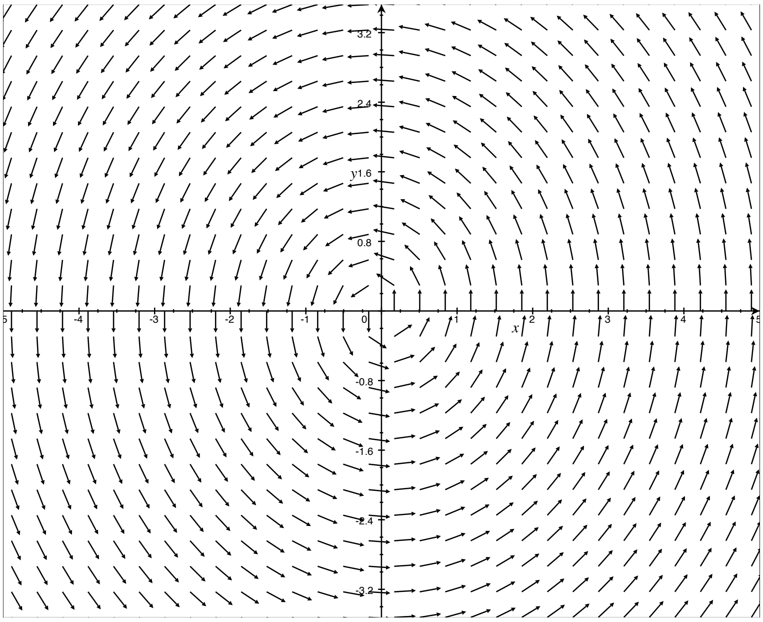
MAST2000 9 S2 2019 Problem Sheets

Vector Calculus (University of Melbourne)

The University of Melbourne
School of Mathematics and Statistics
Semester 2, 2019

MAST20009
Vector Calculus



STUDENT NAME:

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Subject Organisation

Syllabus

This subject studies the fundamental concepts of functions of several variables and vector calculus. It develops the manipulation of partial derivatives and vector differential operators. The gradient vector is used to obtain constrained extrema of functions of several variables. Line, surface and volume integrals are evaluated and related by various integral theorems. Vector differential operators are also studied using curvilinear coordinates.

Functions of several variables topics include limits, continuity, differentiability, the chain rule, Jacobian, Taylor polynomials and Lagrange multipliers. Vector calculus topics include vector fields, flow lines, curvature, torsion, gradient, divergence, curl and Laplacian. Integrals over paths and surfaces topics include line, surface and volume integrals; change of variables; applications including averages, moments of inertia, centre of mass; Green's theorem, Divergence theorem in the plane, Gauss' divergence theorem, Stokes' theorem; and curvilinear coordinates.

On completion of this subject students should:

- Understand calculus of functions of several variables; differential operators; line, surface and volume integrals; curvilinear coordinates; integral theorems.
- Have developed the ability to work with limits and continuity; obtain extrema of functions of several variables; calculate line, surface and volume integrals; work in curvilinear coordinates; apply integral theorems.
- Appreciate the fundamental concepts of vector calculus; the relations between line, surface and volume integrals.

Prerequisites

One of

- MAST10006 Calculus 2
- MAST10009 Accelerated Mathematics 2
- MAST10019 Calculus Extension Studies

and one of

- MAST10007 Linear Algebra
- MAST10008 Accelerated Mathematics 1
- MAST10013 UMEP Mathematics for High Achieving Students (prior to 2017)
- MAST10018 Linear Algebra Extension Studies

Credit Exclusions

Students may only gain credit for one of

- MAST20009 Vector Calculus
- 620-296 Multivariable and Vector Calculus (prior to 2010)

Note:

- Passing MAST20009 Vector Calculus precludes subsequent credit for MAST20029 Engineering Mathematics.
- Enrolment in MAST20009 Vector Calculus is permitted for students who have passed MAST20029 Engineering Mathematics.
- Concurrent enrolment in both MAST20009 Vector Calculus and MAST20029 Engineering Mathematics is not permitted.

Lectures and Practice Classes

The lecturer is Dr Nora Ganter Room 169, Peter Hall Building.

There are 36 lectures (three per week). There are 11 one-hour practice classes (one per week). Practice classes start in the second week of semester. Details of your practice classes are given on your personal timetable.

During practice classes you will be required to work in groups on the whiteboards. A sheet of questions for discussion will be provided at the beginning of each practice class and full solutions will be given out at the end of the practice class. The idea is to discuss the questions and their solution, and to learn about mathematics collaboratively with your fellow students.

The question and solutions are only given out to students who attend the practice classes. They will be not be put on the subject website.

Lecture Notes

Partial lecture notes can be purchased from the Co-op Bookshop at the University of Melbourne. These notes contain the theory, diagrams and a statement of the examples to be covered in lectures - space is left for the completion of the examples during the lectures. Students are expected to bring these partial lecture notes to all lectures and fill in the working of examples in the gaps provided.

The full lecture notes will not be put on the subject website.

Problem Sheets

There are six problem sheets corresponding to the six main topics covered in lectures. The problems can be grouped into two types:

- Questions labelled **Revision** cover material that is assumed knowledge from previous mathematics subjects. You should be able to complete all of these questions.

- Questions that do not have a label **Revision** are the core questions for MAST20009. It is essential that you attempt **all** of these problems. These questions form the examinable material for MAST20009.

Answers are at the back of the problem sheet booklet, but full solutions will not be provided.

Reference Book

The textbook recommended for extra reading and problems is:

- **Vector Calculus**, Fourth Edition, by J. E. Marsden and A. J. Tromba, Freeman.

The book is in the ERC Library, and may be purchased from the Co-op Bookshop at the University of Melbourne. Any earlier or later edition of the textbook is also suitable.

This table gives references to the relevant section(s) of **Vector Calculus**, fourth edition, by Marsden and Tromba and to corresponding questions on the problem sheets.

Topic	**Textbook**	**Questions**
Functions of Several Variables		
Limits and continuity	§2.2	1-4
Partial differentiation revision	§2.3, §3.1	5-6
Differentiability	§2.3	7-9
Matrix version of chain rule	§2.5	10-11
Jacobian	pg 359	12-14
Taylor polynomials	§3.2	15-17
Extrema, constrained extrema	§3.3	18
Lagrange multipliers	§3.4	19-21
Space Curves and Vector Fields		
Vectors revision	§1.1-§1.3	22-24
Parametric paths - velocity, acceleration	§2.4, §4.1	25-28
Arc length	§4.2	29
Tangent vectors, curvature, torsion	pg 263-264	30-32
Vector fields	§4.3	33
Flow lines	§4.3	34-35
Differential operators	§4.4	36-39, 42-46
Basic identities of vector analysis	§4.4	42-46
Scalar and vector potentials	pg 497-500	40-41

Topic	Textbook	Questions
Double and Triple Integrals		
Double integrals	§5.1-§5.3	48-51
Areas and volumes using double integrals	§5.3	51
Change of order of integration	§5.4	52-53
Sketching surfaces revision	§2.1	55
Triple integrals	§5.6	54
Elementary regions	§5.6	56
Volumes using triple integrals	§5.6	57, 67
Polar, cylindrical and spherical coordinates	§1.4	58-60
Change of variables for multiple integrals	§6.2	61-69
Averages, centre of mass, moment of inertia	§6.3	68-69
Integrals over Paths and Surfaces		
Path integrals	§7.1	70-72
Line integrals	§7.2	73-75
Parametrisation of paths and surfaces	§7.3	70, 76-79
Tangent planes to parametrised surfaces	§7.3	78-79
Area of a surface	§7.4	80
Integrals of scalar functions over surfaces	§7.5	80-82
Integrals of vector functions over surfaces	§7.6	83-84
Integral Theorems		
Green's theorem	§8.1	85-88
Divergence theorem in plane	§8.1	89-91
Stokes' theorem	§8.2	92-94
Conservative fields	§8.3	95-96
Gauss' divergence theorem	§8.4	97-99
Mixed integral theorems	§8.1-§8.4	100-101
Applications to physics and engineering	§8.5	102
General Curvilinear Coordinates		
Orthogonal curvilinear coordinates	None	103-106
Differential operators	None	103-106

Website

Information regarding the assignments, past exam papers, and consultation hours will be available from the LMS website at:

`www.lms.unimelb.edu.au/`

Assessment

The assessment is composed of two parts:

- A **three hour exam** at the end of semester;
- **Four assignments** due as follows:
    (1) Assignment 1 - due 11am on Tuesday 20th August;
    (2) Assignment 2 - due 11am on Tuesday 10th September;
    (3) Assignment 3 - due 11am on Tuesday 8th October;
    (4) Assignment 4 - due 11am on Tuesday 22nd October.

Each piece of assessment is **compulsory**.

The **Final Mark** in MAST20009 is computed as:

Final Mark = 80% Exam + 20% Assignments (5% per assignment)

Assignment Submission

• The assignments will be handed out in lectures a week before the due date. The assignments will also be put on the MAST20009 website.

• Make sure that you attach a completed assignment question/cover sheet to each assignment.

• Assignment solutions must be neatly handwritten in blue or black pen - pencil is not acceptable. However, diagrams can be drawn in pencil.

• Your assignment should be placed into the appropriate MAST20009 Vector Calculus assignment box on the ground floor of the Peter Hall building. A list of classes, tutors and boxes is on the noticeboard above the assignment boxes and on the website.

Plagiarism Declaration

• You must complete the online plagiarism declaration on the MAST20009 website before submitting assignment 1.

• The plagiarism declaration will apply to all assignments in MAST20009 during the semester.

Late Assignments

• Submit your late assignment to the special Vector Calculus late pigeonhole which will be identified on the signs on the standard pigeonholes.

• Do NOT place a LATE assignment into your standard Vector Calculus pigeonhole - it will not be collected or marked if you do.

• Late assignments will attract a deduction of 20% of the total marks if submitted after 11am on the due date but before 11am the day after the due date (that is, 24 hours after the deadline).

• Assignments submitted more than 24 hours after the deadline will not be accepted unless a medical certificate is provided.

Medical Certificates and Extensions for Assignments

• Extensions of up to 3 days after the deadline may be granted on submission of a medical certificate.

• Any medical certificates for the assignments should be given to Dr Nora Ganter.

• Do not put your late assignment and medical certificate in the Vector Calculus pigeonholes. Submit your late assignment and medical certificate to Dr Nora Ganter.

• Do not use the Student Portal to apply for special consideration for the assignments; this online application is for the final Vector Calculus exam only.

Return of Marked Assignments

• Marked assignments will not be returned to students until the online plagiarism declaration is completed.

• Your tutor will be marking your assignments. Marked assignments will be returned to students in practice classes in the week after the submission date.

Expectations

In MAST20009 Vector Calculus you are expected to:

- Attend all lectures, and take notes during lectures.
- Attend all practice classes, participate in group work in practice classes, and complete all practice class exercises.
- Work through the problem booklet outside of class in your own time. You should try to keep up-to-date with the problem booklet questions, and aim to have attempted all questions from the problem booklet before the exam.
- Check the announcements on the LMS at least once per week to make sure you do not miss any important subject information.
- Complete all assignments on time.
- Seek help when you need it during consultation sessions.

In total, you are expected to dedicate around 170 hours to this subject, including classes. This equates to an average of about 9 hours of additional study, outside of class, per week over 14 weeks.

Calculators, Dictionaries and Formula Sheets

Students are not permitted to use calculators, computers, dictionaries or mathomats in the end of semester exam.

Students are not permitted to take formula sheets, notes or text books to the end of semester exam. The formulae sheet (Useful Formulae; Basic Identities of Vector Calculus; Grad, Div, Curl, and Laplacian in Orthogonal Curvilinear Coordinates) will be provided in the final exam.

Assessment in this subject concentrates on the testing of concepts and the ability to conduct procedures in simple cases. There is no formal requirement to possess a calculator for this subject. Nonetheless, there are some questions on the problem sheets for which calculator usage is appropriate. If you have a calculator, then you will find it useful occasionally.

Getting Help

The first source of help is the person beside you in lectures and practice classes, who is doing the same problems as you are and having similar but perhaps not exactly the same difficulties. Remember though, that fellow students have no obligation to help you, nor you to help them. Forming a small study group of two to four people is an excellent way of sharing knowledge.

Lecturers and Mathematics and Statistics Learning Centre staff have consultation hours when they will help you on an individual basis with questions from the MAST20009 Vector Calculus lecture notes, problem sheets and practice class sheets. Attendance is on a voluntary basis. Details will be provided on the MAST20009 web site.

