THE UNIVERSITY OF MELBOURNE

Student Number

Semester 1 Assessment, 2018

School of Mathematics and Statistics

MAST30013 Techniques in Operations Research

Writing time: 2 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Students will be provided with a 6-page formulae sheet.
- School approved calculators.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- There are 6 questions with marks as shown. The total number of marks available is 80.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

Blank page (ignored in page numbering)

Question 1 (15 marks)

Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x\log(x),\ x > 0,$$

where log denotes the natural logarithm.

(a) Show that f is continuous and unimodal for $x > 0$.

(b) With a tolerance of $\epsilon = 0.05$, and the initial interval $[0.1,\ 2.1]$, using the Golden Section search, what is the least number of f-calculations required? What is the length of the final interval?

(c) Perform one iteration of the Fibonacci search with the same tolerance and initial interval as in (b). Write down the interval in which the minimum is to be found after one iteration.

(d) Can the false position method be used to find the minimum of f with the initial interval $[0.1, 2.1]$? If so perform one iteration and write down the interval. Otherwise, explain why not.

(e) Is it possible to apply Newton's method in finding the minimum of f on the interval [0.1,2.1], starting from $x_0 = 2$? If so, perform one iteration. Otherwise, explain why not.

Question 2 (10 marks)

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1^3 + 2x_1^2 + x_2^3 - 4x_2^2 - 4x_1 + 12.$$

(a) Using the first-order necessary condition, find all stationary points of f.

(b) Using the second-order sufficient condition, classify all stationary points of f as local minima, local maxima or saddle points.

(c) For the saddle point $\boldsymbol{x}^* = (2/3, 0)$, find a direction in which the value of f increases and another direction in which the value of f decreases.

Question 3 (18 marks)

Consider the unconstrained nonlinear program

$$\min f(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 9x_1 - 3x_2.$$

(a) Perform one iteration of the steepest descent method starting at the point $\boldsymbol{x}^0 = (0, 0, 0)$ to find $\boldsymbol{x}^1$.

(b) Prove that the trajectory of the steepest descent method satisfies the condition that $\boldsymbol{d}^k$ is normal to $\boldsymbol{d}^{k+1}$ for $k = 0, 1, \ldots$.

(c) Perform one iteration of Newton's method starting at the point $\boldsymbol{x}^0 = (0, 0, 0)$ to find $\boldsymbol{x}^1$.

(d) Verify that $\boldsymbol{x}^1$ found in (c) is the local minimum $\boldsymbol{x}^*$ and explain why Newton's method found the minimum in one iteration.

(e) Find the BFGS direction for f at the point $\boldsymbol{x}^0 = (0, 0, 0)$ with $H_0 = 2I_3$, where I_3 is the 3×3 identity matrix.

Question 4 (15 marks)

Consider the constrained nonlinear program

$$\begin{aligned} \min \quad & f(x_1, x_2) = (x_1 - 1)^2 + (x_2 + 1)^2 \\ \text{s.t.} \quad & x_1x_2 \le 1 \\ & x_1 + x_2 = 2. \end{aligned}$$

(a) Write down the Lagrangian for the nonlinear program.

(b) Find the KKT point(s).

(c) Find the critical cone(s) at the KKT point(s).

(d) Check the second-order sufficient condition at the KKT point(s) and determine the minimum value of the objective function.

(e) If the equality constraint in the above nonlinear program is changed to $x_1 + x_2 = 1$, approximately how much would the function value change compared to the minimum found in (d).

Question 5 (10 marks)

Consider the constrained nonlinear program

$$\begin{aligned} \min \quad & f(x_1, x_2) = x_1 x_2 \\ \text{s.t.} \quad & x_1 - x_2 = 1 \\ & x_1 \geq 0. \end{aligned}$$

(a) Write down the l_2 penalty function $P_k(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, x_2)$, with penalty parameter $\alpha_k = k$.

(b) Write down $\boldsymbol{\nabla} P_k(\boldsymbol{x})$, and solve $\boldsymbol{\nabla} P_k(\boldsymbol{x}) = \mathbf{0}$ to find the stationary point(s) $\boldsymbol{x}^k = (x_1^k, x_2^k)$ of $\boldsymbol{\nabla} P_k(\boldsymbol{x})$. Find the limit $\boldsymbol{x}^* = \lim_{k\to\infty} \boldsymbol{x}^k$.

(c) At the stationary point found in (b), find the corresponding Lagrange multipliers (λ^k, η^k). Find the limit $(\lambda^*, \eta^*) = \lim_{k\to\infty}(\lambda^k, \eta^k)$.

Question 6 (12 marks)

Consider the constrained nonlinear program

$$\begin{aligned} \min \quad & f(x_1, x_2) = \frac{x_1^2}{2} - x_1 x_2 + x_2^2 - 7x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 5 \\ & x_1 - x_2 \geq 1. \end{aligned}$$

(a) Show that the above nonlinear program is a convex program.

(b) Write down the Lagrangian for the nonlinear program.

(c) The KKT point for the nonlinear program is $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = ((2, 1), (1, 5))$. Verify that the KKT point satisfies the Lagrangian Saddle Inequalities.

(d) Explain why $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is the only KKT point.

(e) Write down the Wolfe dual of this program.

End of Exam—Total Available Marks = 80