Decision Making
Part 4: n-person games

Mark Fackrell
E-mail: fackrell@unimelb.edu.au
Office: Richard Berry room 148

Topics in this part

- Introduction: coalition, characteristic function
- Games in characteristic function form: TU-games, superadditivity, essential and inessential games, unanimity games
- Core of a TU-game: imputation set, the core of a TU-game
- The Shapley value: Shapley's axioms, Shapley value, marginal vectors
- Cost Games: cost games, dual games

References:
For the introduction to n-person games:
P. Morris, Introduction to Game Theory, Springer-Verlag, 1994, Chapter 6.

Also, Game Theory by G. Owen, 2013.

C. D. Aliprantis and S. K. Chakrabarti, Games and Decision Making, 2nd edition, Oxford University Press, 2011.

Introduction to n-person cooperative games

Setting up

- In an n-person cooperative game, there are $n \geq 2$ players, denoted by $1, 2, \ldots, n$.
- The set of players is denoted by

$$N = \{1, 2, \ldots, n\}.$$

- If $n \geq 3$, then it is no longer possible to represent payoffs by a matrix or bi-matrix.
- Instead we use a function over the set of subsets of N to represent the game.
- In games considered in this part players are allowed to cooperate fully: they can form coalitions and share payoffs in a specific way.

Coalitions

A new idea appears for n-person cooperative games: forming a coalition. When players form a coalition, they act just as one player and they play a jointly agreed set of strategies with the aim of maximising the sum of the payoffs to the players in the coalition. After this is done the next problem is how to share the rewards between the members of the coalition.

The situation is analogous to robbing a bank:

In a 2-person cooperative game, two thieves work together in the robbery, but each keeps the actual money he takes out of the bank.

In an n-person cooperative game, the thieves return to their hideout carrying whatever they took away, put it all in one bag and share it out according to a predetermined scheme.

Definition 1. A coalition is a subset S of the set of players $N = \{1, 2, \ldots, n\}$.

The counter-coalition to S is the complement of S relative to N, namely $N \setminus S = \{i : 1 \leq i \leq n, i \notin S\}$.

Call N the grand coalition and $\emptyset$ the empty coalition.

Define the power set of N by

$$2^N := \{S : S \subseteq N\} = \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}, \{1, 2\}, \ldots, N\}.$$

Every member of 2^N is a coalition.

Since the cardinality $|2^N| = \sum_{i=0}^{n} \binom{n}{i} = 2^n$, there are precisely 2^n coalitions in any n-person cooperative game.

Example 1. (coalitions)

Determine all coalitions of $N = \{1, 2, 3\}$.

Characteristic functions

Let S be a coalition. The worst thing that can happen to S is that the rest of the players form a coalition (i.e. the counter-coalition $N \backslash S$ of S) and try to minimise the payoff that S can obtain.

This can be viewed as a non-cooperative 2-person game with players S and $N \backslash S$. In the payoff bi-matrix for this game, rows correspond to the "joint pure strategies" available to S (i.e. those $(a_i)_{i \in S}$ with each a_i a pure strategy for i), and columns correspond to the "joint pure strategies" available to $N \backslash S$ (i.e. those $(b_j)_{j \in N \backslash S}$ with each b_j a pure strategy for j).

In this bi-matrix the entry in the row indexed by $(a_i)_{i \in S}$ and the column indexed by $(b_j)_{j \in N \backslash S}$ is a pair whose first coordinate is the **sum** of the payoffs to the players in S, and whose second coordinate is the **sum** of the payoffs to the players in $N \backslash S$, if the players in S play according to $(a_i)_{i \in S}$ and those in $N \backslash S$ play according to $(b_j)_{j \in N \backslash S}$.

Denote by $v(S)$ the optimal security level for player S in this non-cooperative 2-person game. Recall that $v(S)$ is the maximum payoff that S can ensure itself.

In theory we could use the techniques developed in previous parts to compute $v(S)$ for every coalition S.

