Analysis of variance
(Module 8)

Statistics (MAST20005) &
Elements of Statistics
(MAST90058)

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Aims of this module

- Introduce the **analysis of variance** technique, which builds upon the variance decomposition ideas in previous modules.
- Revisit linear regression and apply the ideas of hypothesis testing and analysis of variance.
- Discuss ways to derive optimal hypothesis tests.

Overview

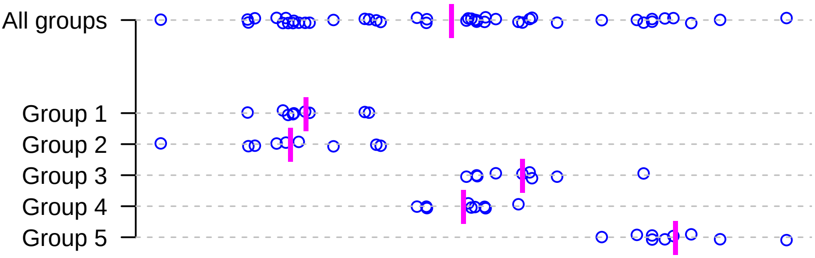
- **Analysis of variance (ANOVA).**
  Comparisons of more than two groups
- **Regression.**
  Hypothesis testing for simple linear regression
- **Likelihood ratio tests.**
  A method for deriving the best test for a given problem

Outline

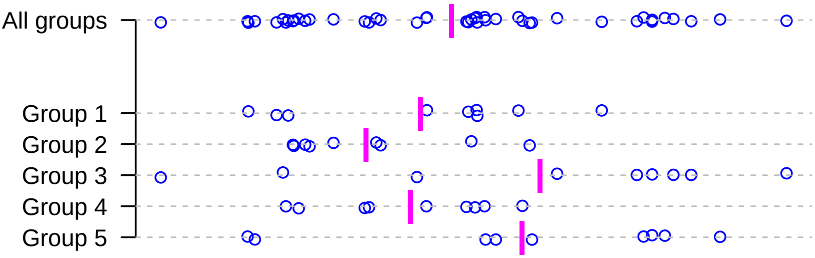
Analysis of variance: introduction

- Initial aim: compare the means of **more than two** populations
- Broader and more advanced aims:
  - Explore components of variation
  - Evaluate the fit a (general) linear model
- Formulated as hypothesis tests
- Referred to as analysis of variance, or ANOVA for short
- Involves comparing different summaries of variation
- Related to the 'analysis of variance' and 'variance decomposition' formulae we derived previously

Example: large variation between groups



Example: smaller variation between groups



ANOVA: setting it up

- We have random samples from k populations, each having a normal distribution
- We sample n_i iid observations from the ith population, which has mean μ_i
- All populations assumed have the **same** variance, σ^2
- Question of interest: do the populations all have the same mean?
- Hypotheses:

$$H_0\colon \mu_1 = \mu_2 = \cdots = \mu_k = \mu \quad \text{versus} \quad H_1\colon \bar{H}_0$$

($\bar{H}_0$ means 'not H_0')

- This model is known as a one-way ANOVA, or single-factor ANOVA

Notation

Population	Sample	Statistics	
$\mathrm{N}(\mu_1, \sigma^2)$	$X_{11}, X_{12}, \ldots, X_{1n_1}$	$\bar{X}_{1\cdot}$	S_1^2
$\mathrm{N}(\mu_2, \sigma^2)$	$X_{21}, X_{22}, \ldots, X_{2n_2}$	$\bar{X}_{2\cdot}$	S_2^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{N}(\mu_k, \sigma^2)$	$X_{k1}, X_{k2}, \ldots, X_{kn_k}$	$\bar{X}_{k\cdot}$	S_k^2
Overall		$\bar{X}_{\cdot\cdot}$	

$$n = n_1 + \cdots + n_k \quad \text{(total sample size)}$$

$$\bar{X}_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} \quad \text{(group means)}$$

$$\bar{X}_{\cdot\cdot} = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} = \frac{1}{n} \sum_{i=1}^{k} n_i \bar{X}_{i\cdot} \quad \text{(grand mean)}$$

Sum of squares (definitions)

- We now define statistics each called a sum of squares (SS)
- The total SS is:

$$SS(TO) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{..})^2$$

- The treatment SS, or between groups SS, is:

$$SS(T) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\bar{X}_{i.} - \bar{X}_{..})^2 = \sum_{i=1}^{k} n_i (\bar{X}_{i.} - \bar{X}_{..})^2$$

- The error SS, or within groups SS, is:

$$SS(E) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})^2 = \sum_{i=1}^{k} (n_i - 1) S_i^2$$

Analysis of variance decomposition

- It turns out that:

$$SS(TO) = SS(T) + SS(E)$$

- This is similar to the analysis of variance formulae we derived earlier, in simpler scenarios (iid model, regression model)
- We will use this relationship as a basis to derive a hypothesis test
- Let's first prove the relationship...

