Linear combinations, spans, kernels and images

Recall that the span of a set of vectors is the set of all linear combinations of those vectors; i.e.,

$$\mathrm{Sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n : a_1, a_2, \ldots, a_n \in \mathbb{R}\}.$$

The set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent (or just independent) if the only solution to

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = \mathbf{0}$$

is $a_1 = a_2 = \ldots = a_n = 0$.
The kernel of a matrix $\mathbf{A}$, denoted $\mathrm{Ker}(\mathbf{A})$, is the solution set to the equation $\mathbf{Ax} = \mathbf{0}$, and can be obtained by applying Gaussian elimination to $\mathbf{A}$ and reading off the associated solution. The image of a matrix $\mathbf{A}$, denote by $\mathrm{Im}(\mathbf{A})$, is the span of its columns, and a basis can be obtained by row reducing $\mathbf{A}^T$ and reading the rows as vectors.

1. Consider the vectors
$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \\ 7 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 7 \\ -5 \\ 7 \end{pmatrix}.$$
   (a) Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is independent.
   (b) Give a geometric description of $\mathrm{Sp}(\mathbf{v}_1)$ and write down two vectors in $\mathrm{Sp}(\mathbf{v}_1)$ that are not $\mathbf{0}$ and not $\mathbf{v}_1$.
   (c) Give a geometric description of $\mathrm{Sp}(\mathbf{v}_2)$ and write down two vectors in $\mathrm{Sp}(\mathbf{v}_2)$ that are not $\mathbf{0}$ and not $\mathbf{v}_2$.
   (d) Give a geometric description of $\mathrm{Sp}(\mathbf{v_1}, \mathbf{v_2})$ and write down two vectors in $\mathrm{Sp}(\mathbf{v}_1, \mathbf{v}_2)$ that are not $\mathbf{0}$ and not $\mathbf{v}_1$ or $\mathbf{v}_2$.

2. Consider the following matrix:
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{pmatrix}.$$
   (a) Use MATLAB's `rref` function to transform $\mathbf{A}$ into reduced row echelon form.
   (b) Based on your answer to (a), are the vectors
$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \mathbf{a}_4 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$
   independent?
   (c) Find a basis for $\mathrm{Ker}(\mathbf{A})$ and $\mathrm{Im}(\mathbf{A})$.

3. Find a basis for $\mathrm{Ker}(\mathbf{B})$ and $\mathrm{Im}(\mathbf{B})$, where
$$\mathbf{B} = \begin{pmatrix} 4 & 3 & 1 \\ -1 & -4 & 3 \\ 1 & -2 & 3 \end{pmatrix}.$$
   Use MATLAB to perform the row reduction. Type `format rational` beforehand to express the output as rational numbers.

4. Find a basis for $\mathrm{Ker}(\mathbf{C})$ and $\mathrm{Im}(\mathbf{C})$, where
$$\mathbf{C} = \begin{pmatrix} 3 & 1 & 5 \\ 4 & -4 & -4 \\ -4 & -2 & -8 \\ 5 & 1 & 7 \end{pmatrix}$$
   Use MATLAB to perform the row reduction.

Tangent and normal spaces

Given a function $\mathbf{h}\colon \mathbb{R}^n \to \mathbb{R}^m$, and an assocated level set

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{h}(\mathbf{x}) = \mathbf{c}\},$$

the tangent space and normal space to $\mathcal{H}$ at a point $\mathbf{p} \in \mathcal{H}$ are given respectiely by

$$T\mathcal{H}(\mathbf{p}) = \operatorname{Ker}(D\mathbf{h}(\mathbf{p})),$$
$$N\mathcal{H}(\mathbf{p}) = \operatorname{Im}(D\mathbf{h}^T(\mathbf{p})),$$

where $D\mathbf{h}(\mathbf{p})$ is the Jacobian, given by

$$D\mathbf{h}(\mathbf{x}) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{pmatrix} (\mathbf{x}),$$

and $D\mathbf{h}^T(\mathbf{p})$ is its transpose. A point $\mathbf{x} \in \mathcal{H}$ is regular if the set

$$\{\nabla h_1(\mathbf{p}), \nabla h_2(\mathbf{p}), \ldots, \nabla h_n(\mathbf{p})\}$$

is linearly independent.

5. Let $\mathbf{h}\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$\mathbf{h}(\mathbf{x}) = \begin{pmatrix} x_1^2 + x_2^2 + x_3^2 \\ x_1^2 + 2x_2x_3 \end{pmatrix},$$

and consider the level set $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{h}(\mathbf{x}) = \begin{pmatrix} 2 & 1 \end{pmatrix}^T\}$. Let $\mathbf{p} = \begin{pmatrix} 0 & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \end{pmatrix}^T$.

   (a) Determine $D\mathbf{h}(\mathbf{x})$.

   (b) Show that $\mathbf{p} \in \mathcal{H}$.

   (c) Determine the normal space of $\mathcal{H}$ at $\mathbf{p}$, and argue that $\mathbf{p}$ is regular.

   (d) Determine the tangent space of $\mathcal{H}$ at $\mathbf{p}$.

6. Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^2 + x_2 + x_3 \\ x_1x_2x_3 \end{pmatrix},$$

and let

$$\mathbf{p}_1 = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}^T, \quad \mathbf{p}_2 = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}^T.$$

Then, for $i = 1, 2$, let

$$\mathcal{F}_i = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{p}_i)\}.$$

   (a) Determine $D\mathbf{f}(\mathbf{x})$.

   (b) Which of the points $\mathbf{p}_1$ and $\mathbf{p}_2$ are regular?

   (c) Find a basis for the tangent space $T\mathcal{F}_1(\mathbf{p}_1)$.

   (d) Find a basis for the normal space $N\mathcal{F}_2(\mathbf{p}_2)$.

   (e) Show that the matrix

$$\mathbf{Q} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

   is negative definite on the Tangent space $T\mathcal{F}_1(\mathbf{p}_1)$ you calculated in (c).