MAST30022 Decision Making
2021
Tutorial 4

1. **(PS2-30)** Solve the 2-person constant-sum game with the payoff matrix below. Identify an optimal strategy pair.

$$\begin{bmatrix} (16,64) & (32,48) & (39,41) \\ (41,39) & (47,33) & (56,24) \\ (29,51) & (50,30) & (26,54) \end{bmatrix}$$

Solution

This is a constant-sum game with $c = 80$. Subtracting 40 (=80/2) from each entry we get a zero-sum game with the following payoff matrix for Player I.

$$\boldsymbol{V} = \begin{bmatrix} -24 & -8 & -1 \\ 1 & 7 & 16 \\ -11 & 10 & -14 \end{bmatrix}$$

The security levels are

$$(s_1, s_2, s_3) = (-24, 1, -14)$$

and

$$(S_1, S_2, S_3) = (1, 10, 16).$$

So $L = U = v_{21} = 1$ and hence (a_2, A_1) is an equilibrium pair for the zero-sum game above. [If this zero-sum game has no saddle point, then we may need to use the Simplex Method to solve it].

The pair (a_2, A_1) is also an equilibrium solution to the original game and the corresponding payoff vector is $(41, 39)$.

2. **(PS3-1)** For the 2-person non-zero-sum game with payoff bi-matrix:

		Player II	
		A_1	A_2
Player I	a_1	(2, 10)	(5, 0)
	a_2	(4, 8)	(8, 3)

(a) Clearly explain why (a_2, A_1) is an equilibrium pair;

(b) Clearly explain why (a_2, A_2) is NOT an equilibrium pair;

(c) Classify the other pairs as equilibrium or not equilibrium.

Solution

(a) We have

$$2 = u_1(a_1, A_1) \leq u_1(a_2, A_1) = 4$$
$$3 = u_2(a_2, A_2) \leq u_2(a_2, A_1) = 8$$

The first inequality means that if Player II keeps using A_1, then Player I receives less if he deviates from a_2. The second inequality means that if Player I keeps using a_2, then Player II receives less if he deviates from A_1. By the definition of an equilibrium, (a_2, A_1) is in equilibrium.

(b) (a_2, A_2) is not in equilibrium because

$$8 = u_2(a_2, A_1) > u_2(a_2, A_2) = 3.$$

(c) Neither (a_1, A_1) nor (a_1, A_2) is in equilibrium.

3. **(PS3-4)**

Describe a systematic way of finding equilibrium pairs of **pure strategies** for 2-person non-zero-sum games, and use it to find all equilibrium pairs of pure strategies for the 2-person non-zero-sum game with the following payoff bi-matrix:

		Player II			
		A_1	A_2	A_3	A_4
	a_1	(5, 2)	(3, 0)	(8, 1)	(2, 3)
Player I	a_2	(6, 3)	(5, 4)	(7, 4)	(1, 1)
	a_3	(7, 5)	(4, 6)	(6, 8)	(0, 2)

Solution

To find all pure strategy equilibria, in the payoff bi-matrix, locate the entries that give the maximum in the column for Player 1 and the maximum in the row for Player 2. In this case the pure strategy equilibria are (a_2, A_2) and (a_1, A_4), or $(\boldsymbol{x}^*, \boldsymbol{y}^*) = ((0, 1, 0), (0, 1, 0, 0))$ and $(\boldsymbol{x}^*, \boldsymbol{y}^*) = ((1, 0, 0), (0, 0, 0, 1))$.

4. **(PS3-17)** Consider the 2-person non-cooperative non-zero-sum game with payoff bi-matrix

$$\begin{bmatrix} (1,1) & (0,0) \\ (2,-1) & (-1,2) \end{bmatrix}.$$

(a) Find a security level mixed strategy for each player.

(b) Check whether the security level strategies found in (a) give an equilibrium pair.

(c) Find all equilibrium pairs by the graphical method.

(d) For each equilibrium pair and the security level pair, give the expected payoff to each player. Compare the payoffs associated with the security level pairs and that associated with the equilibrium pairs, and confirm that the latter is no worse than the former.

Solution

(a) We have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{B}^T = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

We should use $\boldsymbol{A}$ to work out Player I's security levels, and use $\boldsymbol{B}^T$ for Player II's security levels.

Consider $\boldsymbol{A}$ first. Since

$$L = \max\{0, -1\} = 0 = \min\{2, 0\} = U = a_{12},$$

a_{12} is a saddle point and the security level strategy of Player I is

$$\boldsymbol{x}^* = (1, 0)$$

(that is, the first pure strategy of Player I). The optimal value for Player I is $u^* = 0$.

