**MAST30001 Stochastic Modelling**

**Tutorial Sheet 2**

1. Show that the Markov property does not in general imply that for any events $A, B$ and $C$,
$$P(X_{n+1} \in C | X_n \in B, X_{n-1} \in A) = P(X_{n+1} \in C | X_n \in B).$$
(That is, define a Markov chain and events $A, B, C$ where the equality doesn't hold.)

**Ans.** You can see there's a problem since if $S$ denotes the state space and we set $B = S$, then the relation above becomes
$$P(X_{n+1} \in C | X_{n-1} \in A) = P(X_{n+1} \in C);$$
that is, that $X_{n+1}$ and $X_{n-1}$ are independent, which isn't true in general. To be concrete, take simple random walk on $\mathbb{Z}$ and set $A = \{0\}$, $S = \mathbb{Z}$ and $C$ equal to the odd integers. Then the left hand side of the equality is 0 but the right hand side is not.

2. Let $(Y_n)_{n\geq 1}$ be i.i.d. random variables with $P(Y_i = 1) = P(Y_i = -1) = 1/2$ and let $X_n = (Y_{n+1} + Y_n)/2$.

   (a) Find the transition probabilities $P(X_{n+m} = k | X_n = j)$ for $m = 1, 2, \ldots$ and $j, k = 0, \pm 1$.

   (b) Show that $(X_n)_{n\geq 1}$ is *not* a Markov chain.

**Ans.**

(a) Since the $Y_i$ are i.i.d. and $X_n$ is a function of $Y_n, Y_{n+1}$, $X_n$ and $X_{n+m}$ are independent for $m \geq 2$. Thus for $m \geq 2$,
$$P(X_{n+m} = k | X_n = j) = P(X_{n+m} = k) = \begin{cases} 1/4, & k = \pm 1, \\ 1/2, & k = 0. \end{cases}$$
When $m = 1$, the event $X_n = 1$ corresponds to $Y_n = Y_{n+1} = 1$ and so
$$P(X_{n+1} = k | X_n = 1) = P(X_{n+1} = k | Y_{n+1} = 1) = \begin{cases} 1/2, & k = 1, \\ 1/2, & k = 0. \end{cases}$$
Similarly,
$$P(X_{n+1} = k | X_n = -1) = \begin{cases} 1/2, & k = -1, \\ 1/2, & k = 0. \end{cases}$$
When $X_n = 0$, then either $Y_n = 1$ and $Y_{n+1} = -1$ or $Y_n = -1$ and $Y_{n+1} = 1$, and these events have equal probability. So
$$P(X_{n+1} = 1 | X_n = 0) = \frac{P(Y_{n+2} = 1, Y_{n+1} = 1, Y_n = -1)}{P((Y_{n+1} + Y_n)/2 = 0)} = \frac{1/8}{1/2} = 1/4,$$
and similarly
$$P(X_{n+1} = k | X_n = 0) = \begin{cases} 1/4, & k = \pm 1, \\ 1/2, & k = 0. \end{cases}$$
(b) The Markov property is not satisfied since
$$P(X_{n+1} = 1 | X_n = 0, X_{n-1} = 1) = 0 \neq \frac{1}{4} = P(X_{n+1} = 1 | X_n = 0).$$

3. Let $(X_n)$ be a Markov chain with state space $\{1,2,3\}$ and transition matrix

$$\begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/4 & 3/4 & 0 \\ 2/5 & 0 & 3/5 \end{pmatrix}$$

   (a) Compute $P(X_3 = 1, X_2 = 2, X_1 = 2 | X_0 = 1)$.

   (b) If $X_0$ is uniformly distributed on $\{1,2,3\}$, compute $P(X_3 = 1, X_2 = 2, X_1 = 2)$.

   (c) Now assuming $P(X_0 = 1) = P(X_0 = 3) = 2/5$, compute $P(X_1 = 2, X_4 = 2, X_6 = 2)$.

