MAST30001 Stochastic Modelling

Assignment 1

Please complete the Plagiarism Declaration Form on LMS.

Don't forget to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **3pm, Friday Sept 1** in the appropriate assignment box at the north end of Peter Hall building (near Wilson Lab).

Marks may be lost where answers are not clear and concise (or where lacking in explanation).

1. A DTMC with state space $\mathcal{S} = \{1, 2, 3, 4\}$ has the following transition matrix

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

[10 Marks]

(a) Does the chain have any absorbing state(s)?
No, none of the diagonal entries of the matrix are 1

(b) Is the chain irreducible?
Yes, it is possible to reach every state from every state

(c) Is the chain periodic? If so, what is the period?
No, it is irreducible and has a self loop ($p_{1,1} > 0$), so it cannot be periodic.

(d) Is the chain transient, null-recurrent, or positive recurrent?
It is irreducible and finite-state, hence it is positive recurrent

(e) Is the process reversible?
Yes, it is a birth and death chain, hence it is reversible

(f) Find the long run proportion of time spent in state 4.
Since the chain is positive recurrent, this is the stationary probability π_4. Since the chain is reversible, we can solve the detailed balance equations which reduce to

$$\pi_1 = \pi_2, \quad \pi_2 = \pi_3, \quad \frac{1}{2}\pi_3 = \pi_4.$$

Together with $\sum_{i=1}^{4} \pi_i = 1$ this gives $\pi_1 = \pi_2 = \pi_3 = 2/7$ and $\pi_4 = 1/7$.

(g) If the initial distribution is uniform, does the limiting distribution exist? If so, find it.
This chain is ergodic since it is irreducible, positive recurrent and aperiodic, so the limiting distribution exists and is equal to π irrespective of the initial distribution. Therefore the limiting distribution is $(2/7, 2/7, 2/7, 1/7)$.

(h) Starting from state 2, find the probability of hitting state 1 before state 4.
This is the same as the hitting probability $h_{2,1}$ *for the chain where we change* $p_{1,1}$ *and* $p_{4,4}$ *to be 1, i.e. for the chain*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

But this problem is the same as the gambler's ruin problem, so the answer is 2/3.

2. Suppose we roll a fair six sided die repeatedly. **[4 Marks]**

(a) Find the expected number of rolls required to see 3 sixes appear in succession.
Consider a Markov chain with state-space $\{0, 1, 2, 3\}$ *with transition probabilities given by* $p_{i,i+1} = 1/6$ *for* $i = 0, 1, 2$ *and* $p_{i,0} = 5/6$ *for* $i = 0, 1, 2, 3$. *Then we are looking for the expected hitting time of state 3, starting from state 0. We solve the equations*

$$\begin{aligned} m_{0,3} &= 1 + \frac{1}{6}m_{1,3} + \frac{5}{6}m_{0,3} \\ m_{1,3} &= 1 + \frac{1}{6}m_{2,3} + \frac{5}{6}m_{0,3} \\ m_{2,3} &= 1 + \frac{5}{6}m_{0,3}. \end{aligned}$$

Solving gives $m_{0,3} = 258$.

Note that we can solve this kind of problem much more generally. Suppose that the probability of getting a six is p *and we are looking for the expected time to roll* n *sixes in succession, which is* $m_{0,n}$. *Then* $m_{0,n} = m_{0,n-1} + m_{n-1,n}$ *and* $m_{n-1,n} = 1 + (1-p)m_{0,n}$. *It follows that*

$$m_{0,n} = \frac{1}{p}[m_{0,n-1} + 1].$$

Applying this to our problem above we get that $m_{0,1} = 6$, $m_{0,2} = 6 * 7 = 42$, *and* $m_{0,3} = 6 * 43 = 258$.

(b) Find the expected number of rolls required to see 3 of the same number in succession.
This can actually be solved from the previous answer without doing any more Markov chain calculations as $43 = 258/6$.

