MAT4MDS — Practice 7

TAYLOR POLYNOMIALS

The nth Taylor polynomial to f about a is the function $T_n f : \mathbb{R} \to \mathbb{R}$ where

$$(T_n f)(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{1}$$

where $f^{(n)}(x) = \frac{d^n}{dx^n}(f(x))$ is the n^{th} derivative of f.

To calculate the nth Taylor polynomial of a function f,

FIRST, calculate the first n derivatives of f.
SECOND, evaluate f and its first n derivatives at a.
THIRD, substitute these values in the right hand side of equation (1).

Question 1. Find the second Taylor polynomial (about 0) for each of the following functions:

(a) $f(x) = x^2 e^x$

(b) $f(x) = (x+1)\ln(x+1)$

(c) $f(x) = xe^{x^2}$

Question 2. Calculate the 3rd Taylor polynomial (about 0) for the following functions:

(a) $f(x) = e^x$

(b) $g(x) = xe^x$.

Question 3. Three of the following formulas are correct (for arbitrary differentiable functions f and g, and arbitrary n, m, and for $a = 0$), and three are false.
Decide which formulas are correct.

(a) $(T_n(f+g))(x) = T_n f(x) + T_n g(x)$,

(b) $(T_n(f.g))(x) = T_n f(x). T_n g(x)$,

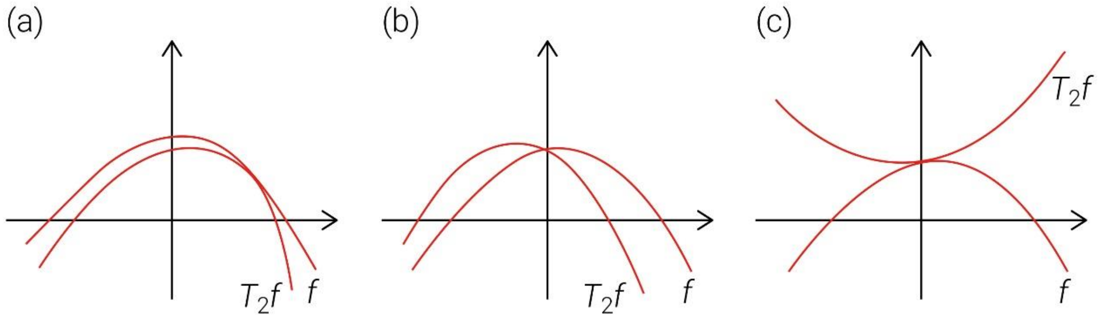
(c) $(T_n(xf))(x) = x(T_n f)(x)$,

(d) $(T_n(xf))(x) = x(T_{n-1} f)(x)$,

(e) $(T_n(T_m f))(x) = (T_{n+m} f)(x)$,

(f) $(T_n(T_n f))(x) = (T_n f)(x)$.

Question 4. Given $(T_2 f)(x) = f(0) + f'(0)\,x + \frac{1}{2!} f''(0)\,x^2$, calculate $(T_2 f)(0)$, $(T_2 f)'(0)$ and $(T_2 f)''(0)$ and use your answers to help you decide (and write down) what is *wrong* with each of the following graphs.



Question 5. In Practice Class 3 we found for the Gaussian function $f(x) = e^{-x^2}$ that

$$f'(x) = -2xe^{-x^2}$$

$$f''(x) = [4x^2 - 2]e^{-x^2}$$

Find the next two derivatives of f, and hence find the 4th Taylor polynomial of the Gaussian function. Compare your answer to the Taylor Series of e^x. What do you suspect?

> The Taylor series of a composite function $f(g(x))$ can be obtained by composing the two Taylor series. That is, in the Taylor series for f, use the Taylor series for g in place of x. When using this property to obtain Taylor polynomials to a particular order n, care must be taken that all terms of this order have been included.

> The Taylor polynomial (about 0) of the function $g(x) = \frac{1}{1-x}$ can be obtained from the first $(n+1)$ terms of the **Geometric Series**
>
> $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$
>
> in which $|x| < 1$.

Question 6. Using the results in the two boxes above, find (with centre 0)

(a) the 5th Taylor polynomial of $g(x) = \frac{1}{1+x}$

(b) the 5th Taylor polynomial of the Cauchy distribution $h(x) = \frac{1}{1+x^2}$.

Question 7. Consider the functions f and $g = f'$. Write down their nth Taylor polynomials. Differentiate $T_n f$ once. Propose a relationship (like those of Question 3) for $(T_n f)'$ and $(T_m f')$.

> The linearisation (or linear approximation) to a function near a is given by the first two terms of the Taylor polynomial, that is
>
> $$f(x) \approx f(a) + f'(a)(x - a)$$

Question 8. Using derivatives from Question 5, find the linearisation of the Gaussian function near $x = 1$.