MAST30001 Stochastic Modelling

Tutorial Sheet 7

1. A two state continuous time Markov chain $(X_t)_{t\geq 0}$ has the following generator with transition rates $\lambda, \mu > 0$:
$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$

   (a) Find the time t transition matrix $P^{(t)}$ with $(P^{(t)})_{i,j} = \mathbb{P}(X_t = j | X_0 = i)$.

   (b) Using your answer to part (a) with $\lambda = \mu$, find a simple expression (i.e., not an infinite sum) for the chance that a random variable having the Poisson distribution with mean λ is an even number.

2. If $(X_t^{(1)})_{t\geq 0}, \ldots, (X_t^{(k)})_{t\geq 0}$ are i.i.d. continuous time Markov chains on $\{0, 1\}$ each having generator
$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$
then what is the generator for the chain determined by $Y_t = \sum_{i=1}^k X_t^{(i)}$?

3. A workshop has two machines and one repairperson. Each machine is either functional or broken. If the the ith machine ($i = 1, 2$) is functional, then it fails after an exponential rate λ_i time. If the ith machine is broken, it takes the repairperson an exponential rate μ_i amount of time to fix it and once it is fixed, it's good as new. Assume the repairperson begins work the instant a machine breaks down, that only one machine can be repaired at a time, and all lifetime and repair times are independent.

   (a) Construct an appropriate continuous time Markov chain to describe the system and find the generator.

   (b) If $\lambda_i = \mu_i = i$ for $i = 1, 2$, find the stationary distribution of the process.

4. (CTMCs as limits of DTMCs) Let P be a one step transition matrix for a discrete time Markov chain on $0, 1 \ldots$ such that $p_{ii} = 0$ for all i. Also let $0 < \lambda_0, \lambda_1, \ldots$ be such that $\max_{i\geq 0} \lambda_i < N$, with N an integer. Define the discrete time Markov chain $Y_0, Y_1, \ldots$ by
$$\mathbb{P}(Y_{n+1}^{(N)} = i | Y_n^{(N)} = i) = \left(1 - \frac{\lambda_i}{N}\right),$$
and for $i \neq j$
$$\mathbb{P}(Y_{n+1}^{(N)} = j | Y_n^{(N)} = i) = \frac{\lambda_i}{N} p_{ij}.$$
We can think of the discrete jumps of $Y^{(N)}$ occurring at times on the lattice $\{0, 1/N, 2/N \ldots\}$ and make a continuous time process by defining
$$X_t^{(N)} = Y_{\lfloor Nt \rfloor}^{(N)},$$
where $\lfloor a \rfloor$ is the greatest integer not bigger than a.

(a) What does a typical trajectory of $X^{(N)}$ look like? Does it have jumps? At what times? How do jumps correspond to $Y^{(N)}$?

(b) Given $X_0^{(N)} = i$, what is the distribution of the random time

$$T^{(N)}(i) = \min\{t \geq 0 : X_t^{(N)} \neq i\}$$

(c) As $N \to \infty$, to what distribution does that of $(T^{(N)}(i) | X_0^{(N)} = i)$ converge?

(d) Based on the previous two items and comparing to the previous problem, do you think that $X^{(N)}$ converges as $N \to \infty$ to a continuous time Markov chain (not worrying about what exactly convergence means)? What is its generator?