5 True or false: For an LP to be unbounded, the LP's feasible region must be unbounded.

6 True or false: Every LP with an unbounded feasible region has an unbounded optimal solution.

7 If an LP's feasible region is not unbounded, we say the LP's feasible region is bounded. Suppose an LP has a bounded feasible region. Explain why you can find the optimal solution to the LP (without an isoprofit or isocost line) by simply checking the z-values at each of the feasible region's extreme points. Why might this method fail if the LP's feasible region is unbounded?

8 Graphically find all optimal solutions to the following LP:

$$\begin{aligned} \min z &= x_1 - x_2 \\ \text{s.t. } x_1 + x_2 &\leq 6 \\ x_1 - x_2 &\geq 0 \\ x_2 - x_1 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

9 Graphically determine two optimal solutions to the following LP:

$$\begin{aligned} \min z &= 3x_1 + 5x_2 \\ \text{s.t. } 3x_1 + 2x_2 &\geq 36 \\ 3x_1 + 5x_2 &\geq 45 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Group B

10 Money manager Boris Milkem deals with French currency (the franc) and American currency (the dollar). At 12 midnight, he can buy francs by paying .25 dollars per franc and dollars by paying three francs per dollar. Let x_1 = number of dollars bought (by paying francs) and x_2 = number of francs bought (by paying dollars). Assume that both types of transactions take place simultaneously, and the only constraint is that at 12:01 A.M., Boris must have a nonnegative number of francs and dollars.

a Formulate an LP that enables Boris to maximize the number of dollars he has after all transactions are completed.

b Graphically solve the LP and comment on the answer.

3.4 A Diet Problem

Many LP formulations (such as Example 2 and the following diet problem) arise from situations in which a decision maker wants to minimize the cost of meeting a set of requirements.

EXAMPLE 6 My diet requires that all the food I eat come from one of the four "basic food groups" (chocolate cake, ice cream, soda, and cheesecake). At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 50¢, each scoop of chocolate ice cream costs 20¢, each bottle of cola costs 30¢, and each piece of pineapple cheesecake costs 80¢. Each day, I must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table 2. Formulate a linear programming model that can be used to satisfy my daily nutritional requirements at minimum cost.

Solution As always, we begin by determining the decisions that must be made by the decision maker: how much of each type of food should be eaten daily. Thus, we define the decision variables:

x_1 = number of brownies eaten daily

x_2 = number of scoops of chocolate ice cream eaten daily to plan institutional menus for a weekly or monthly period.[†] Menu-planning models do contain constraints that reflect tastiness and variety requirements.

Problems

Group A

1 There are three factories on the Momiss River (1, 2, and 3). Each emits two types of pollutants (1 and 2) into the river. If the waste from each factory is processed, the pollution in the river can be reduced. It costs $15 to process a ton of factory 1 waste, and each ton processed reduces the amount of pollutant 1 by 0.10 ton and the amount of pollutant 2 by 0.45 ton. It costs $10 to process a ton of factory 2 waste, and each ton processed will reduce the amount of pollutant 1 by 0.20 ton and the amount of pollutant 2 by 0.25 ton. It costs $20 to process a ton of factory 3 waste, and each ton processed will reduce the amount of pollutant 1 by 0.40 ton and the amount of pollutant 2 by 0.30 ton. The state wants to reduce the amount of pollutant 1 in the river by at least 30 tons and the amount of pollutant 2 in the river by at least 40 tons. Formulate an LP that will minimize the cost of reducing pollution by the desired amounts. Do you think that the LP assumptions (Proportionality, Additivity, Divisibility, and Certainty) are reasonable for this problem?

2[‡] U.S. Labs manufactures mechanical heart valves from the heart valves of pigs. Different heart operations require valves of different sizes. U.S. Labs purchases pig valves from three different suppliers. The cost and size mix of the valves purchased from each supplier are given in Table 3. Each month, U.S. Labs places one order with each supplier. At least 500 large, 300 medium, and 300 small valves must be purchased each month. Because of limited availability of pig valves, at most 500 valves per month can be purchased from each supplier. Formulate an LP that can be used to minimize the cost of acquiring the needed valves.

TABLE 3

	Cost Per Valve	Percent Large	Percent Medium	Percent Small
Supplier 1	$5	40	40	20
Supplier 2	$4	30	35	35
Supplier 3	$3	20	20	60

3 Peg and Al Fundy have a limited food budget, so Peg is trying to feed the family as cheaply as possible. However, she still wants to make sure her family members meet their daily nutritional requirements. Peg can buy two foods. Food 1 sells for $7 per pound, and each pound contains 3 units of vitamin A and 1 unit of vitamin C. Food 2 sells for $1 per pound, and each pound contains 1 unit of each vitamin. Each day, the family needs at least 12 units of vitamin A and 6 units of vitamin C.

a Verify that Peg should purchase 12 units of food 2 each day and thus oversatisfy the vitamin C requirement by 6 units.

b Al has put his foot down and demanded that Peg fulfill the family's daily nutritional requirement exactly by obtaining precisely 12 units of vitamin A and 6 units of vitamin C. The optimal solution to the new problem will involve ingesting less vitamin C, but it will be more expensive. Why?

4 Goldilocks needs to find at least 12 lb of gold and at least 18 lb of silver to pay the monthly rent. There are two mines in which Goldilocks can find gold and silver. Each day that Goldilocks spends in mine 1, she finds 2 lb of gold and 2 lb of silver. Each day that Goldilocks spends in mine 2, she finds 1 lb of gold and 3 lb of silver. Formulate an LP to help Goldilocks meet her requirements while spending as little time as possible in the mines. Graphically solve the LP.

3.5 A Work-Scheduling Problem

Many applications of linear programming involve determining the minimum-cost method for satisfying work-force requirements. The following example illustrates the basic features common to many of these applications.

[†]Balintfy (1976).

[‡]Based on Hilal and Erickson (1981).

$x_5 = 0$, $x_6 = 4$, $x_7 = 3$. Notice that there is no way that the optimal linear programming solution could have been rounded to obtain the optimal all-integer solution.

Baker (1974) has developed an efficient technique (which does not use linear programming) to determine the minimum number of employees required when each worker receives two consecutive days off.

Problems

Group A

1 In the post office example, suppose that each full-time employee works 8 hours per day. Thus, Monday's requirement of 17 workers may be viewed as a requirement of 8(17) = 136 hours. The post office may meet its daily labor requirements by using both full-time and part-time employees. During each week, a full-time employee works 8 hours a day for five consecutive days, and a part-time employee works 4 hours a day for five consecutive days. A full-time employee costs the post office \$15 per hour, whereas a part-time employee (with reduced fringe benefits) costs the post office only \$10 per hour. Union requirements limit part-time labor to 25% of weekly labor requirements. Formulate an LP to minimize the post office's weekly labor costs.

2 During each 4-hour period, the Smalltown police force requires the following number of on-duty police officers: 12 midnight to 4 A.M.—8; 4 to 8 A.M.—7; 8 A.M. to 12 noon—6; 12 noon to 4 P.M.—6; 4 to 8 P.M.—5; 8 P.M. to 12 midnight—4. Each police officer works two consecutive 4-hour shifts. Formulate an LP that can be used to minimize the number of police officers needed to meet Smalltown's daily requirements.

Group B

3 Suppose that the post office can force employees to work one day of overtime each week. For example, an employee whose regular shift is Monday to Friday can also be required to work on Saturday. Each employee is paid \$50 a day for each of the first five days worked during a week and \$62 for the overtime day (if any). Formulate an LP whose solution will enable the post office to minimize the cost of meeting its weekly work requirements.

4 Suppose the post office had 25 full-time employees and was not allowed to hire or fire any employees. Formulate an LP that could be used to schedule the employees in order to maximize the number of weekend days off received by the employees.