A Tutor on Duty service operates from 12noon to 2pm on weekdays during semester in *mathSpace*, which is located on the ground floor of the Peter Hall building. Attendance is on a voluntary basis. The tutors on duty can help you with

- Basic algebra and index laws
- Basic differentiation and integration
- Functions and inverses
- Trigonometric functions and their inverses
- Logarithms and exponentials
- Equations of ellipses and hyperbolae
- Graph sketching
- Elementary vector algebra
- Elementary complex arithmetic
- Elementary probability

Tutors on duty do NOT help with current assignment questions!

Special Consideration

If something major goes wrong during semester or you are sick during the examination period, you should consider applying for *Special Consideration* through the Student Portal. You must submit your online special consideration application no later than 4 days after the date of the final exam in MAST20009 Vector Calculus. You will also need to submit the completed Health Professional Report (HPR) Form with your online application. The HPR Form can only be completed by the professional using the form provided.

For more details see the Special Consideration menu item on the website:

`http://ask.unimelb.edu.au/app/home`

Lecture-by-Lecture Outline

This is subject to change without notice.

Lecture	Topic	Sheet
	Functions of Several Variables	
1	Intuitive idea of limits of functions of several variables	1
2	Limits and continuity of functions of several variables	1
3	Differentiability of functions of several variables, C^N	1
4	Matrix version of chain rule, Jacobian	1
5	Taylor polynomials for functions of several variables, extrema	1
6	Constrained extrema, Lagrange multipliers	1
7	Lagrange multipliers	1
	Space Curves and Vector Fields	
8	Parametric paths, velocity, acceleration, differentiating dot and cross products	2
9	Arc length, tangent and normal vectors, curvature, torsion	2
10	Vector fields, flow lines	2
11	Differential operators: grad, div, curl, Laplacian	2
12	Laplacian, vector analysis identities, scalar potentials	2
13	Using the vector analysis identities	2
	Double and Triple Integrals	
14	Double integrals over rectangular and general domains	3
15	Area using double integrals, changing order of integration	3
16	Triple integrals over rectangular box domains, volume using triple integrals	3
17	Triple integrals over general domains	3
18	Polar coordinates, cylindrical coordinates	3
19	Spherical coordinates, change of variables for multiple integrals	3
20	Change of variables for multiple integrals	3
	Integrals over Paths and Surfaces	
21	Parametrisation of paths, path integrals	4
22	Line integrals, parametrisation of surfaces	4
23	Normals, tangent planes to parametrised surfaces, surface area	4
24	Integrals of scalar functions over surfaces	4
25	Integrals of scalar and vector functions over surfaces	4
26	Integrals of vector functions over surfaces	4
	Integral Theorems	
27	Green's theorem in plane	5
28	Area via line integrals, divergence theorem in plane	5
29	Stokes' theorem	5
30	Stokes' theorem, conservative fields	5
31	Conservative fields, Gauss' divergence theorem	5
32	Gauss' divergence theorem	5
33	Applications of integral theorems to physics and engineering	5
	General Curvilinear Coordinates	
34	General orthogonal curvilinear coordinates, surface area and volume element	6
35	Differential operators, simple PDEs in spherical and cylindrical coordinates	6

Useful Formulae

$$\int \sin x\, dx = -\cos x + C \qquad \int \cos x\, dx = \sin x + C$$

$$\int \sec x\, dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\, dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\, dx = -\cot x + C$$

$$\int \sinh x\, dx = \cosh x + C \qquad \int \cosh x\, dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\, dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\, dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\, dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2-a^2}}\, dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2-x^2}\, dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x \qquad \coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\cos 2x = \cos^2 x - \sin^2 x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cos 2x = 2\cos^2 x - 1 \qquad \cosh 2x = 2\cosh^2 x - 1$$

$$\cos 2x = 1 - 2\sin^2 x \qquad \cosh 2x = 1 + 2\sinh^2 x$$

$$\sin 2x = 2\sin x\cos x \qquad \sinh 2x = 2\sinh x\cosh x$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \qquad \cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y \qquad \sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

$$\cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right)$$

$$\cos x = \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x = \tfrac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$e^{ix} = \cos x + i\sin x \qquad \operatorname{arcsinh} x = \log(x + \sqrt{x^2+1})$$

$$\operatorname{arccosh} x = \log(x + \sqrt{x^2-1}) \qquad \operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

Basic Identities of Vector Calculus

Let f and $g : \mathbb{R}^3 \to \mathbb{R}$ be scalar functions, $\boldsymbol{F}$ and $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be vector fields, and $\beta \in \mathbb{R}$ be any constant.

1. $\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g$
2. $\boldsymbol{\nabla}(\beta f) = \beta \boldsymbol{\nabla} f$
3. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$
4. $\boldsymbol{\nabla}\left(\dfrac{f}{g}\right) = \dfrac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}$ provided $g \neq 0$.
5. $\boldsymbol{\nabla} \cdot (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{\nabla} \cdot \boldsymbol{G}$
6. $\boldsymbol{\nabla} \times (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} \times \boldsymbol{G}$
7. $\boldsymbol{\nabla} \cdot (f\boldsymbol{F}) = f\boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \boldsymbol{\nabla} f$
8. $\boldsymbol{\nabla} \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\boldsymbol{\nabla} \times \boldsymbol{G})$
9. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0$
10. $\boldsymbol{\nabla} \times (f\boldsymbol{F}) = f\boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} f \times \boldsymbol{F}$
11. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$
13. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = 0$
14. $\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = f\nabla^2 g - g\nabla^2 f$
15. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$

Note:
The identities require $f, g, \boldsymbol{F}$ and $\boldsymbol{G}$ to be suitably differentiable, either order C^1 or C^2.

Grad, Div, Curl, and Laplacian in Orthogonal Curvilinear Coordinates

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 scalar function and $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field where

$$\boldsymbol{F}(u_1, u_2, u_3) = F_1(u_1, u_2, u_3)\boldsymbol{e}_1 + F_2(u_1, u_2, u_3)\boldsymbol{e}_2 + F_3(u_1, u_2, u_3)\boldsymbol{e}_3.$$

Then

1. $\boldsymbol{\nabla} f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \dfrac{1}{h_2}\dfrac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \dfrac{1}{h_3}\dfrac{\partial f}{\partial u_3}\mathbf{e_3}$

2. $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial\,(h_2 h_3 F_1)}{\partial u_1} + \dfrac{\partial\,(h_1 h_3 F_2)}{\partial u_2} + \dfrac{\partial\,(h_1 h_2 F_3)}{\partial u_3}\right]$

3. $\boldsymbol{\nabla} \times \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$

4. $\nabla^2 f = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2 h_3}{h_1}\dfrac{\partial f}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_1 h_3}{h_2}\dfrac{\partial f}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1 h_2}{h_3}\dfrac{\partial f}{\partial u_3}\right)\right]$

Note: Equations 1-4 reduce to the usual expressions for cartesian coordinates if

$$h_1 = h_2 = h_3 = 1; \quad \boldsymbol{e}_1 = \boldsymbol{i}, \boldsymbol{e}_2 = \boldsymbol{j}, \boldsymbol{e}_3 = \boldsymbol{k}; \quad (u_1, u_2, u_3) = (x, y, z).$$

Cylindrical Coordinates

Cylindrical coordinates (ρ, ϕ, z) are defined by

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

where $\rho \geq 0$, $0 \leq \phi \leq 2\pi$. Then $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

Equations 1-4 reduce to:

1. $\boldsymbol{\nabla} f = \dfrac{\partial f}{\partial \rho}\,\hat{\boldsymbol{\rho}} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \dfrac{\partial f}{\partial z}\,\hat{\boldsymbol{z}}$

2. $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \dfrac{1}{\rho}\left[\dfrac{\partial(\rho F_1)}{\partial \rho} + \dfrac{\partial F_2}{\partial \phi} + \dfrac{\partial(\rho F_3)}{\partial z}\right] = \dfrac{1}{\rho}\left(F_1 + \rho\,\dfrac{\partial F_1}{\partial \rho}\right) + \dfrac{1}{\rho}\dfrac{\partial F_2}{\partial \phi} + \dfrac{\partial F_3}{\partial z}$

3. $\boldsymbol{\nabla} \times \boldsymbol{F} = \dfrac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_1 & \rho F_2 & F_3 \end{vmatrix}$

$$= \frac{1}{\rho}\left[\left(\frac{\partial F_3}{\partial \phi} - \frac{\partial(\rho F_2)}{\partial z}\right)\hat{\boldsymbol{\rho}} - \left(\frac{\partial F_3}{\partial \rho} - \frac{\partial F_1}{\partial z}\right)\rho\hat{\boldsymbol{\phi}} + \left(\frac{\partial(\rho F_2)}{\partial \rho} - \frac{\partial F_1}{\partial \phi}\right)\hat{\boldsymbol{z}}\right]$$

4. $\nabla^2 f = \dfrac{1}{\rho}\left[\dfrac{\partial}{\partial \rho}\left(\rho\,\dfrac{\partial f}{\partial \rho}\right) + \dfrac{\partial}{\partial \phi}\left(\dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}\right) + \dfrac{\partial}{\partial z}\left(\rho\,\dfrac{\partial f}{\partial z}\right)\right] = \dfrac{1}{\rho}\dfrac{\partial f}{\partial \rho} + \dfrac{\partial^2 f}{\partial \rho^2} + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \phi^2} + \dfrac{\partial^2 f}{\partial z^2}$

Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are defined by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. Then $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.

Equations 1-4 reduce to:

1. $\boldsymbol{\nabla} f = \dfrac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$

2. $$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial\left(r^2\sin\theta F_1\right)}{\partial r} + \frac{\partial\left(r\sin\theta F_2\right)}{\partial \theta} + \frac{\partial\left(rF_3\right)}{\partial \phi}\right]$$
$$= \frac{1}{r^2}\frac{\partial\left(r^2F_1\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta F_2\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial \phi}$$

3. $$\boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_1 & rF_2 & r\sin\theta F_3 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial\left(r\sin\theta F_3\right)}{\partial \theta} - \frac{\partial\left(rF_2\right)}{\partial \phi}\right)\hat{\boldsymbol{r}} - \left(\frac{\partial\left(r\sin\theta F_3\right)}{\partial r} - \frac{\partial F_1}{\partial \phi}\right)r\hat{\boldsymbol{\theta}} + \left(\frac{\partial\left(rF_2\right)}{\partial r} - \frac{\partial F_1}{\partial \theta}\right)r\sin\theta\hat{\boldsymbol{\phi}}\right]$$

4. $$\nabla^2 f = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \phi}\right)\right]$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

MAST20009 Vector Calculus
Problem Sheet 1
Functions of Several Variables

1. Limits. Evaluate the following limits, if they exist.

(a) $\displaystyle\lim_{(x,y)\to(1,0)}(x^3+y^4+3)$; (b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2+2}$;

(c) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3y^2}{2x^2-y^2}$; (d) $\displaystyle\lim_{(x,y)\to(2,1)}\frac{2x^4+3y^3}{3x^2-5}$;

(e) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\cos x-1+\frac{x^2}{2}}{x^4+y^4}$; (f) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{(x-y)^2}{x^2+y^2}$;

(g) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^4-y^4}{x^2+y^2}$; (h) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{y-2x+\sin 2x}{x^3+y}$;

(i) $\displaystyle\lim_{(x,y)\to(1,-1)}\frac{x+y}{2x^2+3xy+y^2}$; (j) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x}{x^2+y^2}$.

2. No Limits. Show that the following limits do not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{(x^2+y)^3}$; (b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{2x^4y^4}{(3x^4+y^2)^3}$.

Hint: You may need to examine the limit along $y=x^2$ as well as the usual lines.

3. Limit Exists. Using the Sandwich Theorem, show that

(a) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{7x^2y^2}{x^2+2y^4}=0$; (b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3yx^2}{x^2+y^2}=0$.

4. Continuity. Consider the following functions f where

(i) $f(x,y)=ye^x+\sin x$; (ii) $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$; (iii) $f(x,y)=\dfrac{xy}{x^2+y^2}$.

(a) Which functions are continuous at $(x,y)=(0,0)$?

(b) Can the functions which are not continuous at $(0,0)$, be made continuous at $(0,0)$ by suitably defining them at the origin?

5. Partial Derivatives Revision. For each function f, find expressions for the partial derivatives listed.

(a) $f(x,y) = \log(x+3y)$. Find $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

(b) $f(x,y) = e^{xy}$. Find $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

(c) $f(x,y,z) = x^2y^3z^5$. Find $f_x, f_y, f_z, f_{yy}, f_{yz}, f_{xyz}$.

(d) $f(x,y,z) = \dfrac{x}{y} + \dfrac{y}{z}$. Find $f_x, f_y, f_z, f_{yz}, f_{yy}, f_{xyy}$.

