Assignment 3 MAT5OPT Michael Le 21689299

"This is my own work. I have not copied any of it from anyone else."

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Q1 a) i. in $\mathcal{H}$?

$h: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by.

$$h(x) = \begin{pmatrix} y^2+2yz+2z^2-2y-4z \\ x^2+3y^2-2xy+4yz-4y \end{pmatrix}$$

$p_1 = (2,0,1)$?

$$h(p_1) = \begin{pmatrix} \cancel{0^2} + \cancel{2(0)(1)} + 2(1)^2 - \cancel{2(0)} - 4(1) \\ 2^2 - \cancel{2(2)(0)} + \cancel{3(0)^2} + \cancel{4(0)(1)} - \cancel{4(0)} \end{pmatrix} = \begin{pmatrix} 2-4 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

Not in $\mathcal{H}$.

$p_2 = (-1,1,0)$?

$$h(p_2) = \begin{pmatrix} 1^2 + \cancel{2(1)(0)} + \cancel{2(0)^2} - 2(1) - \cancel{4(0)} \\ (-1)^2 - 2(-1)(1) + 3(1)^2 + \cancel{4(1)(0)} - 4(1) \end{pmatrix} = \begin{pmatrix} 1-2 \\ 1+2+3-4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

in $\mathcal{H}$.

$p_3 = (3,1,0)$?

$$h(p_3) = \begin{pmatrix} (1)^2 + \cancel{2(1)(0)} + \cancel{2(0)^2} - 2(1) - \cancel{4(0)} \\ 3^2 - 2(3)(1) + 3(1)^2 + \cancel{4(1)(0)} - 4(1) \end{pmatrix} = \begin{pmatrix} 1-2 \\ 9-6+3-4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

in $\mathcal{H}$.

$p_4 = (2,2,-1)$?

$$h(p_4) = \begin{pmatrix} 2^2 + 2(2)(-1) + (-1)^2 2 - 2(2) - 4(-1) \\ 2^2 - 2(2)(2) + 3(2)^2 + 4(2)(-1) - 4(2) \end{pmatrix} = \begin{pmatrix} 4-4+2-4+4 \\ 4-8+12-8+8 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}.$$

Not in $\mathcal{H}$.

Q1aii) Regular points of h?
Suppose, we compute

$$Dh(x) = \begin{pmatrix} 0 & 2y+2z+0-2-0 & 0+2y+4z-0-4 \\ 2x-2y & -2x+6y+4z-4 & 4y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2y+2z-2 & 2y+4z-4 \\ 2x-2y & -2x+6y+4z-4 & 4y \end{pmatrix}$$

$p_1 = (2,0,1)$?

$$Dh(p_1) = \begin{pmatrix} 0 & 2(0)+2(1)-2 & 2(0)+4(1)-4 \\ 2(2)-2(0) & -2(2)+6(0)+4(1)-4 & 4(0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0+2-2 & 0+4-4 \\ 4 & -4+0+4-4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & -4 & 0 \end{pmatrix}$$

Not regular, because the second row is a scalar multiple of 0, from the first row.

$p_2 = (-1,1,0)$?

$$Dh(p_2) = \begin{pmatrix} 0 & 2(1)+\cancel{2(0)}-2 & 2(1)+\cancel{4(0)}-4 \\ 2(-1)-2(1) & -2(-1)+6(1)+\cancel{4(0)}-4 & 4(1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -2 \\ -4 & 4 & 4 \end{pmatrix}, \ p_2 \text{ is regular.}$$

$p_3 = (3,1,0)$?

$$Dh(p_3) = \begin{pmatrix} 0 & 2(1)+\cancel{2(0)}-2 & 2(1)+\cancel{4(0)}-4 \\ 2(3)-2(1) & -2(3)+6(1)+\cancel{4(0)}-4 & 4(1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2-2 & 2-4 \\ 6-2 & -6+6-4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 4 & -4 & 4 \end{pmatrix}, \ p_3 \text{ is } \cancel{\text{not}} \text{ regular.}$$

$p_4 = (2,2,-1)$?

$$Dh(p_4) = \begin{pmatrix} 0 & 2(2)+2(-1)-2 & 2(2)+4(2)-4 \\ 0 & -2(2)+6(2)+4(-1)-4 & 4(2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4-2-2 & 4+8-4 \\ 0 & -4+12-4-4 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 8 \end{pmatrix}, \ p_4 \text{ is not regular.}$$

Q1bi). Is γ a curve in $\mathcal{H}$?

$\gamma : \mathbb{R} \to \mathbb{R}^3,$

$\gamma(t) = (\cos(t) + \sin(t),\ \sin(t) - \cos(t),\ \cos(t) + 1)^T$

where $\mathcal{H} = \{x \in \mathbb{R}^3 : h(x) = (-1\ \ 2)^T\}$

given,

$$h(x) = \begin{pmatrix} y^2 + 2yz + 2z^2 - 2y - 4z \\ x^2 - 2xy + 3y^2 + 4yz - 4y \end{pmatrix}$$

$$h(x = \gamma(t)) = \begin{pmatrix} \sin^2 t - 2\sin t\cos t + \cos^2 t + 2(\sin t - \cos t)(\cos t + 1) + 2(\cos t + 1)^2 - 2(\sin t - \cos t) - 4(\cos t + 1) \\ \cos^2 t + 2\sin t\cos t + \sin^2 t - 2(\sin^2 t - \cos^2 t) + 3(\sin t - \cos t)^2 + 4(\sin t - \cos t)(\cos t + 1) - 4(\sin t - \cos t) \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2 t + \cos^2 t - 2\sin t\cos t + 2(\sin t\cos t + \sin t - \cos^2 t - \cos t) + 2(\cos^2 t + 2\cos t + 1) - 2\sin t + 2\cos t - 4\cos t - 4 \\ \cos^2 t + 2\sin t\cos t + \sin^2 t - 2\sin^2 t + 2\cos^2 t + 3(\sin^2 t - 2\sin t\cos t + \cos^2 t) + 4(\sin t\cos t + \sin t - \cos^2 t - \cos t) - 4\sin t + 4\cos t \end{pmatrix}$$