5 Each day, workers at the Gotham City Police Department work two 6-hour shifts chosen from 12 A.M. to 6 A.M., 6 A.M. to 12 P.M., 12 P.M. to 6 P.M., and 6 P.M. to 12 A.M. The following number of workers are needed during each shift: 12 A.M. to 6 A.M.—15 workers; 6 A.M. to 12 P.M.—5 workers; 12 P.M. to 6 P.M.—12 workers; 6 P.M. to 12 A.M.—6 workers. Workers whose two shifts are consecutive are paid \$12 per hour; workers whose shifts are not consecutive are paid \$18 per hour. Formulate an LP that can be used to minimize the cost of meeting the daily work-force demands of the Gotham City Police Department.

6 During each 6-hour period of the day, the Bloomington Police Department needs at least the number of policemen shown in Table 5. Policemen can be hired to work either 12 consecutive hours or 18 consecutive hours. Policemen are paid \$4 per hour for each of the first 12 hours a day they work and are paid \$6 per hour for each of the next 6 hours they work in a day. Formulate an LP that can be used to minimize the cost of meeting Bloomington's daily police requirements.

TABLE 5

Time Period	Number of Policemen Required
12 A.M.–6 A.M.	12
6 A.M.–12 P.M.	8
12 P.M.–6 P.M.	6
6 P.M.–12 A.M.	15

The optimal solution to this LP is $x_1 = x_3 = x_4 = 1, x_2 = 0.201, x_5 = 0.288, z = 57.449$. Star Oil should purchase 100% of investments 1, 3, and 4; 20.1% of investment 2; and 28.8% of investment 5. A total NPV of $57,449,000 will be obtained from these investments.

It is often impossible to purchase only a fraction of an investment without sacrificing the investment's favorable cash flows. Suppose it costs $12 million to drill an oil well just deep enough to locate a $30-million gusher. If there were a sole investor in this project who invested $6 million to undertake half of the project, he or she would lose the entire investment and receive no positive cash flows. Since, in this example, reducing the money invested by 50% reduces the return by more than 50%, this situation would violate the Proportionality Assumption.

In many capital budgeting problems, it is unreasonable to allow the x_i to be fractions: Each x_i should be restricted to 0 (not investing at all in investment i) or 1 (purchasing all of investment i). Thus, many capital budgeting problems violate the Divisibility Assumption.

A capital budgeting model that allows each x_i to be only 0 or 1 is discussed in Section 9.2.

Problems

Group A

1 Show that if $r = 0.02$, investment 2 has a larger NPV than investment 1.

2 Two investments with varying cash flows (in thousands of dollars) are available, as shown in Table 7. At time 0, $10,000 is available for investment, and at time 1, $7000 is available. Assuming that $r = 0.10$, set up an LP whose solution maximizes the NPV obtained from these investments. Graphically find the optimal solution to the LP. (Assume that any fraction of an investment may be purchased.)

TABLE **7**

	Cash Flow (in thousands) at Time			
	0	1	2	3
Investment 1	−\$6	−\$5	\$7	\$9
Investment 2	−\$8	−\$3	\$9	\$7

3 Suppose that r, the annual interest rate, is 0.20, and that all money in the bank earns 20% interest each year (that is, after being in the bank for one year, $1 will increase to $1.20). If we place $100 in the bank for one year, what is the NPV of this transaction?

Group B

4 Finco must determine how much investment and debt to undertake during the next year. Each dollar invested reduces the NPV of the company by 10¢ and each dollar of debt increases the NPV by 50¢ (due to deductibility of interest payments). Finco can invest at most $1 million during the coming year. Debt can be at most 40% of investment. At present Finco has $800,000 in cash available. All investment must be paid for from current cash or borrowed money. Set up an LP whose solution will tell Finco how to maximize its NPV. Then graphically solve the LP.[†]

[†]Based on Myers and Pogue (1974).

Problems

6 Bullco blends silicon and nitrogen to produce two types of fertilizers. Fertilizer 1 must be at least 40% nitrogen and sells for $70/lb. Fertilizer 2 must be at least 70% silicon and sells for $40/lb. Bullco can purchase up to 80 lb of nitrogen at $15/lb and up to 100 lb of silicon at $10/lb. Assuming that all fertilizer produced can be sold, formulate an LP to help Bullco maximize profits.

7 Eli Daisy uses chemicals 1 and 2 to produce two drugs. Drug 1 must be at least 70% chemical 1, and drug 2 must be at least 60% chemical 2. Up to 40 oz of drug 1 can be sold at $6 per oz; up to 30 oz of drug 2 can be sold at $5 per oz. Up to 45 oz of chemical 1 can be purchased at $6 per oz, and up to 40 oz of chemical 2 can be purchased at $4 per oz. Formulate an LP that can be used to maximize Daisy's profits.

8 Highland's TV-Radio Store must determine how many TVs and radios to keep in stock. A TV requires 10 sq ft of floorspace, whereas a radio requires 4 sq ft; 200 sq ft of floorspace is available. A TV will earn Highland $60 in profits, and a radio will earn $20. The store stocks only TVs and radios. Marketing requirements dictate that at least 60% of all appliances in stock be radios. Finally, a TV ties up $200 in capital, and a radio, $50. Highland wants to have at most $3000 worth of capital tied up at any time. Formulate an LP that can be used to maximize Highland's profit.

9 Linear programming models are used by many Wall Street firms to select a desirable bond portfolio. The following is a simplified version of such a model. Solodrex is considering investing in four bonds; $1,000,000 is available for investment. The expected annual return, the worst-case annual return on each bond, and the "duration" of each bond are given in Table 13. The duration of a bond is a measure of the bond's sensitivity to interest rates. Solodrex wants to maximize the expected return from its bond investments, subject to three constraints.

Constraint 1: The worst-case return of the bond portfolio must be at least 8%.

Constraint 2: The average duration of the portfolio must be at most 6. For example, a portfolio that invested $600,000

Problems

Group B

5 A company produces A, B, and C and can sell these products in unlimited quantities at the following unit prices: A, $10; B, $56; C, $100. Producing a unit of A requires 1 hour of labor; a unit of B, 2 hours of labor plus 2 units of A; and a unit of C, 3 hours of labor plus 1 unit of B. Any A that is used to produce B cannot be sold. Similarly, any B that is used to produce C cannot be sold. A total of 40 hours of labor are available. Formulate an LP to maximize the company's revenues.

6 Daisy Drugs manufactures two drugs: 1 and 2. The drugs are produced by blending two chemicals: 1 and 2. By weight, drug 1 must contain at least 65% chemical 1, and drug 2 must contain at least 55% chemical 1. Drug 1 sells for $6/oz, and drug 2 sells for $4/oz. Chemicals 1 and 2 can be produced by one of two production processes. Running process 1 for an hour requires 3 oz of raw material and 2 hours skilled labor and yields 3 oz of each chemical. Running process 2 for an hour requires 2 oz of raw material and 3 hours of skilled labor and yields 3 oz of chemical 1 and 1 oz of chemical 2. A total of 120 hours of skilled labor and 100 oz of raw material are available. Formulate an LP that can be used to maximize Daisy's sales revenues.

7† Lizzie's Dairy produces cream cheese and cottage cheese. Milk and cream are blended to produce these two products. Both high-fat and low-fat milk can be used to produce cream cheese and cottage cheese. High-fat milk is 60% fat; low-fat milk is 30% fat. The milk used to produce cream cheese must average at least 50% fat and that for cottage cheese, at least 35% fat. At least 40% (by weight) of the inputs to cream cheese and at least 20% (by weight) of the inputs to cottage cheese must be cream. Both cottage cheese and cream cheese are produced by putting milk and cream through the cheese machine. It costs 40¢ to process 1 lb of inputs into a pound of cream cheese. It costs 40¢ to produce 1 lb of cottage cheese, but every pound of input for cottage cheese yields 0.9 lb of cottage cheese and 0.1 lb of waste. Cream can be produced by evaporating high-fat and low-fat

†Based on Sullivan and Secrest (1985).

the worth of the final period's inventory. For example, if Sailco feels that each sailboat left at the end of quarter 4 is worth \$400, a term $-400i_4$ (measuring the worth of quarter 4's inventory) should be added to the objective function.