6. Laplace's Equation. Let

$$V(x,y,z) = -\frac{GmM}{\sqrt{x^2+y^2+z^2}}$$

where $(x,y,z) \neq (0,0,0)$ and G, m, M are constants. Show that V satisfies Laplace's equation, that is,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

7. Differentiable Functions. Find the largest possible domain for each function f. Show that the function is C^1 (and is therefore differentiable) in this domain.

(a) $f(x,y) = \dfrac{x}{y} + \dfrac{y}{x}$; (b) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$; (c) $f(r,\theta) = \dfrac{1}{2} r \sin 2\theta, \quad r > 0$.

8. Continuity and Differentiability. Consider the following function f given by

$$f(x,y) = \begin{cases} \dfrac{4x^2 - y^3}{x^2+5y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

(a) Calculate $\dfrac{\partial f}{\partial y}$ if $(x,y) \neq (0,0)$.

(b) Calculate $\dfrac{\partial f}{\partial y}$ if $(x,y) = (0,0)$.

(c) Is $\dfrac{\partial f}{\partial y}$ continuous at $(0,0)$? Explain.

(d) Is the function C^1 at $(0,0)$?

9. Different Mixed Partials. Let $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$.

(a) If $(x, y) \neq (0, 0)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(b) Show that $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ at $(x, y) = (0, 0)$.

(c) Show that $\dfrac{\partial^2 f}{\partial x\, \partial y} = 1$ and $\dfrac{\partial^2 f}{\partial y\, \partial x} = -1$ at $(x, y) = (0, 0)$. Why are the mixed partials different?

10. Matrix Chain Rule. For each of the following, find the specified derivative using the matrix version of the chain rule. In each case, check your answer by direct substitution.

(a) Let $p(t) = (t, t^2, t^3)$ and $f(x, y, z) = 3x^2 + 4xy + z^2$. Evaluate $\boldsymbol{D}(f \circ p)$ at $t = 1$.

(b) Let $f(x, y) = x^2 - y^2 + x^3 y$ and $(x, y) = (r \cos\theta, r \sin\theta)$. Find the derivative of

$f[x(r, \theta), y(r, \theta)]$ at $(r, \theta) = \left(2, \dfrac{\pi}{4}\right)$.

(c) Let $f(u, v, w) = (u - v, uvw)$ and $g(x, y) = (xy, x^2, y^2)$. Evaluate $\boldsymbol{D}(f \circ g)$ at $(2, 1)$.

11. Matrix Chain Rule Again. Using the matrix chain rule, evaluate the derivative of

(a) $f(g[f(x, y)])$ at $(0, 1)$ where $f(x, y) = (x^2, 2y, x - y)$ and $g(x, y, z) = (x + z, y^2)$.

(b) $g\left[f(g[f(x, y)])\right]$ at $(1, 0)$ where $f(x, y) = (x^2, x^2 - y^2, x^2 + y^2)$, $g(x, y, z) = (x^2 - z^2, y^2 + z^2)$.

12. Cylindrical Coordinates. The relation between three-dimensional Cartesian coordinates (X, Y, Z) and cylindrical coordinates (ρ, ϕ, z) is given by

$$X = \rho\cos\phi, \qquad Y = \rho\sin\phi, \qquad Z = z,$$

where $\rho > 0$, $0 \leq \phi < 2\pi$ and $z \in \mathbb{R}$.

(a) Find the Jacobi matrix of this transformation, and hence find its Jacobian.

(b) Where possible solve these equations for ρ, ϕ and z as functions of X, Y, Z.

(c) Calculate the partial derivatives $\dfrac{\partial \rho}{\partial X}$, $\dfrac{\partial \phi}{\partial Y}$, and $\dfrac{\partial z}{\partial Z}$ in terms of ρ, ϕ, z.

13. Spherical Coordinates. The relation between three-dimensional Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) is given by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

where $r > 0$, $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

(a) Find the Jacobi matrix of this transformation, and hence find its Jacobian.

(b) Where possible solve these equations for r, θ, and ϕ as functions of x, y, z.

(c) Calculate the partial derivatives $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial \theta}{\partial y}$ and $\dfrac{\partial \phi}{\partial z}$ in terms of r, θ, ϕ.

14. Product of Jacobians. Assume that $u = u(r, s)$ and $v = v(r, s)$ are C^1 functions of r and s. Also, assume that $r = r(x, y)$ and $s = s(x, y)$ are C^1 functions of x and y. Prove that

$$\frac{\partial(u, v)}{\partial(r, s)} \frac{\partial(r, s)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)}.$$

(Hint: Use the chain rule and the identity $\det(AB) = \det(A)\det(B)$.)

15. Taylor Polynomials. Determine the Taylor polynomial of order n for each function f about the given point (x_0, y_0).

(a) $f(x, y) = e^{(x+y)}, \quad n = 3, \quad (x_0, y_0) = (0, 0);$

(b) $f(x, y) = \sin(xy) + \cos(xy), \quad n = 2, \quad (x_0, y_0) = (0, 0);$

(c) $f(x, y) = \dfrac{1}{1 + x - y}, \quad n = 2, \quad (x_0, y_0) = (1, 1);$

(d) $f(x, y) = \cos(4x - 3y), \quad n = 3, \quad (x_0, y_0) = \left(\dfrac{\pi}{8}, 0\right).$

16. Taylor Polynomial Application. Let $f(x, y) = e^{(x-1)^2}\cos y$.

(a) Determine the Taylor polynomial of degree 2 for f about the point $(1, 0)$.

(b) Using your answer to part (a), approximate $f(1.1, 0.2)$. Compare your approximate answer to the exact value.

(c) Using your answer to part (a), approximate the double integral:

$$\int_0^1 \int_0^1 f(x, y)\,dx\,dy.$$

17. Taylor Remainder. For a certain C^2 function $f(x, y)$ it is known that

$$f(0,0) = 1, \ f_x(0,0) = 0.25, \ f_y(0,0) = 0.50$$

and that

$$|f_{xx}| \leq 0.15, \ |f_{xy}| \leq 0.05, \ |f_{yy}| \leq 0.05$$

everywhere along that segment of the line $y = x$ which joins $(0, 0)$ and $(0.1, 0.1)$.

(a) Determine the Taylor polynomial of degree 1 for f about the point $(0, 0)$.

(b) Using your answer to part (a), approximate $(0.1, 0.1)$ and determine the error R_1 in your approximation.

(c) Show that

$$1.0735 \leq f(0.1, 0.1) \leq 1.0765.$$

18. Another Taylor Remainder. Consider the function

$$g(x, y) = e^{3-2x+y}.$$

(a) Determine the second order Taylor polynomial for g about the point $(1, -1)$. Hence, approximate $g(0.9, -1.1)$.

(b) Determine an upper bound for the error in your approximation of $g(0.9, -1.1)$ using the remainder formula.

19. Maxima and Minima Revision. Find all the local minima, local maxima and saddle points of the following functions.

(a) $f(x, y) = e^{(1+x^2-y^2)}$;

(b) $f(x, y) = x^3 - 12xy + 8y^3$;

(c) $f(x, y) = x^2 + y^2 + 2xy$.

20. Lagrange Multipliers. Find the extrema of the following functions subject to the given constraints.

(a) $f(x, y, z) = x - y + z, \qquad x^2 + y^2 + z^2 = 2$;

(b) $f(x, y) = x, \qquad x^2 + 2y^2 = 3$;

(c) $f(x, y, z) = xy + yz, \qquad xz = 1$;

(d) $f(x, y, z) = x + y + z, \qquad 2x + z = 1, \qquad 2x^2 + y^2 = 1$;

(e) $f(x, y, z) = x, \qquad x^2 + 2y^2 + 2z^2 = 8, \qquad x + y = z$.

21. More Lagrange Multipliers. (a) Design a cylindrical can with a lid to contain 1 litre (1000 cm^3) of water, using the minimum amount of metal.

(b) Determine the points on the ellipse defined as the intersection of the surfaces $x + y = 1$ and $x^2 + 2y^2 + z^2 = 1$ which are nearest to the origin. (Hint: Minimise the square of the distance from a point on the ellipse to the origin.)

22. Inequality Constraints. (a) Find the absolute maxima and minima of the function f given by $f(x, y) = 5x^2 - 2y^2 + 10$ on the disk $x^2 + y^2 \leq 1$.

(b) A parcel delivery service requires that the dimension of a rectangular box be such that the length plus twice the width plus twice the height be no more than 108 centimetres. What is the volume of the largest box that the company will deliver?

MAST20009 Vector Calculus

Problem Sheet 2

Space Curves and Vector Fields

23. Vectors Revision. (a) Evaluate the dot and cross products

$$\text{(i)} \quad (2\boldsymbol{i} + \boldsymbol{j}) \cdot (-\boldsymbol{j} + \boldsymbol{k}); \qquad \text{(ii)} \quad (2\boldsymbol{i} + \boldsymbol{j}) \times (-\boldsymbol{j} + \boldsymbol{k}).$$

(b) Find a unit vector perpendicular to $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$.

(c) What restrictions must be made on b so that $2\boldsymbol{i} + b\boldsymbol{j}$ is orthogonal to $-3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$?

24. Equation of Lines Revision. Find the equations of the following lines in both vector and parametric form.

(a) The line passing through the point $(0, 1, 2)$ and parallel to $\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$;

(b) The line passing through the points $(1, 2, 1)$ and $(4, -2, 3)$.

25. Equation of Planes Revision. Find the Cartesian equations of the following planes.

(a) The plane passing through the points $(1, 1, 1)$, $(3, 0, 2)$, and $(4, 3, 1)$;

(b) The plane passing through the point $(1, 0, 2)$ and perpendicular to the vector $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

26. Paths. For each path $\boldsymbol{c}$, find the velocity, acceleration, and the equation of the tangent line at the given value of t.

(a) $\boldsymbol{c}(t) = \cos t\boldsymbol{i} + \sin(2t)\boldsymbol{j}, \quad t = 0;$ (b) $\boldsymbol{c}(t) = \sqrt{2}t\boldsymbol{i} + e^t\boldsymbol{j} + e^{-t}\boldsymbol{k}, \quad t = 0.$

27. Rolling Wheel. A wheel of radius R rolls to the right along a straight line at speed v. The path $\boldsymbol{c}$ of a point on the rim is given by

$$\boldsymbol{c}(t) = \left(vt - R\sin\left(\frac{vt}{R}\right), R - R\cos\left(\frac{vt}{R}\right)\right).$$

When is the velocity vector of this particular point horizontal? What is the speed at this point?

28. Velocity Properties. Let $\boldsymbol{c}$ be a path and $\boldsymbol{v}(t)$ its velocity and $\boldsymbol{a}(t)$ the acceleration at time t. Suppose that $\boldsymbol{F}$ is a C^1 vector field, $m > 0$, and $\boldsymbol{F}[\boldsymbol{c}(t)] = m\boldsymbol{a}(t)$. Prove that the rate of change of angular momentum equals the torque, that is,

$$\frac{d}{dt}\left[m\boldsymbol{c}(t) \times \boldsymbol{v}(t)\right] = \boldsymbol{c}(t) \times \boldsymbol{F}\left[\boldsymbol{c}(t)\right].$$

29. Derivative of Triple Products. (a) If $\boldsymbol{F}$ is a differentiable function of t, find the derivative of

$$\boldsymbol{F} \cdot \frac{d\boldsymbol{F}}{dt} \times \frac{d^2\boldsymbol{F}}{dt^2}.$$

(b) If the path $\boldsymbol{u}$ is differentiable at least three times, evaluate and simplify

$$\frac{d}{dt}\left[(\boldsymbol{u} + \boldsymbol{u}'') \cdot (\boldsymbol{u} \times \boldsymbol{u}')\right].$$

30. Arc Length. Find the arc length of the given curve on the specified interval.

(a) $\boldsymbol{c}(t) = (2\cos t, 2\sin t, t); \quad 0 \leq t \leq 2\pi;$ (b) $\boldsymbol{c}(t) = \left(\sin 3t, \cos 3t, 2t^{\frac{3}{2}}\right); \quad 0 \leq t \leq 1.$

31. Calculate Path Properties. Consider the path $\boldsymbol{c}$ given by

$$\boldsymbol{c}(t) = (\sin t - t\cos t, \cos t + t\sin t, 27); \qquad t > 0.$$

(a) Find the unit tangent, unit normal and unit binormal vectors to the path.

(b) Compute the curvature and torsion for the path.

32. Calculate Path Properties Again. Consider the path $\boldsymbol{c}$ given by

$$\mathbf{c}(t) = (e^{2t}\sin t, e^{2t}\cos t, 1).$$

(a) Find the unit tangent, unit normal, and unit binormal vectors to the path.

(b) Compute the curvature and torsion for the path.