Definition 2. The function which assigns the real number $v(S)$ to every coalition S is called the characteristic function of the game.

This is a function from 2^N to $\mathbb{R}$.

Example 2. (3-person game, adapted from P. Morris, pp 151-153)

Consider a 3-person game such that each player has two strategies. Let a_i and b_i be the strategies for player i, $i = 1, 2, 3$. Assume the following payoff vectors:

Strategy triples	Payoff vectors
(a_1, a_2, a_3)	$(-4, 2, 4)$
(a_1, a_2, b_3)	$(2, 2, -2)$
(a_1, b_2, a_3)	$(0, -2, 4)$
(a_1, b_2, b_3)	$(-2, 4, 0)$

Strategy triples	Payoff vectors
(b_1, a_2, a_3)	$(2, -2, 2)$
(b_1, a_2, b_3)	$(0, 0, 2)$
(b_1, b_2, a_3)	$(2, 0, 0)$
(b_1, b_2, b_3)	$(2, 4, -4)$

Determine the characteristic function v.

Theorem 1. Let S and T be **disjoint coalitions** (i.e. $S \cap T = \emptyset$). If the characteristic function is determined by using optimal security levels as described above, then

$$v(S \cup T) \geq v(S) + v(T).$$

This property is called superadditivity of the characteristic function v.

Proof.
For each $i \in N$ let X_i be the set of mixed strategies for player i. By definition of $v(S)$ there exists a joint strategy $(x_i^*)_{i \in S}$ which achieves $v(S)$, where $\forall i$: $x_i^* \in X_i$.
So if S plays $(x_i^*)_{i \in S}$ the sum of payoffs to the players in S is at least $v(S)$ (independent of the strategy of $N \backslash S$).

Similarly there exists a joint strategy $(x_i^*)_{i \in T}$ with the same properties as above for the coalition T.

Note that $(x_i^*)_{i \in S \cup T}$ defines a possible joint strategy for $S \cup T$ in the game of $S \cup T$ vs. $N \backslash (S \cup T)$. Since $S \cap T = \emptyset$ the security level with respect to this joint strategy is at least $v(S) + v(T)$.

However, $v(S \cup T)$ is the optimal (maximum) security level among all possible strategies in the game of $S \cup T$ versus $N \backslash (S \cup T)$. Therefore

$$v(S \cup T) \geq v(S) + v(T).$$

Corollary 1. Let $S_1, \ldots, S_k$ be pairwise disjoint coalitions (ie. $S_i \cap S_j = \emptyset$ whenever $i \neq j$). If v is superadditive, then

$$v(S_1 \cup \cdots \cup S_k) \geq v(S_1) + \cdots + v(S_k).$$

Proof.

Corollary 2. For any n-person superadditive cooperative game with player set $N = \{1, \ldots, n\}$,

$$v(N) \geq v(\{1\}) + \cdots + v(\{n\}).$$

Games in characteristic function form

Games in characteristic function form/TU-games

Definition 3. An n-person game in characteristic function form consists of a set of players $N = \{1, 2, \ldots, n\}$ and a function $v : 2^N \to \mathbb{R}$ such that

$$v(\emptyset) = 0.$$

$v(S)$ is called the worth or value of S and can be interpreted as as the monetary amount a coalition S can jointly generate itself by means of optimal cooperation, without any help of $N \backslash S$.

We will simply say v is an n-person game (as the names of the players are unimportant). The pair (N, v) is also referred to as a transferable utility game (TU-game). The set of all TU-games with player set N is denoted by TU^N.

In the definition above, there is no need anymore to interpret v as the optimal security level for S in the S versus $N \backslash S$ game.

Definition 4. A TU-game (N, v) is superadditive if

$$v(S \cup T) \geq v(S) + v(T)$$

for any disjoint coalitions $S, T \in 2^N$.

Many games that arise from practical situations are superadditive.

