Decision Making
Part 3: 2-person non-zero-sum games

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Topics in this part

- Introduction: problem setting, comparison between zero-sum and non-zero-sum games, examples, Prisoner's dilemma

- Non-cooperative 2-person non-zero-sum games: optimal security levels, equilibrium, equilibrium theorem of John Nash, finding equilibria for non-cooperative 2-person non-zero-sum games, graphical method

- Cooperative 2-person games: cooperative payoff set, Pareto boundary, negotiation set

- Nash's bargaining axioms: Nash's axioms, Nash's theorem

Reference:
P. Morris, Introduction to game theory, Springer-Verlag, 1994, Chapter 5.

Introduction to 2-person non-zero-sum games

Problem setting and notation

A 2-person game in normal form can be represented by a bi-matrix:

		Player II				
		A_1	$\dots$	A_j	$\dots$	A_n
	a_1	(a_{11}, b_{11})	$\dots$	(a_{1j}, b_{1j})	$\dots$	(a_{1n}, b_{1n})
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
Player I	a_i	(a_{i1}, b_{i1})	$\dots$	(a_{ij}, b_{ij})	$\dots$	(a_{in}, b_{in})
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	a_m	(a_{m1}, b_{m1})	$\dots$	(a_{mj}, b_{mj})	$\dots$	(a_{mn}, b_{mn})

$a_1, \dots, a_m$: Pure strategies of Player I
$A_1, \dots, A_n$: Pure strategies of Player II
a_{ij}, b_{ij}: Payoffs to Players I, II respectively if they play (a_i, A_j).

We assume this is a non-constant-sum game, for otherwise we can convert it to a zero-sum game and so the methodologies in the previous part apply.

Denote

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ b_{i1} & \dots & b_{ij} & \dots & b_{in} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \dots & b_{mj} & \dots & b_{mn} \end{bmatrix}$$

It is understood that the rows of these matrices are indexed by $a_1, \dots, a_m$ and columns of them are indexed by $A_1, \dots, A_n$.

Definition 1. Call A and B above the payoff matrices of Players I and II respectively.

Comparison between zero-sum and non-zero-sum games

- 2-person zero-sum games:
  1. One player's loss is the other player's gain. The game is purely competitive.
  2. Linear programming solves such games perfectly.
- 2-person non-zero-sum games:
  1. Not purely competitive.
  2. Both may gain or both may lose if they play certain strategies.
  3. More difficult and more interesting to analyse.

Example 1. (Possible communication between the players)

	A_1	A_2
a_1	$(8, 0)$	$(3, 3)$
a_2	$(3, 3)$	$(0, 8)$

$$A = \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 3 & 8 \end{bmatrix}$$

- Both matrices have saddle points and have security levels 3.
- If the players are allowed to communicate, they may agree to alternate between (a_1, A_1) and (a_2, A_2). In doing so, the average payoff will be equal to 4 for each player.
- What if the first player changes his mind after the first game?

Example 2. (Possible cooperation)

	A_1	A_2
a_1	$(100, 10)$	$(5, 20)$
a_2	$(10, 10)$	$(5, 20)$

$$A = \begin{bmatrix} 100 & 5 \\ 10 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 20 \\ 10 & 20 \end{bmatrix}$$

- Player II can always "beat" Player I and "win" the game by playing A_2.
- However, a better option for both players is that they play (a_1, A_1) and share the total payoff 110 according to some agreed-upon formula.

Example 3. (Threat)

	A_1	A_2
a_1	$(2, 5)$	$(5, 2)$
a_2	$(1, -3)$	$(1, -5)$

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 \\ -3 & -5 \end{bmatrix}$$

- Player I prefers a_1.
- Knowing this, Player II would prefer A_1.
- But, Player I may send the following message to Player II: You'd better use A_2, else I will use a_2!
- Since Player II has much more to lose she may agree to use A_2.

Prisoner's Dilemma

Example 4. The prosecutor makes the following offer to each of the two prisoners, who are being questioned separately.

- If you both plead innocent, you will be convicted and each receives a two year sentence. However, if you help us we will reward you.
- Admitting guilt, it will be easier to convict your friend if he claims innocence. He will get five years and we will let you go.
- If, however, you both plead guilty, you will both get a four year sentence.

Which bi-matrix can be used to model this situation?

What is the dilemma in this problem?

Example (cont.)

Cooperative versus non-cooperative games

If the prisoners are allowed to communicate, they may end up with $(-2, -2)$: both keep silent and receive a two year sentence.

This suggests the concepts of Cooperative Games and Non-cooperative Games.

There is a big difference between these two types. We will study them separately.

Non-cooperative 2-person non-zero-sum games have least departure from 2-person zero-sum games. So we will try to use concepts from zero-sum games to study them.

Non-cooperative 2-person non-zero-sum games

Non-cooperative 2-person non-zero-sum games

- There are two payoff matrices, A for Player I, and B for Player II.
- Since Player II wants to maximize her payoff, we can turn her into a row player by transposing B.
- In the following discussion the payoff matrix for Player II is the transpose B^T of B.
- Treat A as the payoff matrix of a zero-sum game with players I (row) and II (column).
- So we can speak of mixed strategies for Player I and compute the optimal security level for Player I with respect to A.
- Treat B^T as the payoff matrix of a zero-sum game with players II (row) and I (column).
- So we can speak of mixed strategies for Player II and compute the optimal security level for Player II with respect to B^T.

