Exam 6 June 2010, questions

Stochastic Modelling (University of Melbourne)

The University of Melbourne

Semester 2 Assessment — November 2010

Department of Mathematics and Statistics

MAST30001 (620-301) Stochastic Modelling

Reading Time: 15 Minutes

Writing Time: 3 Hours

This paper has 6 pages

Authorised materials:
Hand-held electronic calculators (**except graphic calculators**) may be used.

Instructions to Invigilators:
Students require script books only. No textbooks or lecture notes or graphic calculators are allowed in the examination room. This paper is to remain in the examination room.

Instructions to Students:
This paper has **seven** (7) questions. Attempt as many questions, or parts of questions, as you can. Questions carry marks as shown in the brackets after the question statement. The total of marks available for this examination is 100. Working and/or reasoning must be given to obtain full credit. This paper must be handed in with your script books.

This paper is to be lodged with the Baillieu Library

1. A company has three machines. Each day, independently of each other, any of the machines breaks down with probability 0.2. A repairperson checks the machines each night, and if there are any break-downs, he will repair only one machine at that night. Let X_n be the number of working machines available at the beginning of the nth day. Explain why $\{X_n : \ n \geq 0\}$ is a Markov chain (MC) and find its transition matrix.

[6 marks]

2. A telemarketing saleswoman makes phone calls to potential customers in order to sell a particular product. She is a persistent person who keeps calling the same people again and again. After a while, she concluded that one can use a Markov chain (MC) to model the change of a customer's degree of interest in her product from one call to the next. In the saleswoman's opinion, the transition matrix of the MC has the following form:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.10 & 0.30 & 0 & 0.45 & 0.15 \\ 0.15 & 0.10 & 0 & 0.25 & 0.50 \\ 0.20 & 0.05 & 0 & 0.35 & 0.40 \end{array}\right] \end{array},$$

where 0 = sale completed, 1 = sale lost, 2 = new customer with no history, 3 = customer indicated a low degree of interest, 4 = customer indicated a high degree of interest.

(a) Draw a transition diagram for the MC.

(b) Classify the states of the MC.

(c) Calculate $\mathbb{P}(X_2 = 3, X_3 = 1 | X_1 = 4)$, $\mathbb{P}(X_2 = 3, X_4 = 1 | X_1 = 4)$ and $\mathbb{P}(X_2 = 3 | X_1 = 4, X_3 = 4)$.

(d) What are conditions for a finite MC to be ergodic? Use the conditions to prove or disprove that the MC $\{X_n\}$ is ergodic. If it is ergodic, compute its stationary distribution.

(e) Assuming that a customer shows a high degree of interest now, find the distribution of the number of calls to the customer until he changes his mind.

[17 marks]

3. Customers demanding service at a newly opened central processing facility arrive according to a Poisson process $\{N_t,\ t \geq 0\}$ with intensity $\lambda = 3$ per hour. Each customer, independently of the others, has probability 0.2 of being high priority and probability 0.8 of being low priority.

   (a) What is the probability that there are no customers during the first 30 minutes after openning?

   (b) Compute $\mathbb{P}(N_2 = 5 | N_1 = 3)$ and $\mathbb{P}(N_1 = 3 | N_2 = 5)$.

   (c) What is the expected time from the opening until the third customer arrives?

   (d) What is the probability that, during a given 30 minute period of operation of the facility, there were exactly one high priority customer and one low priority customer?

   (e) What is the probability that the third customer demanding service is a high priority one?

   (f) Given there was one high priority customer in the last hour, what would be the average number of customers arrived at the facility during the hour?

   (g) Suppose that the facility has two servers (A and B), and when a customer arrives, a fair coin is tossed to decide which server will serve the customer. Is it correct to claim that the flow of customers served by server A forms a Poisson process? Explain.

   (h) Suppose that the two servers get tired of tossing coins and decide that, according to the order of the arrivals, server A handles the odd numbered customers and server B serves the even numbered customers. Is it correct to claim that the flow of customers served by server B froms a Poisson process? Explain.

   (i) The central processing facility has a forecast of an average profit of \$200 from a high priority customer and \$120 from a low priority customer, independently of other customers. What is the expected total hourly profit forecasted for the facility?

[22 marks]

4. Suppose that the central processing facility in problem 3 has two servers (A and B) and, regardless of the server used, all the service times are independent and identically distributed exponential random variables with rate $\mu = 4$ per hour. If both servers are idle at the time of a customer's arrival, then server A begins processing the arrived customer immediately. Otherwise, the arriving customer joins a common queue. The facility follows this policy: when there are fewer than 4 customers, server A is active, and for four or more customers, both A and B are active. We model the customer flow in the facility by a continuous time Markov process (MP) by setting $X_t =$ number of customers at the facility at time t and assume that the first-in-first-out rule applies.

