Continuous-Time Markov Chains

A non-negative integer valued stochastic process $\{X_t : \; t \geq 0\}$ in continuous time is said to be a Continuous-Time Markov Chain (CTMC) (also known as a jump Markov process) if, for all $k \geq 1$, $t_1 < t_2 < \cdots < t_{k+1}$ and non-negative integers $i_1, i_2, \ldots, i_{k+1}$,

$$\begin{aligned} & P(X_{t_{k+1}} = i_{k+1} | X_{t_1} = i_1, \cdots, X_{t_k} = i_k) \\ = \; & P(X_{t_{k+1}} = i_{k+1} | X_{t_k} = i_k). \end{aligned}$$

If $P(X_{t+h} = k | X_h = j) = P(X_t = k | X_0 = j) \equiv p_{jk}^{(t)}$ does not depend on h, we say the CTMC is homogeneous. We concentrate on homogeneous CTMCs in this subject.

$(*)$ **Assumption:** $P^{(t)}$ **is stochastically (right) continuous in the sense that** $P^{(t+h)} \to P^{(t)}$ **as** $h \to 0+$**.**

Continuous-Time Markov Chains

Basic example: Poisson process.

Most general example we will consider:

- If $X_0 = i$, wait an exponential rate λ_i time and then jump to a new state.
- When jumping from state i, jump to state $j \neq i$ with probability $b_{i,j}$, where $\sum_{j \neq i} b_{i,j} = 1$.

Why Markovian?

Memoryless property of the exponential + transition probabilities $b_{i,j}$ only depend on states jumping to and from (and not past states).

Example: Continuous analog of the simple random walk.

When in state j wait an exponential rate one time and jump to state $j+1$ with probability p and state $j-1$ with probability $1-p$.

Continuous-Time Markov Chains

This setup is the most general for chains that always spend a positive amount of time at an occupied state:

Suppose $X_0 = j$ and $T(j)$ is the first time the CTMC leaves j. Then

$$
\begin{aligned}
& P(T(j) > t + s | T(j) > s) \\
&= P(X_v = j, 0 \leq v \leq t + s | X_u = j, 0 \leq u \leq s) \\
&= P(X_v = j, s < v \leq t + s | X_s = j) \text{ (Markov)} \\
&= P(X_v = j, 0 < v \leq t | X_0 = j) \text{ (homogeneous)} \\
&= P(T(j) > t)
\end{aligned}
$$

So $T(j)$ has the memoryless property, and $T(j) \sim \exp(\lambda_j)$ for some λ_j which depends on j only.

Continuous-Time Markov Chains

There are more general CTMCs with instantaneous states that are only visited for an instant, though we will not talk about these.

Continuous-Time Markov Chains

While the probabilistic description is intuitive, it isn't always helpful to compute the transitions matrix $P^{(t)}$. For this we need the concept of a generator which we will later relate back to the intuitive description.

The Chapman-Kolmogorov equations

Observe that

$$
\begin{aligned}
p_{ij}^{(s+t)} &= \sum_k P(X_{s+t} = j | X_s = k, X_0 = i) P(X_s = k | X_0 = i) \\
&= \sum_k p_{ik}^{(s)} p_{kj}^{(t)}.
\end{aligned}
$$

These are the Chapman-Kolomogorov equations for a CTMC. In matrix form, we write $P^{(t)} = (p_{jk}^{(t)})$. Then, for $s, t \geq 0$, the Chapman-Kolmogorov equations can be expressed in the form

$$P^{(t+s)} = P^{(t)} P^{(s)}.$$

Continuous-Time Markov Chains

By analogy with the discrete-time case, we might hope that we can write $P^{(t)} = P^t$ for some matrix P.

If $t = m$ (a positive integer), the C-K equations tell us that $P^{(m)} = (P^{(1)})^m$ and our hope is fulfilled, but if $t < 1$?

We want a single object (like $P = P^{(1)}$ in the discrete case) that encodes the information of the chain.

Continuous-Time Markov Chains

If t and h are nonnegative real numbers, we can write

$$\begin{aligned}\frac{P^{(t+h)} - P^{(t)}}{h} &= P^{(t)} \left[\frac{P^{(h)} - I}{h} \right] \\ &= \left[\frac{P^{(h)} - I}{h} \right] P^{(t)}\end{aligned}$$

This suggests that we should investigate the existence of the derivative

$$A \equiv \lim_{h \to 0^+} \frac{P^{(h)} - I}{h}.$$

Continuous-Time Markov Chains

- Amazingly, the continuity assumption on $P^{(t)}$ implies the existence of the matrix A, called the matrix or infinitesimal generator of the CTMC.