Let

$$X := \{\mathbf{x} = (x_1, \ldots, x_m) : x_1, \ldots, x_m \geq 0, x_1 + \cdots + x_m = 1\}$$

$$Y := \{\mathbf{y} = (y_1, \ldots, y_n) : y_1, \ldots, y_n \geq 0, y_1 + \cdots + y_n = 1\}$$

be the sets of mixed strategies for Players I and II, respectively.

It is understood that if Player I plays $\mathbf{x} = (x_1, \ldots, x_m)$, then he uses his pure strategy a_i with probability x_i, $1 \leq i \leq m$.

Similarly, if Player II plays $\mathbf{y} = (y_1, \ldots, y_n)$, then she uses her pure strategy A_j with probability y_j, $1 \leq j \leq n$.

Occasionally we call $(\mathbf{x}, \mathbf{y})$ a joint (mixed) strategy.

If Player I plays $\mathbf{x}$ and Player II plays $\mathbf{y}$, then the expected payoff to Player I is given by

$$\mathbf{x}A\mathbf{y}^T = (x_1, \dots, x_m) \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

and the expected payoff to Player II is given by

$$\mathbf{x}B\mathbf{y}^T = (x_1, \dots, x_m) \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

(Since the payoff matrix for Player II is B^T, the expected payoff to Player II is $\mathbf{y}B^T\mathbf{x}^T = \mathbf{x}B\mathbf{y}^T$.)

Optimal security levels

Define

$$u^* := \max_{\mathbf{x}\in X} \min_{\mathbf{y}\in Y} \mathbf{x}A\mathbf{y}^T, \quad v^* := \max_{\mathbf{y}\in Y} \min_{\mathbf{x}\in X} \mathbf{x}B\mathbf{y}^T.$$

Call u^* the optimal security level for Player I and v^* the optimal security level for Player II. (They are also called the maximin values for Players I and II respectively.)

- u^* is the optimal security level for Player I (row player) in the zero-sum game with payoff matrix A, and v^* is the optimal security level for Player II (row player) in the zero-sum game with payoff matrix B^T.

- $u^* =$ value of the 2-person zero-sum game with payoff matrix A

- $v^* =$ value of the 2-person zero-sum game with payoff matrix B^T

- We can compute u^* and v^* by solving the 2-person zero-sum games with payoff matrices A and B^T, respectively. (We may use the methodologies for zero-sum games to solve these games.)

Example 5. (2-person, non-cooperative, non-zero-sum game)

Consider the 2-person, non-cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (2,2) & (3,3) \\ (1,1) & (4,4) \end{bmatrix}.$$

Determine the optimal security levels u^* and v^*.

Equilibrium for non-cooperative 2-person non-zero-sum games

Definition 2. For a non-cooperative 2-person non-zero-sum game with payoff matrices A and B, a pair of strategies $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ is said to be in equilibrium if

$$\mathbf{x} A {\mathbf{y}^*}^T \le \mathbf{x}^* A {\mathbf{y}^*}^T \ \text{ for all } \ \mathbf{x} \in X$$

and

$$\mathbf{x}^* B \mathbf{y}^T \le \mathbf{x}^* B {\mathbf{y}^*}^T \ \text{ for all } \ \mathbf{y} \in Y$$

We also call $(\mathbf{x}^*, \mathbf{y}^*)$ an equilibrium or an equilibrium pair.

Note: this definition agrees with the one for a general game in normal form.

The definition of equilibria for non-cooperative 2-person non-zero-sum games is also consistent with the definition of an equilibrium for 2-person zero-sum games.

In fact, for a 2-person zero-sum game with payoff matrix V, we have $A = -B = V$ and so the conditions above become

$$\mathbf{x}V\mathbf{y}^{*T} \le \mathbf{x}^*V\mathbf{y}^{*T}, \;\; \mathbf{x}^*(-V)\mathbf{y}^T \le \mathbf{x}^*(-V)\mathbf{y}^{*T},$$

that is,

$$\mathbf{x}V\mathbf{y}^{*T} \le \mathbf{x}^*V\mathbf{y}^{*T} \le \mathbf{x}^*V\mathbf{y}^T.$$

These are exactly the conditions in the definition of an equilibrium for a 2-person zero-sum game.

Example 6. (Equilibria of 2-person non-zero-sum game)

Consider the 2-person non-zero-sum game with payoff bi-matrix:

$$\begin{bmatrix} (2,2) & (3,3) \\ (1,1) & (4,4) \end{bmatrix}.$$

Determine an equilibrium.

Example 7. (Equilibria of 2-person non-zero-sum game)

Consider the 2-person non-zero-sum game with payoff bi-matrix:

$$\begin{bmatrix} (8,13) & (1,5) \\ (6,4) & (13,8) \end{bmatrix}.$$

Determine the equilibria.

Example (cont.)

Example (cont.)

Comments on optimal security levels and equilibria for 2-person non-zero-sum games

In a non-cooperative 2-person non-zero-sum game:

- An optimal security level pair is not necessarily an equilibrium pair (but it may be), and vice versa. (This is different from the zero-sum case.)
- There may be more than one equilibria, and they may give different payoffs.
- Even when there is only one equilibrium, it is not clear whether this equilibrium is exactly what we want. For example, the Prisoners' Dilemma has one equilibrium pair, but it is not a good solution.
- None of the notions of optimal security levels and equilibria provides a satisfactory solution concept for 2-person non-zero-sum games.

Equilibrium versus optimal security levels

Nevertheless, the following result shows that equilibrium pairs are at least as good as optimal security level pairs.

