1. Let $\mathbf{f} = \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 - x_2 + x_3(x_2 - 1) \\ x_1(x_2+1) - x_3 \end{pmatrix}$$

(i) [2 marks] Determine the set of non-regular points of $\mathbf{f}$.

The gradients $\nabla f_1 = (1, x_3 - 1, x_2 - 1)$ and $\nabla f_2 = (x_2 + 1, x_1, -1)$ never vanish. They are proportional, $\nabla f_1 = \lambda \nabla f_2$, when $\lambda = 1$, $x_1 = x_3 - 1$ and $x_2 = 0$, so the set of non-regular points is

$$\{(x_3 - 1, 0, x_3) : x_3 \in \mathbb{R}\}.$$

(ii) [1 mark] Determine which level-set of $\mathbf{f}$ contains the set of non-regular points from (i).

Substituting $\mathbf{p} = (x_3 - 1, 0, x_3)$ into $\mathbf{f}$ gives

$$\begin{pmatrix} (x_3-1) - 0 + x_3(0-1) \\ (x_3-1)(0+1) - x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

So, all non-regular points are contained in the (-1,-1)-level set.

(iii) [3 marks] Find a basis for, and state the dimension of, the tangent space $T\mathcal{F}(\mathbf{p})$ at $\mathbf{p} = (2, 1, 3)$.

The tangent space is $T\mathcal{F}(\mathbf{p}) = \operatorname{Ker} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & -1 \end{pmatrix}$. Applying the Gaussian algorithm we find

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 2 & -1 & 0 \end{array}\right) \equiv \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right) \equiv \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1/2 & 0 \end{array}\right).$$

The solution set is $T\mathcal{F}(\mathbf{p}) = \{(x_3, -x_3/2, x_3)^T : x_3 \in \mathbb{R}\} = \operatorname{Sp}((2, -1, 2))$. The vector $(2, -1, 2)$ forms a basis. The dimension of the tangent space $T\mathcal{F}(\mathbf{p})$ is $\boxed{1}$.

(iv) [2 marks] The curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^3$, given by

$$\boldsymbol{\gamma}(t) = \left(\frac{2}{t}, t, 1 + \frac{2}{t}\right)^T$$

is a curve in $\mathcal{F}$. Find t' such that $\mathbf{p} = (2, 1, 3) = \boldsymbol{\gamma}(t')$ and decide, with reasons, whether $D\boldsymbol{\gamma}(t')$ is contained in $T\mathcal{F}(\mathbf{p})$.

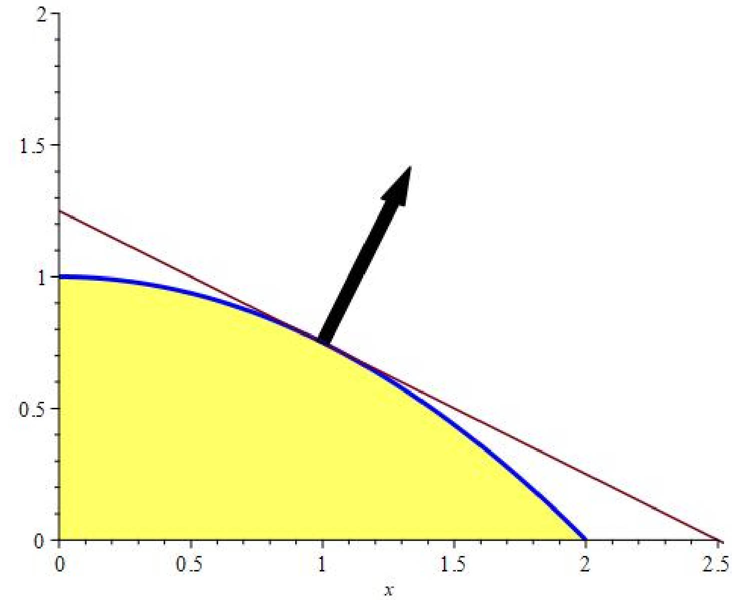
We have $\mathbf{p} = \gamma(1)$ and $D\boldsymbol{\gamma}(t) = \left(-\frac{2}{t^2}, 1, -\frac{2}{t^2}\right)$ at $t = 1$ is $(-2, 1, -2) = -(2, -1, 2) \in \operatorname{Sp}((2, -1, 2))$.

2. Consider the set-constraint problem:

$$\begin{array}{rl} \text{maximize} & x_1^3 + 3x_2^2 - 3x_1x_2 \\ \text{subject to} & \mathbf{x} \in \Omega = \{\mathbf{x} :\ x_1 \geq 0,\ x_2 \geq 0,\ \text{and } 4x_2 \leq (x_1+2)(2-x_1)\} \end{array}$$

(i) [5 marks] Determine and draw in one diagram: the tangent line to the curve $(x_1+2)(2-x_1) - 4x_2 = 0$ at the point $\mathbf{p} = (1, 3/4)^T$, a normal vector at $\mathbf{p}$ and the feasible set Ω.

