Fluid Mechanics

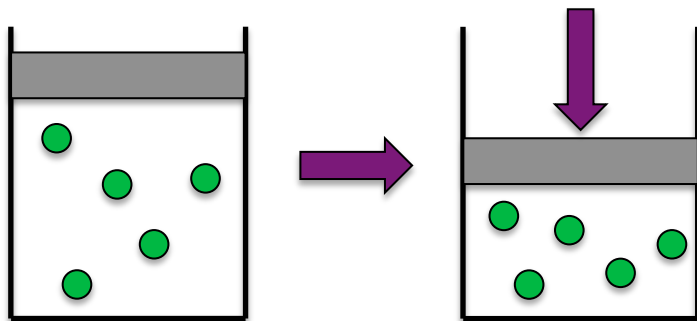density is different alongside with the Pressure!

Compressible flow part 1:

Review of thermodynamics
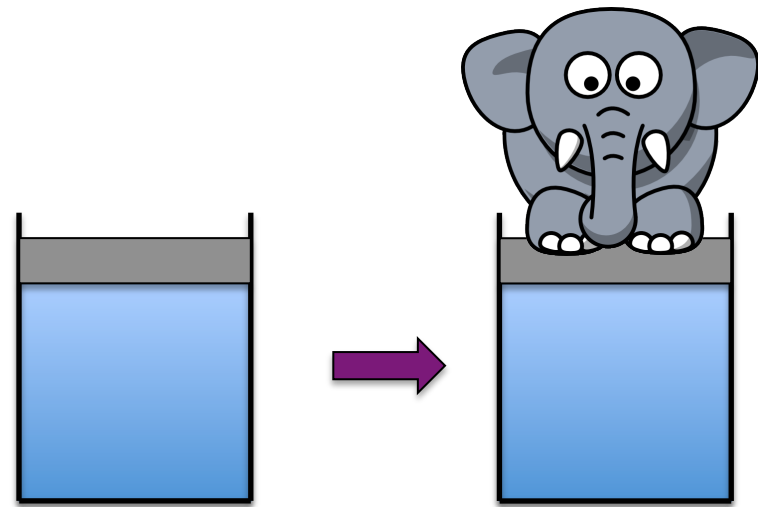
What do we mean by compressibility?

What do we mean when we say **compressible**?

- When a material is compressible, the density changes when pressure is changed
- Mathematically we can say $\dfrac{\partial \rho}{\partial P} \neq 0$ or that $\rho = f(P)$

For Gases   **For Liquids**

What do we mean by compressibility?

What do we mean when we say **compressible**?

- How compressible is sea water?
  - At sea level, ocean water has a density of ~1.03g/cm^3 while 11 km below at the bottom of the Mariana Trench the density of ocean water is ~1.08g/cm^3
  - That is a 5% change in density for a 1,072 atm increase in pressure

- How compressible is air?
  - At sea level, air has a density of ~0.0012g/cm^3 and you would achieve the same 5% change in density by subjecting it to a 1.065 atm of pressure

Compressible versus incompressible flow

Thus far, we've been discussing **liquid flow** which is referred to as **incompressible flow** as the density of the fluid is constant with pressure

- – Pressure drop is linear across the length of the pipe
- – Isothermal
- – Constant density

However, **gas flow** can be (much) more complex as it can be **compressible flow** where the density and temperature will change with pressure

- – Changes in pressure are not as easily predicted
- – Non-isothermal
- – Not constant density

Is the flow of gas always compressible?

The mathematics associated with modeling of compressible flow is much more challenging

However, is all gas flow considered compressible?

While compression of gas will occur any time there is a pressure differential in the system, it is not always significant

– Gas flow can be modeled as **incompressible** whenever the changes in pressure in the system are **less than 20%**

– Gas flow must be modeled as **compressible** whenever the changes in pressure in the system are **greater than 20%**

Compressible flow and thermodynamics

Since we are discussing systems where pressure, density, and temperature are interrelated and change with one another, we must utilize thermodynamics in order to describe the flow of these fluids

In addition to **conservation of mass**, **conservation of energy**, and **conservation of momentum**, we will have to use **equations of state**
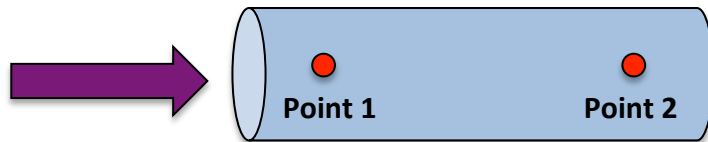
Where are we going in this lesson?