Note: $\sin^2 t + \cos^2 t = 1$.

$$= \begin{pmatrix} \underbrace{\sin^2 t + \cos^2 t}_{1} \cancel{- 2\sin t\cos t} \cancel{+ 2\sin t\cos t} \cancel{+ 2\sin t} \cancel{- 2\cos^2 t} \cancel{- 2\cos t} \cancel{+ 2\cos^2 t} + 4\cos t + 2 \cancel{- 2\sin t} + 2\cos t - 4\cos t - 4 \\ \cos^2 t + 2\sin t\cos t + \sin^2 t \underbrace{- 2\sin^2 t + 2\cos^2 t + 3\sin^2 t}_{\sin^2 t + \cos^2 t} - 6\sin t\cos t + 3\cos^2 t + 4\sin t\cos t \cancel{+ 4\sin t} - 4\cos^2 t \cancel{- 4\cos t} \cancel{- 4\sin t} \cancel{+ 4\cos t} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 2 - 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \therefore \gamma \text{ is in } \mathcal{H}.$$

Q1bii) P_1?

$P_1 = (2, 0, 1)$

$$\gamma = \begin{pmatrix} \cos t + \sin t \\ \sin t - \cos t \\ \cos t + 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{aligned} \cos t + \sin t &= 2 \quad -(1) \\ \sin t - \cos t &= 0 \quad -(2) \\ \cos t + 1 &= 1 \quad -(3) \end{aligned}$$

$\to \cos t = 0$, Then (2) is,

$\to \sin t = 0$

But for (1)

LHS $= 0 + 0 = 0$

RHS $= 2$

LHS $\neq$ RHS, P_1 is not in γ.

P_2? $P_2 = (-1, 1, 0)$

$$\gamma = \begin{pmatrix} \cos t + \sin t \\ \sin t - \cos t \\ \cos t + 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{aligned} \cos t + \sin t &= -1 \quad -(1) \\ \sin t - \cos t &= 1 \quad -(2) \\ \cos t + 1 &= 0 \quad -(3) \end{aligned}$$

Check, $\cos t = -1$, Then sub into (2),
$\rightarrow \sin t + 1 = 1$
$\rightarrow \sin t = 0$ Then,
Sub $\cos t = -1$ and $\sin t = 0$ into (1), to get.
$LHS = -1 + 0 = -1$
$RHS = -1$
$LHS = RHS$ satisfies $\rightarrow P_2$ is on γ.
Solving for t gives us, $t = -n\pi, \ldots, -\pi, \pi, \ldots, (2n+1)\pi, -(2n+1)\pi$.
General solution for t is, $t = \pi(2n+1), n \in \mathbb{Z}$

P_3? $P_3 = (3,1,0)$

$$\gamma = \begin{pmatrix} \cos t + \sin t \\ \sin t - \cos t \\ \cos t + 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \quad \begin{array}{l} \rightarrow \cos t + \sin t = 3 - (1) \\ \rightarrow \sin t - \cos t = 1 - (2) \\ \rightarrow \cos t + 1 = 0 - (3) \end{array}$$

from (3),
$\cos t = -1$
substitute $\cos t = -1$ into (2) gets us,
$\sin t + 1 = 1$, gets us.
$\sin t = 0$
substitute $\cos t = -1$ and $\sin t = 0$ into (1)
from the LHS.
$LHS = -1 + 0 = -1$
But, $RHS = 3$
$LHS \neq RHS$, P_3 is not in γ.

P_4? $P_4 = (2,2,-1)$

$$\gamma = \begin{pmatrix} \cos t + \sin t \\ \sin t - \cos t \\ \cos t + 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \quad \begin{array}{l} \rightarrow \cos t + \sin t = 2 \quad -(1) \\ \rightarrow \sin t - \cos t = 2 \quad -(2) \\ \rightarrow \cos t + 1 = 1 \quad -(3) \end{array}$$

solving $\cos t$ from (3) gives us
$\cos t + 1 = -1$
$\rightarrow \cos t = -2$
substitute $\cos t = -2$ into (2), gives us,
$\rightarrow \sin t - (-2) = 2$
$\rightarrow \sin t + 2 = 2$
$\rightarrow \sin t = 0$

Sub, $\sin t = 0$ and $\cos t = -2$ into (1),
from LHS gives us,

$LHS = -2 + 0 = -2$, But, $RHS = 2$
$LHS \neq RHS$, $\Rightarrow P_4$ is not in γ.