A tangent vector to the curve is orthogonal to the normal $\nabla((x_1+2)(2-x_1) - 4x_2) = (-2x_1, -4)^T$. At $\mathbf{p}$ a normal direction is $(1, 2)^T$ and a tangent vector is $(2, -1)^T$. Alternatively, the tangent vector has slope $dx_2/dx_1 = -x_1/2$, which at $\mathbf{p}$ is -1/2. The equation for the tangent line is $x_2 = 3/4 - (x_1 - 1)/2$.



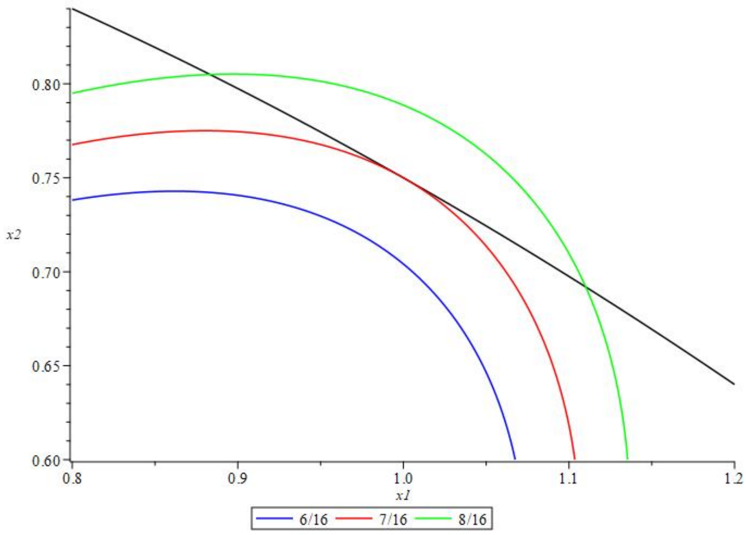
(ii) [2 marks] Describe the set of feasible directions at $\mathbf{p}$ using the normal vector you found in (i). State whether $(2, -1)^T$ is feasible.

Denoting the normal vector from (i) by $\mathbf{n} = (1, 2)^T$ the set of feasible directions is $\{\mathbf{d} \in \mathbb{R}^2 : \mathbf{d}^T\mathbf{n} < 0\}$. The tangent vector $(2, -1)^T$ is not feasible, as the tangent line lies outside the feasible set.

(iii) [2 marks] Is $\mathbf{p}$ a possible maximiser according to the FONC? Justify your answer.

According to the FONC, if $\mathbf{p}$ is a minimiser for $f = -(x_1^3 + 3x_2^2 - 3x_1x_2)$ then $\nabla f(\mathbf{p})^T\mathbf{d} \geq 0$. As $\nabla f(\mathbf{p})^T = -\frac{3}{4}(1, 2)^T$, this condition is satisfied by (ii).

(iv) [1 mark] The plot below contains three level sets of the objective function $x_1^3 + 3x_2^2 - 3x_1x_2 = a/16$, with $a = 6, 7, 8$, as well as the boundary of the constraint set. The point $\mathbf{p} = (1, 3/4)$ is contained in the 7/16-level set.



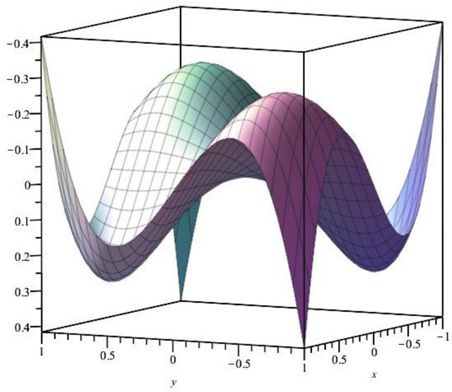
State whether the point $\mathbf{p}$ is a local maximizer. No reasons required.

It is not a local maximizer.

3. [7 marks] Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$, given by

$$\mathbf{f}(\mathbf{x}) = x_1 x_2 \cos(x_1^2 + x_2^2).$$

Its graph



shows several maxima and minima. In particular, there is a minimum close to $\mathbf{q} = (0.7, 0.7)$ (this is the only thing that is relevant about the graph). Perform one step of Newton's method to find a point which is closer to the minimum. Verify that it is closer to the minimum.

