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The University of Melbourne
Semester 1 Assessment 2013

Department of Mathematics and Statistics
MAST10007 Linear Algebra

Reading Time: 15 minutes
Writing Time: 3 hours

This paper has: 8 pages

Identical Examination Papers: None.

Common Content: This examination paper contains questions in common with the paper for MAST10008 Accelerated Mathematics 1.

Authorized Materials:

No materials are authorized.

Calculators and mathematical tables are not permitted.

Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:

Each candidate should be issued with one examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:

This examination consists of 13 questions.

The total number of marks is 120.

All questions may be attempted. All answers should be appropriately justified.

— BEGINNING OF EXAMINATION QUESTIONS —

1. Let

$$A = \begin{bmatrix} -3 & 1 \\ 0 & 4 \\ -2 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$$

Evaluate, if possible:

(a) AB (b) BA (c) $B + A^T$ (d) $B - B^T$

[4 marks]

2. (a) Consider the following linear system:

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & & & - & x_4 & = & 3 \\ x_1 & + & 6x_2 & + & x_3 & & & = & 8 \\ 2x_1 & & & - & x_3 & - & 3x_4 & = & 1 \end{array}$$

(i) Write down the augmented matrix corresponding to the linear system.

(ii) Reduce the matrix to reduced row-echelon form.

(iii) Use the reduced row-echelon form to give all solutions to the linear system.

(b) For

$$A = \begin{bmatrix} \lambda+1 & 3 & -1 \\ 2 & \lambda & 1 \\ \lambda & 4 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 22 \\ 14 \\ 20 \end{bmatrix}$$

the augmented matrix $[A \,|\, \mathbf{b}]$ has row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & \lambda - 4 & 7 - \lambda \\ 0 & 1 & \lambda - 2 & 5 - \lambda \\ 0 & 0 & 9 - \lambda^2 & \lambda(\lambda - 3) \end{array}\right].$$

Determine the values of λ (if any) for which the system of equations $A\mathbf{x} = \mathbf{b}$ has

(i) no solution, (ii) one solution, (iii) infinitely many solutions.

[10 marks]

3. (a) Using row operations, find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -3 & 1 \end{bmatrix}$$

(b) Consider the matrix

$$M = \begin{bmatrix} t+2 & 3t & t+1 \\ 0 & t-1 & 0 \\ 2t+4 & t & 3t+4 \end{bmatrix}$$

where t is a real number.

(i) Evaluate the determinant of M in terms of t.

(ii) For what value(s) of t does M have an inverse?

[11 marks]

4. (a) Consider the three points

$$P = (1, 2, 3), \quad Q = (2, 3, 1), \quad R = (3, 1, 2).$$

(i) Show that there is no line that contains all three points P, Q and R.

(ii) Find a vector equation for the plane that contains P, Q and R.

(iii) Find a Cartesian equation for the plane that contains P, Q and R.

(b) Consider the two lines given by

$$(x, y, z) = (1, 1, 1) + t(1, 2, 2), \qquad t \in \mathbb{R}$$

and

$$(x, y, z) = (0, 1, 1) + s(2, -1, -1), \qquad s \in \mathbb{R}.$$

Determine whether the lines intersect. If they intersect, find the point of intersection

[9 marks]

5. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 & -1 \\ 1 & 1 & 4 & 1 & 1 \\ 3 & -1 & 8 & 1 & -1 \\ 2 & 3 & 9 & 0 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix B is the reduced row echelon form of the matrix A. Using this information, or otherwise, answer the following.

(a) What is the rank of A?

(b) Write down a basis for the row space of A.

(c) Find the dimension of the solution space (i.e. null-space) of A.

(d) Do the columns of A span $\mathbb{R}^4$? Give a reason.

(e) Are the vectors $(1, 1, 3, 2), (-1, 1, -1, 3), (2, 4, 8, 9)$ linearly independent? If not, write one of these vectors as a linear combination of the others.

(f) Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation with standard matrix A. Find bases for the image and kernel of T.

[12 marks]

6. For each of the following, decide whether or not the given set S is a subspace of the vector space V. Justify your answers by either using appropriate theorems, or providing a counter-example.

(a) $V = \mathbb{R}^4$ and

$$S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 1\}.$$

(b) $V = \mathcal{P}_3$ (all real polynomials of degree at most 3) and

$$S = \{p(x) \in P_3 \mid x \cdot p'(x) = p(x)\}.$$

(c) $V = M_{2,2}$ (the set of all real 2×2 matrices) and

$$S = \{A \in M_{2,2} \mid AA^T = A^T A\}.$$

[8 marks]

7. Consider the polynomials

$$p_1(x) = 2x^2 + x + 2, \quad p_2(x) = x^2 - 2x, \quad p_3(x) = 5x^2 - 5x + 2, \quad p_4(x) = x^2 + x + 2$$

in the vector space $\mathcal{P}_2$ of real polynomials of degree at most 2.

Determine whether the polynomials:

(a) are linearly independent, (b) span $\mathcal{P}_2$.

Justify your answers.

[4 marks]

8. Consider the vector space of 2×2 matrices with real entries:

$$M_{2,2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

(a) Show that the determinant function

$$det \colon M_{2,2} \longrightarrow \mathbb{R}$$

is not a linear transformation.