Recall that $\mathbf{x}^* A \mathbf{y}^{*T}$ and $\mathbf{x}^* B \mathbf{y}^{*T}$ are the expected payoffs to Players I and II respectively if they apply the joint strategy $(\mathbf{x}^*, \mathbf{y}^*)$.

Theorem 1. Let u^* and v^* be the optimal security levels for Players I and II respectively in a non-cooperative non-zero-sum game with payoff matrices A and B. Let $(\mathbf{x}^*, \mathbf{y}^*)$ be any equilibrium pair for the same game. Then

$$u^* \leq \mathbf{x}^* A \mathbf{y}^{*T}, \;\; v^* \leq \mathbf{x}^* B \mathbf{y}^{*T}.$$

Proof.

Equilibrium Theorem of John Nash

Theorem 2. (John Nash 1951) Any non-cooperative 2-person game (zero-sum or non-zero-sum) with a finite number of pure strategies for each player has at least one equilibrium pair.

In the zero-sum case, this gives the same result as the Fundamental Theorem of Matrix Game Theory.

In 1994, the Nobel Prize in Economics was awarded jointly to **John Harsanyi, John F. Nash,** and **Reinhard Selten** for their pioneering analysis of equilibria in the theory of non-cooperative games.

Definition 3. A subset A of $\mathbb{R}^k$ is called

- bounded if there exists $r > 0$ such that $\|\mathbf{x}\| \le r$ for any $\mathbf{x} \in A$.

- closed if $\mathbb{R}^k \setminus A$ is open (i.e. for any $\mathbf{x}_0 \in \mathbb{R}^k \setminus A$, there exists $\epsilon > 0$ such that $\left\{\mathbf{x} \in \mathbb{R}^k : \|\mathbf{x} - \mathbf{x}_0\| < \epsilon\right\}$ is contained in $\mathbb{R}^k \setminus A$). Intuitively A is closed if it contains all its boundary points.

- compact if it is both bounded and closed.

- convex if for any two points $\mathbf{x}_1, \mathbf{x}_2 \in A$, the line segment between them, namely $\{\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 : 0 \le \alpha \le 1\}$, is contained in A.

A proof of Nash's theorem requires the following well-known result.

Theorem 3. (Brouwer's Fixed Point Theorem) Let $C \neq \emptyset$ be a compact convex set of $\mathbb{R}^k$. Let $f : C \to C$ be a continuous function. Then there exists a point $\mathbf{x}^* \in C$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$.

Such a point $\mathbf{x}^*$ is called a fixed point.

Finding equilibria in non-cooperative 2-person non-zero-sum games

- In general, it is difficult to find all equilibria in a non-cooperative 2-person non-zero-sum game. However, there are some techniques which are useful in some situations.

- In the case of 2×2 bi-matrices, we can use a graphical method (which is different from the one used for zero-sum games).

- We will illustrate this method by an example.

- We will also learn another approach which works sometimes.

Graphical method

The graphical method discussed here can be used to find all Nash equilibria of a two-person 2×2 non-zero-sum bi-matrix game.

- For both players one can draw so-called best-reply curves.
- An equilibrium pair $(\mathbf{x}^*, \mathbf{y}^*)$ is such that $\mathbf{x}^*$ is a best reply for player I against $\mathbf{y}^*$ and $\mathbf{y}^*$ is a best reply for player II to $\mathbf{x}^*$.
- Nash equilibria correspond to the points of intersection of the best reply curves.

Example 8. (graphical method)

Consider the 2-person, non-cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (3,2) & (2,1) \\ (0,3) & (4,4) \end{bmatrix}.$$

Example (cont.)

Example (cont.)

Example (cont.)

Example 9. (graphical method)

Consider the 2-person, non-cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (2,3) & (2,0) \\ (2,0) & (1,3) \end{bmatrix}.$$

Determine all equilibria of this game by drawing the best-reply curves

Example (cont.)

Example (cont.)

Another approach

Player II thinking:

"If I could find a mixed strategy $\mathbf{y}^*$ such that the expected payoff to Player I is a constant regardless of what he does, then any strategy $\mathbf{x} \in X$ for player I is an optimal strategy against $\mathbf{y}^*$."

To compute $\mathbf{y}^*$, I have to solve:

$$\mathbf{x} A \mathbf{y}^{*T} = \text{constant}, \quad \text{for all} \ \mathbf{x} \in X.$$

Player I may attempt the same and construct a strategy $\mathbf{x}^*$ such that

$$\mathbf{x}^* B \mathbf{y}^T = \text{constant}, \quad \text{for all} \ \mathbf{y} \in Y.$$

The strategy pair $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium (why?).

Try to find $\mathbf{y}^* = (y_1^*, \ldots, y_n^*) \in Y$ and a constant z such that

$$A\mathbf{y}^{*T} = (z, \ldots, z)^T, \;\; y_1^* + \cdots + y_n^* = 1$$

This is a linear system with $m+1$ equations and $n+1$ unknowns (including z).

Try to find $\mathbf{x}^* = (x_1^*, \ldots, x_m^*) \in X$ and a constant w such that

$$\mathbf{x}^* B = (w, \ldots, w), \;\; x_1^* + \cdots + x_m^* = 1$$

This is a linear system with $n+1$ equations and $m+1$ unknowns (including w).

If we are lucky, these equations may be consistent and have non-negative solutions $y_j^* \geq 0$ and $x_i^* \geq 0$. Then we will have
$\mathbf{x}A\mathbf{y}^{*T} = (x_1 + \cdots + x_m)z = z =$ constant and
$\mathbf{x}^* B\mathbf{y}^T = w(y_1 + \cdots + y_n) = w =$ constant, as required.