Q1c).
A basis for the normal space $N\mathcal{H}(p_2) = \mathrm{Im}(Dh^T(p_2))$
$$Dh(p_2) = \begin{pmatrix} 0 & 0 & -2 \\ -4 & 4 & 4 \end{pmatrix}$$

Performing row operations gives us,

$$\begin{pmatrix} 0 & 0 & -2 \\ -4 & 4 & 4 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Strictly in 'that' order.
$R_1' = R_1/-2$
$R_2' = R_2/4$
$R_2' = R_1 - R_2$
$R_2' \longleftrightarrow R_1'$

So, $N\mathcal{H}(p_2) = Sp((1,-1,0),(0,0,1))$

Q1d). Using the result from Q1c,

$\mathrm{Ker}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, obtain that $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$, $x_3 = 0$.

$$\begin{aligned} T\mathcal{H}(p_2) &= \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 - x_2 = 0,\ x_3 = 0\} \\ &= \{(x_2,x_2,0) \in \mathbb{R}^3 : x_2 \in \mathbb{R}\}. \\ &= Sp((1,1,0)) \end{aligned}$$

Q1e).
Compute $D\gamma$.
$$D\gamma = \begin{pmatrix} -\sin t + \cos t \\ \cos t + \sin t \\ -\sin t \end{pmatrix}$$
Using the t-result from Q1bii for $p_2 = (-1,1,0)$ case, $t = (2n+1)\pi$, $n \in \mathbb{Z}$

Choose
$t = \pi$, where $n = 1$
$$D\gamma(t=\pi) = \begin{pmatrix} -\sin(\pi) + \cos(\pi) \\ \cos(\pi) + \sin(\pi) \\ -\sin(\pi) \end{pmatrix} = \begin{pmatrix} 0-1 \\ -1+0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

Using the result from Q1d, for $\mathcal{TH}(p_2)$ is,

$$\mathcal{TH}(p_2) = \{(x_2, x_2, 0) \in \mathbb{R}^3 : x_2 \in \mathbb{R}\}.$$

pick $x_2 = -1$ to satisfy the point at p_2 is in the tangent space,
* To show if it is orthogonal to the normal space.

$$\mathcal{NH}(p_2) = Sp((1,-1,0),(0,0,1)).$$

$V \perp W$: $\forall v \in V, w \in W \quad v \cdot w = 0$
iff $\forall v \in B(V), v \in B(W), \quad v \cdot w = 0$

Where V, W are vector spaces.

Let $v^1 = (-1,-1,0)$ from $\mathcal{TH}(p_2)$ vector space,
Let $w^1 = (1,-1,0)$ and $w^2 = (0,0,1)$.

$$v^1 \cdot w^1 = (-1,-1,0) \cdot (1,-1,0) = -1+1+0 = 0$$
$$v^1 \cdot w^2 = (-1,-1,0) \cdot (0,0,1) = 0+0+0 = 0.$$

Thus, $\mathcal{TH}(p_2)$ is orthogonal to the normal space $\mathcal{NH}(p_2)$.

END of Question 1

Q2).

minimise the distance d,

$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2}$

minimize x^2+y^2

s.t. $(y+x)^2 + y = x + 3/2$

The Lagrangian is,

$\mathcal{L}(x;\lambda) = x^2+y^2+\lambda\left((y+x)^2+y-x-3/2\right).$

This is a linear system of linear equations, which we express as,

$2x+\lambda(2y+2x-1)=0$ — (1)

$2y+\lambda(2y+2x+1)=0$ — (2).

$(y+x)^2+y=x+3/2$ — (3) By Lagrange's multipler theorem.

$\boxed{\text{NOTE: } h(x^*) = (y+x)^2+y-x-3/2=0}$ $\boxed{\text{Also, } Dh(x^*) = \left(2(y+x)-1 \quad 2(y+x)+1\right)}$

Assuming $\lambda \in \mathbb{R}$.

From (1),

$2x(1+\lambda) = \lambda(1-2y)$

$\rightarrow x = \dfrac{\lambda(1-2y)}{2(1+\lambda)}$ substitute into (2),

$\rightarrow 2y+\lambda\left(2y+\dfrac{\lambda(1-2y)}{1+\lambda}+1\right)=0$

$\rightarrow 2y+\lambda\left(\dfrac{2y(1+\lambda)+\lambda-2\lambda y+1+\lambda}{1+\lambda}\right)=0$

$\rightarrow 2y+2y\lambda+\lambda(2y(1+\lambda)+\lambda-2\lambda y+1+\lambda)=0$

$\rightarrow 2y+2y\lambda+\lambda(2y+\cancel{2y\lambda}+\lambda-\cancel{2\lambda y}+1+\lambda)=0$

$\rightarrow 2y+2y\lambda+2y\lambda+\lambda^2+\lambda+\lambda^2=0$

$\rightarrow 2y+4y\lambda+2\lambda^2+\lambda=0$

$\boxed{\text{NOTE: } \lambda \neq -1}$

$\rightarrow 2y(1+2\lambda) = -\lambda(1+2\lambda)$

$\rightarrow y = \dfrac{-\lambda\cancel{(1+2\lambda)}}{2\cancel{(1+2\lambda)}} = -\dfrac{\lambda}{2}$

Substitute into

$$x = \frac{1}{2}\left(\frac{\lambda}{2}\right)\frac{1}{1+\lambda}(1-2y),$$

For $y = -\frac{\lambda}{2}$

$$\Rightarrow x = \frac{\lambda}{4(1+\lambda)}\left(1-2(-\lambda/2)\right) = \frac{\lambda}{4}\frac{\cancel{(1+\lambda)}}{\cancel{(1+\lambda)}} = \frac{\lambda}{4}$$

$$x^* = \left(\frac{\lambda}{4}, \frac{-\lambda}{2}\right) \quad \boxed{\left(x = \frac{\lambda}{4},\ y = -\frac{\lambda}{2}\right)}$$

substitute x^* into (3) gives us,

$$\left(\left(\frac{\lambda}{4}-\frac{\lambda}{2}\right)^2 - \frac{\lambda}{2} - \frac{\lambda}{4} - \frac{3}{2}\right) = 0$$

