# Decision Making
# Part 1: Introduction to Game Theory

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# Topics in this part

- Introduction to game theory: games, aims of game theory
- Trees: graphs and directed graphs, rooted trees, children, parents, descendants, ancestors, cutting, quotient
- Game trees
- Games in extensive form: information sets, choice functions, choice subtrees, game in extensive form, games with perfect information
- Games with chance moves: choice subtrees for games with chance moves, probability of reaching a leaf, expected payoffs
- Equilibrium: Nash equilibrium
- Games in normal form: normal form, equilibrium, constant-sum games, zero-sum games, 2-person games in normal form

Reference:
Chapter 1, P. Morris, Introduction to Game Theory, Springer-Verlag, 1994

# Introduction

**Example 1.** (Matching pennies) There are two players, Alice and Bob, each conceals a penny in her/his closed hands, without the other seeing it. They then open their hands. If both pennies show heads or both show tails, Alice takes both pennies; if one shows head and the other shows tail, Bob takes both pennies.

# Game theory

Game theory is a subject of modelling and analysing problems with two or more decision-makers (called players).

It was originated at the end of World War II with the foundation laid by:

J. von Neumann and O. Morgenstern, Theory of Games and Economic Behaviour Princeton University Press, Princeton, NJ, 1944

Of course playing of games themselves has been one of the enduring activities of human-beings.

In modern times game theory is a dynamic and active field with interests from economics, mathematics, operations research, biology, finance, social sciences, etc.

## Games

There are various kinds of games for entertainment and gambling, e.g. tic-tac-toe, matching pennies, poker, bridge, chess, go, roulette, etc.

There are also many decision making problems from economics, social science, business, etc. which can be modeled as games.

All games considered in this subject have the following features in common:

- There is a finite number of players (who may be human beings, groups of people, organisations, teams, nations, biological species, abstract entities such as nature, etc.).
- Each player has complete knowledge of the rules of the game.
- At different points each player has a set of choices or moves.
- Players make moves in sequence.

- The game ends after a finite number of moves.
- After the game ends, each player receives a numerical payoff, which can be positive, negative, or zero. (We convert non-numerical payoffs such as prestige, satisfaction, etc. to numerical payoffs. However, we will not discuss the methodology of such conversion in this subject.)

The following are properties that a game may or may not have:

- There may be chance moves, e.g. drawing a card, tossing a coin, throwing a dice, etc.
  In a card game, dealing a hand is a chance move.

- We may think of chance moves as made by nature. That is, we may treat nature as a player who makes chance moves.

- In some games, each player knows, at every point of the game, the entire history of the game. (This is the case for tic-tac-toe but not bridge.) A game with this property is said to be of perfect information.

- A game may be non-cooperative or cooperative.

## Aims of game theory

Game theory is not primarily about how to win at games.

Rather, it is mainly about methodologies whereby one may investigate and perhaps resolve conflicts.

The aim of game theory is not to advise individual players on how to choose their strategies.

A game theorist is more like an independent analyst or arbitrator than an advisor to an individual player. (ie. they do not take sides)

However, players may use the analysis to best choose their own strategies

## How can we define a game mathematically?

It is not easy to define a game mathematically. Let us look at an example.

**Example 2.** (Matching pennies cont'd) We may represent this game by a tree structure.

# Trees

## Graphs and directed graphs

**Definition 1.** A graph is a pair $G = (V, E)$, where $V$ is a finite set and $E$ is a family of subsets of $V$ with precisely 2 elements. The elements of $V$ are called the vertices of $G$ and the members of $E$ are edges of $G$.

If $\{v, w\}$ is an edge, we write it as $vw$ (or $wv$) and say that $v$ and $w$ are adjacent.

**Definition 2.** A path in a graph $G$ is a sequence $v_0, v_1, \ldots, v_k$ of distinct vertices of $G$ such that $v_{i-1}v_i$, $1 \leq i \leq k$, are edges of $G$. If in addition $v_kv_0$ is an edge of $G$, then we call the sequence $v_0, v_1, \ldots, v_k, v_0$ a cycle of $G$.

## Example 3.

**Definition 3.** A directed graph is obtained from a graph $G$ by assigning each of its edges an orientation. Such oriented edges are called the edges of the directed graph.

