Assignment 1 MAST30030: Michael Le (998211) 12PM Wednesday Tute
Michael Le (998211). UNIMELB S1 2020.

(Q1) i. let $\boxed{\dot{x} = \frac{dx}{dt} = f(x) \; \ldots\ldots (1)}$

let $\bar{x}$ be a fixed point of the system from the lectures. By using the linear stability analysis.

*Quantitive approach to determine the stability of fixed points.
*which gives the rate a solution approaches/recedes from a fixed point.

Consider a small perturbation away from $\bar{x}$, i.e. let

$\boxed{x(t) = \bar{x} + \varepsilon(t) \; \ldots\ldots (2)}$,

Substitute (2) into (1) gives. $f(\bar{x}) = 0$ (fixed-point).

$\rightarrow \frac{d}{dt}(\bar{x} + \varepsilon(t)) = f(\bar{x} + \varepsilon(t))$

$\boxed{\frac{d\varepsilon}{dt} = f(\bar{x}+\varepsilon) \; \ldots\ldots (3)}$

$\boxed{\therefore \frac{d\varepsilon}{dt} = f(\bar{x}) + \varepsilon f'(\bar{x}) + \cancel{O(\varepsilon^2)} \quad \text{if } x = \bar{x} \ldots. (4)}$

If we let $f(\bar{x}) = 0$, 1st order (4)
is, $\frac{d\varepsilon}{dt} = \underbrace{f'(\bar{x})}_{\text{(constant)}}\varepsilon \leftarrow$ (linear ODE)

Whose general sol^n is,
$\varepsilon(t) = \varepsilon_0 \exp(f'(\bar{x})t) \Rightarrow$ exponential growth decay!!

$\therefore$ if $\varepsilon(t) = 0$, as $t \to \infty$
$\Rightarrow \therefore$ fixed point is inconclusive.
$\Rightarrow$ which satisfies. $f(\bar{x}) = 0$.

(Q1iii). $f(x)=x^4\cos(x)$, find the fixed points for this system.

let $\frac{dx}{dt}=f(x)=x^4\cos(x)$ — (1)

Fixed points are given by:

$\bar{x}^4\cos(\bar{x})=0$, $\bar{x}=0, \pm\pi/2, \pm 3\pi/2, \ldots$ etc

(Q1iv).

first we calculate $f'(x)$

$f'(x)=4x^3\cos(x)-x^4\sin(x)$ — (2)

$f'(x=0)=0$ inconclusive.

$f'(\pi/2)=-(\pi/2)^4\sin(\pi/2)=-(\pi/2)^4$

$f'(-\pi/2)=-(-\pi/2)^4\sin(-\pi/2)=(\pi/2)^4$

$f'(3\pi/2)=-(3\pi/2)^4\sin(3\pi/2)=(3\pi/2)^4$

$f'(-3\pi/2)=-(-3\pi/2)^4\sin(-3\pi/2)=(3\pi/2)^4$

$f'(5\pi/2)=-(5\pi/2)^4\sin(5\pi/2)=+(5\pi/2)^4$

$f'(-5\pi/2)=-(-5\pi/2)^4\sin(-5\pi/2)=(5\pi/2)^4$

etc....

$X_1=\pm((2k-\frac{1}{2})\pi)$, $k\geq 0$ unstable. $k\in\mathbb{Z}_{\{k\geq 0\}}$

$X_2=((2k+\frac{1}{2})\pi)$, $k\geq 0$ stable. $k\in\mathbb{Z}_{\{k\geq 0\}}$

$f''(x)=12x^2\cos(x)-4x^3\sin(x)-(4x^3\sin(x)+x^4\cos(x))$

$=12x^2\cos(x)-4x^3\sin(x)-4x^3\sin(x)-x^4\cos(x)$

$=12x^2\cos(x)-8x^3\sin(x)-x^4\cos(x)$

$f''(x=0)=0$ inconclusive,

$f''(x=\pi/2)=-8(\frac{\pi}{2})^3\cancel{\sin(\pi/2)}_1=-8\times\frac{\pi^3}{8}=-\pi^3$

