THE UNIVERSITY OF MELBOURNE

Semester 1 Assessment, 2015

School of Mathematics and Statistics

MAST30013 Techniques in Operations Research

Writing time: 2 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages of exam questions (including this page), and
6 pages of formulae and algorithms.

Authorised materials:

- Non-programmable calculators are authorised.

Instructions to Students

- You may remove this question paper at the conclusion of the examination
- You should attempt all questions. Marks for individual questions are shown.
- Show all necessary working.
- **Number the questions and question parts clearly, and start each question on a new page.**
- The total number of marks available is 50.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination
- A non-programmable calculator may be brought into the examination.

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Question 1 (8 marks)

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = -x^3 + 6x^2 + 10.$$

(a) Show that the function is unimodal on the interval $[-1, 1]$ and that the derivative of f is increasing on the interval $[-1, 1]$.

(b) If the minimum of f on the interval [-1, 1] is to be found within a tolerance of 0.05, using the Fibonnacci search, how many f-calculations are required ?

(c) Is it possible to perform one complete iteration of the method of false position in finding the minimum of f on the interval $[-1, 1]$, starting with $a = -1$, $b = 1$? Why ? If it is possible, perform the iteration.

(d) Is it possible to perform one complete iteration of the method of Newton in finding the minimum of f on the interval $[-1, 3]$, starting with $x_0 = 2$? Why ? If it is possible, perform the iteration.

Question 2 (8 marks)

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) \;=\; x_1^3 + 6x_1^2 + 2x_2^3 - 6x_2^2 + 30.$$

(a) Using the first-order necessary condition, find all stationary points of f.

(b) Using the second-order sufficiency condition, classify each stationary point as either minima, maxima, saddle points.

(c) Write down the second order Taylor approximation around a saddle point found above (if more than one saddle point was found, just consider one of them).

(d) Using the approximation found in (c), find one direction that increases the value of f, and one direction that decreases the value of f.

Question 3 (8 marks)

Consider the unconstrained nonlinear program

$$\min_{x \in \mathbb{R}^3} \quad f(x_1, x_2, x_3) \;=\; 2x_1^2 + 2x_2^2 + 2x_1x_2 - 6x_1 + 2x_3^2.$$

(a) Starting at point $x_0 = (1, 1, 0)^T$, perform one full iteration of the steepest descent algorithm to find x_1. (In order to compute the step length solve a single-variable minimisation problem).

(b) What is the angle between $d_0 = -\nabla f(x_0)$ and $d_1 = -\nabla f(x_1)$?

(c) Find the Newton direction for f at the point $x_0 = (1, 1, 0)^T$.

(d) Find the BFGS direction for f at the point $x_0 = (1, 1, 0)^T$ for cases:

i) $H_0 = 3 \times I_3$, where I_3 is the 3×3 identity matrix.

ii) $H_0 = 3 \times (\nabla^2 f(1, 1, 0))^{-1}$.

Question 4 (8 marks)

Consider the constrained nonlinear program:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x_1, x_2) \;=\; -x_1 - x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 4 \\ & x_2 \leq 1. \end{aligned}$$

(a) Write the Lagrangian for the nonlinear program.

(b) Find the KKT point(s), (x^*, λ^*).

(c) Prove that one constraint qualification condition holds at the KKT point(s).

(d) Find the critical cone and check the second-order sufficiency condition at the KKT point(s).

Question 5 (8 marks)
Consider the constrained nonlinear program:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x_1, x_2) \;=\; x_1^2 - x_2 \\ \text{subject to} \quad & x_1 \geq 1 \\ & x_2 \leq 3. \end{aligned}$$

(a) Write down the l_2 penalty function $P_\alpha(x)$ for this program with penalty parameter α.

(b) Write down conditions that the minimum of the function $P_\alpha(x)$ above must satisfy, based on the gradient of the function.

(c) Using the conditions in (b), find expressions for x_1 and x_2 in terms of α.

(d) Find x^* and Lagrange multipliers λ^*.

Question 6 (10 marks)
Consider the constrained nonlinear program:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x_1, x_2) \;=\; x_1^2 + x_2^2 + 2x_2 - 3 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 1 \\ & x_1 + x_2 = 1. \end{aligned}$$

(a) Show that the above problem is convex.

(b) Write down the Lagrangian for the nonlinear program.

(c) Verify that $(x^*, \lambda^*, \eta^*) = ((1, 0), 0, -2)$ is a KKT point using the Lagrangian saddle-point inequalities.

(d) Explain why $(x^*, \lambda^*, \eta^*) = ((1, 0), 0, -2)$ is the only KKT point.

(e) Write down the Wolfe dual of this program.

End of Exam—Total Available Marks = 50.