TUTORIAL 2

1. Consider the following LP problem in standard form:

$$\text{Maximise} \quad z = 8x_1 + 6x_2$$

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ 2x_1 + x_2 &\leq 15 \\ x_1, x_2 &\geq 0. \end{aligned}$$

   (a) Convert this problem into canonical form.
   (b) Find all basic solutions to this problem in canonical form by using the definition of a basic solution.
   (c) Which basic solutions that you find in (b) are basic feasible solutions?
   (d) Compare the basic feasible solutions obtained in (c) with the extreme points of the feasible region of the above problem in standard form. What conclusion can you reach from this comparison?

Solutions: (a) The canonical form is

$$\text{Maximise} \quad z = 8x_1 + 6x_2$$

$$\begin{aligned} x_1 + x_2 + x_3 \qquad &= 10 \\ 2x_1 + x_2 \qquad + x_4 &= 15 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

where x_3 and x_4 are slack variables.

(b) The coefficient matrix of the canonical form is $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$. Since A has rank 2, we need to choose two independent columns of A to find basic solutions.

The first two columns of A are independent. So we may set x_1 and x_2 to be basic variables, and x_3 and x_4 to be nonbasic variables. Setting $x_3 = x_4 = 0$, the equations above become

$$\begin{aligned} x_1 + x_2 &= 10 \\ 2x_1 + x_2 &= 15. \end{aligned}$$

Solving these simultaneous equations, we obtain $x_1 = 5, x_2 = 5$. So we get the basic solution $(5, 5, 0, 0)$.

In the same fashion we can obtain all other basic solutions as follows.

Choosing columns 1 and 3 (which are independent), we obtain the basic solution $(15/2, 0, 5/2, 0)$.

Choosing columns 1 and 4 (which are independent), we obtain the basic solution $(10, 0, 0, -5)$.

Choosing columns 2 and 3 (which are independent), we obtain the basic solution $(0, 15, -5, 0)$.

Choosing columns 2 and 4 (which are independent), we obtain the basic solution $(0, 10, 0, 5)$.

Choosing columns 3 and 4 (which are independent), we obtain the basic solution $(0, 0, 10, 15)$.

(c) Among the basic solutions in (b), $(5, 5, 0, 0)$, $(15/2, 0, 5/2, 0)$, $(0, 10, 0, 5)$ and $(0, 0, 10, 15)$ are basic feasible solutions because their coordinates are all nonnegative.

(d) The feasible region of the LP problem in standard form is bound by the quadrilateral with corner points $(0, 0), (15/2, 0), (5, 5)$ and $(0, 10)$. (Draw the feasible region by yourself.) These corner points are in one-to-one correspondence with the basic feasible solutions obtained in (c).

2. To prove the Fundamental Theorem of Linear Programming in lectures, we used the following algorithm:

Input: Coefficient matrix A, vector $\mathbf{b}$, feasible solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$
Output: Basic feasible solution

1: Relabel the variables when necessary so that $\mathbf{x} = (\mathbf{x}^+, \mathbf{0})$ and $\mathbf{x}^+$ consists of the strictly positive components of $\mathbf{x}$. Partition $A = (A^+, A^0)$ such that the number of columns of A^+ is equal to that of $\mathbf{x}^+$.
2: **while** the columns of A^+ are linearly dependent **do**
3:     Choose a non-zero vector $\mathbf{w}^+$ with the same dimension as $\mathbf{x}^+$ such that $A^+\mathbf{w}^+ = \mathbf{0}$ (solve these equations to find $\mathbf{w}^+$). We may assume at least one coordinate of $\mathbf{w}^+$ is strictly positive (otherwise replace $\mathbf{w}^+$ by $-\mathbf{w}^+$).
4:     Let $\epsilon = \min\{\frac{x_j}{w_j} : w_j > 0\}$.
5:     Let $\mathbf{x} := (\mathbf{x}^+ - \epsilon\mathbf{w}^+, \mathbf{0})$.
6:     Relabel the variables when necessary so that $\mathbf{x} = (\mathbf{x}^+, \mathbf{0})$ and $\mathbf{x}^+$ consists of the strictly positive components of $\mathbf{x}$. Partition $A = (A^+, A^0)$ such that the number of columns of A^+ is equal to that of $\mathbf{x}^+$.
7: **end while**
8: Output $\mathbf{x}$ (as a basic feasible solution).