Example 3. (3-person game glove game)

Let $N = \{1, 2, 3\}$ and suppose player 1 owns one left hand glove and player 2 and 3 each own one right hand glove. A single glove is worth nothing, a (left-right) pair of gloves is worth 10 AUD. Determine the TU-game (N, v) that resembles this situation and show that v is superadditive.

Example 4. (voting game)

Consider a parliament of 150 seats with three parties, party 1 having 60 seats, party 2 having 60 seats, and party 3 having 30 seats. The threshold on the number of seats to pass bills is 76. Describe this voting situation with help of a TU-game.

Example 5. (bankruptcy game)

In a bankruptcy problem $(N, E, \boldsymbol{d})$, N represents a finite set of claimants, $E \geq 0$ is the estate which has to be divided among the claimants and $\boldsymbol{d} \in \mathbb{R}^n$, $\boldsymbol{d} \geq 0$, represents the (rightful) claims of the claimants on E. Justifying the term "bankruptcy" it is assumed that

$$E \leq \sum_{i=1}^{n} d_i.$$

Model this situation with help of a TU-game.

Essential and inessential games

Definition 5. A game (N, v) is called inessential (or additive) if

$$v(N) = \sum_{i=1}^{n} v(\{i\}),$$

and v is superadditive. A game which is not inessential is called essential.

Theorem 2. Let (N, v) be an inessential game. Then, for any coalition S,

$$v(S) = \sum_{i \in S} v(\{i\}).$$

This simply means that in an inessential game there is no reason to form a coalition.

Proof.

Being inessential does not mean the game is unimportant, as shown by the following result.

Theorem 3. Any 2-person zero-sum game is inessential when viewed as a game in characteristic function form.

Proof.

Unanimity games

Definition 6. If $v : 2^N \to \mathbb{R}$ is a game in characteristic function form then kv, $k \in \mathbb{R}$, is the game in characteristic function form defined by

$$(kv)(S) = k \cdot v(S)$$

for all $S \in 2^N$.

Definition 7. Let N be a set of players. For each $T \in 2^N \backslash \{\emptyset\}$, the unanimity game (N, u_T) is defined by

$$u_T(S) := \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{if } T \nsubseteq S. \end{cases}$$

for all $S \in 2^N$.

Example 6. (unanimity games)

Let $N = \{1, 2, 3\}$ and $T_1 = \{1, 3\}$, $T_2 = \{1\}$. Provide the characteristic function form of u_{T_1}, $5u_{T_1}$, and $-3u_{T_2}$.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$u_{T_1}(S)$	0	0	0	0	1	0	1
$u_{T_2}(S)$							
$5u_{T_1}(S)$							
$-3u_{T_2}(S)$							

Theorem 4. Every game $v \in TU^N$ can be written in a unique way as a linear combination of unanimity games:

$$v = \sum_{T \in 2^N \backslash \{\emptyset\}} c_T u_T$$

with $c_T \in \mathbb{R}$ for all $T \in 2^N \backslash \{\emptyset\}$.

Proof.
Let N be a set of players. We show that the collection

$$\left\{u_T : T \in 2^N \backslash \emptyset\right\}$$

forms a basis of TU^N.

Proof. (cont.)

Example 7. (writing v as a linear combination of unanimity games)
Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	5	4	7

Write v as a linear combination of unanimity games.

Solution. We start with determining all coefficients c_T, with $|T| = 1$. Then we determine all coefficients c_T with $|T| = 2$, using the coefficients that we already have calculated. In this way we can determine recursively all coefficients.

Example (cont.)