[If there are instantaneous states (excluded in our study) then some $a_{i,i} = -\infty$.]

- In all cases that we shall discuss, $|a_{j,k}| < \infty$ and $\sum_k a_{jk} = 0$ (inherited from the limit) and we shall assume this henceforth.

Continuous-Time Markov Chains

Since

$$\begin{aligned}\frac{P^{(t+h)} - P^{(t)}}{h} &= P^{(t)} \left[\frac{P^{(h)} - I}{h}\right] \\ &= \left[\frac{P^{(h)} - I}{h}\right] P^{(t)}\end{aligned}$$

we hope that

$$\frac{d}{dt} P^{(t)} = A P^{(t)} \quad (\ddagger)$$

and, similarly,

$$\frac{d}{dt} P^{(t)} = P^{(t)} A. \quad (\dagger)$$

Continuous-Time Markov Chains

We write "hope that" since we need to justify pushing the limits through the (possibly infinite) sums

$$\lim_{h\to 0}\sum_{k\in S} p_{i,k}^{(t)}\left[\frac{P^{(h)}-I}{h}\right]_{k,j} = \sum_{k\in S} p_{i,k}^{(t)}\lim_{h\to 0}\left[\frac{P^{(h)}-I}{h}\right]_{k,j}$$

$$\lim_{h\to 0}\sum_{k\in S}\left[\frac{P^{(h)}-I}{h}\right]_{i,k} p_{k,j}^{(t)} = \sum_{k\in S}\lim_{h\to 0}\left[\frac{P^{(h)}-I}{h}\right]_{i,k} p_{k,j}^{(t)}$$

for each $i, j \in S$.

If S is finite, then there is no problem and both (‡) and (†) hold.

Justifying (‡)

In fact, we know from Fatou's Lemma that, for $j, k \in S$,

$$\begin{aligned}
\liminf_{h\to 0^+} \frac{p_{jk}^{(t+h)} - p_{jk}^{(t)}}{h} &= \liminf_{h\to 0^+} \sum_{i\in S} \frac{(p_{ji}^{(h)} - \delta_{ji})p_{ik}^{(t)}}{h} \\
&\geq \sum_{i\in S} \lim_{h\to 0^+} \frac{(p_{ji}^{(h)} - \delta_{ji})p_{ik}^{(t)}}{h} \\
&= \sum_{i\in S} a_{ji} p_{ik}^{(t)}.
\end{aligned}$$

Similarly,

$$\liminf_{h\to 0^+} \frac{p_{jk}^{(t+h)} - p_{jk}^{(t)}}{h} \geq \sum_{i\in S} p_{ji}^{(t)} a_{ik}.$$

Justifying (‡)

We can show that the inequality in the first expression is, in fact, an equality, as follows. For $N > j$,

$$
\begin{aligned}
\sum_{i\in S} \frac{\left[p_{ji}^{(h)} - \delta_{ji}\right] p_{ik}^{(t)}}{h} &= \sum_{i=1}^{N} \frac{\left[p_{ji}^{(h)} - \delta_{ji}\right] p_{ik}^{(t)}}{h} + \sum_{i=N+1}^{\infty} \frac{p_{ji}^{(h)} p_{ik}^{(t)}}{h} \\
&\leq \sum_{i=1}^{N} \frac{\left[p_{ji}^{(h)} - \delta_{ji}\right] p_{ik}^{(t)}}{h} + \sum_{i=N+1}^{\infty} \frac{p_{ji}^{(h)}}{h} \\
&= \sum_{i=1}^{N} \frac{\left[p_{ji}^{(h)} - \delta_{ji}\right] p_{ik}^{(t)}}{h} + \frac{1 - \sum_{i=1}^{N} p_{ji}^{(h)}}{h} \\
&= \sum_{i=1}^{N} \frac{\left[p_{ji}^{(h)} - \delta_{ji}\right] \left[p_{ik}^{(t)} - 1\right]}{h}.
\end{aligned}
$$

Justifying (‡)

