Fluid Mechanics

Topic 9: Compressible flow

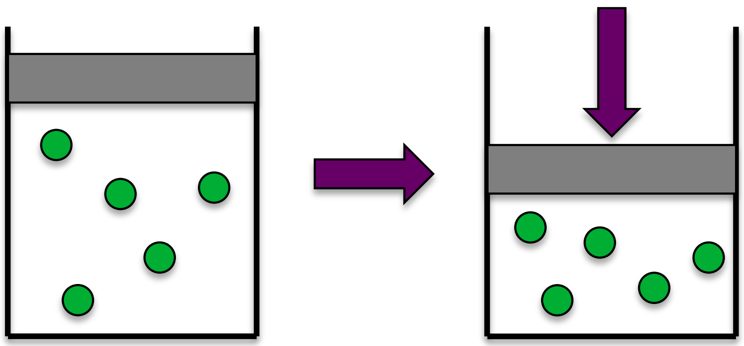
Part 1: Review of thermodynamics

What do we mean by compressibility?

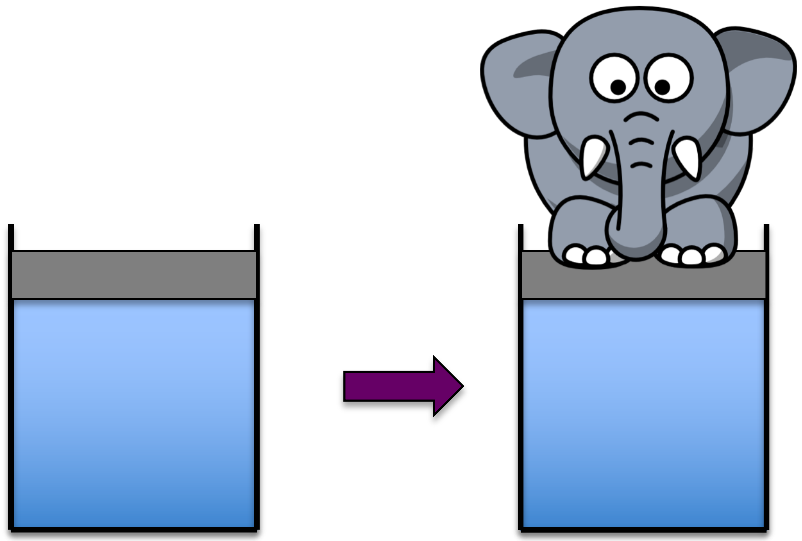
What do we mean when we say **compressible**?

- When a material is compressible, the density changes when pressure is changed
- Mathematically we can say $\frac{\partial \rho}{\partial P} \neq 0$ or that $\rho = f(P)$

For Gases



For Liquids



What do we mean by compressibility?

What do we mean when we say **compressible**?

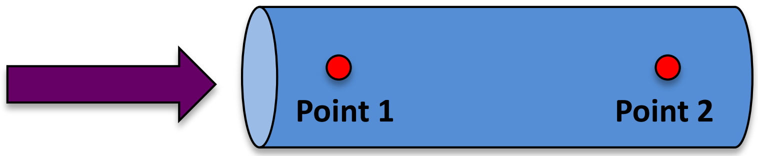
- How compressible is sea water?
  - At sea level, ocean water has a density of ~1.03g/cm^3 while 11 km below at the bottom of the Mariana Trench the density of ocean water is ~1.08g/cm^3
  - That is a 5% change in density for a 1,072 atm increase in pressure

- How compressible is air?
  - At sea level, air has a density of ~0.0012g/cm^3 and you would achieve the same 5% change in density by subjecting it to a 1.065 atm of pressure

Where are we going in this lesson?

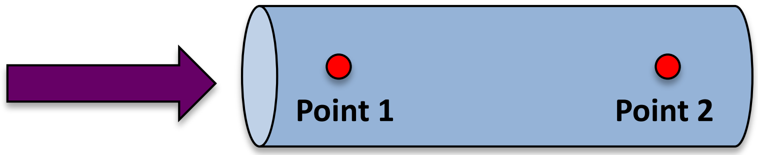
Assume we have a liquid and a gas flowing through horizontal pipes of constant cross section

– If the fluid is an incompressible liquid, the scenario is rather boring



- Pressure drop is linear
- $\rho_1=\rho_2$
- $T_1=T_2$

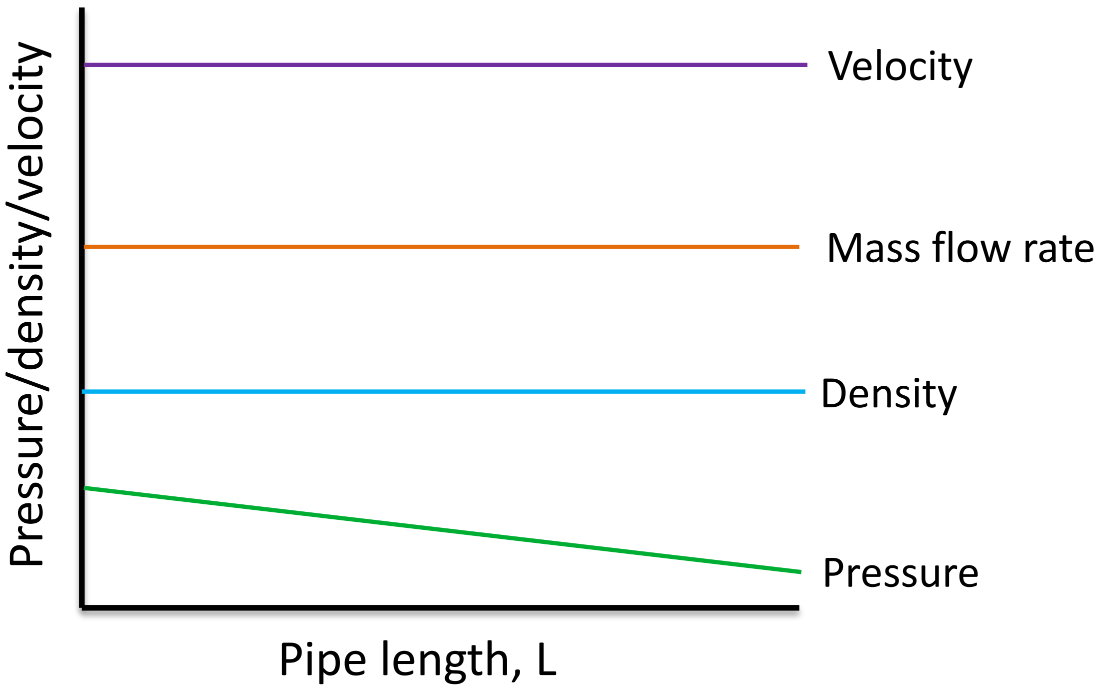
– If the fluid is a compressible gas, the scenario is much more interesting



What would happen here?

P, ρ, V, and Q change with L during liquid flow

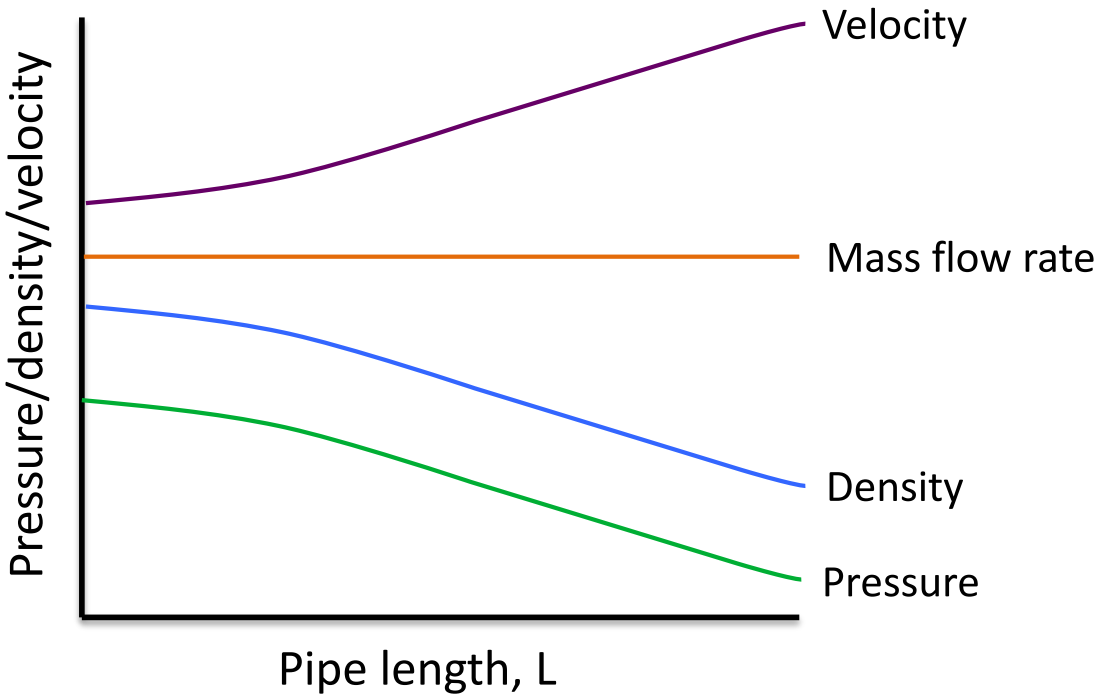
We can qualitatively plot **P**, ρ, **V**, and **G** versus pipe length, and it would look something like the plot below



During flow in a horizontal pipe of constant cross section, the pressure drop is linear, and the density, velocity, and mass flow rate are constant

P, ρ, V, and Q change with L during gaseous pipe flow

We can qualitatively plot **P**, ρ, **V**, and **G** versus pipe length, and it would look something like the plot below



For compressible gas flow through a horizontal pipe of constant cross section:

- The mass flow rate is constant down the length of the pipe
- The pressure decrease in a non-linear manner,
- As the pressure decreases, the gas expands causing the density to also decrease
- To maintain a constant mass flow rate, the velocity of the gas must increase
- The temperature may also vary down the length of the pipe

Is the flow of gas always compressible?

The mathematics associated with modeling of compressible flow is much more challenging

However, is all gas flow considered compressible?

While compression of gas will occur any time there is a pressure differential in the system, it is not always significant

- Gas flow can be modeled as **incompressible** whenever the changes in pressure in the system are **less than 20%**

- Gas flow must be modeled as **compressible** whenever the changes in pressure in the system are **greater than 20%**

Compressible flow and thermodynamics

Since we are discussing systems where pressure, density, and temperature are interrelated and change with one another, we must utilize thermodynamics in order to describe the flow of these fluids

In addition to **conservation of mass**, **conservation of energy**, and **conservation of momentum**, we will have to use **equations of state**

A review of the ideal gas law

As mentioned previously, we will need an equation of state to mathematically describe compressible flow. For the sake of simplicity, we will utilise the **ideal gas law**

$$PV = nRT$$

P =pressure
V = volume
n = number of moles of gas
R = ideal gas constant
T = absolute temperature (K)

However, we will slightly modify the equation so that it works better with our **mechanical energy balance**

A review of the ideal gas law

$$PV = nRT \longrightarrow PV = \frac{m}{M} RT$$

m = mass of gas
M = molecular weight

$$\longrightarrow PV = m\frac{RT}{M} \longrightarrow P = \frac{m}{V}\frac{RT}{M}$$

ρ = m/V

$$\longrightarrow P = \rho\frac{RT}{M} \longrightarrow \boxed{Pv = \frac{RT}{M}}$$

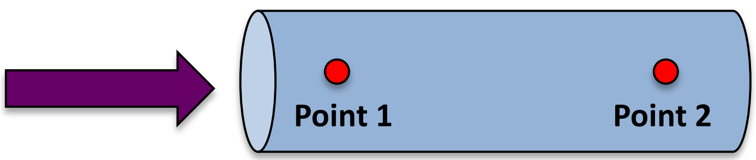
ν = 1/ρ

Isothermal vs adiabatic flow

In this class we will talk about gas flow under two different flow conditions

- **Isothermal**
- **Isentropic**

Isothermal is the easier concept to understand. During isothermal flow, the temperature of the gas is the same throughout the flow process.