Let us think about a fluid flowing through a pipe

- If the fluid is an incompressible liquid, the scenario is rather boring

Point 1 Point 2
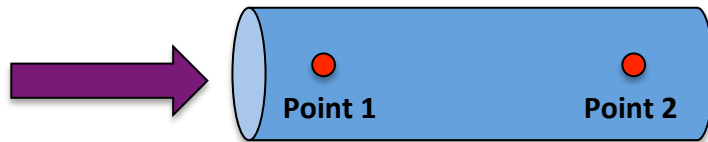
- Pressure drop is linear
- $\rho_1 = \rho_2$
- $T_1 = T_2$

- If the fluid is a compressible gas, the scenario is much more interesting

Point 1 Point 2

- Pressure drop is not necessarily linear
- $\rho_1 \neq \rho_2$
- $T_1 \neq T_2$

In today's class, we will derive an equation that will let us predict the density of a compressible fluid in flow based on the local pressure in **isentropic flow**
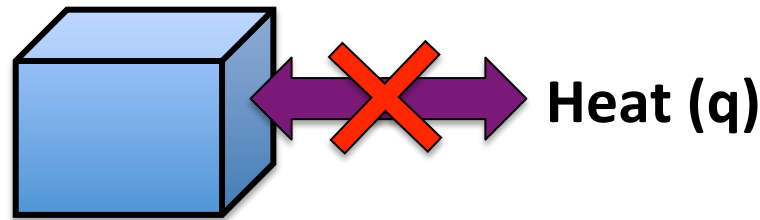
$$\frac{P_1}{P_2} = \left(\frac{v_2}{v_1}\right)^{\gamma} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

In order to get here we're going to have to use thermodynamics

A review of thermodynamic vocabulary

Before we can continue, we have to review two thermodynamic terms: **adiabatic** and **isentropic**

- An **adiabatic process** is one where there is no heat (q) exchanged with the surroundings. It is perfectly insulated

Heat (q)

- An **isentropic** process is one where the entropy within the control volume is constant ($\Delta S = 0$). An isentropic system must be both adiabatic and **reversible**

  - A system is **reversible** when you can alter the system and can retain the original state by retracing the exact same thermodynamic path
  - Said another way, a **reversible** system is always at equilibrium, even during the change in state

A review of the ideal gas law

As mentioned previously, we will need an equation of state to mathematically describe compressible flow. For the sake of simplicity, we will utilise the **ideal gas law**

$$PV = nRT$$

P = pressure
V = volume
n = number of moles of gas
R = ideal gas constant
T = absolute temperature (K)

However, we will slightly modify the equation so that it works better with our **mechanical energy balance**

A review of the ideal gas law

$$PV = nRT$$ ⟶ $$PV = \frac{m}{M}RT$$

m = mass of gas
M = molecular weight

⟶ $$PV = m\frac{RT}{M}$$ ⟶ $$P = \frac{m}{V}\frac{RT}{M}$$

ρ = m/V

⟶ $$P = \rho\frac{RT}{M}$$ ⟶ $$P\nu = \frac{RT}{M}$$

ν = 1/ρ

More thermodynamic vocabulary

In your thermodynamics class you also discussed the **enthalpy**, the **heat content** of the system

$$H = U + PV$$

H = enthalpy
U = internal energy
P = pressure
V = volume
PV = pressure-volume work

P-V work can be thought of as the amount of work the system has to do on its surroundings in order to "make room" for itself

We're going to divide the above equation by the mass of our system (m) in order to arrive at the **specific enthalpy**, or enthalpy per unit mass

$$h = U + Pv$$

h = specific enthalpy
U = internal energy per unit mass
P = pressure
ν = specific volume

Even more thermodynamic vocabulary

Specific heat capacity is the last thermodynamic term we need to remember

Specific heat capacity is the heat required to raise the temperature of a unit mass of fluid by 1^oC

If you remember, you can measure the specific heat capacity in several ways. For instance, there is **constant volume specific heat capacity** (C_v) and **constant pressure specific heat capacity** (C_P)

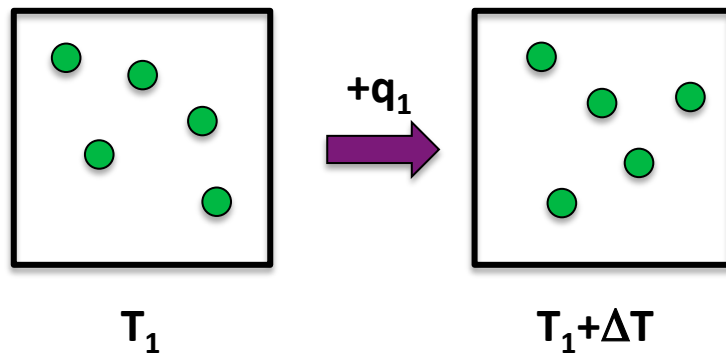# What is the relationship between C_P and C_V?