   (a) Give the state space of the process and sketch a transition diagram indicating transition rates.

   (b) Explain why the MP is ergodic and calculate its steady-state probability distribution.

   (c) Find the expected number of customers at the facility in the steady-state regime.

   (d) Calculate the expected number of customers in the queue for the facility in the steady-state regime.

   (e) What is the long-run proportion of the time that the server B is idle?

   (f) Compute the expected waiting time for a newly arrived customer at the facility.

   (g) Calculate the expected delay time for a newly arrived customer at the facility.

[17 marks]

5. Customers arrive at a service facility with two servers A (with service rate $\mu_1 = 5$ per hour) and B (with service rate $\mu_2 = 6$ per hour) according to a Poisson process with rate $\lambda = 3$ per hour. All arriving customers have to be served by A first, and upon completion of their service, 20% of them will be directed to server B for additional service. Each server has its own queue. Assume also that 5% of the customers will return to the service facility as new customers immediately upon completion of their service at B.

   (a) Specify the routing matrix and sketch a network diagram for the queuing network.

   (b) Calculate the equilibrium arrival rates of customers for the two servers. Is the queuing network ergodic? Explain.

   (c) Assume that the facility operates in the stationary regime.

      i. What is the average number of customers at the facility?

      ii. What is the average delay time for an arriving customer at the facility?

[8 marks]

6. Radioactive particles arrive at a counter according to a Poisson process with rate $\lambda = 1$ per second. After a particle is recorded, the counter must recuperate for the next arrival. Particles arriving during the recuperation period, called dead time or locked time, can't be recorded. We assume that the counter starts with a recuperation period when it is switched on and that the recuperation period for the counter is 0.5 seconds. Let $\{M_t,\ t \geq 0\}$ be the counting process of the detection times of radioactive particles when the counter was switched on at time $t = 0$.

   (a) Explain why $\{M_t,\ t \geq 0\}$ is a renewal process and find the cumulative distribution function of the random time between successive renewals.

   (b) Calculate the mean and variance of the random time between successive renewals.

   (c) Present an asymptotic expression for M_t when t is large. Give an interval capturing the value of M_{9000} with probability 90% (approximately).

   (d) Give the asymptotic distribution for the time Y_t from t until the next radioactive particle is recorded when t is large.

   (e) Determine the approximate probability p_t that the counter is recuperating at time t when t is large.

   (f) Give the definition of the renewal function $\{H(t),\ t \geq 0\}$ and find $H(t)$ for $0 \leq t \leq 1$.

   (*Hint*: If $Z \sim N(0,1)$, then $\mathbb{P}(Z < 1.645) \approx 0.95$ and $\mathbb{P}(Z < 1.96) \approx 0.975$.)

   [18 marks]

7. A person has to sell his car during the next three days. The highest prices he can be offered on days $j = 1, 2, 3$ are independent random variables X_j distributed according to the following table:

x	$\mathbb{P}(X_1 = x)$	$\mathbb{P}(X_2 = x)$	$\mathbb{P}(X_3 = x)$
9000	0.4	0.4	0.3
8000	0.4	0.3	0.2
7500	0.2	0.3	0.5

   Each day the person has to make a decision: either to sell or not to sell. If he does not sell for the best price on the day, the opportunity is lost (he cannot return to the offer later).

   (a) Set this as a stochastic dynamic decision problem: define the decision process, possible actions, reward function etc.

   (b) Write down the optimality equation for the optimal value function and derive the optimal policy for the seller. What is the expected price for the car when he follows the optimal policy?

   [12 marks]

$\boxed{\text{Total marks} = 100}$

End of Questions

Useful formulae

(a) $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ for $|x| < 1$.

(b) $\sum_{i=1}^{\infty} i x^{i-1} = (1-x)^{-2}$ for $|x| < 1$.

(c) For a birth and death process with birth rates $\{\lambda_i : \ i \geq 0\}$ and death rates $\{\mu_i : \ i \geq 1\}$, let $K_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$ for $i \geq 1$. If the stationary distribution $\{\pi_i : \ i \geq 0\}$ exists, then $\pi_i = K_i \pi_0$, $i \geq 1$.

(d) If $\{N_t : \ t \geq 0\}$ is a renewal process with inter-renewal times having mean μ and variance $\sigma^2 > 0$, then

$$\frac{N_t - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{d} N(0,1) \quad \text{as } t \to \infty.$$

End of Examination