- Start with the 'add and subtract' trick:

$$
\begin{aligned}
SS(TO) &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{..})^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{i.} + \bar{X}_{i.} - \bar{X}_{..})^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{i.})^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{X}_{i.} - \bar{X}_{..})^2 \\
&\qquad + 2\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{i.})(\bar{X}_{i.} - \bar{X}_{..}) \\
&= SS(E) + SS(T) + CP
\end{aligned}
$$

- The cross-product term is:

$$
\begin{aligned}
CP &= 2\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_{i\cdot})(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot}) \\
&= 2\sum_{i=1}^{k}(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot})\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_{i\cdot}) \\
&= 2\sum_{i=1}^{k}(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot})(n_i\bar{X}_{i\cdot}-n_i\bar{X}_{i\cdot}) \\
&= 0
\end{aligned}
$$

- Thus, we have:

$$SS(TO) = SS(T) + SS(E)$$

Sampling distribution of $SS(E)$

- The sample variance from the ith group, S_i^2, is an unbiased estimator of σ^2 and we know that $(n_i - 1)S_i^2/\sigma^2 \sim \chi^2_{n_i-1}$
- The samples from each group are independent, so we can usefully combine them,

$$\sum_{i=1}^{k} \frac{(n_i - 1)S_i^2}{\sigma^2} = \frac{SS(E)}{\sigma^2} \sim \chi^2_{n-k}$$

- Note that: $(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = n - k$
- This also gives us an unbiased pooled estimator of σ^2,

$$\hat{\sigma}^2 = \frac{SS(E)}{n-k}$$

- These results are true irrespective of whether H_0 is true or not

Null sampling distribution of $SS(TO)$

- If we assume H_0, we can derive simple expressions for the sampling distributions of the other sums of squares
- The combined data would be a sample of size n from $\mathrm{N}(\mu, \sigma^2)$. Hence $SS(TO)/(n-1)$ is an unbiased estimator of σ^2 and

$$\frac{SS(TO)}{\sigma^2} \sim \chi^2_{n-1}$$

Null sampling distribution of $SS(T)$

- Under H_0, we have $\bar{X}_{i\cdot} \sim \mathrm{N}(\mu, \frac{\sigma^2}{n_i})$
- $\bar{X}_{1\cdot}, \bar{X}_{2\cdot}, \ldots, \bar{X}_{k\cdot}$ are independent
- (Can think of this as a sample of sample means, and then think about what its variance estimator is)
- It is possible to show that (proof not shown):

$$\sum_{i=1}^{k} \frac{n_i(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2}{\sigma^2} = \frac{SS(T)}{\sigma^2} \sim \chi^2_{k-1}$$

and that this is independent of $SS(E)$

Null sampling distributions

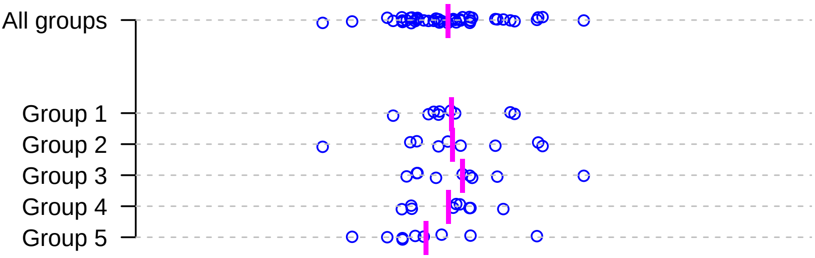
In summary, under H_0:

$$\frac{SS(TO)}{\sigma^2} = \frac{SS(E)}{\sigma^2} + \frac{SS(T)}{\sigma^2}$$

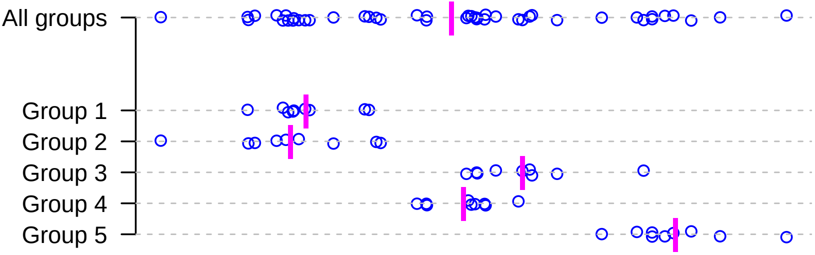
$$\frac{SS(TO)}{\sigma^2} \sim \chi^2_{n-1}, \quad \frac{SS(E)}{\sigma^2} \sim \chi^2_{n-k}, \quad \frac{SS(T)}{\sigma^2} \sim \chi^2_{k-1},$$

$SS(E)$ and $SS(T)$ are independent

H_0 is true



H_1 is true



$SS(T)$ under H_1

- What happens if H_1 is true?
- The population means differ, which will make $SS(T)$ larger
- Let's make this precise...

