

MAST20005/MAST90058: Week 3 Solutions

For all of the solutions below, we use the notation $\ell(\theta) = \ln L(\theta)$ for log-likelihood functions and $s(\theta) = \frac{\partial \ell}{\partial \theta}$ for their first derivatives.

1. (a)

$$L(\mu) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \mu)^2}$$

$$\ell(\mu) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$$

$$s(\mu) = \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu)$$

Setting $s(\mu) = 0$ and solving gives $\hat{\mu} = \bar{X}$.

(b) i.

$$L(\lambda) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i} \frac{1}{\prod_{i=1}^n x_i!}$$

$$\ell(\lambda) = -n\lambda + \left(\sum_{i=1}^n x_i \right) \ln \lambda - \ln \prod_{i=1}^n x_i!$$

$$s(\lambda) = -n + \frac{1}{\lambda} \sum_{i=1}^n x_i$$

Setting $s(\lambda) = 0$ and solving gives $\hat{\lambda} = \bar{X}$.

ii. $\bar{x} = (5 \cdot 0 + 7 \cdot 1 + 12 \cdot 2 + 9 \cdot 3 + 5 \cdot 4 + 1 \cdot 5 + 1 \cdot 6)/40 = 2.225$

(c) i.

$$L(\theta) = \left(\frac{1}{\theta^2} \right)^n \prod_{i=1}^n x_i \exp(-x_i/\theta)$$

$$\ell(\theta) = -2n \ln(\theta) + \sum_{i=1}^n \ln(x_i) - \frac{1}{\theta} \sum_{i=1}^n x_i$$

$$s(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i$$

Setting $s(\theta) = 0$ and solving gives the estimator $\hat{\theta} = \sum_{i=1}^n X_i/(2n) = \bar{X}/2$.

ii.

$$L(\theta) = \left(\frac{1}{2\theta^3} \right)^n \prod_{i=1}^n x_i^2 \exp(-x_i/\theta)$$

$$\ell(\theta) = -n \ln 2 - 3n \ln(\theta) + \sum_{i=1}^n 2 \ln(x_i) - \frac{1}{\theta} \sum_{i=1}^n x_i$$

$$s(\theta) = -\frac{3n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i$$

Setting $s(\theta) = 0$ and solving gives the estimator $\hat{\theta} = \sum_{i=1}^n X_i/(3n) = \bar{X}/3$.

iii.

$$L(\theta) = \left(\frac{1}{2}\right)^n \prod_{i=1}^n \exp(-|x_i - \theta|)$$
$$\ell(\theta) = -n \ln 2 - \sum_{i=1}^n |x_i - \theta|$$
$$s(\theta) = \sum_{i=1}^n \text{sgn}(x_i - \theta)$$

where $\text{sgn}(\cdot)$ is the sign function: $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = 0$. Note that $\ell(\theta)$ is piecewise linear and not differentiable when θ equals any x_i , so $s(\theta)$ is not defined at those points. If n is even, $s(\theta)$ is zero when there are an equal number of positive and negative signs, so $\hat{\theta}$ is between the middle two ordered values (i.e. any value between these will maximise the likelihood), and we would typically pick their average. If n is odd, for $s(\theta)$ to be zero $\hat{\theta}$ must equal the middle value. So, in general, $\hat{\theta}$ is the sample median.

2. The population mean and variance are $\mathbb{E}(X) = 5\theta/4$ and $\text{var}(X) = (7\theta/4) - (5\theta/4)^2$.

(a) We know that $\mathbb{E}(\bar{X}) = (5/4)\theta$ and therefore an unbiased estimator of θ based on \bar{X} is $T_1 = (4/5)\bar{X}$.