An edge with orientation from $v$ to $w$ is denoted as $vw$. We say this edge is from $v$ to $w$, and call $v$ the tail and $w$ the head of $vw$. We also say $vw$ is a leaving edge at $v$ and an entering edge at $w$, or $vw$ leaves $v$ and enters $w$.

Note that $wv$ represents an edge (if it is present in $G$) different from $vw$.

## Example 4.

**Definition 4.** In a directed graph $G$, a (directed) path from a vertex $v$ to a vertex $w$ is a sequence $v = v_0, v_1, \ldots, v_k = w$ of distinct vertices of $G$ such that $v_{i-1}v_i$, $1 \leq i \leq k$, are edges of $G$.

**Example 5.**

**Definition 5.** The underlying graph of a directed graph $G$ is obtained from $G$ by ignoring the orientation of every edge.

**Example 6.**

## Rooted trees

**Definition 6.** A directed graph $T$ satisfying the following conditions is called a rooted tree:

- the underlying graph of $T$ has no cycles;
- $T$ has a distinguished vertex $r$ (called the root) such that there is no entering edge at $r$;
- for every other vertex $v$ there is a unique path in $T$ from $r$ to $v$.

We denote a tree $T$ with root $r$ by $(T, r)$.

We usually draw a rooted tree in such a way that the root is at the top and edges are pointing downwards.

In a rooted tree any vertex other than the root has exactly one entering edge. (Exercise)

## Example 7.

## Children, descendants, parents, and ancestors

**Definition 7.** In a rooted tree, if there is a path from $v$ to $w$, then $v$ is called an ancestor of $w$, and $w$ is called a descendant of $v$. In particular, if $vw$ is an edge, then $v$ is a parent of $w$, and $w$ is a child of $v$.

**Definition 8.** In a rooted tree, a vertex without any child is called a leaf (or a terminal vertex).

**Lemma 1.** In a rooted tree,

(a) every vertex other than the root has exactly one parent;

(b) every vertex other than a leaf has at least one descendant which is a leaf.

## Cutting and quotient

**Definition 9.** Let $(T, r)$ be a rooted tree, and let $v$ be a vertex of $T$.

Define $T_v$ to be the tree whose vertices are $v$ plus all descendants of $v$. The edge set of $T_v$ consists of all edges of $T$ with both end-vertices in $T_v$. Call $T_v$ the cutting of $T$ at $v$.

Define the quotient tree $T/v$ to be the tree obtained from $T$ by removing all descendants of $v$ together with the attached edges.

## Example 8.

## Some observations

- $T_r$ is the same as $T$;
- $T/r$ is the trivial tree with only one vertex (namely $r$) and without any edges;
- If $v$ is a leaf of $T$, then $T_v$ is the trivial tree with vertex $v$ only, and $T/v$ is the same as $T$.

**Lemma 2.** Let $(T, r)$ be a rooted tree. Then

(a) $T_v$ is a rooted tree with root $v$;

(b) $T/v$ is a rooted tree with root $r$.

## Subtrees

**Definition 10.** Let $(T, r)$ be a rooted tree. A subtree $S$ of $T$ is a tree with the following properties:

- the root of $S$ is $r$;
- the vertex set of $S$ is a subset of the vertex set of $T$;
- the edge set of $S$ is a subset of the edge set of $T$;
- the set of leaves of $S$ is a subset of the set of leaves of $T$.

**Example 9.**

**Definition 11.** Given directed graphs $G_1 = (V_1, E_1), \ldots, G_m = (V_m, E_m)$, the union of them, denoted by $\cup_{j=1}^m G_j$, is defined as the graph with vertex set $\cup_{j=1}^m V_j$ and edge set $\cup_{j=1}^m E_j$.

**Definition 12.** The intersection $\cap_{j=1}^m G_j$ of $G_1, \ldots, G_m$ is defined as the graph with vertex set $\cap_{j=1}^m V_j$ and edge set $\cap_{j=1}^m E_j$.

**Example 10.**

**Lemma 3.** Let $(T, r)$ be a rooted tree.

(a) Any subtree $S$ of $T$ is the union of all paths from $r$ to a leaf of $S$;

(b) Conversely, if $L$ is a non-empty set of leaves of $T$, then the union of all paths of $T$ from $r$ to a leaf in $L$ is a subtree of $T$. (The set of leaves of this subtree is exactly $L$.)

**Example 11.**

# Game trees

**Example 12.** (Two-finger Morra) Two players hold up either one finger or two fingers simultaneously, and at the same time predict the number held by the other player by saying "one" or "two".