Consider the linear program in standard form:

$$\text{Maximise} \quad z = 4x_1 + 6x_2$$

$$\begin{aligned} x_1 + x_2 &\leq 40 \\ x_1 + 3x_2 &\leq 90 \\ 2x_1 + x_2 &\leq 70 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(a) Transform this problem into canonical form.

(b) Now use the algorithm described above to find a basic feasible solution, starting with the feasible solution $(10, 20, 10, 20, 30)$.

Solutions: (a) The canonical form is:

$$\text{Maximise} \quad z = 4x_1 + 6x_2$$

$$\begin{aligned} x_1 + x_2 + x_3 \qquad\qquad &= 40 \\ x_1 + 3x_2 \qquad + x_4 \qquad &= 90 \\ 2x_1 + x_2 \qquad\qquad + x_5 &= 70 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

(b) *Step 1:* The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$\mathbf{x} = (10, 20, 10, 20, 30), \quad \mathbf{x}^+ = \mathbf{x}.$$

So

$$A^+ = A$$

and A^0 is empty (without any entries, or a 0×0 matrix).

Step 2: The columns of A are linearly dependent as the first two columns are linear combinations of the other columns.

Step 3: We need to choose a vector $\mathbf{w}^+$ such that:

$$w_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + w_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + w_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + w_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

At any time, there may be more than one vector $\mathbf{w}^+$ that satisfies this system. We choose the most obvious vector, where we add the first two vectors and use the slack columns to remove the values in each row:

$$1\begin{bmatrix}1\\1\\2\end{bmatrix}+1\begin{bmatrix}1\\3\\1\end{bmatrix}-2\begin{bmatrix}1\\0\\0\end{bmatrix}-4\begin{bmatrix}0\\1\\0\end{bmatrix}-3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

That is, $\mathbf{w}^+ = (1, 1, -2, -4, -3)$.
Step 4: We consider only the columns which have an associated w_j that is strictly positive (in the current situation only $w_1 > 0, w_2 > 0$):

$$\epsilon = \min\left\{\frac{x_1}{w_1}, \frac{x_2}{w_2}\right\} = \min\left\{\frac{10}{1}, \frac{20}{1}\right\} = 10.$$

Step 5: Therefore our new feasible solution is $\mathbf{x}^+ - \epsilon\mathbf{w}^+ = (10, 20, 10, 20, 30) - 10\cdot(1, 1, -2, -4, -3) = (0, 10, 30, 60, 60)$.
Step 6: The new $\mathbf{x}^+ = (10, 30, 60, 60)$ and the new A^+ is:

$$\begin{bmatrix}1&1&0&0\\3&0&1&0\\1&0&0&1\end{bmatrix}$$

Step 2: The columns of A^+ are linearly dependent, as we can easily find the first column as a linear combination of the last three.
Step 3: In the same way as above, we find the vector $\mathbf{w}^+ = (1, -1, -3, -1)$.
Step 4: Let $\epsilon = \min\{\frac{10}{1}\} = 10$, as x_2 is the only variable associated with a strictly positive w value.
Step 5: Our new feasible solution is $(\mathbf{x}^+ - \epsilon\mathbf{w}^+, 0) = (0, 10, 30, 60, 60, 0) - 10\cdot(0, 1, -1, -3, -1) = (0, 0, 40, 90, 70)$.
Step 6: Our new A^+ is:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The columns of this matrix are clearly linearly independent. Therefore $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 40, 90, 70)$ is a basic feasible solution.