Simplex algorithm

For your reference, the simplex algorithm has been reproduced here:

1. Determine the entering basic variable by choosing the biggest negative entry in the bottom row.
2. If none of the entries in the column of the entering basic variable are positive, then there are arbitrarily large solutions. STOP.
3. Determine the departing basic variable by applying the minimum ratio test.
4. Apply row operations to determine the new basic feasible solution.
5. If there is a variable with a negative entry in the bottom row, this is not an optimal solution; return to step 1.
6. If there are no negative entries in the bottom row, and there is a zero entry in the column of a *non-basic* variable, and this solution has not yet been recorded, then record this solution as optimal and return to step 3 using this variable as the entering basic variable.
7. If there are no negative entries in the bottom row, the optimal solution has been found. STOP.

For an entering basic variable in column e, and an augmented matrix

$$\left(\begin{array}{ccccc|c} a_{11} & \cdots & a_{1e} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2e} & \cdots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{me} & \cdots & a_{mn} & b_m \end{array}\right),$$

the minimum ratio test works by computing the ratio b_j/a_{jk} for each $j \leqslant m$ with $a_{je} > 0$. The basic variable x_d in the row with smallest such ratio is chosen as the departing basic variable.

1. Consider the following augmented matrix:

$$\left(\begin{array}{cccccc|c} 8 & 6 & 1 & 1 & 0 & 0 & 48 \\ 4 & 2 & 3/2 & 0 & 1 & 0 & 20 \\ 2 & 3/2 & 1/2 & 0 & 0 & 1 & 8 \\ -60 & -35 & -20 & 0 & 0 & 0 & 0 \end{array}\right)$$

   Apply the minimum ratio test to determine which entry to pivot on when applying the first step of the simplex algorithm. *You do not need to apply the simplex algorithm.*

2. One of the three matrices below has arrived at the optimal solution to a linear program, while the other two have not.

$$\text{(a)} \left(\begin{array}{ccccc|c} 5 & -2 & 6 & 1 & 0 & 20 \\ 10 & 4 & -6 & 0 & 1 & 30 \\ -10 & -6 & 8 & 0 & 0 & 0 \end{array}\right) \quad \text{(b)} \left(\begin{array}{ccccc|c} 10 & 0 & 1 & 1 & 1 & 35 \\ 15 & 1 & 0 & 1 & 1 & 25 \\ 25 & 0 & 0 & 1 & 5 & 20 \end{array}\right) \quad \text{(c)} \left(\begin{array}{ccccc|c} -3 & -6 & 0 & 1 & -4 & 35 \\ 2 & -1 & 1 & 0 & 1 & 20 \\ -2 & 11 & 0 & 0 & 4 & 16 \end{array}\right)$$

   For the matrix that has arrived at the optimal solution, state the corresponding solution. For those that have not, indicate which entry will be pivoted on.

3. Use the simplex algorithm to find all optimal solutions to the following linear programs:

$$\begin{array}{rrr} \text{(a)} & \text{maximise} & z = 8x_1 + 4x_2 \\ & \text{subject to} & x_1 + x_2 \leqslant 80 \\ & & x_1 - 4x_2 \leqslant 20 \\ & & \mathbf{x} \geqslant \mathbf{0}. \end{array} \qquad \begin{array}{rrr} \text{(b)} & \text{maximise} & z = 10x_1 + 6x_2 - 8x_3 \\ & \text{subject to} & 5x_1 - 2x_2 + 6x_3 \leqslant 20 \\ & & 10x_1 + 4x_2 - 6x_3 \leqslant 30 \\ & & \mathbf{x} \geqslant \mathbf{0}. \end{array}$$

The 2-phase method

Since the simplex algorithm must start with a basic feasible solution, the algorithm needs to be modified to account for linear programs where the origin is not feasible. The method we use is the 2-phase method, as follows:

Phase 1

1. Introduce slack variables to represent the problem in standard form.
2. For each equality constraint in the standard form that does not have a basic variable, introduce an artificial variable to that constraint.
3. Use the simplex method to minimise the sum of artificial variables.

Phase 2

4. Remove the artificial columns and insert the original objective function.
5. Apply the simplex algorithm to find the optimal solution.

4. Consider the following linear programs:

(a)
$$\begin{aligned} \text{maximise} \quad & z = 75x_1 + 130x_2 \\ \text{subject to} \quad & x_1 + x_2 \leqslant 100 \\ & 10x_1 + 12x_2 \leqslant 1200 \\ & x_2 \leqslant 75 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

(b)
$$\begin{aligned} \text{maximise} \quad & z = x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geqslant 1 \\ & 2x_1 - x_2 \leqslant 2 \\ & -x_1 + 2x_2 \leqslant 3 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

(c)
$$\begin{aligned} \text{maximise} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 6x_2 \leqslant 12 \\ & x_2 \leqslant 2 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

Exactly one would require use of the 2-phase method to solve. State which one would need to use the 2-phase method and then introduce slack variables and artificial variables to construct the initial matrix for phase 1.

You do not need to solve any of these linear programs.

5. Apply the 2-phase method to solve the following linear program.

$$\begin{aligned} \text{minimise} \quad & z = 8x_1 + 9x_2 \\ \text{subject to} \quad & 2x_1 - 3x_2 \geqslant 35 \\ & 3x_1 + 2x_2 \geqslant 100 \\ & 2x_1 + 3x_2 \geqslant 75 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

Beware of the objective function. How should you modify the problem?

6. Write a function `minimumRatio` that takes a matrix `M` as input, which is assumed to be in canonical form and have at least one negative entry in the bottom row, and gives two outputs, `r` and `c`, corresponding to the row number and column number to pivot on according to the minimum ratio test.

7. The following linear program will cycle indefinitely when the simplex method is applied:

$$\begin{aligned} \text{maximise} \quad & z = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} \quad & \tfrac{1}{2}x_1 - \tfrac{11}{2}x_2 - \tfrac{5}{2}x_3 + 9x_4 \leqslant 0 \\ & \tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 \leqslant 0 \\ & x_1 \leqslant 1 \\ & x_1, x_2, x_3, x_4 \geqslant 0. \end{aligned}$$

Use the `minimumRatio` function from Question 6 and the `pivot` function from Question 10 of Workshop 4 to find the number of iterations made before cycling once.

Then, apply Bland's rule to find the optimal solution.