Week 3 Lab (~~Week~~ Tutorial + Lab #2) MAST30025

Q1) let X be a 10×5 matrix of full rank and let
$H = X(X^TX)^{-1}X^T$.

find $tr(H)$ & $r(H)$ ※Recall the exercise in Q6 Tutorial 1 (week 2)

$tr(H) = tr(\underset{10\times5}{X}(\underset{5\times10}{X^T}\underset{10\times5}{X})^{-1}\underset{5\times10}{X^T}) = tr(X^TX(X^TX)^{-1})$
$= tr(I_5) = \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$

$tr(H) = r(H) = 5$!!

H is idempotent and symmetric!!
$H = H^2$ $H = H^T$

can easily prove through definitions

※ H: hat matrix will appear lots of times in the future.

Q2) let,

$A = \begin{bmatrix} 3 & 1 & 8\\ 1 & 0 & 1\\ 2 & 1 & -4\end{bmatrix}, y = \begin{bmatrix} y_1\\y_2\\y_3\end{bmatrix}$.

let $z = y^TAy$. Write out z in full, then find $\frac{\partial z}{\partial y}$ directly & using the matrix formula.

$z = \underset{1\times3}{[y_1\ y_2\ y_3]}\underset{3\times3}{\begin{bmatrix} 3 & 1 & 8\\ 1 & 0 & 1\\ 2 & 1 & -4\end{bmatrix}}\underset{3\times1}{\begin{bmatrix} y_1\\y_2\\y_3\end{bmatrix}}$

$= [3y_1 + y_2 + 2y_3 \quad y_1 + y_3 \quad 8y_1 + y_2 - 4y_3]\begin{bmatrix} y_1\\y_2\\y_3\end{bmatrix}$

$= 3y_1^2 + y_1y_2 + 2y_1y_3 + y_1y_2 + y_2y_3 + 8y_1y_3 + y_2y_3 - 4y_3^2$

$z = 3y_1^2 + 2y_1y_2 + 10y_1y_3 + 2y_2y_3 - 4y_3^2$

$\frac{\partial z}{\partial y} = Ay + A^Ty = \underset{3\times3}{\begin{bmatrix} 3 & 1 & 8\\ 1 & 0 & 1\\ 2 & 1 & -4\end{bmatrix}}\underset{3\times1}{\begin{bmatrix} y_1\\y_2\\y_3\end{bmatrix}} + \begin{bmatrix} 3 & 1 & 2\\ 1 & 0 & 1\\ 8 & 1 & -4\end{bmatrix}\begin{bmatrix} y_1\\y_2\\y_3\end{bmatrix}$

※ A is not symmetric.

$= \begin{bmatrix} 3y_1 + y_2 + 8y_3\\ y_1 + y_3\\ 2y_1 + y_2 - 4y_3\end{bmatrix} + \begin{bmatrix} 3y_1 + y_2 + 2y_3\\ y_1 + y_3\\ 8y_1 + y_2 - 4y_3\end{bmatrix} = \begin{bmatrix} 6y_1 + 2y_2 + 10y_3\\ 2y_1 + 2y_3\\ 10y_1 + 2y_2 - 8y_3\end{bmatrix}$

slide C1 /26

indep $\Rightarrow$ cov

indep $\Leftrightarrow$ cov if Normal

Q3) Let $y = [y_1\ y_2\ y_3]^T$ be a random vector with mean $\mu = [1\ 3\ 2]^T$, and assume that $Var\, y_i = 4$ and $Cov(y_i, y_j) = 0$.

$Cov(y_1, y_3)$

(a) Write down $Var\, y$.

$$V = \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$V = Var\, y = Var[y_1\ y_2\ y_3]^T$

$= cov(y_i, y_j) = 0 = \mathbb{E}[(y_i - \mu_i)(y_j - \mu_j)]$

$= \mathbb{E}[(y-\mu)(y-\mu)^T]$ $\quad \mu = E[y] = [1\ 3\ 2]^T$

$$= \mathbb{E}\left[\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right)\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right)^T\right]$$

$3\times1 \quad 1\times3$

$y_1 - 1$
$y_2 - 3$
$y_3 - 2$

$$= \mathbb{E}\left[\underset{3\times1}{\begin{bmatrix} y_1 - 1 \\ y_2 - 3 \\ y_3 - 2 \end{bmatrix}} \underset{1\times3}{[y_1 - 1\quad y_2 - 3\quad y_3 - 2]}\right]$$

$$= \mathbb{E}\begin{bmatrix} (y_1-1)^2 & (y_1-1)(y_2-3) & (y_1-1)(y_3-2) \\ (y_2-3)(y_1-1) & (y_2-3)^2 & (y_2-3)(y_3-2) \\ (y_3-2)(y_1-1) & (y_3-2)(y_2-3) & (y_3-2)^2 \end{bmatrix}$$