- Let $\bar{\mu} = n^{-1} \sum_{i=1}^{k} n_i \mu_i$, and then,

$$
\begin{aligned}
\mathbb{E}[SS(T)] &= \mathbb{E}\left[\sum_{i=1}^{k} n_i (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2\right] = \mathbb{E}\left[\sum_{i=1}^{k} n_i \bar{X}_{i\cdot}^2 - n\bar{X}_{\cdot\cdot}^2\right] \\
&= \sum_{i=1}^{k} n_i \,\mathbb{E}(\bar{X}_{i\cdot}^2) - n\,\mathbb{E}(\bar{X}_{\cdot\cdot}^2) \\
&= \sum_{i=1}^{k} n_i \left[\mathrm{var}(\bar{X}_{i\cdot}) + \mathbb{E}(\bar{X}_{i\cdot})^2\right] - n\left[\mathrm{var}(\bar{X}_{\cdot\cdot}) + \mathbb{E}(\bar{X}_{\cdot\cdot})^2\right] \\
&= \sum_{i=1}^{k} n_i \left[\frac{\sigma^2}{n_i} + \mu_i^2\right] - n\left[\frac{\sigma^2}{n} + \bar{\mu}^2\right] \\
&= (k-1)\sigma^2 + \sum_{i=1}^{k} n_i (\mu_i - \bar{\mu})^2
\end{aligned}
$$

$$\mathbb{E}[SS(T)] = (k-1)\sigma^2 + \sum_{i=1}^{k} n_i(\mu_i - \bar{\mu})^2$$

- Under H_0 the second term is zero and we have,

$$\frac{\mathbb{E}(SS(T))}{k-1} = \sigma^2$$

- Otherwise (under H_1), the second term is positive and gives,

$$\frac{\mathbb{E}(SS(T))}{k-1} > \sigma^2$$

- In contrast, we always have,

$$\frac{E(SS(E))}{n-k} = \sigma^2$$

F-test statistic

- This movitates using the following as our test statistic:

$$F = \frac{SS(T)/(k-1)}{SS(E)/(n-k)}$$

- Under H_0, we have $F \sim \mathrm{F}_{k-1,n-k}$, since it is the ratio of independent χ^2 random variables
- Under H_1, the numerator will tend to be larger
- Therefore, reject H_0 if $F > c$
- This is known as an F-test

ANOVA table

The test quantities are often summarised using an ANOVA table:

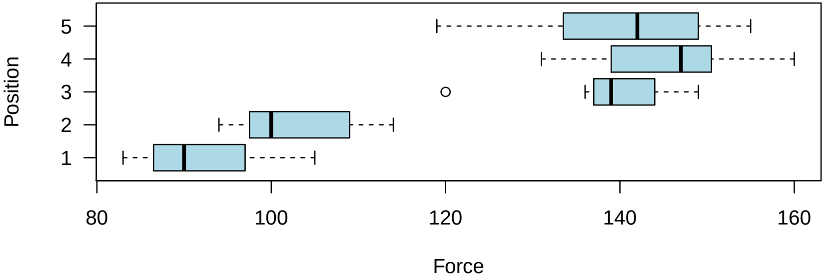
Source	df	SS	MS	F
Treatment	$k-1$	$SS(T)$	$MS(T) = \frac{SS(T)}{k-1}$	$\frac{MS(T)}{MS(E)}$
Error	$n-k$	$SS(E)$	$MS(E) = \frac{SS(E)}{n-k}$	
Total	$n-1$	$SS(TO)$		

Notes:

- MS = 'Mean square'
- $\hat{\sigma}^2 = MS(E)$ is an unbiased estimator

Example (one-way ANOVA)

Force required to pull out window studs in 5 positions on a car window.



```
> head(data1)
  Position Force
1        1    92
2        1    90
3        1    87
4        1   105
5        1    86
6        1    83

> table(data1$Position)

1 2 3 4 5
7 7 7 7 7
```

