**MAST30001 Stochastic Modelling – 2015**

**Assignment 2**

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If you didn't already hand in a completed and signed Plagiarism Declaration Form (available from the LMS or the department's webpage), please do so and attach it to the front of this assignment.

**Don't forget** to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 23 October by 4pm** in the appropriate assignment box at the north end of Richard Berry Building.

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness and style count.

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1. Customers arrive at a shop according to a Poisson process with rate 20 per hour. Arriving customers make a purchase with probability 1/3 and among customers making a purchase, there is 6/7 chance they make a "small" purchase and 1/7 chance they make a "large" purchase. Assume customers make their purchasing decisions independently.

   (a) What is the average amount of time between now and the next time a customer makes a large purchase?

   (b) What is the chance that during a half hour period exactly three customers total enter the store and exactly one of these three makes a large purchase?

   (c) What is the chance that during a one hour period exactly five customers total enter the store and during the first thirty minutes of that period exactly four customers make a small purchase?

   (d) What is the chance that there are at least two customers who make a small purchase before the arrival of the first customer who doesn't make a purchase?

   **Ans.**

   Let $N_t$ ($M_t$) be the number of customers that arrive (make a purchase) $t$ hours into the future and $K_t$ ($L_t$) the number that make a small (large) purchase. Then thinning (twice) says that $N_t - M_t$, $K_t$, $L_t$ are independent Poisson processes with rates 40/3, 40/7, 20/21. Also note that superposition of any of these processes is again a Poisson process and the relevant rates add.

   (a) This is the first point in $L_t$ which is exponential with mean 21/20.

   (b) We want $P(N_{1/2} = 3, L_{1/2} = 1)$. Note that $N_t - M_t + K_t = N_t - L_t$ is Poisson mean $(400/21)t$ and independent of $L_t$. Using this fact, we have

$$\begin{aligned} P(N_{1/2} = 3, L_{1/2} = 1) &= P(N_{1/2} - L_{1/2} = 2)P(L_{1/2} = 1) \\ &= e^{-10/21}(10/21)e^{-200/21}(200/21)^2/2 \\ &= 0.000980454. \end{aligned}$$

(c) We want $P(N_1 = 5, K_{1/2} = 4)$. Using the law of total probability, independent increments and the Bernoulli description of the purchase decisions,

$$\begin{aligned}
P(N_1 = 5, K_{1/2} = 4) &= P(N_1 = 5, N_{1/2} = 4, K_{1/2} = 4) \\
&\quad + P(N_1 = 5, N_{1/2} = 5, K_{1/2} = 4) \\
&= P(N_1 - N_{1/2} = 1)P(N_{1/2} = 4, K_{1/2} = 4) \\
&\quad + P(N_1 - N_{1/2} = 0)P(N_{1/2} = 5, K_{1/2} = 4) \\
&= e^{-10}(10)\frac{e^{-10}(10)^4}{4!}(2/7)^4 + e^{-10}\frac{e^{-10}(10)^5}{5!}\binom{5}{4}(2/7)^4(5/7) \\
&= 9.81093 \times 10^{-8}.
\end{aligned}$$

(d) This is the same as the chance that the time of the second point in the $K_t$ process; denote this time by $S$; is smaller than the time of the first point in the $N_t - M_t$ process; denote this time by $T$. Due to the independence of processes, $S$ and $T$ are independent and by the Poisson process description, $S$ is a gamma distributed variable with parameters $(2, 40/7)$ and $T$ is exponential rate $40/3$.

$$\begin{aligned}
P(S < T) &= \int_0^\infty (40/7)^2 s e^{-(40/7)s} \int_s^\infty (40/3)e^{-40/3} dt ds \\
&= \int_0^\infty (40/7)^2 s e^{-(40/7+40/3)s} ds \\
&= \frac{(40/7)^2}{(40/7 + 40/3)^2} = (3/10)^2.
\end{aligned}$$

Another way to understand this result is that each customer either buys nothing, makes a small purchase, or makes a large purchase, with probabilities, $2/3$, $(1/3)(6/7)$, $(1/3)(1/7)$. The number of customers who make a small purchase before the arrival of a customer who buys nothing is geometric with success probability equal to $P($ buys nothing $|$ buys nothing or small purchase$)$ which equals $(2/3)/(2/3 + 6/21) = 7/10$. The answer above is the probability this variable is at least two.

