Navier-Stokes equations in common coordinate systems

Cartesian Coordinates

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + b_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2} + b_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2} + b_z$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Polar Coordinates (suppressing body force terms)

For *cylindrical polar coordinates* σ, φ, z, with z measured along the axis of the cylinder, σ the distance from the axis of the cylinder, and φ the azimuthal angle:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \sigma} + \frac{v}{\sigma}\frac{\partial u}{\partial \varphi} + w\frac{\partial u}{\partial z} - \frac{v^2}{\sigma} = -\frac{1}{\rho}\frac{\partial p}{\partial \sigma} + \nu(\nabla^2 u - \frac{u}{\sigma^2} - \frac{2}{\sigma^2}\frac{\partial v}{\partial \varphi})$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial \sigma} + \frac{v}{\sigma}\frac{\partial v}{\partial \varphi} + w\frac{\partial v}{\partial z} + \frac{uv}{\sigma} = -\frac{1}{\rho\sigma}\frac{\partial p}{\partial \varphi} + \nu(\nabla^2 v - \frac{v}{\sigma^2} + \frac{2}{\sigma^2}\frac{\partial u}{\partial \varphi})$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial \sigma} + \frac{v}{\sigma}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w$$

$$\frac{1}{\sigma}\frac{\partial}{\partial \sigma}(\sigma u) + \frac{1}{\sigma}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0$$

where

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma\frac{\partial f}{\partial \sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

For *spherical polar coordinates* r, θ, φ, with r the distance from the origin, θ the colatitudinal angle and φ the azimuthal angle:

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}+\frac{w}{r\sin\theta}\frac{\partial u}{\partial \varphi}-\frac{v^2}{r}-\frac{w^2}{r}= \\
&\quad -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu(\nabla^2 u-\frac{2u}{\sigma^2}-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}(v\sin\theta)-\frac{2}{r^2\sin\theta}\frac{\partial w}{\partial \varphi})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{w}{r\sin\theta}\frac{\partial v}{\partial \varphi}+\frac{uv}{r}-\frac{w^2\cot\theta}{r}= \\
&\quad -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\nu(\nabla^2 v-\frac{v}{\sigma^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}-\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial w}{\partial \varphi})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+\frac{w}{r\sin\theta}\frac{\partial w}{\partial \varphi}+\frac{uw}{r}+\frac{vw\cot\theta}{r}= \\
&\quad -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \varphi}+\nu(\nabla^2 w+\frac{2}{r^2\sin\theta}\frac{\partial u}{\partial \varphi}+\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial v}{\partial \varphi}-\frac{w}{r^2\sin^2\theta})
\end{aligned}
$$

$$
\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v)+\frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi}=0
$$

where

$$
\nabla^2 f=\nabla\cdot\nabla f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}
$$