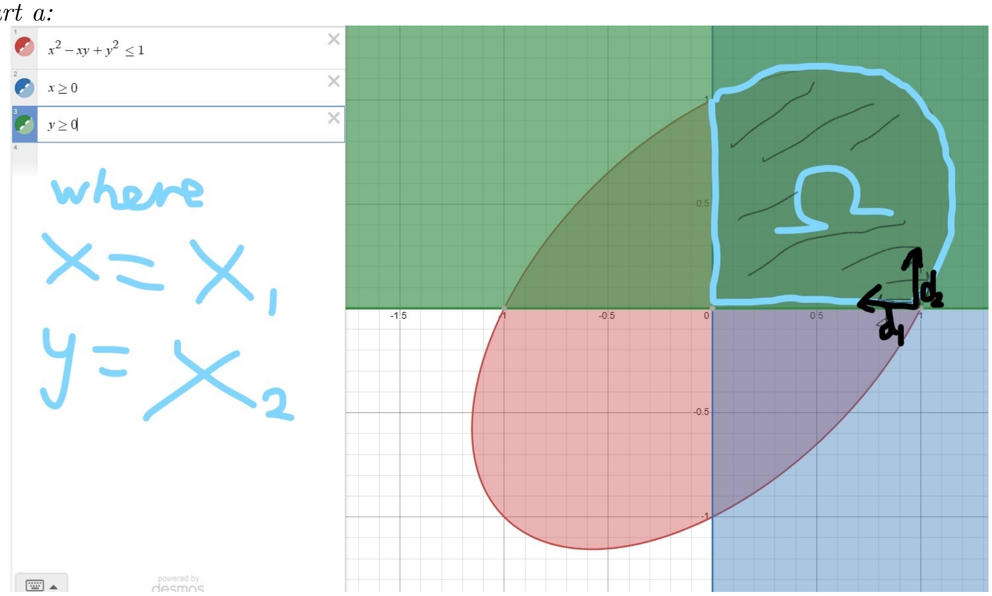
$= \frac{-5}{4}$

$= \frac{-5}{4}$, *smallest possible value.*

Conclude that, $\begin{pmatrix} 1 & \frac{-3}{2} \end{pmatrix}^T$ *is a global minimiser/extermiser since there is only one local minimiser/extermiser.*

6 Conditions for minimisers, a constrained problem.

Part a:



Part b:

The feasible directions are at $x^* = \begin{pmatrix}1 & 0\end{pmatrix}^T$ *where* $d = (d_1, d_2) \neq (0,0)$ *with* $d_1 \leq 0$ *and* $d_2 \geq 0$

Part c:

We than have,

$\nabla^T f(x)d = \begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}d_1\\ d_2\end{pmatrix} = x_1 d_1 + x_2 d_2$

Where, $\nabla^T f(x^*) = \begin{pmatrix}1 & -1\end{pmatrix}$

$\nabla^T f(x^*)d = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}d_1\\ d_2\end{pmatrix} = d_1 - d_2$

Taking $d = \begin{pmatrix}0\\ 1\end{pmatrix}$ *,gives us* $\nabla^T f(x^*)d = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = 1(0) + -1(1) = -1 < 0$. *Thus, by the FONC it is not a local minimiser.*

Part d:

After careful inspection, since we can easily assume that it is convex. But only in 2-dimensional space. If we apply the HINT using implicit differentiation gives us.

$\frac{dy}{dx} = \frac{y-2x}{2y-x}$

Instead if we look from a 3D perspective it is not easier to show at any points inside the feasible region to be compact. Hence it does not satisfy the FOSC the conjecture required.