Problems

4 A manufacturing company produces two types of products: A and B. They have agreed to deliver the products on the schedule shown in Table 30. The company has two assembly lines, 1 and 2, with the available production hours shown in Table 31. The production rates for each assembly line and product combination, in terms of hours per product, are shown in Table 32. It takes 0.15 hour to manufacture 1 unit of product A on line 1, and so on. It costs \$5 per hour of line time to produce any product. The inventory carrying cost per month for each product is 20¢ per unit (charged on each

TABLE **30**

	A	B
March 31	5000	2000
April 30	8000	4000

TABLE **31**

	Production Hours Available	
	Line 1	Line 2
March	800	2000
April	400	1200

TABLE **32**

	Production Rate	
	Line 1	Line 2
Product A	0.15	0.16
Product B	0.12	0.14

TABLE 29

	Month 1		Month 2		Month 3	
	Demand	Cost/Cake	Demand	Cost/Cake	Demand	Cost/Cake
[illegible]	40	\$3.00	30	\$3.40	20	\$3.80
[illegible]	20	\$2.50	30	\$2.80	10	\$3.40

month's ending inventory). At present, there are 500 units of A and 750 units of B in inventory. Management would like at least 1000 units of each product in inventory at the end of April. Formulate an LP to determine the production schedule that minimizes the total cost incurred in meeting demands on time.

5 During the next two months, General Cars must meet (on time) the following demands for trucks and cars: month 1—400 trucks, 800 cars; month 2—300 trucks, 300 cars. During each month, at most 1000 vehicles can be produced. Each truck uses 2 tons of steel, and each car uses 1 ton of steel. During month 1, steel costs $400 per ton; during month 2, steel costs $600 per ton. At most 1500 tons of steel may be purchased each month (steel may only be used during the month in which it is purchased). At the beginning of month 1, 100 trucks and 200 cars are in inventory. At the end of each month, a holding cost of $150 per vehicle is assessed. Each car gets 20 mpg, and each truck gets 10 mpg. During each month, the vehicles produced by the company must average at least 16 mpg. Formulate an LP to meet the demand and mileage requirements at minimum cost (include steel costs and holding costs).

6 Gandhi Clothing Company produces shirts and pants. Each shirt requires 2 sq yd of cloth, each pair of pants, 3. During the next two months, the following demands for shirts and pants must be met (on time): month 1—10 shirts, 15 pairs of pants; month 2—12 shirts, 14 pairs of pants. During each month, the following resources are available: month 1—90 sq yd of cloth; month 2—60 sq yd. (Cloth that is available during month 1 may, if unused during month 1, be used during month 2.)

During each month, it costs $4 to make an article of clothing with regular-time labor and $8 with overtime labor. During each month, a total of at most 25 articles of clothing may be produced with regular-time labor, and an unlimited number of articles of clothing may be produced with overtime labor. At the end of each month, a holding cost of $3 per article of clothing is assessed. Formulate an LP that can be used to meet demands for the next two months (on time) at minimum cost. Assume that at the beginning of month 1, 1 shirt and 2 pairs of pants are available.

7 Each year, Paynothing Shoes faces demands (which must be met on time) for pairs of shoes as shown in Table 33. Workers work three consecutive quarters and then receive one quarter off. For example, a worker may work during quarters 3 and 4 of one year and quarter 1 of the next year. During a quarter in which a worker works, he or she can produce up to 50 pairs of shoes. Each worker is paid $500 per quarter. At the end of each quarter, a holding cost of $50 per pair of shoes is assessed. Formulate an LP that can be used to minimize the cost per year (labor + holding) of meeting the demands for shoes. To simplify matters, assume that at the end of each year, the ending inventory is zero. (*Hint:* It is allowable to assume that a given worker will get the same quarter off during each year.)

TABLE **33**

Quarter 1	Quarter 2	Quarter 3	Quarter 4
600	300	800	100

8 A company must meet (on time) the following demands: quarter 1—30 units; quarter 2—2(..nits; quarter 3—40 units. Each quarter, up to 27 units can be produced with regular-time labor, at a cost of $40 per unit. During each quarter, an unlimited number of units can be produced with overtime labor, at a cost of $60 per unit. Of all units produced, 20% are unsuitable and cannot be used to meet demand. Also, at the end of each quarter, 10% of all units on hand spoil and cannot be used to meet any future demands. After each quarter's demand is satisfied and spoilage is accounted for, a cost of $15 per unit is assessed against the quarter's ending inventory. Formulate an LP that can be used to minimize the total cost of meeting the next three quarters' demands. Assume that 20 usable units are available at the beginning of quarter 1.

9 Donovan Enterprises produces electric mixers. During the next four quarters, the following demands for mixers must be met on time: quarter 1—4000; quarter 2—2000; quarter 3—3000; quarter 4—10,000. Each of Donovan's workers works three quarters of the year and gets one quarter off. Thus, a worker may work during quarters 1, 2, and 4 and get quarter 3 off. Each worker is paid $30,000 per year and (if working) can produce up to 500 mixers during a quarter. At the end of each quarter, Donovan incurs a holding cost of $30 per mixer on each mixer in inventory. Formulate an LP to help Donovan minimize the cost (labor and inventory) of meeting the next year's demand (on time). At the beginning of quarter 1, 600 mixers are available.

3.11 Multiperiod Financial Models

The following example illustrates how linear programming can be used to model multiperiod cash management problems. The key is to determine the relations of cash on hand during different periods.

and for April,

$$y_4 = 0.95y_3 + x_3$$

and for May,

$$y_5 = 0.95y_4 + x_4$$

Adding the sign restrictions $x_t \geq 0$ and $y_t \geq 0$ ($t = 1, 2, 3, 4, 5$), we obtain the following LP:

$$\begin{aligned}
\min z = {} & 1000x_1 + 1000x_2 + 1000x_3 + 1000x_4 + 1000x_5 \\
& + 2000y_1 + 2000y_2 + 2000y_3 + 2000y_4 + 2000y_5 \\
\text{s.t.} \quad & 160y_1 - 50x_1 \geq 6000 \qquad y_1 = 50 \\
& 160y_2 - 50x_2 \geq 7000 \qquad 0.95y_1 + x_1 = y_2 \\
& 160y_3 - 50x_3 \geq 8000 \qquad 0.95y_2 + x_2 = y_3 \\
& 160y_4 - 50x_4 \geq 9500 \qquad 0.95y_3 + x_3 = y_4 \\
& 160y_5 - 50x_5 \geq 11{,}000 \qquad 0.95y_4 + x_4 = y_5 \\
& x_t, y_t \geq 0 \quad (t = 1, 2, 3, 4, 5)
\end{aligned}$$

The optimal solution is $z = 593{,}777$; $x_1 = 0$; $x_2 = 8.45$; $x_3 = 11.45$; $x_4 = 9.52$; $x_5 = 0$; $y_1 = 50$; $y_2 = 47.5$; $y_3 = 53.58$; $y_4 = 62.34$; and $y_5 = 68.75$.

In reality, the y_t's must be integers, so our solution is difficult to interpret. The problem with our formulation is the fact that assuming that exactly 5% of the employees quit each month can cause the number of employees to change from an integer during one month to a fraction during the next month. We might want to assume that the number of employees quitting each month is the integer closest to 5% of the total work force, but then we do not have a linear programming problem!

Problems

Group A

1 If $y_1 = 38$, what would be the optimal solution to CSL's problem?

2 An insurance company believes that it will require the following numbers of personal computers during the next six months: January, 9; February, 5; March, 7; April, 9; May, 10; June, 5. Computers can be rented for a period of one, two, or three months at the following unit rates: one-month rate, \$200; two-month rate, \$350; three-month rate, \$450. Formulate an LP that can be used to minimize the cost of renting the required computers. You may assume that if a machine is rented for a period of time extending beyond June, the cost of the rental should be prorated. For example, if a computer is rented for three months at the beginning of May, then a rental fee of $\frac{2}{3}(450) = \$300$, not \$450, should be assessed in the objective function.