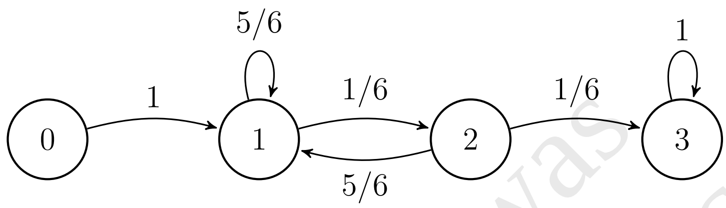
The above is intuitively reasonable, but to understand this formally, note that we roll a die repeatedly until (at time T_1*) we see the same number appearing 3 times in succession. When we do this, with probability 1/6 we have actually seen 3 sixes in succession. The number of times* $\boldsymbol{T}$ *we need to repeat this experiment to see 3 sixes in succession has a Geometric*$(1/6)$ *distribution, which has mean* 6. *The time required to see 3 sixes in succession can be written as*

$\tau = \sum_{i=1}^{\boldsymbol{T}} T_i$, where the $T_i \sim T_1$ are independent random variables, that are also independent of $\boldsymbol{T}$. Therefore

$$\mathbb{E}[\tau] = \mathbb{E}[\sum_{i=1}^{\boldsymbol{T}} T_i] = \mathbb{E}[\boldsymbol{T}]\mathbb{E}[T_1].$$

(It's an exercise to show that this is true if you have independence of $\boldsymbol{T}$ from $(T_i)_{i\in\mathbb{N}}$). In other words, $258 = 6 \times \mathbb{E}[T_1]$, so $\mathbb{E}[T_1] = 258/6$ as claimed.

Of course we can also just compute it directly. This corresponds to finding $m_{0,3}$ for the Markov chain with transition diagram:

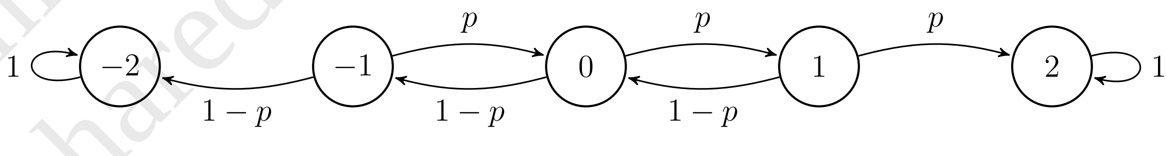


Solving the expected hitting time equations gives $m_{0,3} = 43$.

3. Batman chases the Joker around the vertices of a square. At each time step, Batman steps clockwise with probability $p \in (0,1)$, and anticlockwise with probability $1-p$, while the Joker stays where he is with probability $q \in [0,1]$, and steps clockwise with probability $(1-q)s$ and anticlockwise with probability $(1-q)(1-s)$, where $s \in (0,1)$. Batman catches the Joker if they reach the same vertex at the same time. All steps are taken independently of previous steps.

   Starting from opposite corners of the square: **[6 Marks]**

   (a) what is the expected time until Batman catches the Joker when $q = 1$?
   In this case the Joker doesn't move, so we are looking for the time it takes for Batman to reach distance 2 from his starting point. This is the same as the expected hitting time of $A = \{-2, 2\}$ starting from 0 for a chain with transition diagram



We are looking for $m_{0,A}$, which is found by solving the equations

$$\begin{aligned} m_{0,A} &= pm_{1,A} + (1-p)m_{-1,A} \\ m_{1,A} &= 1 + (1-p)m_{0,A} \\ m_{-1,A} &= 1 + pm_{0,A}, \end{aligned}$$

Solving gives

$$m_{0,A} = \frac{2}{1 - 2p(1-p)}.$$

(b) if $q = 0$, find the expected time until Batman catches the Joker, and for fixed p, find the maximum possible value for this quantity (i.e. optimise over s).
When $q = 0$, both Batman and the Joker move on each step. We are then simply looking for the first time that Batman and the Joker take steps of opposite orientation. This has a geometric distribution with parameter $\delta = p(1-s) + (1-p)s = p + s - 2ps$. Thus the expected time is

