THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS
SEMESTER 1, 2016

MAST30030 APPLIED MATHEMATICAL MODELLING

Exam duration - Three hours

Reading time - 15 minutes

This paper has 8 pages, including this cover sheet.

Instructions to Invigilators:

Initially, students are to receive a 14 page script book.

Authorised Materials:

No materials are authorised.

No calculators or computers are permitted.

No written or printed material may be brought into the examination room.

Instructions to Students:

There are 4 questions on this examination paper.

All questions may be attempted.

All questions are of equal value.

The maximum number of marks available to students in the exam paper is 100.

Formula sheets are attached to this paper on Pages 7 and 8.

This paper must not be removed from the examination room.

Question 1

(a) Give explanations for answers to the following questions:

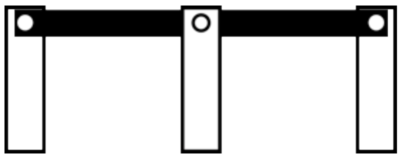
i. What is meant by the term ”dynamical system”?

ii. What is meant by “autonomous” and “non-autonomous” dynamical systems?

iii. Is it possible to achieve periodic motion in one-dimensional deterministic continuous dynamical systems?

iv. How many dimensions are required to achieve chaotic motion in deterministic continuous dynamical systems?

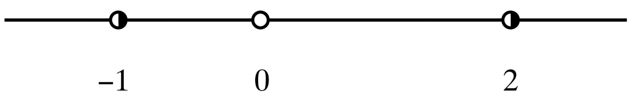
v. Consider the pendulum that consists of (1) a central rigid bar that is hinged at the centre to a rigid wall, and (2) three rigid arms hinged to this central rigid bar (circles indicate hinges):



How many dimensions does its corresponding dynamical system possess?

The pendulum is set in motion by simultaneously moving all the arms and the central rigid bar in a random fashion (as may happen if you were to you accidently bump into the pendulum). What dynamics could you expect from each arm?

vi. The following diagram is given in a textbook as a phase portrait of some unknown dynamical system $\dot{x} = f(x)$:



A student suggests that there is something wrong with this picture. What could the student be referring to?

(b) Consider the following one-dimensional dynamical system

$$\dot{x} = x\left(1 - x^2\right)$$

i. Calculate the potential energy function for this system and plot the result. Identify positions for any local maxima/minima.

ii. Use a linear stability analysis to determine the stability of its fixed points.

iii. How is your result in (ii) related to the potential energy function plotted in (i)?

iv. Plot the vector field for this system and hence independently determine the stability of the fixed points. Explain any difference you may observe in comparison to the result in (ii).

(c) Consider the following two-dimensional dynamical system

$$\frac{dx}{dt} = y(1+y)$$
$$\frac{dy}{dt} = x - e^{-y^2}$$

i. Find the fixed points of this system.

ii. Determine the corresponding eigenvalues and eigenvectors of its Jacobian at each fixed point. Hence draw the local phase portrait of each fixed point. Identify stability of each fixed point?

iii. Calculate the nullclines of the system.

iv. Draw a sketch of its global phase portrait using all the information in (i) – (iii).

Question 2

Traffic along a road obeys the conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

where t and x are scaled time and position, respectively, ρ is the scaled car density distribution (maximum is one), $J = \rho v$ is the scaled flux, and v is the scaled car speed (maximum is one). Measurements show $v = 1 - \rho$. Initially, at time $t = 0$, the car density is

$$\rho(x, 0) = \begin{cases} 0 & : x < 0 \\ 1 & : 0 \le x < 1 \\ 0 & : x \ge 1 \end{cases}$$

(a) How would you expect the car dynamics to evolve in time? Explain your answer with qualitative illustrative sketches of $\rho(x, t)$ vs x at various times.

Note: Do not calculate $\rho(x, t)$ at this stage, but base the sketches on your knowledge of car dynamics (as also explained in lectures).

(b) Convert the above conservation law into two equivalent ordinary differential equations using the method of characteristics. Justify (in words) each step of your derivation.

