THE UNIVERSITY OF MELBOURNE

Student ID

Semester 1 Assessment, 2016

School of Mathematics and Statistics

MAST30013 Techniques in Operations Research

Writing time: 2 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised materials:

- Students will be provided a 6-page formulae sheet.
- School approved calculators.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- The total number of marks available is 60.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

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Question 1 (10 marks)
Let $f(x) = |x - 1.6|, x \in \mathbb{R}$. *We wish to find the minimiser* x^* *of* f *over the interval* $[1, 2]$.

(a) How many **computations of the objective function** would you need to find an interval of size $s \leq 0.2$ that contains x^* using the:

(i) Fibonnacci search.

(ii) Golden search.

b) Use the **Fibonnacci search** (refer to the algorithm in the formulae sheet) to find an interval of size $s \leq 0.2$ that contains x^*. Display all intermediate results using a table such as:

iteration	k	a_k	b_k	p_k	q_k	$f(p_k)$	$f(q_k)$
1	n	1	2	?	?	?	?
2	?	?	?	?	?	?	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

d) Use the **Golden Search** (refer to the algorithm in the formulae sheet) to find an interval of size $s \leq 0.2$ that contains x^*. Display all intermediate results using a table such as Table 2.

iteration	a_k	b_k	p_k	q_k	$f(p_k)$	$f(q_k)$
1	1	2	?	?	?	?
2	?	?	?	?	?	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Question 2 (10 marks)
Given the function $f(x, y) = 2x^3 + 2y^3 - 3x^2 + 9y^2 - 12$

(a) Using the first-order necessary condition, find all stationary points of f.

(b) For each point obtained, use the second-order condition to classify all points found.

Question 3 (10 marks)

For a chemical process, pressure measured at different temperatures is given in the following table. A scientist wants to model the problem using a function $f(t) = \alpha\beta^{(t-10)}$, in which t is the temperature.

Temperature (C)	Pressure (mm of Mercury)
12	2
15	30
16	60

(a) Formulate an unconstrained optimisation problem to find parameters α and β that minimise the sum of the quadratic deviation from the results obtained by the model and the real data available.

(b) Using an initial point $(\alpha_0, \beta_0) = (2, 2)$, find the direction chosen by the steepest descent method and the single variable function you would have to minimise in order to find the step size (no need to simplify the function).

Question 4 (10 marks)

Consider the constrained nonlinear problem:

$$\text{Min } f(x, y) = -20x - 10y \tag{1}$$

subject to:

$$x^2 + y^2 \leq 1 \tag{2}$$
$$x \geq 0 \tag{3}$$
$$y \geq 0 \tag{4}$$
$$\tag{5}$$

(a) Write down the Lagrangian for the problem.

(b) Find the KKT point given that $x^* \geq 0$ and $y^* \geq 0$.

(c) Find the critical cone at the KKT point.

(d) State a second-order sufficiency condition for (x^*, y^*) to be a local minimum of the nonlinear program. Does this condition hold at (x^*, y^*) ? If so, evaluate the minimum value of the objective function.

(e) Give an interpretation for the value of the first Lagrangian multiplier.

Question 5 (10 marks)
Consider the non-linear constrained problem:

$$\text{Min } f(x,y) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + y^2 \tag{6}$$

subject to:

$$-x \leq a \tag{7}$$
$$y \geq b \tag{8}$$
$$\tag{9}$$

with $a = 0$ and $b = 2$.

(a) Write down the l_2-penalty function $P_k(x)$ with penalty parameter k.

(b) Write down $\nabla P_k(x)$ and show that the stationary points for $P_k(x)$ only occur when $x \geq 0$ and $y < 2$.

(c) Find all stationary points for $P_k(x)$.

(d) For each stationary point, what is the approximate change in the objective function if a is changed from 0 to a small value ϵ ?

Question 6 (10 marks)
Given the constrained nonlinear program:

$$\text{Min } f(x,y,z) = x + 2y + 3z \tag{10}$$

subject to:

$$x^2 + y^2 + z^2 \leq 1 \tag{11}$$
$$x \leq 0 \tag{12}$$
$$y \leq 0 \tag{13}$$
$$z \leq 0 \tag{14}$$
$$\tag{15}$$

(a) Explain why the nonlinear program is a convex program.

(b) Write down the Lagrangian for the convex program. Associate multipliers $\lambda_1, \lambda_2, \lambda_3$ and λ_4 to constraints $(11), (12), (13)$ and (14), respectively.

(c) $(x^*, y^*, z^*, \lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*) = (\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{7}{\sqrt{14}}, 0, 0, 0)$ is a KKT point. How many other KKT points exist ?

(d) What is the value of the Lagrangian for point $(x^*, y^*, z^*, \lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*) = (\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{7}{\sqrt{14}}, 0, 0, 0)$.

(e) Show that $(\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*) = (\frac{7}{\sqrt{14}}, 0, 0, 0)$ maximises $L(x, y, z, \lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*)$.

End of Exam—Total Available Marks = 60.