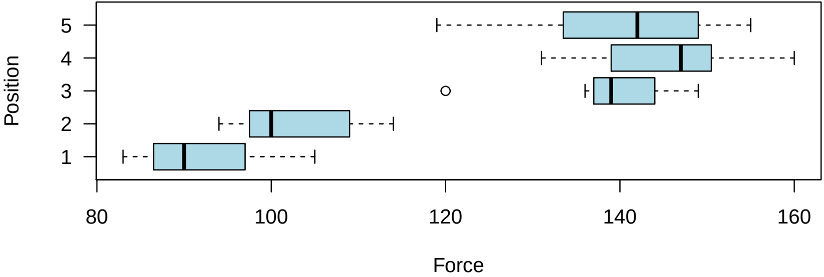
```
> model1 <- lm(Force ~ factor(Position), data = data1)
> anova(model1)
Analysis of Variance Table

Response: Force
                 Df  Sum Sq Mean Sq F value    Pr(>F)
factor(Position)  4 16672.1  4168.0  44.202 3.664e-12 ***
Residuals        30  2828.9    94.3
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Notes:

- Need to use `factor()` to denote categorical variables
- R doesn't provide a 'Total' row, but we don't need it
- `Residuals` is the 'Error' row
- `Pr(>F)` is the p-value for the F-test

We conclude that the mean force required to pull out the window studs varies between the 5 positions on the car window (e.g. p-value < 0.01)

This was obvious from the boxplots: positions 1 & 2 are quite different from 3, 4 & 5



Two factors

- In one-way ANOVA, the observations were partitioned into k groups
- In other words, they were defined by a single categorical variable ('factor')
- What if we had two such variables?
- We can extend the procedure to give two-way ANOVA, or two-factor ANOVA
- For example, the fuel consumption of a car may depend on type of petrol and the brand of tyres

Two-way ANOVA: setting it up

- Factor 1 has a levels, Factor 2 has b levels
- Suppose we have exactly one observation per factor combination
- Observe X_{ij} with factor 1 at level i and factor 2 at level j
- Gives a total of $n = ab$ observations
- Assume $X_{ij} \sim \mathrm{N}(\mu_{ij}, \sigma^2)$, $i = 1, \ldots, a$, $j = 1, \ldots, b$, and that these are independent
- Consider the model:

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

$$\text{with } \sum_{i=1}^{a} \alpha_i = 0,\ \sum_{j=1}^{b} \beta_j = 0$$

- μ is an overall effect, α_i is the effect of the ith row and β_j the effect of the jth column.
- For example, $a = 4$ and $b = 4$,

	1	2	3	4
1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\mu + \alpha_1 + \beta_3$	$\mu + \alpha_1 + \beta_4$
2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$	$\mu + \alpha_2 + \beta_4$
3	$\mu + \alpha_3 + \beta_1$	$\mu + \alpha_3 + \beta_2$	$\mu + \alpha_3 + \beta_3$	$\mu + \alpha_3 + \beta_4$
4	$\mu + \alpha_4 + \beta_1$	$\mu + \alpha_4 + \beta_2$	$\mu + \alpha_4 + \beta_3$	$\mu + \alpha_4 + \beta_4$

- We are usually interested in $H_{0A}\colon \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$ or $H_{0B}\colon \beta_1 = \beta_2 = \cdots = \beta_b = 0$
- Let

$$\bar{X}_{\cdot\cdot} = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} X_{ij}, \quad \bar{X}_{i\cdot} = \frac{1}{b} \sum_{j=1}^{b} X_{ij}, \quad \bar{X}_{\cdot j} = \frac{1}{a} \sum_{i=1}^{a} X_{ij}$$

- Arguing as before,

$$\begin{aligned}
SS(TO) &= \sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij} - \bar{X}_{\cdot\cdot})^2 \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\Big[(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}) + (\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) \\
&\qquad\qquad + (X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot})\Big]^2 \\
&= b\sum_{i=1}^{a}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 + a\sum_{j=1}^{b}(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})^2 \\
&\qquad\qquad + \sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot})^2 \\
&= SS(A) + SS(B) + SS(E)
\end{aligned}$$

- If both $\alpha_1 = \cdots = \alpha_a = 0$ and $\beta_1 = \cdots = \beta_b = 0$, then we have $SS(A)/\sigma^2 \sim \chi^2_{a-1}$, $SS(B)/\sigma^2 \sim \chi^2_{b-1}$ and $SS(E)/\sigma^2 \sim \chi^2_{(a-1)(b-1)}$ and these variables are independent (proof not shown)
- Reject $H_{0A}\colon \alpha_1 = \cdots = \alpha_a = 0$ at significance level α if:

$$F_A = \frac{SS(A)/(a-1)}{SS(E)/((a-1)(b-1))} > c$$

  where c is the $1-\alpha$ quantile of $\mathrm{F}_{a-1,(a-1)(b-1)}$
- Reject $H_{0B}\colon \beta_1 = \cdots = \beta_b = 0$ at significance level α if:

$$F_B = \frac{SS(B)/(b-1)}{SS(E)/((a-1)(b-1))} > c$$

  where c is the $1-\alpha$ quantile of $\mathrm{F}_{b-1,(a-1)(b-1)}$

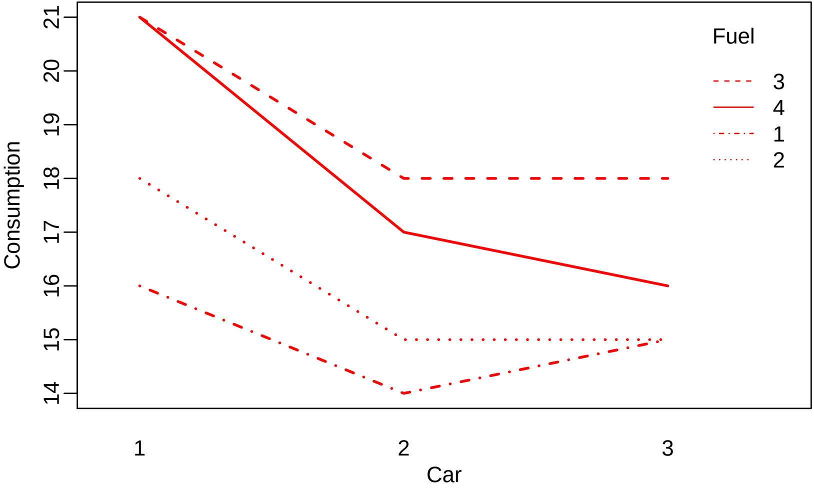
ANOVA table

Source	df	SS	MS	F
Factor A	$a-1$	$SS(A)$	$MS(A)=\frac{SS(A)}{a-1}$	$\frac{MS(A)}{MS(E)}$
Factor B	$b-1$	$SS(B)$	$MS(B)=\frac{SS(B)}{b-1}$	$\frac{MS(B)}{MS(E)}$
Error	$(a-1)(b-1)$	$SS(E)$	$MS(E)=\frac{SS(E)}{(a-1)(b-1)}$	
Total	$ab-1$	$SS(TO)$		

Example (two-way ANOVA)

Data on fuel consumption for three types of car (A) and four types of fuel (B).