This sounds good, but it may not work! Even if it works, it only gives you one equilibrium but not all equilibria!

Example 10. (Equilibrium two-person non-zero-sum bi-matrix game)
Consider the 2-person, non-cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (8,-1) & (4,7) & (-1,3) \\ (7,6) & (-1,2) & (2,-2) \\ (1,4) & (7,0) & (1,8) \end{bmatrix}.$$

Find an equilibrium pair $(\mathbf{x}^*, \mathbf{y}^*)$ with help of the approach described in slides 38-39.

Summary: non-cooperative 2-person non-zero-sum games

- Two solution concepts: optimal security level pairs, equilibrium pairs.
- The security level idea is not really very good here because it assumes players are simultaneously trying to maximize their own payoff, whilst minimizing their opponents payoff. These two objectives are sometimes diametrically opposed. It is no longer true that a player can get rich only by keeping their opponent poor.
- Also, the payoff for security level is less than or equal to that for equilibrium pair.
- Equilibrium pairs are more acceptable solution concept, but there are difficulties with them too (eg. it is difficult to find equilibria in general).
- There is no satisfactory simple notion of “optimal strategy” and “value” as there is with 2-person zero-sum games.

Cooperative 2-person games

Cooperative 2-person games

In a cooperative game:

- Players may not be so much out to “beat” each other, e.g. trading between two nations, negotiations between employer and employee, etc.
- Sometimes an action may benefit both competitors (win-win situation).
- Players are allowed to discuss strategy before the play.
- Thus we have to expand the set of “feasible” strategies.
- We will focus on the payoffs rather than the strategies. This is very different from non-cooperative games!

Example 11. (cooperative 2-person game)

	A_1	A_2	A_3	A_4
a_1	$(4, 10)$	$\ldots$	$\ldots$	$\ldots$
a_2	$\ldots$	$\ldots$	$\ldots$	$\ldots$
a_3	$\ldots$	$\ldots$	$\ldots$	$(10, 6)$

Payoff bi-matrix again

Consider a cooperative 2-person game with payoff bi-matrix:

$$
\begin{array}{cc|ccccc}
 & & & & \text{Player II} & & \\
 & & A_1 & \dots & A_j & \dots & A_n \\
\hline
 & a_1 & (a_{11}, b_{11}) & \dots & (a_{1j}, b_{1j}) & \dots & (a_{1n}, b_{1n}) \\
 & \vdots & \vdots & & \vdots & & \vdots \\
\text{Player I} & a_i & (a_{i1}, b_{i1}) & \dots & (a_{ij}, b_{ij}) & \dots & (a_{in}, b_{in}) \\
 & \vdots & \vdots & & \vdots & & \vdots \\
 & a_m & (a_{m1}, b_{m1}) & \dots & (a_{mj}, b_{mj}) & \dots & (a_{mn}, b_{mn})
\end{array}
$$

Denote $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ as before.

Probability distributions on payoffs

The players may agree to play (a_i, A_j) with a certain probability. Hence there is a probability distribution on the set of joint strategies (a_i, A_j), or equivalently on the set of payoff pairs (a_{ij}, b_{ij}). We treat (a_{ij}, b_{ij}) as a point in $\mathbb{R}^2$.

Definition 4. Let p_{ij} denote the probability that the two players agree to play (a_i, A_j), $1 \leq i \leq m, 1 \leq j \leq n$, where

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1, \ \ 0 \leq p_{ij} \leq 1, \ \ 1 \leq i \leq m, 1 \leq j \leq n.$$

We assume the game is played repeatedly for otherwise it does not make sense to talk about probabilities.

For fixed probabilities p_{ij}, the expected payoffs are given by

$$\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}(a_{ij}, b_{ij}) = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}a_{ij}, \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}b_{ij}\right).$$

This expression is a point in $\mathbb{R}^2$ whose coordinates are the expected payoffs to Players I and II respectively. That is,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}a_{ij} = \text{expected payoff to Player I}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}b_{ij} = \text{expected payoff to Player II}$$

Cooperative payoff set

We consider all probability distributions on the set of points (a_{ij}, b_{ij}) and therefore all possible expected payoff pairs.

Definition 5. Call

$$C := \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(a_{ij}, b_{ij}) : \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1,\ 0 \le p_{ij} \le 1 \text{ for all } i, j \right\}$$

the cooperative payoff set.

Any point (u, v) of C is called a cooperative payoff pair.

For a point $(u, v) \in C$, Player I is rewarded u units and Player II is rewarded v units.

