MAST30001 Stochastic Modelling

Tutorial Sheet 10

1. Show that in in $M/M/1$ queue with arrival rate λ and service rate $\mu > \lambda$, the expected lengths of the idle and busy periods are $1/\lambda$ and $1/(\mu - \lambda)$, respectively. *[Hint: the proportion of time the server is idle is equal to the stationary chance the system is empty.]*

   Ans. Since the arrivals follow a Poisson process (using in particular the memoryless property of the exponential), the time between the moment the system clears and the next arrival is exponential rate λ and so the expected length of an idle period is the expectation of this exponential, that is, $1/\lambda$. If ℓ is the expected length of a busy period and $\pi_0 = 1 - \lambda/\mu$ is the long run proportion of time the system is empty, then
$$\pi_0 = \frac{1/\lambda}{1/\lambda + \ell},$$
   or $\ell = 1/(\mu - \lambda)$.

2. A rent-a-car washing facility can wash one car at a time. Cars arrive to be washed according to a Poisson process with rate 3 per day and the service time to wash a car is exponential with mean 7/24 days. It costs the company \$150 per day to operate the facility and the company loses \$10 per day for each car tied up in the washing facility. The company can increase the rate of washing to get down to a mean service time of 1/4 days at the cost of \$$C$ per day. What's the largest C can be for this upgrade to make economic sense?

   Ans. We can model the number of cars in the wash as an $M/M/1$ queue with arrival rate $\lambda = 3$ and current service rate $\mu = 24/7$ and so with stationary distribution geometric with parameter $1 - 21/24 = 1/8$ having expectation 7. The companies current cost per day is
$$150 + 10 \times 7 = 220.$$
   If the company pays C dollars per day to increase their service rate to 4, then similarly their new cost per day will be
$$150 + C + 10 \times 3 = 180 + C.$$
   Thus they should spend no more than 40 dollars per day to increase their service rates.

3. Customers arrive at a bank according to a Poisson process rate λ. The bank's service policy is that

   - if there are fewer than 4 customers in the bank, then there is 1 teller,
   - if there are $4 - 9$ customers, there are 2 tellers,
   - if there are more than 9 customers, there are 3 tellers.

   Tellers' service times are independent and exponentially distributed with rate μ. Model the number of customers in the bank as a birth and death chain and determine for what values of λ and μ there is stable long run behavior and for these parameters compute the steady state distribution. *[Hint: This is similar to the analysis of the $M/M/a$ queue done in lecture.]*

Ans. We can model the number of customers in the bank as a birth-death process with birth rates $\lambda_i = \lambda$, $i = 0, 1, \ldots$ and death rates

$$\mu_j = \begin{cases} \mu & j = 1, 2, 3, \\ 2\mu & j = 4, \ldots, 9, \\ 3\mu & j > 9. \end{cases}$$

For $K_j = \prod_{\ell=1}^{j} \lambda_{\ell-1}/\mu_\ell$ and writing $\rho = \lambda/\mu$,

$$K_j = \begin{cases} \rho^j & j = 1, 2, 3, \\ 2^{3-j}\rho^j & j = 4, \ldots, 9, \\ 2^{-6}3^{9-j}\rho^j & j > 9, \end{cases}$$

and the system is stable if and only if $\sum_{j\geq 1} K_j < \infty \iff \sum_{j\geq 10} K_j < \infty$ which is the same as $\rho/3 < 1$; that is $\rho < 3$. In this case, the stationary distribution is given by

$$\pi_j = \pi_0 K_j$$

and since the probabilities have to sum to one we have

$$\pi_0^{-1} = \frac{1-\rho^4}{1-\rho} + \frac{\rho^4(1-(\rho/2)^6)}{2-\rho} + \frac{\rho^{10}}{2^6(3-\rho)}.$$

4. (M/M/∞ queue) Assume that in a queuing system customers arrive according to a rate λ Poisson process, customers are always served immediately (for example, customers making purchases on the internet), and the service time of a customer is exponential with rate μ, independent of arrival times and other service times.

   (a) Model this queue as a birth-death chain and write down its generator.

   (b) Describe the long run behaviour of the chain.

   (c) When the queue is in stationary (i.e., after its been running a long time), what is the expected number of customers in the system, number of customers in the queue, number of busy servers, and service time for an arriving customer?

   (d) Let X_t be the number of customers in the system (including those being served) at time t and set $X_0 = 0$. What is $E[X_t]$? *[Hint: if $m(t) = E[X_t]$, consider $m'(t)$.]* You should check your formula makes sense as t tends to infinity.

Ans.

