**MAST30001 Stochastic Modelling**

**Tutorial Sheet 9**

1. Consider a population consisting of particles arriving from outside according to a Poisson process with rate $\lambda$. The lifetime of each particle (after it arrives) is exponential with rate $\alpha$ and the lifetimes are all independent.

   (a) Model the system as a birth-death process and find the birth and death rates.

   (b) Show that the process is ergodic and find its stationary distribution.

   (c) What is the expected number of living particles in the population in stationary?

2. If $(X_t^{(1)})_{t\geq 0}, \dots, (X_t^{(k)})_{t\geq 0}$ are i.i.d. continuous time Markov chains on $\{0,1\}$ each having generator
$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$
then what is the generator for the chain determined by $Y_t = \sum_{i=1}^k X_t^{(i)}$?

3. A workshop has two machines and one repairman. Each machine is either functional or broken. If the the $i$th machine ($i = 1, 2$) is functional, then it fails after an exponential rate $\lambda_i$ time. If the $i$th machine is broken, it takes the repairman an exponential rate $\mu_i$ amount of time to fix it and once it is fixed, it's good as new. Assume the repairman begins work the instant a machine breaks down, that only one machine can be repaired at a time, and all lifetime and repair times are independent.

   (a) Construct an appropriate continuous time Markov chain to describe the system and find the generator.

   (b) If $\lambda_i = \mu_i = i$ for $i = 1, 2$, find the stationary distribution of the process.

4. A system has N particles each of which at any given time are in one of the two energy states $\alpha$ or $\beta$. The particles switch between states $\alpha$ and $\beta$ according to the following rules. When a particle enters state $\alpha$, it switches to state $\beta$ after an exponentially distributed with rate $\mu > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state $\alpha$. Similarly, when a particle enters state $\beta$, it switches to state $\alpha$ after an exponentially distributed with rate $\lambda > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state $\beta$.

   (a) Model the number of particles in the energy state $\alpha$ as a continuous time Markov chain and define its generator.

   (b) Describe the long run behaviour of the chain.

   (c) If the chain starts with $N$ particles in the $\alpha$ energy state and $X_t$ is the number of $\alpha$ particles at time $t$, find the mean and variance of $X_t$ as $t \to \infty$. Your answer should be a tidy formula.

5. The following continuous time Markov chain is used to model population growth without death. The basic assumption of the model is that every member of the population gives birth to a new member with rate $\lambda$ (that is, at times with distribution exponential with rate $\lambda$), independently of the other members of the population. Let $X_t$ be the size of the population at time $t$.

   (a) What is $\mathbb{P}(X_t = n | X_0 = 1)$?

   (b) If $U$ is uniform on the interval $(0, 1)$, independent of $X_t$, find the distribution of $X_U | X_0 = 1$.