```
> head(data2)
  Car Fuel Consumption
1   1    1          16
2   1    2          18
3   1    3          21
4   1    4          21
5   2    1          14
6   2    2          15
```


Fuel
3
4
1
2
Consumption
14
15
16
17
18
19
20
21
1
2
3
Car

```
> model2 <- lm(Consumption ~ factor(Car) + factor(Fuel),
+              data = data2)
> anova(model2)
Analysis of Variance Table

Response: Consumption
             Df Sum Sq Mean Sq F value   Pr(>F)
factor(Car)   2     24 12.0000      18 0.002915 **
factor(Fuel)  3     30 10.0000      15 0.003401 **
Residuals     6      4  0.6667
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

From this we conclude there is a clear difference in fuel consumption between cars (we reject $H_{0A}\colon \alpha_1 = \alpha_2 = \alpha_3$) and also between fuels (we reject $H_{0B}\colon \beta_1 = \beta_2 = \beta_3 = \beta_4$).

Interaction terms

- In the previous example we assumed an additive model:

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

- This assumes, for example, that the relative effect of petrol 1 is the same for all cars.
- If it is not true, then there is a statistical interaction (or simply an interaction) between the factors

- A more general model, which includes interactions, is:

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

where γ_{ij} is the interaction term associated with combination (i, j).

- In addition to our previous assumptions, we also impose:

$$\sum_{i=1}^{a} \gamma_{ij} = 0, \quad \text{and} \quad \sum_{j=1}^{b} \gamma_{ij} = 0$$

- The terms α_i and β_j are called main effects
- When written out as a table they are also often referred to as the row effects and column effects respectively

- Writing this out as a table:

	1	2	$\cdots$
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$	$\cdots$
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$	$\cdots$
3	$\mu + \alpha_3 + \beta_1 + \gamma_{31}$	$\mu + \alpha_3 + \beta_2 + \gamma_{32}$	$\cdots$
4	$\mu + \alpha_4 + \beta_1 + \gamma_{41}$	$\mu + \alpha_4 + \beta_2 + \gamma_{42}$	$\cdots$

- We are now interested in testing whether:
  - the row effects are zero
  - the column effects are zero
  - the interactions are zero (do this first!)
- To make inferences about the interactions we need more than one observation per cell
- Let X_{ijk}, $i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, c$ be the kth observation for combination (i, j)

- Let

$$\bar{X}_{ij.} = \frac{1}{c}\sum_{k=1}^{c} X_{ijk}$$

$$\bar{X}_{i..} = \frac{1}{bc}\sum_{j=1}^{b}\sum_{k=1}^{c} X_{ijk}$$

$$\bar{X}_{.j.} = \frac{1}{ac}\sum_{i=1}^{a}\sum_{k=1}^{c} X_{ijk}$$

$$\bar{X}_{...} = \frac{1}{abc}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} X_{ijk}$$

- and as before

$$
\begin{aligned}
SS(TO) &= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(X_{ijk}-\bar{X}_{...}\right)^2 \\
&= bc\sum_{i=1}^{a}(\bar{X}_{i..}-\bar{X}_{...})^2 + ac\sum_{j=1}^{b}(\bar{X}_{.j.}-\bar{X}_{...})^2 \\
&\quad + c\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{X}_{ij.}-\bar{X}_{i..}-\bar{X}_{.j.}+\bar{X}_{...})^2 \\
&\quad + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(X_{ijk}-\bar{X}_{ij.})^2 \\
&= SS(A)+SS(B)+SS(AB)+SS(E)
\end{aligned}
$$

Test statistics

- Familiar arguments show that to test

$$H_{0AB}\colon \gamma_{ij} = 0, \quad i = 1, \ldots, a, \quad j = 1, \ldots, b$$

we may use the statistic

$$F = \frac{SS(AB)/[(a-1)(b-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(a-1)(b-1)$ and $ab(c-1)$ degrees of freedom.

- To test

$$H_{0A}\colon \alpha_i = 0, \quad i = 1, \ldots, a$$

we may use the statistic

$$F = \frac{SS(A)/[(a-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(a-1)$ and $ab(c-1)$ degrees of freedom.

- To test

$$H_{0B}\colon \beta_j = 0, \quad j = 1, \ldots, b$$

we may use the statistic

$$F = \frac{SS(B)/[(b-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(b-1)$ and $ab(c-1)$ degrees of freedom.

ANOVA table

Source	df	SS	MS	F
Factor A	$a-1$	$SS(A)$	$MS(A) = \frac{SS(A)}{a-1}$	$\frac{MS(A)}{MS(E)}$
Factor B	$b-1$	$SS(B)$	$MS(B) = \frac{SS(B)}{b-1}$	$\frac{MS(B)}{MS(E)}$
Factor AB	$(a-1)(b-1)$	$SS(AB)$	$MS(AB) = \frac{SS(AB)}{(a-1)(b-1)}$	$\frac{MS(AB)}{MS(E)}$
Error	$ab(c-1)$	$SS(E)$	$MS(E) = \frac{SS(E)}{ab(c-1)}$	
Total	$abc-1$	$SS(TO)$		

Example (two-way ANOVA with interaction)

- Six groups of 18 people
- Each person takes an arithmetic test: the task is to add three numbers together
- The numbers are presented either in a down array or an across array; this defines 2 levels of factor A
- The numbers have either one, two or three digits; this defines 3 levels of factor B
- The response variable, X, is the average number of problems completed correctly over two 90-second sessions