First let us remember the first law of thermodynamics

$$\Delta U = \Delta q - \Delta W \qquad \xrightarrow{\text{If only P-V work}} \qquad \Delta U = \Delta q - P\Delta \upsilon$$
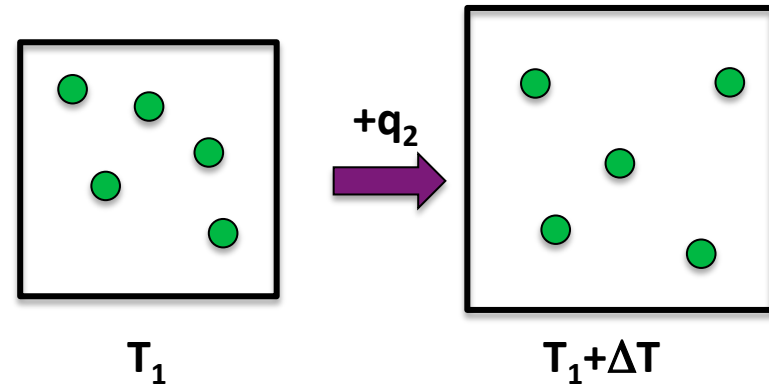
Again, P-V work means that the volume increases against a resisting pressure

It turns out that C_P is greater than C_v for the same ΔT. Why is this?

Constant volume specific heat capacity, C_v Constant pressure specific heat capacity, C_P

T_1 $T_1+\Delta T$ T_1 $T_1+\Delta T$

q_2 is larger than q_1 because you must raise the temperature of the system by ΔT **and** also perform $P\Delta\upsilon$ work. This means C_P is also larger than C_V

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Thermodynamic vocaculary

We just recalled several terms from thermodynamics

- Adiabatic. Perfectly insulated system, no heat is transferred with surroundings

- Isentropic. System where change in entropy is 0, both adiabatic and reversible

- Ideal gas. A gas that follows the ideal gas law, $P\nu=RT/M$

- Specific enthalpy. System's heat content per unit mass, $h = U + P\nu$

- Specific heat capacity (constant volume and constant pressure). The amount of energy required to heat a unit mass of fluid by 1^oC

Why is you making me remember all this horrible thermodynamics???
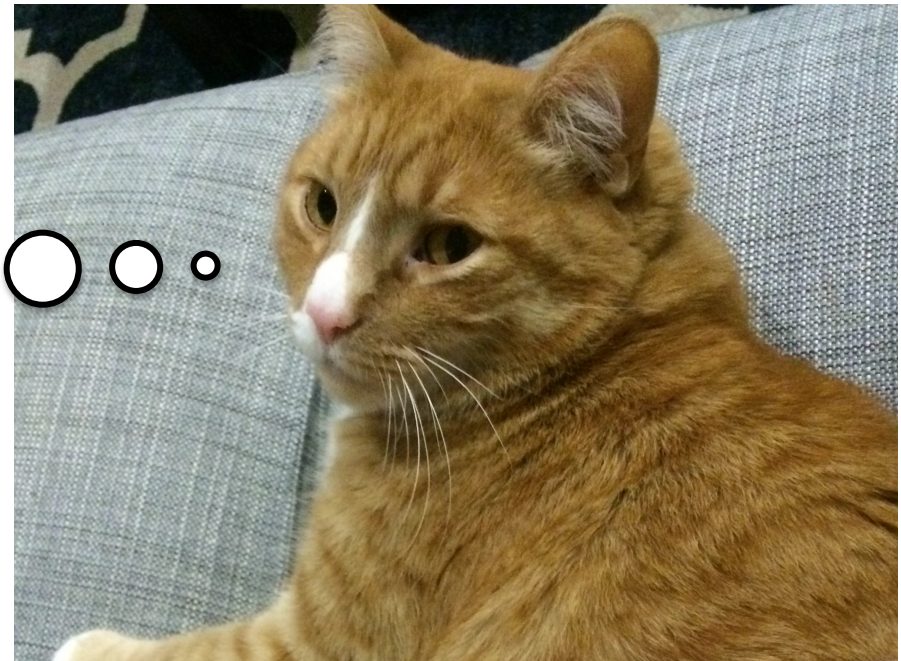
Relating pressure and density in compressible flow