$\mathrm{Var}\ y_i = 4$

$\mathrm{Cov}(y_1, y_2) = 2$

$\mathrm{Cov}(y_3, y_2) = 3$ and the rest of Cov = 0.

$$V = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline y_1 & 4 & 2 & 0 \\ y_2 & 2 & 4 & 3 \\ y_3 & 0 & 3 & 4 \end{array}$$

symmetric

\# $\mathrm{Cov}(y_1, y_2) = \mathrm{Cov}(y_2, y_1)$

off-diagonals : Cov

diagonals : Var

$$\text{Var}\,y = \text{Var}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \text{Var}[y_1] \\ \text{Var}[y_2] \\ \text{Var}[y_3] \end{bmatrix}$$

given $\text{Var}(y_i) = 4$
$\text{Cov}(y_i, y_j) = 0$

$\Rightarrow$, $\text{var}\,y_i = 4$, assume it is independent
given, $\text{Cov}(y_i, y_j) = 0$.

$$V = \text{var}\,y = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3 \qquad \boxed{\sigma^2 = 4}$$

Answer? ✓ correct. Se. 10 linear statistical models

Solution

Q3b). Let, $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$,

and find $\text{Var}\,Ay$ and $E[y^T Ay]$.

Attempt 1

$\text{Var}\,Ay = AVA^T$ ✓ Variance properties.

$$= \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ -3 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -12 & 4 \\ 4 & 8 & 0 \\ -4 & 24 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ -3 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16+36+4=56 & 8-24+0=-16 & -8-72+4=-76 \\ 8-24+0=-16 & 4+16+0=20 & -4+48+0=44 \\ -8-72+4=-76 & -4+48+0=44 & 4+144+4=152 \end{bmatrix}$$

$$= \begin{bmatrix} 56 & -16 & -76 \\ -16 & 20 & 44 \\ -76 & 44 & 152 \end{bmatrix}$$ ✓

$$E[y^TAy] = \underbrace{tr(AV)}_{(1)} + \underbrace{\mu^T A \mu}_{(2)}$$

$$AV = \begin{bmatrix} 8 & -12 & 4 \\ 4 & 8 & 0 \\ -4 & 24 & 4 \end{bmatrix}, \quad tr(AV) = 8+8+4 = 20 \quad —(1)$$

$$\mu^T = ([1\ 3\ 2]^T)^T = [1\ 3\ 2]$$

$$\mu^T A \mu = \underset{1\times3}{[1\ 3\ 2]} \underset{3\times3}{\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}$$

$$= [2+3-2 \quad -3+6+12 \quad 1+0+2] \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \underset{1\times3}{[3 \quad 15 \quad 3]} \underset{3\times1}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} = 3+45+6 = \underbrace{48+6 = 54}_{(2)}$$

$$\Rightarrow (1)+(2) = 20+54 = 74.$$

Q4) Prove corollaries 3.6 and 3.7 from the lectures.

See next page.

Use Theorem 3.5. : $y \sim MVN(\mu, I)$, A is symmetric $\Rightarrow y^T A y \sim \chi^2_{k,\lambda}$, $\lambda = \frac{1}{2}\mu^T A \mu$

Corollary 3.6

$y \sim MVN(0, I_n)$
→ $n \times 1$ random vector, let A be a $n \times n$ symmetric matrix.

$y^T A y$ has a (ordinary) χ^2

Outline

Let $y \sim MVN(0, I_n)$ be a $n \times 1$ random vector and let A be a $n \times n$ symmetric matrix. Then $y^T A y$ has a (ordinary) χ^2 distribution with k degrees of freedom if and only if A is idempotent and has a rank k.

$\mu = 0$

$\lambda = \frac{1}{2}\mu^T A \mu$

$\lambda = 0$

$\chi^2_{k,0} = \chi^2_k$

$y \sim MVN(0, I_n)$

with $\mu = 0$, then $\lambda = \frac{1}{2}\mu^T A \mu = 0$. ✓

from definition of distributions.

Q46) ~~Colla:~~ Corollary 3.7 '11

From the distribution of y, we know that $\frac{1}{\sigma} y \sim MVN(\frac{1}{\sigma}\mu, I)$.

$\therefore \left(\frac{1}{\sigma}y^T\right) A \left(\frac{1}{\sigma}y\right) = \frac{1}{\sigma^2} y^T A y$ has a noncentral χ^2 distribution with k degrees of freedom and non-centrality parameter,

$\lambda = \frac{1}{2}\mu^T A \mu$

$$* \lambda = \frac{1}{2}\left(\frac{1}{\sigma}\mu\right)^T A \left(\frac{1}{\sigma}\mu\right) = \frac{1}{2\sigma^2}\mu^T A \mu$$

if and only if A is idempotent and has rank k.

