#1 (a) $\log l(k) = \log \prod_{i=1}^{n} k(1+x_i)^{-k-1}$

$$= n\log k - (k+1)\sum_{i=1}^{n}\log(1+x_i)$$

(b) $\frac{\partial}{\partial k}\log l(k) = \frac{n}{k} - \sum_{i=1}^{n}\log(1+x_i)$

$\frac{\partial^2}{\partial k^2}\log l(k) = -\frac{n}{k^2}$ $\quad$ $I(k) = -E\left(-\frac{n}{k^2}\right) = \frac{n}{k^2}$

(c) $\frac{\partial}{\partial k}\log l(k) = 0 \Rightarrow$ MLE $\hat{k} = \frac{n}{\sum_{i=1}^{n}\log(1+x_i)}$

$$\hat{k} \overset{d}{\approx} N\left(k, \frac{k^2}{n}\right)$$

#2. See Exam_sol_2017_2.pdf.

#3 (a) $D = -2\log L(\hat{\theta}) + 2\log L(\text{full})$

$AIC = -2\log L(\hat{\theta}) + 2\cdot p$

$-2\log L(\hat{\theta}) = AIC - 2p = 754.49 - 2\times(48-43)$

$= 744.49$

$\log L(\hat{\theta}) = -372.245$

$\log L(\text{full}) = \frac{1}{2}\left(D + 2\log L(\hat{\theta})\right) = \frac{1}{2}(513.75 - 744.49)$

$= -115.37$

(b) Model B: (1) AIC is smaller and (2) chisq test rejects hypothesis that model A is correct.

(c) $\exp(2.88947 - 0.52878 + 1.21614)$

$= 36.1194$

(d) Deviance is large and phi_hat >> 1.

(e) Poisson model assumes Var(Y_i) = Exp(Y_i), but quasi-Poisson model assumes Var(Y_i) = \phi Exp(Y_i) for some \phi.

(f) $0.05279 \times \sqrt{13.93209} = 0.1970424$

(g) ModelC : F test fails to reject the hypothesis that model C is correct.

(h) $F = \dfrac{(D_c - D_b)/(df_c - df_b)}{\hat{\phi}} = \dfrac{7.2729/1}{13.93209} = 0.522$

dfs 1 & 42

#4 See Exam_sol_2017_2.pdf.

#5 For $y \geqq 2$,

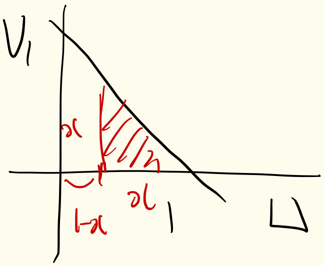
(a) $P(Y=y) = 1-\frac{1}{y}-\left(1-\frac{1}{y-1}\right) = \frac{1}{y-1}-\frac{1}{y} = \frac{\cancel{y}-\cancel{y}+1}{(y-1)y} = \frac{1}{(y-1)y}$

(b) $f(x) = F_X'(x)$

$$F_X(x) = P(X \leqq x) = P(1-U \leqq x \mid U+V<1)$$

$$= P(1-x \leqq U \mid U+V<1)$$

$$= \frac{P(1-x \leqq U,\ U+V<1)}{P(U+V<1)}$$



$$= \frac{\int_0^x \int_{1-x}^{1-v} 1\,du\,dv}{\int_0^1 \int_0^{1-v} 1\,du\,dv} = \frac{\frac{1}{2}x^2}{\frac{1}{2}} = x^2$$

So $f(x) = 2x$

#6 (a) $f(x|\theta) = \theta^{x}(1-\theta)^{1-x}$ $\quad p(\theta) = 1 \quad 0<\theta<1$

$$\theta | x \propto p(x|\theta)\,p(\theta)$$

$$= \theta^{x}(1-\theta)^{1-x} \cdot 1 \qquad 0<\theta<1$$

$$\text{Beta}(x+1, 2-x)$$

(b) $$E(\theta|x) = \frac{x+1}{x+1+2-x} = \frac{x+1}{3} = \hat{\theta}$$

(c) $$E_X\left(\left(\frac{x+1}{3}-\theta\right)^2\right) = E_X\left(\left(\frac{x+1-3\theta}{3}\right)^2\right)$$

