# MAT4MDS — Practice 10 Worked Solutions

## Model Answers to Practice 10

**Question 1.**

(a) Let $u = g(x) = 1 + x^4$ so that $\frac{du}{dx} = 4x^3$ giving $x^3 = \frac{1}{4}\frac{du}{dx}$. Then

$$\int_0^1 (1+x^4)^5 x^3 \; dx = \int_0^1 u^5 \frac{1}{4}\frac{du}{dx} \; dx = \frac{1}{4}\int_0^1 u^5 \frac{du}{dx} \; dx = \frac{1}{4}\int_{g(0)}^{g(1)} u^5 \; du \qquad \text{[substitution rule]}$$

$$= \frac{1}{4}\int_1^2 u^5 \; du = \left[\frac{u^6}{24}\right]_1^2 = \frac{21}{8}.$$

(b) Let $u = g(x) = 1 + x^2$ so that $\frac{du}{dx} = 2x$ giving $x = \frac{1}{2}\frac{du}{dx}$. Then,

$$\int_1^2 \frac{3x}{1+x^2} \; dx = \int_1^2 \frac{3}{u}\frac{1}{2}\frac{du}{dx} \; dx = \frac{3}{2}\int_1^2 \frac{1}{u}\frac{du}{dx} \; dx = \frac{3}{2}\int_{g(1)}^{g(2)} \frac{1}{u} \; du \qquad \text{[substitution rule]}$$

$$= \frac{3}{2}\int_2^5 \frac{1}{u} \; du = \frac{3}{2}[\ln(|u|)]_2^5 = \frac{3}{2}(\ln(5) - \ln(2)).$$

(c) Let $u = g(x) = 1 + e^x$ giving $\frac{du}{dx} = e^x$, so that

$$\int_0^1 \frac{e^x}{1+e^x} \; dx = \int_0^1 \frac{1}{u}\frac{du}{dx} \; dx = \int_{g(0)}^{g(1)} \frac{1}{u} \; du \qquad \text{by substitution}$$

$$= \int_2^{1+e} \frac{1}{u} \; du = [\ln(|u|)]_2^{1+e} = \ln(1+e) - \ln(2).$$

(d) First consider $\int_0^b x^2 e^{-x^3} \; dx$ for an arbitrary real number $b$. Now put $u = x^3$, so that at $x = 0, u = 0$, and at $x = b, u = b^3$ and $\frac{du}{dx} = 3x^2$. Then

$$\int_0^b x^2 e^{-x^3} \; dx = \int_0^b \frac{1}{3}\frac{du}{dx} e^{-u} dx = \int_0^{b^3} \frac{1}{3} e^{-u} \; du \qquad \text{by substitution}$$

$$= \frac{1}{3}[-e^{-u}]_0^{b^3}$$

$$= \frac{1}{3}\left(-e^{-b^3} + e^0\right).$$

Thus

$$\int_0^\infty x^2 e^{-x^3} \; dx := \lim_{b\to\infty} \int_0^b x^2 e^{-x^3} \; dx$$

$$= \lim_{b\to\infty} \frac{1}{3}\left(-e^{-b^3} + e^0\right) = \frac{-1}{3}\lim_{b\to\infty} e^{-b^3} + \frac{1}{3}\lim_{b\to\infty} e^0 = \frac{1}{3},$$

using the sum and constant multiple rules for limits and the known limit $\lim_{b\to\infty} e^{-b^3} = 0$.

**Question 2.** Let $x = g(t) = t + 1$ giving $\frac{dx}{dt} = 1$, so that

$$\int_1^5 x(x-1)^{\frac{1}{2}}\, dx = \int_{g(0)}^{g(4)} x(x-1)^{\frac{1}{2}}\, dx = \int_0^4 (t+1)(t)^{\frac{1}{2}} \frac{dx}{dt}\, dt \quad \text{by substitution .}$$

$$= \int_0^4 (t+1)(t)^{\frac{1}{2}}\, dt \quad as \quad \frac{dx}{dt} = 1$$

$$= \int_0^4 t^{\frac{3}{2}} + t^{\frac{1}{2}}\, dt$$

$$= \left[\frac{2}{5}t^{\frac{5}{2}} + \frac{2}{3}t^{\frac{3}{2}}\right]_0^4 = \frac{64}{5} + \frac{16}{3} = \frac{272}{15}.$$

**Question 3.**

(a) Let $u = \ln(x)$ and $\frac{dv}{dx} = x^3$. Then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{1}{4}x^4$ is an antiderivative of $x^3$.