3 The IRS has determined that during each of the next twelve months they will need the number of supercomputers given in Table 39. To meet these requirements the IRS rents supercomputers for a period of one, two, or three months. It costs \$100 to rent a supercomputer for one month, \$180 for two months, and \$250 for three months. At the beginning of month 1, the IRS has no supercomputers. Determine the rental plan that meets the next twelve months' requirements at minimum cost. *Note:* You may assume that fractional rentals are okay. Thus if your solution says to rent 140.6 computers for one month we can round this up or down (to 141 or 140) without having much effect on the total cost.

7 Steelco manufactures two types of steel at three different steel mills. During a given month, each steel mill has 200 hours of blast furnace time available. Because of differences in the furnaces at each mill, the time and cost to produce a ton of steel differs for each mill. The time and cost for each mill are shown in Table 42. Each month, Steelco must manufacture at least 500 tons of steel 1 and 600 tons of steel 2. Formulate an LP to minimize the cost of manufacturing the desired steel.

8† Walnut Orchard has two farms that grow wheat and corn. Because of differing soil conditions, there are differences in the yields and costs of growing crops on the two farms. The yields and costs are shown in Table 43. Each farm has 100 acres available for cultivation; 11,000 bushels of wheat and 7000 bushels of corn must be grown. Determine a planting plan that will minimize the cost of meeting these demands. How could an extension of this model be used to allocate crop production efficiently throughout a nation?

9 Candy Kane Cosmetics (CKC) produces Leslie Perfume, which requires chemicals and labor. Two production processes are available: Process 1 transforms 1 unit of labor and 2 units of chemicals into 3 oz of perfume. Process 2 transforms 2 units of labor and 3 units of chemicals into 5 oz of perfume. It costs CKC \$3 to purchase a unit of labor and \$2 to purchase a unit of chemicals. Each year, up to 20,000 units of labor and 35,000 units of chemicals can be purchased. In the absence of advertising, CKC believes it can sell 1000 oz of perfume. To stimulate demand for Leslie, CKC can hire the lovely model Jenny Nelson. Jenny is paid \$100/hour. Each hour Jenny works for the company is estimated to increase the demand for Leslie Perfume by 200 oz. Each ounce of Leslie Perfume sells for \$5. Use linear programming to determine how CKC can maximize profits.

TABLE **42** Producing a Ton of Steel

	Steel 1		Steel 2	
	Cost	Time (minutes)	Cost	Time (minutes)
Mill 1	\$10	20	\$11	22
Mill 2	\$12	24	\$9	18
Mill 3	\$14	28	\$10	30

TABLE **43**

	Farm 1	Farm 2
Corn yield/acre	500 bushels	650 bushels
Cost/acre of corn	\$100	\$120
Wheat yield/acre	400 bushels	350 bushels
Cost/acre of wheat	\$90	\$80

†Based on Heady and Egbert (1964).

10 Carco has a \$150,000 advertising budget. To increase automobile sales, the firm is considering advertising in newspapers and on television. The more Carco uses a particular medium, the less effective is each additional ad. Table 44 shows the number of new customers reached by each ad. Each newspaper ad costs \$1000, and each television ad costs 10,000. At most, 30 newspaper ads and 15 television ads can be placed. How can Carco maximize the number of new customers created by advertising?

11 Sunco Oil has refineries in Los Angeles and Chicago. The Los Angeles refinery can refine up to 2 million barrels of oil per year, and the Chicago refinery up to 3 million. Once refined, oil is shipped to two distribution points: Houston and New York City. Sunco estimates that each distribution point can sell up to 5 million barrels per year. Because of differences in shipping and refining costs, the profit earned (in dollars) per million barrels of oil shipped depends on where the oil was refined and on the point of distribution (see Table 45). Sunco is considering expanding the capacity of each refinery. Each million barrels of annual refining capacity that is added will cost \$120,000 for the Los Angeles refinery and \$150,000 for the Chicago refinery. Use linear programming to determine how Sunco can maximize its profits less expansion costs over a ten-year period.

12 For a telephone survey, a marketing research group needs to contact at least 150 wives, 120 husbands, 100 single adult males, and 110 single adult females. It costs \$2 to make a daytime call and (because of higher labor costs) \$5 to make an evening call. Table 46 lists the results. Because of limited staff, at most half of all phone calls can be evening calls. Formulate an LP to minimize the cost of completing the survey.

13 Feedco produces two types of cattle feed, both consisting totally of wheat and alfalfa. Feed 1 must contain at least 80% wheat, and feed 2 must contain at least 60% alfalfa. Feed 1 sells for \$1.50/lb, and feed 2 sells for \$1.30/lb. Feedco can purchase up to 1000 lb of wheat at 50¢/lb and up to 800 lb of alfalfa at 40¢/lb. Demand for each type of feed is unlimited. Formulate an LP to maximize Feedco's profit.

TABLE **44**

	Number of Ads	New Customers
Newspaper	1–10	900
	11–20	600
	21–30	300
Television	1–5	10,000
	6–10	5000
	11–15	2000

TABLE **45**

	Profit per Million Barrels	
	To Houston	To New York
From Los Angeles	\$20,000	\$15,000
From Chicago	\$18,000	\$17,000

TABLE **46**

Person Responding	Percent of Daytime Calls	Percent of Evening Calls
Wife	30	30
Husband	10	30
Single male	10	15
Single female	10	20
None	40	5

14 Feedco (see Problem 13) has decided to give its customer (assume it has only one customer) a quantity discount. If the customer purchases over 300 lb of feed 1, each pound over the first 300 lb will sell for only \$1.25/lb. Similarly, if the customer purchases more than 300 pounds of feed 2, each pound over the first 300 lb will sell for \$1.00/lb. Modify the LP of Problem 13 to account for the presence of quantity discounts. (*Hint:* Define variables for the feed sold at each price.)

15 Chemco produces two chemicals: A and B. These chemicals are produced via two manufacturing processes. Process 1 requires 2 hours of labor and 1 lb of raw material to produce 2 oz of A and 1 oz of B. Process 2 requires 3 hours of labor and 2 lb of raw material to produce 3 oz of A and 2 oz of B. Sixty hours of labor and 40 lb of raw material are available. Demand for A is unlimited, but only 20 oz of B can be sold. A sells for \$16/oz, and B sells for \$14/oz. Any B that is unsold must be disposed of at a cost of \$2/oz. Formulate an LP to maximize Chemco's revenue less disposal costs.

16 Suppose that in the CSL computer example of Section 3.12, it takes two months to train a technician and that during the second month of training, each trainee requires 10 hours of experienced technician time. Modify the formulation in the text to account for these changes.

17 Furnco manufactures tables and chairs. Each table and chair must be made entirely out of oak or entirely out of pine. A total of 150 board ft of oak and 210 board ft of pine are available. A table requires either 17 board ft of oak or 30 board ft of pine, and a chair requires either 5 board ft of oak or 13 board ft of pine. Each table can be sold for \$40, and each chair for \$15. Formulate an LP that can be used to maximize revenue.

†Based on Franklin and Koenigsberg (1973).

