COPY

The University of Melbourne
Semester 1 Assessment 2010

Department of Mathematics and Statistics
620-156 Linear Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.

This paper has: 7 pages.

Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
No materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 11 questions. The total number of marks is 100. All questions may be attempted. All answers should be appropriately justified.

This paper may be held by the Baillieu Library.

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— BEGINNING OF EXAMINATION QUESTIONS —

1. Let

$$A = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

Evaluate, if possible:

(a) AC (b) $B^T B$ (c) BC (d) CB

[7 marks]

2. (a) Consider the following linear system:

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & 2x_4 & = & 10 \\ 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 5 \end{array}$$

(i) Write down the augmented matrix corresponding to the linear system.

(ii) Reduce the matrix to reduced row-echelon form.

(iii) Use the reduced row-echelon form to give the solutions of the system. Your answer should include one solution to the equations and a basis for the solution space of the associated system of homogeneous equations.

(b) Calculate the inverse of

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

[8 marks]

3. (a) Calculate the determinant of

$$M = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 8 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 3 & 4 \end{bmatrix}$$

(b) State whether M has an inverse; if there is an inverse, there is no need to calculate it.

(c) State the determinant of the following matrices.

(i) $N = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$ (ii) $\frac{1}{2}N$

[8 marks]

4. Let $\mathcal{L}$ be the line with equation

$$x - 2 = \frac{y+2}{3} = \frac{z+1}{2}.$$

(a) Find the vector equation of the line $\mathcal{L}$.

(b) Find the cartesian equation of the plane that is perpendicular to the line $\mathcal{L}$ and that contains the point $P(2, -2, 2)$.

(c) Find the cartesian equation of the plane that contains the line $\mathcal{L}$ and the point $Q(3, 1, 0)$.

[10 marks]

5. (a) Let

$$A = \begin{bmatrix} 2 & 7 & -3 & 2 & -5 & 2 \\ 1 & 2 & 0 & 2 & 0 & 1 \\ -1 & 1 & -3 & -4 & -5 & -1 \\ -2 & -5 & 1 & -3 & 2 & -2 \\ 2 & 3 & 1 & 3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this question you may assume that B is obtained from A by applying elementary row operations. Using this information, or otherwise, answer the following:

(i) Write down a basis for the row space of A. What is its dimension?

(ii) Does the set

$$\{(2,1,-1,-2,2), (7,2,1,-5,3), (-3,0,-3,1,1),$$
$$(2,2-4,-3,3), (-5,0,-5,2,0), (2,1,-1,-2,1)\}$$

span $\mathbb{R}^5$? Briefly explain your answer.

(iii) Are the vectors in the set

$$\{(2,1,-1,-2,2), (7,2,1,-5,3), (2,2-4,-3,3), (2,1,-1,-2,1)\}$$

linearly dependent? If so, express one vector as a linear combination of the other vectors.

(iv) Are the vectors in the set

$$\{(7,2,1,-5,3), (2,2-4,-3,3), (-5,0,-5,2,0), (2,1,-1,-2,1)\}$$

linearly dependent? If so, express one vector as a linear combination of the other vectors.

(v) Find a basis for the solution space of A.

Question continued on next page

(b) You are given that

- P is an $m \times n$ matrix,
- Q is an $n \times k$ matrix and
- $PQ = 0$ where 0 is the zero $m \times k$ matrix.

Why is $\text{Rank}(P) + \text{Rank}(Q) \leq n$?

[12 marks]

6. Let

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2} : b = d \right\}.$$

(a) Show that W is a subspace of the vector space $M_{2,2}$ the set of two by two matrices. Justify your answer by appealing to appropriate theorems.

(b) Find a spanning set for W; justify your answer.

(c) Use your answer to (b) to find a basis for W; justify your answer.

(d) What is the dimension of W?

[8 marks]

7. This question concerns linear transformations on $\mathbb{R}^2$.

(a) Find the matrix, with respect to the standard basis, of the rotation R through an angle of $3\pi/2$.

(b) Find the matrix, with respect to the standard basis, of the reflection S in the y-axis.

(c) Find the matrix, with respect to the standard basis, of the composition $T = S \circ R$ of R followed by S .

(d) What are the images, under T, of the vectors $(1, 1)$ and $(1, -1)$?

(e) Give a brief geometric description of T.

[8 marks]

8. (a) Find a 3×3 symmetric matrix A so that

$$X^T AY = x_1y_1 + 8x_2y_2 + 3x_3y_3 - 4x_2y_3 - 4y_3x_2$$

for all $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

(b) Show that $X^TAX > 0$ when $X \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

You may assume that defining

$$\langle (x_1, x_2, x_3,), (y_1, y_2, y_3) \rangle = X^T AY$$

yields an inner product on $\mathbb{R}^3$.

(c) Find a basis for the subspace $V = \{(x, y, z) : x = z\}$ of $\mathbb{R}^3$.

(d) Starting with the basis you found in part (c) of the question, use the Gram-Schmidt orthogonalisation process to find a basis of V which is orthonormal with respect to the inner product given above.

[10 marks]

9. Let $D : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ be the linear transformation given by

$$D(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$$

(a) Find the matrix $[T]_\mathcal{S}$ of T with respect to the standard basis $\mathcal{S} = \{1, x, x^2\}$ for $\mathcal{P}_2$.

(b) Show that the set $\mathcal{B} = \{1 + x, -1 + x, 1 + x + x^2\}$ is a basis of $\mathcal{P}_2$.

(c) Write down the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from the basis $\mathcal{S}$ to the basis $\mathcal{B}$.

(d) Give an expression for $P_{\mathcal{B},\mathcal{S}}$ in terms of $P_{\mathcal{S},\mathcal{B}}$ and use it to calculate the transition matrix $P_{\mathcal{B},\mathcal{S}}$.

(e) Calculate the matrix $[T]_\mathcal{B}$ of T with respect to the basis $\mathcal{B}$.

[11 marks]

10. Let

$$A = \begin{bmatrix} -2 & 2 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Show that $(2,3,0)$, $(1,1,0)$ and $(0,0,1)$ are eigenvectors of A.

(b) Hence write down the eigenvalues of A.

(c) Write down a diagonal matrix D and a change of basis matrix P such that

$$P^{-1}AP = D.$$

(d) Use your answer to (c) to find A^{100}.

[8 marks]

11. You are given that

$$4xy + 3y^2 = 4$$

(a) Identify the conic defined by the above equation.

(b) Find the directions of the principal axes of the conic.

(c) Sketch the conic on the x-y plane, marking the principal axes. Also show the distances between the origin and the points where the conic meets the principle axes.

[10 marks]

— END OF EXAMINATION QUESTIONS —