MAST30013 – Techniques in Operations Research

Semester 1

Tutorial 1 – Solutions

1. We note that $f'(x) = x \cdot (1/x) + 1 \cdot \log x - 1 = \log x = 0 \Longrightarrow x_{\min} = 1$.

   Fibonacci search

Step 1 Solve for n, $\dfrac{1.2 - 0.5}{F_n} < 0.2$. That is, $F_n > 0.7/0.2 = 3.5 \Longrightarrow F_n = 5 \Longrightarrow n = 4$. Therefore, we require 4 f-calculations.

Step 2 $k = 4$

$$\begin{aligned} p &= 1.2 - \tfrac{3}{5}(1.2 - 0.5) &= 0.78 \\ q &= 0.5 + \tfrac{3}{5}(1.2 - 0.5) &= 0.92 \\ f(0.78) &= 4.0262 \\ f(0.92) &= 4.0033 \end{aligned}$$

Step 3 $k = 3$, and $f(0.78) > f(0.92)$, therefore

$$\begin{aligned} a &= 0.78 \\ p &= 0.92 \\ q &= 0.78 + \tfrac{2}{3}(1.2 - 0.78) &= 1.06 \\ (b &= 1.2) \\ f(1.06) &= 4.0018 \end{aligned}$$

Step 4 $k = 2$, $f(0.92) > f(1.06)$, therefore

$$\begin{aligned} a &= 0.92 \\ p &= 1.06 \\ q &= 0.92 + 0.2 &= 1.12 \\ (b &= 1.2) \\ f(1.12) &= 4.0069 \end{aligned}$$

Step 5 $k = 1$, $f(1.06) < f(1.12)$, therefore

$$\begin{aligned} b &= 1.12 \\ (a &= 0.92) \end{aligned}$$

Thus, $x_{\min} \in [0.92, 1.12]$, that is, $x_{\min} = 1.02 \pm 0.1$.

Golden Section search

Step 1 $k = 1$

$$\begin{aligned} p &= 1.2 - 0.618(1.2 - 0.5) &= 0.7674 \\ q &= 0.5 + 0.618(1.2 - 0.5) &= 0.9326 \\ f(0.7674) &= 4.0294 \\ f(0.9326) &= 4.0023 \end{aligned}$$

Step 2 $k = 2$, and $f(0.7674) > f(0.9326)$, therefore

$$\begin{aligned} a &= 0.7674 \\ p &= 0.9326 \\ q &= 0.7674 + 0.618(1.2 - 0.7674) &= 1.0347 \\ (b &= 1.2) \\ f(1.0347) &= 4.0006 \end{aligned}$$

Step 3 $k = 3$, and $f(0.9326) > f(1.0347)$, therefore

$$\begin{aligned} a &= 0.9326 \\ p &= 1.0347 \\ q &= 0.9326 + 0.618(1.2 - 0.9326) &= 1.0979 \\ (b &= 1.2) \\ f(1.0979) &= 4.0046 \end{aligned}$$

Step 4 $k = 4$, $f(1.0347) < f(1.0979)$, therefore

$$\begin{aligned} b &= 1.0979 \\ q &= 1.0347 \\ p &= 1.0979 - 0.618(1.0979 - 0.9326) &= 0.9957 \\ (a &= 0.9326) \\ f(0.9957) &= 4.0000 \end{aligned}$$

Here the calculations of p, q, and $f(p)$ are actually unnecessary as $b - a = 1.0979 - 0.9326 = 0.1653 < 0.2 = 2\epsilon$. Thus, $x_{\min} \in [0.9326, 1.0979]$, that is $x_{\min} = 1.0153 \pm 0.0827$.

Both the Fibonacci and Golden Section search algorithms found $x_{\min}$ using 4 f-calculations. The Golden Section search produced an interval that is slightly smaller than the Fibonacci search because in Step 5 of the Fibonacci search the final interval is of length $\epsilon = 0.2$ rather than half of the previous interval, that is $0.5 \times 0.28 = 0.14$.

2. (a) Substituting $F_n = \lambda^n$ into (1) gives the quadratic equation $\lambda^2 - \lambda - 1 = 0$, which has the two solutions $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_1 = (1-\sqrt{5})/2$. Therefore,

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Using the initial conditions $F_0 = 1$ and $F_1 = 1$ yields $A + B = 1$ and $A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$. Solving these two equations for A and B gives (2).

(b) From (1) we have

$$\begin{aligned}
\frac{F_{n-1}}{F_n} + \frac{F_{n-2}}{F_n} &= \frac{F_n}{F_n} \\
\implies \frac{F_{n-1}}{F_n} + \frac{F_{n-2}}{F_n}\cdot\frac{F_{n-1}}{F_{n-1}} &= 1 \\
\implies \frac{F_{n-1}}{F_n} + \frac{F_{n-2}}{F_{n-1}}\cdot\frac{F_{n-1}}{F_n} &= 1 \\
\implies \gamma_n + \gamma_{n-1}\gamma_n &= 1.
\end{aligned}$$

Letting $n \to \infty$ gives the quadratic equation $\gamma^2 + \gamma - 1 = 0$.

(c) The solutions to the quadratic equation in Part (b) are $\gamma = (-1+\sqrt{5})/2$, $(-1-\sqrt{5})/2$, the first being in the interval $[0, 1]$. This solution is the same as $\lim_{n\to\infty} \frac{F_{n-1}}{F_n}$.

3. Assume, without loss of generality, that the initial interval has length 1. After n calculations (where $n \geq 2$), the Fibonacci search results in an interval of length $1/F_n$, whereas the Golden Section search results in an interval of length γ^{n-1}. Thus, we need to prove that $F_n > 1/\gamma^{n-1}$ for $n \geq 2$. To do this we use induction. Now, $F_2 = 2 > 1/\gamma = (1+\sqrt{5})/2 \approx 1.618$, and $F_3 = 3 > 1/\gamma^2 = (3+\sqrt{5})/2 \approx 2.618$. Assume that for some $n > 3$ that $F_n > 1/\gamma^{n-1}$. Then,

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&> \left(\frac{1}{\gamma}\right)^{n-1} + \left(\frac{1}{\gamma}\right)^{n-2} \\
&= \left(\frac{1}{\gamma}\right)^{n}(\gamma + \gamma^2) \\
&= \left(\frac{1}{\gamma}\right)^{n}\left(\frac{-1+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{2}\right) \\
&= \left(\frac{1}{\gamma}\right)^{n}.
\end{aligned}$$

Thus, by induction, the relation holds for $n \geq 2$.

We note that this may not be the case if, in the Fibonacci search, the width of the final interval is 2ϵ, as is the case in Question 1.