School of Mathematics and Statistics
MAST30030
Applied Mathematical Modelling

Assignment 1. Due: 5pm Friday May 1st

This assignment counts for 10% of the marks for this subject.

Question 1

Consider the general dynamical system

$$\dot{x} = f(x).$$

i. Derive a linear stability analysis of this general system. What does this analysis predict when a fixed point, $\bar{x}$, satisfies $f'(\bar{x}) = 0$.

ii. Repeat the derivation in (a) but continue the expansion to fourth order, for the case of $f'(\bar{x}) = f''(\bar{x}) = f'''(\bar{x}) = 0$. What is the relevant stability condition of f in this case?

   Hint: look for real solutions in the ODE for the small perturbation function, $\epsilon(t)$, and consider the sign of both the initial condition, $\epsilon(0)$, and the fourth derivative, $f^{(4)}(\bar{x})$.

iii. Consider the system where $f(x) = x^4 \cos(x)$. Find the fixed points for this system.

iv. Using the analyses you derived in (i) and (ii), examine the stability of all the fixed points of the system in (iii).

v. Compare your answer in (iv) to the phase portrait of the system $f(x) = x^4 \cos(x)$, as determined using the graphical approach for flow on a line.

Question 2

Consider the following dynamical system

$$\dot{x} = rx + 2x^4 + x^7$$

where $r \in \Re$.

i. Calculate the fixed points for this system.

ii. Determine the stability of the fixed points using linear stability analysis.

iii. Verify your results in (ii) using a graphical approach. What does the graphical approach tell you about the stability for the cases where the linear stability analysis was inconclusive?

iv. Using your analyses from (i) – (iii), draw the bifurcation diagram and discuss what happens to the fixed points as r is varied.

Question 3

The following ODE describes the motion of a swinging pendulum (of mass m) under the influence of friction:

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \alpha\dot{\theta}$$

where $g > 0$, $L > 0$ and $\alpha \geq 0$ are all real constants that represent the gravitational acceleration, distance of the swinging mass from the origin and coefficient of friction respectively. $\theta \in \Re$ is the deflection angle of the pendulum.

i. By referring to the lecture example of a frictionless pendulum, rescale the above ODE with an identical time scale and hence determine a single, unit-less parameter that governs the damped pendulum's behaviour; call this parameter β.

ii. Rewrite the the resulting ODE from (i) as a system of first order ODEs and then find all the fixed points of the dynamical system.

iii. Perform a linearisation around each of the fixed points you found in (ii) and, where possible, determine their stability as a function of the parameter β.

iv. Draw qualitative representations of the phase portrait showing all relevant features such as fixed points, nullclines, direction of trajectories, closed orbits and basins of attraction. Draw as many phase portraits as required to demonstrate the different regimes you identified in (iii).

v. Discuss the physical implications of your results in (i)–(iv). You should discuss each of the different regimes you have identified and the physical interpretation of any special features you drew on the phase portraits.