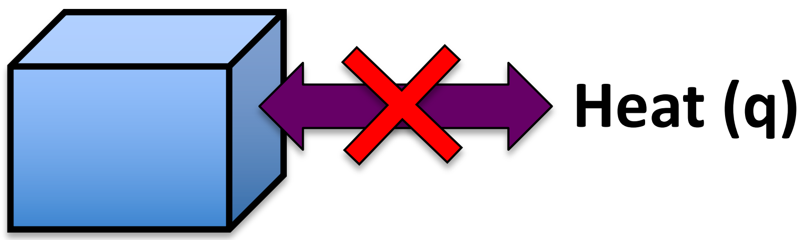


- Pressure drop is not necessarily linear
- $\rho_1 \neq \rho_2$
- $T_1 = T_2$

Isothermal vs isentropic flow

Isentropic flow is a more difficult concept. In this scenario it is the entropy that is constant throughout the flow field. For flow to be isentropic it must be **adiabatic** and **reversible**

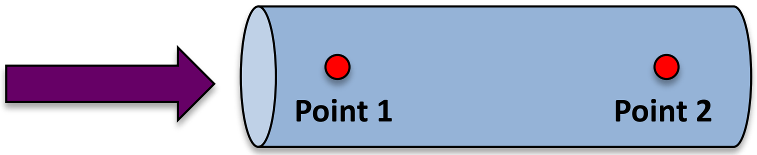
- An **adiabatic process** is one where there is no heat (q) exchanged with the surroundings. It is perfectly insulated



- An **isentropic** process is one where the entropy within the control volume is constant ($\Delta S = 0$). An isentropic system must be both adiabatic and **reversible**
  - A system is **reversible** when you can alter the system and can retain the original state by retracing the exact same thermodynamic path
  - Said another way, a **reversible** system is always at equilibrium, even during the change in state

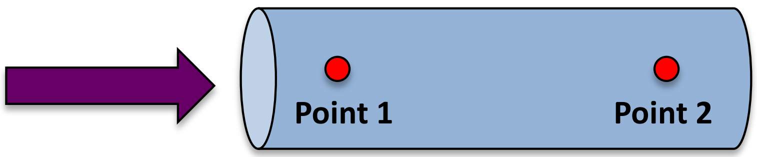
Relating pressure and density in compressible flow

We need an expression that relates the **pressure** and **density** (**specific volume**) of fluid elements in different parts of a compressible flow field assuming. The relationship is simple for **isothermal flow**



$$\frac{P_1}{P_2} = \left(\frac{v_2}{v_1}\right) = \left(\frac{\rho_1}{\rho_2}\right)$$

The relationship is slightly more complicated for **isentropic flow**



$$\frac{P_1}{P_2} = \left(\frac{v_2}{v_1}\right)^{\gamma} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

It turns out that the exponent, γ, is related to the **heat capacities**

How do you know the value for gamma, γ?

The parameter γ is the ratio of the specific heat capacities of the gas

$$\gamma = \frac{C_P}{C_v}$$

In order to use this relationship do you need to know the value of C_P and C_v for you gas? If you're dealing with a complex mixture of gases (say, natural gas) that could be variable and complex…

Interestingly, the value for gamma depends heavily on the number of molecules in the atoms that make up the gas

Atoms per molecule	**Value of γ**
Monatomic (He, Ar)	1.67
Diatomic (H_2, O_2, N_2)	1.40
Triatomic (CO2)	1.30
Complex molecules	~1.05

So what is the γ value for air?

Summary

In this lesson we began discussion of compressible flow

Compressible flow is significant when the pressure changes in the system are > 20%

When that occurs we see pressure, density, and temperature variations in our flow. This means in addition to our energy, mass, and momentum balances, we need an equation of state to mathematically describe the system

We derived an expression that links the pressure and density of an ideal gas undergoing either an isentropic process or an isothermal process

Fluid Mechanics

Compressible flow part 2:
Propagation of pressure waves &
water hammer

Last lesson

During last lesson we introduced the concept of **compressible flow** where the density of the fluid changes with pressure

$$\boxed{\frac{\partial \rho}{\partial P} \neq 0} \qquad \boxed{\rho = f(P)}$$

We then used **thermodynamics** to derive a relationship on how the **pressure** and **density** (**specific volume**) of the compressible fluid are interrelated under **isentropic** or **isothermal** conditions

$$\boxed{\frac{P_1}{P_2} = \left(\frac{v_2}{v_1}\right)^{\gamma} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}} \qquad \boxed{\frac{P_1}{P_2} = \left(\frac{v_2}{v_1}\right) = \left(\frac{\rho_1}{\rho_2}\right)}$$

This lesson

This less will cover two smaller topics and begin one larger topic

1. **Pressure wave** propagation in compressible flow

2. A consequence of pressure waves: **water hammer**; when fluid comes to a sudden halt

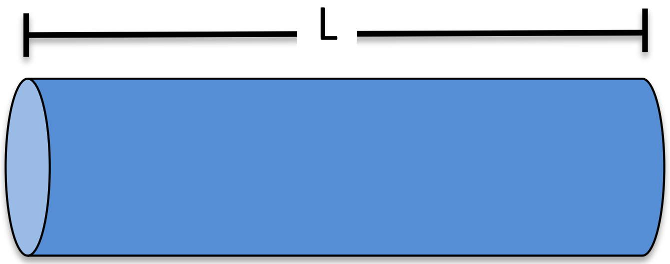
3. Steady state **mechanical energy balance** for gas flow

Pressure distribution and liquid velocity

In the last lesson we alluded to the fact that flow in compressible gases is more complex than in liquids

- Let us first analyze how pressure, density, and fluid velocity are interrelated for an **incompressible liquid**
- We will begin with a pipe of length L filled with liquid initially at rest (pressure = P_0 = 0, velocity = v_0 = 0)

At $t < t_0$, $P_0 = 0$

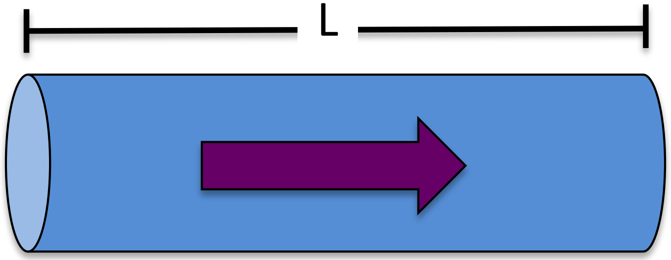


L = 0	L = L
$V_0 = 0$	$V_0 = 0$
$\rho_0 = \rho$	$\rho_0 = \rho$

Pressure distribution and liquid velocity

At some time = t, we increase the pressure at front of the pipe by a fixed amount, ΔP, to initiate flow of the liquid

At $t \geq t_0$, $P_t = \Delta P$



L = 0
V = V
$\rho = \rho$

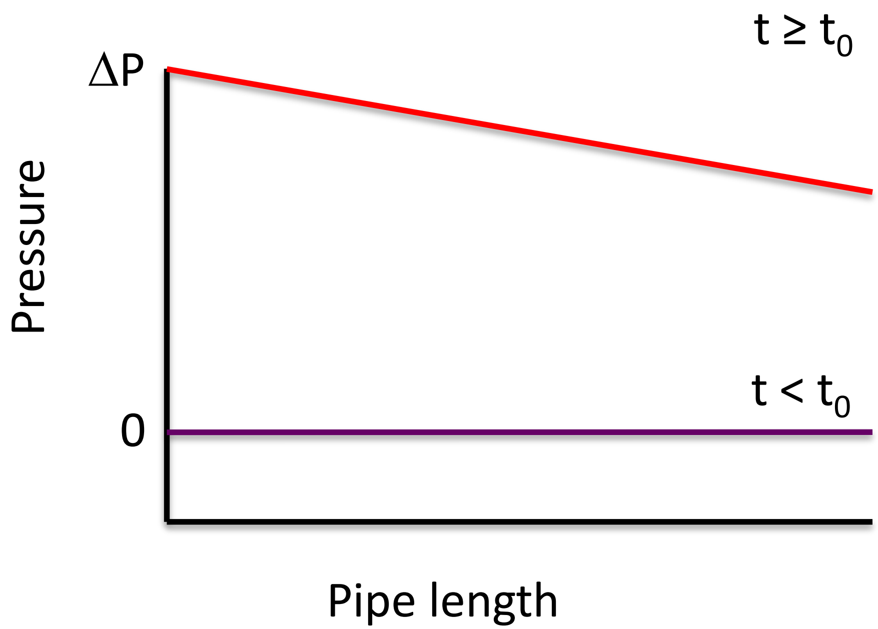
L = L
V =
ρ =

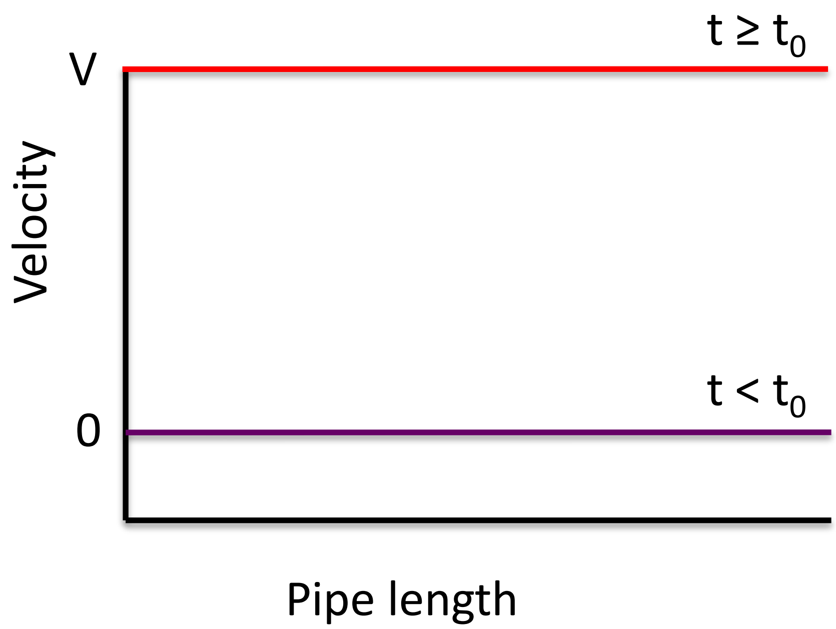
For an incompressible fluid a change in pressure is felt everywhere within the fluid immediately

The pressure change moves through the fluid with infinite velocity so that the fluid velocity changes everywhere instantaneously

Pressure distribution and liquid velocity

At some time = t_0, we increase the pressure at front of the pipe by a fixed amount, ΔP, to initiate flow of the gas





For an incompressible fluid a change in pressure is felt everywhere within the fluid immediately

The pressure change moves through the fluid with infinite velocity so that the fluid velocity changes everywhere instantaneously

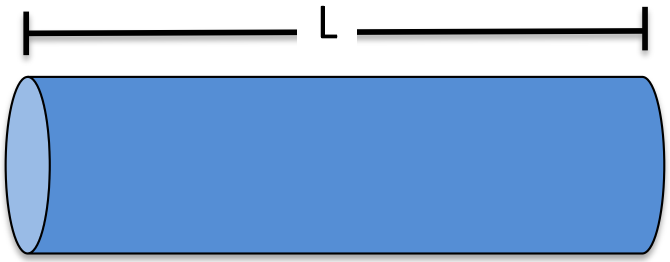
Pressure distribution and gas velocity

Let us explore the same scenario for a **compressible gas**

- We will begin with a pipe of length L filled with gas initially at rest (pressure = P_0 = 0, velocity = v_0 = 0)

What do you think will happen when we increase the pressure at the beginning of the pipe?