$f''(x=-\pi/2)=-8(-\pi/2)^3\sin(-\pi/2)=-8(-\frac{\pi^3}{8})(-1)=-\pi^3$

$f''(x=3\pi/2)=-8(3\pi/2)^3\sin(3\pi/2)=8\times\frac{27\pi^3}{8}=27\pi^3$

$f''(x=-3\pi/2)=-8(-3\pi/2)^3\sin(-3\pi/2)=8\times(-1)\times\frac{-27\pi^3}{8}=27\pi^3$

etc....

$\bar{X}_1=\pm(\frac{1}{2}+2k)\pi$, $k\geq 0$, $k\in\mathbb{Z}_{\{k\geq 0\}}$ stable.

$\bar{X}_2=\pm(\frac{3}{2}+2k)\pi$, $k\geq 0$, $k\in\mathbb{Z}_{\{k\geq 0\}}$ unstable

$X_3=0$, inconclusive,

Q1ii) a' Repeat the derivation in a) but continue the expansion to fourth order, for the case of $f'(\bar{x}) = f''(\bar{x}) = f'''(\bar{x}) = 0$. What is the relevant stability condition of f in this case?

(HINT: Look for real solutions in the ODE for the small perturbation function, $\varepsilon(t)$, and consider the sign of both the initial condition, $\varepsilon(0)$, & the 4th derivative $f^{(4)}(\bar{x})$.)

Solution $\frac{dx}{dt} = f(x) = \dot{x}$

let $\bar{x} + \varepsilon(t) = x(t)$

$\frac{d}{dt}(\underbrace{\bar{x} + \varepsilon(t)}) = f(\bar{x} + \varepsilon(t))$

let $\bar{x} = x$

If $f''''(\bar{x} + \varepsilon(t)) = \frac{d^4}{dt^4}(x + \varepsilon(t))$

$$\boxed{\frac{d\varepsilon}{dt} = f(\bar{x}) + \varepsilon f'(\bar{x}) + \frac{\varepsilon^2}{2} f''(\bar{x}) + \frac{\varepsilon^3}{6} f'''(\bar{x}) + \frac{\varepsilon^4}{24} f''''(\bar{x})}$$

Taylor series expansion

$f'(x) = f''(\bar{x}) = f'''(\bar{x}) = 0$

$$\boxed{\frac{d\varepsilon}{dt} = \frac{\varepsilon^4}{24} f''''(\bar{x}) + f(\bar{x})}$$

If we let $f(\bar{x}) = 0$ (~~first~~)

$\Rightarrow$ becomes,

$$\frac{d\varepsilon}{dt} = \frac{\varepsilon^4}{24} f''''(\bar{x})$$

whose general solution is.

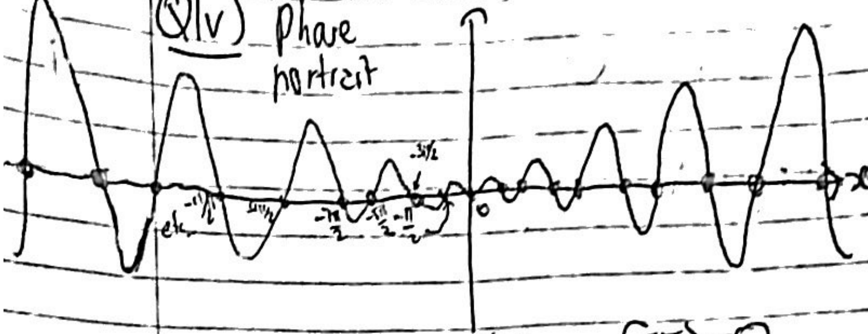
$\varepsilon(t) = \varepsilon_0^4 \exp(f'(\bar{x}) t)$ exponential growth decay!!!

