## Active constraints and regular points

In this workshop, we will deal with optimisation problems of the form

$$\begin{aligned} \text{minimise} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{h}(\mathbf{x}) = 0 \\ & \mathbf{g}(\mathbf{x}) \leqslant 0, \end{aligned}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$, $\mathbf{h}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^k$. An inequality constraint $g_i(\mathbf{x}) \leqslant 0$ is said to be active at $\mathbf{x}^*$ if $g_i(\mathbf{x}^*) = 0$. The equality constraints are always active. Let $J(\mathbf{x}^*)$ be the index set of the active inequality constraints at $\mathbf{x}^*$. A feasible point $\mathbf{x}^*$ is regular if the gradients of the active constraints,

$$\nabla h_i(\mathbf{x}^*), \quad \nabla g_j(\mathbf{x}^*), \quad i \in \{1, \dots, m\}, \quad j \in J(\mathbf{x}^*),$$

are linearly independent.

1. Consider the function $\mathbf{g}\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} x_2 - x_1^2 & x_1 - 1 & -x_2 \end{pmatrix}^T$.
   Let $\mathcal{G} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{g}(\mathbf{x}) \leqslant \mathbf{0}\}$. Sketch $\mathcal{G}$ and determine all points in $\mathcal{G}$ which are not regular.

## The KKT theorem

Given $f\colon \mathbb{R}^n \to \mathbb{R}$, $\mathbf{h}\colon \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$, $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^k$, the Lagrangian is

$$\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x}) + \boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}).$$

The KKT theorem then states that if $\mathbf{x}^*$ is a regular point which is a local minimiser of $f$ subject to $\mathbf{h}(\mathbf{x}) = \mathbf{0}$, $\mathbf{g}(\mathbf{x}) \leqslant \mathbf{0}$, then there exists $\boldsymbol{\lambda} \in \mathbb{R}^m$ and $\boldsymbol{\mu} \in \mathbb{R}^k$ such that

(1) $D\mathcal{L}(\mathbf{x}^*; \boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathbf{0}^T$,

(2) $\boldsymbol{\mu}\mathbf{g}(\mathbf{x}^*) = \mathbf{0}$, and

(3) $\boldsymbol{\mu} \geqslant \mathbf{0}$.

In conjunction with condition (3), condition (2) can be expressed equivalently as (2) $\mu_i g_i(\mathbf{x}^*) = 0$, for all $i \leqslant k$.

2. Show that for the problem of minimising $f(\mathbf{x})$ subject to $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{x}) \geqslant \mathbf{0}$, the conditions are the same but with $\boldsymbol{\mu} \leqslant \mathbf{0}$ instead of $\boldsymbol{\mu} \geqslant \mathbf{0}$.

The SOSC states that if $\mathbf{x}^*$ is a feasible point satisfying the conditions above, and if $D^2\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda})$ is positive definite on $T^a\mathcal{H}(\mathbf{x}^*)$, where $\mathcal{H}$ is the feasible set and $T^a\mathcal{H}(\mathbf{x}^*)$ is the active tangent space,

$$T^a\mathcal{H}(\mathbf{x}^*) = \{\mathbf{x} \in \mathbb{R}^n : D\mathbf{h}(\mathbf{x}^*)\mathbf{x} = \mathbf{0}, \quad Dg_j(\mathbf{x}^*)\mathbf{x} = 0, \quad j \in J(\mathbf{x}^*)\},$$

then $\mathbf{x}^*$ is a strict local minimiser.

Replacing condition (3) with $\boldsymbol{\mu} \leqslant \mathbf{0}$ and positive definite with negative definite will result in a local maximiser instead.

3. Consider the nonlinear problem

$$\begin{aligned} \text{maximise} \quad & 2x_1^2 + 5x_1 - x_2 \\ \text{subject to} \quad & (x_1 + 1)^2 + (x_2 - 3)^2 \leqslant 4 \end{aligned}$$

   (a) Write down the relevant KKT condition.

   (b) Determine, with reasons, whether there is a maximum at $\mu = 0$.

   (c) Assuming that $\mu \neq 0$, derive a polynomial equation for $\mu$. Use MATLAB to find and simplify this polynomial, and determine its roots numerically (using `vpasolve`).

   (d) Three zeros of the polynomial equation for $\mu$ are given by:

   $$\mu_1 = -1.747, \quad \mu_2 = -0.253, \quad \mu_3 = 0.252$$

   Determine, using the SOSC, which $\mu_i$ corresponds to a strict local maximiser.

4. Use the KKT condition to find local minimisers for

$$\begin{aligned} \text{minimise} \quad & x_1^2 + x_2^2 \\ \text{subject to} \quad & x_1^2 + 2x_1x_2 + x_2^2 = 1 \\ & x_1^2 - x_2 \leqslant 0, \end{aligned}$$

by distinguishing the cases $\mu = 0$ and $\mu > 0$. Use the SOSC to conclusively show the point you find is a strict local minimiser.