$$\Rightarrow \left(\frac{-\lambda}{4}\right)^2 - \frac{3\lambda}{4} - 3/2 = 0$$

$$\Rightarrow \lambda^2 - 12\lambda - 24 = 0$$

After solving λ to get,

$$\lambda_{1,2} = \frac{12 \pm \sqrt{144 - 4(1)(-24)}}{2(1)} = \frac{12 \pm \sqrt{144+96}}{2}$$

$$= \frac{12 \pm \sqrt{240}}{2}$$

$$\Rightarrow \lambda_{1,2} = \frac{12 \pm 4\sqrt{15}}{2} = 6 \pm 2\sqrt{15}$$

Case 1: $\lambda_1 = 6 + 2\sqrt{15}$,

Our optimiser is $x^* = \left(\frac{6+2\sqrt{15}}{4}, -\frac{1}{2}(6+2\sqrt{15})\right)$

$$x^* = \left(\frac{3+\sqrt{15}}{2}, -3-\sqrt{15}\right)$$

Case 2: $\lambda_2 = 6-2\sqrt{15}$

Our optimiser is $x^* = \left(\frac{1}{4}(6-2\sqrt{15}), \frac{-1}{2}(6-2\sqrt{15})\right)$

$= \left(\frac{3-\sqrt{15}}{2}, -3+\sqrt{15}\right)$

To classify them, we need the Hessian of $\mathcal{L}$ with respect to x and y.

$$D^2\mathcal{L}(x^*;\lambda) = \begin{pmatrix} 2\lambda+2 & 2 \\ 2 & 2\lambda+2 \end{pmatrix}$$

and the tangent space of the level set where,

$\mathcal{H} = \{x \in \mathbb{R}^2 : h(x) = 0\}$, then

$$T\mathcal{H}(x) = \mathrm{Ker}(Dh(x)) = \mathrm{Ker}\begin{pmatrix} 2y+2x-1 & 2y+2x+1 \end{pmatrix}$$
$$= \mathrm{Sp}(-y, x)$$

Case 1: $\lambda_1 = 6+2\sqrt{15}$, $x_1^* = \left(\frac{3+\sqrt{15}}{2}, -3-\sqrt{15}\right)$

We have,

$$D\mathcal{L}(x^*;\lambda = 6+2\sqrt{15}) = \begin{pmatrix} 2+2(6+2\sqrt{15}) & 2(6+2\sqrt{15}) \\ 2(6+2\sqrt{15}) & 2+2(6+2\sqrt{15}) \end{pmatrix}$$

We need to check on the tangent space

$$T\mathcal{H}(x_1^*) = \mathrm{Sp}\left(3+\sqrt{15}, \frac{3+\sqrt{15}}{2}\right).$$

For $v \in T\mathcal{H}(x_1^*)$, we have $v = a\left(3+\sqrt{15}, (3+\sqrt{15})/2\right)^T$, so

$$v^T D^2\mathcal{L}(x_1^*; 6+2\sqrt{15})v$$

$$= a^2\left(3+\sqrt{15}, (3+\sqrt{15})/2\right)\begin{pmatrix} 2+2(6+2\sqrt{15}) & 2(6+2\sqrt{15}) \\ 2(6+2\sqrt{15}) & 2+2(6+2\sqrt{15}) \end{pmatrix}\begin{pmatrix} 3+\sqrt{15} \\ (3+\sqrt{15})/2 \end{pmatrix}$$

Continuing Case 1,

$$= 2a^2 (3+\sqrt{15})^2 \begin{pmatrix} 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 7+2\sqrt{15} & 6+2\sqrt{15} \\ 2\sqrt{15}+6 & 7+2\sqrt{15} \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$= 2a^2 (3+\sqrt{15})^2 \begin{pmatrix} (7+2\sqrt{15})(1)+\sqrt{15}+3 & 6+2\sqrt{15}+\frac{7}{2}+\sqrt{15} \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$= 2a^2 (3+\sqrt{15})^2 \begin{pmatrix} 7+2\sqrt{15}+\sqrt{15}+3 & 3\sqrt{15}+\frac{19}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$= 2a^2 (3+\sqrt{15})^2 \begin{pmatrix} 10+3\sqrt{15} & \frac{19}{2}+3\sqrt{15} \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$= 2a^2 (3+\sqrt{15})^2 \left(10+3\sqrt{15}+\frac{19}{4}+\frac{3\sqrt{15}}{2} \right)$$

$$= 2a^2 (3+\sqrt{15})^2 \left(\frac{59}{4}+\frac{9\sqrt{15}}{2} \right) > 0.$$

$$= \frac{a^2 (3+\sqrt{15})^2}{2} \left(18\sqrt{15}+59 \right) > 0$$

Hence, $D^2 \mathcal{L}(x_1^*; 6+2\sqrt{15})$ is positive definite on $T\mathcal{H}(x_1^*)$,

$\Rightarrow x_1^*$ is a strict local minimiser. (a), closest to 0.