Therefore

$$\begin{aligned}
\limsup_{h\to 0^+} \sum_{i\in S} \frac{\left[p_{ji}^{(h)} - \delta_{ji}\right] p_{ik}^{(t)}}{h} &\leq \sum_{i=1}^{N} \lim_{h\to 0^+} \frac{\left[p_{ji}^{(h)} - \delta_{ji}\right]\left[p_{ik}^{(t)} - 1\right]}{h} \\
&= \sum_{i=1}^{N} a_{ji} p_{ik}^{(t)} - \sum_{i=1}^{N} a_{ji}.
\end{aligned}$$

Now we let $N \to \infty$ and use the fact that $\sum_{i=1}^{\infty} a_{ji} = 0$ to derive

$$\limsup_{h\to 0^+} \frac{p_{jk}^{(t+h)} - p_{jk}^{(t)}}{h} \leq \sum_{i\in S} a_{ji} p_{ik}^{(t)},$$

which proves that $p_{jk}^{(t)}$ is differentiable (since $\liminf \geq \limsup$) and

$$\frac{dp_{jk}^{(t)}}{dt} = \sum_{i\in S} a_{ji} p_{ik}^{(t)}.$$

So we have justified the interchange leading to (‡). However, the interchange leading to (†) cannot be justified in general, and there is a class of CTMCs, known as explosive CTMCs for which equation (†) does not hold.

Example: Explosive CTMC

Consider a CTMC on the nonnegative integers $j = 0, 1, \ldots$ with $\lambda_j = \lambda^j$ for some $\lambda > 1$ and which always increases by one; $b_{j,j+1} = 1$. Then the expected total time that the CTMC takes to 'reach infinity' is finite. This is an example of an explosive CTMC.

Continuous-Time Markov Chains

Equations (‡) are known as the Kolmogorov backward equations and equations (†) are known as the Kolmogorov forward equations.

For non-explosive CTMCs, the matrix A determines the transition probability completely by solving the backward or forward equations to get

$$\begin{aligned} P^{(t)} &= \exp(tA) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k, \end{aligned}$$

subject to $P^{(0)} = I$.

Example: The Poisson process

The Poisson process can be thought of as a CTMC. For $j, k \geq 0$, the transition probabilities are

$$p_{j,j+k}^{(t)} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

We can verify that

$$\frac{d}{dt} p_{j,j+k}^{(t)}|_{t=0} = \begin{cases} -\lambda, & \text{for } k = 0, \\ \lambda, & \text{for } k = 1, \\ 0, & \text{for } k \geq 2. \end{cases}$$

Continuous-Time Markov Chains

So

$$A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \ddots \\ 0 & -\lambda & \lambda & 0 & 0 & 0 & \ddots \\ 0 & 0 & -\lambda & \lambda & 0 & 0 & \ddots \\ 0 & 0 & 0 & -\lambda & \lambda & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Continuous-Time Markov Chains

Can we derive the transition probabilities from A? We could

- Compute $\exp(tA)$.
- Solve the Kolmogorov backward or forward differential equations.

We solve the forward equations to find $p_{0k}^{(t)}$.

Continuous-Time Markov Chains

Now,

$$\begin{aligned} \frac{d}{dt} p_{00}^{(t)} &= -\lambda p_{00}^{(t)} \\ \Longrightarrow p_{00}^{(t)} &= e^{-\lambda t}. \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} p_{01}^{(t)} &= \lambda (p_{00}^{(t)} - p_{01}^{(t)}) \\ \Longrightarrow p_{01}^{(t)} &= \lambda t e^{-\lambda t}. \end{aligned}$$

By mathematical induction, we can show that $p_{0k}^{(t)} = e^{-\lambda t} (\lambda t)^k / k!$.

Interpretation of the generator

For small h,

$$\begin{aligned} P(X_{t+h} = k | X_t = j) &= p_{jk}^{(h)} \\ &\approx (I + hA)_{jk} \\ &= \begin{cases} ha_{jk}, & \text{if } j \neq k, \\ 1 + ha_{jj}, & \text{if } j = k. \end{cases} \end{aligned}$$

So we can think of a_{jk} as the rate of transition from j to k, with

$$a_{jk} \begin{cases} \geq 0 \text{ if } k \neq j \\ \leq 0 \text{ if } k = j \end{cases}$$

and, since each row sums to 0, $\sum_{k \neq j} a_{jk} = -a_{jj}$.

