The University of Melbourne
Semester 2 Assessment 2012

Department of Mathematics and Statistics
MAST10007 Linear Algebra

Reading Time: 15 minutes
Writing Time: 3 hours

This paper has: 7 pages

Identical Examination Papers: None
Common Content Papers: None

Authorized Materials:

No materials are authorized.

Calculators and mathematical tables are not permitted.

Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:

This examination consists of 13 questions.

The total number of marks is 100.

All questions may be attempted. All answers should be appropriately justified.

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) Consider the following linear system:

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 3 \\ 2x_1 & - & x_2 & + & 3x_3 & + & 7x_4 & = & 1 \\ 3x_1 & & & + & 3x_3 & + & 7x_4 & = & 5 \end{array}$$

(i) Write down the augmented matrix corresponding to the linear system.

(ii) Reduce the matrix in (i) to reduced row-echelon form.

(iii) Use the reduced row-echelon form to give all solutions in $\mathbb{R}^4$ to the linear system.

(b) Determine the values (if any) of k in $\mathbb{R}$ for which the following linear system has:

(i) no real solution,

(ii) infinitely many real solutions,

(iii) a unique real solution.

$$\begin{array}{rcrcrcl} x & + & ky & - & 2z & = & -1 \\ x & + & 2ky & - & 3kz & = & -2 \\ 7x & + & 6ky & + & (k^2-14)z & = & k^2-7 \end{array}$$

[10 marks]

2. Consider the matrices

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad D = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

Evaluate, if possible:

(a) BA

(b) $CD + A^T$

(c) CB^2A

(d) $B^TB - 5B$

[4 marks]

3. Consider the matrix

$$M = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

(a) Use cofactor expansion to calculate its determinant $\det(M)$.

(b) Use row-reduction to find the inverse of the matrix M given above or explain why it does not exist.

(c) Suppose that B is a 3×3 matrix with $\det(B) = 4$. Calculate $\det(2M(B^{-3})^{\mathrm{T}})$, where M is the matrix given above.

[6 marks]

4. Let L be the line in $\mathbb{R}^3$ given by the following Cartesian equation:

$$\frac{x}{-2} = \frac{1-y}{3}, \qquad z = 2.$$

(a) Find the intersection of the line L with the plane whose vector equation is given by:

$$(x, y, z) = (1, -1, 0) + s(1, 1, 2) + t(-1, 2, 0), \qquad s, t \in \mathbb{R}.$$

(b) Find the Cartesian equation of the plane that contains the point $(1, 0, 2)$ and is perpendicular to the line L.

[6 marks]

5. Let

$$A = \begin{bmatrix} 2 & -6 & 1 & 0 & 2 \\ 1 & -3 & 0 & 1 & 1 \\ -1 & 3 & -1 & 1 & -1 \\ -2 & 6 & -3 & 3 & -1 \\ 2 & -6 & 1 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B is the reduced row-echelon form of the matrix A. Using this information, or otherwise, answer the following questions, giving reasons for your answers.

(a) What is the rank of A?

(b) Write down a basis for the row space of A.

(c) Write down a basis for the column space of A.

(d) Is the set

$$\{(2,1,-1,-2,2),(1,0,-1,-3,1),(0,1,1,3,0),(2,1,-1,-1,2)\}$$

linearly independent? If yes, give a reason. If not, write one of these vectors as a linear combination of the others.

(e) Find a basis for the solution space of A.

[8 marks]

6. (a) For each of the following, decide whether or not the given set S is a subspace of the vector space V. Justify your answers by either using appropriate theorems, or providing a counter-example.

(i) $V = \mathcal{P}_2$ (all polynomials of degree at most two) and

$$S = \{a_0 + a_1 x + a_2 x^2 \mid 2a_0 - 3a_1 + 5a_2 = 0\}$$

(ii) $V = M_{2,2}$ (all 2×2 matrices) and

$$S = \{A \in M_{2,2} \mid \det(A) = 0\}$$

(b) Let A be an $n \times m$ matrix, prove that the nullspace of A is a subspace of $\mathbb{R}^m$.

[8 marks]

7. (a) Consider the subspace V of $M_{2,2}$ given by

$$V = \left\{ \begin{bmatrix} a & a \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

and let S be the subset of V given by

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 3 & 9 \end{bmatrix} \right\}$$

(i) Determine whether or not the set S is linearly dependent and if it is, express one of its vectors as a linear combination of the other vectors in the set S.

(ii) Determine whether or not the set S is a spanning set for the vector space V. If S is a spanning set, find a subset of S that is a basis for V.

(b) Let $\mathcal{P}_3$ be the vector space of all polynomials of degree at most three, and let T be the subset of $\mathcal{P}_3$ given by

$$T = \{1 - x + 2x^2,\ 2 - x - x^3,\ 1 + x - 3x^2\}$$

(i) Determine whether or not the set T is linearly dependent and if it is, express one of its vectors as a linear combination of the other vectors in the set T.

