1. A linear programming student is sitting an exam with three questions. He is interested in optimising the quantity of time he spends on each question. Each question is allocated marks as outlined in the table below:

Q 1	Q 2	Q 3
20	40	60

The student estimates that he can attain full marks for $Q1$ if he spends 20 minutes on it. For $Q2$, he could attain full marks if he spends 30 minutes on it and, finally, for $Q3$ he could attain full marks if he works on it for 25 minutes.

The student estimates that he should not spend more than 30 minutes questions 1 and 2 together, and not more than 50 minutes on questions 2 and 3 together. The total length of the exam is 55 minutes.

The student makes the assumption that the marks he will receive will be proportional to the time he spends on a question up to his estimate required to attain full marks. Formulate the problem of deciding the amount of time that the student should spend on each question to maximise his overall mark for the exam as a linear program. You **do not** need to write this in standard or canonical form.

[7 marks]

Solutions:
EASY with minor trick (5 MIN)
Let x_i be the quantity of time spent on question i. Then,

$$\max \frac{20}{20}x_1 + \frac{40}{30}x_2 + \frac{60}{25}x_3$$

subject to

$$\begin{aligned}
x_1 + x_2 + x_3 &\leq 55 \\
x_1 + x_2 \qquad &\leq 30 \\
x_2 + x_3 &\leq 50 \\
x_1 \qquad\qquad &\leq 20 \\
x_2 \qquad &\leq 30 \\
x_3 &\leq 25 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

2. Consider the linear programming problem:

$$\min \quad z = 2x_1 + 3x_2$$

subject to

$$\begin{aligned}
2x_1 + x_2 &\leq 5 \\
5x_1 + 3x_2 &\leq 12 \\
x_1, x_2 &\geq 0
\end{aligned}$$

with x_1 and x_2 non-negative. Adding slack, surplus and artificial variables as necessary, apply the Simplex Algorithm to solve this problem.

[8 marks]

Solutions:
EASY
To attain canonical form we must add 2 slack variables.

$$\min \quad z = 2x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 \qquad &= 5 \\ 5x_1 + 3x_2 \qquad + x_4 &= 12 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The initial tableau is:

BV	Eq.	z	x_1	x_2	x_3	x_4	RHS
x_3	1	0	2	1	1	0	5
x_4	2	0	5	3	0	1	12
	z	1	-2	-3	0	0	0

We choose the most negative reduced cost variable to enter the basis. This corresponds to x_2. We perform the ratio test on this column. This yields $\frac{5}{1}$ for the first row and $\frac{12}{3}$ for the second row. Therefore the second basic variable leaves the basis.

BV	Eq.	z	x_1	x_2	x_3	x_4	RHS
x_3	1	0	$\frac{1}{3}$	0	1	-$\frac{1}{3}$	1
x_2	2	0	$\frac{5}{3}$	1	0	$\frac{1}{3}$	4
	z	1	3	0	0	1	12

Therefore the solution is optimal and $x = (3, 0, 0, 1)$ and $f(x) = 12$.

6. You are organising a dinner party where you would like to allow as much time as possible **after** the meal for playing boardgames. You have a list of activities and their precedence constraints:

Activities	*Time*	*Precedence*
A. Clean the house	60	—
B. Create playlist and turn on music	10	—
C. Prepare meal	60	—
D. Greet guests	20	A, B, C
E. Serve main course	60	C, D
F. Prepare dessert	20	C, E
G. Serve dessert	30	F

Formulate the problem of deciding the time to start each activity assuming that you will be able to find friends to help you perform tasks simultaneously, as a **linear program**.

You are most interested in finding the earliest finishing time of the complete meal. You **do not** need to attain standard or canonical form. [7 marks]

Solutions:

Let x_i be the start time of each activity. Then,

$$\min x_G$$

subject to

$$\begin{aligned}
x_D &\geq x_A + 60 \\
x_D &\geq x_B + 10 \\
x_D &\geq x_C + 60 \quad (\textit{redundant}) \\
x_E &\geq x_C + 60 \\
x_E &\geq x_D + 20 \\
x_F &\geq x_D + 20 \quad (\textit{redundant}) \\
x_F &\geq x_E + 60 \\
x_G &\geq x_F + 20 \\
x_i &\geq 0 \; \forall \; i
\end{aligned}$$

7. Consider the following algorithm:

The Simplex Method

Require: Linear program in *standard form.*
Ensure: Optimal solution.