At t = 0, P_0 = 0

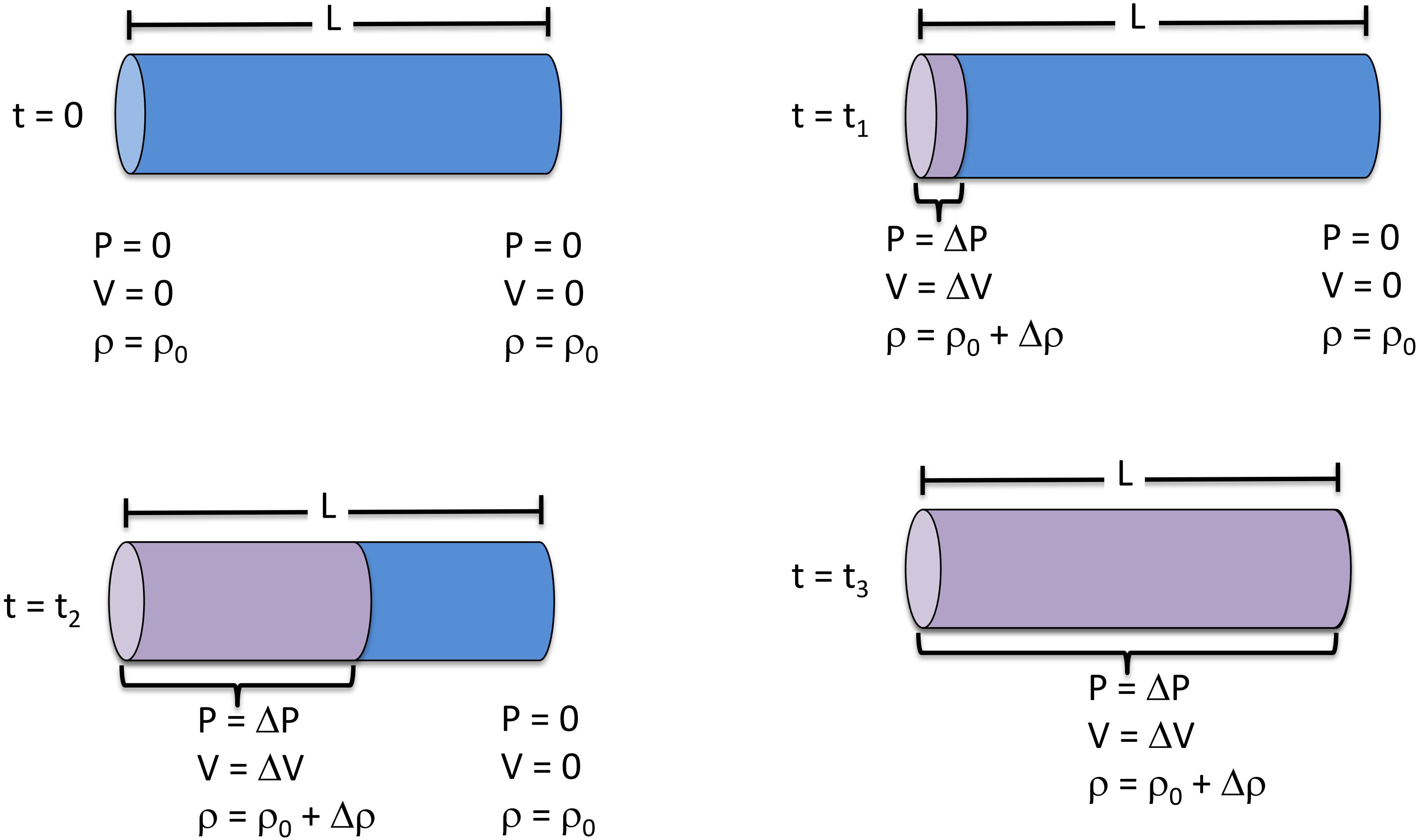


L = 0
$V_0 = 0$
$\rho_0 = \rho$

L = L
$V_0 = 0$
$\rho_0 = \rho$

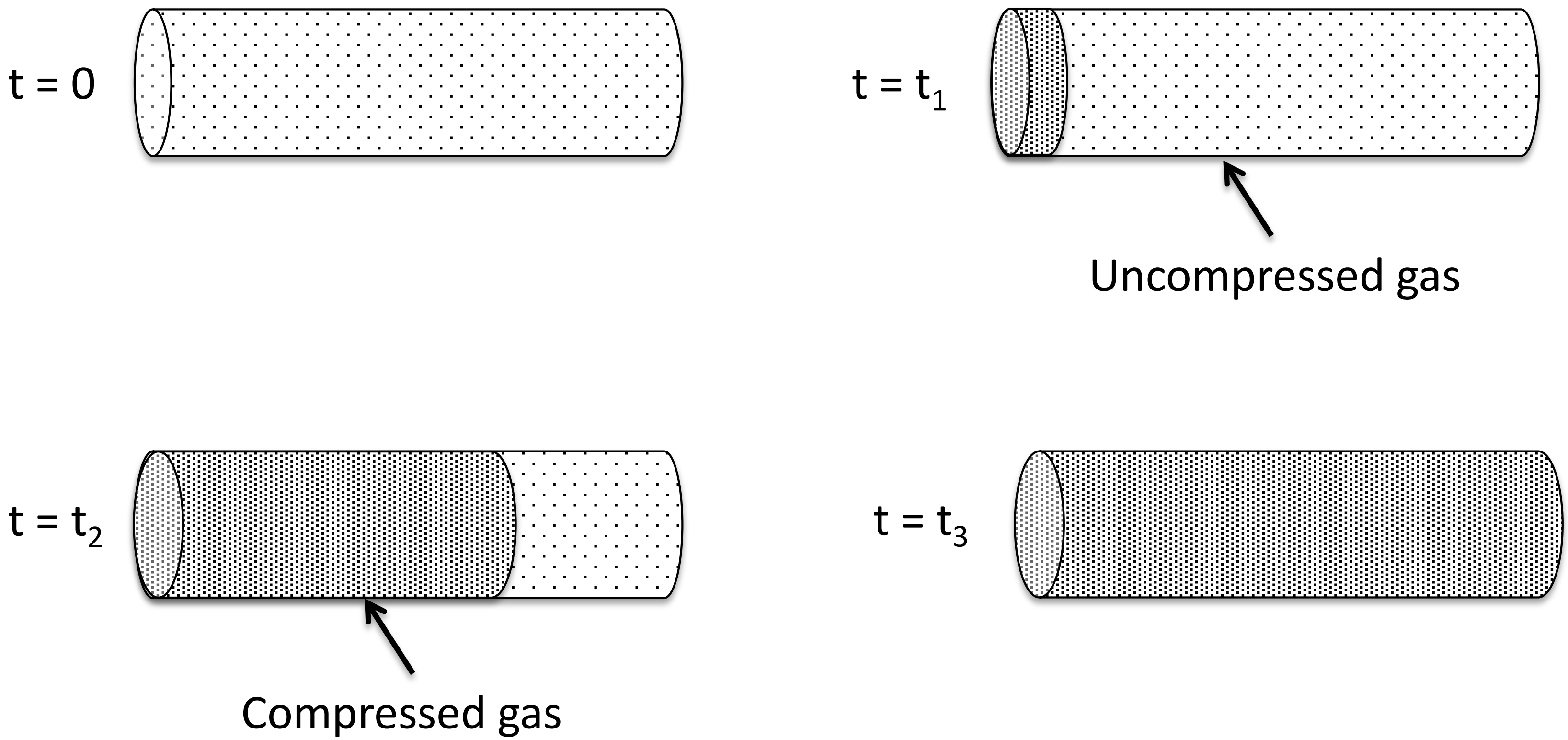
Pressure distribution and gas velocity

The change in gas pressure, density, and velocity moves down the length of pipe as a pressure wave



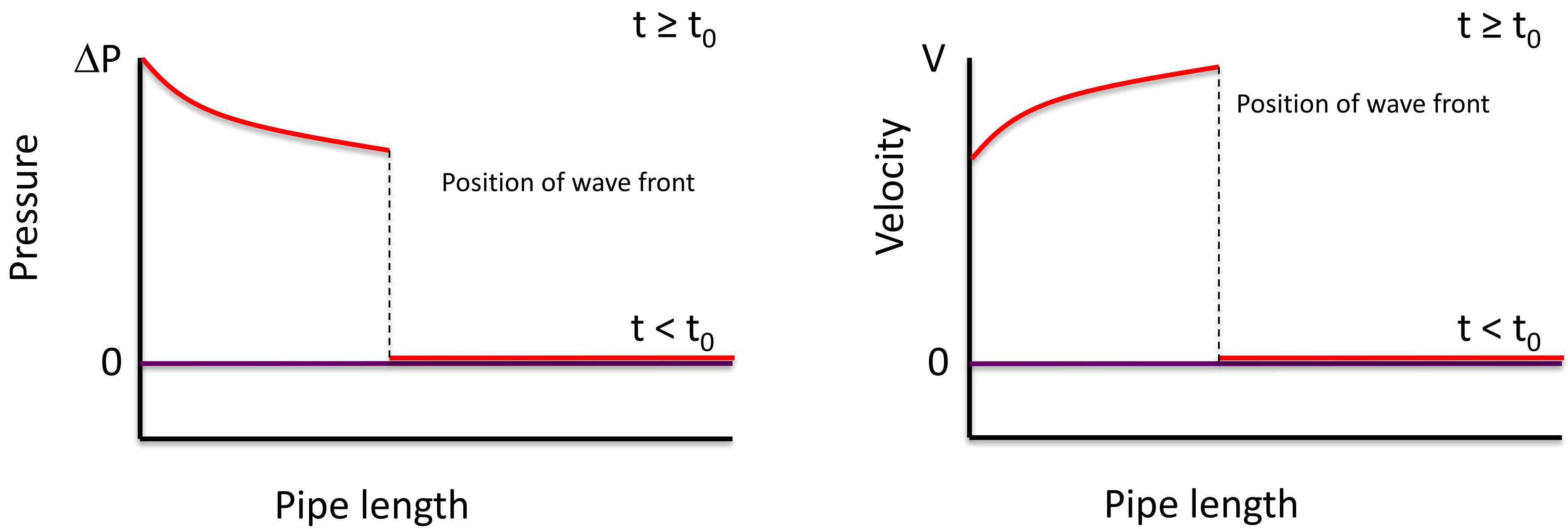
Why does this occur?

The pressure wave is transmitted by molecular collision. It is not instantaneous because an increase in pressure at one point (i.e. the beginning of the pipe) first compress the local fluid without affecting the fluid elsewhere in the pipe



Pressure distribution and gas velocity

At some time = t_0, we increase the pressure at front of the pipe by a fixed amount, ΔP, to initiate flow of the gas

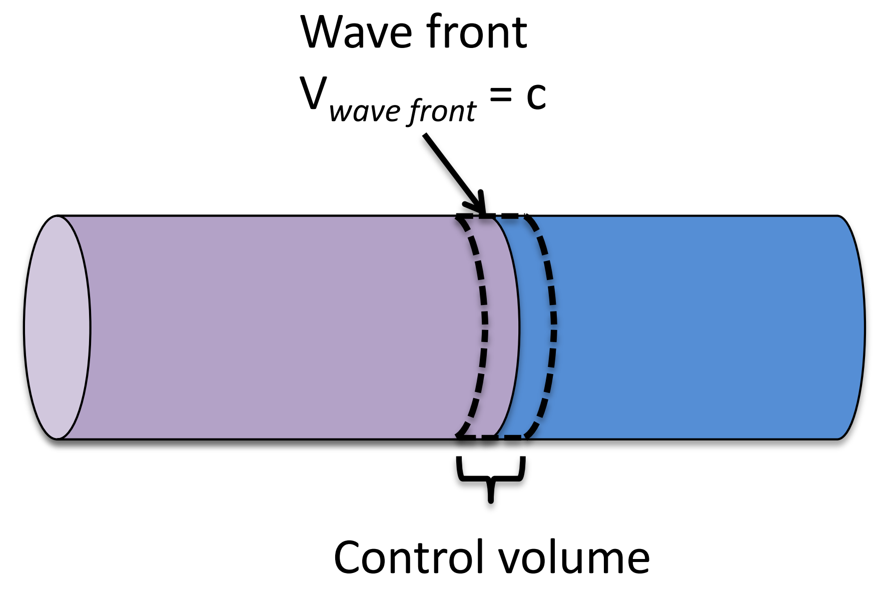


For a compressible gas, changes in pressure travel as waves

Additionally the pressure drop is non-linear and the velocity of the gas increases down the length of the pipe

How fast does this pressure wave travel?