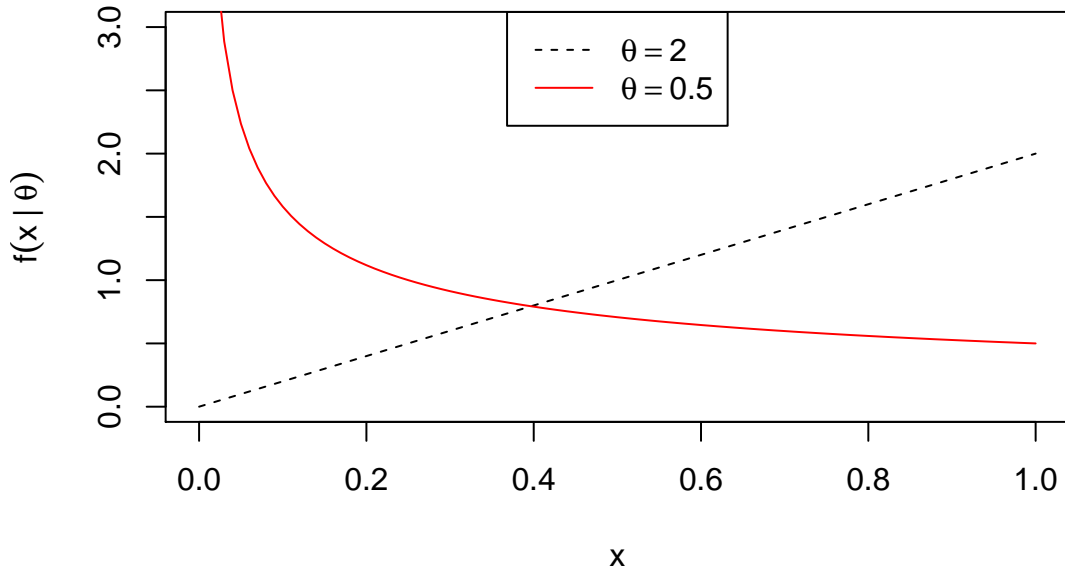
Note that $Z \sim \text{Bi}(n, 1 - \theta)$, which means that $\mathbb{E}(Z) = n(1 - \theta)$ and $\mathbb{E}(Z/n) = 1 - \theta$. Therefore, an unbiased estimator of θ based on Z is $T_2 = 1 - Z/n$.

(b) Calculating the variance of the above estimators gives:

$$\text{var}(T_1) = \frac{\theta \left(\frac{28}{25} - \theta \right)}{n}, \quad \text{var}(T_2) = \frac{\theta(1 - \theta)}{n}$$

We can therefore see that $\text{var}(T_1) > \text{var}(T_2)$.

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3. (a) f1 <- function(x) {2 * x^(2 - 1)}
      f2 <- function(x) {0.5 * x^(0.5 - 1)}
      curve(f1, 0, 1, col = 1, lty = 2, ylim = c(0, 3),
            ylab = expression(f(x ~ "|" ~ theta)))
      curve(f2, 0, 1, col = 2, lty = 1, add = TRUE)
      legend("top", c(expression(theta == 2), expression(theta == 0.5)),
            col = c(1, 2), lty = c(2, 1))
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(b)

$$L(\theta) = \theta^n \prod_{i=1}^n x_i^{\theta-1}$$

$$\ell(\theta) = n \ln \theta + (\theta - 1) \sum_{i=1}^n \ln(x_i)$$

$$s(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \ln(x_i)$$

Setting $s(\theta) = 0$ and solving gives the estimator:

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln X_i}$$

(c) The MLEs are: $\hat{\theta}_X = 0.549$, $\hat{\theta}_Y = 2.210$, $\hat{\theta}_Z = 0.959$.

To find the method of moments estimator, we need to solve $\bar{X} = \theta/(\theta + 1)$ which gives $\tilde{\theta} = \bar{X}/(1 - \bar{X})$. Therefore, the MM estimates are: $\tilde{\theta}_X = 0.598$, $\tilde{\theta}_Y = 2.400$ and $\tilde{\theta}_Z = 0.865$.

4. Recall that for a random sample X_1, \dots, X_n ,

$$E(\bar{X}) = \frac{1}{n} \sum_{i=1}^n E(X_i) = E(X_i)$$

and

$$\text{var}(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^n \text{var}(X_i) = \frac{\text{var}(X_i)}{n}.$$

(a) The X_i are iid exponential random variables with mean θ . Therefore, $\mathbb{E}(\bar{X}) = \mathbb{E}(X_i) = \theta$ and \bar{X} is unbiased.