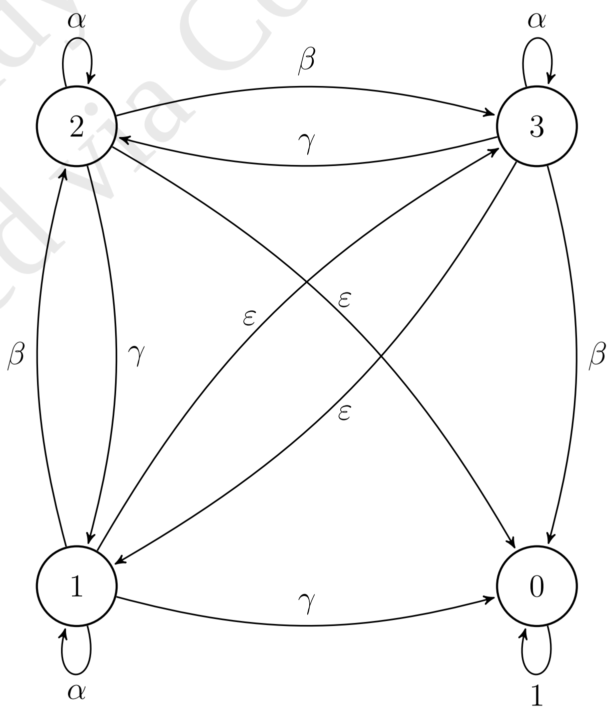
$$1/\delta = 1/(p + s - 2ps).$$

If $p = 1/2$ then $1/\delta = 2$ (so there is nothing to optimise) Otherwise δ is minimised by taking $s = 1$ if $p > 1/2$ and $s = 0$ if $p < 1/2$.
In each case the maximum is

$$\frac{2}{1 - |2p - 1|}.$$

(c) When $p = 1/2$, find the expected time until Batman catches the Joker.
For fun I write out the solution for general parameters:
Let X_n denote the clockwise number of steps from Batman to the Joker. This takes values in $\mathcal{S} = \{0, 1, 2, 3\}$. We start with $X_0 = 2$ and we are looking for the expected hitting time of 0. The transition probabilities are

$$\begin{aligned}
p_{i,i} &= p(1-q)s + (1-p)(1-q)(1-s) =: \alpha \\
p_{i,(i+1) \bmod 4} &= (1-p)q =: \beta \\
p_{i,(i-1) \bmod 4} &= pq =: \gamma \\
p_{i,(i+2) \bmod 4} &= 1 - \alpha - \beta - \gamma =: \varepsilon
\end{aligned}$$



We are asked for $m_{2,0}$, where

$$\begin{aligned} m_{2,0} &= 1 + \alpha m_{2,0} + \beta m_{3,0} + \gamma m_{1,0} \\ m_{3,0} &= 1 + \alpha m_{3,0} + \varepsilon m_{1,0} + \gamma m_{2,0} \\ m_{1,0} &= 1 + \alpha m_{1,0} + \varepsilon m_{3,0} + \beta m_{2,0}. \end{aligned}$$

Solving gives

$$m_{2,0} = \frac{(-\varepsilon + \alpha - 1)\,\gamma + \varepsilon^2 + \beta\,(-\varepsilon + \alpha - 1) - \alpha^2 + 2\alpha - 1}{\varepsilon\gamma^2 + (2 - 2\alpha)\,\beta\gamma + \varepsilon^2 + \alpha\,(3 - \varepsilon^2) + \beta^2\varepsilon + \alpha^3 - 3\alpha^2 - 1},$$

provided that the denominator is non-zero (note that the denominator is zero when $q = 0$).

Let us now turn to the problem at hand. When $p = 1/2$,

$$\alpha = \frac{1-q}{2} = \gamma, \qquad \beta = \frac{q}{2} = \varepsilon,$$

and in particular the equations do not depend on s at all.
If $q = 0$ then the first equation becomes $m_{2,0} = 1 + \frac{1}{2} m_{2,0}$, so $m_{2,0} = 2$ (as we saw above when $q = 0$).
If $q > 0$ then symmetry gives $m_{1,0} = m_{3,0}$, and solving the equations gives $m_{2,0} = 4$.
In summary, when $p = 1/2$,

$$m_{2,0} = \begin{cases} 2, & \text{if } q = 0 \\ 4, & \text{otherwise,} \end{cases}$$

which is discontinuous at $q = 0$.