$\therefore$ if $\varepsilon(0) = \frac{\varepsilon_0^4}{24}$, as $t \to \infty$, as $\varepsilon(t) \to 0$

$\Rightarrow \therefore$ Fixed point is unstable, If $f''''(x) > 0$.

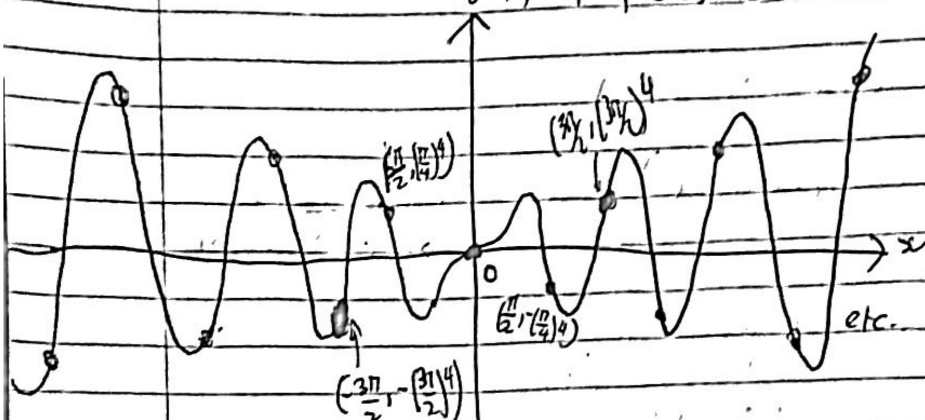
• fixed points

(Q1v) Phase portrait



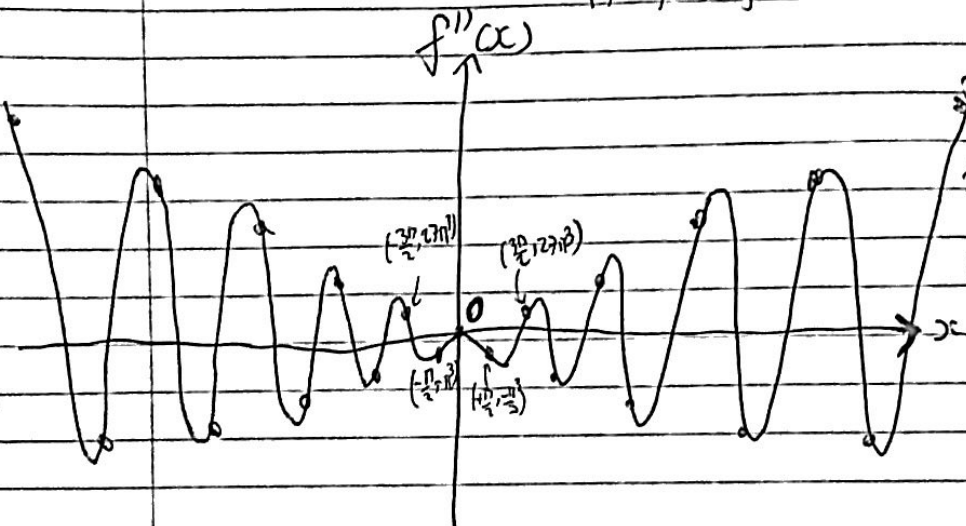
$\pm\frac{\pi}{2}$ increments.

