Techniques in Operations Research

Exam 2016/1

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(a) (i) we want the smallest n such that $\frac{(2-1)}{F_n} \leq 0.2 \Rightarrow \boxed{n=4}$

(ii) we want the smallest n such that $(2-1)\cdot\gamma^n \leq 0.2 \Rightarrow \boxed{n=4}$

(b)

Iter	K	a_k	b_k	p_k	q_k	$f(p_k)$	$f(q_k)$
1	4	1	2	1.4	1.6	0.2	0
2	3	1.4	2	1.6	1.8	0	0.2
3	2	1.4	1.8	1.6	1.6	0	0

- the final interval is [1.4, 1.6] (or [1.6, 1.8])
- the table underlines the new computations of the obj. function.

(c) Using $\gamma = 0.62$

Iter	a_k	b_k	p_k	q_k	$f(p_k)$	$f(q_k)$
1	1	2	1.38	1.62	0.22	0.02
2	1.38	2	1.62	1.76	0.02	0.16
3	1.38	1.76	1.52	1.62	0.08	0.02

$\Rightarrow$ final interval [1.52, 1.76]

②

$$f(x,y) = 2x^3 + 2y^3 - 3x^2 + 9y^2 - 12$$

(a) First order conditions :

$$\nabla f(x,y) = \begin{bmatrix} 6x^2 - 6x \\ 6y^2 + 18y \end{bmatrix} = 0$$

$$6x(x-1) = 0 \quad \begin{cases} x = 0 \\ x = 1 \end{cases}$$

$$6y(y+3) = 0 \quad \begin{cases} y = 0 \\ y = -3 \end{cases}$$

$\Rightarrow$ Stationnary points

P_1 $(0,0)$

P_2 $(0,-3)$

P_3 $(1,0)$

P_4 $(1,-3)$

(b)

$$\nabla^2 f(x,y) = \begin{bmatrix} 12x - 6 & 0 \\ 0 & 12y + 18 \end{bmatrix}$$

Reason : The hessian has :

$P_1 \rightarrow \nabla^2 f = \begin{bmatrix} -6 & 0 \\ 0 & 18 \end{bmatrix}$ saddle point - (one negative and one positive eigenvalue)

$P_2 \rightarrow \nabla^2 f = \begin{bmatrix} -6 & 0 \\ 0 & -24 \end{bmatrix}$ maximum - (two negative eigenvalues)

$P_3 \rightarrow \nabla^2 f = \begin{bmatrix} 6 & 0 \\ 0 & 18 \end{bmatrix}$ minimum - (two positive eigenvalues)

$P_4 \rightarrow \nabla^2 f = \begin{bmatrix} 6 & 0 \\ 0 & -24 \end{bmatrix}$ saddle point - (one negative and one positive eigenvalue)

③ We want to minimise

(a) $$f(\alpha,\beta) = (\alpha\beta^{2} - 3)^2 + (\alpha\beta^5 - 30)^2 + (\alpha\beta^6 - 60)^2 \quad ?$$

(b)
$$\nabla f(\alpha,\beta) = \begin{bmatrix} 2\cdot\beta^2(\alpha\beta^2-3) + 2\beta^5\cdot(\alpha\beta^5-30) + 2\beta^6\cdot(\alpha\beta^6-60) \\ 4\alpha\beta(\alpha\beta^2-3) + \overset{10}{8}\alpha\beta^4(\alpha\beta^5-30) + 12\alpha\beta^5(\alpha\beta^6-60) \end{bmatrix}$$

at $(\alpha,\beta) =$
$$\nabla f(2,2) = \begin{bmatrix} 2\cdot4\cdot(8-3) + 2\cdot32(64-30) + 2\cdot64(128-60) \\ 8(8-3) + \overset{10}{8}\cdot32(64-30) + 12\cdot64(128-60) \end{bmatrix}$$

$$= \begin{bmatrix} 40 + 64\cdot34 + 128\cdot68 \\ 40 + \underset{10}{8}\cdot32\cdot34 + 12\cdot64\cdot68 \end{bmatrix} = \begin{bmatrix} 10920 \checkmark \\ 60968 \\ 63144 \end{bmatrix}$$

$$\simeq \begin{bmatrix} 1 \\ 5.58 \\ 5.78 \end{bmatrix} \Rightarrow \text{steepest direction} = \begin{bmatrix} -1 \\ -5.58 \end{bmatrix}$$

$$f(t) = (-t\cdot(-5.58t)^2 - 3)^2 + (-t\cdot(-5.58t)^5 - 30)^2 + (-t\cdot(-5.58t)^6 - 60)^2$$
$$= (-5.58^2\cdot t^2 - 3)^2 + (5.58^6 t^6 - 30)^2 + (-5.58^6 t^7 - 60)^2$$

4 (i)

$$L(\underset{\sim}{x},\underset{\sim}{\lambda}) = -20x_1 - 10x_2 + \lambda_1(x_1^2 + x_2^2 - 1) + \lambda_2(-x_1) + \lambda_3(-x_2)$$