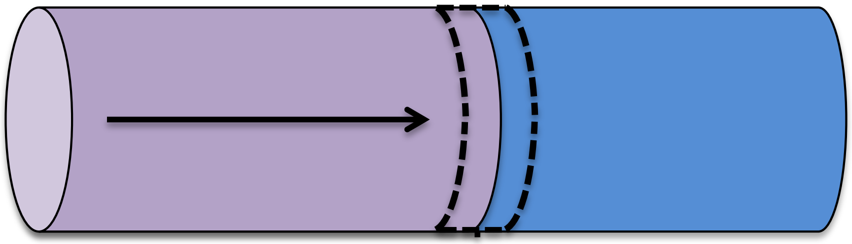
We can solve for the **velocity of the pressure wave**, **c**, by applying **conservation of mass** and **conservation of linear momentum** to a control volume centered on the wave front



The control volume is moving with the pressure wave. When writing the conservation equations we will use the frame of reference of the control volume that is moving at a velocity of c.

Frame of reference: the pressure wave

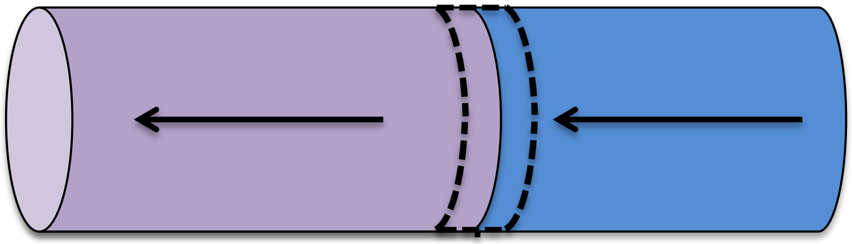
From an external frame of reference



$V = \Delta V$
$P = P_0 + \Delta P$
$\rho = \rho_0 + \Delta\rho$

$V = 0$
$P = P_0$
$\rho = \rho_0$

From the moving wave's frame of reference



$V = -(c - \Delta V)$
$P = P_0 + \Delta P$
$\rho = \rho_0 + \Delta\rho$

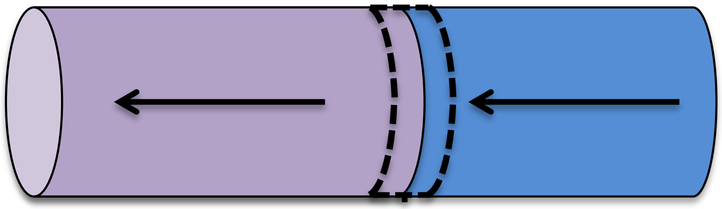
$V = -c$
$P = P_0$
$\rho = \rho_0$

Think of riding in a car. In reality you are approaching your surroundings at a given velocity, but it looks like the surroundings are approaching you with the opposite velocity

Similarly, car behind you approaches at a velocity that is the difference between yours and its

Continuity (mass balance) around wave

Continuity tells us that as the pressure wave passes a certain point the total mass must remain the same.



$V = -(c - \Delta V)$
$P = P_0 + \Delta P$
$\rho = \rho + \Delta\rho$

$V = -c$
$P = P_0$
$\rho = \rho$

Let us write a mass balance around the wave in a pipe with constant cross section, A.

– The mass can be expressed as **velocity*****density*****area**$_{cross\ section}$ [=] mass/time

$$-(c-\Delta V)*(\rho+\Delta\rho)*A = -c*\rho*A$$

Continuity (mass balance) around wave

We can now simplify the mass balance

$$-(c-\Delta V)*(\rho+\Delta\rho)*A=-c*\rho*A$$

We can cancel the negative velocity signs and the cross section area

$$(c-\Delta V)(\rho+\Delta\rho)=c\rho$$

Expand left hand side

Negligible for small changes

$$\cancel{c\rho}+c\Delta\rho-\rho\Delta V-\cancel{\Delta V\Delta\rho}=\cancel{c\rho}$$

$$\boxed{c\Delta\rho=\Delta V\rho}$$

Result from the mass balance

Linear momentum balance around wave

Earlier in the semester you learned the mathematical definition for the conservation of linear momentum

$$\underbrace{\sum \vec{F}}_{\text{Sum of forces on cross section}} = \underbrace{\int_{CS} \rho \vec{V} (\vec{V}_r \bullet \hat{n}) dA}_{\text{Rate change of momentum across cross section}}$$

We can re-write this expression in terms of our system parameters

- The forces on our control volume will be in terms of the pressure of our system
- The rate change of momentum will be in terms of the velocity of the fluid

$$PA - (P + \Delta P)A = -\rho c c A + \rho c (c - \Delta V) A$$

Linear momentum balance across wave

We can now simplify the momentum balance

$$PA - (P + \Delta P)A = -\rho ccA + \rho c(c - \Delta V)A$$

Expand the relationship and simplify

$$\cancel{P}A - \cancel{P}A - \Delta P\cancel{A} = -\rho c\cancel{c}A + \rho c\cancel{c}A - \rho c\Delta V\cancel{A}$$

$$\Delta P = \rho c \Delta V \quad \Rightarrow \quad \boxed{\Delta V \rho = \frac{\Delta P}{c}}$$

Combining mass and momentum balance

We derived the following relationships from our **mass balance** and **momentum balance**

$$c\Delta\rho = \Delta V\rho \qquad\qquad \Delta V\rho = \frac{\Delta P}{c}$$

We can combine these two expressions through substitution of $\Delta v\rho$

$$c^2 = \frac{dP}{d\rho}$$

In order to utilize this expression we must use an **equation of state** that relates pressure and density within our system

Combining mass and momentum balance

We must use an equation of state that relates pressure and density

Equation of state & $c^2 = \dfrac{dP}{d\rho}$ → ***Velocity of wave front***

Ideal gas and isothermal

$$\underbrace{\frac{P}{\rho} = \frac{RT}{M}}_{\text{Ideal gas law}} = \underbrace{\text{constant}}_{\text{Isothermal}}$$

$$\Rightarrow \frac{dP}{d\rho} = \frac{RT}{M}$$

$$\Rightarrow \boxed{c = \sqrt{\frac{RT}{M}}}$$

Ideal gas and isentropic

$$\boxed{c = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma RT}{M}}}$$

Where $\gamma = C_P/C_V$

How fast does the pressure wave travel?

How fast would the pressure wave travel if the fluid of interest was air with a temperature of 20°C?

Ideal gas and isothermal

$$c = \sqrt{\frac{RT}{M}}$$

R = 8.314 J/K*mol
T = 293.14 K
M_{air} = 28.97 g/mol

$$c = \sqrt{84127\, m^2/s^2}$$

$$\boxed{c = 290\, m/s}$$

Ideal gas and isentropic

$$c = \sqrt{\frac{\gamma RT}{M}}$$

Same values for R, T, and M
γ = 1.4

$$\boxed{c = 343.18\, m/s}$$

To put this into perspective the speed of sound at 20°C is 343.2 m/s

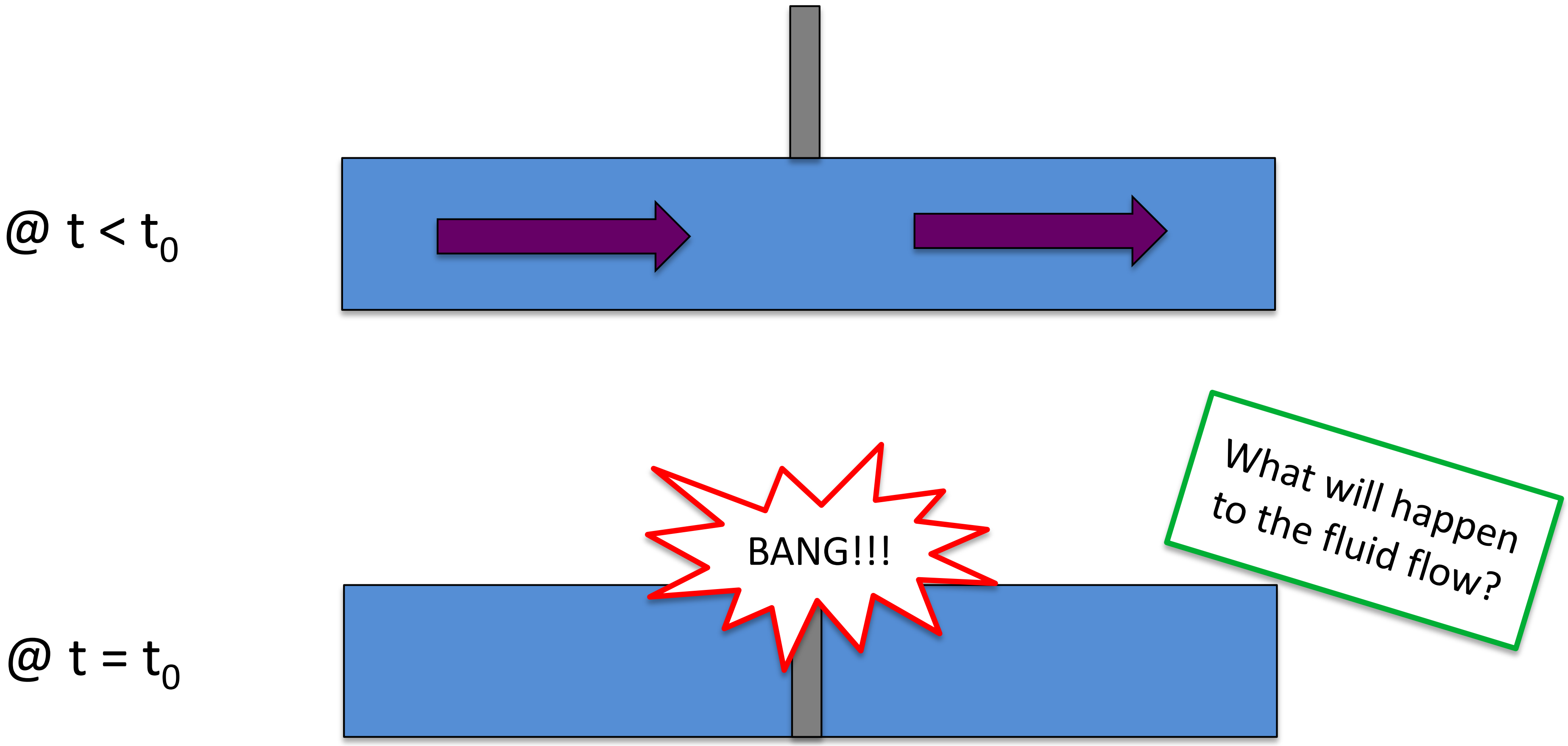
Water hammer

This water bending technique should be called **water hammer** because flowing water can carry a lot of momentum along with it and cause quite a lot of damage – **especially when it becomes compressed!!!**

Water hammer

You have a fluid flowing through a pipe, it carries considerable momentum. Suddenly closing a valve, halting the flow of fluid, can produce very large pressure changes

- Say a fluid is flowing in the below pipe and we suddenly close the valve at t = t0



Consequences of compressible flow

One of the consequences of compressible flow is the phenomena referred to as **water hammer**

Water hammer occurs when a flowing fluid is brought to a sudden halt such when a valve is suddenly closed

The name water hammer is derived because of the sound generated in water pipes. When you close a valve (for instance, turn off a faucet) you can hear a banging sound

What hammer

Previously stated that **liquids** are **not compressible**. However, the pressure changes that occur during **water hammer** are enough to cause **liquid compression**. Water hammer can occur in **both liquids and gases**

Upstream. The flowing column of fluid rams into a solid barrier, the fluid compresses, pressure rises, and can be great enough to expand the cross section of the pipe. A positive pressure wave will then propagate upstream

Downstream. Fluid will continue to flow; however, a vacuum will form near the valve (cavitation), pressure decreases, and can be enough to contract the cross section of the pipe. A negative pressure wave then propagates downstream

Consequences of water hammer

Is water hammer actually important or does it just cause annoying noises in pipes?