$f'(x)$, $f(\bar{x})=0$



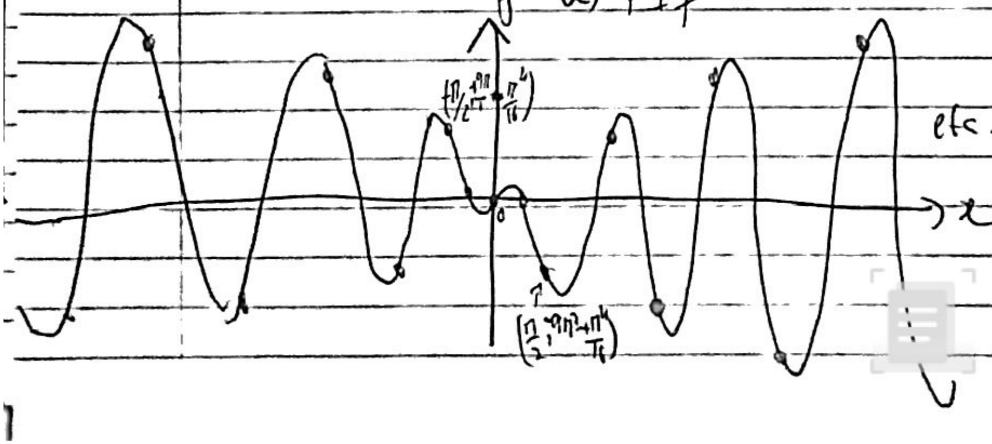
$f'(x)=0$
Inconclusive.
$f'(x)>0$
UNSTABLE
$f'(x)<0$
STABLE.

If, $f(\bar{x})=f'(\bar{x})=0$



$f''(x)>0$
unstable.
$f''(x)<0$
STABLE
$f''(x)=0$
Inconclusive

$f'''(x)$, If



$f'''(x)>0$
unstable
etc.. $f'''(x)<0$
stable
$f'''(x)=0$
Inconclusive

Q2A)

$\dot{x} = rx + 2x^4 + x^7$
where $r \in \mathbb{R}$.

i). Calculate the fixed points for this system.
First, let $f(x) = rx + 2x^4 + x^7$.
If $\bar{x}$ is a fixed point $\therefore f(\bar{x}) = 0$.
$\Rightarrow$ We need to solve the eq; $rx + 2x^4 + x^7 = 0$ ..... (1)

$\Rightarrow \bar{x}_1 = 0$ is a solution.

We then use the long division to find the other ~~ones~~ roots.

$$\bar{x}\,\overline{)\ r\bar{x} + 2\bar{x}^4 + \bar{x}^7}\quad \bar{x}^6 + 2\bar{x}^3 + r.$$

$\therefore$ (1) becomes.
$\Rightarrow \bar{x}(\bar{x}^6 + 2\bar{x}^3 + r) = 0$
$\Rightarrow \bar{x}((\bar{x}^3)^2 + 2(\bar{x}^3) + r) = 0$

let $B = \bar{x}^3$ - (2)
$B^2 + 2B + r = 0$
$B = \dfrac{-2 \pm \sqrt{4 - 4r}}{2}$

$B = \dfrac{-2 \pm 2\sqrt{1-r}}{2}$

$= -1 \pm \sqrt{1-r}$

Then $\bar{x}_{2,3} = (-1 \pm \sqrt{1-r})^{1/3}$

So the fixed points.
$\bar{x}_1 = 0$, $\bar{x}_{2,3} = (-1 \pm \sqrt{1-r})^{1/3}$

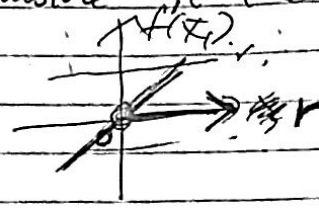
ii) Determine the stability of the fixed points using linear stability analysis.

Sl'n Linear Stability Analysis.

We first calculate $f'(x)$ out, where $f(x) = rx + 2x^4 + x^7$

$\Rightarrow f'(x) = r + 8x^3 + 7x^6$

① for $\bar{x}_1 = 0$, $f'(\bar{x}_1) = r$, notes that $r=0$, i.e. the linear stability of $\bar{x}_1$ is inconclusive at those points.