21[§] Chandler Enterprises produces two competing products: A and B. The company wants to sell these products to two groups of customers: group 1 and group 2. The value each customer places on a unit of A and B is as shown in Table 50. Each customer will buy either product A or product B, but not both. A customer is willing to buy product A if she believes that

$$\text{Value of product A} - \text{price of product A} \geq \text{Value of product B} - \text{price of product B}$$

and

$$\text{Value of product A} - \text{price of product A} \geq 0$$

A customer is willing to buy product B if she believes that

$$\text{Value of product B} - \text{price of product B} \geq \text{value of product A} - \text{price of product A}$$

and

$$\text{Value of product B} - \text{price of product B} \geq 0$$

Group 1 has 1000 members, and group 2 has 1500 members. Chandler wants to set prices for each product that ensure that group 1 members purchase product A and group 2 members

TABLE **49**

	Capital	Labor	Profit
Tract 1 Spruce	3	0.1	0.2
Tract 1 Hunting	3	0.2	0.4
Tract 1 Both	4	0.2	0.5
Tract 2 Spruce	1	0.05	0.06
Tract 2 Camping	30	5	0.09
Tract 2 Both	10	1.01	1.1

TABLE **50**

	Group 1 Customer	Group 2 Customer
Value of A to	\$10	\$12
Value of B to	\$8	\$15

[§]Based on Dobson and Kalish (1988).

purchase product B. Formulate an LP that will help Chandler maximize revenues.

22[†] Alden Enterprises produces two products. Each product can be produced on one of two machines. The length of time needed to produce each product (in hours), on each machine is as shown in Table 51. Each month, 500 hours of time are available on each machine. Each month, customers are willing to buy up to the quantities of each product at the prices given in Table 52. The company's goal is to maximize the revenue obtained from selling units during the next two months. Formulate an LP to help meet this goal.

23 Kiriakis Electronics produces three products. Each product must be processed on each of three types of machines. When a machine is in use, it must be manned by a worker. The time (in hours) required to process each product on each machine and the profit associated with each product are shown in Table 53. At present, five type 1 machines, three type 2 machines, and four type 3 machines are available. The company has ten workers available and must determine how many workers to assign to each machine. The plant is open 40 hours per week, and each worker works 35 hours per week. Formulate an LP that will enable Kiriakis to assign workers to machines in a way that maximizes weekly profits. (*Note:* A worker need not spend the entire work week manning a single machine.)

24 Gotham City Hospital serves cases from four diagnostic related groups (DRGs). The profit contribution, diagnostic service use (in hours), bed-day use (in days), nursing care use (in hours) and drug use (in dollars) are given in Table 54. At present the hospital has available each week 570 hours of diagnostic services, 1000 bed-days, 50,000 nursing hours, and $50,000 worth of drugs. To meet the community's minimum health care demands at least 10 DRG1, 15 DRG2, 40 DRG3, and 160 DRG4 cases must be handled each week. Use LP to determine the hospital's optimal mix of DRGs.[‡]

25 Oliver Winery produces four award-winning wines in Bloomington, Indiana. The profit contribution, labor hours, and tank usage (in hours) per gallon for each type of wine are given in Table 55. By law at most 100,000 gallons of wine can be produced each year. A maximum of 12,000 labor hours and 32,000 tank hours are available annually. Each gallon of wine 1 spends an average of $\frac{1}{3}$ year in inventory; wine 2, an average of 1 year; wine 3, an average of 2 years; wine 4, an average of 3.333 years. The winery's warehouse can handle an average inventory level of 50,000 gallons. Determine how much of each type of wine should be produced annually to maximize Oliver Winery's profit.

26 Graphically solve the following LP:

$$\begin{aligned} \min z = 5x_1 + \ &x_2 \\ \text{s.t.} \quad 2x_1 + \ &x_2 \geq 6 \\ x_1 + \ &x_2 \geq 4 \\ 2x_1 + 10&x_2 \geq 20 \\ x_1, &x_2 \geq 0 \end{aligned}$$

27 Grummins Engine produces diesel trucks. New government emission standards have dictated that the average pollution emissions of all trucks produced in the next three years cannot exceed 10 grams per truck. Grummins produces two types of trucks. Each type 1 truck sells for $20,000,

TABLE **51**

	Machine 1	Machine 2
Product 1	4	3
Product 2	7	4

TABLE **52**

	Demands		Prices	
	Month 1	Month 2	Month 1	Month 2
Product 1	100	190	$55	$12
Product 2	140	130	$65	$32

TABLE **53**

	Product 1	Product 2	Product 3
Machine 1	2	3	4
Machine 2	3	5	6
Machine 3	4	7	9
Profit	$6	$8	$10

[†]Based on Jain, Stott, and Vasold (1978).

TABLE **54**

	Profit	Diagnostic Services	Bed-day	Nursing Use	Drugs
DRG 1	2000	7	5	30	800
DRG 2	1500	4	2	10	500
DRG 3	500	2	1	5	150
DRG 4	300	1	0	1	50

TABLE **55**

	Profit	Labor	Tank
Wine 1	$6	.2 hr	.5 hr
Wine 2	$12	.3 hr	.5 hr
Wine 3	$20	.3 hr	1 hr
Wine 4	$30	.5 hr	1.5 hr

[‡]Based on Robbins and Tuntiwonpiboon (1989).

30 Graphically solve the following linear programming problem:

$$\begin{aligned} \max z &= 5x_1 - x_2 \\ \text{s.t.} \quad 2x_1 + 3x_2 &\geq 12 \\ x_1 - 3x_2 &\geq 0 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

31 Graphically find all solutions to the following LP:

$$\begin{aligned} \min z &= x_1 - 2x_2 \\ \text{s.t.} \quad x_1 \qquad &\geq 4 \\ x_1 + x_2 &\geq 8 \\ x_1 - x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

32 Each day Eastinghouse produces capacitors during three shifts: 8 A.M.–4 P.M., 4 P.M.–midnight, midnight–8 A.M. The hourly salary paid to the employees on each shift, the price charged for each capacitor made during each shift, and the number of defects in each capacitor produced during a given shift are shown in Table 57. Each of the company's 25 workers can be assigned to one of the three shifts. A worker produces 10 capacitors during a shift, but due to machinery limitations, no more than ten workers can be assigned to any shift. Each day at most 250 capacitors can be sold, and the average number of defects per capacitor for the day's production cannot exceed three. Formulate an LP to maximize Eastinghouse's daily profit (sales revenue – labor cost).

33 Graphically find all solutions to the following LP:

$$\begin{aligned} \max z &= 4x_1 + x_2 \\ \text{s.t.} \quad 8x_1 + 2x_2 &\leq 16 \\ x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

34 During the next three months Airco must meet (on time) the following demands for air conditioners: month 1, 300; month 2, 400; month 3, 500. Air conditioners can be produced in either New York or Los Angeles. It takes 1.5 hours of skilled labor to produce an air conditioner in Los Angeles, and 2 hours in New York. It costs \$400 to produce an air conditioner in Los Angeles, and \$350 in New York. During each month each city has 420 hours of skilled labor available. It costs \$100 to hold an air conditioner in inventory for a month. At the beginning of month 1 Airco has 200 air

TABLE **57**

Shift	Hourly Salary	Defects (per capacitor)	Price
8 A.M.–4 P.M.	\$12	4	\$18
4 P.M.–Midnight	\$16	3	\$22
Midnight–8 A.M.	\$20	2	\$24

conditioners in stock. Formulate an LP whose solution will tell Airco how to minimize the cost of meeting air conditioner demands for the next three months.

35 Formulate the following as a linear programming problem: A greenhouse operator plans to bid for the job of providing flowers for city parks. He will use tulips, daffodils, and flowering shrubs in three types of layouts. A type 1 layout uses 30 tulips, 20 daffodils, and 4 flowering shrubs. A type 2 layout uses 10 tulips, 40 daffodils, and 3 flowering shrubs. A type 3 layout uses 20 tulips, 50 daffodils, and 2 flowering shrubs. The net profit is \$50 for each type 1 layout, \$30 for each type 2 layout, and \$60 for each type 3 layout. He has 1000 tulips, 800 daffodils, and 100 flowering shrubs. How many layouts of each type should be used to yield maximum profit?

36 Explain how your formulation in problem 35 changes if both of the following conditions are added:

a The number of type 1 layouts cannot exceed the number of type 2 layouts.

b There must be at least five layouts of each type.

