THE UNIVERSITY OF MELBOURNE

Student ID

Semester 1 Assessment, 2016

School of Mathematics and Statistics

MAST10007 Linear Algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 13 pages (including this page)

Authorised materials:

- No materials are authorised.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- All answers should be appropriately justified.
- Some notation used in this exam:

  $\mathcal{P}_n$ denotes the (real) vector space of all polynomials of degree at most n.
  $M_{m,n}$ denotes the (real) vector space of all $m \times n$ matrices.
  $\mathcal{F}(S, \mathbb{R})$ denotes the (real) vector space of functions from a set S to $\mathbb{R}$.

- There are 12 questions. You should attempt all questions.
- The total number of marks available is 100.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

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Question 1 (8 marks)

(a) Do the following planes have a common point of intersection?

$$\begin{aligned} 2x - 2z &= 2, \\ x + 2y + z &= 1, \\ -2y - 2z &= 2. \end{aligned}$$

If so, find the point.

(b) Define

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 2 & 1 \\ 0 & -2 & -2 \end{bmatrix}$$

From your working in part (a), find $\det A$.

(c) Does A^{-1} exist? If so, find A^{-1}.

Question 2 (10 marks)

(a) What is the cosine of the angle between the vectors $\mathbf{u} = (-1, 1, 2)$ and $\mathbf{v} = (1, 0, -1)$? Are the vectors parallel? Are the vectors perpendicular? Is the angle acute (at most $\pi/2$) or obtuse (bigger than $\pi/2$)?

(b) Write down the vector and Cartesian forms of the plane P that contains the vectors $\mathbf{u}$ and $\mathbf{v}$ as well as the point $(1, 0, 3)$.

(c) Does the line

$$L_1 : (x, y, z) = (1, -1, 1) + t(1, -2, 0), \quad t \in \mathbb{R}$$

intersect with P? If so, write down the point of intersection.

(d) Write down the vector equation of the line

$$L_2 : -\frac{x}{3} = y = \frac{z-1}{3}.$$

(e) For any two lines L_1, L_2 we can always find two parallel planes P_1, P_2 such that L_1 is in P_1 and L_2 is in P_2. Find the cartesian equations for the planes P_1 and P_2 with L_1 from part (c) and (d). (*Hint: You need to find a single vector that is normal to both planes — think cross product.*)

Question 3 (8 marks)

(a) Use co-factor expansion to find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

(b) Find A^{-1} if possible.

(c) Express the following linear system as a matrix equation:

$$\begin{aligned} 3x + y + z &= 2 \\ 2x + 2y - z &= 1 \\ x + z &= 0 \end{aligned}$$

What does part (b) tell you about the solution of this linear system?

(d) Write down all solutions to the linear system in part (c).

Question 4 (8 marks)

Define

$$B = \begin{bmatrix} 1 & 1 & 2 & -1 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 & 0 \end{bmatrix}$$

(a) Is $\{(1,1,0),(1,0,1),(2,1,1),(-1,2,3),(2,2,0)\}$ a linearly independent subset of $\mathbb{R}^3$?

(b) What is the column space of B? Find a basis for it.

(c) What is the row space of B? Find a basis for it.

(d) What are the dimensions of the column space, row space and null space?

(e) Find a basis for the null space of B.

(f) Is $(2,0,3,-1,0)$ in the null space of B? If so, express it as a linear combination of the basis vectors from part (e).

Question 5 (10 marks)

(a) Is the set $Q_1 = \{a_0 + a_1x + a_2x^2 : a_0 + a_1 + a_2 = -1, a_0, a_1, a_2 \in \mathbb{R}\}$ a subspace of $\mathcal{P}_2$? Why/why not?

(b) Is the set $Q_2 = \{a_0 + a_1x + a_2x^2 : a_0 + a_1 + a_2 = 0, a_0, a_1, a_2 \in \mathbb{R}\}$ a subspace of $\mathcal{P}_2$? Why/why not?

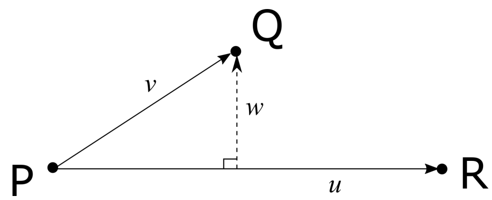
(c) Show that $C = \{1 + x - 2x^2, 2 - 2x^2\}$ is a basis for Q_2. You may want to use the fact that for any $a_0, a_1 \in \mathbb{R}$ we have

$$\left[\begin{array}{cc|c} 1 & 2 & a_0 \\ 1 & 0 & a_1 \\ -2 & -2 & -a_0 - a_1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & a_1 \\ 0 & 1 & \frac{a_0 - a_1}{2} \\ 0 & 0 & 0 \end{array}\right].$$

(d) Find the co-ordinate vector $[4 + x - 5x^2]_C$ where C is the basis from part (c).

