Fluid Mechanics

Topic 10.3

Application of the Navier-Stokes Equations

Goals for this class

Apply the Navier-Stokes equations in order to

– Derive flow profiles in different geometries

– Determine flow rates and forces

– Analyse unique 2D solutions to the N.S. equations

Conservation laws - mass

We previously derived the differential forms of the Navier Stokes equations (NSE), which are expressions of mass and momentum conservation.

For the continuity equation (mass conservation), this was:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

This fundamentally states that the rate of mass change (the time derivative term) in a control volume is equal to the rate of mass that enters/leaves across the surface of the control volume.

Conservation laws - reformulation

To make use of this equation, however, we need to put it in a form that is appropriate for a given coordinate system.

The divergence term $\nabla \cdot \mathbf{F}$, where $\mathbf{F}$ is some vector, yields:

We can use these to construct the expanded form of the continuity equation in Cartesian and Cylindrical coordinates.

Cartesian coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Conservation laws - momentum

For the conservation of momentum, we had:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{F},$$

which is an expression that says that the rate of momentum change in a control volume and the rate of momentum entering/leaving a control volume is equal to the sum of all the forces (including, pressure and viscous forces) acting on that volume.

Conservation laws - momentum

The gradient term $\nabla \mathbf{F}$, where $\mathbf{F}$ is some vector, yields:

We can use these to construct the expanded form of the momentum conservation equation in Cartesian and Cylindrical coordinates.

Cartesian coordinates (x, y, z):[a]

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} - \left[\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right] + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} - \left[\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right] + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left[\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right] + \rho g_z$$

Conservation laws – momentum (continued)

Cylindrical coordinates (r, θ, z):[b]

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz}\right] + \rho g_z$$

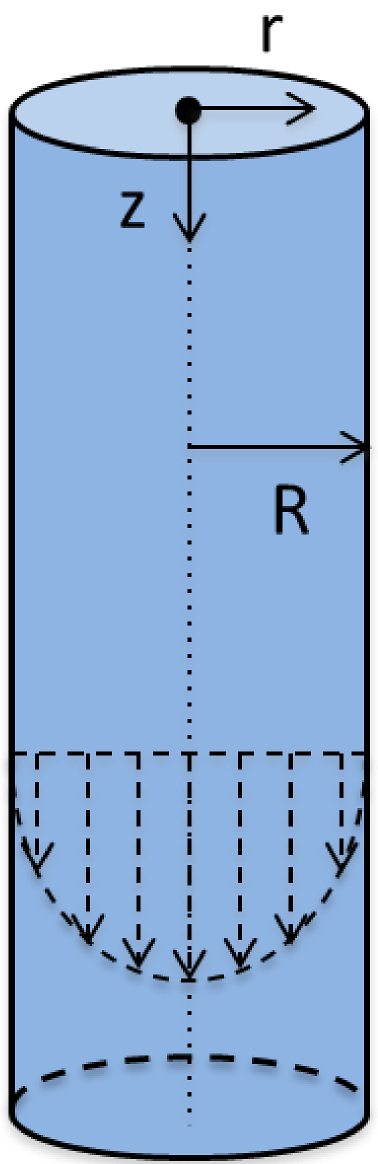
These are also called the ‘equations of motion’.

Using the N.S. equations

To solve fluid dynamics problems we must often utilize both the **continuity equation** and the **equations of motion**. Together these are sometimes referred to as the **equations of change**.

Let's work through an example problem where we will use the equations of change to derive an analytical expression for the velocity of a Newtonian fluid flowing through a pipe.

Since a pipe is a cylinder, we will use cylindrical coordinates. We will define pipe flow as being in the +z direction along a pipe with radius R. Coordinates within this pipe can be expressed in cylindrical coordinates in terms of radius, theta and height (r, θ, z).

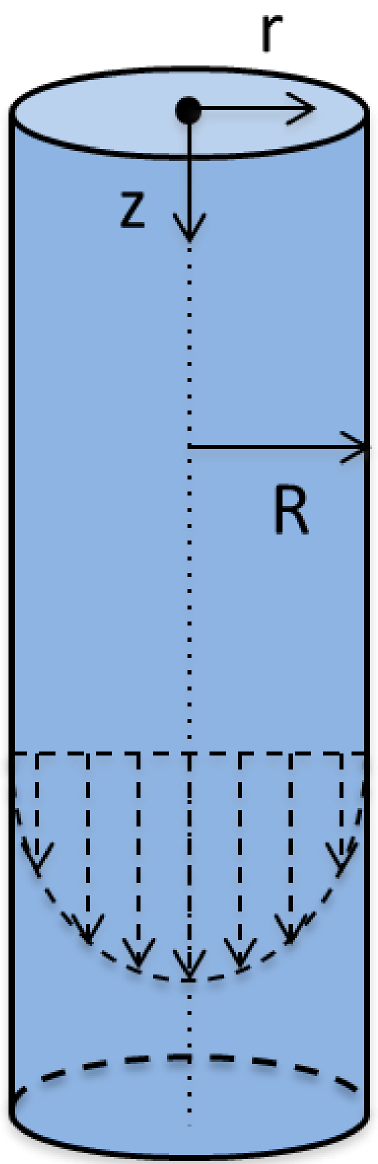


Using the N.S. equations

What we will do now is determine what an analytical expression for the flow profile is in the pipe (what the fluid velocity is across the pipe cross-section).