$$= \frac{1}{9}\left[E_X(X^2) + 2(1-3\theta)\underset{\theta}{E_X(X)} + (1-3\theta)^2\right]$$

$$= \frac{1}{9}\left[\underbrace{\theta(1-\theta)+\cancel{\theta^2}}_{\cancel{\theta}-\cancel{\theta^2}} \cancel{+2\theta} - \cancel{6\theta^2} + 1 - \cancel{6\theta} + \cancel{9\theta^2}\right]$$

$$= \frac{1}{9}\left[3\theta^2 - 3\theta + 1\right] = \frac{1}{3}\theta^2 - \frac{1}{3}\theta + \frac{1}{9}$$

(d) $$\int_0^1 \left[\frac{1}{3}\theta^2 - \frac{1}{3}\theta + \frac{1}{9}\right] d\theta = \frac{1}{9} - \frac{1}{6} + \frac{1}{9}$$

$$= \frac{2-3+2}{18} = \frac{1}{18}$$

#7 (a) μ_1, μ_2, τ

(b) $$p(\mu_1, \mu_2, \tau, X, Y) \propto p(X \mid \mu_1, \tau)\, p(Y \mid \mu_2, \tau)\, p(\mu_1)\, p(\mu_2)\, p(\tau)$$

$$= \tau^{\frac{n_1}{2}} \exp\left(-\frac{\tau}{2} \underbrace{\sum_{i=1}^{n_1} (x_i - \mu_1)^2}_{[n_1\mu_1^2 - 2\mu_1 n_1 \bar{x} + x_i^2]}\right) \times \tau^{\frac{n_2}{2}} \exp\left(-\frac{\tau}{2} \sum_{i=1}^{n_2} (y_i - \mu_2)^2\right)$$

$$\times \frac{1}{\tau}$$

$$p(\mu_1 \mid \mu_2, \tau, X, Y) \propto \exp\left(-\frac{\tau n_1}{2}(\mu_1 - \bar{x})^2\right)$$

$$\sim N\left(\bar{x}, \frac{1}{\tau n_1}\right)$$

$$p(\mu_2 \mid \mu_1, x, y, \tau) \propto \exp\left(-\frac{\tau}{2} n^2 (\mu_2 - \bar{y})^2\right)$$

$$\sim N\left(\bar{Y}, \frac{1}{\tau n_2}\right)$$

$$p(\tau \mid \mu_1, \mu_2, x, y) \propto \tau^{\frac{1}{2}(n_1+n_2)-1} \exp\left(-\frac{\tau}{2}\left[\sum_{i=1}^{n_1}(x_i-\mu_1)^2 + \sum_{i=1}^{n}(y_i-\mu_2)^2\right]\right)$$

$$\sim \Gamma\left(\frac{1}{2}(n_1+n_2), \ \frac{1}{2}\left[\qquad\qquad\right]\right)$$

(c) Start with \tau(0), \mu_1(0), \mu_2(0). Given \tau(n), \mu_1(n), \mu_2(n), generate \tau(n+1) from \tau | \mu_1(n), \mu_2(n), \mu_1(n+1) ~ \mu_1 | \tau(n+1), \mu_2(n), \mu_2(n+1) from \mu_2 | \mu_1(n+1), \tau(n+1): do this step for n = 0 to M.

(d) Run multiple chains with widely separated initial values. Informal check - make a trace plot and see whether multiple chains come together and start to behave similarly. Formal method - compute BGR statistic which compare within-chain variability to between-chain variability. If BGR statistic < 1.05, it’s good.

#8 for $l = 1, \dots, L$

$$\left[\begin{array}{l} \text{① Simulate } \theta^{(l)} \sim p(\theta \mid y) \\ \text{② Simulate } \tilde{y}^{(l)} \sim p(\tilde{y} \mid \theta^{(l)}) \end{array}\right.$$

Then, $\tilde{y}^{(1)}, \tilde{y}^{(2)}, \dots \tilde{y}^{(L)}$ are samples from $p(\tilde{y} \mid y)$.

$\frac{1}{L} \sum_{l=1}^{L} \tilde{y}^{(l)} \Rightarrow$ estimator for the mean of the posterior predictive distribution.