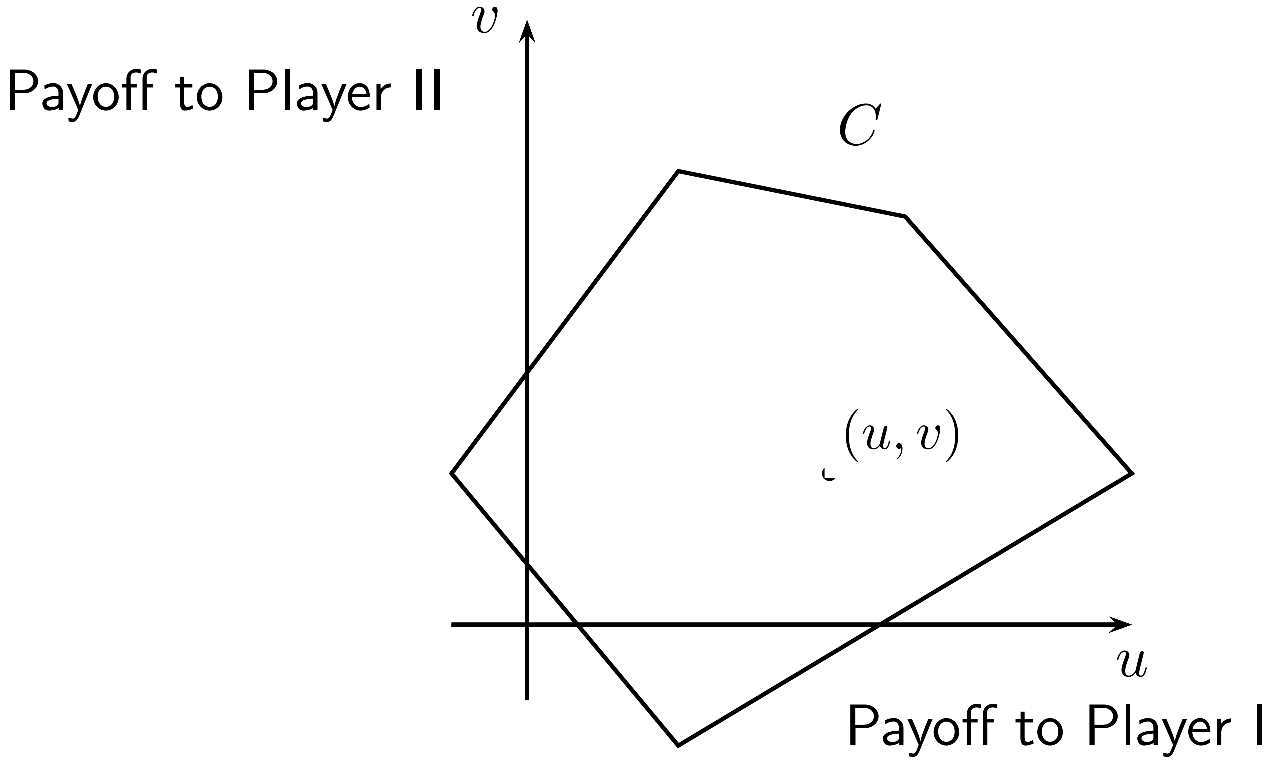
C is the convex hull of the set of points (a_{ij}, b_{ij}), $1 \le i \le m, 1 \le j \le n$, in $\mathbb{R}^2$. Some of these points are extreme points of C.

Major task for cooperative 2-person games

Find a cooperative payoff pair $(u, v) \in C$ which is "good" for both players and reflects their contributions to the game.

Once we find (u, v), we can work out the corresponding probabilities p_{ij} (ie. those p_{ij} such that $(u, v) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(a_{ij}, b_{ij})$) by linear algebra.

In general, C is an infinite set. Which point should we choose?



Example 12. (cooperative payoff set)

Consider the 2-person, cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (0,5) & (5,2) \\ (1,-3) & (1,-5) \end{bmatrix}.$$

Draw the cooperative payoff set.

Pareto boundary

Let us try to reduce the size of the solution space so as to make it easier for the players to agree on a solution to the game.

Definition 6. A pair $(u', v') \in C$ is called dominated by another pair $(u, v) \in C$ if

$$u \geq u', v \geq v'$$

and at least one inequality is strict (i.e. either $u > u'$ or $v > v'$).

A pair $(u, v) \in C$ is called Pareto optimal if it is **not dominated** by any other pair in C.

The set of Pareto optimal payoff pairs is called the Pareto boundary of the game, denoted by $PB(C)$.

Example 13. (Pareto boundary)

Consider the 2-person, cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (0,5) & (5,2) \\ (1,-3) & (1,-5) \end{bmatrix}.$$

Determine the Pareto boundary of this game.

Further cut by using optimal security levels

Let us pretend the two players do not cooperate. Then each of them is guaranteed to receive no less than what she/he can get by playing herself/himself, i.e. their respective optimal security levels:

$$u^* = \max_{\mathbf{x}\in X}\min_{\mathbf{y}\in Y} \mathbf{x}A\mathbf{y}^T, \quad v^* = \max_{\mathbf{y}\in Y}\min_{\mathbf{x}\in X} \mathbf{x}B\mathbf{y}^T.$$

If one of the players receives less than her/his optimal security level, then there is no incentive for her/him to cooperate. Any such cooperative payoff pair should be rejected.

Recall that

$$u^* = \text{value of the 2-person zero-sum game with payoff matrix } A$$

$$v^* = \text{value of the 2-person zero-sum game with payoff matrix } B^T$$

We can simply solve these two zero-sum games to compute u^* and v^*.

Negotiation set

Definition 7. The negotiation set of a cooperative 2-person game is defined as

$$NS(C) = \{(u, v) \in C : u \geq u^*, v \geq v^*\} \cap PB(C).$$

$NS(C)$ is also called the bargaining set.

Example 14. (negotiation set)

Consider the 2-person, cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (0,5) & (5,2) \\ (1,-3) & (1,-5) \end{bmatrix}.$$

Draw the negotiation set.

- The idea of "negotiation set" gets rid of pairs that are clearly unwanted as solutions to the game.
- The reduction can be substantial.
- However, in general the negotiation set is infinite.
- Can we reduce further to get a finite number of reasonably fair payoff pairs or even a unique pair?
- John Nash did this more than 50 years ago.