It turns out that water hammer can generate very large forces which can cause physical damage to your process and be dangerous to workers

Expansion joints destroyed by steam hammer

Heat exchanger piping destroyed by water hammer

Consequences of water hammer

Videos that illustrate the physics of water hammer and the consequences of the phenomenon

- Video 1. High speed video to illustrate what happens during water hammer
- Video 2. How water hammer can easily cause damage
- Video 3. Industrial scale accident occurring due to water hammer
- Video 4. A method that can be used to mitigate water hammer

Summary

Pressure waves

- We discovered how pressure changes in gas flow to not affect the entire system instantaneously as happens with liquid systems
- Instead, the pressure changes travel as a wave that takes a finite amount of time to propagate through the system
- The speed of the pressure waves is high, almost the speed of sound

Water hammer

- Usually fluids are incompressible. One scenario where they display compressible flow is water hammer (which can occur in both liquids and gases)
- This occurs when a flowing liquid is brought to a sudden halt generating large pressure changes that can damage the system and surroundings

Fluid Mechanics

Compressible flow part 3:

Mechanical Energy Balance & Choked flow

Last lesson

Pressure waves occur in compressible flow

- In liquids a change in system pressure is felt everywhere immediately. In compressible flow it takes a finite amount of time for the pressure change to propagate through the system
- This is because the gas compresses due to changes in pressure and this results in a pressure wave traveling through the system
- The speed of these waves is very rapid, near the speed of sound

Water hammer occurs when flow – either liquid or gas – is suddenly halted such as by the rapid closing of a valve

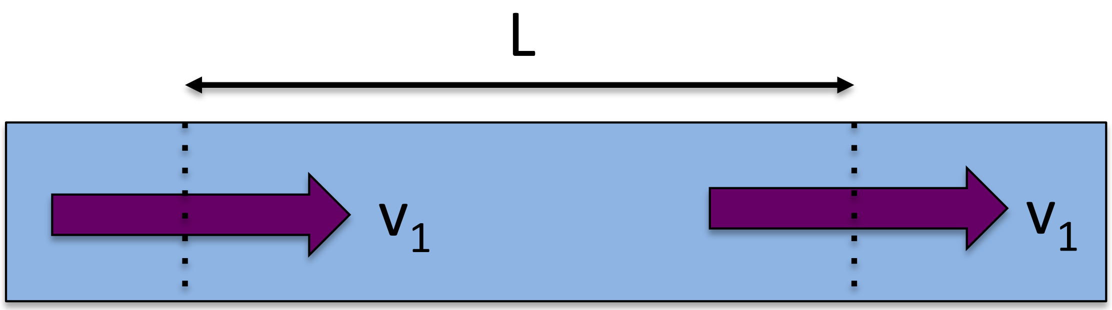
- Water hammer occurs when a fluid compresses and creates a pressure wave that propagates along the pipe
- This can cause damage to the system and surroundings

Mechanical energy balance for compressible fluids

Now let us (begin) to derive the mechanical energy balance for a compressible gas

Let us analyze the schematic of the pipe of constant cross section below

- For an **incompressible liquid**, **velocity is constant** mean that v_1 must equal v_2 because of continuity
- However, **velocity is not constant** for a **compressible gas**. As the density can change within the flow, the velocities can also vary
- Instead, for a **compressible gas** the **mass flow rate is constant** $\boxed{G = \rho V A}$



Mechanical energy balance for compressible fluids

Let us begin with the mechanical energy balance that you derived earlier in the course

$$\frac{\Delta P}{\rho} + \Delta\left(\frac{V^2}{2\alpha}\right) + g\Delta z + W_s + F = 0$$

(for constant density)

Now we will simplify the expression somewhat

Turbulent flow regime → $\alpha = 1$

No shaft work → $W_s = 0$

Horizontal pipe → $\Delta z = 0$

$$\frac{\Delta P}{\rho} + \Delta\left(\frac{V^2}{2}\right) + F = 0$$

Mechanical energy balance for compressible fluids

Apply mechanical energy balance for constant density over a small length of pipe Δx (eventually we will integrate over the length of pipe) and including the expression for the frictional work in terms of phi

$$\frac{\Delta P}{\rho} + \Delta\left(\frac{V^2}{2}\right) + F = 0 \quad \Longrightarrow \quad \frac{\Delta P}{\rho} + \Delta\left(\frac{V^2}{2}\right) + 4\phi\frac{\Delta x V^2}{D} = 0$$

However, our velocity is not constant, it is the mass flow rate, G, that is constant. We need to manipulate our expression to put in terms of G

Previously we related V and G $\quad V = \frac{G}{\rho A} = \frac{Gv}{A}$

$$\Longrightarrow \quad v\Delta P + \left(\frac{G}{A}\right)^2 \Delta\left(\frac{v^2}{2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 v^2 \Delta x = 0$$

Mechanical energy balance for compressible fluids

We can simplify the expression a little more by dividing by v^2

$$\frac{\Delta P}{\rho} + \Delta\left(\frac{V^2}{2}\right) + 4\phi\frac{\Delta x V^2}{D} = 0 \quad \Rightarrow \quad \frac{\Delta P}{v} + \left(\frac{G}{A}\right)^2 \frac{\Delta v}{v} + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 \Delta x = 0$$

Now we can integrate this expression over the length of the pipe

$$\int_{P_1}^{P_2} \frac{dP}{v} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{v_2}{v_1}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

In order to complete the integration, we will need an equation of state

M.E.B. will depend on system

In this lesson, we will derive the mechanical energy balance for compressible flow for two distinct conditions

- Case 1. Compressible flow of an **ideal gas** under **isothermal conditions**
- Case 2. Compressible flow of an **ideal gas** under **adiabatic conditions**

$$\int_{P_1}^{P_2} \frac{dP}{v} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{v_2}{v_1}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

Case 1

Isothermal flow of a compressible ideal gas

Case 2

Adiabatic flow of a compressible ideal gas

Along the way will also discover the phenomenon of **choked flow**

Case 1. Isothermal flow

Derive the mechanical energy balance for isothermal flow of a compressible ideal gas

$$\underline{\int_{P_1}^{P_2} \frac{dP}{v}} + \underline{\left(\frac{G}{A}\right)^2 \ln\left(\frac{v_2}{v_1}\right)} + \underline{\frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L} = 0$$

We want to put the second term in terms of pressure instead of specific volume. We derived an expression that related these two properties for an ideal gas during our first lesson on compressible flow

$$\ln\left(\frac{v_2}{v_1}\right) = \ln\left(\frac{P_1}{P_2}\right) \xrightarrow{\text{Substitution}} \underline{\int_{P_1}^{P_2} \frac{dP}{v}} + \underline{\left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right)} + \underline{\frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L} = 0$$

Now we will employ an equation of state, the ideal gas law, to simplify the first term

Case 1. Isothermal flow

Derive the mech eng balance for compressible isothermal ideal gas flow

$$\int_{P_1}^{P_2} \frac{dP}{v} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

Previously, we wrote the ideal gas law in terms of the specific volume, ν

$$PV = nRT \quad \Rightarrow \quad Pv = \frac{RT}{M} \quad \Rightarrow \quad \frac{1}{v} = P\frac{M}{RT}$$

Substitute 1/ν into our mechanical energy balance

$$\int_{P_1}^{P_2} \frac{dP}{v} = \frac{M}{RT}\int_{P_1}^{P_2} P\,dP = \frac{P_2^2 - P_1^2}{2(RT/M)}$$

$$\Rightarrow \quad \boxed{\frac{P_2^2 - P_1^2}{2(RT/M)} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0}$$

An inconsistency in the mech energy balance?

Derive the mechanical energy balance for compressible isothermal ideal gas flow in a horizontal pipe of constant cross section

$$\frac{P_2^2 - P_1^2}{2(RT/M)} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

This expression can only be used when you have an ideal gas under isothermal conditions in a horizontal pipe of constant cross section

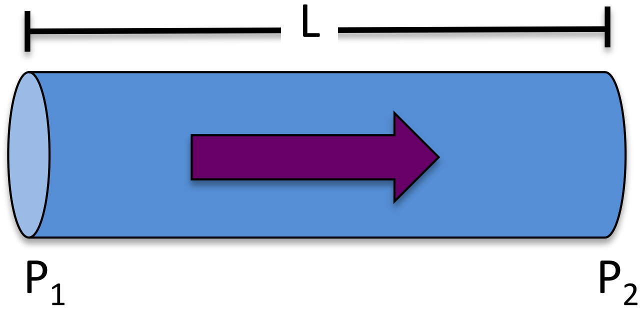
We will derive more complete expressions later that can be used under a wider variety of conditions

However, first let us analyze this equation some to see what else we can discover about gas flow

An inconsistency in the mech energy balance?

The mechanical energy balance for compressible isothermal ideal gas flow in a horizontal pipe of constant cross section

$$\frac{P_2^2 - P_1^2}{2(RT/M)} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$



How does pressure drop across the pipe affects the mass flow rate, G

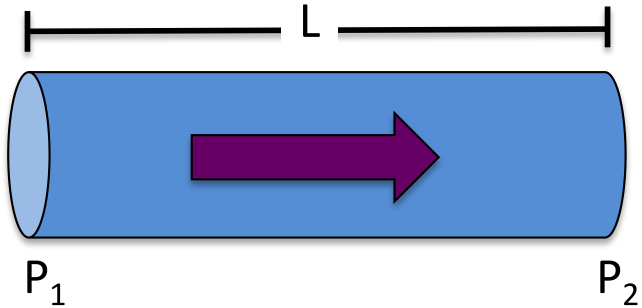
- Scenario 1. If $P_1 = P_2$

- Scenario 2. If P_1 is a positive and non-zero value and $P_2 = 0$

An inconsistency in the mech energy balance?