Newton's method is captured by

$$\mathbf{x}_{n+1} = \mathbf{x}_n - D^2 f(\mathbf{x}_n)^{-1} \nabla f(\mathbf{x}_n).$$

At $\mathbf{x} = \mathbf{q}$ we have

$$\nabla f(\mathbf{q}) \approx \begin{pmatrix} -0.1798054134 \\ -0.1798054134 \end{pmatrix}, \quad D^2 f(\mathbf{q}) \approx \begin{pmatrix} -2.976626723 & -1.605716753 \\ -1.605716753 & -2.976626723 \end{pmatrix}.$$

This gives, with $\mathbf{x_0} = \mathbf{q}$,

$$\mathbf{x_1} = \begin{pmatrix} .7 \\ .7 \end{pmatrix} - \begin{pmatrix} -0.473835725955795 & 0.255606777113899 \\ 0.255606777113899 & -0.473835725955795 \end{pmatrix} \begin{pmatrix} -0.1798054134 \\ -0.1798054134 \end{pmatrix} \approx \begin{pmatrix} 0.660761253637635 \\ 0.660761253637635 \end{pmatrix}$$

This is closer to the minimum as one verifies that

$$\nabla f(\mathbf{x_1}) \approx \begin{pmatrix} -0.0177452497 \\ -0.0177452497 \end{pmatrix}$$

is closer to $(0,0)^T$ than $\nabla f(\mathbf{x_0})$.

4. [5 marks] Solve the LP problem

Maximise $\quad z = x_1 + x_2$

Subject to $\quad \frac{1}{2}x_1 - x_2 \leq -1$

$\qquad \frac{1}{2}x_1 + x_2 \leq 4$

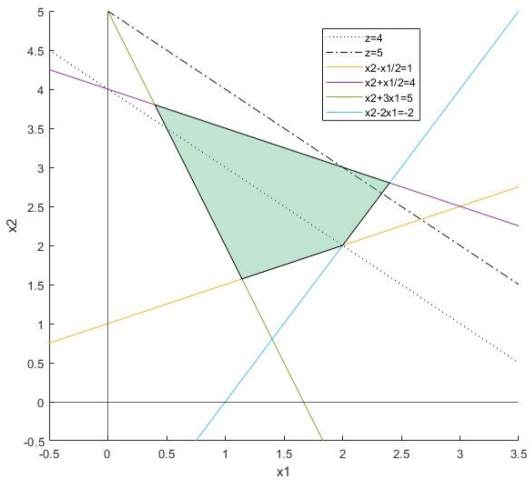
$\qquad 3x_1 + x_2 \geq 5$

$\qquad 2x_1 - x_2 \leq 2$

$\qquad \mathbf{x} \geq \mathbf{0}$

by sketching the feasible region and drawing some level sets of the objective function. State the maximum and the corner at which the maximum occurs.

The feasible region is bounded by the four constraints, not the axes. The corner points are (2,19)/5, (8,11)/7, (2,2), and 2(6,7)/5. The 4,5-level set are drawn, the maximum will be slightly larger than 5.



Adding the equation $\frac{1}{2}x_1 + x_2 = 4$ to $2x_1 - x_2 = 2$ gives $x_1 = 12/5$. Then $x_2 = 2x_1 - 2 = 14/5$, and we obtain that the corner at which the maximum $x_1 + x_2 = \boxed{26/5}$ occurs is (12/5,14/5).

5. [4 marks] After having applied the simplex algorithm to an LP problem, we have arrived at the following (augmented) matrix

$$\left(\begin{array}{cccc|c} 2 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

where the 3rd and 4th columns correspond to slack variables and the bottom row represents the objective function. Write down the set of all optimal solutions.

The basic solution of the matrix is $(x_1, x_2) = (0, 3)$. As there is a zero in the bottom row in a non-basic column (the 1st) we perform an extra pivot

$$\left(\begin{array}{cccc|c} \boxed{2} & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \equiv \left(\begin{array}{cccc|c} 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 1 & -1/2 & 2 & 5/2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right),$$

which gives rise to another solution $(1/2, 5/2)$. The full set of solutions is the convex hull of these points, namely $\{(t, 3-t) \in \mathbb{R}^2 : \ t \in [0, 1/2]\}$.

6. Consider the LP problem

$$\begin{aligned} \text{maximize} \quad & z = x_1 + 5x_2 \\ \text{subject to} \quad & 5x_1 + x_2 \geq 5 \\ & x_2 \leq 3 \\ & -x_1 + x_2 \geq 1 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

(i) [3 marks] By introducing slack variables and artificial variables as appropriate, and using the 2-phase method, write down the augmented matrix for the first phase.
Indicate the entry to pivot on in this matrix.