Nash's bargaining axioms

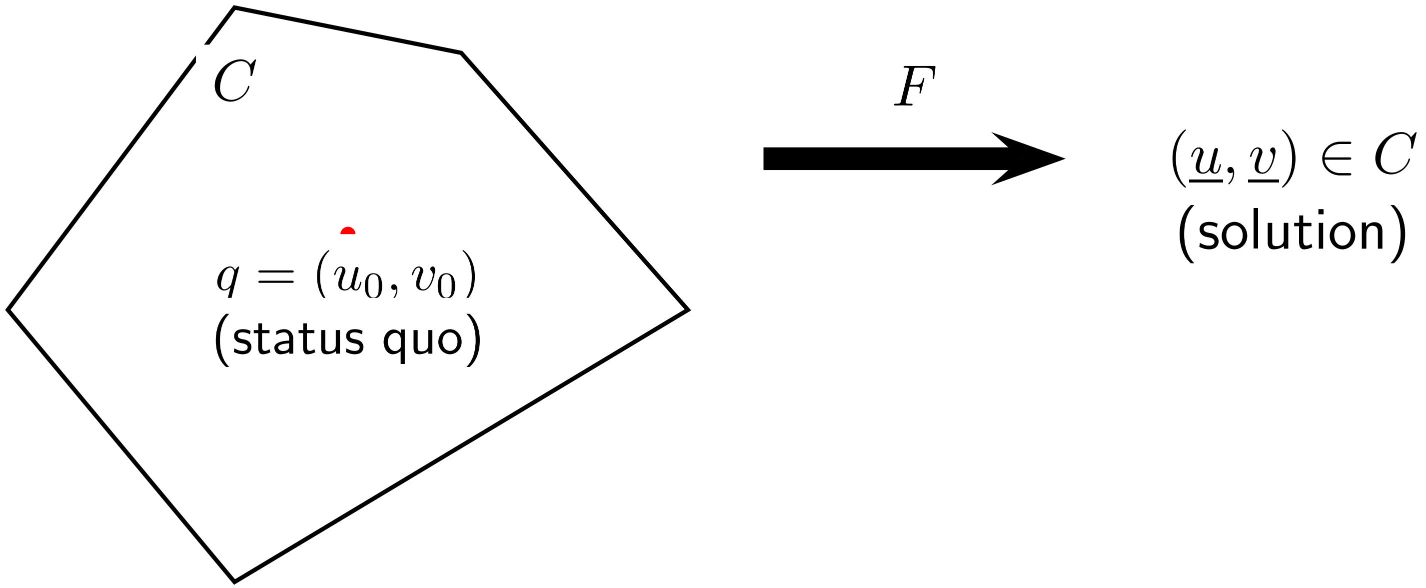
Overview of Nash's theory

Nash proved the existence of a bargaining procedure F which, when presented with a cooperative payoff set C together with a status quo point $q = (u_0, v_0)$, outputs a unique payoff pair $(\underline{u}, \underline{v})$ which is fair to both players.

Nash proposed a set of axioms that he believed to be "reasonable" conditions for such a payoff pair $(\underline{u}, \underline{v})$ to satisfy.

Denote

$$(\underline{u}, \underline{v}) = F(C, (u_0, v_0)).$$



The status quo point $q = (u_0, v_0) \in C$ should be a payoff pair which is acceptable by both players to initiate the negotiation.

We typically choose (u_0, v_0) to be the pair (u^*, v^*) of optimal security levels. But other points can be used also.

We may assume that there is at least one point in C that dominates the status quo point for otherwise the players would just use the status quo point.

Nash's Axioms

Nash proposed the following conditions for $(\underline{u}, \underline{v}) = F(C, (u_0, v_0))$ to satisfy.

1: Individual Rationality

2: Feasibility

3: Pareto Optimality

4: Independence of Irrelevant Alternatives

5: Invariance under Linear Transformations

6: Symmetry

Axiom 1. (Individual Rationality) The point $(\underline{u}, \underline{v})$ is as good as the status quo point $q = (u_0, v_0)$, that is,

$$\underline{u} \geq u_0, \ \underline{v} \geq v_0.$$

Comment: This makes a lot of sense! Why should a player be willing to accept less than his part of the status quo point?

Axiom 2 requires that $(\underline{u}, \underline{v})$ is feasible:

Axiom 2. (Feasibility) The point $(\underline{u}, \underline{v})$ is in the cooperative payoff set C.

Axiom 3. (Pareto Optimality) The point $(\underline{u}, \underline{v})$ is on the Pareto boundary $PB(C)$ of C. That is, if a point $(u, v) \in C$ is such that $u \geq \underline{u}, v \geq \underline{v}$, then $u = \underline{u}, v = \underline{v}$.

Comment: These two axioms sounds reasonable. Axiom 3 says that there exists no point in C which dominates $(\underline{u}, \underline{v})$. (If otherwise, why not go with that point?)

Axiom 4. (Independence of Irrelevant Alternatives) If D is a subset of C and both (u_0, v_0) and $(\underline{u}, \underline{v})$ are in D, then

$$F(D, (u_0, v_0)) = (\underline{u}, \underline{v}).$$

Comment: This is a controversial axiom. See the example below.