Similarly, if we change $S_p(y, -x)$
By symmetry, we also
get $x_2^* = \left(\frac{-3-\sqrt{15}}{2}, 3+\sqrt{15} \right)$

Case 2. $\lambda_2 = 6-2\sqrt{15}$, then $x_2^* = \left(\frac{3-\sqrt{15}}{2}, -3+\sqrt{15}\right)$

We have,

$$D\mathcal{L}(x_2^*; \lambda_2 = 6-2\sqrt{15}) = \begin{pmatrix} 2+2(6-2\sqrt{15}) & 2(6-2\sqrt{15}) \\ 2(6-2\sqrt{15}) & 2+2(6-2\sqrt{15}) \end{pmatrix}$$

We need to check on the tangent space.

$$TH(x_2^*) = \left(3-\sqrt{15}, \frac{3-\sqrt{15}}{2}\right).$$

For $v \in TH(x_2^*)$, we have $v = a\left(3-\sqrt{15} \;\; (3-\sqrt{15})/2\right)^T$,
So,

$$v^T D^2\mathcal{L}(x_2^*; \lambda_2 = 6-2\sqrt{15}) v$$

$$= a^2\left(3-\sqrt{15} \;\; (3-\sqrt{15})/2\right)\begin{pmatrix} 2+2(6-2\sqrt{15}) & 2(6-2\sqrt{15}) \\ 2(6-2\sqrt{15}) & 2+2(6-2\sqrt{15}) \end{pmatrix}\begin{pmatrix} 3-\sqrt{15} \\ (3-\sqrt{15})/2 \end{pmatrix}$$

$$= 2a^2(3-\sqrt{15})^2 \left(1 \;\; 1/2\right)\begin{pmatrix} 7-2\sqrt{15} & 6-2\sqrt{15} \\ 6-2\sqrt{15} & 7-2\sqrt{15} \end{pmatrix}\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$= 2(a(3-\sqrt{15}))^2\left(7-2\sqrt{15}+3-\sqrt{15} \quad 6-2\sqrt{15}+\frac{7}{2}-\sqrt{15}\right)\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$= 2(a(3-\sqrt{15}))^2\left(10-3\sqrt{15} \quad \frac{19}{2}-3\sqrt{15}\right)\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$= 2(a^2(3-\sqrt{15})^2)\left((10-3\sqrt{15})(1) + \frac{19}{4} - \frac{3\sqrt{15}}{2}\right)$$

$$= 2(a^2(3-\sqrt{15})^2)\left(10+\frac{19}{4} - \frac{9\sqrt{15}}{2}\right) = 2a^2(3-\sqrt{15})^2\left(\frac{59}{4} - \frac{9\sqrt{15}}{2}\right)$$

$$= 2a^2(3-\sqrt{15})^2\left(\frac{59}{4}-\frac{9\sqrt{15}}{2}\right)$$

$$= \frac{a^2}{2}(3-\sqrt{15})^2(59-18\sqrt{15})$$

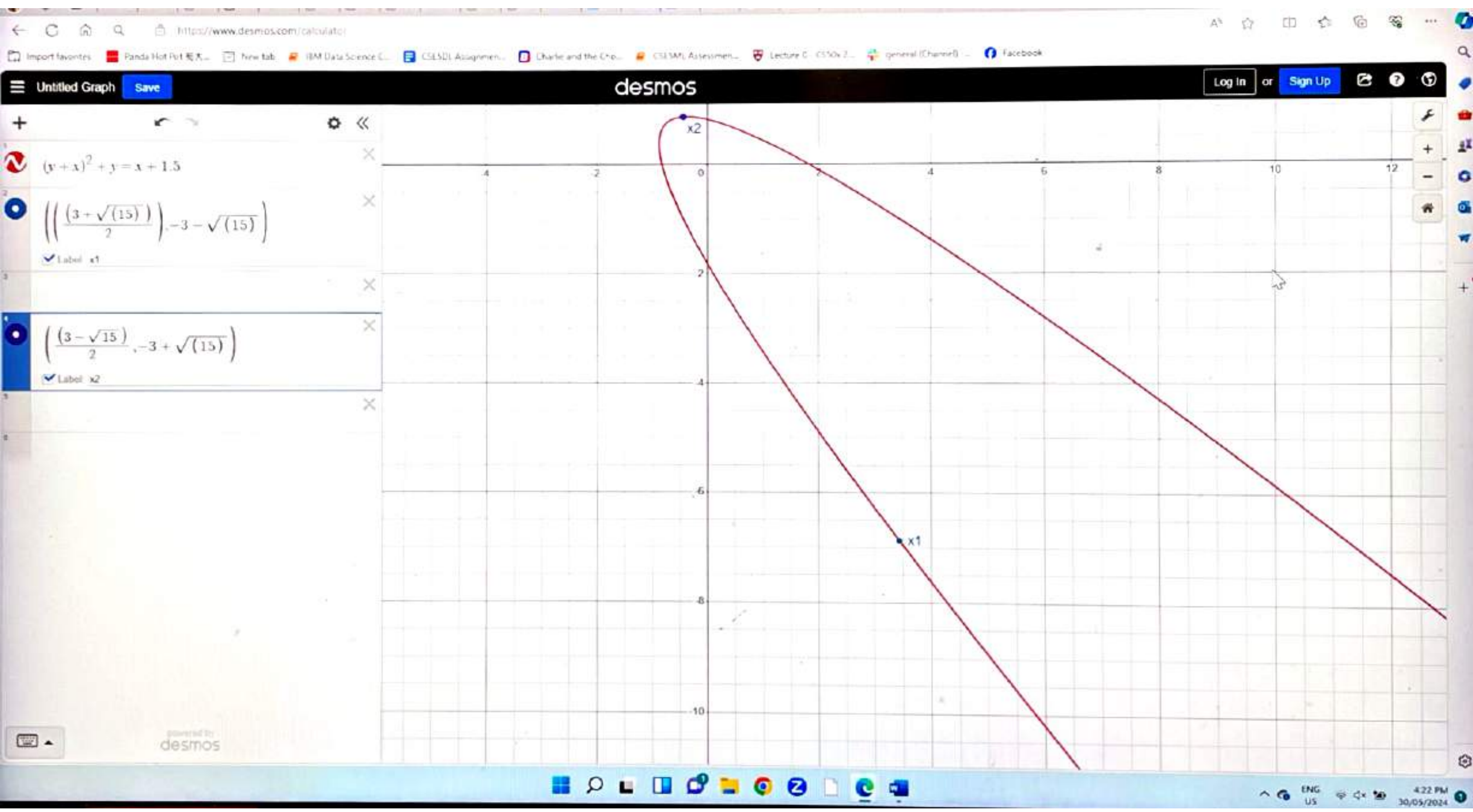
$$= \frac{a^2}{2}(0.7621\ldots)(-10.7137\ldots)$$