I can understand how scenario 1 leads to no flow, but why does scenario 2 lead to no flow?

$$\frac{P_2^2 - P_1^2}{2(RT/M)} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$



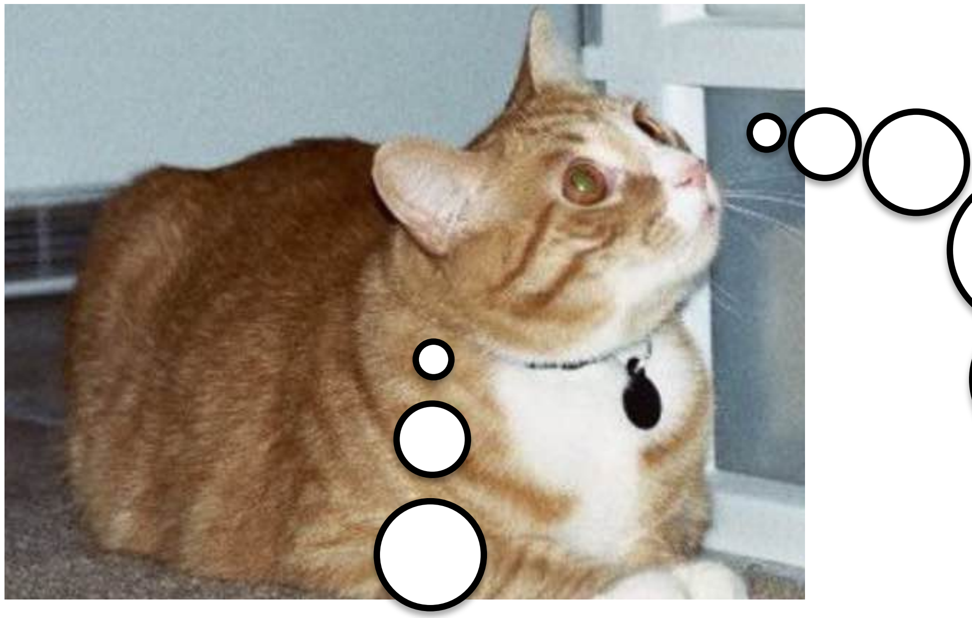
If P_1 is a non-zero value and $P_2 = 0$, G = 0. Why?

- Evaluate expression at the above conditions

$$\frac{-P_1^2}{2(RT/M)} + \underline{\left(\frac{G}{A}\right)^2 \ln(\infty)} + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

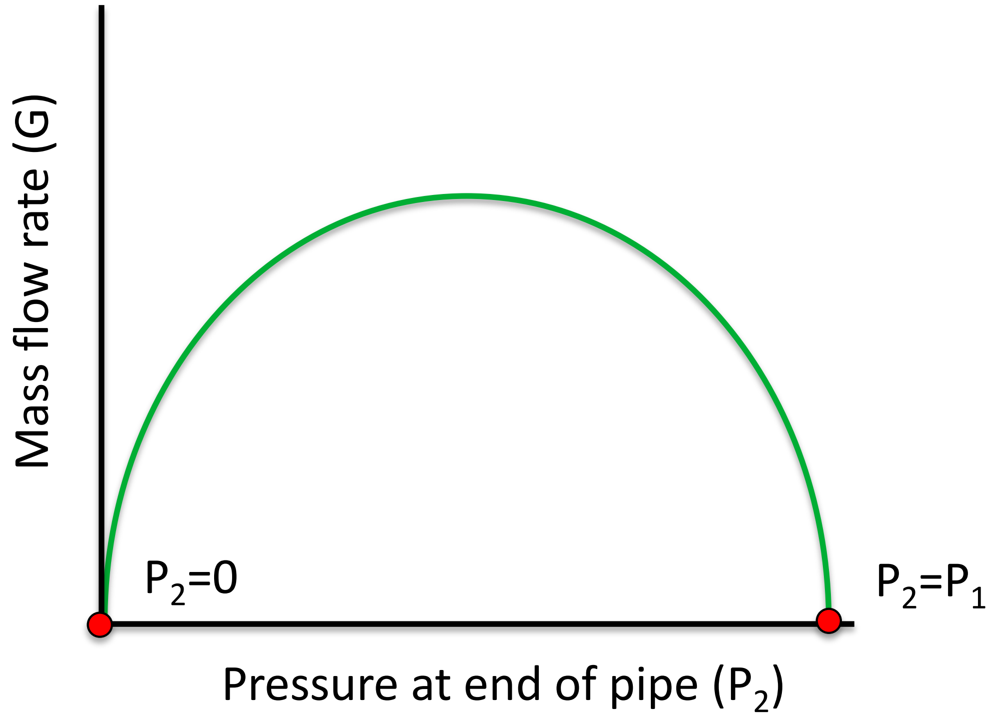
The second term in this expression goes to infinite. The only way for this equality to hold is if G = 0

An inconsistency in the mech energy balance?



I is confused. Flow rate is supposed to increase with increasing pressure drop. How can the flow rate possibly be zero when the pressure drop is non-zero?

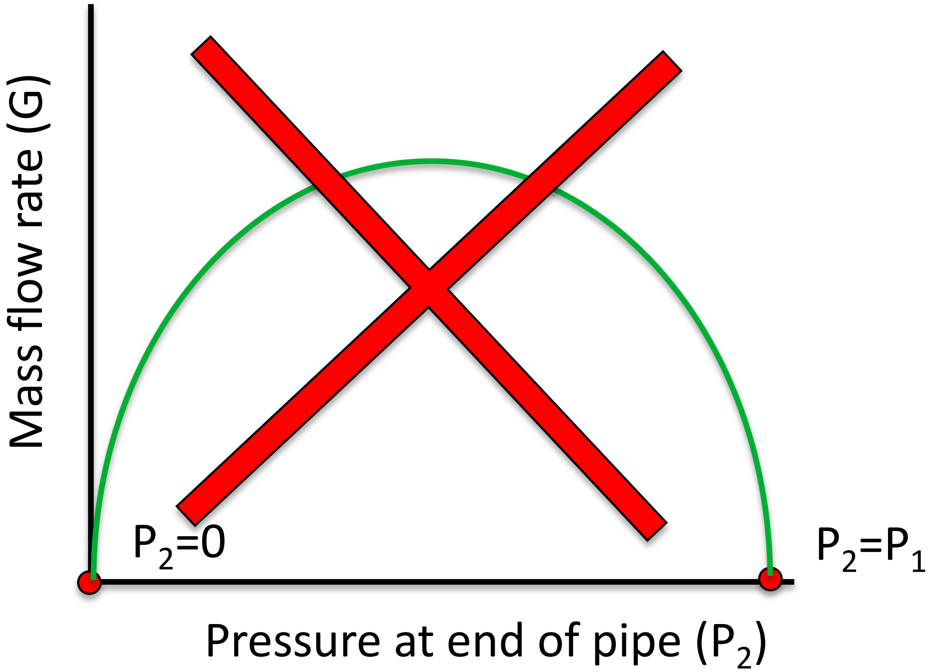
We know that initially flow rate does increase with increasing pressure drop. If both these scenarios are true, that means G must go through a maximum?!

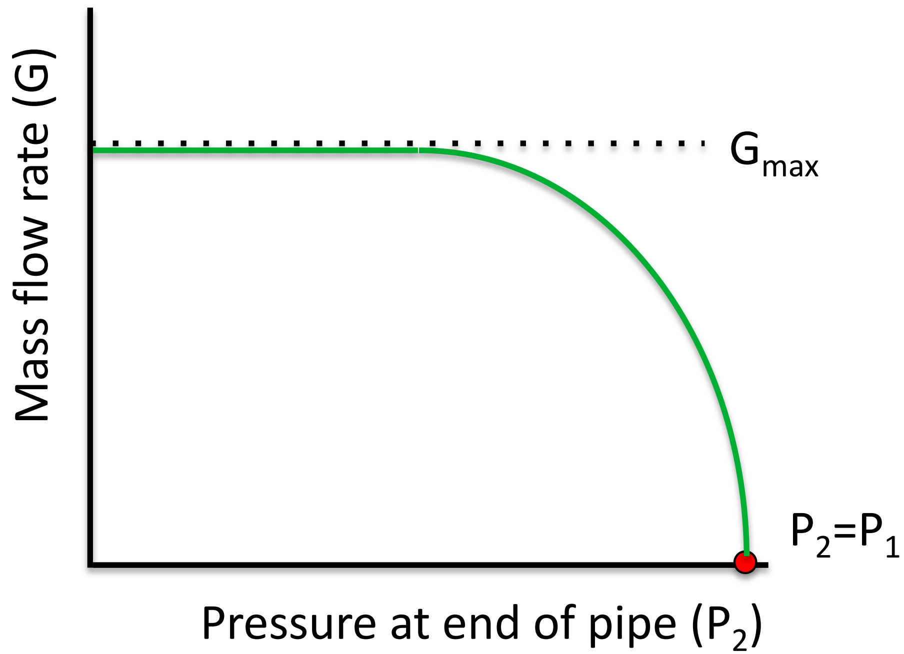


An inconsistency in the mech energy balance?

Fudge the Cat is correct, this is physically not a reasonable scenario. Furthermore, this phenomenon is not supported by experimental data

What is experimentally observed is that G will increase as P_2 decreases, but only to a certain point. **There is a maximum flow rate of G** beyond which further decreases in P_2 will not cause more flow





Finding the maximum flow velocity

The mechanical energy balance for compressible flow will only predict the correct mass flow rate, G, until it reaches its maximum value

- We can use calculus in order to determine the maximum value of flow velocity
- The maximum (or minimum) of a function occurs when the first derivative is zero, $dG/dP_2 = 0$
- You will need to use the product rule: (f*g)' = f'*g + f*g'

$$\frac{P_2^2 - P_1^2}{2(RT/M)} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

Divide both sides by $(G/A)^2$

$$\left(\frac{A}{G}\right)^2 \frac{P_2^2 - P_1^2}{2(RT/M)} + \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D} L = 0$$

Differentiate with respect to P_2

Remember $G=f(P_2)$

Finding the maximum flow velocity

Continue taking the derivative of $G = f(P_2)$ in order to find G_{max}

$$A^2\left(\frac{-2}{G^3}\frac{dG}{dP_2}\right)\frac{P_2^2 - P_1^2}{2(RT/M)} + \left(\frac{A}{G}\right)^2\frac{2P_2}{2(RT/M)} - \frac{1}{P_2} = 0$$

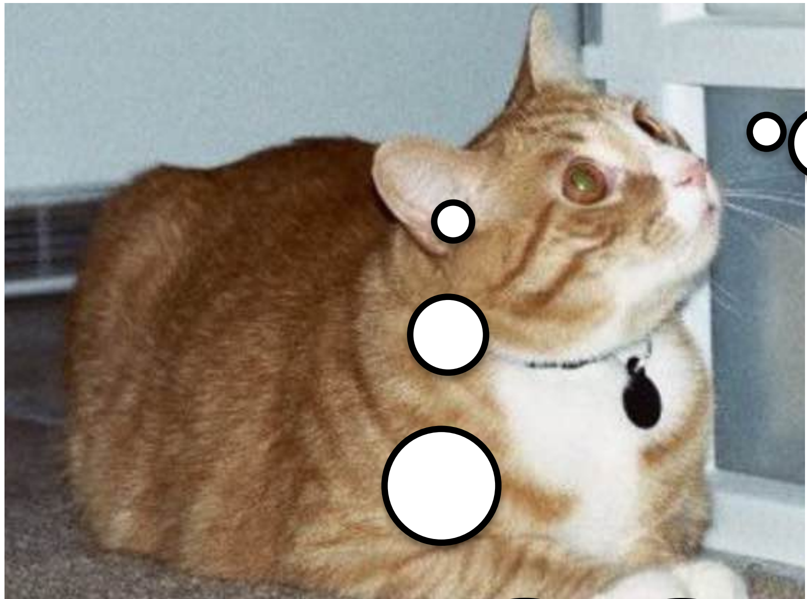
Let $dG/dP_2 = 0$. This corresponds to $G = G_{max}$. At this point let $P_2 = P_w$

$$\left(\frac{A}{G_{max}}\right)^2\frac{P_w}{(RT/M)} = \frac{1}{P_w} \xrightarrow{\text{Rearrange}} \left(\frac{G_{max}}{A}\right)^2 = \frac{P_w^2}{(RT/M)}$$

Substitute in expression for $G_{max} = \rho_w V_w A$ and ideal gas law $P_w = \rho_w RT/M$

$$(\rho_w V_w)^2 = \frac{(\rho_w RT/M)^2}{(RT/M)} \longrightarrow V_w = \sqrt{RT/M}$$

Finding the maximum flow velocity



Wait a minute… we've seen that expression before…

Wasn't that the expression for c, speed of the pressure wave in an isothermal ideal gas?!