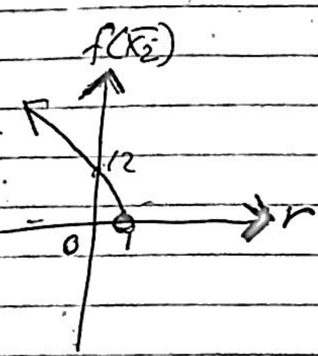


$\begin{cases} r<0 & \text{stable} \\ r>0 & \text{is unstable.} \end{cases}$

②. for $\bar{x}_{2,3} = (-1 \pm \sqrt{1-r})^{1/3}$ where $r \in \mathbb{R}$

$\cancel{f'(\bar{x}_2)=r}$

Case 1 $\bar{x}_2 = (-1-\sqrt{1-r})^{1/3}$



$f'(\bar{x}_2) = r + 8(-1-\sqrt{2-r}) + 7(-1-\sqrt{1-r})^2$

$= r - 8 - 8\sqrt{1-r} + 7(1 + 2\sqrt{1-r} + 1 - r)$

$= \cancel{r} - \cancel{8} - \cancel{8\sqrt{1-r}} + \cancel{7} + 14\sqrt{1-r} + \cancel{7} - \cancel{7r}$

$= -6r + 6\sqrt{1-r} + 6$

$f'(\bar{x}_2) = 0$, $\cancel{6r}$ $r - \sqrt{1-r} = 1$

$-1 + r = \sqrt{1-r}$

$1 - r = -\sqrt{1-r}$

$\sqrt{1-r} = -1$

Square both sides $(\sqrt{1-r})^2 = (-1)^2$

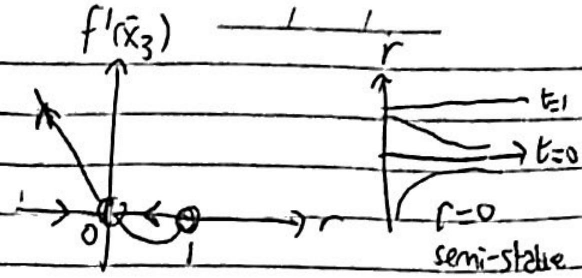
$1 - r = 1$

$r = 0.$ ← inconclusive.

if $\begin{cases} r<0 & \text{stable} \\ \cancel{r>0} & \text{unstable.} \end{cases}$

$0 < r < 1$

(3). $\bar{x}_3 = (-1+\sqrt{1-r})^{1/3}$



$r=0$ semi-stable

$r=1$ unstable.

$$f'(x_3) = r + 8(-1+\sqrt{1-r}) + 7(-1+\sqrt{1-r})^2$$
$$= r - 8 + 8\sqrt{1-r} + 7(1 + 1 - r - 2\sqrt{1-r})$$
$$= r - 8 + 8\sqrt{1-r} + 14 - 14\sqrt{1-r} - 7r$$
$$= -6r - 6\sqrt{1-r} + 6$$

$f'(x_3) = 0$, $\quad 0 = -6r - 6\sqrt{1-r} + 6$

$$0 = -r + 1 - \sqrt{1-r}$$
$$\sqrt{1-r} = (1-r)$$
$$1 = (1-r)^{1/2}, \text{ divide } (1-r)^{1/2}$$
$$1^2 = 1-r \quad \text{, square both sides.}$$
$$1 = 1-r$$
$$r = 0, \quad \text{is inconclusive.}$$

If $f'(\bar{x}_3) > 0$ unstable

$f'(\bar{x}_3) < 0$ stable

$f'(\bar{x}_3) = 0$ inconclusive.