- Example of adding **one-digit** numbers in an **across** array:

$$2 + 5 + 1 = ?$$

- Example of adding **two-digit** numbers in an **down** array:

$$\begin{array}{rr} & 13 \\ & 87 \\ + & 51 \\ \hline & ? \end{array}$$

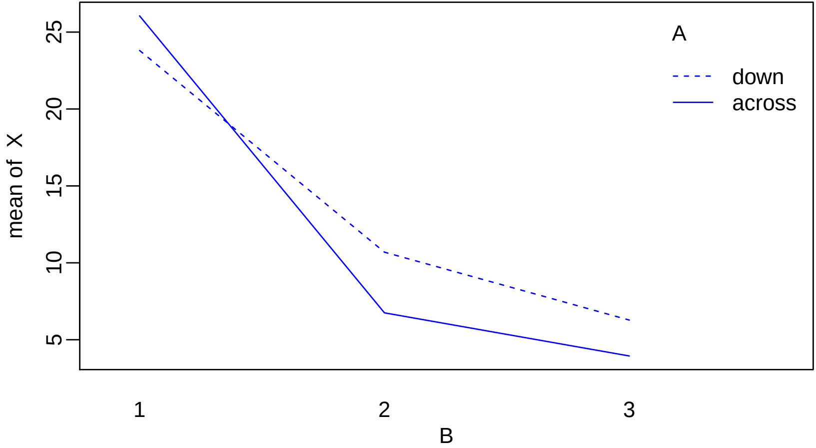
```
> head(data3)
     A B    X
1 down 1 19.5
2 down 1 18.5
3 down 1 32.0
4 down 1 21.5
5 down 1 28.5
6 down 1 33.0

> table(data3[, 1:2])
        B
A         1  2  3
  down   18 18 18
  across 18 18 18
```

```
> model3 <- lm(X ~ factor(A) * factor(B), data = data3)
> anova(model3)
Analysis of Variance Table

Response: X
                     Df Sum Sq Mean Sq  F value  Pr(>F)
factor(A)             1   48.7    48.7   2.8849 0.09246 .
factor(B)             2 8022.7  4011.4 237.7776 < 2e-16 ***
factor(A):factor(B)   2  185.9    93.0   5.5103 0.00534 **
Residuals           102 1720.8    16.9
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Note the use of '∗' in the model formula.
The interaction is significant at a 5% level (or even at 1%).

Interaction plot

```
with(data3, interaction.plot(B, A, X, col = "blue"))
```



Beyond the F-test

- We have rejected the null... now what?
- This is often only the beginning of a statistical analysis of this type of data
- Will be interested in more detailed inferences, e.g. CIs/tests about individual parameters
- You know enough to be able to work some of this out...
- ...and later subjects will go into this in more detail (e.g. MAST30025)

Outline

Recap of simple linear regression

- Y a response variable, e.g. student's grade in first-year calculus
- x a predictor variable, e.g. student's high school mathematics mark
- Data: pairs $(x_1, y_1), \ldots, (x_n, y_n)$
- Linear regression model:

$$Y_i = \alpha + \beta(x_i - \bar{x}) + \epsilon_i$$

  where $\epsilon_i \sim \mathrm{N}(0, \sigma^2)$ is a random error
- **Note:** α here plays the same role as α_0 from Module 5. We have dropped the '0' subscript for convenience, and also to avoid confusion with its use to denote null hypotheses.

- The MLE (and OLS) estimators are:

$$\hat{\alpha} = \bar{Y}, \quad \hat{\beta} = \frac{\sum_{i=1}^{n} Y_i(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- and

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}[Y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x})]^2$$

- We also derived:

$$\hat{\alpha} \sim \mathrm{N}\left(\alpha, \frac{\sigma^2}{n}\right)$$
$$\hat{\beta} \sim \mathrm{N}\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)$$

- and

$$\frac{(n-2)\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n}\left[Y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x})\right]^2}{\sigma^2} \sim \chi^2_{n-2}$$

- From these we obtain,

$$T_\alpha = \frac{\hat{\alpha} - \alpha}{\hat{\sigma}/\sqrt{n}} \sim t_{n-2}$$

$$T_\beta = \frac{\hat{\beta} - \beta}{\hat{\sigma}/\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}} \sim t_{n-2}$$

- We used these previously to construct confidence intervals
- We can also use them to construct hypothesis tests
- For example, to test $H_0\colon \beta = \beta_0$ versus $H_1\colon \beta \neq \beta_0$ (or $\beta > \beta_0$ or $\beta < \beta_0$), we use T_β as the test statistic

Example: testing the slope parameter (β)

- Data: 10 pairs of scores on a preliminary test and a final exam
- Estimates: $\hat{\alpha} = 81.3$, $\hat{\beta} = 0.742$, $\hat{\sigma}^2 = 27.21$
- Test $H_0\colon \beta = 0$ versus $H_1\colon \beta \neq 0$ with a 1% significance level
- Reject H_0 if:

$$|T_\beta| \geqslant 3.36 \quad \text{(0.995 quantile of } t_8\text{)}$$

- For the observed data,

$$t_\beta = \frac{0.742 - 0}{\sqrt{27.21/756.1}} = 3.91$$

so we reject H_0, concluding there is sufficient evidence that the slope differs from zero.

Note regarding the intercept parameter (α)

- Software packages (such as R) will typically fit the model:

$$Y_i = \alpha + \beta x_i + \epsilon_i$$

- This is equivalent to

$$Y_i = \alpha^* + \beta(x_i - \bar{x}) + \epsilon_i$$

  where $\alpha = \alpha^* - \beta\bar{x}$

- The formulation $Y_i = \alpha^* + \beta(x - \bar{x}) + \epsilon$ is easier to examine theoretically.
- We saw that

$$\hat{\alpha}^* = \bar{Y}, \quad \text{and} \quad \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x}$$

- $\hat{\alpha}$ or $\hat{\alpha}^*$ are rarely of direct interest

Using R

Use R to fit the regression model for the slope example:

```
> m1 <- lm(final_exam ~ prelim_test)
> summary(m1)

Call:
lm(formula = final_exam ~ prelim_test)

Residuals:
   Min     1Q Median     3Q    Max
-6.883 -3.264 -0.530  3.438  8.470
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  30.6147    13.0622   2.344  0.04714 *
prelim_test   0.7421     0.1897   3.912  0.00447 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.217 on 8 degrees of freedom
Multiple R-Squared: 0.6567,  Adjusted R-squared: 0.6137
F-statistic:  15.3 on 1 and 8 DF,  p-value: 0.004471
```

The t-value and the p-value are for testing $H_0 \colon \alpha = 0$ and $H_0 \colon \beta = 0$ respectively.

Interpreting the R output

- Usually most interested in testing $H_0\colon \beta = 0$ versus $H_1\colon \beta \neq 0$
- If we reject H_0 then we conclude there is sufficient evidence of (at least) a linear relationship between the mean response and x
- In the example,
$$t = \frac{0.7421}{0.1897} = 3.912$$
- This test statistic has a t-distribution with $10 - 2 = 8$ degrees of freedom, and the associated p-value is $0.00447 < 0.05$ so at the 5% level of significance we reject H_0
- It is also possible to represent this test using an ANOVA table

Deriving the variance decomposition formula

- Independent pairs $(x_1, Y_1), \ldots, (x_n, Y_n)$
- Parameter estimates,

$$\hat{\alpha} = \bar{Y}, \quad \hat{\beta} = \frac{\sum_{i=1}^{n} Y_i(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- Fitted value (estimated mean),

$$\hat{Y}_i = \bar{Y} + \hat{\beta}(x_i - \bar{x})$$

- Do the 'add and subtract' trick again:

$$
\begin{aligned}
\sum_{i=1}^{n}(Y_i - \bar{Y})^2 &= \sum_{i=1}^{n}(Y_i - \hat{Y}_i + \hat{Y}_i - \bar{Y})^2 \\
&= \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 + \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2 \\
&\quad + 2\sum_{i=1}^{n}(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y})
\end{aligned}
$$

- Deal with the cross-product term,

$$\begin{aligned}
\sum_{i=1}^{n}(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) &= \sum_{i=1}^{n}\left[Y_i - \bar{Y} - \hat{\beta}(x_i - \bar{x})\right]\hat{\beta}(x_i - \bar{x}) \\
&= \hat{\beta}\sum_{i=1}^{n}\left[Y_i - \bar{Y} - \hat{\beta}(x_i - \bar{x})\right](x_i - \bar{x}) \\
&= \hat{\beta}\left[\sum_{i=1}^{n}(Y_i - \bar{Y})(x_i - \bar{x}) - \hat{\beta}\sum_{i=1}^{n}(x_i - \bar{x})^2\right] \\
&= \hat{\beta}\left[\sum_{i=1}^{n}Y_i(x_i - \bar{x}) - \hat{\beta}\sum_{i=1}^{n}(x_i - \bar{x})^2\right] \\
&= 0
\end{aligned}$$

- That gives us,

$$\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 + \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$$

- We can write this as follows,

$$SS(TO) = SS(E) + SS(R)$$

where $SS(R)$ is the regression SS or model SS

- The regression SS quantifies the variation **due to** the straight line
- The error SS quantifies the variation **around** the straight line

- To complete the specification,

$$MS(E) = \frac{SS(E)}{n-2} = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 = \hat{\sigma}^2$$

$$MS(R) = \frac{SS(R)}{1} = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$$

- Then we have the test statistic,

$$F = \frac{MS(R)}{MS(E)} \sim \mathrm{F}_{1,n-2}$$

ANOVA table

Source	df	SS	MS	F
Model	1	$SS(R)$	$MS(R) = \frac{SS(R)}{1}$	$\frac{MS(R)}{MS(E)}$
Error	$n-2$	$SS(E)$	$MS(E) = \frac{SS(E)}{n-2}$	
Total	$n-1$	$SS(TO)$		

Using R

```
> anova(m1)
Analysis of Variance Table

Response: final_exam
           Df Sum Sq Mean Sq F value   Pr(>F)
prelim_test  1 416.39  416.39  15.301 0.004471 **
Residuals    8 217.71   27.21
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Notes:

- The F-statistic tests the 'significance of the regression'
- That is, $H_0\colon \beta = 0$ versus $H_1\colon \beta \neq 0$

Outline

Is there a 'best' test?