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	5	4	7

Let $i \in N$, then

$$v(\{i\}) = 0, \text{ and } v(\{i\}) = c_{\{i\}} \Rightarrow c_{\{i\}} = 0.$$

Taking into account the coalitions of size two, we get

$$\begin{aligned} v(\{1,2\}) = 2, \text{ and } v(\{1,2\}) &= c_{\{1\}} + c_{\{2\}} + c_{\{1,2\}} = c_{\{1,2\}} \Rightarrow c_{\{1,2\}} = 2, \\ v(\{1,3\}) = 5, \text{ and } v(\{1,3\}) &= c_{\{1\}} + c_{\{3\}} + c_{\{1,3\}} = c_{\{1,3\}} \Rightarrow c_{\{1,3\}} = 5, \\ v(\{2,3\}) = 4, \text{ and } v(\{2,3\}) &= c_{\{2\}} + c_{\{3\}} + c_{\{2,3\}} = c_{\{2,3\}} \Rightarrow c_{\{2,3\}} = 4. \end{aligned}$$

Finally,

$$\begin{aligned} v(N) = 7, \text{ and } v(N) &= c_{\{1\}} + c_{\{2\}} + c_{\{3\}} + c_{\{1,2\}} + c_{\{1,3\}} + c_{\{2,3\}} + c_{\{1,2,3\}} \\ &= 11 + c_{\{1,2,3\}} \Rightarrow c_{\{1,2,3\}} = -4. \end{aligned}$$

We conclude that $v = 2u_{\{1,2\}} + 5u_{\{1,3\}} + 4u_{\{2,3\}} - 4u_{\{1,2,3\}}$.

Example 8. (writing v as a linear combination of unanimity games)
Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	0	0	3	4	2	6

Write v as a linear combination of unanimity games.

Core of a TU-game

Imputation

To solve an n-person game we have to formulate a rule that governs the allocation of payoffs to the individual players.

Let x_i denote the reward to player i. The reward vector (or allocation vector) $\mathbf{x} = (x_1, \ldots, x_n)$ must satisfy certain conditions if it is to make sense.

Definition 8. A vector $\mathbf{x} = (x_1, \ldots, x_n)$ is called an imputation of a game $v \in \mathsf{TU}^N$ if it satisfies the following conditions:

(a) (Efficiency)

$$\sum_{i=1}^{n} x_i = v(N);$$

(b) (Individual Rationality)

$$x_i \geq v(\{i\}), \quad \text{for every } i = 1, 2, \ldots, n.$$

The set of all imputations of a game $v \in \mathsf{TU}^N$ is called the imputation set and is denoted by $I(v)$.

Example 9. (imputation set)

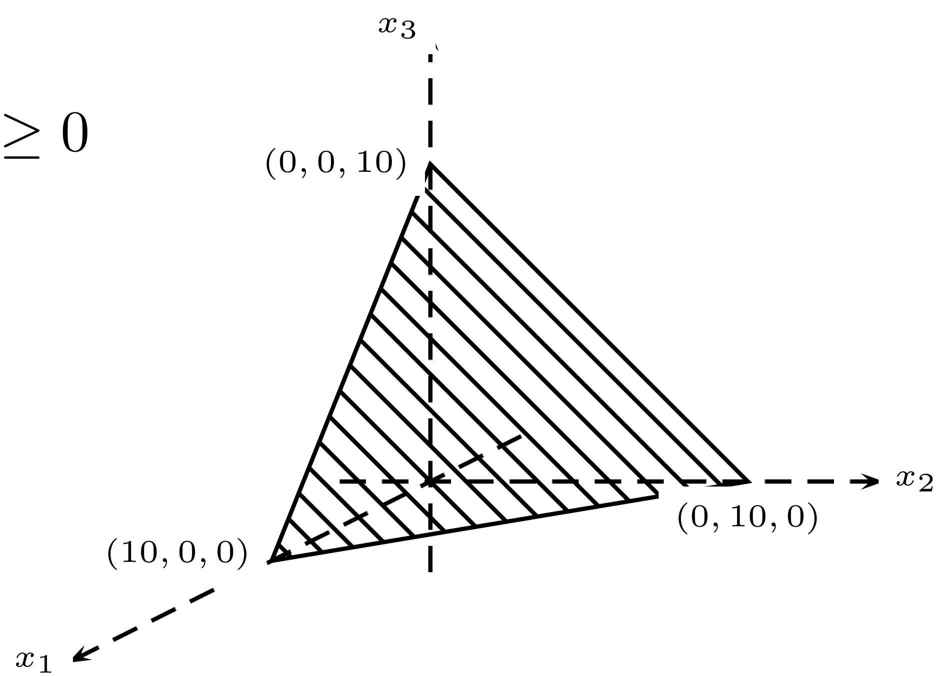
Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	10	10	0	10

