Q1 Solve the following equation for y in terms of x

$$\log_2(y+2) - 3\log_2(2-2x) = 3 \quad (*)$$

Using log - laws we have:

$$(*) \Leftrightarrow \log_2(y+2) - \log_2(2-2x)^3 = 3$$

$$\Leftrightarrow \log_2\left((y+2)(2-2x)^3\right) = \log_2 8 = \log_2 2^3$$

$$\Leftrightarrow (y+2)(2-2x)^3 = 8 = 2^3$$

$$\Leftrightarrow y+2 = \frac{8}{(2-2x)^3}$$

$$\Leftrightarrow \boxed{y = \frac{8}{(2-2x)^3} - 2} \text{ - Answer}$$

Q2 Solve the following equation for x. Express your answer using $\ln x$ or $\log_{10} x$:

$$3^{5-x} = 11$$

$$\Leftrightarrow \ln 3^{5-x} = \ln 11$$

(log. - laws) $\Leftrightarrow (5-x)\ln 3 = \ln 11$

(basic algebra) $\Leftrightarrow (5-x) = \frac{\ln 11}{\ln 3}$

$$-x = \frac{\ln 11}{\ln 3} - 5 \Leftrightarrow \boxed{x = -\frac{\ln 11}{\ln 3} + 5}$$

Answer

Q3 Given the equation

(i) $\log_8(x^8) + \log_{64}(x) = 14, \quad (*)$

determine the value of $\log_8 x$

(as fraction, not evaluate)

Using the change of base rule, eq. (*) can be written as follows:

$$\log_8 x^8 + \frac{\log_8 x}{\underbrace{\log_8 64}_{=2}} = 14$$

$\Rightarrow$ (log. law) $$8 \log_8 x + \frac{1}{2} \log_8 x = 14$$

$\Rightarrow$ ×2 both sides of the last eq. $$16 \log_8 x + \log_8 x = 28$$

$\Rightarrow$ collecting like terms $$17 \log_8 x = 28$$

$\Rightarrow$ dividing both sides of the last eq. by 17. $$(**) \quad \boxed{\log_8 x = \frac{28}{17}}$$ - Answer for part (i)

(ii) Determine the value of x (exact algebraic expression).

$(**) \Leftrightarrow$ (def. of log) $$\boxed{x = 8^{\frac{28}{17}}}$$ Answer for part (ii)

Q4. Evaluate following logarithms, giving exact answer:

(i) $\log_6 6^1 = 1$

(ii) $\log_{10}(0.00001) = -5$

(iii) $\ln(\sqrt{e^{13}}) = \ln e^{\frac{13}{2}} = \frac{13}{2}$

(iv) $\ln e^3 = 3\underbrace{\ln e}_{=1} = 3$

(v) $\log_2\left(\frac{1}{4}\right) = \log_2 4^{-1} = \log_2 (2^2)^{-1}$

$= \log_2 2^{-2} = -2\underbrace{\log_2 2}_{=1} = -2$

Q5 Given: $f(x) = \frac{1}{x^2}$, $h(x) = x+5$

Give the expression for the composite functions:

$f(h(x)) = f(x+5) = \frac{1}{(x+5)^2}$

$h(f(x)) = h\left(\frac{1}{x^2}\right) = \frac{1}{x^2} + 5$

$f(f(x)) = f\left(\frac{1}{x^2}\right) = \frac{1}{\left(\frac{1}{x^2}\right)^2} = \frac{1}{\frac{1}{x^4}} = x^4$

(Working for the last step:
$1 \div \frac{1}{x^4} = \frac{1}{1} \times \frac{x^4}{1} = x^4$)

Q6 Consider the function $y = x^2$

(1) Give a new function, obtained from y by translating it 4 units upwards

Answer: $\boxed{x^2 + 4}$ - Answ.

(2). Give the new function, obtained from your answer to part (1) by translating it by 8 units to the right:

$\boxed{(x-8)^2 + 4}$ - Answ.