(c) Identity all the key features of the problem, including any fans or shocks, and sketch the characteristics on a space-time diagram.

i. What is the time, $t = T_{\text{move}}$, for all cars on the road to either be on the verge of moving faster, or actually moving faster than their initial velocities? Explain your answer. Hint: Examine your space-time diagram and identify the time for which the densities along all characteristics are all less than unity.

Important: Do not solve the problem for $t > T_{\text{move}}$.

ii. Accounting for all of these details, give a mathematical expression for the density distribution $\rho(x, t)$ for $t \le T_{\text{move}}$.

iii. Calculate the density distributions for $t = 0$, $T_{\text{move}}/2$, T_{move} and hence plot these distributions.

How do your calculated density distributions compare with (a)? Explain your answer.

Question 3

(a) Define what is meant by a Lagrangian and an Eulerian description of fluid flow.

(b) Explain what the "material derivative" describes and why it is used in studying fluid dynamics.

(c) Give the continuity equation for a general compressible flow. What conservation principle is used to derive it?

(d) What is the primary physical condition for a flow to be considered incompressible? Use this definition to simplify the equation in (c) for this special case.

(e) Are any flows in real life truly incompressible? Explain your answer with two examples.

(f) Explain what is meant by "dynamic similarity" for a steady incompressible flow. In your answer include the definition of the relevant dimensionless parameter.

(g) Consider a flat solid surface that bounds a viscous incompressible flow. Derive an expression for the normal viscous stress at the solid surface. Would you expect the same expression hold for a general compressible viscous flow that is bounded by the same surface? Explain your answer.

(h) Describe in words what the streamlines and particle paths of a flow represent. Explain if they give an Eulerian or Langrangian description of the flow, and

i. Derive the governing equations for the streamlines, $\boldsymbol{r}^{st}(s)$, where s is a parameter.

ii. Derive the governing equation for the particle paths, $\boldsymbol{r}^{p}(t)$, where t is time.

iii. Under what condition are the streamlines and particle paths identical and why?

(i) Using Cartesian tensor methods, prove the following vector identities

i. $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$

ii. $\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$

The following relationship may be useful in proving at least one of these identities

$$\varepsilon_{ijk}\,\varepsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

Question 4

(a) Starting from the Stokes equations (ignoring body forces), show that

$$\nabla^2 p = 0 \quad \text{and} \quad \nabla^2 \boldsymbol{\omega} = \mathbf{0}$$

(b) Express the second equation in terms of the stream function ψ for a plane flow. Derive the corresponding equation for ψ of a two-dimensional irrotational flow.

(c) Consider a solid sphere of radius a immersed in an unbounded fluid of viscosity μ; fluid is stationary far from the sphere. The sphere rotates with constant angular velocity $\Omega\boldsymbol{k}$.

i. Show that the velocity of the fluid at the surface of the sphere is

$$\boldsymbol{u}(a,\theta) = \Omega a \sin\theta\, \hat{\boldsymbol{\phi}}$$

in spherical polar coordinates.

ii. Guided by the form of the boundary conditions, consider a velocity field of the form

$$\boldsymbol{u}(r,\theta) = f(r)\sin\theta\, \hat{\boldsymbol{\phi}}$$

where $f(r)$ is a function of r. Show that such a form satisfies the continuity equation of incompressible flow.

iii. Using the result of (ii), calculate the velocity field in the fluid, under creeping flow conditions. (Hint: Derive the following ODE

$$r^2 f''(r) + 2r f'(r) - 2f(r) = 0$$

and look for solutions of the form r^n, where n is a constant).

iv. The torque (about the origin) acting on a rigid particle is $\boldsymbol{\Lambda} = \boldsymbol{r} \times \boldsymbol{F}$, where $\boldsymbol{r}$ is the position vector and $\boldsymbol{F}$ is the applied force. Show that the torque exerted by the fluid on the rotating sphere (about its centre) is given by the general formula

$$\boldsymbol{\Lambda} = -2\mu a \int_S e_{r\phi} \hat{\boldsymbol{\theta}}\, dS$$

where $e_{r\phi}$ is a rate-of-strain component and the integral is over the sphere's surface.