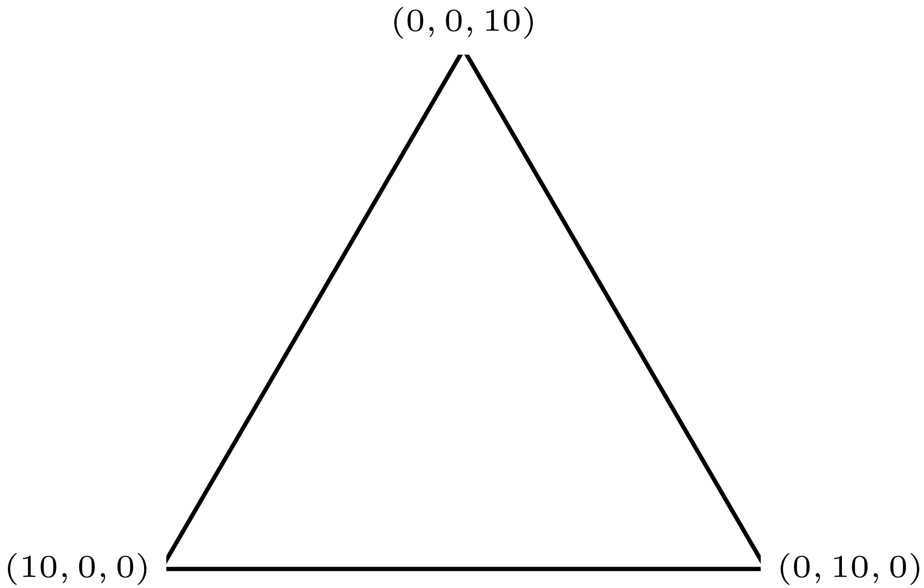
Draw the imputation set and write $I(v)$ as the convex hull of suitable vectors.

Solution.

$\mathbf{x} \in I(v) \iff$
$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$
$x_1 + x_2 + x_3 = 10.$



Or equivalently this can be drawn in two dimensions as



We see that $I(v) = \mathsf{conv}\big\{(10,0,0),(0,10,0),(0,0,10)\big\}$.

Example 10. (imputation set)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	0	0	3	4	2	6

Draw the imputation set and write $I(v)$ as the convex hull of suitable vectors.

Theorem 5. Let $v \in \mathsf{TU}^N$. Then

$$I(v) \neq \emptyset \iff v(N) \geq \sum_{i=1}^{n} v(\{i\}).$$

Furthermore, if

$$\sum_{i=1}^{n} v(\{i\}) = v(N),$$

then $I(v) = \big((v(\{1\}), \ldots, v(\{n\})\big)$.

Proof.

The core

Now we introduce our first solution concept for n-person games, which is one of the most fundamental concepts within the theory of cooperative games.

Definition 9. The core $C(v)$ of a game $v \in \mathsf{TU}^N$ is defined by

$$C(v) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\}.$$

Core elements are imputations. Furthermore, core elements are stable against coalitional deviations: no coalition can rightfully object to a proposal $\mathbf{x} \in C(v)$ (why not?). Or in other words, core allocations are stable, since no coalition has an incentive to split off of the grand coalition.

Remark: In general, the core of a game is bounded and is determined by a finite system of linear inequalities. Therefore, the core is a polytope: the convex hull of finitely many points.

Example 11. (core of a game)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	10	10	0	10

Determine $C(v)$.