(b) Show that the trace function

$$tr \colon M_{2,2} \longrightarrow \mathbb{R}$$

defined by

$$tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + d$$

is a linear transformation

(c) Write down the matrix representation $[tr]_{\mathcal{S}_1,\mathcal{S}_2}$ of the trace function, where the ordered bases are the standard bases

$$\mathcal{S}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \qquad \mathcal{S}_1 = \{1\}.$$

(d) Determine the rank of tr.

(e) What is the dimension of the kernel of tr?

(f) Is tr invertible? Explain your answer.

(g) Find a basis for the kernel of tr.

[11 marks]

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the xy-plane given by a stretch with factor 2 along the line $y = x$ and a compression by $\frac{1}{2}$ along the line $y = -x$ (see Figure 1).

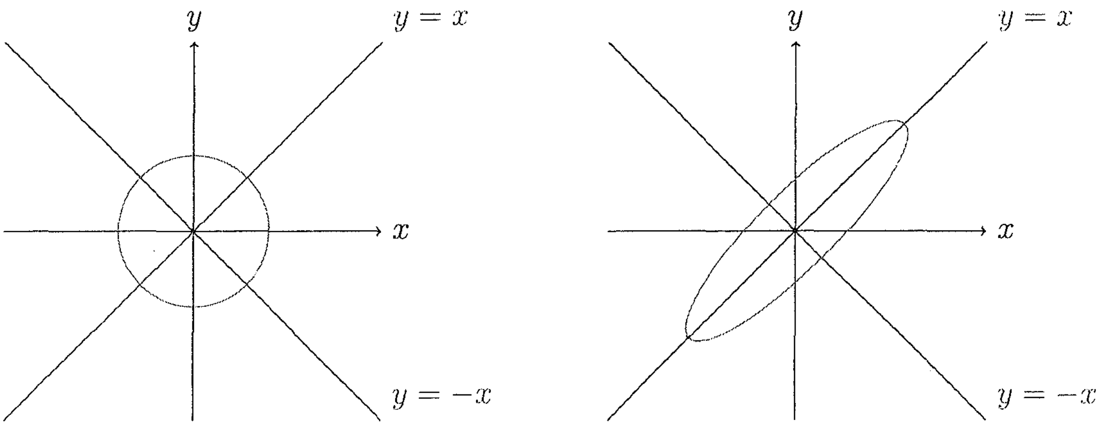

Figure 1: The unit circle in $\mathbb{R}^2$ (left) and its image under T (right).

Let $\mathcal{S} = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ be the standard (ordered) basis of $\mathbb{R}^2$, and let $\mathcal{C}$ be the ordered basis for $\mathbb{R}^2$ consisting of the vectors $\mathbf{c}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{c}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$.

(a) Express the lines defined by the equations $y = x$ and $y = -x$ in terms of the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$.

(b) Write down the matrix representation $[T]_\mathcal{C}$ of T **with respect to the basis** $\mathcal{C}$.

(c) Find the change of basis matrices

$$P_{\mathcal{S},\mathcal{C}} \quad \text{and} \quad P_{\mathcal{C},\mathcal{S}}.$$

(d) Assume $\mathbf{v} \in \mathbb{R}^2$ has standard coordinate matrix $[\mathbf{v}]_\mathcal{S} = \begin{bmatrix}7\\3\end{bmatrix}$.
Find the coordinate matrix $[\mathbf{v}]_\mathcal{C}$.

(e) Calculate the standard matrix representation $[T]_\mathcal{S}$ of T.

(f) To check your answers, calculate $[T]_\mathcal{S} \begin{bmatrix}1\\1\end{bmatrix}$ and $[T]_\mathcal{S} \begin{bmatrix}1\\-1\end{bmatrix}$.

[13 marks]

10. (a) (i) Apply the Gram-Schmidt procedure to the vectors

$$\mathbf{v}_1 = (1, 1, 1, 1), \quad \mathbf{v}_2 = (5, 3, 5, 3), \quad \mathbf{v}_3 = (13, 9, -3, -7)$$

in $\mathbb{R}^4$, using the dot product as inner product, to obtain an orthonormal basis for the subspace W spanned by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(ii) Calculate the orthogonal projection of $(4, 0, 0, 0)$ onto W.

(b) Calculate the cosine of the angle between the vectors x and $x + 1$ in the inner product space

$$\mathcal{P}_2 \quad = \quad \{p(x) = a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

(polynomials up to degree 2) with inner product

$$\langle p, q \rangle \quad = \quad \int_0^1 p(x) q(x) dx.$$

[13 marks]

11. Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}.$$

(a) What are the characteristic polynomials of A and B?

(b) Determine the eigenvalues of A and of B.

(c) For each eigenvalue, find the dimension of the corresponding eigenspace.

(d) Which of the matrices A and B is/are diagonalizable? Justify your answers.

[8 marks]

12. Consider the matrix

$$C = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}.$$

(a) Find all eigenvalues of the matrix C.

(b) Find a corresponding eigenvector for each eigenvalue.

(c) Find invertible matrices P, P^{-1} and a diagonal matrix D such that $C = PDP^{-1}$.

(d) Use your results from (c) to find a formula for C^n valid for each integer $n \geq 1$.

(e) Find the limit of $\frac{1}{5^n}C^n$ as $n \to \infty$.

[13 marks]

13. Let $T : V \to W$ be a linear transformation and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. Assume that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is a basis for W and that T is injective (i.e. one-to-one). Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for V. [4 marks]

— END OF EXAMINATION QUESTIONS —