33. Path Properties. Let $\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}$ be the unit tangent, normal and binormal vectors to a C^3 path. Prove that

(a) $\dfrac{d\boldsymbol{B}}{dt} \cdot \boldsymbol{B} = 0;$ (b) $\dfrac{d\boldsymbol{B}}{dt} \cdot \boldsymbol{T} = 0;$ (c) $\dfrac{d\boldsymbol{B}}{dt}$ is a scalar multiple of $\boldsymbol{N}$.

34. Flow Fields. Sketch the following vector fields $\boldsymbol{F}$ where

(a) $\boldsymbol{F}(x, y) = (2, 2);$ (b) $\boldsymbol{F}(x, y) = (x, y);$ (b) $\boldsymbol{F}(x, y) = (2y, x).$

35. Flow Lines. Sketch a few flow lines of the following vector fields $\boldsymbol{F}$. Derive the differential equations for the flow lines and solve them to obtain an expression for the flow lines.

(a) $\boldsymbol{F}(x, y) = (y, -x);$ (b) $\boldsymbol{F}(x, y) = (x, -y);$ (c) $\boldsymbol{F}(x, y, z) = (y, -x, 0).$

36. Flow Line Properties. Prove that the given curve $\boldsymbol{c}(t)$ is a flow line of the given velocity vector field $\boldsymbol{F}$.

$$\text{(a) } \boldsymbol{c}(t) = \left(e^{2t}, \log|t|, \frac{1}{t}\right), \quad t \neq 0, \quad \boldsymbol{F}(x,y,z) = (2x, z, -z^2);$$

$$\text{(b) } \boldsymbol{c}(t) = (\sin t, \cos t, e^t), \quad \boldsymbol{F}(x,y,z) = (y, -x, z).$$

37. Divergence. Find the divergence of the following vector fields $\boldsymbol{V}$.

$$\text{(a) } \boldsymbol{V}(x,y,z) = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k};$$

$$\text{(b) } \boldsymbol{V}(x,y,z) = x\boldsymbol{i} + (y + \cos x)\boldsymbol{j} + (z + e^{xy})\boldsymbol{k};$$

$$\text{(c) } \boldsymbol{V}(x,y) = \sin(xy)\boldsymbol{i} - \cos(x^2y)\boldsymbol{j}.$$

38. Curl. Find the curl of the following vector fields $\boldsymbol{V}$.

$$\text{(a) } \boldsymbol{V}(x,y,z) = xy\boldsymbol{i} - \sin z\boldsymbol{j} + \boldsymbol{k};$$

$$\text{(b) } \boldsymbol{V}(x,y,z) = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k};$$

$$\text{(c) } \boldsymbol{V}(x,y,z) = (x^2 + y^2 + z^2)(3\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k});$$

39. Div, Grad, and Curl. Let $\boldsymbol{F}(x,y,z) = 2xz^2\boldsymbol{i} + \boldsymbol{j} + y^3zx\boldsymbol{k}$, $\quad f(x,y,z) = x^2y$. Find expressions for the following quantities where they are defined.

(a) $\boldsymbol{\nabla} f$; (b) $\boldsymbol{\nabla} \times \boldsymbol{F}$; (c) $\boldsymbol{\nabla} \cdot \boldsymbol{F}$;
(d) $\boldsymbol{\nabla} \times f$; (e) $\boldsymbol{F} \times \boldsymbol{\nabla} f$; (f) $\boldsymbol{F} \cdot \boldsymbol{\nabla} f$;
(g) $\boldsymbol{\nabla} \cdot f$; (h) $\nabla^2 f$; (i) $\nabla^2 \boldsymbol{F}$.

40. Curl of Grad. Show that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{0}$ for the following functions f.

(a) $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$; (b) $f(x,y,z) = xy + yz + xz$.

41. Scalar Potentials. Consider the following vector fields $\boldsymbol{F}$. Which vector fields are irrotational? For each irrotational vector field, find a scalar function ϕ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$.

$$\text{(a) } \boldsymbol{F}(x,y,z) = 2x\boldsymbol{i} + 3y\boldsymbol{j} + 4z\boldsymbol{k};$$

$$\text{(b) } \boldsymbol{F}(x,y,z) = y\cos x\boldsymbol{i} + x\sin y\boldsymbol{j};$$

$$\text{(c) } \boldsymbol{F}(x,y,z) = 2xye^z\boldsymbol{i} + x^2e^z\boldsymbol{j} + (x^2ye^z + z^2)\boldsymbol{k}.$$

42. Finding Vector Fields. Consider the following vector fields $\boldsymbol{F}$. Which vector fields are incompressible? For each incompressible vector field, find a vector $\boldsymbol{G}$ such that $\boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{G}$.

(a) $\boldsymbol{F}(x, y, z) = xz\boldsymbol{i} - yz\boldsymbol{j} + y\boldsymbol{k}$;

(b) $\boldsymbol{F}(x, y, z) = x\cos y\boldsymbol{i} - \sin y\boldsymbol{j} + \sin x\boldsymbol{k}$;

(c) $\boldsymbol{F}(x, y, z) = 2xye^z\boldsymbol{i} + x^2e^z\mathbf{j} + (x^2ye^z + z^2)\boldsymbol{k}$.

43. Vector Identity One. (a) Let f and g be C^1 scalar functions. Prove that

$$\boldsymbol{\nabla}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(g\boldsymbol{\nabla}f - f\boldsymbol{\nabla}g\right), \quad g \neq 0.$$

(b) Using (a), evaluate $\boldsymbol{\nabla}\left(\dfrac{e^{r^2}}{r}\right)$ where $\boldsymbol{r}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = |\boldsymbol{r}|$.

44. Vector Identity Two. (a) Let $\boldsymbol{F}$ be a C^1 vector field and f be a C^1 scalar function. Prove that

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{F}) = f\boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \boldsymbol{\nabla}f.$$

(b) Using (a), evaluate $\boldsymbol{\nabla} \cdot \left(\dfrac{\boldsymbol{r}}{r^5}\right)$ where $\boldsymbol{r}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = |\boldsymbol{r}|$.

45. Vector Identity Three. (a) Let $\boldsymbol{F}$ be a C^1 vector field and f be a C^1 scalar function. Prove that

$$\boldsymbol{\nabla} \times (f\boldsymbol{F}) = f\boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla}f \times \boldsymbol{F}.$$

(b) Using (a), evaluate $\boldsymbol{\nabla} \times \left(\dfrac{\boldsymbol{r}}{r^3}\right)$ where $\boldsymbol{r}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = |\boldsymbol{r}|$.

46. Using Vector Identities. Let f, g, and h be any C^2 scalar functions. Using the standard identities of vector calculus, prove that

$$\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla}g \times \boldsymbol{\nabla}h) = \boldsymbol{\nabla}f \cdot (\boldsymbol{\nabla}g \times \boldsymbol{\nabla}h).$$

47. Vector Properties. Let $\boldsymbol{r}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = |\boldsymbol{r}|$. Using the standard identities of vector calculus, or otherwise, prove that

(a) $\boldsymbol{\nabla}(r^n) = nr^{n-2}\boldsymbol{r}$;

(b) $\boldsymbol{\nabla} \cdot (r^n\boldsymbol{r}) = (n+3)r^n$;

(c) $\nabla^2(r^n) = n(n+1)r^{n-2}$;

(d) $\boldsymbol{\nabla} \times (r^n\boldsymbol{r}) = \boldsymbol{0}$;

where $n \in \mathbb{R}$.

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Problem Sheet 3
Double and Triple Integrals

48. Revision Integrals. Find the indefinite integrals of the following functions.

(a) $\cos^4 x$; (b) $\dfrac{1}{x^2(x+1)}$; (c) $\sin^6 x \cos^3 x$;

(d) $x \cosh x$; (e) $x^2 e^{x^3}$; (f) $\sqrt{1+4x^2}$.

49. Double Integrals Revision. Evaluate the following double integrals.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 (y \cos x + 2) dy dx$; (b) $\displaystyle\int_{-1}^0 \int_1^2 (-x \log y) dy dx$.

50. Double Integrals. Evaluate the following double integrals.

(a) $\displaystyle\int_0^1 \int_0^{3x} x e^y dy dx$; (b) $\displaystyle\int_0^1 \int_0^y 2x \cos(y^3) dx dy$.

51. Double Integrals over Regions. (a) Let R be the region enclosed by the lines $y = 0, y = x$ and $y = 6 - 2x$. Evaluate

$$\iint_R x\, dR.$$

(b) Let D be the interior of the triangle with vertices $(0,0), (1,3)$ and $(2,2)$. Evaluate

$$\iint_D e^{x-y} dA.$$

52. Double Integral Applications. Using double integrals find

(a) the volume of the solid bounded by the graph $z = f(x,y) = 1 + 2x + 3y$, the rectangle $R = [1,2] \times [0,1]$ and the vertical sides of R;

(b) the area of the region enclosed by $x = 1 - y^2$ and $y = -x - 1$;

(c) the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a, b > 0$.

53. Repeated Integrals. For each of the following integrals sketch the region of integration, evaluate the integral as given, and then check your answer by interchanging the order of integration.

(a) $\int_0^2 \int_0^x x^2 y \, dy dx$; (b) $\int_0^2 \int_0^{x^2} xy \, dy dx$;

(c) $\int_0^4 \int_{\sqrt{y}}^2 (x+y) \, dx dy$.

54. Look Before You Leap. Evaluate the following double integrals.

(a) $\int_0^9 \int_{\sqrt{y}}^3 \sin(x^3) \, dx dy$; (b) $\int_1^e \int_0^{\log x} y \, dy dx$.

55. Triple Integrals. Evaluate the following triple integrals.

(a) $\iiint_B y e^{-xy} dx dy dz$ where $B = [0,1] \times [0,1] \times [0,1]$;

(b) $\iiint_B (2x + 3y + 4z) dx dy dz$ where $B = [0,2] \times [-1,1] \times [0,1]$;

(c) $\int_0^1 \int_0^x \int_0^y (y + xz) dz dy dx$;

(d) $\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz dy dx$.

56. Surfaces Revision. Describe and sketch the following surfaces.

(a) $z = \sqrt{1 - x^2 - y^2}$; (b) $z = 2\sqrt{x^2 + y^2}$;

(c) $z = 6 - 2x - 3y$; (d) $(x-1)^2 + y^2 + z^2 = 16$;

(e) $x^2 + y^2 = 9, \ -2 \leq z \leq 4$; (f) $z = 3 - x^2 - y^2$.

57. Elementary Regions. Describe the given region as an elementary region, that is, determine the range of x, y, z specifying the region.

(a) The region between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$.

(b) The region inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.

(c) The region cut out of the ball $x^2 + y^2 + z^2 \leq 4$ by the elliptic cylinder $2x^2 + y^2 = 1$; that is the region inside the cylinder and the ball.

(d) The region bounded by the elliptic cylinder $x^2 + 3z^2 = 9$ and the planes $y = 0$ and $y = 5$.

58. Volumes via Triple Integrals. Using triple integrals and cartesian coordinates, find the volume of the

(a) solid bounded by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ $(a, b, c > 0)$ and the coordinate planes $x = 0, y = 0, z = 0$.

(b) solid bounded by $z = 3x^2 + 3y^2 - 7$ and $z = 9 - x^2 - y^2$.

59. Convert from Polar. (a) The following points are given in polar coordinates (r, θ). Express each point in cartesian coordinates (x, y).

$$\text{(i) } \left(7, \frac{2\pi}{3}\right); \qquad \text{(ii) } \left(6, \frac{-5\pi}{6}\right); \qquad \text{(iii) } (0, -\pi).$$

(b) The following points are given in cartesian coordinates (x, y). Express each point in polar coordinates (r, θ).

$$\text{(i) } \left(2\sqrt{3}, -2\right); \qquad \text{(ii) } (0, -2); \qquad \text{(iii) } (-8, -8).$$

60. Convert from Cylindrical. The following points are given in cylindrical coordinates (ρ, ϕ, z). Express each point in cartesian coordinates (x, y, z) and spherical coordinates (r, θ, ϕ).

$$\text{(a) } (1, \frac{\pi}{4}, 1); \qquad \text{(b) } \left(2, \frac{\pi}{2}, -4\right); \qquad \text{(c) } \left(3, \frac{\pi}{6}, 4\right); \qquad \text{(d) } \left(2, \frac{3\pi}{4}, -2\right).$$

61. Convert from Cartesian. The following points are given in cartesian coordinates (x, y, z). Express each point in spherical coordinates (r, θ, ϕ) and cylindrical coordinates (ρ, ϕ, z).

$$\text{(a) } (2, 1, -2); \qquad \text{(b) } (0, 3, 4); \qquad \text{(c) } (\sqrt{2}, 1, 1); \qquad \text{(d) } (-2\sqrt{3}, -2, 3).$$

62. Change of Variables. (a) Let D be the triangle with vertices $(0, 0), (1, 0)$ and $(0, 1)$. Evaluate

$$\iint_D \exp\left(\frac{y - x}{y + x}\right) dx dy$$

by making the substitutions $u = y - x$ and $v = y + x$.