37 Graphically solve the following LP problem:

$$\begin{aligned} \min z &= 6x_1 + 2x_2 \\ \text{s.t.} \quad 3x_1 + 2x_2 &\geq 12 \\ 2x_1 + 4x_2 &\geq 12 \\ x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Group B

38 Gotham City National Bank is open Monday–Friday from 9 A.M. to 5 P.M. From past experience, the bank knows that it needs the number of tellers shown in Table 58. The bank hires two types of tellers. Full-time tellers work 9–5 five days a week, except for 1 hour off for lunch. (The bank determines when a full-time employee takes lunch hour, but each teller must go between noon and 1 P.M. or between 1 P.M. and 2 P.M.) Full-time employees are paid (including fringe benefits) \$8/hour (this includes payment for lunch hour). The bank may also hire part-time tellers. Each part-time teller must work exactly 3 consecutive hours each day. A part-time teller is paid \$5/hour (and receives no fringe benefits). To maintain adequate quality of service, the bank has decided that at most five part-time tellers can be hired. Formulate an LP to meet the teller requirements at minimum cost. Solve the LP on a computer. Experiment with the LP answer to determine an employment policy that comes close to minimizing labor cost.

39[†] The Gotham City Police Department employs 30 police officers. Each officer works 5 days per week. The crime rate fluctuates with the day of the week, so the number of police officers required each day depends on which day of the week it is: Saturday, 28; Sunday, 18; Monday, 18; Tuesday, 24; Wednesday, 25; Thursday, 16; Friday, 21. The police department wants to schedule police officers to minimize the number whose days off are not consecutive. Formulate an LP that will accomplish this goal. (*Hint:* Have a constraint for each day of the week that ensures that the proper number of officers are *not* working on the given day.)

40[‡] Alexis Cornby makes her living buying and selling corn. On January 1, she has 50 tons of corn and \$1000. On the first day of each month Alexis can buy corn at the following prices per ton: January, \$300; February, \$350; March, \$400; April, \$500. On the last day of each month Alexis can sell corn at the following prices per ton: January, \$250; February, \$400; March, \$350; April, \$550. Alexis stores her corn in a warehouse that can hold at most 100 tons of corn. She must be able to pay cash for all corn at the time of purchase. Use linear programming to determine how Alexis can maximize her cash on hand at the end of April.

41[§] At the beginning of month 1, Finco has \$400 in cash. At the beginning of months 1, 2, 3, and 4, Finco receives certain revenues, after which it pays bills (see Table 59). Any money left over may be invested for one month at the interest rate of 0.1% per month; for two months at 0.5% per month; for three months at 1% per month; or for four months at 2% per month. Use linear programming to determine an investment strategy that maximizes cash on hand at the beginning of month 5.

42 City 1 produces 500 tons of waste per day, and city 2 produces 400 tons of waste per day. Waste must be incinerated at incinerator 1 or 2, and each incinerator can process up to 500 tons of waste per day. The cost to incinerate waste

TABLE **58**

Time Period	Tellers Required
9–10	4
10–11	3
11–Noon	4
Noon–1	6
1–2	5
2–3	6
3–4	8
4–5	8

TABLE **59**

	Revenues	Bills
Month 1	\$400	\$600
Month 2	\$800	\$500
Month 3	\$300	\$500
Month 4	\$300	\$250

[†] Based on Rothstein (1973).

[‡] Based on Charnes and Cooper (1955).

[§] Based on Robichek, Teichroew, and Jones (1965).

TABLE 62

Refired Grade of Germanium	Percent Yielded by Refiring			
	Defective	Grade 1	Grade 2	Grade 3
Defective	30	0	0	0
Grade 1	25	30	0	0
Grade 2	15	30	40	0
Grade 3	20	20	30	50
Grade 4	10	20	30	50

44[§] A paper recycling plant processes box board, tissue paper, newsprint, and book paper into pulp that can be used to produce three grades of recycled paper (grades 1, 2, and 3). The prices per ton and the pulp contents of the four inputs are shown in Table 63. Two methods, de-inking and asphalt dispersion, can be used to process the four inputs into pulp. It costs \$20 to de-ink a ton of any input. The process of de-inking removes 10 percent of the input's pulp, leaving 90% of the original pulp. It costs \$15 to apply asphalt dispersion to a ton of material. The asphalt dispersion process removes 20% of the input's pulp. At most 3000 tons of input can be run through the asphalt dispersion process or the de-inking process. Grade 1 paper can only be produced with newsprint or book paper pulp; grade 2 paper, only with book paper, tissue paper, or box board pulp; and grade 3 paper, only with newsprint, tissue paper, or box board pulp. To meet its current demands, the company needs 500 tons of pulp for grade 1 paper, 500 tons of pulp for grade 2 paper, and 600 tons of pulp for grade 3 paper. Formulate an LP to minimize the cost of meeting the demands for pulp.

45 Turkeyco produces two types of turkey cutlets for sale to fast food restaurants. Each type of cutlet consists of white meat and dark meat. Cutlet 1 sells for \$4/lb and must consist of at least 70% white meat. Cutlet 2 sells for \$3/lb and must consist of at least 60% white meat. At most 50 lb of cutlet 1 and 30 lb of cutlet 2 can be sold. The two types of turkey used to manufacture the cutlets are purchased from the GobbleGobble Turkey Farm. Each type 1 turkey costs \$10 and yields 5 lb of white meat and 2 lb of dark meat. Each type 2 turkey costs \$8 and yields 3 lb of white meat and 3 lb of dark meat. Formulate an LP to maximize Turkeyco's profit.

TABLE 63

	Cost	Pulp Content
Box board	\$5	15%
Tissue paper	\$6	20%
Newsprint	\$8	30%
Book paper	\$10	40%

[§]Based on Glassey and Gupta (1975).

46 Priceler manufactures sedans and wagons. The number of vehicles that can be sold each of the next three months are listed in Table 64. Each sedan sells for \$8000, and each wagon sells for \$9000. It costs \$6000 to produce a sedan and \$7500 to produce a wagon. To hold a vehicle in inventory for one month costs \$150 per sedan and \$200 per wagon. During each month, at most 1500 vehicles can be produced. Production line restrictions dictate that during month 1, at least two thirds of all cars produced must be sedans. At the beginning of month 1, 200 sedans and 100 wagons are available. Formulate an LP that can be used to maximize Priceler's profit during the next three months.

47 The production-line employees at Grummins Engine work four days a week, ten hours a day. Each day of the week, (at least) the following numbers of line employees are needed: Monday–Friday, 7 employees; Saturday and Sunday, 3 employees. Grummins has 11 production-line employees. Formulate an LP that can be used to maximize the number of consecutive days off received by the employees. For example, a worker who gets Sunday, Monday, and Wednesday off receives two consecutive days off.

48 Bank 24 is open 24 hours per day. Tellers work two consecutive 6-hour shifts and are paid \$10 per hour. The possible shifts are as follows: midnight–6 A.M., 6 A.M.–noon, noon–6 P.M., 6 P.M.–midnight. During each shift, the following numbers of customers enter the bank: midnight–6 A.M., 100; 6 A.M.–noon, 200; noon–6 P.M., 300; 6 P.M.–midnight, 200. Each teller can serve up to 50 customers per shift. To model a cost for customer impatience, we assume that any customer who is present at the end of a shift "costs" the bank \$5. We assume that by midnight of each day, all customers must be served, so each day's midnight–6 A.M. shift begins with 0 customers in the bank. Formulate an LP that can be used to minimize the sum of the bank's labor and customer impatience costs.

49[†] Transeast Airlines flies planes on the following route: L.A.–Houston–N.Y.–Miami–L.A. The length (in miles) of each segment of this trip is as follows: L.A.–Houston, 1500 miles; Houston–N.Y., 1700 miles; N.Y.–Miami, 1300 miles; Miami–L.A., 2700 miles. At each stop, the plane may purchase up to 10,000 gallons of fuel. The price of fuel at each city is as follows: L.A., 88¢; Houston, 15¢; N.Y., \$1.05; Miami, 95¢. The plane's fuel tank can hold at most 12,000 gallons. To allow for the possibility of circling over a landing site, we require that the ending fuel level for each leg of the flight be at least 600 gallons. The number of gallons used per mile on each leg of the flight is

$$1 + (\text{average fuel level on leg of flight}/2000)$$

To simplify matters, assume that the average fuel level on any leg of the flight is

$$\frac{(\text{Fuel level at start of leg}) + (\text{fuel level at end of leg})}{2}$$

Formulate an LP that can be used to minimize the fuel cost incurred in completing the schedule.