Question 6 (8 marks)

Consider the figure



(a) If $\mathbf{v} = \mathbf{w} + \text{Proj}_{\mathbf{u}}\mathbf{v}$ in the diagram above, then show that $\mathbf{w}$ is perpendicular to $\mathbf{u}$ with any inner product $\langle \cdot, \cdot \rangle$. (*Hint: Recall that* $\text{Proj}_{\mathbf{u}}\mathbf{v} = \langle \hat{\mathbf{u}}, \mathbf{v} \rangle \hat{\mathbf{u}}$.)

(b) If we have the points $P = (0, 1, 3), Q = (1, 1, 2), R = (-1, 3, 3)$, find $\mathbf{u}$ and $\mathbf{v}$.

(c) Calculate $\mathbf{u} \times \mathbf{v}$.

(d) Using the cross product, what is the area of the parallelogram formed by the vectors $\mathbf{u}$ and $\mathbf{v}$?

(e) Use the formula Area(parallelogram) = base $\times$ height to confirm your answer from part (d).

(f) Introduce a new point $S = (1, 3, -1)$. Find the vector $\mathbf{z} = \overrightarrow{PS}$.

(g) Find the volume of the parrallelpiped formed by the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{z}$.

Question 7 (8 marks)

Let V be a real vector space and $P : V \to V$ a linear transformation such that $P \circ P = P$.

(a) Show that the only possible eigenvalues of P are 0 or 1.

(b) Show that every $v \in V$ is a sum of eigenvectors of P. (Hint: Consider the vectors $w = P(v)$ and $u = v - w$.)

Question 8 (8 marks)
Consider the function $T : \mathcal{P}_2 \to \mathbb{R}^3$ given by $T(f) = (f(0), f(1), f(-1))$.

(a) Show that T is a linear transformation

(b) Compute the matrix representation of T with respect to the standard bases of $\mathcal{P}_2$ and $\mathbb{R}^3$.

(c) Compute the rank and nullity of T.

Question 9 (8 marks)

Consider the ordered basis $\mathcal{B} = \{\mathbf{v}_1 = (1,1,1), \mathbf{v}_2 = (0,1,1), \mathbf{v}_3 = (0,0,1)\}$ of $\mathbb{R}^3$. (You do not have to show this is a basis.)

(a) Find a matrix A such that $\mathbf{v}_1^T$ is an eigenvector with eigenvalue 0, $\mathbf{v}_2^T$ is an eigenvector with eigenvalue 1 and $\mathbf{v}_3^T$ is an eigenvector with eigenvalue -1 for A.

(b) Compute the characteristic polynomial $\det(\lambda I - A)$ of A and verify that 0, 1 and -1 are zeroes of this polynomial.

(c) Find an invertible matrix B such that BAB^{-1} is diagonal.

Question 10 (8 marks)

(a) State the definition of an *inner product*.

(b) Verify that $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + 2x_2y_2 - x_1y_2 - x_2y_1$ defines an inner product on $\mathbb{R}^2$.

(c) Find an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product.

Question 11 (8 marks)

Consider the ordered bases $\mathcal{B} = \{(2,1,0),(0,1,0),(0,1,2)\}$ and $\mathcal{C} = \{(0,1,1),(0,1,0),(1,1,0)\}$ of $\mathbb{R}^3$ and the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (z,y,x)$

(a) Find the matrix representation $[T]_{\mathcal{C},\mathcal{B}}$.

(b) Find the transition matrix $P_{\mathcal{B},\mathcal{C}}$

(c) Using the matrices from parts (a) and (b) find $[T]_{\mathcal{B}}$.

(d) Compute $[T]_{\mathcal{B}}$ directly to check your results.

Question 12 (8 marks)

Let $T : V \to W$ be an injective linear transformation between finite-dimensional vector spaces of the same dimension n. Suppose $\mathcal{B} = \{v_1, \ldots, v_n\}$ is an ordered basis of V.

(a) Show that T is surjective.

(b) Show that there is an ordered basis $\mathcal{C}$ of W such that $[T]_{\mathcal{C},\mathcal{B}}$ is the $n \times n$-identity matrix.

End of Exam—Total Available Marks = 100.