Birth & Death Process

At state n. $T_n^+ \sim Exp(\lambda_n)$, $T_n^- \sim Exp(\mu_n)$ $\Big\}$ Independent.

wait $\min\{T_n^+, T_n^-\}$ to state $\begin{cases} n+1 & \text{if } T_n^+ < T_n^- \\ n-1 & \text{if } T_n^+ > T_n^- \end{cases}$

$\min\{T_n^+, T_n^-\} \sim Exp(\lambda_n + \mu_n)$,

continous. $P(T_n^+ = T_n^-) = 0$

$P(X=Y) = \iint_{x=y} f(x,y)\, dx\, dy = 0.$

$$\begin{cases} P(T_n^+ < T_n^-) = \dfrac{\lambda_n}{\lambda_n + \mu_n} \\ P(T_n^- < T_n^+) = \dfrac{\mu_n}{\lambda_n + \mu_n} \end{cases}$$

$\Rightarrow$ Generator of CTMC. $-a_{nn} = \lambda_n + \mu_n$.

$a_{n\,n-1} = \mu_n$

$a_{n\,n+1} = \lambda_n$.

$$A = \begin{bmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & (\mu_1+\lambda_1) & \lambda_1 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

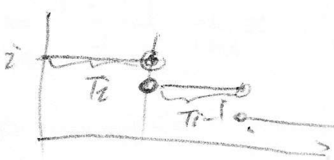
Consider a population with no birth $(\lambda_i = 0)$. death rates μ_i.

$X_t \to$ population. $\{X_t\}$ CTMC.

$P_{ij}(t) := P(X_{t+s} = j \mid X_s = i)$

$\Rightarrow P_{ii}(t) = P(T_i > t) = e^{-\mu_i t}$

$P_{i\,i-1}(t) = P(T_i < t,\ T_i + T_{i-1} > t)$.



eg. Compute distribution of birth rate $\lambda_i = \lambda$ $(i \geq 0)$
unit per capita death rate $\mu_i = i$ $(i \geq 0)$

$\pi_k = \pi_0 \prod_{i=1}^{k} \dfrac{\lambda_{i-1}}{\mu_i} = \pi_0 \dfrac{\lambda^k}{k!}$.

the stationary distribution $\exists$ if $\sum_{k=0} \dfrac{\lambda^k}{k!} < \infty$
$\underset{\text{=}}{\ } e^{\lambda}$ ✓

$\Rightarrow \pi_0 = e^{-\lambda}$. $\underbrace{\pi \sim Po(\lambda)}$

just $P\left((N_{11.25}-M_{11.25}) - (N_{11}-M_{11}) = 0 \,\middle|\, (M_{11.25}-M_{11}) \geq 2 \right)$.

Queue.

customers arriving according to a Poisson process N_t.

customers get on queue / service

X_t = # of customers in queue at time t.

$\{N_t\}$ poisson process. $\{X_i\}$ iid. & independent of $\{N_t\}$.

for $t \geq 0$ $\quad Y_t := \sum_{j \leq N_t} X_j$ called compound Poisson process.

X_i. μ. σ^2.

independent

$$\Rightarrow E[Y_t] = E[\sum_{j=1}^{N_t} X_j] = E[E[\sum_{j=1}^{N_t} X_j | N_t]]$$

$$= E\{N_t \cdot E[X_1]\} = \mu(\lambda t)$$

$$Var(Y_t) = E\{Var[Y_t|N_t]\} + Var(E(Y_t|N_t))$$

Moment generating Function of Y_t

$$E[e^{\theta Y_t}] = E\{E[e^{\theta Y_t}|N_t]\} = E\{M_X(\theta)^{N_t}\}$$

$$= M_{N_t}(\log(M_X(\theta))).$$

$\downarrow$ $Po(\lambda t)$.

$$M_{N_t}(s) = \cancel{e^{\lambda t}(e^s - 1)} \quad e^{\lambda t(e^s-1)} = e^{\lambda t(M_X(s)-1)}$$

fact

$X_i = 1$ w.p. 1. $\quad Y_t = N_t$. $\quad M_X(\theta) = e^{\theta}$