You know, that makes sense. Both the pressure wave and sound are propagated by molecular collisions. It make sense that the maximum gas flow rate is equal to the speed of sound.

Fudge is correct. The maximum value of gas flow is the same speed at which our pressure wave propagated in yesterday's class.

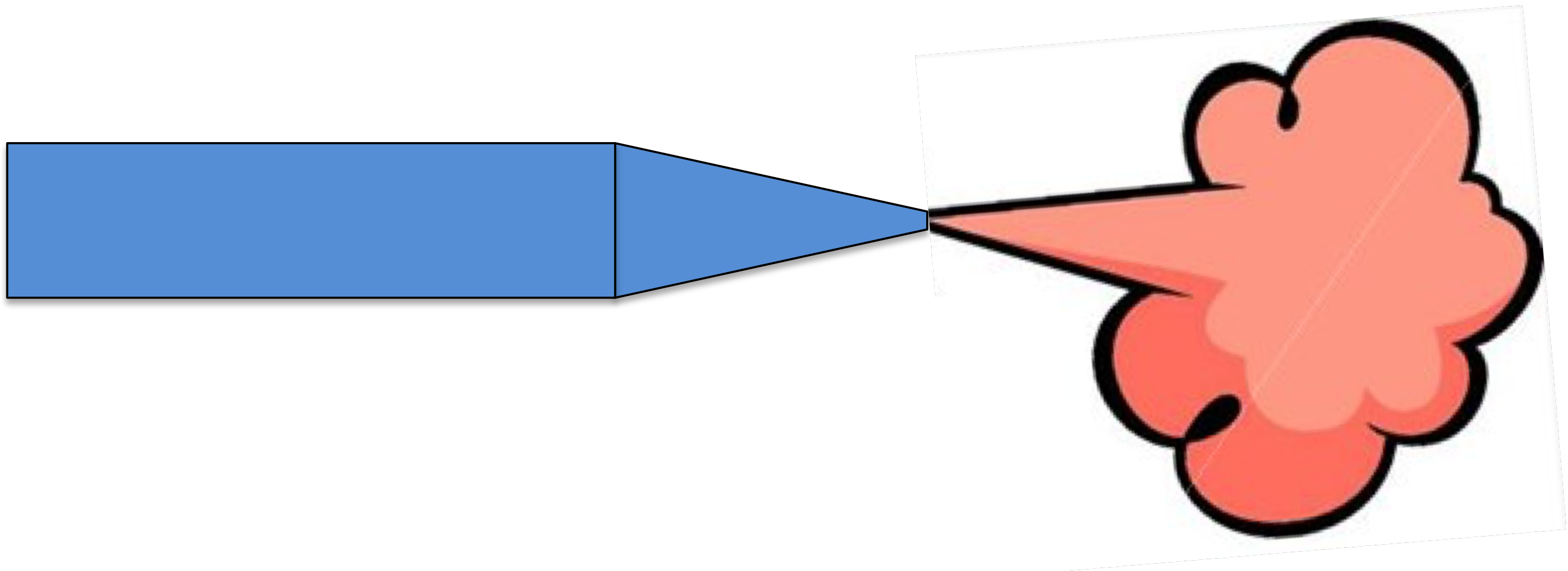
This value has yet another significance. It is also the speed of sound in isothermal flow

If we performed the same derivation for isentropic flow you would arrive at the same expression for the pressure wave speed for isentropic flow

Is this actually important in real life?

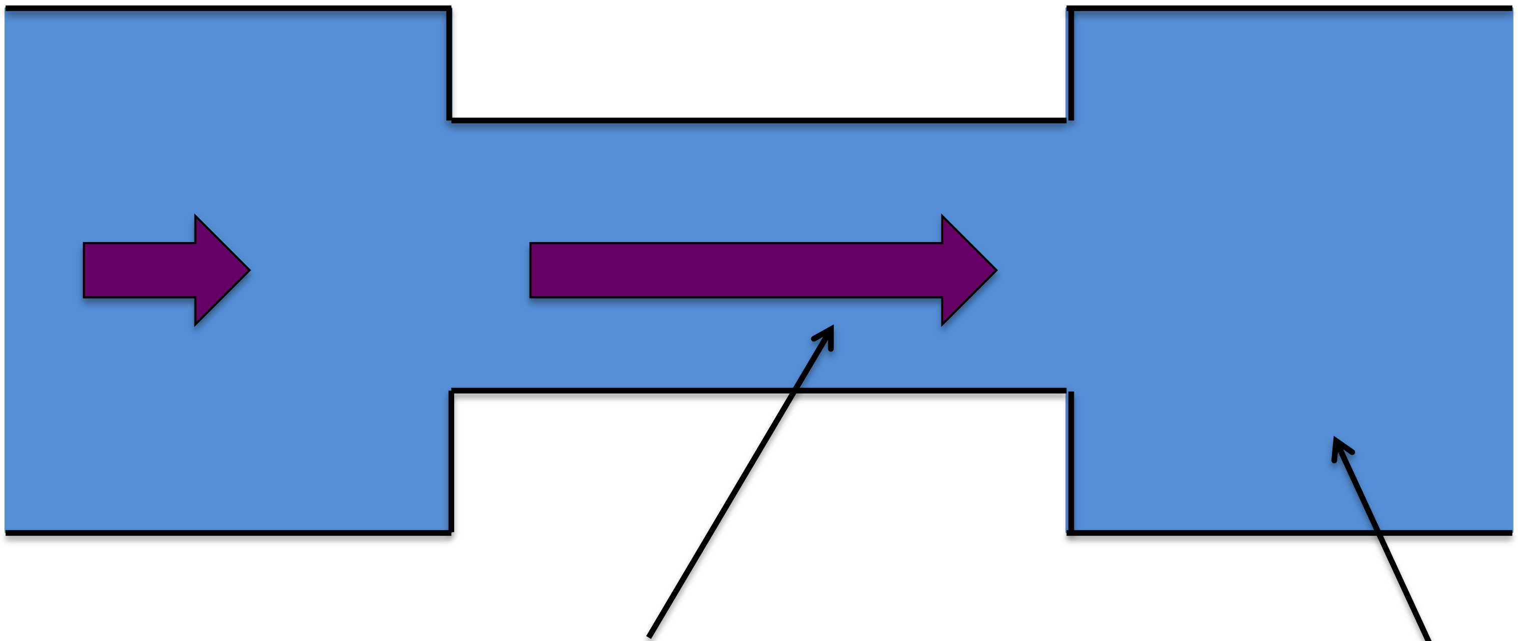
Is this limitation of sonic velocity a real concern in the chemical engineering field?

- The flow rate of your gas reaching its maximum velocity does happen, most often when a gas is flowing through a constriction such as a nozzle. Remember, as a fluid passes through a constriction the velocity must increase rapidly to maintain the mass flow rate, G. In this situation the fluid velocity can approach sonic velocity
- Think of aerosol applications where you have solid or liquid particles being carried on by a gas stream. The maximum rate at which you can spray you suspension is limited by sonic speed



Is this actually important in real life?

Similarly, depending on the velocity of the fluid, any time your flow goes through a constriction in a pipe (even a narrow fitting in your pipeline) the limit of sonic velocity may become a limiting factor



Does this phenomenon have a name?

If your gas flow rate is at its maximum (i.e. the speed of sound) your flow is said to be **choked**

This term, **choked flow**, make sense. Even if you want to deliver a higher flow rate of gas, even if the end pressure in your system is less than P_w, you cannot do so. Your system has reached its maximum gas flow rate

You must always check if your flow is choked!

When pumping a gas, you must always check to see if the flow is choked

If the flow becomes choked, you will create a larger pressure drop than is required for the maximum flow rate of gas, a waste of money

How can you guarantee that your system won't enter the choked flow regime?

A relationship between G and P at choked flow

The M.E.B for compressible flow

$$\frac{P_2^2 - P_1^2}{2(RT/M)} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

(Mechanical energy balance)

We previously derived an expression relating G and P and choked flow conditions

$$\left(\frac{G_{\max}}{A}\right)^2 = \frac{P_w^2}{(RT/M)}$$

Relationship between G and P

By substitution, we can arrive at the following expression

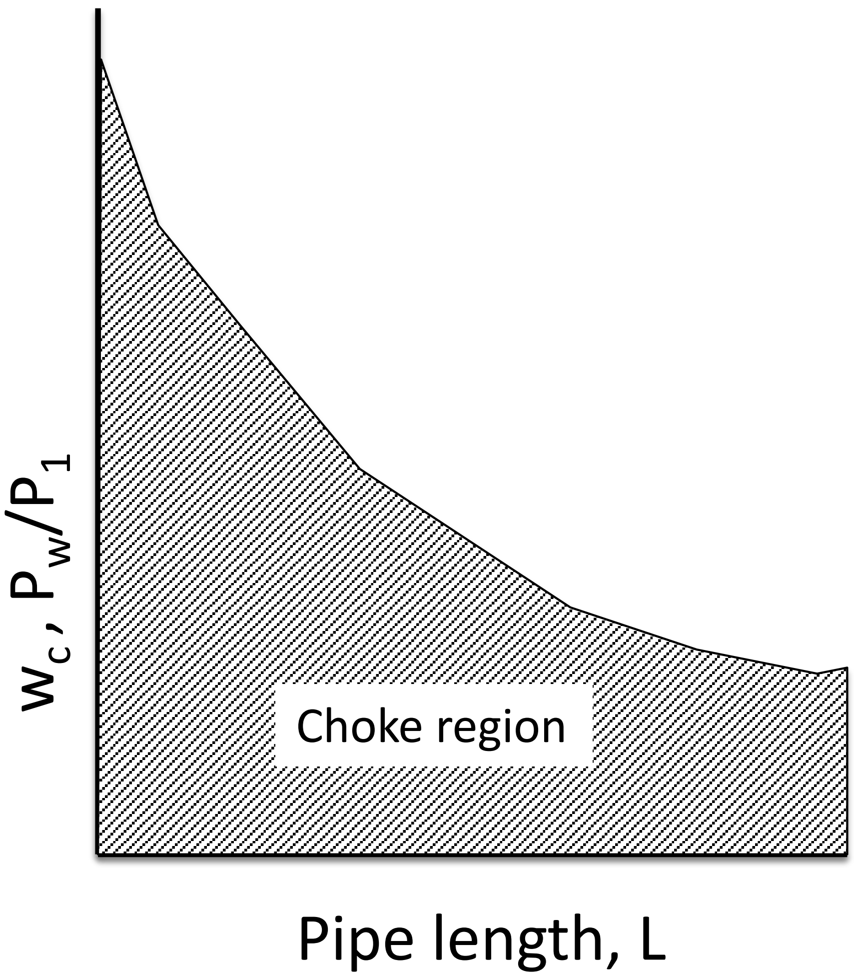
$$\boxed{8\phi\frac{L}{D} = \left(\frac{1}{w_c}\right)^2 - \ln\left(\frac{1}{w_c}\right)^2 - 1}$$

where

$$w_c = \left(\frac{P_w}{P_1}\right)$$

If we were to plot this equation

Let's plot w_c versus pipe length, L. You discover that the value of P_2 at which choke begins is drastically affected by the length of the pipe



Without this knowledge you would have prescribed a larger pressure drop than needed. That would have cost the company more money. You would have been fired for your mistake. The company would have gone bankrupt. All of your former coworkers would be jobless. You would have been sad ☹

However, by discovering this relationship you were able to perform an economic analysis, design a pipe length and pressure drop that minimizes cost. You will receive promotion. You are happy ☺

Summary

In this lesson we derived the mechanical energy balance for a isothermal and compressible flow

We observed an apparent inconsistency in the equation, and from that discovered the phenomenon of choked flow

We discussed how this is a real life industrial problem and explored an example problem on how design of your system can avoid choked flow conditions

Fluid Mechanics

Compressible flow part 4:

Isothermal versus Adiabatic

Last lesson

Choked flow occurs in compressible flow

- This is the maximum velocity of the compressible fluid
- It is also the speed at which a pressure wave propagates through compressible flow
- This value is the same as the speed of sound
- Compressible fluid can flow slower than the speed of sound, but even for larger pressure drops it will not travel faster
- The critical downstream pressure at which choked flow begins is a function of flow variables such as pipe length, diameter, and roughness
- It has applications in pumping processes, especially for nozzle flow and flow through constrictions where the velocity increases drastically to maintain flow rate, G

Isothermal and compressible flow

So we're talking about isothermal and compressible flow

Yep. The crazy flow behaviours of those constant temperature ideal gasses

Isothermal implies that a certain amount of heat, q, is being added or removed from the system to maintain temperature

Yeah, you must be right. I don't think the flow would stay at a constant temperature by itself

So which is it? Does heat need to be added to the flow or removed from the flow to maintain isothermal conditions?