If you remember, our goal for this class is to derive an expression that will enable us to relate the **pressure** and **density** (**specific volume**) of fluid elements in different parts of a compressible flow field assuming **isentropic flow** (adiabatic and reversible of an **ideal gas**)
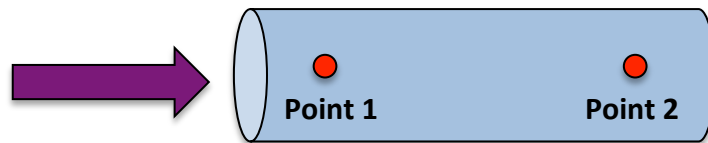
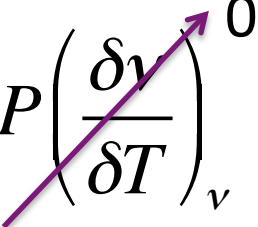$$\frac{P_1}{P_2} = \left(\frac{v_2}{v_1}\right)^{\gamma} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

It turns out that the exponent, γ, is related to the **heat capacities**

Mathematically defining constant vol heat capacity

We've defined the specific heat capacities verbally, but we have yet to define them mathematically

- Mathematically, the specific heat capacity at constant volume is defined as the derivative of the specific internal energy (U) with respect to temperature

$$C_v = \left(\frac{\delta U}{\delta T} \right)_v \qquad \text{where} \qquad \Delta U = \Delta q - P\Delta v$$

$$C_v = \left(\frac{\delta U}{\delta T} \right)_v = \left(\frac{\delta q}{\delta T} \right)_v - P \left(\frac{\delta v}{\delta T} \right)_v^{\,0}$$

$$C_v = \left(\frac{\delta q}{\delta T} \right)_v$$

Mathematically defining constant pres heat capacity

We've defined the specific heat capacities verbally, but we have yet to define them mathematically

- Mathematically, the specific heat capacity at constant pressure is defined as the derivative of specific enthalpy (h) with respect to temperature

$$C_P = \left(\frac{\delta h}{\delta T}\right)_P \quad \text{where} \quad \Delta h = \Delta U + \Delta(Pv)$$

$$\Delta h = \Delta U + \Delta(Pv) \quad \xrightarrow{\text{expand}} \quad \Delta h = \Delta U + \Delta P v + P \Delta v$$

Remember $\quad \Delta U = \Delta q - P \Delta v \quad$ and plug in value of U

$$C_P = \left(\frac{\delta q}{\delta T}\right)_P + v \left(\frac{\delta P}{\delta T}\right)_P^{\;\;0} \quad \longrightarrow \quad C_P = \left(\frac{\delta q}{\delta T}\right)_P$$

Mathematically definitions of heat capacities

We've now mathematically defined the **constant volume specific heat capacity** and the **constant pressure specific heat capacity**

$$C_v = \left(\frac{\delta U}{\delta T}\right)_v = \left(\frac{\delta q}{\delta T}\right)_v \qquad C_P = \left(\frac{\delta h}{\delta T}\right)_P = \left(\frac{\delta q}{\delta T}\right)_P$$

However, I told you earlier that the heat capacities will be related the pressure and density of in incompressible fluid through the parameter γ, the exponent

This means we will need to perform further mathematical manipulation in order to relate these parameters in terms of state variables such as pressure, temperature, and specific volume

Putting C_v in terms of measurable variables

First we will put C_v in terms of our desirable variables

- We'll begin with our expression for internal energy

$$\Delta U = \Delta q - P\Delta v \quad \xrightarrow{\text{adiabatic}} \quad \Delta U = -P\Delta v$$

- And then manipulate our expression for constant volume specific heat capacity. Assuming a constant value for heat capacity over the relevant range of temperatures we can integrate the expression

$$C_v = \left(\frac{\delta U}{\delta T}\right)_v \quad \xrightarrow{\text{integrate}} \quad \Delta U = C_v \Delta T$$

- We can substitute for ΔU and get our heat capacity in terms of P, v, and T

$$C_v \Delta T = -P\Delta v$$

Putting C_P in terms of measurable variables

Now we will put C_P in terms of our desirable variables

- We'll begin with our expression specific enthalpy and heat

$$\Delta h = \Delta U - \Delta(Pv)$$
$$\Delta q = \Delta U + P\Delta v$$

combine $\longrightarrow$ $\Delta h = \Delta q + v\Delta P$ **adiabatic** $\longrightarrow$ $\Delta h = v\Delta P$