50[‡] To process income tax forms, the IRS first sends each form through the data preparation (DP) department, where information is coded for computer entry. Then the form is sent to data entry (DE), where it is entered into the computer. During the next three weeks, the following number of forms will arrive: week 1, 40,000; week 2, 30,000; week 3, 60,000. The IRS meets the crunch by hiring employees who work 40 hours per week and are paid \$200 per week. Data preparation of a form requires 15 minutes, and data entry of a form requires 10 minutes. Each week, an employee is assigned to either data entry or data preparation. The IRS must complete processing of all forms by the end of week 5 and wants to minimize the cost of accomplishing this goal. Formulate an LP that will determine how many workers should be working each week and how the workers should be assigned over the next five weeks.

51 In the electrical circuit of Figure 9, I_t = current (in amperes) flowing through resistor t, V_t = voltage drop (in volts) across resistor t, and R_t = resistance (in ohms) of resistor t. Kirchoff's Voltage and Current Laws imply that $V_1 = V_2 = V_3$ and $I_1 + I_2 + I_3 = I_4$. The power dissipated by the current flowing through resistor t is $I_t^2 R_t$. Ohm's Law implies that $V_t = I_t R_t$. The two parts of this problem should be solved independently.

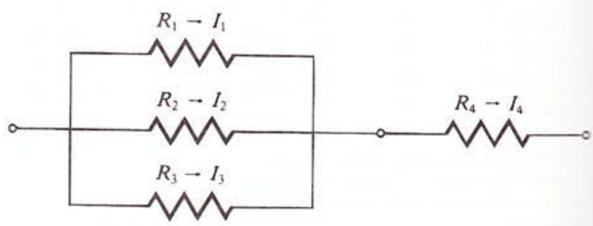
a Suppose you are told that $I_1 = 4$, $I_2 = 6$, $I_3 = 8$, and $I_4 = 18$ are required. Also, the voltage drop across each resistor must be between 2 and 10 volts. Choose the R_t's to minimize the total dissipated power. Formulate an LP whose solution will solve your problem.

b Suppose you are told that $V_1 = 6$, $V_2 = 6$, $V_3 = 6$, and $V_4 = 4$ are required. Also, the current flowing through each resistor must be between 2 and 6 amperes.

TABLE **64**

	Sedans	Wagons
Month 1	1100	600
Month 2	1500	700
Month 3	1200	500

FIGURE **9**



[†] Based on Darnell and Loflin (1977).

[‡] Based on Lanzenauer et al. (1987).

54[‡] Olé Oil produces three products: heating oil, gasoline, and jet fuel. The average octane levels must be at least 4.5 for heating oil, 8.5 for gas, and 7.0 for jet fuel. To produce these products Olé purchases two types of oil: crude 1 (at \$12 per barrel) and crude 2 (at \$10 per barrel). Each day, at most 10,000 barrels of each type of oil can be purchased.

Before crude can be used to produce products for sale, it must be distilled. Each day, at most 15,000 barrels of oil can be distilled. It costs 10¢ to distill a barrel of oil. The result of distillation is as follows: (1) Each barrel of crude 1 yields 0.6 barrel of naphtha, 0.3 barrel of distilled 1, and 0.1 barrel of distilled 2. (2) Each barrel of crude 2 yields 0.4 barrel of naphtha, 0.2 barrel of distilled 1, and 0.4 barrel of distilled 2. Distilled naphtha can be used only to produce gasoline or jet fuel. Distilled oil can be used to produce heating oil or it can be sent through the catalytic cracker (at a cost of 15¢ per barrel). Each day, at most 5000 barrels of distilled oil can be sent through the cracker. Each barrel of distilled 1 sent through the cracker yields 0.8 barrel of cracked 1 and 0.2 barrel of cracked 2. Each barrel of distilled 2 sent through the cracker yields 0.7 barrel of cracked 1 and 0.3 barrel of cracked 2. Cracked oil can be used to produce gasoline and jet fuel but not to produce heating oil.

The octane level of each type of oil is as follows: naphtha, 8; distilled 1, 4; distilled 2, 5; cracked 1, 9; cracked 2, 6.

All heating oil produced can be sold at \$14 per barrel; all gasoline produced, \$18 per barrel; and all jet fuel produced, \$16 per barrel. Marketing considerations dictate that at least 3000 barrels of each product must be produced daily. Formulate an LP to maximize Olé's daily profit.

55 Donald Rump is the international funds manager for Countribank (he needs the money to support Marva!). Each-day Donald's job is to determine how the bank's current holdings of dollars, pounds, marks, and yen should be adjusted to meet the day's currency needs. Today the exchange rates between the various currencies are given in Table 67. For example, one dollar can be converted to .58928 pounds, or one pound can be converted to 1.697 dollars.

At the beginning of the day Countribank has the currency holdings given in Table 68.

At the end of the day, Countribank must have at least the amounts of each currency given in Table 69.

[‡]Based on Garvin et al. (1957).

Donald's goal is to each day transfer funds in a way that makes currency holdings satisfy the previously listed minimums, and maximizes the dollar value of the currency holdings at the end of the day.

To figure out the dollar value of, say, one pound, average the two conversion rates. Thus, one pound is worth about

$$\frac{1.697 + (1/.58928)}{2} = 1.696993 \text{ dollars}$$

TABLE 67

From	To: Dollars	Pounds	Marks	Yen
Dollars	1	.58928	1.743	138.3
Pounds	1.697	1	2.9579	234.7
Marks	.57372	.33808	1	79.346
Yen	.007233	.00426	.0126	1

TABLE 68

Currency	Units (in billions)
Dollars	8
Pounds	1
Marks	8
Yen	0

TABLE 69

Currency	Units (in billions)
Dollars	6
Pounds	3
Marks	1
Yen	10

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Each of the following six books is a cornucopia of interesting LP formulations:

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Dobson, G., and S. Kalish "Positioning and Pricing a Product Line," *Marketing Science* 7(1988):107–126.

Forgionne, G. "Corporate MS Activities: An Update," *Interfaces* 13(1983):20–23. Concerns the fraction of large firms using linear programming (and other operations research techniques).

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Step 4 For now, ignore the original LP's objective function. Instead, solve an LP whose objective function is min w' = (sum of all the artificial variables). This is called the **Phase I LP**.

Since each $a_i \geq 0$, solving the Phase I LP will result in one of the following three cases:

Case 1 The optimal value of w' is greater than zero. In this case, the original LP has no feasible solution.

Case 2 The optimal value of w' is equal to zero, and no artificial variables are in the optimal Phase I basis. In this case, drop all columns in the optimal Phase I tableau that correspond to the artificial variables and combine the original objective function with the constraints from the optimal Phase I tableau. This yields the **Phase II LP**. The optimal solution to the Phase II LP and the original LP are the same.

Case 3 The optimal value of w' is equal to zero, and at least one artificial variable is in the optimal Phase I basis.

Solving Minimization Problems

To solve a minimization problem by the simplex, choose as the entering variable the nonbasic variable in row 0 with the most positive coefficient. A tableau or canonical form is optimal if each variable in row 0 has a nonpositive coefficient.

Alternative Optimal Solutions

If a nonbasic variable has a zero coefficient in row 0 of an optimal tableau and the nonbasic variable can be pivoted into the basis, the LP may have **alternative optimal solutions**. If two basic feasible solutions are optimal, any point on the line segment joining the two optimal basic feasible solutions is also an optimal solution to the LP.