**Ans.**

(a) $p_{1,2}p_{2,2}p_{2,1} = (1/3)(3/4)(1/4)$.

(b)

$$\sum_{i=1}^{3}(1/3)p_{i,2}p_{2,2}p_{2,1} = (1/3)(1/3)(3/4)(1/4) + (1/3)(3/4)(3/4)(1/4).$$

(c)

$$\begin{aligned}&(2/5)p_{1,2}\left(P^3\right)_{2,2}\left(P^2\right)_{2,2} + (1/5)p_{2,2}\left(P^3\right)_{2,2}\left(P^2\right)_{2,2} + (2/5)p_{3,2}\left(P^3\right)_{2,2}\left(P^2\right)_{2,2} \\ &\quad = \left[(2/5)(1/3) + (1/5)(3/4)\right](31/48)\,(35/64)\,.\end{aligned}$$

4. A simplified model for the spread of a contagion in a small population of size 4 is as follows. At each discrete time unit, two individuals in the population are chosen uniformly at random to meet. If one of these persons is healthy and the other has the contagion, then with probability 1/4 the healthy person becomes sick. Otherwise the system stays the same.

   (a) If $X_n$ is the number of healthy people at step $n$, then explain why $X_0, X_1 \ldots$ is a Markov chain.

   (b) Specify the transition probabilities of $X_n$.

   (c) If initially the chance that a given person in the population has the disease equals 1/2, determined independently, then what is the chance everyone has the disease after two steps in the process?

   (d) Assume now that the process begins with exactly one person infected. Given that not everyone is infected after three steps of the process, what is the chance exactly one person is infected?

**Ans.**

(a) The sequence of variables $X_0, X_1 \ldots$ either stays the same or decreases by one at each step. The chance that the chain decreases by one from step $n$ to $n+1$ given the entire history up to $n$ is the chance a healthy person and a sick person are chosen times 1/4:

$$\mathbb{P}(X_{n+1} = X_n - 1 | X_n, \ldots, X_1) = \frac{X_n(4 - X_n)}{6} \times \frac{1}{4},$$

which only depends on the history through $X_n$.

(b) Using the formula above,

$$p_{i,i-1} = 1 - p_{i,i} = i(4-i)/24, i = 0, \dots, 4,$$

and the remaining probabilities are zero.

(c) We're looking for $\mathbb{P}(X_2 = 0)$ which according to the Chap-Kol equations and the fact that the initial distribution is binomial$(4, 1/2)$ is

$$\sum_{k=0}^{4} (P^2)_{k0} \binom{4}{k} \frac{1}{2^4},$$

where $P$ is the one step transition matrix of the chain, given by the previous part of the problem.

(d) We want

$$\begin{aligned}
\mathbb{P}(X_3 = 3 | X_3 > 0, X_0 = 3) &= \frac{\mathbb{P}(X_3 = 3 | X_0 = 3)}{\mathbb{P}(X_3 > 0 | X_0 = 3)} \\
&= \frac{\mathbb{P}(X_3 = 3 | X_0 = 3)}{1 - \mathbb{P}(X_3 = 0 | X_0 = 3)} \\
&= \frac{p_{3,3}^3}{1 - p_{3,2} p_{2,1} p_{1,0}},
\end{aligned}$$

where the $p_{i,j}$ are given above.

5. A Markov chain on $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.3 & 0.2 \end{pmatrix}$$

starts with initial distribution uniform on the states $1, 2, 3, 4$. For each $i = 1, 2, 3, 4$, and $n \geq 0$, compute the chance the chain is in state $i$ at step $n$.

**Ans.** If $\mathbf{x} = (1/4, 1/4, 1/4, 1/4)$, then the chance the chain is in state $i$ at step $n$ is the $i$th component of $\mathbf{x}P^n$. But notice that $\mathbf{x}P = \mathbf{x}$, and so by induction $\mathbf{x}P^n = \mathbf{x}$. That is, the chance the chain is in state $i$ at step $n$ is $1/4$.