$$= -a^2(0.7621\ldots)(10.7137\ldots)/2 < 0$$

Hence, $D^2\mathcal{L}(x_i^*; \lambda = 6-2\sqrt{15})$ is negative definite on $T\mathcal{H}(x_i^*)$,

$\rightarrow x_i^*$ is a strict local maximiser.

(b). locally furthest.



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$(y+x)^2 + y = x + 1.5$
$\left(\left(\frac{(3+\sqrt{(15)})}{2}\right), -3-\sqrt{(15)}\right)$
Label x1
$\left(\frac{(3-\sqrt{15})}{2}, -3+\sqrt{(15)}\right)$
Label x2
x2
x1

Case ..

Q3).

$f(x) = (x+y-3)^2 + (y-z+2)^2$, $h(x) = -x^2+y+z-5$
$g(x) = y-z-1$
where $\Omega = \{x \in \mathbb{R}^3 : h(x)=0,\ g(x) \le 0\}$

NOTE: $x^* = (x, y, z)$.

$$\mathcal{L}(x^*; \lambda, \mu) = (x+y-3)^2 + (y-z+2)^2 + \lambda(-x^2+y+z-5) + \mu(y-z-1).$$

Our K.K.T Conditions are

$2(x+y-3) - 2\lambda x = 0$ —— (1)
$2(x+y-3) + 2(y-z+2) + \lambda + \mu = 0$ —— (2)
$-2(y-z+2) + \lambda - \mu = 0$ —— (3).
$\lambda(-x^2+y+z-5) = 0$ —— (4)
$\mu(y-z-1) = 0$ —— (5)
$g(x^*) = y-z-1 \le 0$ —— (6)
$h(x^*) = -x^2+y+z-5 = 0$ —— (7).

where, $\lambda, \mu \in \mathbb{R}$.

Using the hint, Solving equations (1), (2) and (3).

$(-1, 1, 1) \cdot D(f + \lambda h + \mu g)$

$= (-1, 1, 1) \cdot \left(2(x+y-3) - 2x\lambda,\ 2(x+y-3) + 2(y-z+2) + \lambda + \mu,\ 2(y-z+2)(-1) + \lambda - \mu\right)$
$= -(2(x+y-3) - 2\lambda x) + 2(x+y-3) + 2(y-z+2) + \lambda + \mu + 2(-1)(y-z+2)$
$+ \lambda - \mu = 0$

$\Rightarrow -2(x+y-3) + 2\lambda x + 2(x+y-3) + 2(y-z+2) + \lambda + \mu - 2(y-z+2) + \lambda - \mu = 0$
gives us,

$\Rightarrow 2\lambda x + 2\lambda = 0$
$\Rightarrow 2\lambda(x+1) = 0 \Rightarrow \lambda(x+1) = 0$
$\lambda = 0$ or $x = -1$.

Case 1 $\lambda = 0$. (Case 1a, Case 1b)

Case 2 $x = -1$. ((Case 2a, Case 2b, Case 2c, Case 2d).

Case 1 $\lambda=0$.

Gives us,

$2(x+y-3)=0$ (1)

$2(x+y-3)+2(y-z+2)+\mu=0$ (2)

$-2(y-z+2)+0-\mu=0$ (3)

$0=0$ ✓ (4)

$\mu(y-z-1)=0$ (5)

$g(x^*)=y-z-1\leq 0$ (6)

$h(x^*)=-x^2+y+z-5=0$ } conditions. (7)

Solving

$2x+2y-6=0$ and $-2y+2z-4=\mu$. simultaneously,

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 0 & -2 & 2 & \mu+4 \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & \frac{\mu+4}{2} \end{array}\right] \quad \begin{array}{l} R_1' = R_1/2 \\ R_2' = R_2/2. \end{array}$$

After Row operations.

gives us, $-y+z=\frac{\mu+4}{2}$

let $z=s$, then $y=s-\left(\frac{\mu+4}{2}\right)$

then $x+y=3 \Rightarrow x=3-y=3-\left(s-\left(\frac{\mu+4}{2}\right)\right)$

$\Rightarrow x=3-s+\left(\frac{\mu+4}{2}\right)$

$$x^*=\left(3-s+\frac{\mu}{2}+2,\ s-\frac{\mu}{2}-2,\ s\right) \qquad s\in\mathbb{R}.$$

Checking conditions (5), (6) & (7).

$\mu\left(\cancel{s}-\frac{\mu}{2}-2-\cancel{s}-1\right)=\mu\left(-\frac{\mu}{2}-3\right)=0$ from (5).