(3) Give the new function obtained from answer to (2) by expanding it by factor 9 along the y-axis

$$9\left((x-8)^2+4\right) = \boxed{9(x-8)^2 + 36}$$ - Answ

Q7. Consider: $f(x) = \left(\dfrac{\ln(x-10)}{6}\right)^4$

$$g(x) = \ln\left(\left(\frac{x}{6} - 10\right)^4\right)$$

and: $p(x) = \ln(x-10)$, $q(x) = x^4$;

$r(x) = \dfrac{x}{6}$

(i) Write $f(x)$ as a composition of $p(x)$, $q(x)$, $r(x)$:

$$f(x) = q\left(\frac{\ln(x-10)}{6}\right) = q\left(r\left(\underbrace{\ln(x-10)}_{p(x)}\right)\right)$$

$= q(r(p(x)))$, i.e $\boxed{f(x) = q(r(p(x)))}$

Q7 (continue)

(ii) Write $g(x)$ as a composition of $p(x)$, $q(x)$ and $r(x)$.

Method 1

$$g(x) = \ln\Big(\underbrace{\left(\frac{x}{6}-10\right)^4+10}_{u} - 10\Big)$$

$$= \ln(u-10) = p(u-10)$$

$$= p\Big(\underbrace{\left(\frac{x}{6}-10\right)^4}_{q\left(\frac{x}{6}-10\right)}+10\Big)$$

$$= p\Big(q\Big(\underbrace{\frac{x}{6}}_{r(x)}-10\Big)+10\Big)$$

$$\Rightarrow g(x) = p(q(r(x)-10)+10)$$

Method 2

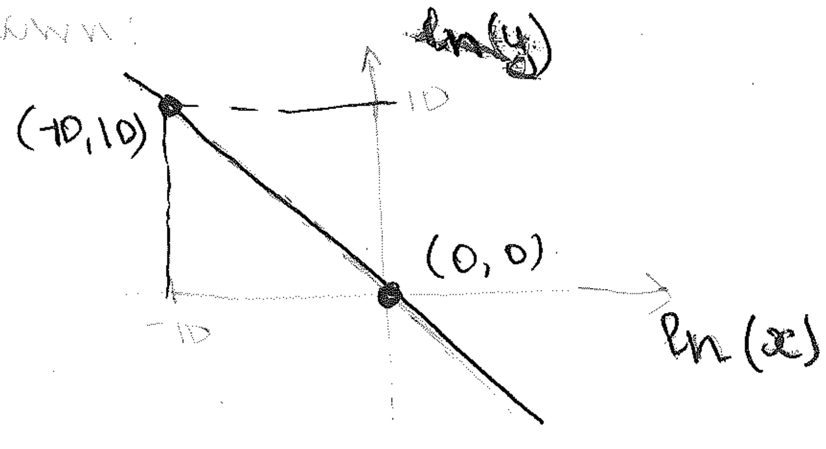
$$g(x) = 4\ln\left(\frac{x}{6}-10\right) =$$

$$= 4\,p\left(\frac{x}{6}\right) = 4\,p(r(x))$$

So $\boxed{g(x) = 4\,p(r(x))}$

Q8 A log-log plot of $y(x)$ is shown:

(i)



Reconstruct the function $y(x)$:

From the graph, grad of the given line is $= \frac{\text{rise}}{\text{run}} = \frac{-10}{10}$

$= -1$

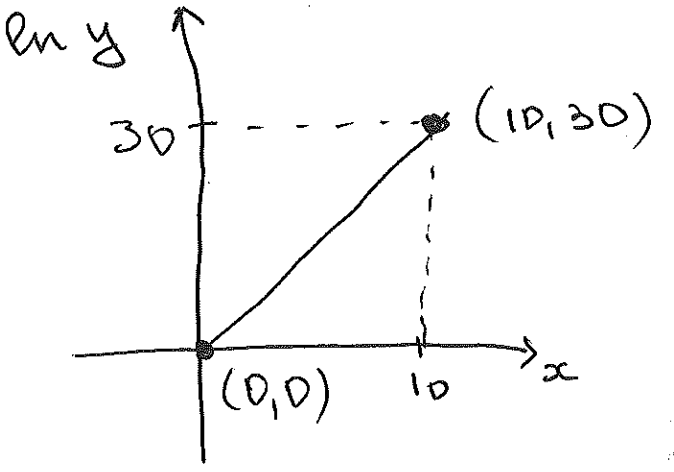
(y-int is $(0,0)$)

That means the eq. of the line is: $\ln y = -\ln x$

$\Rightarrow \ln y = \ln x^{-1}$

$\Rightarrow \boxed{y = x^{-1}}$ Answ.