$\chi^2_{k,\lambda}$

Note that $y \sim MVN(\mu, V) \Rightarrow y^TAy, y^TBy$ are χ^2

$\rightarrow$ df depends on rank of A, B.

$\rightarrow \lambda = \frac{1}{2}\mu^T A \mu$

Q5) let $y = [y_1 \ y_2]^T$ be a normal random vector with mean & variance.

$$\mu = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \text{ and } V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

idempotent

let $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

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(a) Find the distributions of y^TAy and y^TBy.

Attempt 1

$$y^TAy = \frac{1}{2}\underset{1\times 2}{\begin{bmatrix} y_1 & y_2 \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2}\underset{1\times 2}{\begin{bmatrix} y_1+y_2 & y_1+y_2 \end{bmatrix}}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2}\left[y_1^2 + y_1y_2 + y_2y_1 + y_2^2\right]$$

$$= \frac{1}{2}(y_1^2 + 2y_1y_2 + y_2^2) = \frac{1}{2}(y_1+y_2)^2$$

$$y^TBy = \frac{1}{2}\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} y_1-y_2 & -y_1+y_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2}(y_1^2 - y_1y_2 - y_1y_2 + y_2^2)$$

$$= \frac{1}{2}(y_1^2 - 2y_1y_2 + y_2^2) = \frac{1}{2}(y_1-y_2)^2$$

MISSING,

Both have non central χ^2 distribution w/ 1 degree of freedom & noncentrality parameter,

where A and B has rank 1 (Both symmetric & idempotent) eigenvalue

$\lambda = \frac{1}{2}\mu^T A \mu$ & $\lambda = \frac{1}{2}\mu^T B \mu$

(1) (2).

$\lambda = \frac{1}{2}[2 \quad 4]\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}$

$= \frac{1}{2}\times\frac{1}{2}[6 \quad 6]\begin{bmatrix}2\\4\end{bmatrix} = \frac{1}{4}(12+24) = 9$

(1)

$\lambda = \frac{1}{2}[2 \; 4]\frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}$

$y^TAy \sim \chi^2_{k,\lambda}$ where $k=1$, $\lambda = 9$

$= \frac{1}{4}\begin{bmatrix}-2 & 2\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = \frac{1}{4}[-4+8] = \frac{4}{4} = 1$

(2).

Q5b) Are y^TAy and y^TBy independent?

Thm 5.11

$AVB = \frac{1}{4}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\overset{I}{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

Show that C, D is not indep.

$CVD \neq 0$

$\therefore$ not indep.

$= \frac{1}{4}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

$AIB = AB$

$= \frac{1}{4}\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0$

$\therefore y^TAy$ & y^TBy are independent

QED

in answer you can go straight to AB where, $AB = 0$

Q5c) What is the distribution of $y^TAy + y^TBy$?

1 degree + 1 degree = 2 degrees of freedom?

Yeap!

$$y^TAy + y^TBy = \frac{1}{2}(y_1+y_2)^2 + \frac{1}{2}(y_1-y_2)^2$$

$$= \frac{1}{2}(y_1^2 + 2y_1y_2 + y_2^2) + \frac{1}{2}(y_1^2 - 2y_1y_2 + y_2^2)$$

$$= \frac{1}{2}(2y_1^2 + 2y_2^2) = y_1^2 + y_2^2$$

distribution.

has a non central χ^2 distribution with 2 degree of free dom and non centrality parameter w/ rank 1, A, B are idempotents both have rank 1!!

that was wrong?

$$\lambda = 9 + 1 = 10$$ ✓

Prove that: $X_1 \sim \chi^2_{k_1, \lambda_1}$; $X_2 \sim \chi^2_{k_2, \lambda_2}$ leads to $X_1 + X_2 \sim \chi^2_{k_1+k_2, \lambda_1+\lambda_2}$

Q6) let $y_1, \ldots, y_n$ be an i.i.d. normal sample. Show that

$$\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i \text{ and, } s^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2$$

are independent. (HINT: Express them as a random "vector" and quadratic form respectively.)