(b) Let D be the region bounded by $x = 0, y = 0, x + y = 1$ and $x + y = 4$. Evaluate

$$\iint_D \frac{dx dy}{x + y}$$

by making the change of variables $x = u - uv, y = uv$.

63. More Change of Variables. Let D be the parallelogram with vertices $(0,0),(1,0),(1,2)$ and $(2,2)$. Let P be the parallelogram with vertices $(1,1),(2,1),(2,3)$ and $(3,3)$. Let R be the rectangle $[0,1]\times[0,2]$.

(a) Determine a linear transformation that maps D in the x-y plane to R in the u-v plane.

(b) Using the change of variables in part (a), evaluate the double integral

$$\iint_D (x+y)dxdy.$$

(c) By modifying your answer to part (a), determine a transformation that maps P in the xy-plane to R in the uv-plane.

(d) Using the change of variables in part (c), evaluate the double integral

$$\iint_P (x+y)dxdy.$$

64. Polar Coordinates. (a) Let D be the unit disk $x^2+y^2\leq 1$. Evaluate

$$\iint_D e^{x^2+y^2}dA.$$

(b) Let R be the region bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$. Evaluate

$$\iint_R \left(3x+8y^2\right)dA.$$

65. Elliptical Coordinates. To evaluate double integrals over elliptical regions, it is useful to change variables by modifying the definition of polar coordinates. Let D be the region enclosed by the ellipse $2x^2+3y^2=1$ and the line $y=0$, for $y\leq 0$.

Evaluate the double integral

$$\iint_D \sinh\left(4x^2+6y^2\right)dxdy$$

by making the change of variables $x=\dfrac{r}{\sqrt{2}}\cos\theta$ and $y=\dfrac{r}{\sqrt{3}}\sin\theta$.

66. Change of Coordinates. Sketch the regions of integration. Write down the integrals obtained after making the required change of coordinates. Do NOT evaluate the integrals.

(a) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} xyz\,dzdxdy$, cylindrical coordinates.

(b) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z^2\,dzdxdy$, spherical coordinates.

67. Change Coordinates. Make an appropriate change of coordinates to evaluate the following integrals.

(a) $\displaystyle\iiint_S \frac{dxdydz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ where S is the solid bounded by two spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$ where $0<b<a$.

(b) $\displaystyle\iiint_B z\,dxdydz$ where B is the region within the cylinder $x^2+y^2=1$ that is above the xy plane and below the cone $z=\sqrt{x^2+y^2}$.

68. Volumes. Using either cylindrical or spherical coordinates, as appropriate, find the volume

(a) inside the surfaces $z=x^2+y^2$ and $z=\sqrt{2-x^2-y^2}$.

(b) of the solid bounded by $z=3x^2+3y^2-7$ and $z=9-x^2-y^2$.

69. Centre of Mass. (a) Find the centre of mass of the region between $y=x^2$ and $y=x$ if the mass density is $x+y$.

(b) Find the centre of mass of the solid bounded by $z=3x^2+3y^2-7$ and $z=9-x^2-y^2$ if the mass density of the solid is constant.

70. Other Applications. (a) A gold plate D is defined by $0\le x\le 2\pi$ cm and $0\le y\le\pi$ cm and has mass density ρ given by $\rho(x,y)=y^2\sin^2(4x)+2\,\mathrm{g/cm}^2$. If gold sells for \$7 per gram, how much is the gold plate worth?

(b) The temperature at points in the cube $C=[-1,1]\times[-1,1]\times[-1,1]$ is $32d^2$, where d is the distance to the origin

(i) What is the average temperature?
(ii) At what points in the cube is the temperature equal to the average temperature?

(c) Find the moment of inertia around the y axis for the ball $x^2+y^2+z^2\le R^2$ if the mass density is a constant ρ.

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Problem Sheet 4

Integrals over Paths and Surfaces

71. Parametric Curves. Find a parametrisation for the part of the circle $x^2+y^2=a^2$ in the first quadrant of the x-y plane, in terms of the

(a) y-coordinate, oriented anti-clockwise;

(b) x-coordinate, oriented clockwise;

(c) polar angle θ, oriented anti-clockwise;

(d) polar angle θ, oriented clockwise.

72. Path Integrals. Evaluate the following path integrals $\int_{\boldsymbol{c}} f ds$ where

(a) $f(x,y,z)=x+y+z, \quad \boldsymbol{c}(t)=(\sin t, \cos t, t), \quad 0 \le t \le 2\pi;$

(b) $f(x,y,z)=e^{\sqrt{z}}, \quad \boldsymbol{c}(t)=(1,2,t^2), \quad 0 \le t \le 1;$

(c) $f(x,y,z)=\dfrac{x+y}{y+z}, \quad \boldsymbol{c}(t)=\left(t, \dfrac{2}{3}t^{\frac{3}{2}}, t\right), \quad 1 \le t \le 2.$

73. More Path Integrals. (a) Evaluate the path integral

$$\int_C x \cos z \, ds$$

where C is the parabola $y=x^2, z=0$ from $(0,0,0)$ to $(1,1,0)$.

(b) Find the mass of a wire along the curve

$$\boldsymbol{c}(t)=3t\boldsymbol{i}+3t^2\boldsymbol{j}+2t^3\boldsymbol{k} \quad \text{for} \quad 0 \le t \le 1$$

if the mass density at $\boldsymbol{c}(t)$ is $1+t$ grams per unit length.

(c) Let the path C traverse part of the ellipse $9x^2+y^2=9$ from $(0,-3)$ to $(1,0)$, in a clockwise direction. Determine the total charge of a wire in the shape of C, if the charge density of the wire is $f(x,y)=\sqrt{x^2+\dfrac{y^2}{81}}$ Coulombs per unit length.

74. Line Integrals. Let $\boldsymbol{F}(x,y,z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Evaluate the following line integrals $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$ where

(a) $\boldsymbol{c}(t) = (t,t,t), \quad 0 \le t \le 1;$

(b) $\boldsymbol{c}(t) = (\cos t, \sin t, 0), \quad 0 \le t \le 2\pi;$

(c) $\boldsymbol{c}(t) = (t^2, 3t, 2t^3), \quad -1 \le t \le 2.$

75. More Line Integrals. Evaluate the following integrals.

(a) $\int_C yzdx + xzdy + xydz$ where C consists of straight line segments joining (1,0,0) to (0,1,0) to (0,0,1).

(b) $\int_C x^2dx - xydy + dz$ where C is the parabola $z = x^2, y = 0$ from $(-1,0,1)$ to $(1,0,1)$.

(c) $\int_C 2xdx + ydy$ where C is the closed curve consisting of the semi circle $y = \sqrt{1-x^2}$ and the line segment $-1 \le x \le 1$ traversed in the anticlockwise direction.

(d) $\int_C ydx - xdy$ where C is the boundary of the square $[-1,1] \times [-1,1]$ traversed in the clockwise direction.

76. Line Integral Properties. Let $\boldsymbol{c}$ be a differentiable path in $\mathbb{R}^3$.

(a) If $\boldsymbol{F}(\boldsymbol{c}(t))$ is perpendicular to $\boldsymbol{c}'(t)$, show that $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.

(b) If $\boldsymbol{F}(\boldsymbol{c}(t))$ is parallel to $\boldsymbol{c}'(t)$, show that $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{c}} |\boldsymbol{F}| ds$.

77. Parametrised Surfaces. Find a parametrisation for the following surfaces based on either cylindrical or spherical coordinates, as appropriate.

(a) The cylinder $x^2 + z^2 = 2,\ 0 \le y \le 2$.

(b) The cone $z = 15 - 3\sqrt{x^2+y^2}$ for $z \ge 0$.

(c) The hemisphere $z = -\sqrt{9 - x^2 - y^2}$.

78. Parametrised Normals. Find a unit vector normal to the following parametrised surfaces in terms of x, y and z. Identify each surface.

(a) $x = \cos v \sin u, \quad y = \sin v \sin u, \quad z = \cos u \quad$ where $0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$.

(b) $x = \sin v, \quad y = u, \quad z = \cos v \quad$ where $-1 \leq u \leq 3, \quad 0 \leq v \leq 2\pi$.

79. Parametrised Surface 1. Consider the parametrised surface defined by:

$$x = 2u, \quad y = u^2 + v, \quad z = v^2.$$

(a) Find a vector normal to the surface in terms of u and v.

(b) For what values of u and v is the surface smooth?

(c) Find the equation of the tangent plane to the surface at $(0, 1, 1)$.

80. Parametrised Surface 2. Consider the parametrised surface defined by:

$$x = u^2 - v^2, \quad y = u + v, \quad z = u^2 + 4v.$$

(a) Find a vector normal to the surface in terms of u and v.

(b) For what values of u and v is the surface smooth?

(c) Find the equation of the tangent plane to the surface at $\left(-\frac{1}{4}, \frac{1}{2}, 2\right)$.

81. Surface Area. (a) Let $\Phi(u, v) = (u - v, u + v, uv)$ and D be the unit disk in the uv-plane. Find the area of $\Phi(D)$.

(b) Find the surface area of the torus represented parametrically by

$$x = (R + \cos\phi)\cos\theta, \quad y = (R + \cos\phi)\sin\theta, \quad z = \sin\phi$$

where $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq 2\pi$ and $R > 1$ is fixed.

82. Surface Integrals of Scalars. Evaluate the following surface integrals.

(a) $\displaystyle\iint_S z\, dS$ where S is the upper hemisphere $(z \geq 0)$ of radius a, centred at $(0, 0, 0)$.

(b) $\displaystyle\iint_S x\, dS$ where S is the triangular plate with vertices (1,0,0), (0,2,0) and (0,1,1).

(c) $\displaystyle\iint_S z^2\, dS$ where S is the surface of the cube $[-1, 1] \times [-1, 1] \times [-1, 1]$.

83. Mass of Sphere. A metallic surface S is in the shape of a hemisphere

$$z = \sqrt{R^2 - x^2 - y^2}, \quad 0 \leq x^2 + y^2 \leq R^2.$$

If the mass of the metal per unit area of S is given by $m(x, y, z) = x^2 + y^2$, find the total mass of S.

84. Flux Across a Surface. In (a) - (c) take the normal pointing outwards from the surface.

(a) Find the flux of $\boldsymbol{F}(x,y,z) = 3xy^2\boldsymbol{i} + 3x^2y\boldsymbol{j} + z^3\boldsymbol{k}$ out of the unit sphere centred at $(0,0,0)$.

(b) A uniform fluid that flows vertically downward is described by the vector field $\boldsymbol{F}(x,y,z) = -\boldsymbol{k}$. Find the flux through the cone $z = \sqrt{x^2+y^2}, \quad x^2+y^2 \leq 1$.

(c) Let S be the closed surface that consists of the upper hemisphere $(z \geq 0)$ of radius 1, centred at $(0,0,0)$, and its base $x^2+y^2 \leq 1, \quad z = 0$. Let the electric field be defined by $\boldsymbol{E}(x,y,z) = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$. Find the electric flux across S.

85. Surface Integrals of Vectors. Evaluate the following surface integrals.

(a) $\displaystyle\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = 18z\boldsymbol{i} - 12\boldsymbol{j} + 3y\boldsymbol{k}$ and S is that part of the plane $2x+3y+6z = 12$ where $x \geq 0, y \geq 0, z \geq 0$. Take the normal pointing upwards.

(b) $\displaystyle\iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x,y,z) = (x^2+y-4)\boldsymbol{i} + 3xy\boldsymbol{j} + (2xz+z^2)\boldsymbol{k}$ and S is the upper hemisphere $(z \geq 0)$ of radius 4, centred at $(0,0,0)$. Take the normal pointing upwards.

MAST20009 Vector Calculus

Problem Sheet 5

Integral Theorems

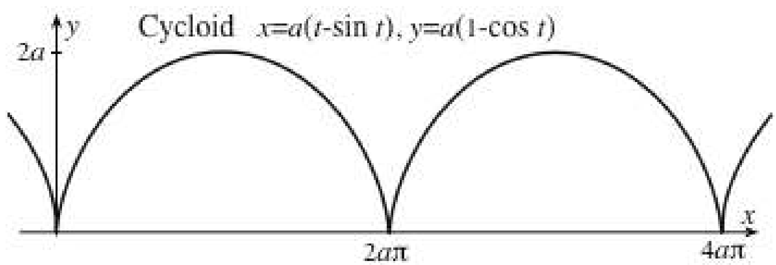
86. Green's Theorem. Verify Green's theorem for the disk D with centre (0,0) and radius R and the functions P and Q.