$r = 3/4$, $f'(x_3) = \frac{-18}{4} - 6\times\frac{1}{2} + 6 = -\frac{9}{2} + 3 = -\frac{3}{2}$

$r = 1/2$, $f'(x_3) = -3 - \frac{6\sqrt{2}}{2} + 6 = 3 - 3\sqrt{2}$

Hence, $\bar{x}_1 = 0 \to$ unstable: $r>0$

stable: $r<0$

Inconclusive: $r=0$

$\bar{x}_2 = (-1-\sqrt{1-r})^{1/3}$

$\hookrightarrow$ stable: $r<0$

unstable: $0<r<1$

Inconclusive: $r=0$

$\bar{x}_3 = (-1+\sqrt{1-r})^{1/3}$, stable $0<r<1$ or $r<0$. unstable: $r>0$ Inconclusive. $r=0$

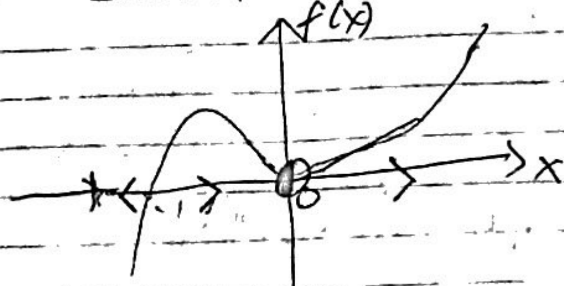
iii) Verify your results in (ii) using a graphical approach. What does the graphical approach tell you about the stability for the cases where the linear stability analysis was inconclusive?

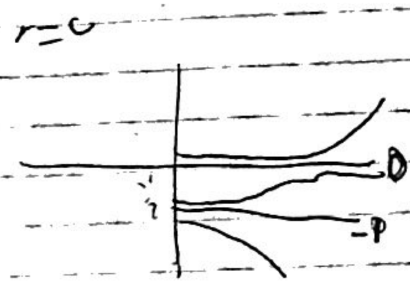
what is our graphical approach?

1

When $r=0$, Then $\dot{x}=2x^4+x^7$

$x^4(x^3$





$x=-1$ is stable

$x=0$ is semi-stable

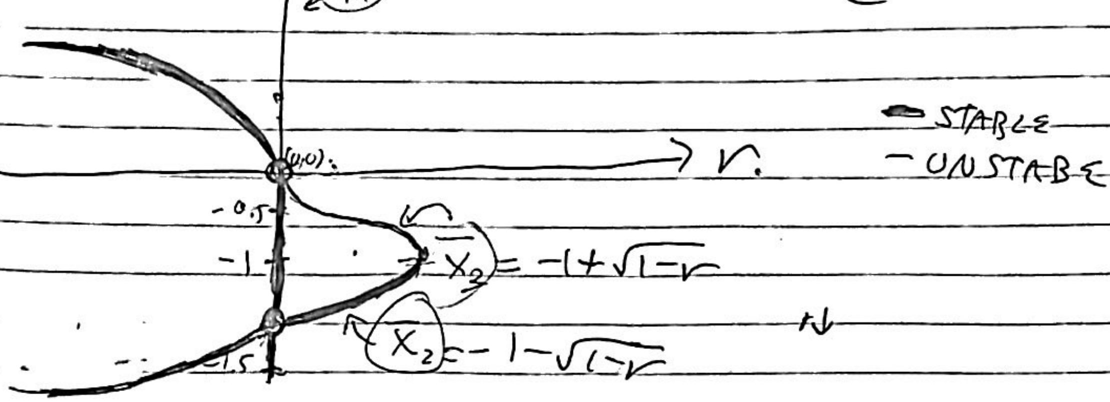
2, & 3 have the same r-value 0 when the linear stability analysis is zero.

iv) Using your analyses from (i) - (iii), draw the diagram (bifurcation) & discuss what happens to the fixed points as r is varied.

Sol^n.

$\bar{x}_1 = 0,$

$\bar{X}_{2,3} = \left(-1 \pm \sqrt{1-r}\right)^{1/3}$



What happens to the fixed points as the the parameter r varies? The # of fixed points

- when $0<r<1$, there is 2 stable fixed points $\bar{x}_2$ and $\bar{x}_3$.

The value of $\bar{x}_2$ increases when r is between 0 & 1.
The value of $\bar{x}_3$ decreases when r is between $0<r<1$.