(ii) $\ln(y(x))$ is given as:



The line is going through points $(0,0)$ and $(10,30)$

$\Rightarrow$ grad of this line $= \frac{30-0}{10-0} = \frac{30}{10} = 3$

y-intercept of the line is $(0, 0)$,

therefore the eq of the line is:

$\ln y = 3x \Rightarrow \boxed{y = e^{3x}}$ Answ

Q9 Given: $f(x) = \sqrt{2\ln(x)-3}$
for $x \in [e^{3/2}, \infty)$

Find $f^{-1}(x)$:

Due to the def. of inverse function,

$$f(f^{-1}(x)) = x$$

Let $f^{-1}(x) = y \Rightarrow$

$$f(f^{-1}(x)) = x \Leftrightarrow f(y) = x$$

$$\Leftrightarrow \left(\sqrt{2\ln(y)-3}\right) = x$$

$$\Leftrightarrow 2\ln y - 3 = x^2$$

(take both sides to the power of 2)

$$\Leftrightarrow 2\ln y = x^2 + 3$$

$$\Leftrightarrow \ln y = \frac{x^2+3}{2}$$

$$\Leftrightarrow y = e^{\frac{x^2+3}{2}}$$

$$\Rightarrow \boxed{f^{-1}(x) = e^{\frac{x^2+3}{2}}}$$ - inverse of f.

Thereby the range of $f^{-1}(x)$ is $[e^{3/2}, \infty)$ since range$(f^{-1}(x))$ is the same as dom$(f(x))$

(Dom(f) is given as $[e^{3/2}, \infty)$).

Range of f^{-1} is in the form $[A, +\infty) \Rightarrow A = e^{3/2}$

Q10 Given is: $\frac{1}{2}10^{15x} = \frac{3}{2} - 10^{-15x}$ (*)

Determine the largest possible value of 10^{15x} and the smallest possible value of 10^{15x}

Eq. (*) can be written as follows:

$$\frac{1}{2}10^{15x} = \frac{3}{2} - \frac{1}{10^{15x}}$$

$$\Rightarrow \quad (10^{15x})^2 = 3 \times 10^{15x} - 2$$

(Multiply both sides by 10^{15x} and by 2)

$$\Rightarrow \quad (10^{15x})^2 - 3 \times 10^{15x} + 2 = 0 \quad (**)$$

Let $u = 10^{15x}$ $(u > 0)$

$\Rightarrow$ (**) can be written as:

$$u^2 - 3u + 2 = 0$$

$$\Rightarrow (u-2)(u-1) = 0$$

$$\underset{>0}{u = 2} \quad \text{or} \quad \underset{>0}{u = 1}$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{largest} \\ \text{possible}}}{10^{15x} = 2} \quad \text{or} \quad \underset{\substack{\uparrow \\ \text{smallest} \\ \text{possible}}}{10^{15x} = 1}$$

Determine the largest value of x:

$$10^{15x} = 2$$

$$\Leftrightarrow \log_e 10^{15x} = \log_e 2 \Leftrightarrow 15x \log_e 10 = \log_e 2$$

$$\Leftrightarrow 15x = \frac{\log_e 2}{\log_e 10} \Leftrightarrow x = \frac{\ln 2}{15 \ln 10}$$

Q11 Simplify the given expression. Give the answer in the form $x^a y^b$.

$$\frac{(x^5y^5)^2}{(x^{-5}y^{-2})^5} = \frac{x^{10}y^{10}}{x^{-25}y^{-10}} \underset{\text{exponetial laws}}{=} x^{10-(-25)} \times y^{10-(-10)}$$

$$= x^{10+25} \times y^{10+10} = x^{35}y^{20}$$