TUTORIAL 1

1. A company assembles both cars and trucks. The assembly is carried out in three departments: sheet metal stamping, engine assembly and final assembly. The number of person hours required for assembling cars and trucks in each of these departments is shown below:

	sheet metal	*engine assembly*	*final assembly*
cars	50	50	25
trucks	25	50	100

Each department has a capacity of 200,000 person hours per year. Cars earn a profit of \$6000 each, and trucks \$10,000 each, for the company. Formulate a linear program that can be used to determine how many cars and trucks the company should manufacture each year to maximise its profit.

Solution:

Let x_1 be the number of cars manufactured each year and x_2 be the number of trucks manufactured each year. Then we can formulate the problem as

$$\max 6,000x_1 + 10,000x_2$$

such that

$$\begin{aligned} 50x_1 + 25x_2 &\leq 200,000 \\ 50x_1 + 50x_2 &\leq 200,000 \\ 25x_1 + 100x_2 &\leq 200,000 \\ x_1, x_2 &\geq 0. \end{aligned}$$

2. Solve the following optimisation problems using the "graphical" method outlined in class.

(a)

$$\max 3x_1 + 5x_2$$

such that

$$\begin{aligned} 2x_1 + 3x_2 &\leq 12 \\ 3x_1 + 4x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b)

$$\max 3x_1 + 2x_2$$

such that

$$\begin{aligned} 4x_1 + 3x_2 &\leq 12 \\ 3x_1 + 4x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(c)

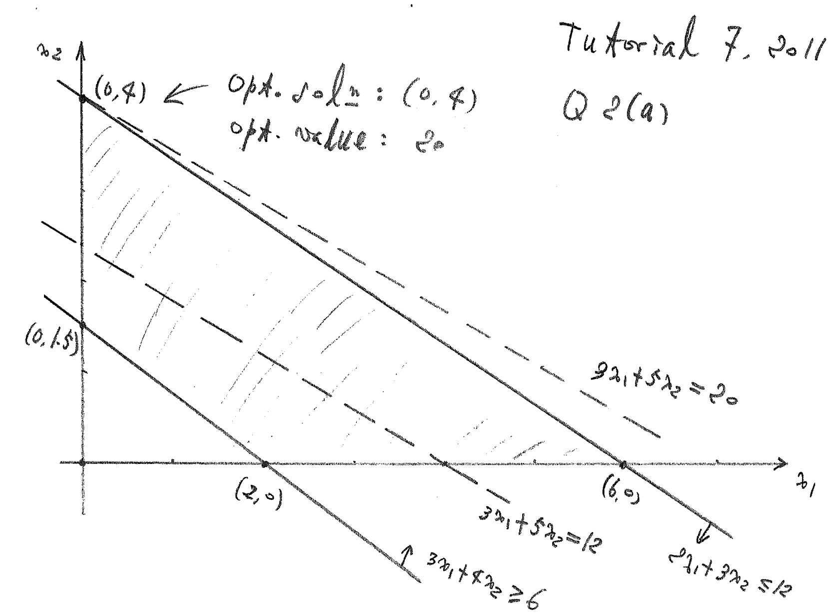
$$\max 3x_1 + 2x_2 + x_3$$

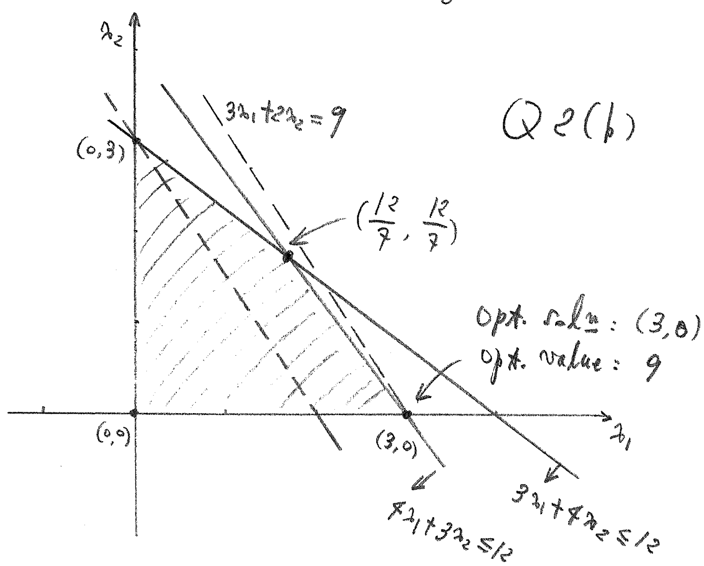
such that

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 4 \\ 3x_1 + x_2 &\leq 5 \\ x_1 + 4x_2 &\leq 6 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution:

(a) Optimal solution $x^* = (0, 4)$; optimal value $z^* = 20$.
(b) Optimal solution $x^* = (3, 0)$; optimal value $z^* = 9$.


Tutorial 7, 2011
Q 2(a)
x_2
(0,4)
Opt. soln : (0, 4)
opt. value : 20
(0, 1.5)
$2x_1+5x_2=20$
(2,0)
(6,0)
x_1
$3x_1+5x_2=12$
$2x_1+3x_2 \le 12$
$3x_1+4x_2 \ge 6$


x_2
$3x_1+2x_2=9$
Q 2(b)
(0,3)
$(\frac{12}{7}, \frac{12}{7})$
Opt. soln : (3,0)
opt. value : 9
(0,0)
(3,0)
x_1
$4x_1+3x_2 \le 12$
$3x_1+4x_2 \le 12$

(c) From the first functional constraint, we have $x_3 = 4 - 3x_1 - x_2$. Since $x_3 \geq 0$, we must have $3x_1 + x_2 \leq 4$.