If one player is correct in her prediction while the other one is wrong, then the one who is right wins from the other an amount of money equal to the total number of fingers held up by both players. If neither player is correct or both are correct in prediction, neither wins anything.

We may represent this game by a game tree. We may pretend a neutral third party is present and the two players whisper their moves to him successively.

**Game trees** We denote players in an $n$-player game by $1, 2, \ldots, n$.

**Definition 13.** An $n$-person game tree is a non-trivial rooted tree $(T, r)$ such that each non-terminal vertex is labelled by one of the players, and each leaf $v$ is assigned an $n$-tuple of numbers (called the payoff vector):

$$\mathbf{p}(v) = (p_1(v), p_2(v), \ldots, p_n(v)).$$

- If a non-terminal vertex $v$ is labelled $i$, then we say that $v$ belongs to player $i$ or player $i$ owns $v$.
- The game is played sequentially, beginning with the root.
- The player who owns the root makes the first move by choosing one of the children of the root.
- If that child is not a leaf, the player who owns it chooses one of its children.
- In general, if at a point the current vertex is not a leaf, then the player who owns it chooses one of its children.
- Continue until a leaf, say, $v$ is reached. Player $i$ receives the $i$th coordinate $p_i(v)$ of $\mathbf{p}(v) = (p_1(v), p_2(v), \ldots, p_n(v))$ as payoff, $i = 1, 2, \ldots, n$.

**Example 13.** ($2 \times 2$ Nim)

- Initial state: 4 matches are set out in 2 piles of 2 matches each.
- Play: Two players take alternative turns. At each turn the player selects a pile that has at least one match and removes at least one match from this pile. The player may take several matches, but only from one pile.
- Stopping rule: When both piles have no more matches the game is ended.
- Winning rule: The player who removes the last match loses.

- A game described above is called a tree game.
- It does not incorporate some important notions such as chance moves, perfect/imperfect information, strategies, etc., which will be discussed later.
- At any point choosing a child $v$ of the current vertex $u$ is equivalent to choosing the edge $uv$.

  We may think a choice is associated with each edge.
- If $u$ is a vertex of $T$, then $(T_u, u)$ is also a game tree, and $T/u$ is a game tree if $u \neq r$ and an $n$-tuple $\mathbf{p}(u)$ is assigned to $u$.

# Games in extensive form

## Information sets

**Definition 14.** In a game tree, a set of vertices $A$ is called an information set for player $i$ if:

- all the vertices in $A$ belong to player $i$;
- no vertex of $A$ is an ancestor or descendant of another vertex of $A$;
- all vertices of $A$ are equivalent for player $i$: they all have the same number of children, and for any two vertices $u, v \in A$ there is a one-to-one correspondence between the set of edges leaving from $u$ and that leaving from $v$ such that player $i$ views the corresponding edges (i.e. choices) as identical.

Thus player $i$ has the same choices at every vertex of $A$.

The edges leaving from every vertex of $A$ are labelled by such choices.

The information sets for player $i$ constitute a partition of the set of vertices owned by $i$.

Thus, in an $n$-player tree game, the set of non-terminal vertices is partitioned into $n$ subsets $V_1, V_2, \ldots, V_n$, where $V_i$ is the set of vertices owned by player $i$.

Each $V_i$ is further partitioned into information sets.

## Choice functions

**Definition 15.** A choice function for player $i$ is a function $c$ defined on the set of vertices owned by player $i$ such that $c(u)$ is a child of $u$ for every vertex $u$ owned by player $i$.

**Example 14.**

## Choice subtrees

**Definition 16.** Let $(T, r)$ be a game tree, $i$ a player and $c$ a choice function for $i$. An $(i, c)$-path is defined as a path $P$ of $T$ from $r$ to a leaf such that if $u$ is a vertex on $P$ owned by $i$ then the edge from $u$ to $c(u)$ is an edge of $P$.

**Example 15.**

**Definition 17.** Let $(T, r)$ be a game tree, $i$ a player and $c$ a choice function for $i$. The choice subtree determined by $i$ and $c$ is defined as the union of all the $(i, c)$-paths.

The choice subtree determined by $i$ and $c$ contains "essential information" needed for player $i$ to make decision according to the choice function $c$.

A choice subtree is a subtree of $T$. But not every subtree of $T$ is a choice subtree.