(ii) Determine whether or not the set T is a spanning set for the vector space $\mathcal{P}_3$. If T is a spanning set, find a subset of T that is a basis for $\mathcal{P}_3$.

[8 marks]

8. Let $\mathcal{P}_2$ denote the vector space of all real polynomials of degree at most 2 and let V denote the vector space of all 2×2 matrices with real entries and zero trace:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2} \mid a + d = 0 \right\}$$

Define a linear transformation $T\colon \mathcal{P}_2 \to V$ by

$$T(a + bx + cx^2) = \left(\begin{bmatrix} a+b & c \\ -c & -a-b \end{bmatrix} \right)$$

(a) Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

Find $[T]_{\mathcal{C},\mathcal{B}}$, the matrix that represents T relative to the bases $\mathcal{B}$ for $\mathcal{P}_2$ and $\mathcal{C}$ for V.

(b) Find bases for the image and kernel of T.

[8 marks]

9. (a) Let V be a vector space and let S be a subset of V. Define what it means to say that S is a spanning set for V.

   (b) Let $T : V \to W$ be a linear transformation, and let $S = \{v_1, \ldots, v_k\}$ be a subset of V. Show that if S is a spanning set for V, then $\{T(v_1), \ldots, T(v_k)\}$ is a spanning set for the image of T.

[6 marks]

10. Consider the following bases for $\mathbb{R}^2$:

$$\mathcal{S} = \{(1,0),(0,1)\} \quad \mathcal{B} = \{(2,1),(2,2)\}$$

   (a) (i) Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{S}$.

   (ii) Find the transition matrix $P_{\mathcal{B},\mathcal{S}}$ from $\mathcal{S}$ to $\mathcal{B}$.

   (iii) A third basis $\mathcal{C}$ for $\mathbb{R}^2$ is such that $P_{\mathcal{B},\mathcal{C}} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$. Calculate $P_{\mathcal{S},\mathcal{C}}$.

   (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Given that

$$T(2,1) = (2,1) \quad \text{and} \quad T(2,2) = (2,1) + (2,2)$$

   (i) Find the matrix $[T]_{\mathcal{B}}$ of the transformation T with respect to the basis $\mathcal{B}$.

   (ii) Find the matrix $[T]_{\mathcal{S}}$ of the transformation T with respect to the basis $\mathcal{S}$.

   (iii) Find $T(3,-1)$.

[8 marks]

11. (a) Let V be a real vector space. State the definition of an inner product on V.

   (b) Show that the following defines an inner product on $\mathbb{R}^3$.

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = 5x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 + x_3y_3$$

   (c) Let W be the subspace of $\mathbb{R}^3$ that has basis $\mathcal{B} = \{(1,1,1),(1,-1,-1)\}$. Apply the Gram-Schmidt procedure to the basis $\mathcal{B}$ to obtain a basis for W that is orthonormal with respect to the inner product defined above.

   (d) Find the point of W that is closest, *with respect to the distance given by the above inner product*, to the point $(1,1,0)$.

[10 marks]

12. Let

$$M = \begin{bmatrix} 4 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) Find the characteristic polynomial of M.

(b) Find all eigenvalues of M.

(c) For each eigenvalue find a basis for the corresponding eigenspace.

(d) State the Cayley-Hamilton Theorem and use it to write M^3 as a linear combination of the matrices I, M and M^2.

[8 marks]

13. (a) For each of the following three matrices decide whether or not the matrix is diagonalizable over $\mathbb{R}$. The eigenvalues for each matrix are given. You should justify your answers.

(i) $\begin{bmatrix} 7 & -14 & -10 \\ -2 & 7 & 4 \\ 7 & -19 & -12 \end{bmatrix}$ has eigenvalues: $-1, 1, 2$

(ii) $\begin{bmatrix} 3 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ has eigenvalues: $1, 1, 2$

(iii) $\begin{bmatrix} 0 & 0 & 1 \\ 6 & 4 & -6 \\ 1 & 1 & 0 \end{bmatrix}$ has eigenvalues: $1, 1, 2$

(b) (i) Write down a symmetric 2×2 matrix A such that

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = [8x^2 + 6xy]$$

(ii) Find an orthogonal matrix Q and a diagonal matrix D such that

$$D = Q^T A Q$$

(iii) Consider the conic

$$8x^2 + 6xy = 1$$

Use your answer to part (ii) to find a simplified equation for the conic. Hence identify and sketch the conic using both the x-y axes and your new coordinate axes (i.e., the principal axes for the conic).

[10 marks]

— END OF EXAMINATION QUESTIONS —