$$\text{So,} \quad \int_2^4 x^3 \ln(x)\, dx = \int_2^4 \frac{dv}{dx} u\, dx = \int_2^4 u\frac{dv}{dx}\, dx = uv|_2^4 - \int_2^4 v\frac{du}{dx}\, dx \qquad \text{(by parts rule)}$$

$$= \left[\ln(x)\frac{1}{4}x^4\right]_2^4 - \int_2^4 \frac{1}{4}x^4 \cdot \frac{1}{x}\, dx$$

$$= \ln(4)\cdot 64 - \ln(2)\cdot 4 - \int_2^4 \frac{1}{4}x^3\, dx$$

$$= 64\ln(2^2) - 4\ln(2) - \left[\frac{1}{16}x^4\right]_2^4$$

$$= (128-4)\ln(2) - (16-1) = 124\ln(2) - 15.$$

(b) Let $u = x$ and $\frac{dv}{dx} = e^{-x}$. Then $\frac{du}{dx} = 1$ and $v = -e^{-x}$ is an antiderivative of $e^{-x}$.

$$\text{So,} \quad \int_0^1 xe^{-x}\, dx = \int_0^1 u\frac{dv}{dx}\, dx = uv|_0^1 - \int_0^1 v\frac{du}{dx}\, dx \quad \text{(by parts rule)}$$

$$= -xe^{-x}|_0^1 - \int_0^1 (-e^{-x})\cdot 1\, dx$$

$$= -e^{-1} - e^{-x}|_0^1 = -e^{-1} - e^{-1} + 1 = 1 - 2e^{-1}.$$

(c) Let $u = x^2$ and $\frac{dv}{dx} = e^{-x}$. Then $\frac{du}{dx} = 2x$ and $v = -e^{-x}$ is an antiderivative of $e^{-x}$.

$$\text{So,} \quad \int_0^1 x^2e^{-x}\, dx = \int_0^1 u\frac{dv}{dx}\, dx = uv|_0^1 - \int_0^1 v\frac{du}{dx}\, dx \quad \text{(by parts rule)}$$

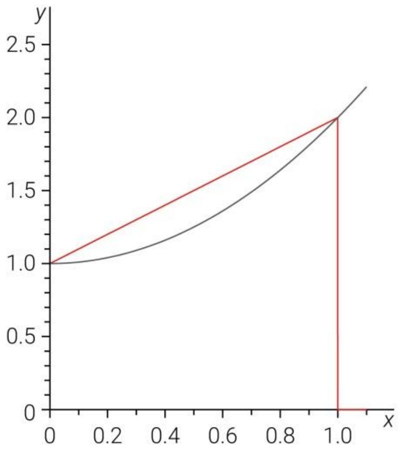
$$= -x^2e^{-x}|_0^1 - \int_0^1 (-e^{-x})2x\, dx$$

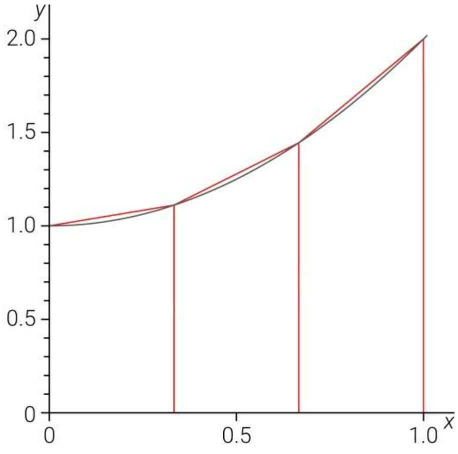
$$= -e^{-1} + 2\int_0^1 xe^{-x}\, dx$$

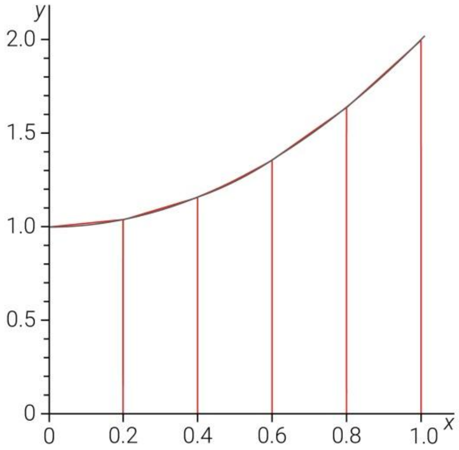
$$= -e^{-1} + 2(1 - 2e^{-1}) = 2 - 5e^{-1} \quad \text{by (c).}$$

**Question 4.** $f(x) = 1 + x^2$

(a) The figures are:




(b) (i) With one trapezium:

$$\text{Area} = \frac{1}{2}(f(0) + f(1)) = \frac{1}{2}(1+2) = \frac{3}{2} = 1.5$$