Continuous-Time Markov Chains

The total rate of leaving state j is $\sum_{k \neq j} a_{jk} = -a_{jj}$, so $\lambda_j = -a_{jj}$.

To see where the CTMC moves upon leaving state j, observe that, for $k \neq j$,

$$P(X_h = k | X_h \neq j, X_0 = j) = \frac{p_{jk}^{(h)}}{\sum_{l \neq j} p_{jl}^{(h)}} \to \frac{-a_{jk}}{a_{jj}}$$

as $h \to 0$.

That is, when the CTMC leaves state j, it has probability $-a_{jk}/a_{jj}$ of moving to state k.

The Evolution of a CTMC

Starting at an initial state $X_0 = j$, the CTMC stays in j for a random time, exponentially distributed with parameter $-a_{jj}$.

Then it jumps to a state $k \neq j$ with probability $-a_{jk}/a_{jj}$,

and stays there for a random time which is exponentially distributed with parameter $-a_{kk}$ and independent of anything that has happened previously,

and then it jumps to ℓ, and so on ...

Examples

Determine whether each of the following matrices is the generator of a Markov chain and, if yes, describe how the CTMC evolves (λ and μ are nonnegative):

$$A_1 = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}; \; A_2 = \begin{pmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & -3 \end{pmatrix};$$

$$A_3 = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & -3 \end{pmatrix}.$$

Ergodicity

When does $\pi_j = \lim_{t\to\infty} p_{ij}^{(t)}$ exist, independently of i?

The conditions are similar to those for DTMCs, but we don't have to deal with periodicity.

- We classify the states and identify whether there is only one class of essential states.
- If yes, and the class is finite, then the CTMC is ergodic.
- If yes, and the class is not finite, the we need to check the mean return times.

Continuous-Time Markov Chains

An irreducible CTMC is positive recurrent if and only if there is a stationary distribution satisfying

$$\boldsymbol{\pi} = \boldsymbol{\pi} P^{(t)}$$

with $\sum_j \pi_j = 1$ for any $t \geq 0$.

For non explosive CTMCs, this occurs if and only if $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi} A = 0,$$

with $\sum_j \pi_j = 1$.

For explosive CTMCs, it is possible to have a solution to

$$\boldsymbol{\pi} A = 0,$$

with $\sum_j \pi_j = 1$ that is not the stationary distribution.

Example (Explosive CTMC with stationary distribution)

- Take the CTMC with generator with $a_{i,i+1} = \lambda_i p$ for $i \geq 0$, and $a_{i,i-1} = (1-p)\lambda_i$ for $i \geq 1$ and all other off-diagonal entries zero.
- If $p > 1-p$, then embedded DTMC is transient, but choosing $\lambda_i = \lambda^i$ where $\lambda > p/(1-p)$, there is a solution to

$$\boldsymbol{\pi} A = 0$$

with $\sum_{i \geq 0} \pi_i = 1$.

[We know the chain is transient since the discrete time chain is, so it must be explosive if there is a stationary distribution.]

Summary

	Discrete time MC	**Cont. time MC**
Unit of time	One step	"dt"
Basic info	P	A
Distribution	$P^{(n)} = P^n$	$\frac{d}{dt}P^{(t)} = P^{(t)}A = AP^{(t)}$
Evolution	geometric times+jumps	exponential times+jumps
Stationarity	$\boldsymbol{\pi} P = \boldsymbol{\pi}$	$\boldsymbol{\pi} A = 0$

Birth and Death Processes

Let

- X_t be the number of 'people' in a system at time t.
- Whenever there are n 'people' in the system
  - new arrivals enter (by birth or immigration) the system at an exponential rate λ_n
  - 'people' leave (or die from) the system at an exponential rate μ_n
  - arrivals and departures occur independently of one another
- $\{X(t) : \ t \geq 0\}$ is a birth-and-death process with arrival (or birth) rates $\{\lambda_n\}$ and departure (or death) rates $\{\mu_n\}$.

Continuous-Time Markov Chains

The generator of a birth and death process has the form

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \ddots \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & 0 & 0 & \ddots \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & 0 & \ddots \\ 0 & 0 & \mu_3 & -(\mu_3+\lambda_3) & \lambda_3 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The CTMC evolves by remaining in state k for an exponentially distributed time with rate $\lambda_k + \mu_k$,

then it moves to state $k+1$ with probability $\lambda_k/(\lambda_k+\mu_k)$ and state $k-1$ with probability $\mu_k/(\lambda_k+\mu_k)$, and so on.