(a) $P(x,y) = xy^2, \quad Q(x,y) = -yx^2;$ (b) $P(x,y) = 2y, \quad Q(x,y) = x.$

87. Apply Green's Theorem. Redo questions 75(c) and 75(d) using Green's theorem.

88. Area via Line Integrals. Find the area of the following regions using Green's theorem and an appropriate line integral.

(a) The region bounded by one arc of the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ where $a > 0$ and $0 \leq \theta \leq 2\pi$ and the x axis.



(b) The region inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a, b > 0$.

89. Harmonic Functions. Let D be a region for which Green's theorem holds and ∂D is traversed anticlockwise. Suppose that f is harmonic, that is $f_{xx} + f_{yy} = 0$ on D. Prove that

$$\int_{\partial D} \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = 0.$$

90. Divergence Theorem in Plane. Verify the divergence theorem in the plane for the

(a) unit disk $x^2 + y^2 \leq 1$ if $\boldsymbol{F}(x,y) = x\boldsymbol{i} + y\boldsymbol{j}$;

(b) unit square $[0,1] \times [0,1]$ if $\boldsymbol{F}(x,y) = 10x\boldsymbol{i} - 5y\boldsymbol{j}$.

91. Apply Divergence Theorem in Plane. Let $\hat{\boldsymbol{n}}$ be the unit outward normal to the curve C in the x-y plane. Evaluate

$$\int_C \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds$$

where

(a) $\boldsymbol{F}(x, y) = \big(2 \sin y + e^y + 3x, 3 \cos x - e^{4x} - y\big)$ and C is the rectangle $[-1, 2] \times [0, 4]$ traversed anticlockwise.

(b) $\boldsymbol{F}(x, y) = \left(x^2 + 5y + y^3, \dfrac{1}{2}y^2 + 3x - x^5\right)$ and C is the triangle with vertices $(-1, 0), (0, 0), (0, 2)$ traversed anticlockwise.

92. Integral Theorem Proof. Let D be a region in the xy-plane bounded by a simple closed curve C which is traversed anticlockwise. Let $\phi(x, y)$ and $\psi(x, y)$ be C^2 scalar functions. Let $\hat{\mathbf{n}}$ be the unit outward normal to C in the xy-plane. Prove that

$$\int_C (\phi \nabla \psi - \psi \nabla \phi) \cdot \hat{\mathbf{n}}\, ds = \iint_D \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right)\, dxdy.$$

93. Stokes' Theorem. Verify Stokes' theorem for the upper hemisphere of radius 1, centred at $(0, 0, 0)$, where the vector field is

(a) $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$; (b) $\boldsymbol{F}(x, y, z) = \left(x^3 + \dfrac{z^4}{4}, 4x, xz^3 + z^2\right)$.

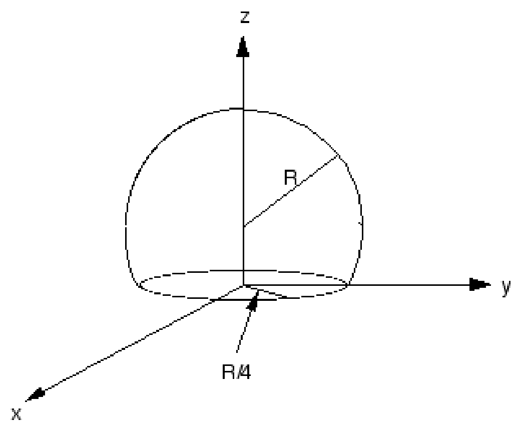
94. Apply Stokes' Theorem. (a) Redo question 85(b) using Stokes' theorem.

(b) Using Stokes' theorem evaluate

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$$

where $\boldsymbol{F}(x, y, z) = y\boldsymbol{i} - x\boldsymbol{j} + zx^3y^2\boldsymbol{k}$ and S is the surface $x^2 + y^2 + 3z^2 = 1, z \leq 0$. Take normal pointing downwards.

(c) A hot air balloon has the truncated spherical shape shown below. The hot gases escape through the porous surface with a velocity field $\boldsymbol{F}(x, y, z) = \boldsymbol{\nabla} \times (-y, x, 0)$. If $R = 5$, compute the volume flow rate (flux) of the gases through the surface. Take normal pointing outwards from balloon.



95. Integral Proofs. If C is a closed curve which is the boundary of an open surface S, and f and g are C^2 functions, show that

(a) $\displaystyle\int_C f\boldsymbol{\nabla} g \cdot d\mathbf{s} = \iint_S (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) \cdot d\boldsymbol{S}$; (b) $\displaystyle\int_C (f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f) \cdot d\boldsymbol{s} = 0.$

96. Conservative Fields. In each case, show that $\boldsymbol{F}$ is a conservative vector field and find a scalar function ϕ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$. Evaluate $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ along paths joining $(1, -2, 1)$ to $(3, 1, 4)$.

(a) $\boldsymbol{F}(x, y, z) = (2xyz + \sin x)\boldsymbol{i} + x^2 z\boldsymbol{j} + x^2 y\boldsymbol{k}$;

(b) $\boldsymbol{F}(x, y, z) = (2xy + z^3)\boldsymbol{i} + x^2\boldsymbol{j} + 3xz^2\boldsymbol{k}$.

97. Further Line Integrals. Evaluate $\displaystyle\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$ along the given path.

(a) $\boldsymbol{F}(x, y, z) = e^x \sin y\boldsymbol{i} + e^x \cos y\boldsymbol{j} + z^2\boldsymbol{k}$; $\quad \boldsymbol{c}(t) = (\sqrt{t}, t^3, e^{\sqrt{t}}), \quad 0 \le t \le 1.$

(b) $\boldsymbol{F}(x, y, z) = (xy^2 + 3x^2 y)\boldsymbol{i} + (x^3 + x^2 y)\boldsymbol{j}$; $\boldsymbol{c}$ is the path consisting of line segments from (1,1) to (0,2) to (3,0).

98. Divergence Theorem. Verify Gauss' Divergence theorem for the cube $[0, 1] \times [0, 1] \times [0, 1]$ when

$$\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

99. Apply Divergence Theorem. (a) Redo questions 84(a) and 84(c) using Gauss' Divergence theorem.

(b) Let S be the surface of the closed region $x^2 + y^2 \leq z \leq 1$ and $x \geq 0$. Find the flux of the velocity field $\boldsymbol{F}(x, y, z) = (x - y, y - z, z - x)$ in the direction of the outward normal to S.

100. More Integral Proofs. Let $\boldsymbol{F}$ be a smooth vector field in $\mathbb{R}^3$. Let ∂W be an oriented closed surface that bounds a region W.

(a) Suppose $\boldsymbol{F}$ is tangent to ∂W. Prove that

$$\iiint_W \boldsymbol{\nabla} \cdot \boldsymbol{F} \, dV = 0.$$

(b) Prove that

$$\iiint_W (\boldsymbol{\nabla} f) \cdot \boldsymbol{F} \, dV = \iint_{\partial W} f\boldsymbol{F} \cdot d\boldsymbol{S} - \iiint_W f\boldsymbol{\nabla} \cdot \boldsymbol{F} \, dV.$$

101. Mixed Integral Theorems. In (a), (b), (d), and (e) take normal pointing outwards from the surface. By applying an appropriate integral theorem, evaluate the following integrals.

(a) $\displaystyle\iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S}$ over the portion of the surface $2z = x^2 + y^2$ below the plane $z = 2$ when $\boldsymbol{F}(x, y, z) = (3y, -xz, -yz^2)$.

(b) $\displaystyle\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where S is the surface of the closed cylinder $x^2 + y^2 \leq 1, \quad 0 \leq z \leq 1$ and $\boldsymbol{F}(x, y, z) = (1, 1, z(x^2 + y^2)^2)$.

(c) $\displaystyle\int_C x^3 dy - y^3 dx$ where C is the unit circle $x^2 + y^2 = 1$ traversed in the clockwise direction.

(d) $\displaystyle\iint_S \boldsymbol{F} \cdot d\mathbf{S}$ where $\boldsymbol{F}(x, y, z) = (2xy + z, y^2, -x - 3y)$ and S is surface of the tetrahedron bounded by $2x + 2y + z = 6, x = 0, y = 0, z = 0$.

(e) $\displaystyle\iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x, y, z) = (zx + z^2 y + x, z^3 yx + y, z^4 x^2)$. Let S be the capped cylindrical surface given by the union of two surfaces S_1 and S_2 where S_1 is $x^2+y^2 = 1, 0 \leq z \leq 1$ and

S_2 is $x^2 + y^2 + (z - 1)^2 = 1, z \geq 1$.

(f) $\displaystyle\int_C \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\ ds$ where $\boldsymbol{F}(x, y) = 2xy\boldsymbol{i} - y^2\boldsymbol{j}$, $\hat{\boldsymbol{n}}$ is the unit outward normal to the curve C in the xy-plane and C is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(a, b > 0)$, traversed in the anticlockwise direction.

102. Think Carefully. In (a) and (b) take the normal pointing outwards from the surface. In each case, use an appropriate integral theorem to evaluate the integral.

(a) $\displaystyle\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x, y, z) = (x^3, 3yz^2, 3y^2z + 10)$ and S is the surface $z = -\sqrt{4 - x^2 - y^2}$.

(b) $\displaystyle\iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S}$ where S is that part of the sphere $x^2 + y^2 + (z - 2)^2 = 5$ for $z \geq 1$ and $\boldsymbol{F}(x, y, z) = (e^{y+z} - y^3\,, xe^{y+z}\,, e^{-xyz} - z^2\,)$.

(c) $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ where $\boldsymbol{F}(x, y) = \dfrac{y}{x^2 + y^2}\boldsymbol{i} - \dfrac{x}{x^2 + y^2}\boldsymbol{j}$ and C is the curve $x^2 + 2y^2 = 1$ oriented anticlockwise.

103. Maxwells Equations. Verify that Maxwell's equations imply the equation of continuity for $\boldsymbol{J}$ and ρ.

Hint: Take the divergence of Ampere's Law and differentiate Gauss' Law.

MAST20009 Vector Calculus

Problem Sheet 6

General Curvilinear Coordinates

104. Elliptical Cylindrical Coordinates. Define curvilinear coordinates (u, v, z) by

$$x = a\cosh u \cos v, \qquad y = a \sinh u \sin v, \qquad z = z$$

where $u \geq 0, 0 \leq v \leq 2\pi$ and $a > 0$ is a real constant.

(a) Find the unit vectors $\boldsymbol{e}_u, \boldsymbol{e}_v, \boldsymbol{e}_z$.

(b) Show that the system is orthogonal.

(c) Show that the element of volume is of the form

$$dV = a^2(\sinh^2 u + \sin^2 v)dudvdz.$$

105. Cartesian to Cylindrical. Represent the vector $\mathbf{A}(x, y, z) = z\boldsymbol{i} - 2x\boldsymbol{j} + y\boldsymbol{k}$ in cylindrical coordinates by writing it in the form

$$\mathbf{A}(\rho, \phi, z) = A_1(\rho, \phi, z)\hat{\boldsymbol{\rho}} + A_2(\rho, \phi, z)\hat{\boldsymbol{\phi}} + A_3(\rho, \phi, z)\hat{\boldsymbol{z}}.$$

106. Cylindrical Coordinates. Define curvilinear coordinates (ρ, ϕ, z) by

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z.$$

where $\rho \geq 0, 0 \leq \phi < 2\pi$. If n is an integer, find expressions for the following quantities in terms of ρ, ϕ, z and $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}$.

(a) $\boldsymbol{\nabla}\phi$;

(b) $\boldsymbol{\nabla}\rho^n$;

(c) $\nabla^2(\rho^2\cos\phi)$;

(d) $\boldsymbol{\nabla}\cdot(\rho\hat{\boldsymbol{\rho}} + \rho\phi\hat{\boldsymbol{\phi}} + z\hat{\boldsymbol{z}})$;

(e) $\nabla \times \hat{\boldsymbol{\phi}}$;

(f) $\boldsymbol{\nabla}\cdot(\rho^{n-1}\sin(n\phi)\hat{\boldsymbol{\rho}} + \rho^{n-1}\cos(n\phi)\hat{\boldsymbol{\phi}})$.

107. Spherical Coordinates. Define curvilinear coordinates (r, θ, ϕ) by

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta.$$

where $r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$. Find expressions for the following quantities in terms of r, θ, ϕ and $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$.

