# School of Mathematics and Statistics
# MAST30030
# Applied Mathematical Modelling

## Problem Sheet 3.
## First-Order Partial Differential Equations

### Question 1

For each of the following partial differential equations, indicate whether the equation is linear or nonlinear, homogeneous or inhomogeneous, and give its order:

(a) $yu_{xx} + u_{yy} = 0$

(b) $au_x^2 + 2bu_xu_{yy} + cu_y = 0$, where $a$, $b$ and $c$ are constants.

(c) $\dfrac{\partial\phi}{\partial x}\dfrac{\partial^2\phi}{\partial y^2} = 1$

(d) $\phi\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial\phi}{\partial y} = \dfrac{\partial^3\phi}{\partial x\partial z^2}$

(e) Schrödinger equation: $i\hbar u_t + \dfrac{\hbar}{2m}\nabla^2 u = V(x,y,z)u$

(f) Biharmonic equation: $\nabla^4 u = f(x,y)$

(g) Eikonal equation: $u_x^2 + u_y^2 = n^2(x,y)$

(h) Navier-Stokes equation: $\rho\left(\mathbf{u}_t + [\mathbf{u}\cdot\nabla]\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u}$, $\nabla\cdot\mathbf{u} = 0$

(i) Korteweg-de Vries equation: $u_t + 6uu_x + u_{xxx} = 0$

(j) Black-Scholes equation: $u_t + \dfrac{1}{2}\sigma^2x^2u_{xx} + rxu_x - ru = 0$

### Question 2

Use the Method of Characteristics to solve:

$$\frac{\partial\phi}{\partial t} + 3\frac{\partial\phi}{\partial x} = 0, \quad -\infty < x < \infty,\ t > 0$$

subject to the initial condition

$$\phi(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

Sketch the space-time diagram and the solutions for: (a) $\phi(x,1)$; (b) $\phi(x,2)$.

### Question 3

Solve the partial differential equation

$$u_t + (xu)_x = 0, \quad -\infty < x < \infty,\ t > 0$$

subject to the initial condition

$$u(x,0) = g(x)$$

## Question 4

Find the solution to the partial differential equation, $u_t + 2u_x = u + t$, $-\infty < x < \infty$, $t > 0$, with initial condition $u(x, 0) = u_0(x)$.

## Question 5

Consider the partial differential equation

$$\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial x} = 0, \quad -\infty < t < \infty, \ x > 0$$

(a) Write down the equations of the characteristics for this equation.

(b) Sketch the graph of a few typical characteristics in the $xt$-plane.

(c) Could the solution contain a shock or a fan? Explain your answer.

(d) Solve the equation for $\phi$ as a function of a characteristic, i.e., find the general solution.

(e) Solve for $\phi$ subject to the boundary conditions

$$\phi(x, 0) = A_0, \quad \phi(0, t) = B_0$$

where $A_0$ and $B_0$ are constants.

(f) Write down the value of $\phi$ at the point $(3, -5)$.

## Question 6

Solve the differential equation

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = 0, \quad -\infty < x < \infty, \ t > 0$$

subject to the following initial conditions:

(a) $\phi(x, 0) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}$

(b) $\phi(x, 0) = \begin{cases} 0, & x < 0 \\ 1, & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$

In each case, sketch the space-time diagram and the solution $\phi(x, t)$ for various times $t$.

## Question 7

Consider the nonlinear equation

$$\frac{\partial \phi}{\partial t} + \phi^2 \frac{\partial \phi}{\partial x} = 0, \quad -\infty < x < \infty, \ t \geq 0$$

(a) Solve the equation in the $xt$-plane, subject to the initial condition:

$$\phi(x, 0) = \begin{cases} 0, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

sketching the characteristics in the $xt$-plane and indicating any regions that contain a shock or a fan.

(b) Solve for $\phi(x, t)$ in the fan region.

(c) Find the value of $\phi$ at the point $(1, 2)$.