v. Using the results in (iii) and (iv) calculate the torque exerted by the fluid on the rotating sphere. The following identity may be useful

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\, \hat{\boldsymbol{x}} + \cos\theta\sin\phi\, \hat{\boldsymbol{y}} - \sin\theta\, \hat{\boldsymbol{z}}$$

In spherical polar coordinates, the Stokes equations are:

$$\hat{\boldsymbol{r}}: \quad 0 = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)$$

$$\hat{\boldsymbol{\theta}}: \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right)$$

$$\hat{\boldsymbol{\phi}}: \quad 0 = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \mu\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\right)$$

END OF EXAMINATION

Vector Identities

$\nabla \cdot (\phi\, \mathbf{q}) = (\nabla\phi) \cdot \mathbf{q} + \phi\nabla \cdot \mathbf{q}$ $\qquad$ $\nabla \times (\phi\, \mathbf{q}) = (\nabla\phi) \times \mathbf{q} + \phi\nabla \times \mathbf{q}$

$\nabla \cdot (\nabla \times \mathbf{q}) = 0 \qquad \nabla \times (\nabla\phi) = \mathbf{0}$ $\qquad$ $\nabla \times (\nabla \times \mathbf{q}) = \nabla(\nabla \cdot \mathbf{q}) - \nabla^2\mathbf{q}$

$\nabla \times (\mathbf{p} \times \mathbf{q}) = \mathbf{p}(\nabla \cdot \mathbf{q}) - \mathbf{q}(\nabla \cdot \mathbf{p}) + (\mathbf{q} \cdot \nabla)\mathbf{p} - (\mathbf{p} \cdot \nabla)\mathbf{q}$

$\nabla(\mathbf{p} \cdot \mathbf{q}) = (\mathbf{p} \cdot \nabla)\mathbf{q} + (\mathbf{q} \cdot \nabla)\mathbf{p} + \mathbf{p} \times (\nabla \times \mathbf{q}) + \mathbf{q} \times (\nabla \times \mathbf{p})$

Polar Coordinates

For *cylindrical polar coordinates* σ, φ, z, with z measured along the axis of the cylinder, σ the distance from the axis of the cylinder, and φ the azimuthal angle:

$$\nabla f = \hat{\boldsymbol{\sigma}}\,\frac{\partial f}{\partial \sigma} + \hat{\boldsymbol{\varphi}}\,\frac{1}{\sigma}\frac{\partial f}{\partial \varphi} + \hat{\mathbf{z}}\,\frac{\partial f}{\partial z} \qquad \nabla \cdot (u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}) = \frac{1}{\sigma}\,\frac{\partial}{\partial \sigma}(\sigma u) + \frac{1}{\sigma}\,\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z}$$

$$\nabla \times (u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}) = \left\{\frac{1}{\sigma}\,\frac{\partial w}{\partial \varphi} - \frac{\partial v}{\partial z}\right\}\hat{\boldsymbol{\sigma}} + \left\{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial \sigma}\right\}\hat{\boldsymbol{\varphi}} + \left\{\frac{1}{\sigma}\,\frac{\partial}{\partial \sigma}(\sigma v) - \frac{1}{\sigma}\,\frac{\partial u}{\partial \varphi}\right\}\hat{\mathbf{z}}$$

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{\sigma}\,\frac{\partial}{\partial \sigma}\left(\sigma\frac{\partial f}{\partial \sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

For *spherical polar coordinates* r, θ, φ, with r the distance from the origin, θ the colatitudinal angle and φ the azimuthal angle:

$$\nabla f = \hat{\mathbf{r}}\,\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\varphi}}\,\frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \varphi}$$

$$\nabla \cdot (u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}}) = \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \theta}\,(\sin\theta\, v) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi}$$

$$\nabla\times(u\hat{\mathbf{r}}+v\hat{\boldsymbol{\theta}}+w\hat{\boldsymbol{\varphi}}) = \left\{\frac{\partial}{\partial \theta}(w\sin\theta) - \frac{\partial v}{\partial \varphi}\right\}\frac{\hat{\mathbf{r}}}{r\sin\theta} + \left\{\frac{1}{\sin\theta}\frac{\partial u}{\partial \varphi} - \frac{\partial}{\partial r}(rw)\right\}\frac{\hat{\boldsymbol{\theta}}}{r} + \left\{\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right\}\frac{\hat{\boldsymbol{\varphi}}}{r}$$

