The University of Melbourne
Semester 2 Assessment 2009

Department of Mathematics and Statistics
620-156 Linear Algebra

Reading Time: 15 minutes
Writing Time: 3 hours

This paper has: 5 pages

Identical Examination Papers: None
Common Content Papers: None

Authorized Materials:

No materials are authorized.

Calculators and mathematical tables are not permitted.

Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:

This examination consists of 11 questions.

The total number of marks is 100.

All questions may be attempted. All answers should be appropriately justified.

— BEGINNING OF EXAMINATION QUESTIONS —

1. (a) Consider the following linear system:

$$\begin{array}{rcrcrcrcl} 3x_1 & + & x_2 & + & x_3 & + & x_4 & = & 4 \\ -2x_1 & - & x_2 & - & x_3 & + & x_4 & = & 6 \\ x_1 & + & x_2 & & & + & x_4 & = & 1 \end{array}$$

   (i) Write down the augmented matrix corresponding the the linear system.

   (ii) Reduce the matrix to reduced row-echelon form.

   (iii) Use the reduced row-echelon form to give all solutions to the linear system.

   (b) Determine the values of k for which the following linear system has (i) no solution (ii) a unique solution (iii) infinitely many solutions.

$$\begin{array}{rcrcrcl} x & + & 2y & + & kz & = & 1 \\ 2x & + & ky & + & 8z & = & 3 \end{array}$$

[10 marks]

2. Let

$$A = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 2 & 0 \\ 1 & 5 & 6 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Evaluate, if possible:

(a) AB

(b) BA

(c) AA^T

(d) $C + CB$

[6 marks]

3. Consider the matrix

$$M = \begin{bmatrix} 4 & -2 & 5 \\ 1 & 1 & -2 \\ 6 & 1 & -1 \end{bmatrix}$$

(a) Calculate its determinant $\det(M)$.

(b) Find M^{-1} or explain why it does not exist.

(c) Suppose that N is a 3×3 matrix with $\det(N) = 5$. What is $\det(2NM)$?

[10 marks]

4. (a) Let L be the line in $\mathbb{R}^3$ that contains the points $(-1, 1, 5)$ and $(6, -3, 0)$.
   (i) What is the vector equation of the line ?
   (ii) What is the cartesian equation of the line ?

   (b) Find the Cartesian equation of the plane Π that contains the point $(2, 3, 7)$ and is perpendicular to the line L.

   (c) Find the point at which L and Π intersect.

[6 marks]

5. For each of the following, decide whether or not the given set S is a subspace of the vector space V. Justify your answers by either appealing to appropriate theorems, or providing a counter-example.

   (a) $V = \mathcal{P}_2$ (all polynomials of degree at most two) and S is the collection of all polynomials of the form $p(x) = (a - c) + (5b + c)x$ where $a, b, c \in \mathbb{R}$.

   (b) $V = M_{2,2}$ (all 2×2 matrices) and S is the collection of all matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with either $a + d = 0$ or $b + c = 0$.

[6 marks]

6. Let V be a vector space and S a subset of V.

   (a) What does it mean to say that S is linearly independent?

   (b) Show that if $\mathbf{0} \in S$, then S is linearly dependent.

   (c) Define Span(S), the span of the set S.

   (d) Suppose that $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\} \subseteq V$ is linearly independent. Show that for any $\boldsymbol{v} \in V$,

$$\boldsymbol{v} \in \text{Span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\} \iff \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k, \boldsymbol{v}\} \text{ is linearly dependent}$$

[10 marks]

7. (a) Let $W \subseteq \mathbb{R}^4$ the span of the vectors in the set

$$S = \{(-3, 1, 0, -4), (4, -3, 5, 2), (1, -7, 20, -12), (11, 3, -2, 8)\}$$

(i) Find a basis B for W satisfying $B \subseteq S$.

(ii) What is the dimension of W ?

(iii) Are the vectors in S linearly independent ?

(b) The following two matrices are row-equivalent:

$$A = \begin{bmatrix} -1 & 5 & 3 & 0 & 7 \\ 3 & 1 & 7 & 1 & 12 \\ 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & 1 & 4 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) What is the rank of A ?

(ii) Write one of the columns of A as a linear combination of the others.

(iii) Write down a basis for the column space of A.

(iv) Find a basis for the solution space of A.

[12 marks]

8. Let $T : \mathcal{P}_2 \to \mathcal{P}_1$ be the linear transformation given by

$$T(a_0 + a_1x + a_2x^2) = (-3a_0 + 14a_1 - 6a_2) + (a_0 - 4a_1 + 2a_2)x$$

(a) Find the matrix $[T]_{\mathcal{C},\mathcal{B}}$ of T with respect to the bases $\mathcal{B} = \{1, x, x^2\}$ for $\mathcal{P}_2$ and $\mathcal{C} = \{1, x\}$ for $\mathcal{P}_1$.

(b) The set $\mathcal{D} = \{-3 + x, 7 - 2x\}$ is also a basis for $\mathcal{P}_1$.
Write down the transition matrix $P_{\mathcal{C},\mathcal{D}}$ from the basis $\mathcal{D}$ to the basis $\mathcal{C}$.

(c) Give an expression for $P_{\mathcal{D},\mathcal{C}}$ in terms of $P_{\mathcal{C},\mathcal{D}}$ and use it to calculate the transition matrix $P_{\mathcal{D},\mathcal{C}}$.

(d) Calculate the matrix $[T]_{\mathcal{D},\mathcal{B}}$ of T with respect to the bases $\mathcal{B}$ and $\mathcal{D}$.

[10 marks]

9. (a) Show that the following defines an inner product on $\mathbb{R}^3$:

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + 3x_2 y_2 + 7x_3 y_3 - x_1 y_2 - x_2 y_1 + 2x_1 y_3 + 2x_3 y_1$$

(b) Starting with the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$, apply the Gram-Schmidt procedure to obtain a basis of $\mathbb{R}^3$ that is orthonormal with respect to the inner product of part (a).

(c) Let V be an inner product space, and $S \subseteq V$ a subset. Show that if S is orthonormal, then it is linearly independent.

[12 marks]

10. (a) Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 5 & 1 & -4 \end{bmatrix}$$

(i) Find the characteristic polynomial of M.

(ii) What are the eigenvalues of M ?

(b) Consider the matrix

$$A = \begin{bmatrix} -1 & -1 & 1 & -1 \\ -2 & 0 & 1 & -1 \\ -2 & 3 & -2 & -1 \\ 4 & -10 & 10 & 3 \end{bmatrix}$$

It is known that A has two distinct eigenvalues. One of the eigenvalues is equal to 1, and the other has associated eigenvector $\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T$. Using this information answer the following questions.

(i) What is the other eigenvalue of A?

(ii) Find a basis for the eigenspace associated with each (distinct) eigenvalue.

(iii) Is A diagonalizable ? Justify your answer.

[12 marks]

11. Consider the curve C defined by the equation $5x^2 - 2xy + 5y^2 = 1$.

(a) Identify the curve C.

(b) Give the directions of the principal axes.

(c) Sketch the curve, and give the co-ordinates of the points at which it crosses the principal axes.

[6 marks]

— END OF EXAMINATION QUESTIONS —