That's a good question. I think it must be taken away…

No way, it needs to be added!

Added!

Added!

Taken away!

Taken away!

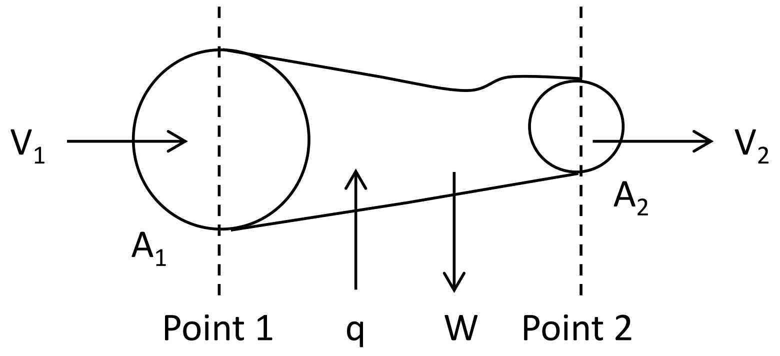
So who is correct? Me or a marsupial?!

What do you think? Does heat need to be removed or added to maintain isothermal conditions?

Adding or removing heat for isothermal flow?

To answer this question, we need to look at the energy balance you derived in the semester

$$E_2 - E_1 = q - W$$



We have to balance our energy terms which can be in the form of internal energy, kinetic energy, or potential energy

$$E_2 - E_1 = \underline{\Delta U} + \underline{\frac{1}{2}\Delta\left(V^2\right)} + \underline{g\Delta z}$$

The energy change between points 1 & 2 will equal heat exchanged with the surroundings and the work performed (both flow work and shaft work)

$$q - W = \underline{q} - \left(\underline{P_2/\rho_2 - P_1/\rho_1} + \underline{W_s}\right)$$

Adding or removing heat for isothermal flow?

From this analysis, we can write out steady flow energy balance

$$\Delta U + \frac{1}{2}\Delta\left(V^2\right) + g\Delta z = q - \left(P_2/\rho_2 - P_1/\rho_1 + W_s\right)$$

Rearrange

$$\Delta U + \Delta\left(\frac{P}{\rho}\right) + \frac{1}{2}\Delta\left(V^2\right) + \underbrace{\cancel{g\Delta z}}_{\text{Horizontal}} = q - \underbrace{\cancel{W_s}}_{\text{No } W_s}$$

$\nu = 1/\rho$

$$q = \Delta U + \Delta\left(P\nu\right) + \frac{1}{2}\Delta\left(V^2\right)$$

We're getting closer to solving for the amount of energy that must be transferred to/from the system to maintain isothermal conditions

However, we need to somehow get rid of our reliance on ΔU...

Adding or removing heat for isothermal flow?

Again, if you pull some thermodynamics relationships from your brain, we can make the necessary simplifications to the system

$$q = \Delta U + \Delta(Pv) + \frac{1}{2}\Delta\left(V^2\right)$$

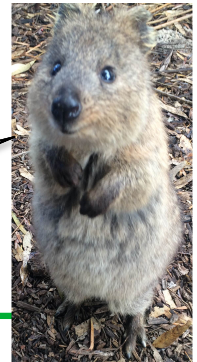
Recall our definition for specific enthalpy, Δh

$$\Delta h = \Delta U + \Delta(Pv) \xrightarrow{\text{Ideal gas}} \Delta h = \Delta U + \Delta(Pv) = \Delta U + \Delta\left(\frac{RT}{M}\right)$$

Through substitution we arrive at the following expression

$$q = \underbrace{\cancel{\Delta U}}_{\text{Isothermal}} + \underbrace{\cancel{\Delta\left(\frac{RT}{M}\right)}}_{\text{Isothermal}} + \frac{1}{2}\Delta\left(V^2\right) \longrightarrow \boxed{q = \frac{1}{2}\Delta\left(V^2\right)}$$

Told you so!



The quokka was right, the value for q is positive, so heat must be added to our system for it to remain isothermal. The amount of heat is proportional to the change in velocity of the compressible fluid

Isothermal versus adiabatic flow

In the last few lessons we derived the mechanical energy balance for **ideal gas flow** under **isothermal conditions**

Now we will look at the mechanical energy balance for **ideal gas flow** under **adiabatic conditions**

$$\int_{P_1}^{P_2} \frac{dP}{v} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{v_2}{v_1}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

Case 1

Isothermal flow of a compressible ideal gas

Case 2

Adiabatic flow of a compressible ideal gas

Isothermal vs. adiabatic flow

	Isothermal	***Isentropic***
P vs. ρ relationship	$$\frac{P_1}{P_2}=\left(\frac{v_2}{v_1}\right)=\left(\frac{\rho_1}{\rho_2}\right)$$	$$\frac{P_1}{P_2}=\left(\frac{v_2}{v_1}\right)^{\gamma}=\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$
Max /sonic speed	$$c=\sqrt{\frac{RT}{M}}$$	$$c=\sqrt{\frac{\gamma RT}{M}}$$
Mechanical energy balance	$$\frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L=\frac{P_1^2-P_2^2}{2(RT/M)}-\left(\frac{G}{A}\right)^2\ln\left(\frac{P_1}{P_2}\right)$$	$$8\phi\frac{L}{D}=\left[\frac{\gamma-1}{2\gamma}+\frac{P_1}{v_1}\left(\frac{A}{G}\right)^2\right]\left(1-\left(\frac{v_1}{v_2}\right)^2\right)-\frac{\gamma+1}{\gamma}\ln\left(\frac{v_2}{v_1}\right)$$ This expression is for adiabatic flow, not isentropic. Why?

Isothermal vs. adiabatic flow

We've made many comparisons between the flow of a compressible fluid in **isothermal** versus **adiabatic** conditions

- Pressure versus density
- Maximum/sonic velocity
- Mechanical energy balance

We can also draw comparisons between the **flow rate** of gas under each of these conditions as well as the **Reynolds Number** of the flow

- **Flow rate** of adiabatic flow is greater than that of isothermal flow for the same pressure drop; however, the difference is <20%. For longer pipes (when L/D > 1000) the difference is < 5%

- **Reynolds Number** is relatively constant for compressible flow under either isothermal or adiabatic conditions

$$\mathrm{Re} = \frac{\rho V D}{\mu} = \frac{G D}{A \mu}$$

For a given mass flow rate, Re will vary only from the temperature dependence of the viscosity. This affect is usually small

Compressible flow example problem #1

You are doing great work in your new position at ***Bob's Big Boy Plastic Producers***, and your ***CEO*** has you in mind for the position when he retires in a year. To verify your aptitude, he asks you to calculate the upstream pressure, P_1, for the isothermal flow (24^{o}C) of methane in a horizontal pipe (68.5 meters in length, G = 2.35 kg/s, μ = 1.08x10^{-5} Pa s, MW = 16 g/mol). All the piping at BBBPP is standardized (8.4cm inner diameter, e = 0.26 cm). The downstream pressure is 560 kPa

Since you took fluid mechanics with the ***amazing***, ***stupendous***, ***awesome Lecturer Daniel Heath*** (and ***Fudge the Cat***), you know that this problem can be solved through simple application of the isothermal mechanical energy balance.

Compressible flow: Example Problem 2

You have two gas streams (stream A and stream B) converging into a single flow stream (stream C). The diameter of all pipes is 20cm with a constant value of f = 0.005. Pipe A and B are both 10km long and pipe C is 5km long. The mass flow rate in stream A is 0.75 kg/s with a initial pressure in stream A of 800kPa and an initial pressure in stream B of 1000 kPa. The flow is isothermal at 30^{o}C and the molecular weight of the gas is 16 g/mol

A. What is the pressure where the pipes join?

B. What is the gas flow rate at the exit of stream C?

C. What is the pressure and velocity of the flow at the exit of stream C?

D. Is the flow choked?

Compressible flow: Example Problem 3

You have nitrogen gas with a mass flow rate of 0.32 kg/s flowing through a pipe with a length of 145 meters and constant friction (f = 0.006). The pressure drops across the length of the pipe from 800 kPa to 500 kPa. The molecular weight of nitrogen is 28, flow is isothermal at 15^{o}C and the viscosity at this temperature is 1.8x10^{-5} Pa s.

Assume kinetic energy is small enough to ignore.

A. What is the diameter of the pipe?

B. What is the entry and exit velocity as % of sonic velocity?

C. What is the pressure halfway along the length of the pipe?

D. What is the absolute roughness of the pipe?

Summary

We determined that in order to maintain isothermal conditions, heat must be added to an isothermal ideal gas flow system

The total amount of energy is related to the change in velocity of the gas

We then determined that the density and velocity of the gas are changing at each instant down the length of the pipe (I had told you a wee lie a few days ago in class)

We then compared isothermal and adiabatic flow conditions

- Flow rate in adiabatic flow is greater, though only be <5% for long pipes
- The Reynolds Number for the flow is largely unaffected

Summary

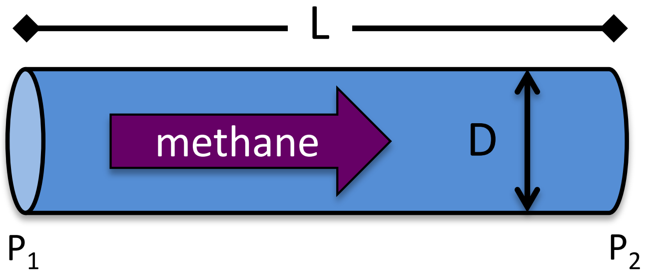
This concludes our section on compressible flow!

A few of the key pieces of knowledge we discovered are:

- That changes in pressure do not occur instantaneously in the system; instead, they propagate as pressure waves that move at sonic speed
- The speed of sound is not constant, it depends on the environment
- Compression of fluid can lead to large pressure spikes, and these large changes in pressure can damage equipment and surroundings. This is referred to as water hammer
- We derived the mechanical energy balance for both isothermal and adiabatic flow conditions
- We discovered that sonic speed is the maximum speed of the flow of a gas, even if the mechanical energy balance predicts a higher flow rate. This is referred to as choked flow

Compressible flow example problem #1

As any good engineer (or physicist), we will begin solving this problem by sketching the system and tabulating known values. We will then determine our unknown values



$$\frac{P_2^2 - P_1^2}{2(RT/M)} + \left(\frac{G}{A}\right)^2 \ln\left(\frac{P_1}{P_2}\right) + \frac{4\phi}{D}\left(\frac{G}{A}\right)^2 L = 0$$

By looking at our mechanical energy balance, we are given values for most of the necessary parameters in our problem statement. What remains unknown is P_1, A, and ϕ

Parameter	Value	Units
L	68.5	m
D	8.4	cm
T	24	^{o}C
G	2.35	kg/s
P_2	560	kPa
R	8.314	J/(mol*K)
M	1.08×10^{-5}	Pa s
e	0.026	cm
M	16	g/mol