(a) $\boldsymbol{\nabla}\phi$;

(b) $\boldsymbol{\nabla}\theta$;

(c) $\boldsymbol{\nabla} \cdot (\hat{\boldsymbol{r}} \cot\phi - 2\hat{\boldsymbol{\phi}})$;

(d) $\boldsymbol{\nabla} \times (\hat{\boldsymbol{r}} \cot\phi - 2\hat{\boldsymbol{\phi}})$;

(e) $\nabla^2(r + \theta + \phi)$;

(f) $\boldsymbol{\nabla} \cdot (\sin\theta \hat{\boldsymbol{\theta}})$.

MAST20009 Vector Calculus
Problem Sheet 1 Answers
Functions of Several Variables

1. Limits.

(a) 4 (b) 0 (c) does not exist (d) 5

(e) does not exist (f) does not exist (g) 0

(h) does not exist (i) 1 (j) does not exist

2. No Limits.

3. Limit Exists.

4. Continuity.

(a) (i) is continous at $(x, y) = (0, 0)$.

(b) (ii) can be made continuous by defining $f(x, y) = 1$ at $(x, y) = (0, 0)$.

5. Partial Derivatives Revision.

In order listed.

(a) $\dfrac{1}{x+3y}, \quad \dfrac{3}{x+3y}, \quad \dfrac{-1}{(x+3y)^2}, \quad \dfrac{-3}{(x+3y)^2}, \quad \dfrac{-3}{(x+3y)^2}, \quad \dfrac{-9}{(x+3y)^2}$

(b) $ye^{xy}, \quad xe^{xy}, \quad y^2e^{xy}, \quad e^{xy} + xye^{xy}, \quad e^{xy} + xye^{xy}, \quad x^2e^{xy}$

(c) $2xy^3z^5, \quad 3x^2y^2z^5, \quad 5x^2y^3z^4, \quad 6x^2yz^5, \quad 15x^2y^2z^4, \quad 30xy^2z^4$

(d) $\dfrac{1}{y}, \quad \dfrac{-x}{y^2} + \dfrac{1}{z}, \quad \dfrac{-y}{z^2}, \quad \dfrac{-1}{z^2}, \quad \dfrac{2x}{y^3}, \quad \dfrac{2}{y^3}$

6. Laplace's Equation.

$$V_{xx} = GmM\frac{y^2 + z^2 - 2x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}, \quad V_{yy} = GmM\frac{x^2 + z^2 - 2y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}, \quad V_{zz} = GmM\frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

7. Differentiable Functions.

(a) $x \neq 0, y \neq 0$ (b) $(x, y) \neq (0, 0)$ (c) $r > 0$, all θ

8. Continuity and Differentiability.

(a) $\dfrac{\partial f}{\partial y} = \dfrac{-3x^2y^2 - 5y^4 - 40x^2y}{(x^2+5y^2)^2}$ (b) $-\dfrac{1}{5}$

(c) No, since $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}$ does not exist (d) No

9. Different Mixed Partials.

(a) $\dfrac{\partial f}{\partial x} = \dfrac{x^4y - y^5 + 4x^2y^3}{(x^2+y^2)^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$

(c) $f(x,y)$ is not C^2

10. Matrix Chain Rule.

(a) 24 (b) $(8,-16)$ (c) $\begin{bmatrix} -3 & 2 \\ 12 & 24 \end{bmatrix}$

11. Matrix Chain Rule Again.

(a) $\begin{bmatrix} -2 & 2 \\ 0 & 16 \\ 1 & -9 \end{bmatrix}$ (b) $\begin{bmatrix} -256 & 0 \\ 512 & 0 \end{bmatrix}$

12. Cylindrical Coordinates.

(a) ρ (b) $\rho = \sqrt{X^2+Y^2}$, $\tan\phi = \dfrac{Y}{X}$

(c) $\dfrac{\partial \rho}{\partial X} = \cos\phi$, $\dfrac{\partial \phi}{\partial Y} = \dfrac{\cos\phi}{\rho}$, $\dfrac{\partial z}{\partial Z} = 1$

13. Spherical Coordinates.

(a) $r^2\sin\theta$ (b) $r = \sqrt{x^2+y^2+z^2}, \theta = \arccos\left(\dfrac{z}{r}\right)$, $\tan\phi = \dfrac{y}{x}$

(c) $\dfrac{\partial r}{\partial x} = \sin\theta\cos\phi$, $\dfrac{\partial \theta}{\partial y} = \dfrac{\sin\phi\cos\theta}{r}$, $\dfrac{\partial \phi}{\partial z} = 0$

14. Product of Jacobians.

15. Taylor Polynomials.

(a) $1 + x + y + \dfrac{1}{2}(x^2+2xy+y^2) + \dfrac{1}{6}\left(x^3+3x^2y+3xy^2+y^3\right)$

(b) $1 + xy$

(c) $1 - (x-1) + (y-1) + (x-1)^2 - 2(x-1)(y-1) + (y-1)^2$

(d) $-4\left(x-\frac{\pi}{8}\right)+3y+\frac{1}{6}\left[64\left(x-\frac{\pi}{8}\right)^3-144\left(x-\frac{\pi}{8}\right)^2y+108\left(x-\frac{\pi}{8}\right)y^2-27y^3\right]$

16. Taylor Polynomial Application.

(a) $1+(x-1)^2-\frac{1}{2}y^2$

(b) approximate answer is 0.99, exact answer is 0.98992

(c) $\frac{7}{6}$

17. Taylor Remainder.

(a) $1+\frac{1}{4}x+\frac{1}{2}y$ (b) 1.075, $|R_1|\leq 0.0015$

18. Another Taylor Remainder.

(a) $1-2(x-1)+(y+1)+2(x-1)^2-2(x-1)(y+1)+\frac{1}{2}(y+1)^2$, 1.105

(b) $\frac{e^{0.1}}{6000}<0.000184$

19. Maxima and Minima Revision.

(a) saddle point at $(0,0)$

(b) minimum at $(2,1)$, saddle point at $(0,0)$

(c) mimimum on $y=-x$.

20. Lagrange Multipliers.

(a) maximum at $\left(\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$, minimum at $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{3}}\right)$

(b) maximum at $(\sqrt{3},0)$, minimum at $(-\sqrt{3},0)$,

(c) no extrema

(d) maximum at $(\frac{-1}{\sqrt{6}},\frac{2}{\sqrt{6}},1+\frac{2}{\sqrt{6}})$, minimum at $(\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},1-\frac{2}{\sqrt{6}})$.

(e) maximum at $(2,-1,1)$, minimum at $(-2,1,-1)$.

21. More Lagrange Multipliers.

(a) Problem is to minimise $S=2\pi r^2+2\pi rh$ given $\pi r^2h=1000$.
radius $=\dfrac{10}{(2\pi)^{\frac{1}{3}}}$ cm, height $=\dfrac{20}{(2\pi)^{\frac{1}{3}}}$ cm

(b) Problem is to minimise $F=x^2+y^2+z^2$ given $x+y=1$ and $x^2+2y^2+z^2=1$.
$(x,y,z)=\left(\frac{1}{3},\frac{2}{3},0\right)$

22. Inequality Constraints.

(a) $f = 8$ is minimum at $(0, 1), (0, -1)$, $f = 15$ is maximum at $(-1, 0), (1, 0)$

(b) Problem is to maximise $V = lwh$ given $l + 2w + 2h \leq 108$.

volume $= 11664$ cm^3

MAST20009 Vector Calculus
Problem Sheet 2 Answers
Space Curves and Vector Fields

23. Vectors Revision.

(a) (i) -1 (ii) $(1,-2,-2)$ (b) $\frac{1}{\sqrt{78}}(7,-2,-5)$ (c) $b=3$.

24. Equation of Lines Revision.

(a) $\boldsymbol{r}=(0,1,2)+t(1,-2,1)$; $x=t,\ y=1-2t,\ z=2+t$
(b) $\boldsymbol{r}=(1,2,1)+t(3,-4,2)$; $x=1+3t,\ y=2-4t,\ z=1+2t$

25. Equation of Planes Revision.

(a) $2x-3y-7z=-8$; (b) $x+2y-z=-1$.

26. Paths.

(a) $v=2\boldsymbol{j},\ a=-\boldsymbol{i}$, tangent$=\boldsymbol{i}+2t\boldsymbol{j}$
(b) $v=\sqrt{2}\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k},\ a=\boldsymbol{j}+\boldsymbol{k}$, tangent$=\sqrt{2}t\boldsymbol{i}+(1+t)\boldsymbol{j}+(1-t)\boldsymbol{k}$

27. Rolling Wheel.

$t=\frac{nR\pi}{v}$ where n is an integer. If n is even speed is 0. If n is odd speed is $2v$.

28. Velocity Properties.

29. Derivative of Triple Products.

(a) $\mathbf{F}\cdot\frac{d\mathbf{F}}{dt}\times\frac{d^3\mathbf{F}}{dt^3}$ (b) $\mathbf{u}'''\cdot\mathbf{u}\times\mathbf{u}'$

30. Arc Length.

(a) $2\pi\sqrt{5}$ (b) $2(2\sqrt{2}-1)$

31. Calculate Path Properties.

(a) $\mathbf{T}=(\sin t,\cos t,0)$, $\mathbf{N}=(\cos t,-\sin t,0)$, $\mathbf{B}=(0,0,-1)$,
(b) $\kappa=\frac{1}{t},\tau=0$

32. Calculate Path Properties Again.

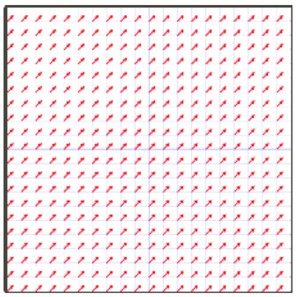
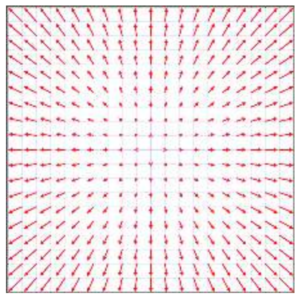
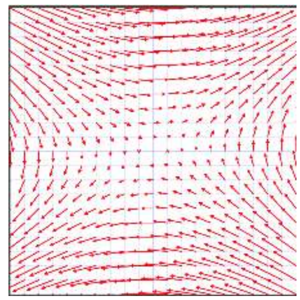
(a) $\mathbf{T} = \frac{1}{\sqrt{5}}(2\sin t + \cos t, 2\cos t - \sin t, 0)$,

$\mathbf{N} = \frac{1}{\sqrt{5}}(2\cos t - \sin t, -2\sin t - \cos t, 0)$, $\quad \mathbf{B} = (0, 0, -1)$,

(b) $\kappa = \frac{e^{-2t}}{\sqrt{5}}, \tau = 0$

33. Path Properties.

34. Flow Fields.

(a)  (b)  (c) 

35. Flow Lines.

(a) $x^2 + y^2 = r^2$ where r is a constant

(b) $y = \frac{c}{x}$ where c is a constant

(c) level curves are concentric circles $x^2 + y^2 = r^2$ where r is a constant

36. Flow Line Properties.

37. Divergence.

(a) 0 $\quad$ (b) 3 $\quad$ (c) $y\cos(xy) + x^2 \sin(x^2 y)$

38. Curl.

(a) $\cos z\boldsymbol{i} - x\boldsymbol{k}$ $\quad$ (b) $\mathbf{0}$ $\quad$ (c) $(10y - 8z)\boldsymbol{i} - (10x - 6z)\boldsymbol{j} + (8x - 6y)\boldsymbol{k}$

39. Div, Grad, and Curl.

(a) $2xy\boldsymbol{i} + x^2\boldsymbol{j}$ $\quad$ (b) $3xy^2z\boldsymbol{i} + (4xz - y^3z)\boldsymbol{j}$ $\quad$ (c) $2z^2 + xy^3$

(d) undefined $\quad$ (e) $-y^3zx^3\boldsymbol{i} + 2y^4x^2z\boldsymbol{j} + (2x^3z^2 - 2xy)\boldsymbol{k}$ $\quad$ (f) $4x^2yz^2 + x^2$

(g) undefined $\quad$ (h) 2y $\quad$ (i) $4x\boldsymbol{i} + 6xyz\boldsymbol{k}$

40. Curl of Grad.

41. Scalar Potentials.

(a) yes, $\phi = x^2 + \frac{3}{2}y^2 + 2z^2 + c$ (b) no (c) yes, $\phi = x^2 y e^z + \frac{1}{3}z^3 + c$

42. Finding Vector Fields.

(a) yes, $\mathbf{G} = -\frac{1}{2}(yz^2 + y^2)\mathbf{i} - \frac{1}{2}xz^2\mathbf{j}$
(b) yes, $\mathbf{G} = -(z\sin y + y\sin x)\mathbf{i} - xz\cos y\mathbf{j}$ (c) no