Variables That Are Unrestricted in Sign

If we replace a urs variable x_i by $x_i' - x_i''$, the LP's optimal solution will have x_i', x_i'' or both x_i' and x_i'' equal to zero.

Review Problems

Group A

1 Use the simplex algorithm to find *two* optimal solutions to the following LP:

$$
\begin{aligned}
\max z &= 5x_1 + 3x_2 + x_3 \\
\text{s.t.} \quad x_1 + x_2 + 3x_3 &\leq 6 \\
5x_1 + 3x_2 + 6x_3 &\leq 15 \\
x_3, x_1, x_2 &\geq 0
\end{aligned}
$$

2 Use the simplex algorithm to find the optimal solution to the following LP:

$$
\begin{aligned}
\min z &= -4x_1 + x_2 \\
\text{s.t.} \quad 3x_1 + x_2 &\leq 6 \\
-x_1 + 2x_2 &\leq 0 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

... and the two-phase method to find ... the following LP:

$$\begin{aligned} \max z &= 5x_1 - x_2 \\ \text{s.t.}\quad 2x_1 + x_2 &= 6 \\ x_1 + x_2 &\le 4 \\ x_1 + 2x_2 &\le 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

... algorithm to find the optimal solution ...

$$\begin{aligned} \max z &= 5x_1 - x_2 \\ \text{s.t.}\quad x_1 - 3x_2 &\le 1 \\ x_1 - 4x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

... algorithm to find the optimal solution ...

$$\begin{aligned} \min z &= -x_1 - 2x_2 \\ \text{s.t.}\quad 2x_1 + x_2 &\le 5 \\ x_1 + x_2 &\le 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

... and the two-phase method to find ... the following LP:

$$\begin{aligned} \max z &= x_1 + x_2 \\ \text{s.t.}\quad 2x_1 + x_2 &\ge 3 \\ 3x_1 + x_2 &\le 3.5 \\ x_1 + x_2 &\le 1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

... algorithm to find *two* optimal solutions ... optimal solutions does this ... solution.

$$\begin{aligned} \max z &= 4x_1 + x_2 \\ \text{s.t.}\quad 2x_1 + 3x_2 &\le 4 \\ x_1 + x_2 &\le 1 \\ 4x_1 + x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

... method to find the optimal solution to ...

$$\begin{aligned} \max z &= 5x_1 + x_2 \\ \text{s.t.}\quad 2x_1 + x_2 &\le 6 \\ x_1 - x_2 &\le 0 \\ x_1, x_2 &\ge 0 \end{aligned}$$

9 Use the two-phase method to find the optimal solution to the following LP:

$$\begin{aligned} \min z &= -3x_1 + x_2 \\ \text{s.t.}\quad x_1 - 2x_2 &\ge 2 \\ -x_1 + x_2 &\ge 3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

10 Suppose that in the Dakota Furniture problem, there were ten types of furniture that could be manufactured. To obtain an optimal solution, how many types of furniture (at the most) would have to be manufactured?

11 Consider the following LP:

$$\begin{aligned} \max z &= 10x_1 + x_2 \\ \text{s.t.}\quad x_1 \qquad &\le 1 \\ 20x_1 + x_2 &\le 100 \\ x_1, x_2 &\ge 0 \end{aligned}$$

a Find all the basic feasible solutions for this LP.

b Show that when the simplex is used to solve this LP, every basic feasible solution must be examined before the optimal solution is found.

By generalizing this example, Klee and Minty (1972) constructed (for $n = 2, 3, \ldots$) an LP with n decision variables and n constraints for which the simplex algorithm examines $2^n - 1$ basic feasible solutions before the optimal solution is found. Thus, there exists an LP with 10 variables and 10 constraints for which the simplex requires $2^{10} - 1 = 1023$ pivots to find the optimal solution. Fortunately, such "pathological" LPs rarely occur in practical applications.

Group B

12 Consider a maximization problem with the optimal tableau in Table 43. The optimal solution to this LP is $z = 10, x_3 = 3, x_4 = 5, x_1 = x_2 = 0$. Determine the second-best bfs to this LP. (*Hint*: Show that the second-best solution must be a bfs that is one pivot away from the optimal solution.)

13 A camper is considering taking two types of items on a camping trip. Item 1 weighs a_1 lb, and item 2 weighs a_2 lb. Each type 1 item earns the camper a benefit of c_1 units, and each type 2 item earns the camper c_2 units. The knapsack can hold items weighing at most b lb.

a Assuming that the camper can carry a fractional number of items along on the trip, formulate an LP to maximize benefit.

TABLE **43**

z	x_1	x_2	x_3	x_4	rhs
1	2	1	0	0	10
0	3	2	1	0	3
0	4	3	0	1	5

since the fourth primal constraint is a $\le$ constraint, the fourth dual variable x_4 must satisfy $x_4 \le 0$. We can now complete the table (see Table 20). Reading the dual across, we obtain

$$
\begin{aligned}
\max z = 2x_1 + x_2 + x_3 + 3x_4 & \\
\text{s.t.} \quad x_1 + x_2 \qquad + 2x_4 &= 2 \\
2x_1 \qquad + x_3 + \ x_4 &\le 4 \\
x_1 - x_2 + x_3 \qquad &\le 6 \\
x_1, x_2 \ge 0, x_3 \text{ urs}, x_4 &\le 0
\end{aligned}
$$

The reader may verify that with these rules, the dual of the dual is always the primal. This is easily seen from the Table 14 format, because when you take the dual of the dual you are changing the LP back to its original position.

TABLE 19
Finding the Dual of LP (19)

min w		max z				
		$(x_1 \ge 0)$	$(x_2 \ge 0)$			
		x_1	x_2	x_3	x_4	
$(y_1$ urs$)^*$	y_1	1	1	0	2	2
$(y_2 \ge 0)$	y_2	2	0	1	1	≤ 4
$(y_3 \ge 0)$	y_3	1	-1	1	0	≤ 6
		≥ 2	≥ 1	$= 1^*$	$\le 3^*$	

TABLE 20
Finding the Dual of LP (19) (Continued)

min w		max z				
		$(x_1 \ge 0)$	$(x_2 \ge 0)$	$(x_3$ urs$)$	$(x_4 \le 0)$	
		x_1	x_2	x_3	x_4	
$(y_1$ urs$)$	y_1	1	1	0	2	$= 2$
$(y_2 \ge 0)$	y_2	2	0	1	1	≤ 4
$(y_3 \ge 0)$	y_3	1	-1	1	0	≤ 6
		≥ 2	≥ 1	$= 1$	≤ 3	

Problems

Group A

Find the duals of the following LPs:

1
$$
\begin{aligned}
\max z = 2x_1 + x_2 & \\
\text{s.t.} \quad -x_1 + x_2 &\le 1 \\
x_1 + x_2 &\le 3 \\
x_1 - 2x_2 &\le 4 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

2
$$
\begin{aligned}
\min w = y_1 - y_2 & \\
\text{s.t.} \quad 2y_1 + y_2 &\ge 4 \\
y_1 + y_2 &\ge 1 \\
y_1 + 2y_2 &\ge 3 \\
y_1, y_2 &\ge 0
\end{aligned}
$$

3
$$
\begin{aligned}
\max z = 4x_1 - x_2 + 2x_3 & \\
\text{s.t.} \quad x_1 + x_2 \qquad &\le 5 \\
2x_1 + x_2 \qquad &\le 7 \\
2x_2 + x_3 &\ge 6 \\
x_1 \qquad + x_3 &= 4 \\
x_1 \ge 0, x_2, x_3 & \text{ urs}
\end{aligned}
$$

4
$$
\begin{aligned}
\min w = 4y_1 + 2y_2 - y_3 & \\
\text{s.t.} \quad y_1 + 2y_2 \qquad &\le 6 \\
y_1 - y_2 + 2y_3 &= 8 \\
y_1, y_2 \ge 0, y_3 & \text{ urs}
\end{aligned}
$$