MAT4MDS — Practice 1 Worked Solutions

Model Answers to Practice 1

Question 1.

$$\left(\frac{(75x^3y^{-2}z)^2}{3z^9y^4x^2}\right)\left(\frac{5xyz}{15x^2z^{-1}}\right)^{-1} = \frac{3^2 5^4 x^6 y^{-4} z^2}{3z^9y^4x^2}\ \frac{3x^2z^{-1}}{xyz} = 3^2 5^4 x^5 y^{-9} z^{-9}$$

Question 2. (Note: part (a) is a revision of quadratics, and does not use index laws!)

(a) $x^2 + 4x = 21 \ \Rightarrow\ x^2 + 4x - 21 = (x-3)(x+7) = 0$, so that $x = 3$ or $x = -7$

(b) This is a quadratic equation for e^x.

$$\begin{aligned} & e^{2x} - 4e^x + 3 = 0 \\ \Rightarrow\ & (e^x - 3)(e^x - 1) = 0 \\ \Rightarrow\ & e^x = 3 \ \text{ or } \ e^x = 1 \\ \Rightarrow\ & x = 0 \ \text{ or } \ x = \log_e(3) \end{aligned}$$

(c) Multiplying by 6^x converts this to a quadratic equation:

$$\begin{aligned} & 6^x - 1 = 6^{1-x} \\ \Rightarrow\ & 6^{2x} - 6^x - 6 = 0 \\ \Rightarrow\ & (6^x - 3)(6^x + 2) = 0 \end{aligned}$$

Since $6^x > 0$, the second factor can never be zero and there is only one solution $x = \log_6(3)$.

(d) Noting that $4^x = 2^{2x}$ and $2^{x+3} = 2^3 \cdot 2^x$, this is a quadratic equation

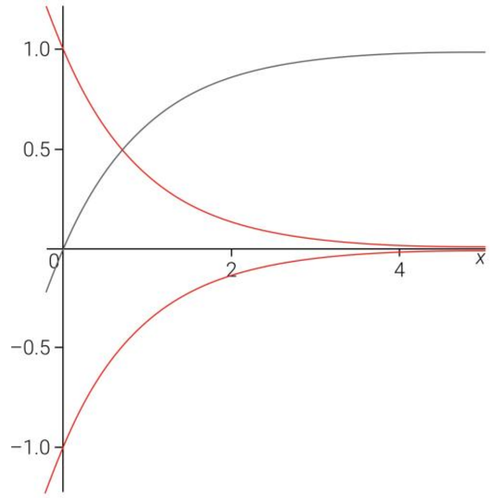
$$\begin{aligned} & 4^x - 2^{x+3} + 12 = 0 \\ \Rightarrow\ & 2^{2x} - 8 \cdot 2^x + 12 = 0 \\ \Rightarrow\ & (2^x - 2)(2^x - 6) = 0 \\ \Rightarrow\ & 2^x = 2 \ \text{ or } \ 2^x = 6 \\ \Rightarrow\ & x = 1 \ \text{ or } \ x = \log_2(6) \end{aligned}$$

(e) This is a quadratic that can't be factorised by inspection:

$$\begin{aligned} & 9^x = 3^x + 1 \\ \Rightarrow\ & 3^{2x} - 3^x - 1 = 0 \\ \Rightarrow\ & 3^x = \frac{1 \pm \sqrt{1+4}}{2} \end{aligned}$$

Since $3^x > 0$ again there is only one solution, and $x = \log_3\left(\frac{1+\sqrt{5}}{2}\right)$.

Question 3. The bold line is the final graph of $y = 1 - e^{-x}$. The other two graphs shown are e^{-x} and $-e^{-x}$. This graph arises both in logistic growth, and as the cumulative distribution function of the exponential distribution.



Question 4. Using an arbitrary base, $\log_d(a) = \log_d(b^m) = m\log_d(b)$. Thus, as required

$$\log_b(a) = m = \frac{\log_d(a)}{\log_d(b)}.$$

If we know logarithms in one base, d, we can obtain logarithms in any other base b using the number $\log_d(b)$ i.e. the log of the new base in the old base. Calculators usually have buttons both for $\log_{10}$ and $\ln$ or $\log_e$. This is not really necessary because of this change of base rule.

Question 5.

(a) We use the change of base rule from Question 4. $3 = \sqrt{9}$, so that

$$2\log_3(x) + \log_9(x) = 10$$

$$\Rightarrow \quad 2\frac{\log_9(x)}{\log_9(3)} + \log_9(x) = 10$$

$$\Rightarrow \quad (4+1)\log_9(x) = 10$$

$$\Rightarrow \quad \log_9(x) = 2$$

$$\Rightarrow \quad x = 9^2 = 81$$

(b) Note that we must have $x > 0$ and $2x - 1 > 0$ for the expression to be defined. Then

$$\log_e(x) = 2\log_e(2x-1) = \log_e(2x-1)^2$$

$$\Rightarrow \quad x = 4x^2 - 4x + 1$$

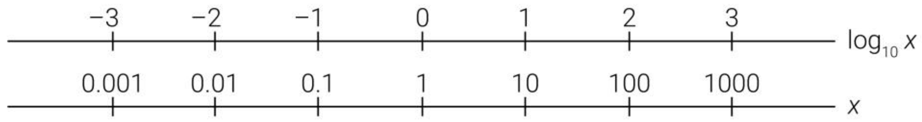
$$\Rightarrow \quad 4x^2 - 5x + 1 = 0$$

$$\Rightarrow \quad (4x-1)(x-1) = 0$$

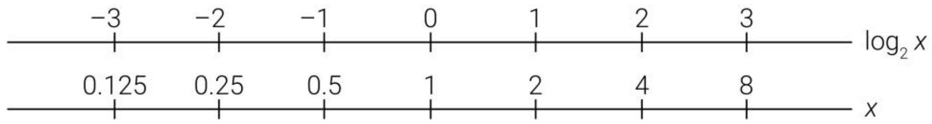
$$\Rightarrow \quad x = 1 \text{ or } x = \frac{1}{4}.$$

However, since we must have $x > \frac{1}{2}$, the only solution is $x = 1$.

Question 6.



Question 7.



Question 8.

(a) The graph is a log-linear graph, although the vertical axis has been marked with the powers of 2 evenly spaced, rather than with logarithms base 2.

(b) Because of the logarithmic nature of the vertical axis marking, the graph indicates that around 2^{16} people were infected, and 2^8 people died. That is only $\frac{1}{256}$ or 0.4% of the total, not half, died.

(c) The dark green line (USA) is about one vertical scale mark below the blue line (total) for all of the graph. Because the scale is in powers of 2, this means that half of all infections occurred in the USA.

Question 9.

(a) Figure 1 uses a linear scale on both axes. The small values on the right hand end of the graph are hard to distinguish, and around 6000 data points near zero on the horizontal axis have been left out, due to scale problems.

(b) Figure 2 is a log-log graph. The negative slope indicates an inverse relationship between frequency and size of the event being measured. That is, light rainfall (in a 5 minute period) occurred more often than heavy rainfall (in a 5 minute period).

(c) Figure 3 is a bar chart, that uses a logarithmic scale on the horizontal axis. This is called a linear-log chart. Because the shape of the bars is approximately the shape of a bell-curve, the author proposes that the data follows a log-normal distribution.

(d) The author says that the plot in Figure 5 shows frequency against the logarithm of rainfall amount. In fact, it is a log-log graph, not linear-log.