Attempt 1

$$\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i = \frac{1}{n}\mathbb{1}^T y$$

* get comfortable with $\mathbb{1}$

$$\frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \frac{1}{n}(y_1 + \cdots y_n)$$

$$y - \bar{y}\mathbb{1} = (I - \frac{1}{n}\mathbb{1}\mathbb{1}^T)y$$

$$s^2 = \frac{1}{n-1}[(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T)y]^T(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T)y$$

$$= \frac{1}{n-1}y^T(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T)^T(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T)y$$

$$= \frac{1}{n-1}y^T(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T)y.$$

$I - \frac{1}{n}\mathbb{1}\mathbb{1}^T$ is symmetric & idempotent.

$$y - \bar{y}\mathbb{1} = \left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)y = A$$

$$A^2 = A^T A$$

$$\frac{1}{n}\mathbb{1}\mathbb{1}^T = \frac{1}{n}\underset{n \times 1}{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}} \underset{1 \times n}{\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}}$$

$$= \frac{1}{n}\underset{n \times n}{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & 1 \\ 1 & \cdots & & 1 \end{bmatrix}}$$

$$I - \frac{1}{n}\mathbb{1}\mathbb{1}^T = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ & 1-\frac{1}{n} & & \\ & & & \\ & & & \end{bmatrix}$$

$$1 - \frac{1}{n} = \frac{n-1}{n}$$

$$I - \frac{1}{n}\mathbb{1}\mathbb{1}^T = \frac{1}{n}\begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ & n-1 & & & \\ & & n-1 & & \\ & & & n-1 & \ddots \end{bmatrix}$$

$$s^2 = \frac{1}{n-1}\left[\left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)y\right]^T\left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)y$$

$$s^2 = \frac{1}{n-1}\sum(y - \bar{y})^2$$

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Prove that: $X_1 \sim \chi^2_{k_1, \lambda_1}$; $X_2 \sim \chi^2_{k_2, \lambda_2}$ leads to $X_1 + X_2 \sim \chi^2_{k_1+k_2, \lambda_1+\lambda_2}$

$X_1 = y_1^T y_1$ where $y_1 \sim MVN(\mu_1, I)$ $\Rightarrow \lambda_1 = \frac{1}{2}\mu_1^T \mu_1$

$X_2 = y_2^T y_2$ where $y_2 \sim MVN(\mu_2, I)$ $\Rightarrow \lambda_2 = \frac{1}{2}\mu_2^T \mu_2$

$$X_1 + X_2 = y_1^T y_1 + y_2^T y_2 = \sum_{i=1}^{k_1} y_i^2 + \sum_{j=1}^{k_2} y_j^2 = \sum_{i=1}^{k_1+k_2} y_i^2$$

$\therefore$ since $X_1 + X_2$ is the sum of squares of $k_1 + k_2$ indep. normals,

then d.f of $X_1 + X_2$ is $k = k_1 + k_2$

$$X_1 + X_2 = \sum_{i=1}^{k_1+k_2} y_i^2 = (y_1+y_2)^T (y_1+y_2)$$

$(y_1 + y_2) \sim MVN(\mu_1 + \mu_2, I)$

$$\lambda = \frac{1}{2}\mu^T \mu$$

$$= \frac{1}{2}(\mu_1+\mu_2)^T(\mu_1+\mu_2)$$

$4y_1 + 3y_2 \sim MVN(4\mu_1 + 3\mu_2,$

$$= \frac{1}{2}\left(\mu_1^T\mu_1 + \underbrace{\mu_2^T\mu_1 + \mu_1^T\mu_2}_{=0} + \mu_2^T\mu_2\right)$$

$\vdots$

$$= \frac{1}{2}\mu_1^T\mu_1 + \frac{1}{2}\mu_2^T\mu_2$$

$$\lambda = \lambda_1 + \lambda_2$$

$$\mu_1 = \begin{bmatrix} \mu_{11} \\ \vdots \\ \mu_{1,k_1} \end{bmatrix} \quad \mu_2 = \begin{bmatrix} \mu_{21} \\ \vdots \\ \mu_{2,k_2} \end{bmatrix}$$

$$\mu_2^T \mu_1 = \begin{bmatrix} \mu_{21} \cdots \mu_{2k_2} \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \vdots \\ \mu_{1k_1} \end{bmatrix}$$

$$= \mu_{21}\mu_{11} + \cdots \mu_{2k_2}\mu_{1k_1}$$

$$\mu_1^T \mu_2 = \begin{bmatrix} \mu_{11} \cdots \mu_{1k_1} \end{bmatrix} \begin{bmatrix} \mu_{21} \\ \vdots \\ \mu_{2k_2} \end{bmatrix}$$

$$= \mu_{11}\mu_{21} + \cdots - \mu_{2k_2}\mu_{1k_1}$$

$$\mu_1 = \begin{bmatrix} \mu_{11} \\ \vdots \\ \mu_{1k_1} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mu_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline \mu_{21} \\ \vdots \\ \mu_{2k} \end{bmatrix}$$

$$\mu_1 + \mu_2 = \begin{bmatrix} \mu \\ \vdots \\ \mu_{k_1+k_2} \end{bmatrix}$$

$$\mu_2^T \mu_1 = 0$$
$$\mu_1^T \mu_2 = 0$$