Construct canonical form and obtain a *basic feasible solution.*
while There are negative *reduced costs* **do**
  Select entering variable with most negative reduced cost.
  Select leaving variable using the *ratio test.*
  Pivot the tableau.
end while

Briefly explain the reasoning for the following steps in the algorithm:

(a) Constructing canonical form;

(b) Obtaining a basic feasible solution;

(c) Choosing an entering variable based on most negative reduced cost;

(d) Choosing a leaving variable using the minimum ratio test;

(e) Looking at the reduced costs to indicate optimality.

[10 marks]

Solutions:

(a) Constructing canonical form;
Canonical form provides us with slack variables and a subsequent basic solution which is usually feasible. It also creates a set of linear equalities which we can solve easily to find canonical form for a new solution.

(b) Obtaining a basic feasible solution;
The starting basic feasible solution gives us a starting point from which we can find an improved basic feasible solution, iteratively, until we find the optimal solution (fundamental theorem of linear programming).

(c) Choosing an entering variable based on most negative reduced cost;
The most negative reduced cost indicates the steepest gradient obtainable from the current BFS for a maximisation problem. Selecting the variable associated with this gradient returns the best improvement per unit in the objective function.

(d) Choosing a leaving variable using the minimum ratio test;
The minimum ratio test indicates which variable most restricts travel along the edge found by the most negative reduced cost.

(e) Looking at the reduced costs to indicate optimality.
If there are no negative reduced costs, then there are no edges leading from the current BFS that will improve the objective function.

8. Consider the following primal linear program:

$$\max_{x} z = x_1 + x_2 + x_3$$

$$\begin{array}{rcl} 2x_2 - x_3 & \geq & 4 \\ x_1 - 3x_2 + 4x_3 & = & 5 \\ x_1 - 2x_2 \qquad & \leq & 3 \end{array}$$

$$x_1, x_2 \geq 0, x_3 \; urs.$$

(a) Obtain canonical form for the primal;

(b) Obtain the dual of the primal.

[9 marks]

Solutions:

(a)

$$\max z = x_1 + x_2 + x_3$$

$$\begin{array}{lcl} 2x_2 - x_4 + x_5 - x_6 + \underline{x}_7 & = & 4 \\ x_1 - 3x_2 + 4x_4 - 4x_5 \qquad + \underline{x}_8 & = & 5 \\ x_1 - 2x_2 \qquad\qquad\qquad\qquad + x_9 & = & 3 \end{array}$$

$$x_1, x_2, x_4, x_5, x_6, \underline{x}_7, \underline{x}_8, x_9 \geq 0.$$

(b) .

Primal (equivalent non-standard form)

$$\max_{x} z = x_1 + x_2 + x_3^{(1)} - x_3^{(2)}$$

$$\begin{array}{rcl} -2x_2 + x_3^{(1)} - x_3^{(2)} & \leq & -4 \\ x_1 - 3x_2 + 4x_3^{(1)} - 4x_3^{(2)} & \leq & 5 \\ -x_1 + 3x_2 - 4x_3^{(1)} + 4x_3^{(2)} & \leq & -5 \\ x_1 - 2x_2 \qquad\qquad & \leq & 3 \end{array}$$

$$x_1, x_2, x_3^{(1)}, x_3^{(2)} \geq 0.$$

Dual (equivalent non-standard)

$$\min_y w = -4y_1 + 5y_2^{(1)} - 5y_2^{(2)} + 3y_3$$

$$\begin{aligned}
y_2^{(1)} - y_2^{(2)} + y_3 &\geq 1 \\
-2y_1 - 3y_2^{(1)} + 3y_2^{(2)} - 2y_3 &\geq 1 \\
y_1 + 4y_2^{(1)} - 4y_2^{(2)} &\geq 1 \\
-y_1 - 4y_2^{(1)} + 4y_2^{(2)} &\geq -1 \\
y_1, y_2^{(1)}, y_2^{(2)}, y_3 &\geq 0.
\end{aligned}$$

9. The Fundamental Theorem of Linear Programming is:

Consider the linear programming problem:

$$\max_x z = \sum_{j=1}^{k} c_j x_j$$

subject to

$$\begin{array}{rcl} a_{11}x_1 + ... + a_{1n}x_n + ... + a_{1,n+m}x_{n+m} & = & b_1 \\ a_{21}x_1 + ... + a_{2n}x_n + ... + a_{2,n+m}x_{n+m} & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n + ... + a_{m,n+m}x_{n+m} & = & b_m \end{array}$$

$x_j \geq 0$, $j = 1, ..., n+m$, where $b_i \geq 0$, for all i, and the coefficient matrix has m linearly independent columns.