Examples

- The Poisson process is a pure birth process with constant birth rates. Thus $\lambda_n = \lambda$ and $\mu_n = 0$ for all n.
- If $\lambda_n = 0$ for all n, then the CTMC is a pure death process.
- A binary continuous time branching process is a birth and death process: each particle lives for an exponentially distributed time with parameter z, then either splits into two new particles, with probability p or just dies, with probability $q = 1 - p$. All particles behave independently.
  The rates are $\lambda_n = znp$ and $\mu_n = znq$ for all n.

Continuous-Time Markov Chains

The generator for the binary continuous-time branching process is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ddots \\ qz & -z & pz & 0 & 0 & \ddots \\ 0 & 2qz & -2z & 2pz & 0 & \ddots \\ 0 & 0 & 3qz & -3z & 3pz & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

If $p \leq q$, then eventual extinction is guaranteed. If $p > q$ then extinction occurs with probability less than one.

From generator to distribution

For a birth and death process with a given initial distribution, one can find $p_j(t) = P(X_t = j)$ by solving the system of differential equations

$$\frac{d}{dt}\boldsymbol{p}^{(t)} = \boldsymbol{p}^{(t)}A,$$

that is

$$\begin{aligned}
\frac{d}{dt}p_0(t) &= -\lambda_0 p_0(t) + \mu_1 p_1(t) \\
\frac{d}{dt}p_k(t) &= \lambda_{k-1}p_{k-1}(t) - (\lambda_k + \mu_k)p_k(t) + \mu_{k+1}p_{k+1}(t), \;\; k \geq 1.
\end{aligned}$$

This system of equations governs the 'redistribution' of 'probability mass' as time passes. For finite state space birth and death process, it can be solved numerically. For infinite state space CTMCs there are techniques of solution that involve orthogonal polynomials.

Example

Consider a population in which there are no births ($\lambda_i = 0$ for $i \geq 0$) and, for $i \geq 1$, the death rates are μ_i.

Suppose X_t is the population size at time t and so $\{X_t, t \geq 0\}$ is a CTMC. Define $p_{ij}(t) = P(X_{t+s} = j | X_s = i)$.

1. Find expressions for $p_{ii}(t)$ and $p_{i,i-1}(t)$ for $i \geq 1$.
2. Given that the population size is two at a particular time, calculate the probability that no more than one death will occur within the next one unit of time.

Stationary distribution of a Birth and Death process

Assume $\lambda_i > 0$ and $\mu_i > 0$ for all i. We derive the stationary distribution (if it exists) by solving $\boldsymbol{\pi} A = 0$. This leads to

$$\begin{aligned} 0 &= -\lambda_0 \pi_0 + \mu_1 \pi_1 \\ 0 &= \lambda_{k-1}\pi_{k-1} - (\lambda_k + \mu_k)\pi_k + \mu_{k+1}\pi_{k+1}, \ k \geq 1. \end{aligned}$$

It turns out that this is equivalent to

$$\lambda_k \pi_k = \mu_{k+1}\pi_{k+1}$$

for $k \geq 0$.

Continuous-Time Markov Chains

The first-order recursion has solution

$$\pi_k = \pi_0 \prod_{\ell=1}^{k} \frac{\lambda_{\ell-1}}{\mu_\ell}.$$

So a stationary distribution exists if and only if

$$\sum_{k=0}^{\infty} \prod_{\ell=1}^{k} \frac{\lambda_{\ell-1}}{\mu_\ell} < \infty$$

in which case

$$\pi_0 = \left(\sum_{k=0}^{\infty} \prod_{\ell=1}^{k} \frac{\lambda_{\ell-1}}{\mu_\ell} \right)^{-1}.$$

Examples

- Compute the stationary distribution of an 'M/M/1 queue', which is a birth and death process with $\lambda_i = \lambda$, $i \geq 0$, and $\mu_i = \mu$, $i \geq 1$.
- Compute the stationary distribution of a birth and death process with constant birth rate $\lambda_i = \lambda$, $i \geq 0$, and unit per capita death rate $\mu_i = i$, $i \geq 1$.