eg.

$\{X_n\}$ describes the weather at a particular location, with $X_n = 1$. if it is sunny ; $X_n = 2$ if it is rainy on day n.

Suppose that the weather on day $n+1$ depends on the weather conditions on days $n-1$ & n

$$\begin{cases} P(X_{n+1}=2 \mid X_n = X_{n-1} = 2) = 0.6 \\ P(\quad =1 \mid \cdots \quad =1) = 0.8 \\ P(\cdots =2 \mid X_n = 2,\ X_{n-1}=1) = 0.5 \\ P(\cdots =1 \mid X_n=1,\ X_{n-1}=2) = 0.75 \end{cases}$$

put $Y_n = (X_{n-1}, X_n)$. $\Rightarrow$ Y_n DTMC. states are $1' = (1,1)$ $2' = (1,2)$ $3' = (2,1)$ $4' = (2,2)$.

Transition matrix of Y_n is

$$\begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}.$$

Markov Decision Theory.

$E_a :=$ expectation of under choosing action a. R: reward

then for $n = 0, 1, \cdots, N-1$

$$V_n(i) = \max_a E\left[\sum_{t=N-n}^{N} R(X_t, a_t) \,\middle|\, X_{N-n} = i\right]$$

given state i

$$= \max_a \left[R(i,a) + E_a\left[V_{n-1}(X_{N-n+1}) \mid X_{N-n} = i\right]\right]$$

$$= \max_a \left(R(i,a) + \sum_j P_{ij}(a) \cdot V_{n-1}(j)\right)$$

Periodic DTMC

- P_r : period $d>1$. P_r is recurrent with a finite expected recurrence time

then $\{X_{nd+k} \mid X_0 \in S_r\}_{0 \le k \le d-1}$ is ergodic DTMC with state space $S_r^{(k)}$

S_r : $\boxed{S_r^{(1)}}$ $\boxed{S_r^{(2)}}$ - - - $\boxed{S_r^{(d)}}$ (d step return to the class).

- $\forall\, l, k \in \{0,\dots,d-1\}$ $P(X_{nd+k}=j \mid X_0 \in S_r^{(l)}) \begin{cases} \to \tilde{\pi}_j^{(r)} \ (n\to\infty) & j \in S_r^{(l+k) \ (\mathrm{mod}\ d)} \\ = 0 & j \notin \cdots \end{cases}$

周期性

$$\sum_{j \in S_r^{(l)}} \tilde{\pi}_j^{(r)} = 1$$

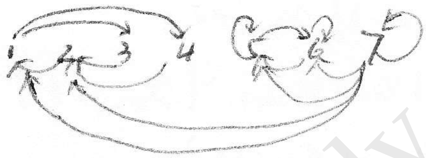
$$P=\begin{pmatrix} 0 & 0 & 0.1 & 0.9 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}.$$

pd = 3

$S_1^e = \{1, 2, 3, 4\}$. positive recurrent as is finite

$S_2^e = \{5, 6\}$. positive recurrent aperodic.

$S_1^n = \{7\}$.



$\{7\}$ non-essential. start here $\to$ end up in S_1^e ~~&~~ S_2^e

- Give end up / start in S_2^e unique long run stationary distribution.

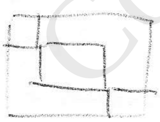
π : $\pi \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} = \pi$ $\Rightarrow \frac{1}{3}\pi_1 + \frac{1}{5}\pi_2 = \pi_1$

$\pi_1 + \pi_2 = 1$

- in S_1^e $\{1\} \to \{3, 4\} \to \{2\}$ period = 3.

The chain viewed at this a multiple of 3 is an irreducible aperodic M-chain.

$\pi_1^{(1)} = 1$



$\pi_2^{(3)} = 1$.

$\{\pi_3^{(2)}, \pi_4^{(2)}\}$ statisfi

$$(\pi_3^{(2)}, \pi_4^{(2)}) \begin{pmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{pmatrix} = (\pi_3^{(2)}, \pi_4^{(2)})$$

$\Rightarrow \pi_3^{(2)} = 0.1$. $\pi_4^{(2)} = 0.9$

eg for example starts at 1

$$P(X_{3n+k}=j \mid X_0=1) = \begin{cases} 1 & k=0,\ j=1;\ k=2,\ j=2 \\ 0.1 & k=1,\ j=3 \\ 0.9 & (k=1,\ j=4) \end{cases}$$