MAST30001 Stochastic Modelling

Tutorial Sheet 4

1. Refer to Tutorial Sheet 3, Problem 3 and also assume that in Corner 2 there is a bigger spider ready to eat the little spider and in Corner 3, there is a hole leading to the outside through which the spider can escape.

   (a) If the spider starts in Corner 1, what is the chance the spider will escape before being eaten?

   (b) If the spider starts in Corner 1, what is the expected number of steps before the spider exits (either the box or this world)?

   (c) If initially the spider is dropped in the middle of the box and it chooses a corner uniformly, what is the chance the spider will escape before being eaten?

   (d) If initially the spider is dropped in the middle of the box and it chooses a corner uniformly, what is the expected number of steps before the spider exits (either the box or this world)?

2. A simplified model for the spread of a contagion in a small population of size 5 is as follows. At each discrete time unit, two individuals in the population are chosen uniformly at random to meet. If one of these persons is healthy and the other has the contagion, then with probability $1/4$ the healthy person becomes sick. Otherwise the system stays the same. If initially one person has the disease, what is the average amount of time before everyone in the population has the disease? What about if the population is of size N?

3. A Markov chain has transition matrix

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

   (a) Analyse the state space of the chain (reducibility, periodicity, null/pos recurrence, etc).

   (b) Find the stationary distribution of the chain.

   (c) If the initial state of the chain is uniformly distributed, find $\lim_{n\to\infty} P(X_n = 2)$.

4. A Markov chain has transition matrix

$$\begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix}$$

   Analyse the state space (reducibility, periodicity, null/pos recurrence, etc), and discuss the chain's long run behaviour.

5. A machine produces two items per day. The probability that an item is *not* defective is p, with all items produced independently, and defective items are thrown away immediately. The demand for items is one per day, and any demand not met by the end of the day is lost, while any extra item is stored. Let X_n be the number of items in storage just before the beginning of the nth day.

   (a) Model X_n as a Markov chain, draw its transition diagram and compute its transition probabilities.

   (b) When is the Markov chain ergodic? Compute the limiting distribution when it exists.

   (c) Suppose it costs \$$c$ to store an item for one night and \$$d$ for every demanded item that cannot be supplied. Compute the long run cost of the production facility when the chain is ergodic.