**Example 16.**

## Characterisation of choice subtrees

**Theorem 1.** Let $(T, r)$ be a game tree, $i$ a player and $c$ a choice function for $i$. If $S$ is the choice subtree determined by $i$ and $c$, then the following hold:

(a) if $u$ is a vertex in $S$ owned by $i$, then exactly one of the children of $u$ is in $S$;

(b) if $u$ is a vertex in $S$ not owned by $i$, then all children of $u$ are in $S$.

Conversely, if a subtree $S$ of $T$ has the two properties above, then there exists a choice function $c$ such that $S$ is the choice subtree determined by $i$ and $c$.

## Choice subtrees which respect an information set

**Definition 18.** A choice subtree $S$ for a player $i$ is said to respect an information set $A$ for player $i$ if for any two vertices $u, v \in A$ which are in $S$, the choices at $u$ and $v$ by $S$ are the same, i.e. $S$ calls for the same move at every vertex in $A$.

**Definition 19.** A choice subtree $S$ for a player $i$ is called a strategy for $i$ if it respects every information set of player $i$.

Denote by $\Sigma_i$ the set of strategies for player $i$, for $i = 1, 2, \ldots, n$.

**Definition 20.** Call $(S_1, S_2, \ldots, S_n)$ a strategy profile if each $S_i \in \Sigma_i$.

The game can be thought of consisting of the game tree $(T, r)$ together with $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$.

**Example 17.**

## Games in extensive form

**Definition 21.** An $n$-person game in extensive form consists of a game tree $(T, r)$ with $n$ players, $1, 2, \ldots, n$, and a non-empty set $\Sigma_i$ of choice subtrees for each player $i$. We denote such a game by

$$((T, r), \{1, 2, \ldots, n\}, \{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}).$$

Information sets are implied by this definition:

An information set for player $i$ can be defined as a subset of vertices owned by $i$ and respected by all choice subtrees in $\Sigma_i$.

**Lemma 4.** If a strategy $S_i$ is chosen from $\Sigma_i$ for each $i$, then the intersection of all choice trees $S_i$'s is a path from the root to a leaf.

This path determined uniquely by $(S_1, \ldots, S_n)$ represents the history of moves if the players follow their strategies $S_1, \ldots, S_n$ respectively.

If the leaf of this path is $v$, then we denote the payoff vector $\mathbf{p}(v) = (p_1(v), \ldots, p_n(v))$ by

$$\mathbf{u}(S_1, \ldots, S_n) = (u_1(S_1, \ldots, S_n), \ldots, u_n(S_1, \ldots, S_n))$$

where $u_i(S_1, \ldots, S_n) = p_i(v)$, $i = 1, \ldots, n$, for this particular $v$.

## Example 18.

## Games with perfect information

**Definition 22.** A game in extensive form is said to be of perfect information if every information set for every player contains one vertex only. Otherwise it is called a game of imperfect information.

If a game is of perfect information, then at every point in the game every player knows entirely which moves have been made, and at every point every player knows precisely the vertex at which she has to make her move.

# Games with chance moves

## Games with chance moves

Now we consider games with chance moves. In terms of game trees, this means that there are non-terminal vertices which belong to “chance”. We may think of “nature” as a player who makes chance moves.

Notation: As before we use $1, \ldots, n$ to denote non-chance players. No symbol will be given to the chance player (“nature”).

For every chance vertex $u$, there is a probability distribution on the edges leaving from $u$. That is, each edge $uv$ leaving from $u$ is assigned a probability $\Pr(u, v)$ such that

$$\Pr(u, v) \geq 0 \text{ and } \sum_{uv} \Pr(u, v) = 1,$$

where the sum is taken over all edges $uv$ of the game tree leaving from $u$.

A chance vertex $u$ makes decision according to its probability distribution. That is, whenever it is $u$'s turn to move, for each edge $uv$ leaving from $u$, $v$ is chosen with probability $\Pr(u, v)$.

## Example 19.

## Choice subtrees for games with chance moves

For games with chance moves, choice functions and choice subtrees are defined in the same way as before.

However, $\mathbf{u}(S_1, \ldots, S_n)$ no longer makes sense now because $(S_1, \ldots, S_n)$ does not determine a unique leaf due to chance moves.

Suppose player $i$ plays according to choice subtree $S_i$. Whenever $S_i$ runs into a chance vertex, we put all edges leaving from this chance vertex into $S_i$.