Example 12. (core of a game)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	5	4	7

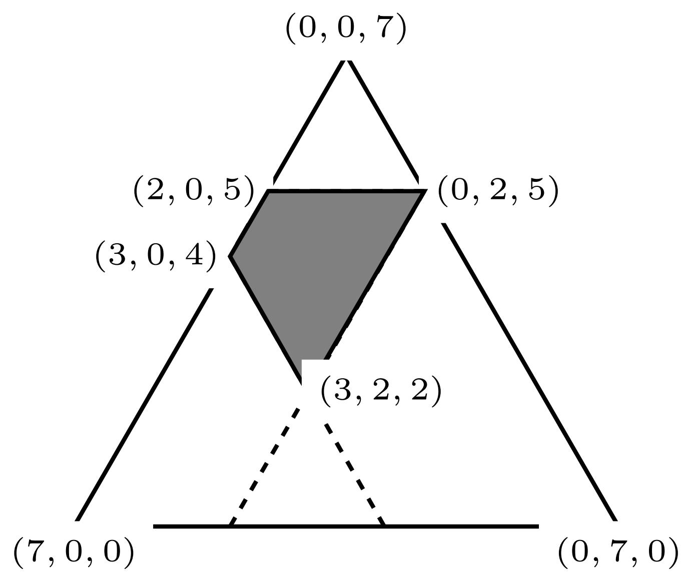
Draw $C(v)$ and write it as the convex hull of suitable vectors.

Solution. Suppose $\mathbf{x} \in C(v)$, then

$$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ x_1 &\geq 0, \\ x_2 &\geq 0, \\ x_3 &\geq 0, \\ x_1 + x_2 \geq 2 &\Leftrightarrow x_3 \leq 5, \\ x_1 + x_3 \geq 5 &\Leftrightarrow x_2 \leq 2, \\ x_2 + x_3 \geq 4 &\Leftrightarrow x_1 \leq 3. \end{aligned}$$

Since $C(v) \subseteq I(v)$, we first draw $I(v)$ and then indicate the points that satisfy each of the inequalities.

The core is the shaded area in the figure below.



We see that

$$C(v) = \mathsf{conv}\big\{(3,0,4),(2,0,5),(0,2,5),(3,2,2)\big\}.$$

Example 13. (core of a game)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	0	0	3	4	2	6

Draw $C(v)$ and determine its extreme points.

Example 14. (core of a game)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	6	7	8	10

Determine $C(v)$.

Flaws of the core as a solution concept

There are two serious problems by using the core as a solution concept:

(a) the core can be an empty set;

(b) if it is nonempty, usually it contains infinitely many imputations and so we do not know which one should be chosen as a solution.

The Shapley value

The Shapley value

Another approach towards resolving an n-person game was proposed by Shapley in 1950's. He looked at what each player could reasonably expect to get before the start of the game. The Shapley value is an example of a one-point solution concept for TU-games: it assigns to each game $v \in \mathsf{TU}^N$ a unique vector $\Phi(v) \in \mathbb{R}^n$.

We introduce the Shapley value with a so-called axiomatic approach.

Shapley's axioms

Let $f : \mathsf{TU}^N \to \mathbb{R}^n$ be a one-point solution concept. Then f satisfies

- efficiency if $\sum_{i=1}^n f_i(v) = v(N)$ for all $v \in \mathsf{TU}^N$;
- symmetry, if $f_i(v) = f_j(v)$ for all $v \in \mathsf{TU}^N$ and $i, j \in N$ satisfying

  $$v(S \cup \{i\}) = v(S \cup \{j\})$$

  for any $S \subseteq N \backslash \{i, j\}$. In this case i and j are called symmetric players in the game (N, v);
- the dummy property, if $f_i(v) = v(\{i\})$ for all $v \in \mathsf{TU}^N$ and $i \in N$ satisfying

  $$v(S \cup \{i\}) = v(S) + v(\{i\})$$

  for any $S \subseteq N \backslash \{i\}$;
- additivity, if $f(v + w) = f(v) + f(w)$, for all $v, w \in \mathsf{TU}^N$, where the game $(v + w) \in \mathsf{TU}^N$ is defined as

  $$(v + w)(S) = v(S) + w(S)$$

  for all $S \in 2^N$.