- And then manipulate our expression for constant pressure specific heat capacity. Assuming a constant value for heat we can integrate the expression

$$C_P = \left(\frac{\delta h}{\delta T}\right)_P$$

$\longrightarrow$

$$\Delta h = C_P \Delta T$$

- We can substitute for ΔU and get our heat capacity in terms of P, v, and T

$$C_P \Delta T = v\Delta P$$

Relating P & ν in isentropic flow for an ideal gas

We have put C_P and C_ν in terms of measurable variables

$$C_P \Delta T = \nu \Delta P \qquad C_\nu \Delta T = -P \Delta \nu$$

We can now ratio these two expressions and define $\gamma = C_P/C_\nu$

$$\frac{\Delta P}{P} = -\gamma \frac{\Delta \nu}{\nu}$$

Now let us convert this to a differential by focusing on a narrow range of pressure and specific volume

$$\frac{dP}{P} = -\gamma \frac{d\nu}{\nu} \quad \Longrightarrow \quad \ln\left(\frac{P_1}{P_2}\right) = \ln\left(\frac{\nu_2}{\nu_1}\right)^\gamma$$

$$\Longrightarrow \quad \boxed{\frac{P_1}{P_2} = \left(\frac{\nu_2}{\nu_1}\right)^\gamma = \left(\frac{\rho_1}{\rho_2}\right)^\gamma}$$

How do you know the value for gamma, γ?

In order to use this relationship do you need to know the value of C_P and C_v for you gas? If you're dealing with a complex mixture of gases (say, natural gas) that could be variable and complex…

Interestingly, the value for gamma depends heavily on the number of molecules in the atoms that make up the gas

| Atoms per molecule	Value of γ
Monatomic (He, Ar)	1.67
Diatomic (H_2, O_2, N_2)	1.40
Triatomic (CO2)	1.30
Complex molecules	~1.05

So what is the γ value for air?

How versatile is this equation?

So this relationship was derived assuming isentropic flow of an ideal gas? Can it be used in any other conditions?

This particular expression can only be used for an isentropic system of an ideal gas; however, a very similar expression can be written for an **isothermal** process of an ideal gas

For an isothermal system $\gamma=1$

$$\frac{P_1}{P_2} = \left(\frac{v_2}{v_1}\right) = \left(\frac{\rho_1}{\rho_2}\right)$$
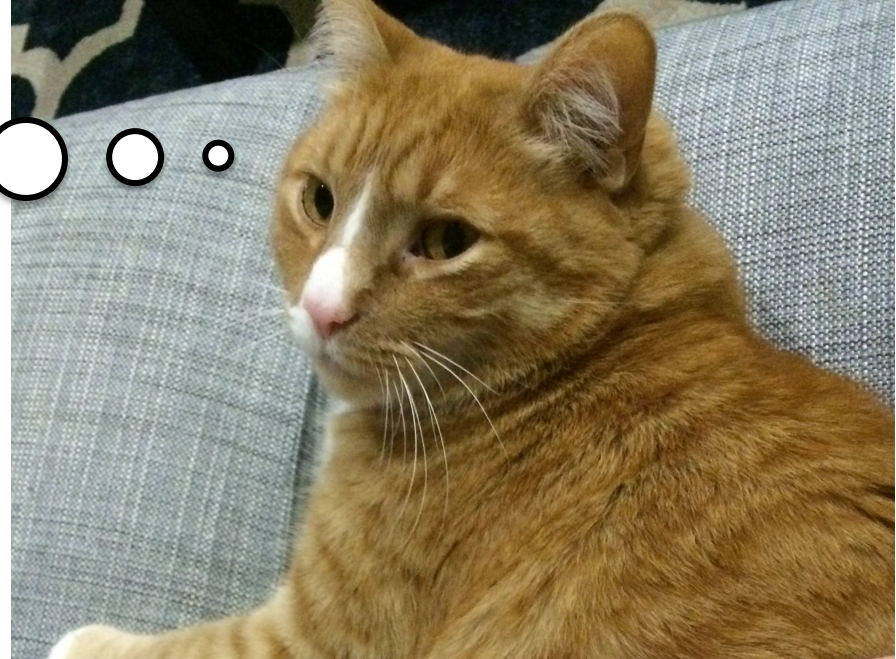
Summary

In this lesson we began discussion of compressible flow

Compressible flow is significant when the pressure changes in the system are > 20%

When that occurs we see pressure, density, and temperature variations in our flow. This means in addition to our energy, mass, and momentum balances, we need an equation of state to mathematically describe the system

We derived an expression that links the pressure and density of an ideal gas undergoing an isentropic process