As a result, the intersection $\cap_{i=1}^{n} S_i$ of $S_1, \ldots, S_n$ may not be a path of the game tree from $r$ to a leaf. In fact, it may have many leaves.

## Probability of reaching a leaf

If $w$ is a leaf of $\cap_{i=1}^{n} S_i$ and $P$ is the unique path in $\cap_{i=1}^{n} S_i$ from the root to $w$, then the probability that $w$ is reached is equal to the product of the probabilities of the edges on $P$ leaving from chance vertices encountered by $P$. (Note that $P$ relies on $S_1, \ldots, S_n$ and $w$.)

**Definition 23.** With the notation above, define

$$\Pr(S_1, \ldots, S_n; w) = \prod_{uv} \Pr(u, v),$$

where the product is made over all edges $uv$ of $P$ leaving from chance vertices $u$ on $P$.

It is understood that, if there are no chance vertices on $P$, then $\Pr(S_1, \ldots, S_n; w) = 1$.

## Expected payoffs

**Definition 24.** With the notation on Slide 46, the expected payoff to player $i$ associated with choice subtrees $(S_1, \ldots, S_n)$ is defined as

$$\mathbf{u}_i(S_1, \ldots, S_n) = \sum_w \Pr(S_1, \ldots, S_n; w)\ p_i(w),$$

where the sum is taken over all leaves $w$ of $\cap_{i=1}^n S_i$.

It is understood that a game with chance moves is played repeatedly, for otherwise it does not make sense to talk about expect payoffs.

**Lemma 5.** Let $(T, r)$ be an $n$-player game tree with chance moves. Let $S_1, \ldots, S_n$ be choice subtrees for players $1, \ldots, n$, respectively.

(a) If $r$ belongs to a player $i$ and $(r, u)$ is an edge of $S_i$, then for each $j = 1, \ldots, n$, $S_j \cap T_u$ is a choice subtree of $T_u$ for player $j$;

(b) If $r$ belongs to chance and $u$ is any child of $r$, then for each $j = 1, \ldots, n$, $S_j \cap T_u$ is a choice subtree of $T_u$ for player $j$.

**Theorem 2.** Let $(T, r)$ be an $n$-player tree game with chance moves. Let $S_1, \ldots, S_n$ be choice subtrees for players $1, \ldots, n$, respectively.

(a) If $r$ belongs to a player $i$ and $(r, u)$ is an edge of $S_i$, then for each $j = 1, \ldots, n$,

$$\mathbf{u}_j(S_1, \ldots, S_n) = \mathbf{u}_j(S_1 \cap T_u, \ldots, S_n \cap T_u).$$

(b) If $r$ belongs to chance, then for each $j = 1, \ldots, n$,

$$\mathbf{u}_j(S_1, \ldots, S_n) = \sum_u \Pr(r, u)\ \mathbf{u}_j(S_1 \cap T_u, \ldots, S_n \cap T_u),$$

where the sum is taken over all children $u$ of $r$.

# Equilibrium

## Equilibrium

A strategy profile is said to be in equilibrium if no single player can get more payoff by deviating from it while others are faithful to it.

**Definition 25.** Let $(T, r)$ be an $n$-player tree game in extensive form. A strategy profile $(S_1^*, \ldots, S_n^*)$ is called an equilibrium strategy profile (an equilibrium, a Nash equilibrium, or in Nash equilibrium) if for each player $i$ and any strategy $S_i \in \Sigma_i$,

$$\mathbf{u}_i(S_1^*, \ldots, S_i, \ldots, S_n^*) \leq \mathbf{u}_i(S_1^*, \ldots, S_i^*, \ldots, S_n^*).$$

Not every game in extensive form has equilibria.

The following important result was essentially proved by von Neumann and Morgenstern. It was rediscovered by Kuhn.

**Theorem 3.** Every game in extensive form with perfect information has at least one Nash equilibrium.

**Example 20.**

# Normal form

## Games in normal form

Let $((T, r), \{1, \ldots, n\}, \{\Sigma_1, \ldots, \Sigma_n\})$ be a game in extensive form.
Then every strategy profile

$$(S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \times \Sigma_n$$

corresponds to a payoff vector

$$\mathbf{u}(S_1, \ldots, S_n) = (u_1(S_1, \ldots, S_n), \ldots, u_n(S_1, \ldots, S_n)).$$

This defines a function from $\Sigma_1 \times \cdots \times \Sigma_n$ to $\mathbb{R}^n$, which is called the normal form of $((T, r), \{1, \ldots, n\}, \{\Sigma_1, \ldots, \Sigma_n\})$.

