MAST30001 Stochastic Modelling

Tutorial Sheet 8

1. Customers enter a bank according to a Poisson process $(N_t)_{t\geq 0}$ with rate $\lambda = 10$ per hour and each customer makes a deposit or withdrawal. If X_j is the amount brought in by the jth customer, assume that the X_j are i.i.d. and independent of the arrivals of customers with distribution uniform on $\{-4, -3, \ldots, 4, 5\}$ (negative amounts correspond to withdrawals). Then the balance of the bank over t hours is given by a compound Poisson process

$$Y_t = \sum_{j=1}^{N_t} X_j.$$

   (a) Draw a typical trajectory of the process Y_t.

   (b) Calculate the mean and variance of the money brought into the bank over an eight hour business day.

   (c) Use the central limit theorem to approximate the probability that the bank has a total balance greater than \$4500 over 100 business days.

   Ans.

   (a) The process is piecewise constant with jumps of size distributed according to X_j at times of the jumps of a Poisson process.

   (b) If $E[X_j] = \mu$ and $Var(X_j) = \sigma^2$, then formulas from lecture imply

$$E[Y_t] = \lambda t \mu, \quad Var(Y_t) = \lambda t(\sigma^2 + \mu^2).$$

   Simple calculations show $\mu = 0.5$ and $\sigma^2 + \mu^2 = 8.5$. Thus setting $t = 8$ and $\lambda = 10$ in the formulas above, we have

$$E[Y_8] = 40, \quad Var(Y_8) = 680.$$

   (c) If W is the balance over 100 business days, then we can represent

$$W = \sum_{j=1}^{100} Y_8^{(j)},$$

   where the $Y_8^{(j)}$ are i.i.d. having distribution Y_8. Thus the CLT says that W is approximately normal with mean and variance given in (b) and so if Z is standard normal, then

$$P(W > 4500) = P\left(\frac{W - 4000}{\sqrt{68000}} > \frac{500}{\sqrt{68000}}\right) \approx P(Z > 1.917) \approx 0.035.$$

2. For $r > 0$ and $0 < p < 1$, let N_t be a Poisson process with rate $\lambda = r\log(1/p)$ and $X_1, X_2, \ldots$ be i.i.d. with distribution

$$P(X_1 = k) = \frac{(1-p)^k}{k\log(1/p)}, \quad k = 1, 2, \ldots$$

Use moment generating functions to show that the compound Poisson variable

$$Y_t = \sum_{j=1}^{N_t} X_j$$

has the negative binomial distribution (started from zero) with parameters rt and p; that is, that

$$P(Y_t = k) = \binom{k + rt - 1}{k} (1-p)^k p^{rt}, \quad k = 0, 1, 2, \dots$$

Ans. By conditioning on the X_i and taking expectations, a computation shows that if ϕ_X is the moment generating function of X_1, then the moment generating function φ_t of Y_t is

$$\varphi_t(\theta) = \exp\{\lambda t(\varphi_X(\theta) - 1)\}.$$

We compute

$$\varphi_X(\theta) = \sum_{k\geq 1} \frac{e^{\theta k}(1-p)^k}{k\log(1/p)} = \frac{1}{\log(1/p)} \sum_{k\geq 1} \frac{(e^\theta(1-p))^k}{k} = \frac{-\log(1-e^\theta(1-p))}{\log(1/p)};$$

the last equality is by Taylor series or integrating the geometric series and φ_X is defined for $|e^\theta(1-p)| < 1$. Thus we find that over the same range of θ,

$$\varphi_t(\theta) = \exp\left\{ rt\log(1/p) \left(\frac{-\log(1-e^\theta(1-p))}{\log(1/p)} - 1 \right) \right\} = \left(\frac{p}{1-e^\theta(1-p)} \right)^{rt}.$$

On the other this is the same as the moment generating function of the negative binomial distribution in the problem, shown by computing Taylor series.

3. A two state continuous time Markov chain $(X_t)_{t\geq 0}$ has the following generator with transition rates $\lambda, \mu > 0$:

$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$

   (a) Find the time t transition matrix $P^{(t)}$ with $(P^{(t)})_{i,j} = \mathbb{P}(X_t = j | X_0 = i)$.

   (b) Using your answer to part (a) with $\lambda = \mu$, find a simple expression (i.e., not an infinite sum) for the chance that a random variable having the Poisson distribution with mean λ is an even number.

Ans.