- We have examined a variety of commonly used tests
- We used test statistics that:
  - Seemed useful
  - We were familiar with
- Did we use the 'best' one?
- Is there a general procedure for finding a good/best test statistic?
- We will introduce a general procedure now, and discuss why it is optimal later in the semester

Likelihood ratio test

- The likelihood ratio test (LRT) is a general procedure that can find the best test for a given problem
- Suppose we have H_0 and H_1 and both are composite and of the form:

$$H_0 \colon \theta \in A_0 \quad \text{versus} \quad H_1 \colon \theta \in A_1$$

  where A_0 and A_1 are sets of possible parameter values consistent with each of the hypotheses.
- Note: we have mostly dealt with A_0 that has only one element (simple null hypothesis)
- The likelihood ratio is:

$$\lambda = \frac{L_0}{L_1} = \frac{\max_{\theta \in A_0} L(\theta)}{\max_{\theta \in A_1} L(\theta)}$$

$$\lambda = \frac{L_0}{L_1} = \frac{\max_{\theta \in A_0} L(\theta)}{\max_{\theta \in A_1} L(\theta)}$$

- L is the likelihood function
- Clearly $\lambda \geqslant 0$
- Large $\lambda \Rightarrow$ more support for H_0 over H_1
- λ near zero $\Rightarrow$ more support for H_1 over H_0
- Therefore, we want a critical region of the form,

$$\lambda \leqslant k$$

- Choose k to give the desired significance level

Example 1 (likelihood ratio test)

- $X_i \sim \mathrm{N}(\mu, \sigma^2 = 5)$, i.e. σ is known
- $H_0\colon \mu = 162$ versus $H_1\colon \mu \neq 162$
- When H_0 is true, $\mu = 162$ so $L_0 = L(162)$
- When H_1 is true, need to maximise the likelihood, $L_1 = L(\hat{\theta}) = L(\bar{x})$
- The likelihood ratio is,

$$
\begin{aligned}
\lambda = \frac{L_0}{L_1} = \frac{L(162)}{L(\bar{x})} &= \frac{(10\pi)^{-n/2} \exp\left[-\frac{1}{10}\sum_{i=1}^{n}(x_i - 162)^2\right]}{(10\pi)^{-n/2} \exp\left[-\frac{1}{10}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]} \\
&= \exp\left[-\frac{n}{10}(\bar{x} - 162)^2\right]
\end{aligned}
$$

$$\lambda = \exp\left[-\frac{n}{10}(\bar{x} - 162)^2\right]$$

- $\lambda \leqslant k$ same as

$$\frac{|\bar{x} - 162|}{\sigma/\sqrt{n}} \geqslant c$$

- A critical region for a size α test is

$$\frac{|\bar{x} - 162|}{\sigma/\sqrt{n}} \geqslant \Phi^{-1}(1 - \alpha/2)$$

- Note: this required knowledge of the distribution of $\bar{X}$!

Example 2 (likelihood ratio test)

- $X_i \sim \mathrm{N}(\mu, \sigma^2)$, i.e. σ is unknown
- $H_0\colon \mu = \mu_0$ versus $H_1\colon \mu \neq \mu_0$
- Under H_0 we have $\mu = \mu_0$, and under H_1 we need to use its MLE
- Under either hypothesis, σ^2 is unspecified, so in both cases we need its MLE (conditional on the specified value of μ).
- So, under H_0 we use:

$$\hat{\mu} = \mu_0, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2$$

- And under H_1 we use:

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

- Some simplification yields

$$\lambda = \left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \mu_0)^2} \right]^{n/2}$$

- and

$$\sum_{i=1}^{n}(x_i - \mu_0)^2 = \sum_{i=1}^{n}(x_i - \bar{x} + \bar{x} - \mu_0)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2$$

- Substitute and rearrange to get

$$\lambda = \left[\frac{1}{1 + \frac{n(\bar{x}-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}} \right]^{n/2}$$

- Therefore, we have $\lambda \leqslant k$ when,

$$\frac{n(\bar{x}-\mu_0)^2}{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2} \geqslant c$$

- When H_0 is true, $\sqrt{n}(\bar{X}-\mu_0)/\sigma \sim \mathrm{N}(0,1)$ and $\sum_{i=1}^{n}(X_i-\bar{X})^2/\sigma^2 \sim \chi^2_{n-1}$, and is independent of $\bar{X}$.
- Therefore,

$$\begin{aligned} T &= \frac{\sqrt{n}(\bar{X}-\mu_0)/\sigma}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2/\sigma^2}} \\ &= \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2}} = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \sim t_{n-1} \end{aligned}$$

- So we reject H_0 when $|T|$ is too large, with the following critical region for a test with significance level α,

$$|T| \geqslant d, \quad \text{where } d \text{ is the } 1-\frac{\alpha}{2} \text{ quantile of } t_{n-1}$$

Remarks

- Usually easy to find the **form** of the test
- What is harder is to find the corresponding sampling distribution
- Manipulating λ until we have something whose distribution we know can be tricky!
- Many of the standard tests arise from the likelihood ratio

Asymptotic distribution & optimality

- The likelihood ratio itself is a statistic and therefore has a sampling distribution.
- For large sample sizes, this approaches a known distribution
- Also, the LRT gives the optimal test
- We will cover this theory later in the semester