**Definition 26.** Let $X_1, \ldots, X_n$ be finite non-empty sets. Any function

$$\mathbf{u} : X_1 \times \cdots \times X_n \to \mathbb{R}^n$$

is called an $n$-person game in normal form with strategy sets $X_1, \ldots, X_n$.

The game is played as follows.

Each player $i$ chooses a strategy $x_i \in X_i$, and all players make choices simultaneously and independently. The $n$-tuple $(x_1, \ldots, x_n)$ is called a strategy profile.

If the players choose $(x_1, \ldots, x_n)$, then the payoff to player $i$ is given by the $i$th coordinate $u_i(x_1, \ldots, x_n)$ of

$$\mathbf{u}(x_1, \ldots, x_n) = (u_1(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n)).$$

**Example 21.** Normal form of Matching Pennies:

## Equilibrium

**Definition 27.** Let

$$\mathbf{u} : X_1 \times \cdots \times X_n \to \mathbb{R}^n$$

be a game in normal form. A strategy profile $(x_1^*, \ldots, x_n^*) \in X_1 \times \cdots \times X_n$ is

called an equilibrium (or a Nash equilibrium, or in equilibrium) if for every $i$ and any $x_i \in X_i$,

$$u_i(x_1^*, \ldots, x_i, \ldots, x_n^*) \leq u_i(x_1^*, \ldots, x_i^*, \ldots, x_n^*).$$

If a game in extensive form has an equilibrium, then so does the corresponding game in normal form.

So the definition above is consistent with that of equilibria for games in extensive form.

## Example 22.

## Constant-sum games and zero-sum games

**Definition 28.** A game

$$\mathbf{u} : X_1 \times \cdots \times X_n \to \mathbb{R}^n$$

in normal form is called a constant-sum game if there exists a constant $c$ such that, for every strategy profile $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$,

$$\sum_{i=1}^{n} u_i(x_1, \ldots, x_n) = c.$$

In particular, a constant-sum game such that $c = 0$ is called a zero-sum game.

## Example 23.

## Two-person games in normal form

In a 2-person game we call the two players I and II.

Suppose player I has $m$ strategies $a_1, \ldots, a_m$, and player II has $n$ strategies $A_1, \ldots, A_n$. That is, $X_1 = \{a_1, \ldots, a_m\}$ and $X_2 = \{A_1, \ldots, A_n\}$. Then the game in normal form is defined by a function

$$\mathbf{u} : \{a_1, \ldots, a_m\} \times \{A_1, \ldots, A_n\} \to \mathbb{R}^2.$$

Write

$$\mathbf{u}(a_i, A_j) = (a_{ij}, b_{ij}), 1 \leq i \leq m, 1 \leq j \leq n.$$

This means that, if player I plays $a_i$ and player II plays $A_j$, then player I receives $a_{ij}$ and player II receives $b_{ij}$.

We can represent the above 2-person game by a bi-matrix:

$$
\begin{array}{cc|ccccc}
 & & & & \text{Player II} & & \\
 & & A_1 & \ldots & A_j & \ldots & A_n \\
\hline
 & a_1 & (a_{11}, b_{11}) & \ldots & (a_{1j}, b_{1j}) & \ldots & (a_{1n}, b_{1n}) \\
 & \vdots & \vdots & & \vdots & & \vdots \\
\text{Player I} & a_i & (a_{i1}, b_{i1}) & \ldots & (a_{ij}, b_{ij}) & \ldots & (a_{in}, b_{in}) \\
 & \vdots & \vdots & & \vdots & & \vdots \\
 & a_m & (a_{m1}, b_{m1}) & \ldots & (a_{mj}, b_{mj}) & \ldots & (a_{mn}, b_{mn})
\end{array}
$$

Table 1: Payoff bi-matrix for a 2-person game

This game is a constant-sum game iff there exists a constant $c$ such that, for every pair $1 \leq i \leq m, 1 \leq j \leq n$, $a_{ij} + b_{ij} = c$.

Similarly, this game is a zero-sum game iff for every pair $i, j$, $a_{ij} + b_{ij} = 0$.

## Example 24.