(ii) With three trapezia:

$$\text{Area} = \frac{1}{3}\cdot\frac{1}{2}\left(f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + f(1)\right)$$

$$= \frac{1}{6}\left(1 + \frac{20}{9} + \frac{26}{9} + 2\right) = \frac{73}{54} \approx 1.3519$$

(iii) With five trapezia:

$$\text{Area} = \frac{1}{5}\cdot\frac{1}{2}\left(f(0) + 2f\left(\frac{1}{5}\right) + 2f\left(\frac{2}{5}\right) + 2f\left(\frac{3}{5}\right) + 2f\left(\frac{4}{5}\right) + f(1)\right)$$

$$= \frac{1}{10}\left(1 + \frac{52}{25} + \frac{58}{25} + \frac{68}{25} + \frac{82}{25} + 2\right) = \frac{335}{250} = \frac{67}{50} = 1.34$$

**Question 5.**

(a) $\int_0^1 x^2+1\,dx \approx \text{area } A_1 \; + \text{area } A_2 + \text{area } A_3 + \cdots + \text{area } A_{n-1} + \text{area } A_n$

$$= \frac{1}{n}\cdot((\frac{1}{n})^2+1) + \frac{1}{n}\cdot((\frac{2}{n})^2+1) + \cdots + \frac{1}{n}\cdot((\frac{n-2}{n})^2+1) + \frac{1}{n}\cdot((\frac{n-1}{n})^2+1) + \frac{1}{n}\cdot((\frac{n}{n})^2+1)$$

(b) $= \frac{1}{n}\left[((\frac{1}{n})^2+1) + ((\frac{2}{n})^2+1) + \cdots + ((\frac{n-2}{n})^2+1) + ((\frac{n-1}{n})^2+1) + ((\frac{n}{n})^2+1)\right]$

$$= \frac{1}{n}\left[\underbrace{(1+\cdots+1)}_{n\text{terms}} + (\frac{1}{n})^2 + (\frac{2}{n})^2 + \cdots + (\frac{n-2}{n})^2 + (\frac{n-1}{n})^2 + (\frac{n}{n})^2\right]$$

$$= \frac{1}{n}\left[n + \frac{1}{n^2}(1^2+2^2+\cdots+(n-2)^2+(n-1)^2+n^2)\right]$$

(c) $= 1 + \frac{1}{n^3}(1^2+2^2+\cdots+(n-2)^2+(n-1)^2+n^2)$

$$= 1 + \frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)$$

$$= 1 + \frac{1}{6}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} = 1 + \frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n})$$

(d) $n = 10$ gives $\int_0^1 x^2+1\,dx \approx 1 + \frac{1}{6}(1+\frac{1}{10})(2+\frac{1}{10}) \approx 1.3852.$

$n = 100$ gives $\int_0^1 x^2+1\,dx \approx 1 + \frac{1}{6}(1+\frac{1}{100})(2+\frac{1}{100}) = 1.3384.$

(e) $\int_0^1 x^2+1\,dx := \lim_{n\to\infty}\left(1+\frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n})\right)$

$$= \lim_{n\to\infty} 1 + \frac{1}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\lim_{n\to\infty}\left(2+\frac{1}{n}\right)$$

$$= 1 + \frac{1}{6}\cdot(1=0)\cdot(2+0) = \frac{4}{3}\text{basic limits.}$$

(f) $\int_0^1 x^2+1\,dx = \left[\frac{1}{3}x^3+x\right]_0^1 = \frac{4}{3}$. The rules give the same answer as (e).

**Question 6.**

(a) Using substitution with $u = x^2$, so that $\frac{du}{dx} = 2x$,

$$\int_{-b}^{b} xe^{-x^2}dx = \frac{1}{2}\int_{b^2}^{b^2} e^{-u}du = 0$$

Then, taking the limit as $b \to \infty$, the mean value is found to be $0$.

(b) Using the same substitution for $f(x) = \frac{1}{1+x^2}$,

$$\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{1}{1+u}du = \ln(1+u) = \ln(1+x^2).$$

As this does not have a limit as $x$ gets large, the Cauchy distribution does not have a well-defined mean.

(c) Using Question 3(b)

$$\int_0^b \lambda xe^{-\lambda x}dx = -\frac{1}{\lambda}(\lambda x+1)e^{-\lambda x}\Big|_0^b = -\frac{1}{\lambda}\left((\lambda b+1)e^{-\lambda b}-1\right)$$

Taking $b \to \infty$, the mean value is $\frac{1}{\lambda}$.