$\mu=0$ or $\mu=-6$

$g(x^*)=y-z-1=\cancel{s}-\frac{\mu}{2}-2-\cancel{s}-1\leq 0$

$\Rightarrow -\frac{\mu}{2}-3\leq 0$

$\Rightarrow -\frac{\mu}{2}\leq 3$

$\Rightarrow \mu \geq -6$ from (6)

$$h(x^*) = -\left(3 - s + \frac{\mu}{2} + 2\right)^2 + s - \frac{\mu}{2} - 2 + s - 5$$

$$= -\left(5 - s + \frac{\mu}{2}\right)^2 + s - \frac{\mu}{2} - 2 + s - 5$$

$$= -\left(5 - s + \frac{\mu}{2}\right)\left(5 - s + \frac{\mu}{2}\right) + s - \frac{\mu}{2} + s - 5 - 2$$

$$= -\left(25 - 5s + \frac{5}{2}\mu - 5s + s^2 - \frac{\mu s}{2} + \frac{5\mu}{2} - \frac{\mu s}{2} + \frac{\mu^2}{2}\right) + 2s - \frac{\mu}{2} - 7$$

$$= +\left(-\cancel{25} + \cancel{5s} - \cancel{\frac{5}{2}\mu} + \cancel{5s} - \cancel{s^2} + \cancel{\frac{\mu s}{2}} - \frac{5\mu}{2} + \cancel{\frac{\mu s}{2}} - \frac{\mu^2}{2} + \cancel{2s} - \cancel{\frac{\mu}{2}} - \cancel{7}\right)$$

$$= -32 - s^2 + 12s - 3\mu + \mu s - \frac{5\mu}{2} - \frac{\mu^2}{2}$$

$$= -32 - s^2 + 12s - 3\mu - \frac{5\mu}{2} + \mu s - \frac{\mu^2}{2}, \quad s \in \mathbb{R}.$$

$$= -32 - s^2 + 12s - \frac{11\mu}{2} + \mu s - \frac{\mu^2}{2} = 0 \quad (s \in \mathbb{R}). \text{ from (7).}$$

<u>Case 1a)</u> $\mu = 0$.

Then, $-32 - s^2 + 12s = 0$

$\rightarrow s^2 + 32 - 12s = 0$

$\rightarrow s^2 - 12s + 32 = 0$

$s \quad -8$

$s \quad -4$ $\qquad s = 4, 8$.

Then, $x^* = (3 - 4 + 0 + 2, 4 - 2, 4)$

$= (1, 2, 4)$ $\mu = 0$ and $s = 4$. is a possible extremiser.

and $x^* = (3 - 8 + 0 + 2, 8 - 0 - 2, 8)$

$= (-3, 6, 8)$ $\mu = 0$ and $s = 8$. is a possible extremiser.

Case 1b). $\mu = -6$.

Then in condition (7).

$$\Rightarrow -32 - s^2 + 12s + \frac{11}{2}(6) + (-6)s - \frac{(-6)^2}{2} = 0$$

$$\Rightarrow -32 - s^2 + 12s + 33 - 6s - 18 = 0$$

$$\Rightarrow -s^2 + 6s + 1 - 18 = 0$$

$$\Rightarrow -s^2 + 6s - 17 = 0.$$

Using the quadratic equation,

$$s = \frac{-6 \pm \sqrt{36 - 4(-1)(-17)}}{2(-1)} = \frac{-6 \pm \sqrt{36+68}}{-2}$$

$$= \frac{-6 \pm \sqrt{104}}{-2} = \frac{-6 \pm 2\sqrt{26}}{-2}$$

$$= 3 \pm \sqrt{26}$$

Then,

$$x' = \left(3 - (3+\sqrt{26}) - \frac{6}{2} + 2,\ (3+\sqrt{26}) + 3 - 2,\ 3 + \sqrt{26}\right) \qquad s = 3+\sqrt{26}$$

$= (-\sqrt{26} - 1,\ \sqrt{26} + 4,\ 3 + \sqrt{26})$ is a possible extremiser.

and,

$$x' = (3 - (3 - \sqrt{26}) - 1,\ (3 - \sqrt{26}) + 1,\ 3 - \sqrt{26})$$

$$= (3 - 3 + \sqrt{26} - 1,\ 3 - \sqrt{26} + 1,\ 3 - \sqrt{26})$$

$= (\sqrt{26} - 1,\ 4 - \sqrt{26},\ 3 - \sqrt{26})$ is a possible extremiser.

Case 2 : $\lambda = -1$.

Substitute $x = -1$ into (1)

$-2\lambda = 2(-1+y-3)$

$-2\lambda = -8+2y$

$\rightarrow -2y = 2\lambda - 8$

$\rightarrow \quad y = -\lambda + 4$

Substitute $y = -\lambda + 4$ into (3),

$2(-\lambda+4-z+2) = \lambda - \mu$

$\rightarrow -2\lambda + 8 - 2z + 4 = \lambda - \mu$

$\rightarrow -2z = 3\lambda - \mu - 12$

$\rightarrow z = \frac{1}{2}(12 - 3\lambda + \mu)$

Hence,

$x^* = (-1, -\lambda+4, \frac{1}{2}(12-3\lambda+\mu))$

Check the conditions for (4), (5), (6) and (7).
First Check (5) and (4).

$$\lambda(-(-1)^2 + (4-\lambda) + (6 - \frac{3}{2}\lambda + \frac{\mu}{2}) - 5) = 0 \quad -(4)$$

$$\mu(-\lambda + 4 - 6 + \frac{3}{2}\lambda - \frac{\mu}{2} - 1) = 0 \quad -(5)$$