Example 15. (independence of irrelevant alternatives)

Axiom 5. (Invariance under Linear Transformations) The solution is invariant under any linear transformation

$$L(u, v) = (a_1 u + b_1, a_2 v + b_2), \text{ where } a_1 > 0, a_2 > 0.$$

That is, if $(\underline{u}, \underline{v}) = F(C, (u_0, v_0))$ is the solution of the procedure with respect to $(C, (u_0, v_0))$, then $L(\underline{u}, \underline{v}) = (a_1 \underline{u} + b_1, a_2 \underline{v} + b_2)$ is the solution of the procedure with respect to $(L(C), L(u_0, v_0))$, where $L(C)$ is the image of C under L. In other words,

$$F(L(C), L(u_0, v_0)) = L(F(C, (u_0, v_0))).$$

Comment: Is the World of Games linear? This axiom is controversial.

Definition 8. A subset A of $\mathbb{R}^2$ is said to be symmetric if it is symmetric about the line $y = x$, i.e. $(x, y) \in A$ if and only if $(y, x) \in A$.

Axiom 6. (Symmetry) If the cooperative payoff set C is symmetric and the status quo point is symmetric, i.e. $u_0 = v_0$, then the solution is symmetric, i.e.

$$\underline{u} = \underline{v}.$$

Comment: This axiom guarantees that the two players are treated equally fairly. It says that if the game is "symmetric" with regard to the two players, then the solution should also be "symmetric".

Example 16. (symmetry)

Consider the 2-person, cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (8,0) & (3,3) \\ (3,3) & (0,8) \end{bmatrix}.$$

Using the optimal security level pair $(3,3)$ as status quo point, which solution is implied by axioms 1-3 and 6? Also determine the corresponding probability distribution on the set of joint strategies (a_i, A_j), $1 \le i \le m$, $1 \le j \le n$.

Nash's Theorem

Theorem 4. There exists a unique point $(\underline{u}, \underline{v})$ which satisfies Nash's Axioms 1–6. Moreover, if there exists at least one point $(u_1, v_1) \in C$ such that $u_1 > u_0, v_1 > v_0$, then $(\underline{u}, \underline{v})$ is the unique optimal solution of the following optimization problem:

$$\max_{(u,v) \in PB(C),\, u > u_0, v > v_0} (u - u_0)(v - v_0).$$

This point $(\underline{u}, \underline{v})$ is called Nash's solution.

In the general case where $u_1 > u_0, v_1 > v_0$ holds for at least one $(u_1, v_1) \in C$, we can find Nash's solution $(\underline{u}, \underline{v})$ by solving the above nonlinear optimization problem.

Proof of Nash's theorem. The proof is not discussed in this year's lectures. You can find the proof in 'Introduction to Game Theory' by Peter Morris.

Computing Nash's solution

- The objective function $(u - u_0)(v - v_0)$ is monotonically increasing with u and v as we move from the status quo point.
- Hence the maximum of $(u - u_0)(v - v_0)$ will be achieved at a point in

$$A = \{(u, v) \in C,\ u > u_0, v > v_0\} \cap PB(C) = NS(C).$$

  As $PB(C)$ is a union of line segments, so is A.
  Each line segment contained in A should have the form
  $v = ku + l,\ c \le u \le d$ (or $u = kv + l,\ c \le v \le d$) for some k, l, c, d.
  Substituting this into $(u - u_0)(v - v_0)$ we obtain a quadratic function of u (or v) whose maximum value over $[c, d]$ can be easily found.
  Among the maximum values for all line segments in A, choose the maximum one. The corresponding maximum point is $(\underline{u}, \underline{v})$.
- Usually we choose the status quo point (u_0, v_0) to be the pair (u^*, v^*) of optimal security levels. In this case, $A = NS(C)$ and the optimization problem becomes $\max_{(u,v)\in NS(C)}(u - u^*)(v - v^*)$.

Example 17. (Nash's solution)

Consider the 2-person, cooperative, non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (0,5) & (5,2) \\ (1,-3) & (1,-5) \end{bmatrix}.$$

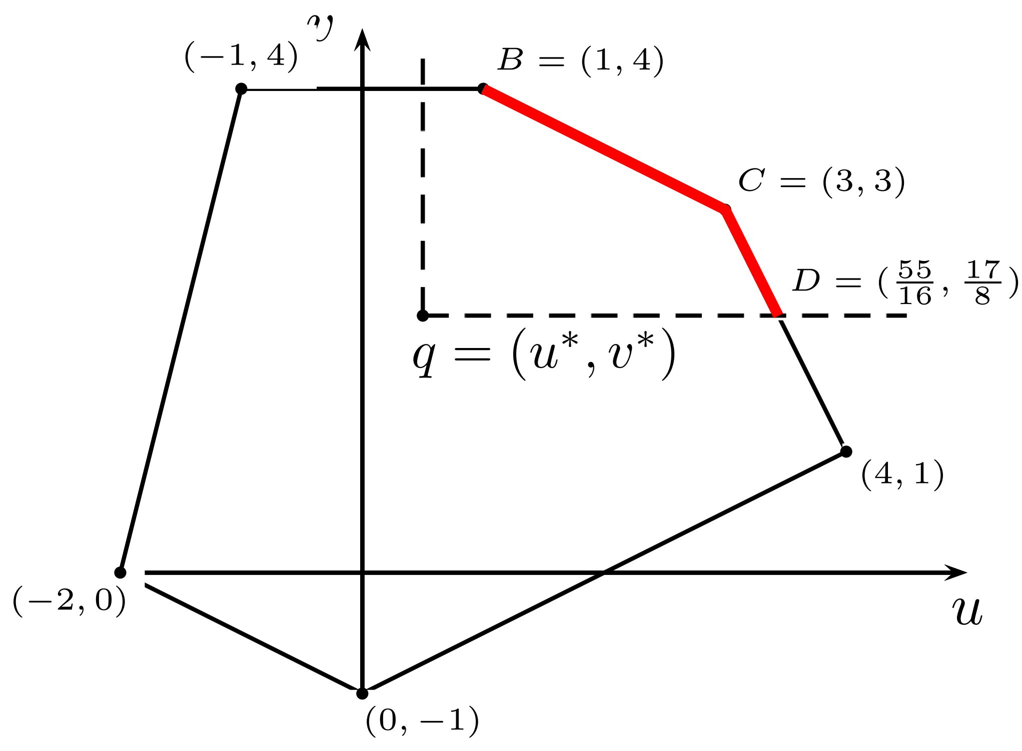
Compute Nash's solution.