Theorem 6. Let $u_T \in \mathsf{TU}^N$, $T \in 2^N \backslash \emptyset$, be a unanimity game, $c \in \mathbb{R}$ and f be a one-point solution that satisfies efficiency, symmetry, and the dummy property. Then

$$f_i(cu_T) := \begin{cases} \frac{c}{|T|} & \text{if } i \in T, \\ 0 & \text{otherwise} \end{cases}$$

for $i \in N$.

Proof.

Since every $v \in \mathsf{TU}^N$ can be written as a unique linear combination of unanimity games, the above result implies that there can only be at most one solution that satisfies efficiency, symmetry, the dummy property, and additivity.

Definition 10. Let $v = \sum_{T \in 2^N \setminus \emptyset} c_T u_T \in \mathsf{TU}^N$, then the Shapley value $\Phi(v)$ is given by

$$\Phi_i(v) = \sum_{T \in 2^N ; i \in T} \frac{c_T}{|T|}$$

for all $i \in N$.

It can be shown that the Shapley value indeed satisfies efficiency, symmetry, the dummy property, and additivity. Hence, these properties provide a full characterization of Φ.

Example 15. (Shapley value)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	5	4	7

Determine the Shapley value.

Solution. Write

$$v = 2u_{\{1,2\}} + 5u_{\{1,3\}} + 4u_{\{2,3\}} - 4u_{\{1,2,3\}}.$$

Then

$$\begin{aligned}\Phi(v) &= (1,1,0) + \left(\tfrac{5}{2}, 0, \tfrac{5}{2}\right) + (0,2,2) - \left(\tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}\right) \\ &= \left(\tfrac{13}{6}, \tfrac{10}{6}, \tfrac{19}{6}\right).\end{aligned}$$

Example 16. (Shapley value)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	0	0	3	4	2	6

Determine the Shapley value.

Marginal vectors

An order of N is a bijective function $\sigma : \{1, \ldots, n\} \to N$. The player at position k in the order σ is denoted by $\sigma(k)$. The set of all orders of N is denoted by $\Pi(N)$.

Definition 11. Let $v \in \mathsf{TU}^N$ and $\sigma \in \Pi(N)$. The marginal vector $m^\sigma(v) \in \mathbb{R}^n$ is defined by

$$m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$$

for all $k \in \{1, \ldots, n\}$.

A marginal vector can be interpreted as follows: assume that the players "enter the game" one by one in the order prescribed by σ. The corresponding marginal vector assigns to each player its marginal contribution that he creates at the moment that he joins the group of players that is already present.

Example 17. (marginal vector)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	5	4	7

Let σ be defined by $\sigma(1) = 3$, $\sigma(2) = 1$ and $\sigma(3) = 2$, which is shortly denoted as $\sigma = (3\ 1\ 2)$. Then

$$m_3^\sigma (= m_{\sigma(1)}) = v(\{3\}) - v(\emptyset) = 0,$$
$$m_1^\sigma (= m_{\sigma(2)}) = v(\{1,3\}) - v(\{3\}) = 5 - 0 = 5,$$

and

$$m_2^\sigma (= m_{\sigma(3)}) = v(\{1,2,3\}) - v(\{1,3\}) = 7 - 5 = 2,$$

hence $m^\sigma(v) = (5, 2, 0)$.

Another explicit formula for the Shapley value is given by means of the average of all marginal vectors.

Theorem 7.

$$\Phi(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v) \tag{1}$$

Proof. Show that the rule as defined in equation (1) satisfies efficiency, symmetry, the dummy property, and additivity.

Example 18. (Shapley value and marginal vectors)
Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	5	4	7

Determine the Shapley value.

Solution. We first determine all marginal vectors.