Substituting $x_3 = 4 - 3x_1 - x_2$ into the objective function, we get $x_2 + 4$.

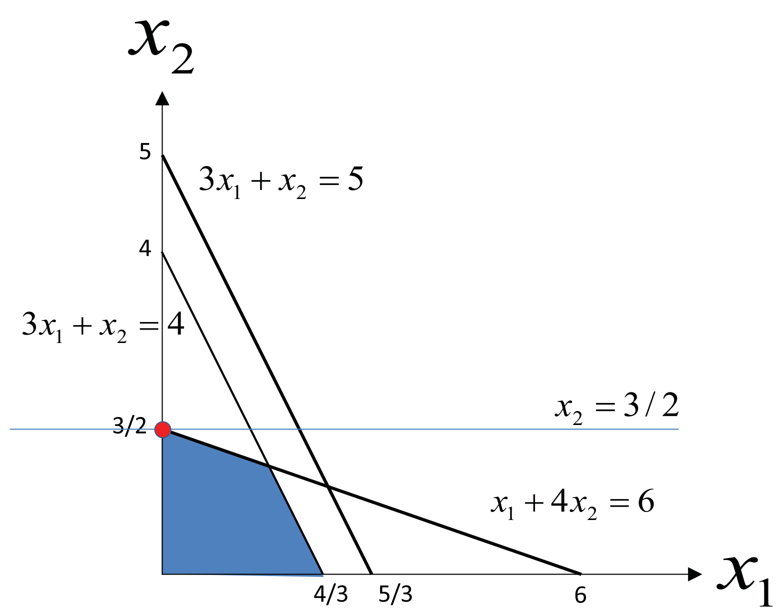
We can rewrite the LP as:

$$\max x_2 + 4$$

such that

$$\begin{aligned} 3x_1 + x_2 &\leq 5 \\ x_1 + 4x_2 &\leq 6 \\ 3x_1 + x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The new LP's feasible region is shown below:



From the sketch, the optimal solution is $x_1 = 0$ and $x_2 = \frac{3}{2}$ and the objective value is $\frac{3}{2} + 4 = \frac{11}{2} = 5.5$. The optimal value of $x_3 = 4 - 3(0) - \frac{3}{2} = \frac{5}{2}$.

So the optimal solution to the original LP is $(x_1, x_2, x_3) = (0, \frac{3}{2}, \frac{5}{2})$ with objective value of 5.5.

3. **Transportation problem**

Kantorovich, Hitchcock, Koopman and Monge all independently formulated versions of the transportation problem during WWII. Therefore I think it is apt to use a war themed problem:

You are a Field-Marshal with the job of distributing Instant Mash Potato mix to the camps you command. You have 4 of these. The mix can be sent from one of two distribution centres. You may send any weight of mix (you are allowed to open the bags and alter the quantity). You have been given two stipulations: (a) you must try to conserve fuel in the transportation of the mix and (b) you must send all of the mix you have at the two distributions centres. As you do not have any information regarding the cost of fuel and transportation, you may assume that the distance travelled is a sound estimate of cost of travel (i.e., x kg transported for y km has a cost of xy units). You have scribbled down approximate distances (in km) on a crude map. The orders are in the telegraph below:

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Hawk has 40kgs mash. Pigeon has 35kgs  mash.

G needs 15kg. B needs 20kg. R needs 5kg. Y needs 35kg.
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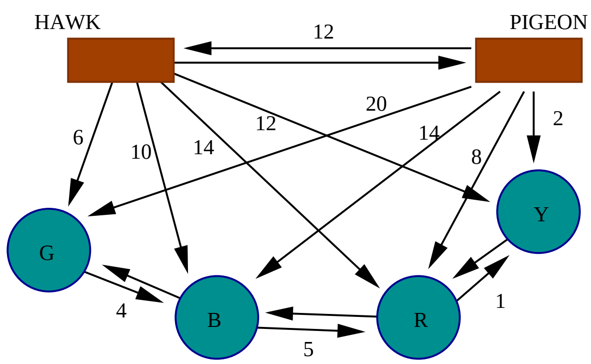

Assume that the mash mix can be delivered from a node u to another node v if an arc from u to v exists in the directed graph above. Formulate this problem as a linear program to determine the cheapest way to deliver the mash mix.

Solution:
We first prepare the distance matrix. The question says that the Field-Marshal has scribbled down approximate distances on a crude map. Note that the distances do not satisfy the triangle inequality. This could make sense if they are measured along roads, rather than in Euclidean space. The solution below uses distances that are given by the shortest route from each distribution centre to each camp.

	1. G	2. B	3. R	4. Y
1. Hawk	6	10	13 (via Y)	12
2. Pigeon	12 (via Y, R, B)	8 (via Y, R)	3 (via Y)	2

For $i = 1, 2$ and $j = 1, 2, 3, 4$, let the decision variables x_{ij} be the number of kgs of mix moved from distribution centre i to camp j. The linear program is:

$$\min \quad 6x_{1,1} + 10x_{1,2} + 13x_{1,3} + 12x_{1,4} + 12x_{2,1} + 8x_{2,2} + 3x_{2,3} + 2x_{2,4}$$

subject to

$$x_{1,1} + x_{2,1} \geq 15 \tag{1}$$
$$x_{1,2} + x_{2,2} \geq 20 \tag{2}$$
$$x_{1,3} + x_{2,3} \geq 5 \tag{3}$$
$$x_{1,4} + x_{2,4} \geq 35 \tag{4}$$
$$\sum_{j=1}^{4} x_{1,j} \leq 40 \tag{5}$$
$$\sum_{j=1}^{4} x_{2,j} \leq 35 \tag{6}$$
$$x_{i,j} \geq 0, \text{ for } i = 1, 2, j = 1, \ldots, 4. \tag{7}$$