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(ii) $x_1 > 0 \Rightarrow \lambda_2 = 0$
$x_2 > 0 \Rightarrow \lambda_3 = 0$ KKTb 1

$$\underset{\sim}{\nabla} L(\underset{\sim}{x},\underset{\sim}{\lambda}) = \begin{pmatrix} -20 + 2\lambda_1 x_1 - \lambda_2 \\ -10 + 2\lambda_1 x_2 - \lambda_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\therefore$ $-20 + 2\lambda_1 x_1 = 0$
$-10 + 2\lambda_1 x_2 = 0$ 1 KKTa

& $x_1 = \dfrac{10}{\lambda_1}$

1

$x_2 = \dfrac{5}{\lambda_1}$

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$\Rightarrow$ Critical cone is

$$\left\{ (d_1, d_2) \,\middle|\, 4d_1 + 2d_2 = 0 \right\}$$

$$\left\{ (d_1, -2d_1) \right\} \quad 2$$

(v) For $\underset{\sim}{x}^*$ to be a local minimum

$\underset{\sim}{\nabla}^2 L(\underset{\sim}{x}^*, \underset{\sim}{\lambda}^*)$ needs to be positive definite on the critical cone. 1

$$\underset{\sim}{\nabla}^2 L(\underline{x}^*, \underline{\lambda}) = \begin{bmatrix} 2\lambda_1 & 0 \\ 0 & 2\lambda_2 \end{bmatrix}$$

$$\underset{\sim}{\nabla}^2 L\left(\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 5\sqrt{5}, 0, 0\right) = \begin{bmatrix} 10\sqrt{5} & 0 \\ 0 & 10\sqrt{5} \end{bmatrix} \quad 1$$

which is positive definite 1

$\Rightarrow x^*$ is a local min. 1 $f(x^*) = -\frac{50}{\sqrt{5}}$ 1

⑤

We rewrite the problem as :

$$\text{Min } f(x,y) = \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2 + y^2$$

$$\text{s.t.} \quad -x - a \leq 0 \xrightarrow{a=0} -x \leq 0$$

$$b - y \leq 0 \xrightarrow{b=2} 2 - y \leq 0$$

(a) $$P_k(x,y) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + y^2 + \frac{k}{2}\left[\left((-x)_+\right)^2 + \left((2-y)_+\right)^2\right]$$

(b) $$\nabla P_k(x,y) = \begin{bmatrix} x^3 - x - k(-x)_+ \\ 2y - k(2-y)_+ \end{bmatrix}$$

$(-x)_+$
$= \max(0, -x)$
$= -x$ if $x < 0$

$\nabla P_k(x,y) = 0$ (first order necessary condition)

$x^2 = k-1$

Let's assume $x < 0$:

$$x^3 - x \overset{+}{-} kx = 0 \Rightarrow x(x^2 - k \overset{+}{-} 1) = 0$$

$x = 0$

or

$x = \pm\sqrt{k+1}$ (divergent)

therefore, the unique possibility for stationnary point occurs for $x = 0$, which contradicts the hypothesis

Let's assume $\cancel{x}\, y \geq 2$

$$\Rightarrow \quad 2y - k\underbrace{\cancel{(2-y)_+}}_{=0 \text{ since } y \geq 2} = 0$$

$$2y = 0 \quad \Rightarrow \quad y = 0$$ which contradicts the hypothesis

therefore, stationnary points only occur for

$x \geq 0, \quad y < 2$.

c)

$x^3 - x = 0 \quad \rightarrow \quad x = 0$ or $x^2 - 1 = 0$

$x = 1$

(the other solution is discarded since we know $x \geq 0$)

$2y - k(2-y) \quad \rightarrow \quad 2y - 2k + ky = 0$

$y(2+k) = 2k$

$y = \dfrac{2k}{2+k}$

stationnary points:

$x^* = \lim_{k \to \infty} x_k = 0, \quad 1$

$y^* = \lim_{k \to \infty} y_k = 2$

$P_1 : \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$P_2 : \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

6 a) f is convex (its linear) and all 4 constraints are convex sets.

b) $L(\underset{\sim}{x}, \underset{\sim}{\lambda}) = x_1 + 2x_2 + 3x_3 + \lambda_1(x_1^2 + x_2^2 + x_3^2 - 1) + \lambda_2 x_1 + \lambda_3 x_2 + \lambda_4 x_3$

c) It is the only KKT point as the program is convex.

d) $$L(\underset{\sim}{x}^*, \underset{\sim}{\lambda}^*) = -\frac{1}{\sqrt{14}} - \frac{4}{\sqrt{14}} - \frac{9}{\sqrt{14}}$$

$$= -\frac{14}{\sqrt{14}}.$$

$$L(\underset{\sim}{x}, \lambda^*) = x_1 + 2x_2 + 3x_3 + \frac{7}{\sqrt{14}}(x_1^2 + x_2^2 + x_3^2 - 1)$$

$$= \frac{7}{\sqrt{14}}\left(x_1^2 + \frac{\sqrt{14}}{7}x_1 + x_2^2 + \frac{2\sqrt{14}}{7}x_2 + x_3^2 + \frac{3\sqrt{14}}{7}x_3 - 1\right)$$

$$= \frac{7}{\sqrt{14}}\left(\left(x_1 + \frac{\sqrt{14}}{14}\right)^2 - \frac{1}{14} + \left(x_2 + \frac{\sqrt{14}}{7}\right)^2 - \frac{2}{7} + \left(x_3 + \frac{3\sqrt{14}}{14}\right)^2 - \frac{9}{14} - 1\right)$$