TUTORIAL 4

1. Recall that the Simplex Algorithm generally has two phases: the first phase is concerned with obtaining a basic feasible solution and the second is concerned with finding an optimal basic feasible solution. The first phase is trivial if we have a slack variable in each constraint of our canonical form. However, if we have artificial variables then additional work is required in order to force the the artificial variables to become zero. We remove the artificial variables from the solution by performing the Simplex Algorithm with objective function $\min w = \sum_i y_i$, where the y_i are the artificial variables.

   Consider the following linear program:

$$\min z = -x_1 - x_2$$

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

   (a) Transform this problem to canonical form.

   (b) Obtain canonical form for the Phase 1 problem.

   (c) Solve Phase 1 for this problem.

   (d) Once Phase 1 is complete, you substitute the reduced costs from z into the bottom row. Do this, and explain the immediate problem with the tableau and how to fix it.

Solutions:

(a) Since the right-hand sides of all constraints are nonnegative, we do not need to multiply any constraint by -1.

Since the first constraint is a $\geq$ inequality, we need to introduce a surplus variable x_3 for it. So we obtain:

$$\min z = -x_1 - x_2$$

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 \qquad &\leq 2 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The first constraint is now an equality. So we need to introduce an artificial variable y_1 for it. The linear program now reads

$$\min z = -x_1 - x_2$$

$$\begin{aligned} x_1 + x_2 - x_3 + y_1 &= 1 \\ x_1 + x_2 \qquad\qquad &\leq 2 \\ x_1, x_2, x_3, y_1 &\geq 0. \end{aligned}$$

The second constraint is a given by a $\leq$ inequality and so it needs a slack variable x_4.

$$\min z = -x_1 - x_2$$

$$\begin{aligned} x_1 + x_2 - x_3 + y_1 \qquad &= 1 \\ x_1 + x_2 \qquad\qquad + x_4 &= 2 \\ x_1, x_2, x_3, x_4, y_1 &\geq 0. \end{aligned}$$

Now the problem is in canonical form with basic variables y_1 and x_4.

(b) Since y_1 is the only artificial variable, we have $w = y_1$. The Phase 1 linear program is:

$$\min w = y_1$$

$$\begin{aligned} x_1 + x_2 - x_3 + y_1 \qquad &= 1 \\ x_1 + x_2 \qquad\qquad + x_4 &= 2 \\ x_1, x_2, x_3, x_4, y_1 &\geq 0. \end{aligned}$$

(c) The initial tableau is:

BV	x_1	x_2	x_3	y_1	x_4	RHS
y_1	1	1	-1	1	0	1
x_4	1	1	0	0	1	2
w	0	0	0	-1	0	0

Note that this is not in canonical form since the basic variable y_1 has a nonzero reduced cost. To obtain a canonical tableau we apply the elementary row operation (new Row 3 = Row 3 + Row 1).

BV	x_1	x_2	x_3	y_1	x_4	RHS
y_1	1	1	-1	1	0	1
x_4	1	1	0	0	1	2
w	1	1	-1	0	0	1

We have two options for the entering variable. Arbitrarily choose x_1 as the entering variable. The leaving variable is the artificial variable by the ratio test.

BV	x_1	x_2	x_3	y_1	x_4	RHS
x_1	1	1	-1	1	0	1
x_4	0	0	1	-1	1	1
w	0	0	0	-1	0	0

There are no positive reduced costs, so this tableau is optimal and we have finished Phase 1. Since $w = 0$, we have obtained a feasible solution.

(d) Again we need to use the minimisation version of the Simplex Algorithm.

We can remove the y_1 column from the above tableau and continue with Phase 2. On the other hand we should put the costs for z into the tableau. So the starting tableau for Phase 2 is:

BV	x_1	x_2	x_3	x_4	RHS
x_1	1	1	-1	0	1
x_4	0	0	1	1	1
z	1	1	0	0	0

Notice that this tableau is not in canonical form since the basic variable x_1 has a nonzero reduced cost. We must perform a row operation to obtain a zero in that cell, i.e. (new Row 3 = Row 3 – Row 1). After this we obtain the first canonical tableau in Phase 2:

BV	x_1	x_2	x_3	x_4	RHS
x_1	1	1	-1	0	1
x_4	0	0	1	1	1
z	0	0	1	0	-1

Starting with this tableau you can now apply the minimisation version of the Simplex Algorithm to Phase 2.

2. Rewrite the following linear programming problems in canonical form:

(a)

$$\begin{aligned} \min \quad & z = 4x_1 + 4x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 2 \\ & 2x_1 + x_2 \leq 3 \\ & 2x_1 + x_2 + 3x_3 \geq 3 \\ & x_1 \geq 0, x_2 \geq 0\ x_3 \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} \min \quad & z = 4x_1 + 4x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 2 \\ & 2x_1 + x_2 \leq 3 \\ & 2x_1 + x_2 + 3x_3 \geq 3 \\ & x_1 \geq 0, x_2 \geq 0\ x_3 \geq 0. \end{aligned}$$

(c)

$$\begin{aligned} \min \quad & z = 4x_1 + 2x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 11 \\ & 2x_1 + 3x_2 + x_3 \leq -20 \\ & x_1 + 3x_2 + 2x_3 \geq 10 \\ & x_1 \in \mathbb{R},\ x_2 \geq 0\ x_3 \geq 0. \end{aligned}$$

Solutions:

(a)

$$\begin{aligned} \min \quad & z = 4x_1 + 4x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_5 = 2 \\ & 2x_1 + x_2 + x_6 = 3 \\ & 2x_1 + x_2 + 3x_3 - x_4 + y_1 = 3 \\ & x_1, x_2, x_3, x_4, x_5, x_6, y_1 \geq 0 \end{aligned}$$

where x_4 is a surplus variable, x_5 and x_6 are slack variables, and y_1 is an artificial variable.