(b) We know that $\text{var}(X_i) = \theta^2$. Therefore, $\text{var}(\bar{X}) = \text{var}(X_i)/n = \theta^2/n$.

(c) Based on the above, an estimate of θ is $\hat{\theta} = \bar{x} = 3.48$.

5. From question 4(a)(i) from week 2, we know that:

$$S^2 = \frac{1}{n-1} \left(\sum_{i=1}^n X_i^2 - n\bar{X}^2 \right).$$

Let $\mu = \mathbb{E}(X)$. Since $\sigma^2 = \text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{E}(X^2) - \mu^2$, we see that $\mathbb{E}(X^2) = \sigma^2 + \mu^2$. A similar argument shows that $\mathbb{E}(\bar{X}^2) = \sigma^2/n + \mu^2$. Using the above expression for the sample variance and taking expectations of both sides,

$$\mathbb{E}(S^2) = \frac{1}{n-1} \left\{ n(\sigma^2 + \mu^2) - n \left(\frac{\sigma^2}{n} + \mu^2 \right) \right\} = \frac{(n-1)\sigma^2}{n-1} = \sigma^2$$

which shows that S^2 is unbiased for σ^2 .

6. We already know that $\mathbb{E}(S^2) = \theta^2$, meaning that it is unbiased. Note that,

$$\theta^2 = \text{var}(X_i) = \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2 = \mathbb{E}(X_i^2) - 0 = \mathbb{E}(X_i^2).$$

Therefore,

$$\mathbb{E}(\hat{\theta}^2) = \mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n X_i^2 \right) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}(X_i^2) = \frac{1}{n} \sum_{i=1}^n \theta^2 = \theta^2,$$

meaning that it is also unbiased.

To derive the variance of the estimator, first note that,

$$\text{var}(X_i^2) = \mathbb{E}(X_i^4) - \mathbb{E}(X_i^2)^2 = \mathbb{E}(X_i^4) - \theta^4.$$

The 4th moment for a normal distribution (easy to look up) is,

$$\mathbb{E}(X^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.$$

Here we have $\mu = 0$ and $\sigma^2 = \theta^2$, so we have,

$$\mathbb{E}(X_i^4) = 3\theta^4.$$

Therefore we have,

$$\text{var}(X_i^2) = 2\theta^4.$$

(After covering Module 3, you will learn that another way to do this is to note that $X_i^2/\theta^2 \sim \chi_1^2$ and therefore $\text{var}(X_i^2/\theta^2) = 2$.) Now we derive the variance of the estimator,

$$\text{var}(\hat{\theta}^2) = \frac{1}{n^2} \sum_i \text{var}(X_i^2) = \frac{2\theta^4}{n}.$$

Also, we know that,

$$\text{var}(S^2) = \frac{2\theta^4}{n-1}.$$

Therefore, $\text{var}(\hat{\theta}^2) < \text{var}(S^2)$ for any $n > 1$.

7. (a) We know this result from the lectures, but to show you some of the derivation: since $\text{var}(X_i) = \sigma^2$ for all $i = 1, \dots, n$ and the observations are independent, we have $\text{var}(\bar{X}) = (\sigma^2 + \dots + \sigma^2)/n^2 = \sigma^2/n$.

- (b) According to the information given, $\text{var}(\hat{\pi}_{0.5}) \approx \pi/2 \times \sigma^2/n$. Since $\sigma^2/n < \pi/2 \times \sigma^2/n$, we see that the sample mean has smaller variance than the sample median.
- (c) We already know that the sample mean is unbiased. According to the information given, $\mathbb{E}(\hat{\pi}_{0.5}) \approx \pi_{0.5} = \mu$, so the sample median is at least approximately unbiased.
- (d) Both estimators are unbiased (exactly or approximately) but the sample mean has smaller variance, so we would expect it to be more accurate.