2. A car repair shop has two oil change service bays. Cars take an exponential amount of time (independent between different cars) to have their oil changed. Cars in the first service bay take about 20 minutes to have their oil changed and in the second service bay an oil change takes about 1 hour. Cars arrive for an oil change according to a Poisson process (independent of service times) with rate 2 per hour. If both service bays are empty, then an arriving car begins service in one of the two bays, chosen uniformly at random. If there is only one bay free, then an arriving car starts service in that bay. When both bays are full, cars can form a queue. Assume that if the length of the queue is two, then cars from the arrival process don't want to wait and so don't join (so the total number of cars in the system is capped at four).

   (a) What is the long run proportion of time both service bays are in use?

   (b) What is the average number of cars in the system?

   (c) What is the average amount of time that cars that enter the system have to wait for service?

(d) What is the long run proportion of cars that enter the system that receive service from the first bay?

**Ans.**

We can view this system as a CTMC with states $0, (1,0), (0,1), (1,1), (1,1,1), (1,1,2)$ corresponding to the system being empty, a car in Bay 1 but not in Bay 2, a car in Bay 2 but not in Bay 1, a car in both bays, one and two in the queue, respectively. For this CTMC the generator matrix $A$ is given by, for $\lambda = 2, \mu_1 = 3, \mu_2 = 1$,

$$\begin{pmatrix} -\lambda & \lambda/2 & \lambda/2 & 0 & 0 & 0 \\ \mu_1 & -(\mu_1+\lambda) & 0 & \lambda & 0 & 0 \\ \mu_2 & 0 & -(\mu_2+\lambda) & \lambda & 0 & 0 \\ 0 & \mu_2 & \mu_1 & -(\mu_1+\mu_2+\lambda) & \lambda & 0 \\ 0 & 0 & 0 & \mu_1+\mu_2 & -(\mu_1+\mu_2+\lambda) & \lambda \\ 0 & 0 & 0 & 0 & \mu_1+\mu_2 & -(\mu_1+\mu_2) \end{pmatrix},$$

and it is ergodic since it's irreducible with a finite state space. The steady state regime is given by the stationary distribution $\pi$ satisfying $\pi A = 0$, which is solved by (now putting in numbers)

$$\pi = \frac{1}{21}(6, 2, 6, 4, 2, 1).$$

(a) Both service bays are in use in states $(1,1), (1,1,1), (1,1,2)$ and the chain is in these states in the long run with probability

$$\frac{4+2+1}{21} = 1/3.$$

(b) The average number of cars in the system is

$$\left(\frac{2+6}{21}\right) + 2\left(\frac{4}{21}\right) + 3\left(\frac{2}{21}\right) + 4\left(\frac{1}{21}\right) = 26/21 = 1.238.$$

(c) *Arriving* cars find the system in stationary due to PASTA. If a car enters the system in states $0, (1,0), (0,1)$ then it is serviced immediately. If the car enters the system in state $(1,1)$ then it takes and exponential rate $\mu_1 + \mu_2 = 4$ time for a bay to become free and if the car enters the system in state $(1,1,1)$, then it must wait the sum of two exponential rate $\mu_1 + \mu_2 = 4$ variables to receive service. Thus averaging over the state the arriving car enters the system in and conditioning on cars entering the system, we find the average waiting time for service is

$$\frac{(4/21)(1/4) + (2/21)(2/4)}{1 - 1/21} = 2/20 = 0.1.$$

Note that the expected length of the queue is $L_q = 2/21 + 2/21 = 4/21$ and so using Little's law naively gives for the average waiting time $L_q/\lambda = 2/21$, which is not right. However the rate of cars entering the system is $\lambda(20/21)$ and with this correct rate, Little's law gives $L_q/(\lambda(20/21)) = 2/20$ which is the right answer.

(d) Again arriving cars find the system in stationary due to PASTA. In order for an arriving car to get serviced by the first bay, it must either find the system in state $(0,1)$, or in state $(0,0)$ and be assigned to Bay 1 (with probability $1/2$), or in states $(1,1),(1,1,1)$ and have the service just before its be finished by Bay 1, which happens with probability $\mu_1/(\mu_1+\mu_2)=3/4$. Thus conditioning on a car entering the system, the proportion of cars that enter the system and receive service in the first bay is

$$\frac{\frac{6}{21}+\frac{1}{2}\left(\frac{6}{21}\right)+\frac{3}{4}\left(\frac{4}{21}+\frac{2}{21}\right)}{1-1/21}=27/40=0.675.$$