(a) The number of customers in the system is a birth-death chain that has birth rate λ in every state. If the chain is in state i, there are i customers each getting served at rate μ so the total rate of service is $i\mu$, and this is the death rate. Its generator has for $i \geq 1$

$$a_{ii+1} = \lambda, \;\; a_{ii-1} = \mu i, \;\; a_{ii} = -(\lambda + \mu i).$$

And $a_{01} = -a_{00} = \lambda$.

(b) The chain is irreducible and has unique stationary distribution (by directly solving $\pi A = 0$) $\pi_k = e^{-\rho}\rho^k/k!$ with $\rho = \lambda/\mu$ so it is ergodic with long run frequencies given by π.

(c) The expected number of customers in the system is just the expectation against π which is ρ, the number of customers in the queue is 0 since customers are served instantaneously, the number of busy servers is equal to the number of customers in the system which as already said has mean ρ, and since customers are served instantaneously and at rate μ, the expected time in the system of an arriving customer is $1/\mu$.

(d) Since for $k \geq 1$,

$$\frac{d}{dt}P_{0,k}(t) = \lambda P_{0,k-1}(t) - (\lambda + k\mu)P_{0,k}(t) + (k+1)\mu P_{0,k+1}(t),$$

multiplying by k and summing we find

$$m'(t) = \lambda - \mu m(t),$$

and $m(0) = 0$. This ODE has solution $m(t) = (\lambda/\mu)(1 - e^{-\mu t})$, and note that as t tends to infinity, this quantity tends to the mean of the stationary number of customers in the queue.

5. (M/G/∞ queue) In a certain communications system, information packets arrive according to a Poisson process with rate λ per second and each packet is processed in one second with probability p and in two seconds with probability $1-p$, independent of the arrival times and other service times. Let N_t be the number of packets that have entered the system up to time t and X_t be the number of packets in the system (including those being served) at time t.

   (a) Is $(X_t)_{t\geq 0}$ a Markov chain? (No detailed argument is necessary here, just think about it heuristically.)

   (b) If $X_0 = 0$, what is the distribution of X_2?

   (c) If $X_0 = 0$, is there a "stationary" limiting distribution $\pi_k = \lim_{t\to\infty} P(X_t = k)$? If so, what is it?

   (d) If $X_0 = N_0 = 0$, what is the joint distribution of X_t and N_t?

   Ans.

   (a) X_t is not a Markov chain because the chance of the chain decreasing by one in the interval $(t, t+h)$ given the value of the chain at time t also depends on the times of the arrivals in the past.

   (b) If A_t are the arrivals that require one second of service, and B_t are the arrivals requiring two seconds of service, then A_t and B_t are independent Poisson processes with rates $p\lambda$ and $(1-p)\lambda$. And $X_2 = (N_2 - N_1) + B_1$; the sum of two independent Poisson variables (using independent increments) with respective means λ and $(1-p)\lambda$. So X_2 is Poisson with mean $\lambda(2-p)$.

   (c) X_t only depends on the number of arrivals of the two different types in the in the interval $(t-2, t)$ since all arrivals previous to this time have left the system. As in part (b), we can write $X_t = (N_t - N_{t-1}) + (B_{t-1} - B_{t-2})$, and the two variables in parentheses are independent Poisson with respective means λ and $(1-p)\lambda$. So for $t \geq 2$, X_t is Poisson with mean $\lambda(2-p)$.

(d) When $0 < t \leq 1$, then $X_t = N_t$ and they're both distributed as Poisson mean t. The case $1 < t < 2$ is similar but easier than $t \geq 2$; the latter case we show here. Assuming $t \geq 2$, then as above we write $X_t = (N_t - N_{t-1}) + (B_{t-1} - B_{t-2})$ and also $N_t = X_t + (A_{t-1} - A_{t-2}) + N_{t-2}$, and note that by the comments of part (b), X_t is independent of $(A_{t-1} - A_{t-2})$ and these variables are both independent of N_{t-2}. So we can write $N_t = X_t + Y_t$, where Y_t is a Poisson variable with mean $\lambda(p + t - 2)$, independent of X_t which implies that for $0 \leq j \leq n$,

$$
\begin{aligned}
\mathbb{P}(X_t = j, N_t = n) &= \mathbb{P}(X_t = j, Y_t = n - j) \\
&= \mathbb{P}(X_t = j)\mathbb{P}(Y_t = n - j) \\
&= e^{-\lambda t} \frac{\lambda^n}{n!} \binom{n}{j} (2-p)^j (p + t - 2)^{n-j}.
\end{aligned}
$$