To start, we can make some simplifications based on what we know about flow in this pipe.

1. incompressible flow, therefore derivatives of ρ are equal to _____
2. fluid flow is only in the z-direction, therefore v_r = ____, v_θ = ____
3. shear stress (τ_{ij}) only occurs where there are velocity gradients. The only velocity gradient in the system is v_z changing with respect to r. Therefore, all τ_{ij} = ____ except for τ_{rz}



Note: we define τ_{ij} as the viscous stress in the $\hat{j}$-direction acting on a surface with normal in the $\hat{\imath}$-direction.

Using the N.S. equations

Finally,

4. There is only gravity in the z-direction, so g_r= ____, g_θ = ____

Continuity equation

Let’s analyse the pipe flow first using the continuity equation. In cylindrical coordinates, this is given by:

To which we apply our simplifications:

Which yields:

This shows us that the z-velocity is constant over the length of the pipe.

Equations of motion

Let's examine this now for the three equations of motion (momentum conservation.

r-component of momentum:

θ-component of momentum:

Equations of motion (continued)

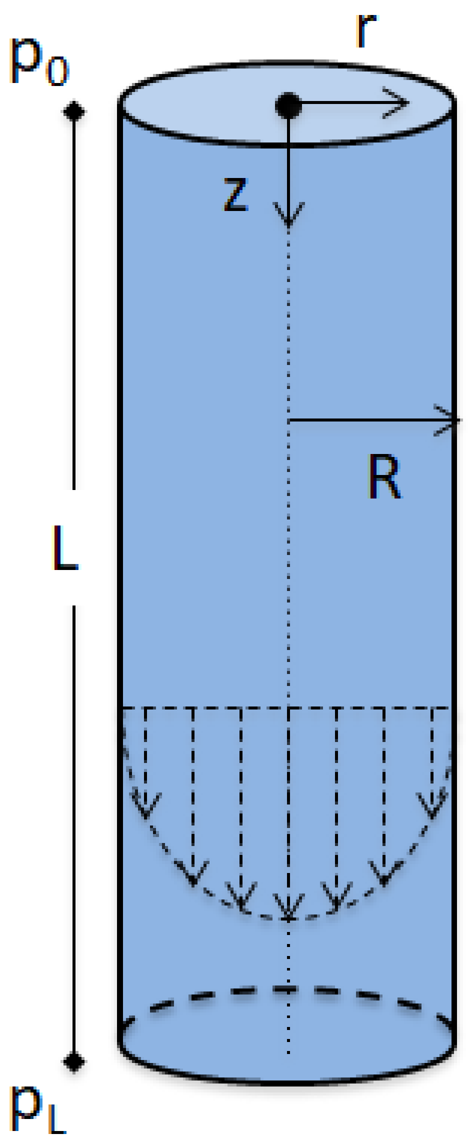
z-component of momentum:

re-arranging this result from our z-component analysis, we have:

where $\partial p/\partial z$ is a constant, meaning the pressure drop along the pipe is linear.

Deriving flow profiles

Let's add some additional terms to our equation – start at some p_0, travel a distance L, and end up at a pressure p_L.



This gives us:

Where we can integrate this equation to find an expression for τ_{rz}

Deriving flow profiles (continued)

Deriving flow profiles (continued)

Finally, we can use the relationship relating shear stress to velocity gradients to find the value of velocity as a function of radius.

This is an expression for the parabolic flow profile in laminar flow.

Deriving flow profiles (continued)

Given this equation, what is the maximum velocity and average velocity in the pipe?

Deriving flow profiles (continued)

Deriving flow rates

Finally, what is the volumetric flow rate through a pipe? The result of this derivation is called the <u>Hagen-Poiseuille equation</u>

Example Problem 9.4

Water is flowing through a pipe. The pressure drop along the pipe is 2Pa. Calculate the velocity in the centre of the pipe if it is (a) 100 m long and (b) 1000 m long. R = 0.5 m.

Example Problem 9.5

Derive an expression for the velocity in the x-direction for flow between horizontal flat plates (Poiseuille flow).

Example Problem 9.5

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Example Problem 9.5

Example Problem 9.6

A moving plate is positioned above a stationary plate, inducing flow in the fluid between these plates. Derive an expression for velocity in the x-direction for this case. This is called "Couette flow".

Example Problem 9.6