

MAST30001 Stochastic Modelling

Tutorial Sheet 2

1. Show that the Markov property does not in general imply that for any events A, B and C ,

$$P(X_{n+1} \in C | X_n \in B, X_{n-1} \in A) = P(X_{n+1} \in C | X_n \in B).$$

(That is, define a Markov chain and events A, B, C where the equality doesn't hold.)

2. Let $(Y_n)_{n \geq 1}$ be i.i.d. random variables with $P(Y_i = 1) = P(Y_i = -1) = 1/2$ and let $X_n = (Y_{n+1} + Y_n)/2$.

- (a) Find the transition probabilities $P(X_{n+m} = k | X_n = j)$ for $m = 1, 2, \dots$ and $j, k = 0, \pm 1$.
(b) Show that $(X_n)_{n \geq 1}$ is *not* a Markov chain.

3. Let (X_n) be a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/4 & 3/4 & 0 \\ 2/5 & 0 & 3/5 \end{pmatrix}$$

- (a) Compute $P(X_3 = 1, X_2 = 2, X_1 = 2 | X_0 = 1)$.
(b) If X_0 is uniformly distributed on $\{1, 2, 3\}$, compute $P(X_3 = 1, X_2 = 2, X_1 = 2)$.
(c) Now assuming $P(X_0 = 1) = P(X_0 = 3) = 2/5$, compute $P(X_1 = 2, X_4 = 2, X_6 = 2)$.
4. A simplified model for the spread of a contagion in a small population of size 4 is as follows. At each discrete time unit, two individuals in the population are chosen uniformly at random to meet. If one of these persons is healthy and the other has the contagion, then with probability $1/4$ the healthy person becomes sick. Otherwise the system stays the same.
- (a) If X_n is the number of healthy people at step n , then explain why X_0, X_1, \dots is a Markov chain.
(b) Specify the transition probabilities of X_n .
(c) If initially the chance that a given person in the population has the disease equals $1/2$, determined independently, then what is the chance everyone has the disease after two steps in the process?
(d) Assume now that the process begins with exactly one person infected. Given that not everyone is infected after three steps of the process, what is the chance exactly one person is infected?

5. A Markov chain on $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.3 & 0.2 \end{pmatrix}$$

starts with initial distribution uniform on the states $1, 2, 3, 4$. For each $i = 1, 2, 3, 4$, and $n \geq 0$, compute the chance the chain is in state i at step n .