Now consider $\boldsymbol{B}^T$. Since

$$L = \max\{-1, 0\} = 0 < 1 = \min\{1, 2\} = U,$$

$\boldsymbol{B}^T$ has no saddle point. We may use the 2×2 formulae to find a security level mixed strategy for Player II. Note that Player II is the row player with respect to $\boldsymbol{B}^T$, and that for $\boldsymbol{B}^T$ we have $r = 1 + 2 - (-1) - 0 = 4$. So his security level mixed strategy is

$$\begin{aligned} \boldsymbol{y}^* &= ((d-c)/r, (a-b)/r) \\ &= (2/4, (1-(-1)/4) \\ &= (1/2, 1/2). \end{aligned}$$

[Warning: This corresponds to the row security level mixed strategy in the 2×2 formulae!] And the optimal value for Player II is

$$v^* = (ad - bc)/r = 1/2.$$

(b) Let $\boldsymbol{x} = (x_1, 1-x_1) \in X$ and $\boldsymbol{y} = (y_1, 1-y_1) \in Y$. Then

$$\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{y}^{*T} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}1/2\\ 1/2\end{bmatrix} = 1/2$$

$$\begin{aligned}\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} &= \begin{bmatrix}x_1 & 1-x_1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}1/2\\ 1/2\end{bmatrix}\\ &= \begin{bmatrix}x_1 & 1-x_1\end{bmatrix}\begin{bmatrix}1/2\\ 1/2\end{bmatrix}\\ &= 1/2.\end{aligned}$$

Hence $\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} \le \boldsymbol{x}^*\boldsymbol{A}\boldsymbol{y}^{*T}$ for any $\boldsymbol{x} \in X$.

We have

$$\boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^{*T} = (1,0)\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}1/2\\ 1/2\end{bmatrix} = 1/2$$

$$\begin{aligned}\boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^{T} &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}y_1\\ 1-y_1\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}y_1\\ 1-y_1\end{bmatrix}\\ &= y_1.\end{aligned}$$

If $1/2 < y_1 \le 1$, then $\boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^T = y_1 \not\le \mathbf{x}^*\boldsymbol{B}\boldsymbol{y}^{*T} = 1/2$.

Hence the security level pair $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ found in (a) is NOT an equilibrium pair.

(c)

$$\begin{aligned}\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{T} &= \begin{bmatrix}x_1 & 1-x_1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}\begin{bmatrix}y_1\\ 1-y_1\end{bmatrix}\\ &= x_1(-2y_1+1) + (3y_1-1)\\ \boldsymbol{x}\boldsymbol{B}\boldsymbol{y}^{T} &= \begin{bmatrix}x_1 & 1-x_1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}y_1\\ 1-y_1\end{bmatrix}\\ &= y_1(4x_1-3) + (-2x_1+2).\end{aligned}$$

For fixed y_1, $\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^T$ is maximised when

$x_1 = 0$, for $-2y_1+1 < 0$, i.e. $y_1 > 1/2$
$x_1 = 1$, for $-2y_1+1 > 0$, i.e. $y_1 < 1/2$
$0 \le x_1 \le 1$, for $-2y_1+1 = 0$, i.e. $y_1 = 1/2$.

For fixed x_1, $\boldsymbol{x}\boldsymbol{B}\boldsymbol{y}^T$ is maximised when

$y_1 = 0$, for $4x_1-3 < 0$, i.e. $x_1 < 3/4$
$y_1 = 1$, for $4x_1-3 > 0$, i.e. $x_1 > 3/4$
$0 \le y_1 \le 1$, for $4x_1-3 = 0$, i.e. $x_1 = 3/4$.

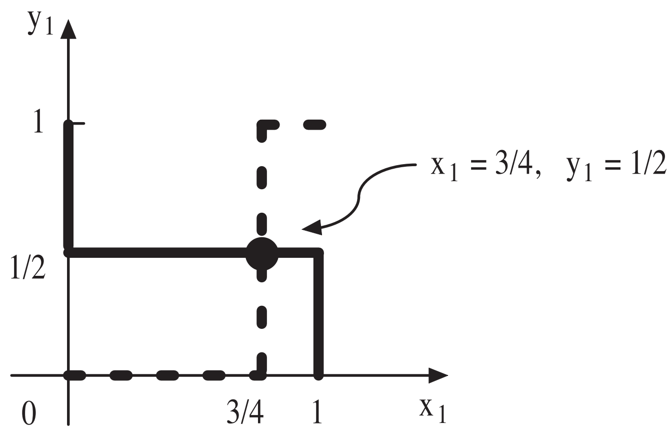

Figure 1: PS3-17

From Figure 1 we can see that there is a unique equilibrium pair, given by

$$\boldsymbol{x}^* = (3/4, 1/4), \boldsymbol{y}^* = (1/2, 1/2).$$

(d) Recall that for any pair $(\boldsymbol{x}, \boldsymbol{y})$ of strategies, the expected payoffs to the two players are $\boldsymbol{xAy}^T$ and $\boldsymbol{xB}\mathbf{y}^{\mathbf{T}}$ respectively.

For the equilibrium pair $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ found in (c), the expected payoffs to the two players are

$$\boldsymbol{x}^*\boldsymbol{Ay}^{*T} = (3/4, 1/4)\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 1/2$$

$$\boldsymbol{x}^*\boldsymbol{By}^{*T} = (3/4, 1/4)\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 1/2$$

For the security level pair $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ found in (a), from the computation in (b) we have

$$\boldsymbol{x}^*\boldsymbol{Ay}^{*T} = 1/2 \text{ (Player I)}$$

$$\boldsymbol{x}^*\boldsymbol{By}^{*T} = 1/2 \text{ (Player II)}$$

Comparison: The payoff to each player associated with the unique security level pair is 1/2, and that with the equilibrium pair is also 1/2. Therefore, the latter is no worse than the former.