(a) We solve the equation

$$\frac{d}{dt} P^{(t)} = P^{(t)} A,$$

where A is the generator in the problem. Thus

$$p_{11}'(t) = -\lambda p_{11}(t) + \mu p_{12}(t)$$

and also use $p_{11}(t) = 1 - p_{12}(t)$. We find

$$p_{11}(t) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} \quad \text{and} \quad p_{12}(t) = \frac{\lambda}{\lambda+\mu} \left(1 - e^{-(\lambda+\mu)t}\right)$$

and by symmetry

$$p_{22}(t) = \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} \quad \text{and} \quad p_{12}(t) = \frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right).$$

(b) If $\lambda = \mu$, then the exponential waiting time description of the chain with generator in the problem implies that that number of times the chain switches states in $(0,t)$ has a Poisson distribution with mean λt. If the chain starts at 0 then the chain is at 0 at time t is the same as there being an even number of jumps. So $p_{00}(1)$ is the chance a Poisson variable with mean λ is even and according to (a) this is

$$(1/2)(1 + e^{-2\lambda}).$$

4. (CTMCs as limits of DTMCs) Let P be a one step transition matrix for a discrete time Markov chain on $0, 1\ldots$ such that $p_{ii} = 0$ for all i. Also let $0 < \lambda_0, \lambda_1, \ldots$ be such that $\max_{i\geq 0} \lambda_i < N$, with N an integer. Define the discrete time Markov chain $Y_0, Y_1, \ldots$ by

$$\mathbb{P}(Y_{n+1}^{(N)} = i | Y_n^{(N)} = i) = \left(1 - \frac{\lambda_i}{N}\right),$$

and for $i \neq j$

$$\mathbb{P}(Y_{n+1}^{(N)} = j | Y_n^{(N)} = i) = \frac{\lambda_i}{N} p_{ij}.$$

We can think of the discrete jumps of $Y^{(N)}$ occurring at times on the lattice $\{0, 1/N, 2/N \ldots\}$ and make a continuous time process by defining

$$X_t^{(N)} = Y_{\lfloor Nt \rfloor}^{(N)},$$

where $\lfloor a \rfloor$ is the greatest integer not bigger than a.

(a) What does a typical trajectory of $X^{(N)}$ look like? Does it have jumps? At what times? How do jumps correspond to $Y^{(N)}$?

(b) Given $X_0^{(N)} = i$, what is the distribution of the random time

$$T^{(N)}(i) = \min\{t \geq 0 : X_t^{(N)} \neq i\}$$

(c) As $N \to \infty$, to what distribution does that of the previous item converge?

(d) Based on the previous two items and comparing to the previous problem, do you think that $X^{(N)}$ converges as $N \to \infty$ to a continuous time Markov chain (not worrying about what exactly convergence means)? What is its generator?

Ans.

(a) The chain only has jumps at times k/N for k an integer. Given the chain is in state i, it stays there for a geometric λ_i/N (> 0) number of $1/N$ time units and then jumps according to the one step transition matrix P. The number of these time units are the number of integer time units between jumps in the $Y^{(N)}$ chain.

(b) As mentioned in the previous problem, the $Y^{(N)}$ chain stays at state i for a geometric (λ_i/N) number of time units before jumping. Then considering the time change to get from $Y^{(N)}$ to $X^{(N)}$, the variable $NT^{(N)}(i)$ is geometric λ_i/N (> 0); that is for $k = 1, 2, \ldots$

$$\mathbb{P}(T^{(N)}(i) = k/N) = \frac{\lambda_i}{N}\left(1 - \frac{\lambda_i}{N}\right)^{k-1}.$$

(c) A standard calculation (do it!) shows that if Z_p is geometric p, then pZ_p converges in distribution to an exponential distribution with mean 1 as $p \to 0$. Since $NT^{(N)}(i)$ is geometric λ_i/N, $T^{(N)}(i)$ converges to an exponential variable with rate λ_i (or what's the same an exponential rate one variable divided by λ_i).

(d) Since the holding times converge to exponential variables as N goes to infinity, the description of the limiting chain is as follows. Given $X_0 = i$ the chain waits an exponential with rate λ_i time and then jumps to state j with probability p_{ij} (the state jumped to is independent of the time of the jump). Then the chain stays in state j and exponential λ_j amount of time and jumps to state k with probability p_{jk}, and so on. According to our interpretation of the entries of the generator from lecture, the (i, j)th entry of the generator is $\lambda_i p_{ij}$ for $i \neq j$ and $p_{ii} = -\lambda_i$.