One would introduce slack variables x_3, x_4, x_5 and artificial variables x_6, x_7: $5x_1 + x_2 - x_3 + x_6 = 5$, $x_2 + x_4 = 3$, $-x_1 + x_2 - x_5 + x_7 = 1$ and maximise

$$w = -x_6 - x_7 = 5x_1 + x_2 - x_3 - 5 - x_1 + x_2 - x_5 - 1 = 4x_1 + 2x_2 - x_3 - x_5 - 6.$$

The augmented matrix for the first phase is

$$\left(\begin{array}{ccccccc|c} 5 & 1 & -1 & 0 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 \\ -1 & \boxed{1} & 0 & 0 & -1 & 0 & 1 & 1 \\ -4 & -2 & 1 & 0 & 1 & 0 & 0 & -6 \end{array}\right)$$

(ii) [3 marks] Solve the first phase using the simplex method (this requires two steps), and write down the basic solution it provides.

Note: You do not have to solve the second phase.

The simplex method gives

$$\left(\begin{array}{ccccccc|c} \boxed{5} & 1 & -1 & 0 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 \\ -1 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ -4 & -2 & 1 & 0 & 1 & 0 & 0 & -6 \end{array}\right) \equiv \left(\begin{array}{ccccccc|c} 1 & 1/5 & -1/5 & 0 & 0 & 1/5 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & \boxed{6/5} & -1/5 & 0 & -1 & 1/5 & 1 & 2 \\ 0 & -6/5 & 1/5 & 0 & 1 & 4/5 & 0 & -2 \end{array}\right)$$
$$\equiv \left(\begin{array}{ccccccc|c} 1 & 0 & -1/6 & 0 & 1/6 & 1/6 & -1/6 & 2/3 \\ 0 & 0 & 1/6 & 1 & 5/6 & -1/6 & -5/6 & 4/3 \\ 0 & 1 & -1/6 & 0 & -5/6 & 1/6 & 5/6 & 5/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right),$$

which yields $w = 0$ as required, and basic solution $(x_1, x_2) = (2/3, 5/3)$.

7. Consider the nonlinear problem

Minimise $\quad f(\mathbf{x}) = x_1^2 + x_2^3 - 2x_1x_2$

Subject to $\quad g(\mathbf{x}) = x_1^2 + x_2^2 \leq 1$

(i) [1 mark] Write down the relevant KKT condition.

The KKT condition is the existence of a multiplier μ such that

$$Df(\mathbf{x}) + \mu Dg(\mathbf{x}) = \mathbf{0}^T, \;\; \mu(g(\mathbf{x}) - 1) = 0, \;\; g(\mathbf{x}) - 1 \leq 0, \;\; \mu \geq 0.$$

(ii) [3 marks] Decide, with reasons, whether there is a minimum in the interior of the feasible region.

Since $Df(\mathbf{x}) = \mathbf{0}^T$ implies $x_1 = x_2$ and $(3x_1 - 2)x_1 = 0$, possible minimisers are $(0,0)$ and $(2/3, 2/3)$. The Hessian matrix is

$$D^2 f = \begin{pmatrix} 2 & -2 \\ -2 & 6x_2 \end{pmatrix}.$$

At $x_2 = 0$ the eigenvalues are $-1.2361, 3.2361$, whereas at $x_2 = 2/3$ the eigenvalues are $0.7639, 5.2361$. Thus the Hessian is positive definite only at $(2/3, 2/3)$, which tells us there is a minimum.

(iii) [2 marks] Assuming that $\mu \neq 0$, solve the KKT-condition using the MATLAB routine `vpasolve`. How many points are possibly minimisers?

The code

```
syms x1 x2 mu
f=x1^2+x2^3-2*x1*x2;
g=x1^2+x2^2-1;
L=f+mu*g;
Y=vpasolve([diff(L,x1),diff(L,x2),g]);
Y.x1
Y.x2
Y.mu
```

gives us 6 solutions. Two of them are complex, and only one of them has a positive value of $\mu = 1.772$. This is $(-0.33937, -0.94065)$, the only possible minimiser.

(iv) [3 marks] For the possible minimisers identified in (iii), does the SOSC imply they are strict local minimisers? Give reasons.

The Jacobian of the Lagrangian is $D^2\mathcal{L} = \begin{pmatrix} 2\mu+2 & -2 \\ -2 & 2\mu+6x_2 \end{pmatrix}$, which at $\mu = 1.772$ and $x_2 = -0.94065$ is $\begin{pmatrix} 5.544 & -2 \\ -2 & -2.1002 \end{pmatrix}$. The tangent space to the constraint is $\mathrm{Ker}(Dg)$ which is $\mathrm{Sp}(\mathbf{v})$ with $\mathbf{v} = (0.94065, -0.33937)$ (orthogonal to the position vector). As $\mathbf{v}^T D^2\mathcal{L}\mathbf{v} = 5.940 > 0$, this implies that $(-0.33937, -0.94065)$ is a minimum indeed.