MAT4MDS — Practice 9

INTEGRATION I

Question 1. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x - 4$. Draw the graph of f and hence calculate each of the following integrals exactly using areas. **Do not use anti-differentiation.**

(a) $\int_2^5 f(x)\ dx$,

(b) $\int_3^5 f(x)\ dx$,

(c) $\int_2^x f(t)\,dt$, where $x \in [2,5]$.

Question 2.

Let f be a function for which: $\int_{-1}^{2} f(x)\ dx = 6$, $\int_0^1 f(x)\ dx = 8$ and $\int_{-1}^{1} f(x)\ dx = 2$.

Use these three values and the additive property of integrals to find

(a) $\int_1^2 f(x)\ dx$

(b) $\int_{-1}^{0} f(x)\ dx$

(c) $\int_0^2 f(x)\ dx$.

Question 3.

(a) Sketch the graph of $\frac{1}{x}$ for $x > 0$.

(b) On your sketch, shade the area given by $\int_1^a \frac{1}{x} dx$, where $a > 1$.

(c) If the shaded area has value 1, what is a?

An antiderivative of a function f is a function F whose derivative is f, i.e.

$$F'(x) = f(x).$$

The antiderivative is often denoted $\int f(x)\ dx$ and is also called the indefinite integral.

Question 4. By **differentiating** the right hand side, verify the following antiderivatives (where c is a constant).

(a) $\int x\ dx = \frac{1}{2}x^2 + c.$

(b) $\int e^{2x}\ dx = \frac{1}{2}e^{2x} + c$

(c) $\int (3x+1)^{1/2}\ dx = \frac{2}{9}(3x+1)^{3/2} + c$

Table of Common Antiderivatives.

$f(x)$	Anti-derivative $F(x)$	Comments
x^k	$\frac{1}{k+1}x^{k+1}$	$k \neq -1, x > 0.$
e^{ax}	$\frac{1}{a}e^{ax}$	
$\frac{1}{x}$	$\log_e(x)$	$x > 0$
$\log_e(x)$	$x\log_e(x) - x$	$x > 0$

SUM AND DIFFERENCE, AND CONSTANT MULTIPLE PROPERTIES

$$\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx \quad and \quad \int \alpha f(x)\, dx = \alpha \int f(x)\, dx \text{ for all } \alpha \in \mathbb{R}.$$

Question 5. Calculate the following indefinite integrals.

(a) $\int (t^2 + 3t + t^{-1})\ dt$

(b) $\int (5\log_e(x) + 2e^x)\ dx$

(c) $\int 3p^{0.2}\ dp$

(d) $\int dx = \int 1\ dx$

THE FUNDAMENTAL THEOREM OF CALCULUS

Suppose that f is continuous on an interval I and let $a, b \in I$.

PART 1. The function F defined by $(x) = \int_a^x f(t)\ dt$ for each $x \in I$ is an antiderivative of f on I. That is, $F'(x) = f(x)$ for all $x \in I$.

PART 2. If F is **any** antiderivative of f on I then $\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a)$.

Question 6. Calculate the following definite integrals.

(a) $\int_{-1}^{1} x^2(x+2)\ dx$

(b) $\int_1^3 (x + x^{-1})\ dx$

(c) $\int_1^e (2x - \log_e(x))\ dx$

For all constants $a, b \in \mathbb{R}$ with $a \neq 0$: If $\int f(x)\ dx = F(x)$ then $\int f(ax+b)\ dx = \frac{1}{a}F(ax+b)$

Question 7.

(a) Calculate the following antiderivatives (where c and d are constants with $c \neq 0$).

(i) $\int (4t+1)^7 \ dt$

(ii) $\int \frac{1}{(cx+d)^3} \ dx$

(iii) $\int_0^x e^{-3x+2} \ dx$

(b) Calculate the following integrals.

(i) $\int_1^2 (1+3x)^{-1} \ dx$

(ii) $\int_{-2}^2 \sqrt{2x+5} \ dx$

(iii) $\int_0^x (x^2 + e^{-x}) \ dx$

We cannot calculate an integral directly when one of the terminals is infinite. We need to first calculate a finite integral and then take a limit. (If the limit does not exist, then the integral is not defined.)

Example: Calculate $\int_2^\infty x^{-2} \ dx$.

$$\int_2^b x^{-2} \ dx = [-x^{-1}]_2^b = (-b^{-1}) - (-2^{-1}) = 2^{-1} - b^{-1} = \frac{1}{2} - \frac{1}{b}.$$

Now $\int_2^\infty x^{-2} \ dx = \lim_{b\to\infty} \int_2^b x^{-2} \ dx = \lim_{b\to\infty} \left[\frac{1}{2} - \frac{1}{b}\right] = \frac{1}{2}.$

Question 8. Calculate the following integrals. Show clearly how limits are used in your calculations.

(a) $\int_1^\infty x^{-5} \ dx$

(b) $\int_0^\infty e^{-x} \ dx$

(c) $\int_{-\infty}^{-1} x^{-4} \ dx$

The cumulative distribution function F is an anti-derivative of the probability density function f for continuous data. That is:

$$P(X \leqslant x) = F(x) = \int_{-\infty}^{x} f(t)dt$$

The **mean** value is given by

$$\int_{-\infty}^{\infty} xf(x)dx$$

Question 9. Let us consider the cdf of the Pareto distribution (we met this distribution before):

$$F: [a, \infty) \to \mathbb{R} \qquad F(x) = 1 - \left(\frac{a}{x}\right)^b$$

(a) What is the probability density function of the Pareto distribution?

(b) Calculate the mean value of the Pareto distribution.

Question 10. The function

$$F: [0,1] \to \mathbb{R} \;\; F(x) = 1 - (1 - x^a)^b$$

is also a cdf, of a distribution called the Kumaraswamy distribution. (Here a, b are non-negative.) Find the associated probability density function.