(b)

$$\min \qquad z = 4x_1 + 4x_2 + x_3$$

$$\begin{aligned} \text{subject to} \qquad x_1 + x_2 + x_3 + y_1 &= 2 \\ 2x_1 + x_2 + x_5 &= 3 \\ 2x_1 + x_2 + 3x_3 - x_4 + y_2 &= 3 \\ x_1, x_2, x_3, x_4, x_5, y_1, y_2 &\geq 0 \end{aligned}$$

where x_4 is a surplus variable, x_5 is a slack variable, and y_1 and y_2 are artificial variables.

(c)

$$\min \qquad z = 4x_1^{(1)} - 4x_1^{(2)} + 2x_2 + 3x_3$$

$$\begin{aligned} \text{subject to} \qquad x_1^{(1)} - x_1^{(2)} + x_2 + x_3 + y_1 &= 11 \\ -2x_1^{(1)} + 2x_1^{(2)} - 3x_2 - x_3 - x_4 + y_2 &= 20 \\ x_1^{(1)} - x_1^{(2)} + 3x_2 + 2x_3 - x_5 + y_3 &= 10 \\ x_1^{(1)}, x_1^{(2)}, x_2, x_3, x_4, x_5, y_1, y_2 &\geq 0 \end{aligned}$$

3. Consider the following LP problem:

$$\min \qquad z = x_1 + x_2 - x_4$$

$$\begin{array}{lcl} 2x_1 + x_2 - x_3 & = & 14 \\ x_1 - x_2 \qquad\qquad -x_4 & = & 6 \\ x_1 + x_2 & \leq & 10 \\ x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

(a) Transform this problem to canonical form.
(b) Complete Phase 1 for the two-phase method.
(c) Set up the first canonical tableau for Phase 2.

Solutions:

(a) Convert the given problem to canonical form:

$$\begin{array}{lcr} \min x_1 + x_2 - x_4 & & \\ 2x_1 + x_2 - x_3 + y_1 & = & 14 \\ x_1 - x_2 - x_4 + y_2 & = & 6 \\ x_1 + x_2 + x_5 & = & 10 \\ x_1, x_2, x_3, x_4, x_5, y_1, y_2 & \geq & 0 \end{array}$$

where y_1 and y_2 are artificial variables.

(b) In Phase 1 we minimise $w = y_1 + y_2$. The first tableau is:

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	RHS
y_1	2	1	−1	0	0	1	0	14
y_2	1	−1	0	−1	0	0	1	6
x_5	1	1	0	0	1	0	0	10
w	0	0	0	0	0	−1	−1	0

This tableau is not in canonical form since the basic variables y_1, y_2 have nonzero reduced costs. We restore the canonical form by adding Rows 1 and 2 to the w-row.

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	RHS
y_1	2	1	−1	0	0	1	0	14
y_2	1	−1	0	−1	0	0	1	6
x_5	1	1	0	0	1	0	0	10
w	3	0	−1	−1	0	0	0	20

Pivoting on the $(i = 2, j = 1)$-entry, we obtain:

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	RHS
y_1	0	3	−1	2	0	1	−2	2
x_1	1	−1	0	−1	0	0	1	6
x_5	0	2	0	1	1	0	−1	4
w	0	3	−1	2	0	0	−3	2

Pivoting on the $(i = 1, j = 2)$-entry, we obtain:

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	RHS
x_2	0	1	−1/3	2/3	0	1/3	−2/3	2/3
x_1	1	0	−1/3	−1/3	0	1/3	1/3	20/3
x_5	0	0	2/3	−1/3	1	−2/3	1/3	8/3
w	0	0	0	0	0	−1	−1	0

Since all artificial variables have been driven out of the basis, this is the last tableau in Phase 1.

(c) The starting tableau in Phase 2 is:

	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	$-1/3$	$2/3$	0	$2/3$
x_1	1	0	$-1/3$	$-1/3$	0	$20/3$
x_5	0	0	$2/3$	$-1/3$	1	$8/3$
z	-1	-1	0	1	0	0

Adding the first two rows to the z we obtain the initial canonical tableau in Phase 2:

	x_1	x_2	x_3	x_4	x_5	RHS
x_2	0	1	$-1/3$	$2/3$	0	$2/3$
x_1	1	0	$-1/3$	$-1/3$	0	$20/3$
x_5	0	0	$2/3$	$-1/3$	1	$8/3$
z	0	0	$-2/3$	$4/3$	0	$22/3$