- When $r=0$ The number of fixed points is 2, with $\bar{x}_2 = (-2)^{1/3}$ and $\bar{x}_3 = 0$, which is stable and inconclusive respectively.

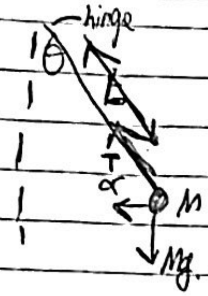
- When $r<0$, the number of fixed points is 2 with $\bar{x}_3$ unstable & $\bar{x}_2$ stable

as r decrease $\bar{x}_3$ increases and $\bar{x}_2$ decreases.

Q3i).

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \alpha\dot{\theta}$$

Refer to the Motion of the Pendulum.



A dampled Pendulum.

From Newton's 2ND Law.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = -\alpha\frac{d\theta}{dt}, \quad \alpha > 0. \quad (1)$$

Now we change the variable

$$\tau = wt \quad (2)$$

Then (1) becomes.

$$w^2\frac{d^2\theta}{d\tau^2} + \alpha w\frac{d\theta}{d\tau} = -\frac{g}{L}\sin(\theta) \quad (3)$$

where: $w = \sqrt{g/L}$ (4)

$$\frac{g}{L}\frac{d^2\theta}{d\tau^2} + \alpha\sqrt{\frac{g}{L}}\frac{d\theta}{d\tau} = -\frac{g}{L}\sin(\theta)$$

$$\Rightarrow \frac{d^2\theta}{d\tau^2} + \sin(\theta) = -\alpha\left(g/L\right)^{-1/2}\frac{d\theta}{d\tau} \quad (5)$$

⇒ The equation has now been simplied through the choice of scale.

let $\beta = \frac{d\theta}{d\tau}$ (6), $\frac{d\beta}{d\tau} = -\alpha\left(g/L\right)^{-1/2} v - \sin(\theta)$ (7)

Q3(ii) $\bar{\beta}=0$
$-\alpha(g/L)^{-1/2}\bar{\beta}-\sin(\bar{\theta})=0$ $\Big\}$ (8)

sub $\bar{\beta}=0$ into (2), $-\sin(\bar{\theta})=0$
$\Rightarrow (\bar{\theta},\bar{\beta})=(k\pi,0),\ k\in\mathbb{Z}$

Q3(iii) Use the Jacobian of the system to linearise non-linear system.

at (θ,β) is

$$J=\begin{pmatrix}0 & 1\\ -\cos(\theta) & -\alpha(g/L)^{-1/2}\end{pmatrix}$$

At $\overset{(\bar{\theta},\bar{\beta})}{(k\pi,0)}$ we have

$$J\Big|_{(k\pi,0)}=\begin{bmatrix}0 & 1\\ -\cos(k\pi) & -\alpha(g/L)^{-1/2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ (-1)^{k+1} & -\alpha\underbrace{(g/L)^{-1/2}}_{=\text{ constant.}}\end{bmatrix}$$

Case 1:

k is odd, $\lambda_1=-1$ eigenvalue

$$\begin{bmatrix}0 & 1\\ +1 & -\alpha(g/L)^{-1/2}\end{bmatrix}\xrightarrow{(1)\times-\alpha(g/L)^{-1/2}}\begin{bmatrix}0 & -\alpha(g/L)^{-1/2}\\ +1 & -\alpha(g/L)^{-1/2}\end{bmatrix}$$

$$\xrightarrow{(2):(1)-(2)}\begin{bmatrix}0 & -\alpha(g/L)^{-1/2}\\ +1 & 0\end{bmatrix}\xrightarrow{(1)\leftrightarrow(2)}\begin{bmatrix}+1 & 0\\ 0 & \underset{=r}{-\alpha(g/L)^{-1/2}}\end{bmatrix}$$

$$=r\begin{bmatrix}+1\\ -\alpha\left(\frac{g}{L}\right)^{-1/2}\end{bmatrix}$$ eigenvector. TAKE $r=1$, $\begin{bmatrix}1\\ -\alpha(g/L)^{-1/2}\end{bmatrix}$

