The University of Melbourne
Semester 1 Assessment 2009

Department of Mathematics and Statistics
620-156 Linear Algebra

Reading Time: 15 minutes.
Writing Time: 3 hours.

This paper has: 6 pages.

Identical Examination Papers: None.
Common Content Papers: None.

Authorised Materials:
No materials are authorised. Calculators and mathematical tables are not permitted. Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators:
Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

Instructions to Students:
This examination consists of 13 questions. The total number of marks is 100. All questions may be attempted. All answers should be appropriately justified.

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— BEGINNING OF EXAMINATION QUESTIONS —

1. Let

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Evaluate, if possible:

(a) AB

(b) BA

(c) BU

(d) $U^T U$

[6 marks]

2. (a) Let

$$G = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 1 & 2 & 0 \end{bmatrix}$$

Calculate the determinant of G using either row operations or cofactor expansion.

(b) State the determinant of the following matrices.

(i) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$

(iii) $3I_4$ where I_4 denotes the 4×4 identity matrix.

[7 marks]

3. (a) Use row operations on a suitable matrix to find all solutions to the equations

$$\begin{array}{rcrcrcrcl} x & & & + & z & - & w & = & 1 \\ 2x & + & y & & & + & w & = & 2 \\ x & + & y & & & & & = & 2 \end{array}$$

(b) Calculate the inverse of

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

[8 marks]

4. (a) (i) Let L denote the line in $\mathbb{R}^3$ which contains the points with position coordinates $P(1,2,1)$ and $Q(3,1,-1)$. What is the equation of the plane H which is perpendicular to L and contains the point $R(3,0,1)$?

   (ii) What is the point of intersection of the plane H and the line L.

   (b) What is the area of the triangle with vertices $A(0,1,1), B(1,0,1), C(1,1,0)$?

[7 marks]

5. (a) Let $\mathcal{Q}$ be the set of all vectors (x, y) in $\mathbb{R}^2$ so that $y \geq x^2$. Is $\mathcal{Q}$ a subspace of $\mathbb{R}^2$? Justify your answer either by appealing to appropriate theorems or by providing a counter-example.

   (b) Let $M_{2\times 3}$ denote the real vector space of all 2×3 matrices with real entries. Write down a set of matrices which form a basis for $M_{2\times 3}$. (You need not prove that these matrices form a basis.) Do the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   span $M_{2\times 3}$? Justify carefully your answer to this question.

[6 marks]

6. Let

$$A = \begin{bmatrix} -2 & 5 & 2 & 10 & -2 \\ 2 & -5 & -1 & -9 & 4 \\ 2 & -5 & -2 & -10 & 3 \\ -1 & 3 & 1 & 6 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In this question you may assume the fact that the matrix B is obtained from the matrix A by applying elementary row operations. Using this information, or otherwise, answer the following:

   (a) What is the rank of A?

   (b) Are the columns of A linearly independent? Explain your answer.

   (c) Write down a basis for the row space of A.

   (d) Write down a basis, consisting of columns of A, for the column space of A.

   (e) Do the rows of A form a basis for the row-space of A? Explain your answer.

   (f) Write $(10, -9, -10, 6)$ as a linear combination of

   $\{(-2, 2, 2, -1), (5, -5, -5, 3), (2, -1, -2, 1)\}$.

   (g) Find a basis for the solution space of A.

[10 marks]

7. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $x + y = 0$ and let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the x-axis.

   (a) Write down the matrix representations for S and R with respect to the standard basis.

   (b) Use part (a) to find the matrix representation, with respect to the standard basis, for $S \circ R$ (R followed by S).

   (c) Write down the images of the standard basis vectors under $S \circ R$ and identify the linear transformation as a familiar geometric transformation.

[6 marks]

8. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation given by

$$T(x, y, z, w) = (x - 2y - z - 4w, 2x - 3y - z - 6w, 2x - 5y - 3z - 10w).$$

   (a) Write down the standard matrix of T.

   (b) Find a basis $\mathcal{B}$ for $\mathrm{Ker}(T)$, the kernel of T and write down the nullity of T.

   (c) Find a basis $\mathcal{C}$ for $\mathrm{Im}(T)$, the image of T and write down the rank of T.

[8 marks]

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by:

$$T(x_1, x_2, x_3) = (-\frac{3}{5}x_1 + \frac{4}{5}x_3, x_2, \frac{4}{5}x_1 + \frac{3}{5}x_3)$$

   (a) Write down the matrix $[T]_{\mathcal{S}}$ representing T with respect to the standard basis

$$\mathcal{S} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

   (b) Find the transition matrix $P_{\mathcal{S},\mathcal{B}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{S}$, where

$$\mathcal{B} = \{(1, 0, 2), (0, 1, 0), (2, 0, -1)\}.$$

   (c) Use your answer to the previous part to find the transition matrix $P_{\mathcal{B},\mathcal{S}}$.

   (d) Calculate the matrix $[T]_{\mathcal{B}}$ representing T with respect to the basis $\mathcal{B}$.

   (e) Give a geometric description of T.

[10 marks]

10. Consider the matrix $A = \begin{bmatrix} 0 & -2 \\ 3 & 5 \end{bmatrix}$.

(a) Calculate the characteristic polynomial of A.

(b) Find the eigenvalues and eigenvectors of the matrix

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

[8 marks]

11. (a) You may assume that for exactly one of the values $a = 2$, $a = -2$, the following definition *does not* define an inner product on $\mathbb{R}^3$.

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + 3ax_2y_2 + x_3y_3 + 2x_1y_2 + ax_2y_1$$

State, giving reasons, which value of a *does not* yield an inner product. (The other value of a does give an inner product; you need not show this.)

(b) Write down a basis for the solution space of the equation $2x + 3y - z = 0$. From part (a), choose the value of a that *does* give an inner product and, using this inner product and the Gram-Schmidt procedure, find a basis for this solution space which is orthonormal.

[8 marks]

12. (a) Use the method of least squares to find the line of best fit $y = a + bx$ to the data:

x	-2	-1	0	1
y	4	3	1	0

(b) With this line of best fit, estimate a value for y when $x = 0.5$.

(c) With this line of best fit, what is the error at $x = 1$?

[8 marks]

13. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Find an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$.

[8 marks]

— END OF EXAMINATION QUESTIONS —