Example 18. (Nash's solution)

Determine the solution generated by Nash's axioms for the game with payoff matrix below, using the security level pair as the status quo point.

$$\begin{bmatrix} (1,4) & (-2,0) & (4,1) \\ (0,-1) & (3,3) & (-1,4) \end{bmatrix}.$$

Solution: the security level pair is $(u^*, v^*) = \left(\frac{1}{2}, \frac{17}{8}\right)$, hence $q = \left(\frac{1}{2}, \frac{17}{8}\right)$.



The set A (in red) consists of the line segments L_{BC} and L_{CD}.

Example (cont.)

For each line segment of A we determine the maximum of the function

$$(u-u^*)(v-v^*)=\left(u-\tfrac{1}{2}\right)\left(v-\tfrac{17}{8}\right).$$

$L_{BC}: v=-\frac{1}{2}u+\frac{9}{2}$, $1\leq u\leq 3$.

$$\begin{aligned}\max_{(u,v)\in L_{BC}}\left(u-\tfrac{1}{2}\right)\left(v-\tfrac{17}{8}\right) &= \max_{1\leq u\leq 3}\left(u-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}u+\tfrac{9}{2}-\tfrac{17}{8}\right)\\ &= \max_{1\leq u\leq 3}\left(u-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}u+\tfrac{19}{8}\right)\\ &= \max_{1\leq u\leq 3}\left(-\tfrac{1}{2}u^2+\tfrac{21}{8}u-\tfrac{19}{16}\right).\end{aligned}$$

Let $g(u)=-\frac{1}{2}u^2+\frac{21}{8}u-\frac{19}{16}$, then $g'(u)=-u+\frac{21}{8}$. Setting $g'(u)=0$, we get $u=\frac{21}{8}$ which is in the interval $[1,3]$. So the maximum value on the line segment L_{BC} is $g\left(\frac{21}{8}\right)=\frac{17^2}{2^7}(\approx 2.2578)$.

Example (cont.)
$L_{CD}: v = -2u + 9$, $3 \le u \le \frac{55}{16}$.

$$\max_{(u,v)\in L_{CD}} \left(u - \tfrac{1}{2}\right)\left(v - \tfrac{17}{8}\right) = \max_{3\le u\le \frac{55}{16}} \left(u - \tfrac{1}{2}\right)\left(-2u + \tfrac{55}{8}\right)$$
$$= \max_{3\le u\le \frac{55}{16}} \left(-2u^2 + \tfrac{63}{8}u - \tfrac{55}{16}\right).$$

Let $h(u) = -2u^2 + \frac{63}{8}u - \frac{55}{16}$, then $h'(u) = -4u + \frac{63}{8}$. Setting $h'(u) = 0$, we get $u = \frac{63}{32} < 3$. So the maximum value on the line segment L_{CD} is achieved at $u = 3$ or $u = \frac{55}{16}$.

$$h(3) = \tfrac{35}{16},\ h\left(\tfrac{55}{16}\right) = 0.$$

Hence the maximum on the line segment L_{CD} is $h(3) = \frac{35}{16}(= 2.1875) < g\left(\frac{21}{8}\right)$.

Example (cont.)

Conclusion: Nash's solution is $(\underline{u}, \underline{v}) = \left(\frac{21}{8}, \frac{51}{16}\right)$.

Since $\left(\frac{21}{8}, \frac{51}{16}\right) = \frac{3}{16}(1,4) + \frac{13}{16}(3,3)$, this solution can be achieved by the players by playing the strategy pair (a_1, A_1) with probability $\frac{3}{16}$ and the strategy pair (a_2, A_2) with probability $\frac{13}{16}$.

Special case in Nash's Theorem

In the special case when there exists no $(u_1, v_1) \in C$ such that $u_1 > u_0, v_1 > v_0$, the negotiation set contains a unique point, and this point is Nash's solution $(\underline{u}, \underline{v})$.

Proof. We assumed that (u_0, v_0) is dominated by at least one point $(u, v) \in C$, i.e. one of the following occurs:

- Case 1: $u > u_0,\ v \geq v_0$
- Case 2: $u \geq u_0,\ v > v_0$

Since we assume no $(u, v) \in C$ satisfies $u > u_0$ and $v > v_0$ simultaneously, in Case 1 we have $v = v_0$, and in Case 2 we have $u = u_0$.

- In Case 1, Nash's solution is $(\underline{u}, v_0)$, where

$$\underline{u} = \max_{(u,v_0)\in C,\, u\geq u_0} u$$

- In Case 2, Nash's solution is $(u_0, \underline{v})$, where

$$\underline{v} = \max_{(u_0,v)\in C,\, v\geq v_0} v$$

Hence, in both cases Nash's solution is the unique point of the negotiation set.

Example 19. (special case in Nash's theorem)