$\lambda_1=1$, eigenvalue

$$\begin{bmatrix}0 & 1\\ -1 & -\alpha(g/L)^{-1/2}\end{bmatrix}\xrightarrow{(2)\leftrightarrow(1)}\begin{bmatrix}-1 & -\alpha(g/L)^{-1/2}\\ 0 & 1\end{bmatrix}$$

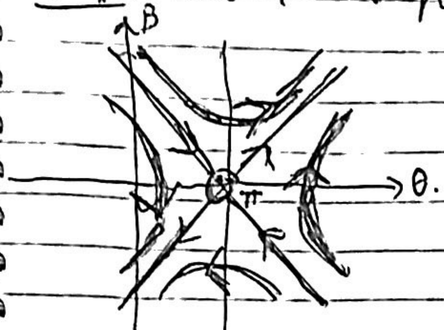
(2) : (2) $\times -\alpha(g/L)^{-1/2}$

$$\longrightarrow\begin{bmatrix}-1 & -\alpha(g/L)^{-1/2}\\ 0 & -\alpha(g/L)^{-1/2}\end{bmatrix}\xrightarrow[(1):(1)-(2)]{}\begin{bmatrix}-1 & 0\\ 0 & \underset{=r.}{-\alpha(g/L)^{-1/2}}\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1\\ \alpha(g/L)^{-1/2}\end{bmatrix}$$ eigenvector.

Q3iii) where $k\pi = \theta$, $k \in \mathbb{Z}$ are ~~stable~~.
↳ SADDLE POINTS.

Q3iv) local phase portrait.



(A) trajectories.

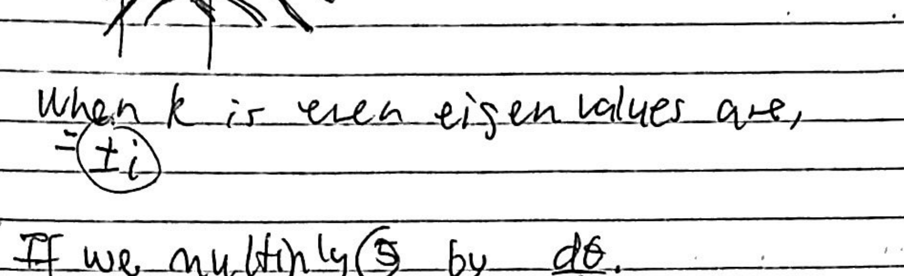
When k is even eigenvalues are, $= \pm i$

If we multiply (5) by $\frac{d\theta}{d\tau}$.

$$\frac{d\theta}{d\tau}\frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau}\sin\theta = -\frac{d\theta}{d\tau}\frac{d\theta}{d\tau}\alpha\left(\frac{g}{L}\right)^{-1/2}$$

$$\Rightarrow \frac{d}{d\tau}\left\{\frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2\alpha\left(\frac{g}{L}\right)^{-1/2} - \cos\theta\right\} = 0$$

$$\Rightarrow E(\theta) = \frac{1}{2}\left(\frac{d\theta}{d\tau}\right)^2 - \cos(\theta) + \alpha\left(\frac{g}{L}\right)^{-1/2} = C. \leftarrow C \in \mathbb{R}$$



• where $\pi k = \theta$, $k \in \mathbb{Z}$
The phase diagram has ⇒ SADDLE & attractors POINTS.

Q3v)

Since $\theta = \theta_0$ is identical, to $\theta = \theta_0 + 2\pi$, the phase portrait is also periodic.

* Oscillations with exist around $\theta = 2k\pi$, $k \in \mathbb{Z}$
* Non-periodic trajectories separate small & large velocities.

→ Motion is from 1 fixed point to another (Orbit) → (Orbital spiral)

* → with contours surrounding around the phase portraits.
(trajectories) → converge to one of some of the equilibrium points.