43. Vector Identity One.

(b) $\frac{\mathbf{r}e^{r^2}}{r^3}(2r^2 - 1)$

44. Vector Identity Two.

(b) $\dfrac{-2}{r^5}$

45. Vector Identity Three.

(b) 0

46. Using Vector Identities.

47. Vector Properties.

MAST20009 Vector Calculus

Problem Sheet 3 Answers

Double and Triple Integrals

48. Revision Integrals.

(a) $\frac{1}{32}\sin 4x + \frac{3}{8}x + \frac{1}{4}\sin 2x + C$

(b) $-\log|x| - \frac{1}{x} + \log|x+1| + C$

(c) $\frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$

(d) $x\sinh x - \cosh x + C$

(e) $\frac{1}{3}e^{x^3} + C$

(f) $\frac{1}{4}(\operatorname{arcsinh} 2x + 2x\sqrt{1+4x^2}) + C$

49. Double Integrals Revision.

(a) $\frac{1}{2} + \pi$ (b) $\log 2 - \frac{1}{2}$

50. Double Integrals.

(a) $\frac{2}{9}e^3 - \frac{7}{18}$ (b) $\frac{1}{3}\sin 1$

51. Double Integrals over Regions.

(a) 5 (b) $1 + e^{-2}$

52. Double Integral Applications.

(a) $\frac{11}{2}$ (b) 4.5 (c) πab

53. Repeated Integrals.

(a) $\dfrac{16}{5}$ (b) $\dfrac{16}{3}$ (c) $\dfrac{36}{5}$

54. Look Before You Leap.

(a) $\frac{1}{3}(1 - \cos 27)$ (b) $\frac{e}{2} - 1$

55. Triple Integrals.

(a) e^{-1} (b) 16 (c) $\frac{7}{60}$ (d) $\frac{1}{6}$

56. Surfaces Revision.

(a) hemisphere radius 1 centred at the origin for $z \geq 0$
(b) cone with the $z-$axis as the axis of symmetry
(c) plane intercepts $(3,0,0),(0,2,0),(0,0,6)$
(d) sphere radius 4 centred at $(1,0,0)$
(e) cylinder of radius 3 and height 6 units with the $z-$axis as the axis of symmetry
(f) inverted parabolic bowl, z intercept (0, 0, 3), with the $z-$axis as the axis of symmetry

57. Elementary Regions.

(a) $-1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \quad x^2+y^2 \leq z \leq \sqrt{x^2+y^2}$
(b) $-1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \quad 0 \leq z \leq \sqrt{1-x^2-y^2}$
(c) $\frac{-1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}, \quad -\sqrt{1-2x^2} \leq y \leq \sqrt{1-2x^2}, \quad -\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}$
(d) $-\sqrt{9-3z^2} \leq x \leq \sqrt{9-3z^2}, \quad 0 \leq y \leq 5, \quad -\sqrt{3} \leq z \leq \sqrt{3}$

58. Volumes via Triple Integrals.

(a) $\dfrac{abc}{6}$ (b) 32π

59. Convert from Polar.

(a)(i) $(x,y) = \left(\dfrac{-7}{2}, \dfrac{7\sqrt{3}}{2}\right)$ (ii) $(x,y) = \left(-3\sqrt{3}, -3\right)$ (iii) $(x,y) = (0,0)$

(b)(i) $(r,\theta) = \left(4, \dfrac{11\pi}{6}\right)$ (ii) $(r,\theta) = \left(2, \dfrac{3\pi}{2}\right)$ (iii) $(r,\theta) = \left(8\sqrt{2}, \dfrac{5\pi}{4}\right)$

60. Convert from Cylindrical.

(a) $(x,y,z) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right), \quad (r,\theta,\phi) = \left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{4}\right)$
(b) $(x,y,z) = (0,2,-4), \quad (r,\theta,\phi) = \left(2\sqrt{5}, 2.678, \frac{\pi}{2}\right)$
(c) $(x,y,z) = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 4\right), \quad (r,\theta,\phi) = \left(5, 0.644, \frac{\pi}{6}\right)$
(d) $(x,y,z) = \left(-\sqrt{2}, \sqrt{2}, -2\right), \quad (r,\theta,\phi) = \left(2\sqrt{2}, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$

61. Convert from Cartesian.

(a) $(r,\theta,\phi) = (3, 2.301, 0.464), \quad (\rho,\phi,z) = \left(\sqrt{5}, 0.464, -2\right)$
(b) $(r,\theta,\phi) = \left(5, 0.644, \frac{\pi}{2}\right), \quad (\rho,\phi,z) = \left(3, \frac{\pi}{2}, 4\right)$
(c) $(r,\theta,\phi) = \left(2, \frac{\pi}{3}, 0.615\right), \quad (\rho,\phi,z) = \left(\sqrt{3}, 0.615, 1\right)$
(d) $(r,\theta,\phi) = \left(5, 0.927, \frac{7\pi}{6}\right), \quad (\rho,\phi,z) = \left(4, \frac{7\pi}{6}, 3\right)$

62. Change of Variables.

(a) $\dfrac{1}{4}\left(e - \dfrac{1}{e}\right)$ (b) 3

63. More Change of Variables.

(a) $x = u + \frac{1}{2}v, y = v$ (b) 4 (c) $x = u + \frac{1}{2}v + 1, y = v + 1$ (d) 8

64. Polar Coordinates.

(a) $\pi(e - 1)$ (b) 30π

65. Elliptical Coordinates.

$\frac{\pi}{4\sqrt{6}}\left(\cosh(2) - 1\right)$

66. Change of Coordinates.

(a) $\displaystyle\int_0^1 \int_0^{2\pi} \int_{-\sqrt{4-\rho^2}}^{\sqrt{4-\rho^2}} z\rho^3 \cos\phi \sin\phi \, dz d\phi d\rho$ (b) $\displaystyle\int_0^2 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} r^4 \cos^2\theta \sin\theta \, d\theta d\phi dr$

67. Change Coordinates.

(a) $4\pi \log\left(\frac{a}{b}\right)$ (b)$\frac{\pi}{4}$

68. Volumes.

(a) $\frac{\pi}{3}(4\sqrt{2} - \frac{7}{2})$ (b) 32π

69. Centre of Mass.

(a) $(\frac{11}{18}, \frac{65}{126})$ (b) $(0, 0, \frac{7}{3})$

70. Other Applications.

(a) $7\left(\frac{\pi^4}{3} + 4\pi^2\right) \approx \$503.64$ (b) (i) 32 (ii) on $x^2 + y^2 + z^2 = 1$ (c) $\dfrac{8\pi\rho R^5}{15}$

MAST20009 Vector Calculus
Problem Sheet 4 Answers
Integrals over Paths and Surfaces

71. Parametric Curves.

(a) $\mathbf{c} = \sqrt{a^2 - y^2}\mathbf{i} + y\mathbf{j}, \quad 0 \le y \le a;$ (b) $\mathbf{c} = x\mathbf{i} + \sqrt{a^2 - x^2}\mathbf{j}, \quad 0 \le x \le a;$
(c) $\mathbf{c} = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j}, \quad 0 \le \theta \le \dfrac{\pi}{2};$ (d) $\mathbf{c} = a\cos\theta\mathbf{i} - a\sin\theta\mathbf{j}, \quad \dfrac{3\pi}{2} \le \theta \le 2\pi.$

72. Path Integrals.

(a) $2\sqrt{2}\pi^2$ (b) 2 (c) $\frac{2}{3}(8 - 3\sqrt{3})$

73. More Path Integrals.

(a) $\frac{1}{12}(5\sqrt{5} - 1)$ (b) 8 grams (c) $\frac{5\pi}{2}$ Coulombs

74. Line Integrals.

(a) $\frac{3}{2}$ (b) 0 (c) 147

75. More Line Integrals.

(a) 0 (b) $\frac{2}{3}$ (c) 0 (d) 8

76. Line Integral Properties.

77. Parametrised Surfaces.

(a) $x = \sqrt{2}\cos\phi, y = y, z = \sqrt{2}\sin\phi,\ 0 \le \phi \le 2\pi,\ 0 \le y \le 2$
(b) $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = 15 - 3\rho,\ 0 \le \phi \le 2\pi,\ 0 \le \rho \le 5$
(c) $x = 3\sin\theta\cos\phi,\ y = 3\sin\theta\sin\phi,\ z = 3\cos\theta,\ 0 \le \phi \le 2\pi,\ \frac{\pi}{2} \le \theta \le \pi$

78. Parametrised Normals.

(a) (x, y, z), unit sphere centred at origin
(b) $(-x, 0, -z)$ cylinder with axis on y axis extending from $-1 \le y \le 3$

79. Parametrised Surface 1.

(a) $(4uv, -4v, 2)$ (b) smooth for all u, v (c) $2y - z = 1$

80. Parametrised Surface 2.

(a) $(4 - 2u, -8u - 4uv, 2u + 2v)$ (b) not smooth at $(u, v) = (2, -2)$ (c) $4x + z = 1$

81. Surface Area.

(a) $\frac{\pi}{3}(6\sqrt{6} - 8)$ (b) $4\pi^2 R$

82. Surface Integrals of Scalars.

(a) πa^3 (b) $\frac{1}{\sqrt{6}}$ (c) $\frac{40}{3}$

83. Mass of Sphere.

$\frac{4}{3}\pi R^4$

84. Flux Across a Surface.

(a) $\frac{12\pi}{5}$ (b) π (c) 4π

85. Surface Integrals of Vectors.

(a) 24 (b) -16π

MAST20009 Vector Calculus

Problem Sheet 5 Answers

Integral Theorems

86. Green's Theorem.

(a) 0 (b) $-\pi R^2$

87. Apply Green's Theorem.

0, 8

88. Area via Line Integrals.

(a) $3\pi a^2$ (b) πab

89. Harmonic Functions.

90. Divergence Theorem in Plane.

(a) 2π (b) 5

91. Apply Divergence Theorem in Plane.

(a) 24 (b) 0

92. Integral Theorem Proof.

93. Stokes' Theorem.

(a) 0 (b) 4π

94. Apply Stokes' Theorem.

(a) -16π (b) 2π (c) $\frac{25\pi}{8}$

95. Integral Proofs.

96. Conservative Fields.

(a) $\phi = x^2yz - \cos x + c,\ 38 - \cos 3 + \cos 1$ (b) $\phi = xz^3 + x^2y + c,\ 202$

97. Further Line Integrals.

(a) $e \sin 1 + \frac{1}{3}e^3 - \frac{1}{3}$, (b) $-\frac{3}{2}$

98. Divergence Theorem.

3

99. Apply Divergence Theorem.

(a) $\frac{12\pi}{5}$, 4π (b) $\frac{3\pi}{4}$

100. More Integral Proofs.

101. Mixed Integral Theorems.

(a) 20π (b) $\frac{\pi}{3}$ (c) $\frac{-3\pi}{2}$ (d) 27 (e) 0 (f) 0

102. Think Carefully.

(a) $\frac{-8\pi}{5}$ (b) 12π (c) -2π

103. Maxwells Equations.

MAST20009 Vector Calculus

Problem Sheet 6 Answers

General Curvilinear Coordinates

104. Elliptical Cylindrical Coordinates.

(a) $\mathbf{e}_u = \dfrac{(\sinh u \cos v, \cosh u \sin v, 0)}{\sqrt{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v}}$, $\quad \mathbf{e}_v = \dfrac{(-\cosh u \sin v, \sinh u \cos v, 0)}{\sqrt{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v}}$,
$\mathbf{e}_z = (0, 0, 1)$

105. Cartesian to Cylindrical.

$(z\cos\phi - \rho\sin 2\phi)\hat{\boldsymbol{\rho}} - (z\sin\phi + 2\rho\cos^2\phi)\hat{\boldsymbol{\phi}} + \rho\sin\phi\,\hat{\boldsymbol{z}}$

106. Cylindrical Coordinates.

(a) $\frac{1}{\rho}\hat{\boldsymbol{\phi}}$ (b) $n\rho^{n-1}\hat{\boldsymbol{\rho}}$ (c) $3\cos\phi$ (d) 4 (e) $\frac{1}{\rho}\hat{\boldsymbol{z}}$ (f) 0

107. Spherical Coordinates.

(a) $\dfrac{1}{r\sin\theta}\hat{\boldsymbol{\phi}}$ (b) $\dfrac{1}{r}\hat{\boldsymbol{\theta}}$ (c) $\dfrac{2\cot\phi}{r}$ (d) $\dfrac{-2\cot\theta}{r}\hat{\boldsymbol{r}} + \left(\dfrac{2}{r} - \dfrac{\operatorname{cosec}^2\phi}{r\sin\theta}\right)\hat{\boldsymbol{\theta}}$
(e) $\dfrac{2}{r} + \dfrac{\cot\theta}{r^2}$ (f) $\dfrac{2\cos\theta}{r}$