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial \theta}\,\left(\sin\theta\,\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

Stream Functions and Potentials

If $\nabla \times \mathbf{q} = \mathbf{0}$ in a simply-connected domain, then $\mathbf{q} = \nabla\phi$.

If $\nabla \cdot \mathbf{q} = 0$, then $\mathbf{q} = \nabla \times \mathbf{A}$.

In two-dimensional flow, if $\nabla \cdot \mathbf{q} = 0$, then $\mathbf{q} = \nabla \times (\psi\,\hat{\mathbf{k}})$, where $\hat{\mathbf{k}}$ is the unit basis vector associated with the z direction, and ψ is independent of z.

In axisymmetric three-dimensional flow, if $\nabla \cdot \mathbf{q} = 0$, then with Λ and χ independent of the azimuthal angle φ,

$$\mathbf{q} = \nabla \times \left(\frac{\Lambda}{\sigma}\,\hat{\boldsymbol{\varphi}}\right) + \chi\,\hat{\boldsymbol{\varphi}} = \left\{\frac{1}{r^2\sin\theta}\,\frac{\partial\Lambda}{\partial\theta}\right\}\hat{\mathbf{r}} - \left\{\frac{1}{r\sin\theta}\,\frac{\partial\Lambda}{\partial r}\right\}\hat{\boldsymbol{\theta}} + \chi\,\hat{\boldsymbol{\varphi}}.$$

The rate-of-strain tensor

The rate-of-strain tensor $\mathbf{e}$ is related to the velocity field $\mathbf{q}$ by the equation $\mathbf{e} = \frac{1}{2}\{\nabla\mathbf{q}+(\nabla\mathbf{q})^{\mathrm{T}}\}$, where $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of the tensor $\mathbf{A}$.

For *cylindrical polar coordinates* σ, φ, z, if $\mathbf{q} = u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}$, the components of $\mathbf{e}$ are

$$\begin{aligned}
e_{\sigma\sigma} &= \frac{\partial u}{\partial \sigma} \\
e_{\varphi\varphi} &= \frac{1}{\sigma}\frac{\partial v}{\partial \varphi} + \frac{u}{\sigma} \\
e_{zz} &= \frac{\partial w}{\partial z} \\
e_{\sigma\varphi} &= e_{\varphi\sigma} = \frac{\sigma}{2}\frac{\partial}{\partial \sigma}\left(\frac{v}{\sigma}\right) + \frac{1}{2\sigma}\frac{\partial u}{\partial \varphi} \\
e_{\sigma z} &= e_{z\sigma} = \frac{1}{2}\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial w}{\partial \sigma} \\
e_{z\varphi} &= e_{\varphi z} = \frac{1}{2\sigma}\frac{\partial w}{\partial \varphi} + \frac{1}{2}\frac{\partial v}{\partial z}
\end{aligned}$$

For *spherical polar coordinates* r, θ, φ, if $\mathbf{q} = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}}$, the components of $\mathbf{e}$ are

$$\begin{aligned}
e_{rr} &= \frac{\partial u}{\partial r} \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \\
e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{u}{r} + \frac{v\cot\theta}{r} \\
e_{r\theta} &= e_{\theta r} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{2r}\frac{\partial u}{\partial \theta} \\
e_{r\varphi} &= e_{\varphi r} = \frac{1}{2r\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{w}{r}\right) \\
e_{\theta\varphi} &= e_{\varphi\theta} = \frac{\sin\theta}{2r}\frac{\partial}{\partial \theta}\left(\frac{w}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial v}{\partial \varphi}
\end{aligned}$$

These formulae may also be used to deduce the polar coordinate representations of the linear strain tensor in the linear theory of elasticity.