From (4),

$$\lambda(\cancel{-1} + 4 - \lambda + \cancel{6} - \frac{3}{2}\lambda + \frac{\mu}{2} \cancel{-5})$$

$$= \lambda(4 - \lambda - \frac{3}{2}\lambda + \frac{\mu}{2})$$

$$= \lambda(4 - \frac{5}{2}\lambda + \frac{\mu}{2})$$

$$= 4\lambda - \frac{5}{2}\lambda^2 + \frac{\mu\lambda}{2}$$

$$= -\frac{5}{2}\lambda^2 + 4\lambda + \frac{\mu\lambda}{2} = 0 \quad — \boxed{4}$$

Similarly for (5),

$$\mu(+\frac{1}{2}\lambda - 3 - \frac{\mu}{2}) = -\frac{\mu^2}{2} + \frac{\mu\lambda}{2} - 3\mu = 0 \quad — \boxed{5}$$

Solving λ from $\boxed{4}$ using the quadratic formula,

$$\lambda_{1,2} = \frac{-\left(\frac{\mu}{2}+4\right) \pm \sqrt{\left(\frac{\mu}{2}+4\right)^2 - 4\left(\frac{-5}{2}\right)\cancel{(0)}}}{2(-5/2)}$$

$\Rightarrow \lambda_1 = 0$ or $\lambda_2 = \frac{1}{5}(\mu+8)$

Similarly for μ from $\boxed{5}$ using the quadratic formula.

$$\mu_{1,2} = \frac{(3+\lambda/2) \pm \sqrt{(-(3+\lambda/2))^2 - \cancel{4(-1/2)(0)}}}{2(-1/2)}$$

$$= -3-\lambda/2 \pm (3+\lambda/2)$$

$\mu_1 = 0$ or $\mu_2 = -6-\lambda$

<u>Case 2a.</u> $\lambda_1 = 0,\ \mu_1 = 0$

<u>Case 2b.</u> $\lambda_1 = 0,\ \mu_2 = -6-\lambda$

<u>Case 2c</u> $\lambda_2 = \frac{1}{5}(\mu+8),\ \mu_1 = 0$

<u>Case 2d</u> $\lambda_2 = \frac{1}{5}(\mu+8),\ \mu_2 = -6-\lambda.$

$\boxed{x^* = (-1, -\lambda+4, \frac{1}{2}(12-3\lambda+\mu))}$

For Case 2a Check conditions (6) and (7) for the optimiser $x^* = (-1, 4, 6)$

$g(x^*) = 4-6-1 = -3 \leq 0$ ✓ satisfies (6).
$h(x^*) = -(-1)^2+4+6-5 = -1+10-5 = 9-5 = 4,\quad 4 \neq 0.$
Does not satisfy.

$\therefore$ $x^* = (-1, 4, 6)$ is not an extremiser.

Case 2b $\lambda_1 = 0, \mu_2 = -6-\lambda$.

Then,

$\lambda = 0$, sub into $\mu_2 = -6-\lambda$ to get $\mu = -6$.

Then $x^* = (-1, 4, 6-3)$

$= (-1, 4, 3)$

Check conditions (6) and (7).

$4-3-1 = 0 \leq 0$ satisfies (6).

$-(-1)^2 + 4 + 3 - 5 = -1 + 7 - 5 = 1$, $1 \neq 0$. Does not satisfy.

$\therefore x^* = (-1, 4, 3)$ is not an extremiser.

Case 2c $\lambda_2 = \frac{1}{5}(\mu + 8), \mu_1 = 0$

If $\mu = 0$, sub $\lambda_2 = \frac{1}{5}(\mu+8)$, to get $\lambda = \frac{8}{5}$

Then $x^* = (-1, -8/5 + 4, \frac{1}{2}(12 - 3(8/5)))$

$= (-1, 12/5, \frac{1}{2}(\frac{36}{5}))$

$= (-1, \frac{12}{5}, \frac{18}{5})$.

Check the conditions (6) and (7).

$\frac{12}{5} - \frac{18}{5} - 1 = -\frac{11}{5} \leq 0$ satisfies (6).

$-(-1)^2 + \frac{12}{5} + \frac{18}{5} - 5 = -1 + 6 - 5 = 0 = 0$ ✓

satisfies (7).

$\therefore x^* = (-1, 12/5, 18/5)$ is an extremiser.

Case 2d $\lambda_2 = \frac{1}{5}(\mu + 8), \mu_2 = -6-\lambda$

If $\lambda = \frac{\mu+8}{5}$ sub into $\mu = -6-\lambda$, to get

$\Rightarrow \mu = -6 - \left(\frac{\mu+8}{5}\right) \Rightarrow \mu + \left(\frac{\mu+8}{5}\right) = -6$

$\Rightarrow 5\mu + \mu + 8 = -30$

(see in 2 pages). $\Rightarrow 6\mu = -38$, $\mu = -19/3$

Then,

$\lambda = \frac{1}{5}(\mu + 8)$ is,

$\rightarrow \lambda = \frac{1}{5}\left(-\frac{19}{3}+8\right) = \frac{1}{5}\left(-\frac{19}{3}+\frac{24}{3}\right) = \frac{1}{5}\left(\frac{5}{3}\right) = \frac{1}{3}.$

Then our optimiser would be,

$x^* = (-1, -1/3 + 4, \frac{1}{2}(12 - 1 - 19/3))$

$= (-1, 11/3, \frac{1}{2}(14/3))$

$= (-1, 11/3, 7/3).$

Check conditions (6) and (7).

$\frac{11}{3} - \frac{7}{3} - 1 = \frac{1}{3} \nleq 0.$ ~~contradiction~~ violates (6).

Then,

$x^* = (-1, 11/3, 7/3)$ is not an extremiser.

END of Question 3