- If this problem has a feasible solution then it must have a basic feasible solution.
- If this problem has an optimal solution then it must have an optimal basic feasible solution.

The following parts refer to the proof provided in the lecture notes:

(a) We assume we have a feasible solution, $\mathbf{x}$, to the linear program. We partition $\mathbf{x} = [\mathbf{x}^+ \mathbf{x}^0]$. Why do we do this?
Solutions:
We are looking for a feasible solution which is also basic. To do this, we look at the columns which are contributing to the solution. That is, only the columns $A^+\mathbf{x}^+ = \mathbf{b}$.

(b) We then partition the original columns of the coefficient matrix, A, in the same way. How can we tell if the partition A^+ is linearly independent? Why do we want it to be linearly independent?
Solutions:
Since we are looking at the partition of A^+ that contributes to the equation $A^+\mathbf{x}^+ = \mathbf{b}$, we know that if A^+ is linearly independent then we have a basic feasible solution to the equation. The columns of A^+ are linearly independent if no column can be obtained from a linear combination of the others.

(c) We find $\mathbf{w}^+$ such that $A^+\mathbf{w}^+ = \mathbf{0}$ and ϵ such that $\epsilon = \min\{\frac{x_j}{w_j} : w_j > 0\}$ which leads to a new solution $[\mathbf{x}^+ - \epsilon\mathbf{w}^+]$. How is this new solution different to the last? Why have we taken this step?
Solutions:
We take this step when the columns of A^+ are linearly dependent, so that we can find a new feasible solution that is also basic. We have increased ϵ to the first point where one or more components becomes zero using the minimum ratio rule. This removes a column from the set of strictly positive x_i^+'s. That is, we have one more zero than the previous iteration.

10. Consider the following Primal tableau:

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_4	0	$-2/5$	$-12/5$	1	$-8/5$	4
x_1	1	$4/5$	$4/5$	0	$1/5$	12
z	0	4	14	0	16	960

(a) Write down the Primal basic feasible solution and the corresponding objective function value.
Solutions:
$\mathbf{x} = (12, 0, 0, 4, 0)$. The basis is (x_1, x_4). The optimal objective function value is $z^* = 960$.

(b) Write down the Dual basic feasible solution and the corresponding objective function value.
Solutions:
$\mathbf{y} = (0, 16, 0, 4, 14)$. The basis is (y_2, y_4, y_5). The optimal objective function value is $w^* = 960$.

(c) Write down the solution obtained from the above tableau. Show, using the complementary slackness relation, that this solution is optimal.
Solutions:
The complementary slackness relation states:
$\mathbf{sy} = 0$ and $\mathbf{tx} = 0$.

$$\begin{array}{lccccc} (s,x) & = & \{4, & 0, & 12, & 0, & 0\} \\ (y,t) & = & \{0, & 16, & 0, & 4, & 14\} \\ \hline x_i \times y_i & & 0 & 0 & 0 & 0 & 0 \end{array}$$

From the complementary slackness theorem we have that both primal and dual are optimal feasible.

(d) Write down the corresponding Dual tableau.
Solutions:

BV	y_1	y_2	y_3	y_4	y_5	RHS
y_2	$8/5$	1	$-1/5$	0	0	16
y_4	$2/5$	0	$-4/5$	1	0	4
y_5	$-4/5$	0	$12/5$	0	1	14
w	4	0	12	0	0	960

(e) From the Primal tableau, name the row operations that have occurred to get to this form.
Solutions:
Row 2 has been divided by 5. Row 1 has had 8/5 copies of Row 2 subtracted from it.

(f) Suppose that the original tableau is as follows:

BV	Eq. #	z	x_1	x_2	x_3	x_4	x_5	RHS
x_4	1	0	8	6	4	1	0	100
x_5	2	0	5	4	4	0	1	60
z	z	1	-80	-60	-50	0	0	0

How much can we change the first component of $\mathbf{b}$ without changing the basis for the optimal solution?

Solutions:

$A_b = \begin{pmatrix} 8 & 1 \\ 5 & 0 \end{pmatrix}$. The inverse is:

$A_b^{-1} = \begin{pmatrix} 0 & 1/5 \\ 1 & \text{-}8/5 \end{pmatrix}$.

$A_b^{-1}(\mathbf{b} - \delta e_1) = \begin{pmatrix} 12 \\ \delta + 4 \end{pmatrix}$. That is, the current solution will remain optimal if the value of b_1 is greater than or equal to 96.

[15 marks]