σ	$m^\sigma(v)$
$(1\ 2\ 3)$	$(0,2,5)$
$(1\ 3\ 2)$	$(0,2,5)$
$(2\ 1\ 3)$	$(2,0,5)$
$(2\ 3\ 1)$	$(3,0,4)$
$(3\ 1\ 2)$	$(5,2,0)$
$(3\ 2\ 1)$	$(3,4,0)$

The Shapley value equals

$$\begin{aligned}\Phi(v) &= \tfrac{1}{3!}\big((0,2,5)+(0,2,5)+(2,0,5)+(3,0,4)+(5,2,0)+(3,4,0)\big)\\ &= \tfrac{1}{6}(13,10,19) = \left(\tfrac{13}{6},\tfrac{10}{6},\tfrac{19}{6}\right).\end{aligned}$$

Example 19. (Shapley value and marginal vectors)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	0	0	3	4	2	6

Determine the Shapley value.

Cost Games

Up until now we have considered "profit" or "reward" games where $v(S)$ is the profit made, or reward earned, by coalition S.

Definition 12. A TU-game (N, c) is a *cost* game if the characteristic function $c : 2^N \to \mathbb{R}$ represents a cost that coalitions incur.

Example 20. Norway (1), Sweden (2), and Finland (3) need to build a dam near where the three countries meet in order to generate 3 gigawatts. The cost of constructing the dam is €180 million. However, Norway and Sweden can build a smaller dam that generates 2 gigawatts at a cost of €100 million. If Norway and Finland build a dam that generates 2 gigawatts it costs €130 million, and if Sweden and Finland build such a dam it costs €100 million. Any country that chooses not to join the grand coalition can build a dam that generates 1 gigawatt. This will cost €90 million for Norway, €80 million for Sweden, and €85 million for Finland.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	90	80	85	100	130	100	180

Definition 13. A TU-game (N, c) is *subadditive* if

$$c(S \cup T) \leq c(S) + c(T)$$

for any disjoint coalitions $S, T \in 2^N$.

In Example 20, (N, c) is subadditive.

The core of a cost game

The core of a cost game is defined in a similar manner to that of a profit game with the inequalities reversed.

Definition 14. The core $C(c)$ of a cost game $c \in \mathsf{TU}^N$ is defined by

$$C(c) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S \in 2^N \right\}.$$

An imputation $\mathbf{x} \in C(c)$ ensures that individuals in each coalition S pay no more than the cost of the coalition $c(S)$.

The dual of a profit game

Definition 15. Let (N, v) be a profit game. The *dual game* (N, v^*) is the cost game

$$v^* : 2^N \to \mathbb{R}$$

such that

$$v^*(S) \;=\; v(N) - v(N \backslash S).$$

In some sense, the dual game represents the (missed) opportunity cost for coalition S in not joining the grand coalition N.

If (N, c) is a cost game then its dual game (N, c^*) is defined similarly and is a profit game.

Example 21. (3-person game glove game)

Let $N = \{1, 2, 3\}$ and suppose player 1 owns one left hand glove and player 2 and 3 each own one right hand glove. A single glove is worth nothing, a (left-right) pair of gloves is worth 10 AUD.

Calculate and interpret v^*.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	10	10	0	10
$v^*(S)$								

The dual game of a dual game

Theorem 8. Let (N, v) be a profit game and (N, v^*) its dual game. The dual game of (N, v^*) is the game (N, v).

Similarly, the dual of the dual of a cost game (N, c), is (N, c).

Proof.

The core of a dual game

Theorem 9. Let (N, v) be a profit game and (N, v^*) its dual game. Then $C(v) = C(v^*)$.

Similarly, if (N, c) is a a cost game, $C(c) = C(c^*)$.

Proof.

Example 22. (core of a game)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	5	4	7
$v^*(S)$							

Calculate v^* and draw $C(v^*)$ and determine its extreme points.

Compare with Example 12 on Slide 36.

Example (cont.)

Example 23. (Shapley value)

Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	2	5	4	7
$v^*(S)$							

Calculate v^* and determine the Shapley value.

Compare with Example 15 on Slide 45.

Theorem 10. The Shapley value for v^* is equal to the Shapley value for v.