MAT5OPT Solutions to Assignment 1, 2024

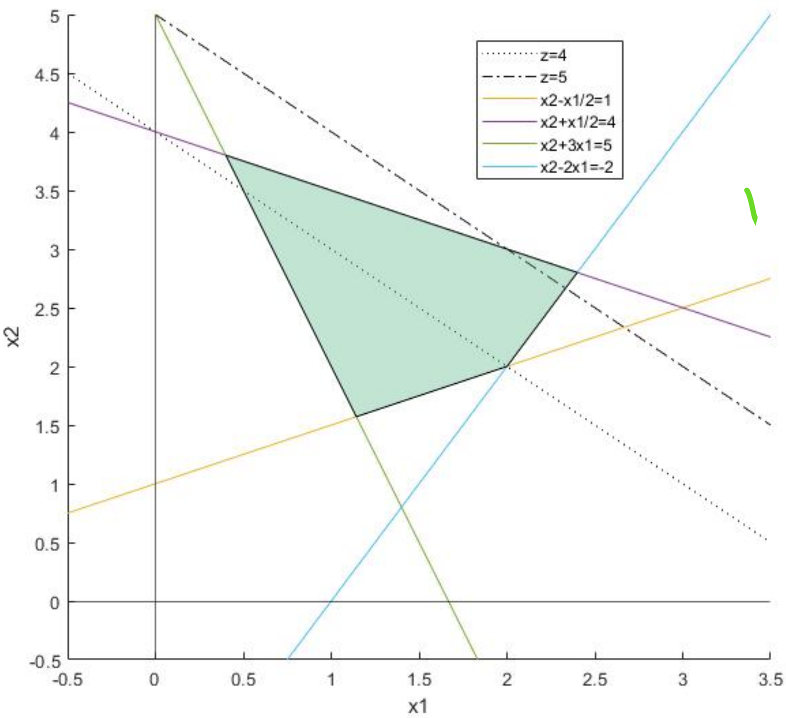
You can obtain 10 marks. For full marks you should explain your answers. This assignment is worth 10%. Please self-mark the unmarked questions!

1. *Wordy scenario.* Let x_1, x_2, x_3 respectively be the number of bottles of home made, cheap, and expensive wine. Then the wine-maker is advised to solve the following problem.

$$\begin{aligned}
\text{Minimise} \quad & 5x_1 + 4x_2 + 9x_3 \quad \text{(note that scaling doesn't matter)} \quad 2/5\\
\text{Subject to} \quad & x_1 + x_2 + x_3 \geq 2750 \quad 2/5\\
& \left.\begin{aligned} & x_1 \leq 1250 \\ & x_2 \leq 2000 \\ & x_3 \leq 800 \end{aligned}\right\} 2/5\\
& 5x_2 - x_1 - 2x_3 \leq 0 \quad \left(\text{which is equiv. to } \frac{80x_1 + 200x_2 + 60x_3}{x_1 + x_2 + x_3} < 100\right) \quad 2/5\\
& \mathbf{x} \geq \mathbf{0}. \quad 2/5
\end{aligned}$$

2

2. *Visual method.* The feasible region is bounded by the four constraints, not the axes. The corner points are (2,19)/5, (8,11)/7, (2,2), and 2(6,7)/5. The 4,5-level set are drawn, the maximum will be slightly larger than 5.



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Adding the equation $\frac{1}{2}x_1 + x_2 = 4$ to $2x_1 - x_2 = 2$ gives $x_1 = 12/5$. Then $x_2 = 2x_1 - 2 = 14/5$, and we obtain that the corner at which the maximum $x_1 + x_2 = \boxed{26/5}$ occurs is (12/5,14/5).

1/2 1/2

3. *MATLAB functions manipulating arrays.*

(a) `[1 2 3]^2` gives an error because MATLAB tries to matrix multiply. `[1 2 3].^2` will give `[1 4 9]`, MATLAB squares each component.

(b) `X=2+3*[1:9]` Note that this was unintentionally tricky as for the question to make sense the elements in the array are counting starting from 0 (which some languages do, but not MATLAB).

(c)
```
function y=F(x)
y=x;
y(2:2:end)=x(2:2:end).^2 + 1
end
```

2

(d)
```
function y=D(x)
l=length(x);
y=zeros(1,l-1)
    for i=1:l-1
        y(i)=x(i+1)-x(i);
    end
end
```

1

3/4 1/4

(e) With `A=@(x) sum(x(mod(x,12)==0))` we find `A(D(F(X)))=-384` Note, if your answer to (b) was `X=5+3*[1:9]` then `A(D(F(X)))=480`.

4. Definiteness.

1. The quadratic form corresponding to $\mathbf{A}$ is

$$A(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + x_3^3.$$

2. If $\mathbf{x}^T = \begin{pmatrix} 0 & a & 1 \end{pmatrix}$ then $A(\mathbf{x}) = 1 + 2a$. So a can be chosen so that the quadratic form $A(\mathbf{x})$ is either positive or negative. Therefore, the matrix $\mathbf{A}$ is indefinite. 1

3. As $x_3 - x_1 + 2x_2 + = 0$, we have $x_3 = x_1 - 2x_2$. Then,

$$\begin{aligned} A(\mathbf{x}) = A(x_1, x_2, x_1 - 2x_2) &= x_1^2 + 2x_1x_2 + 2x_1(x_1 - 2x_2) + 2x_2(x_1 - 2x_2) + (x_1 - 2x_2)^2 \\ &= 4x_1^2 - 4x_1x_2 \\ &= 4x_1(x_1 - x_2). \end{aligned}$$ 1/2

2

1/2 This is positive for $(x_1, x_2) = (1, 0)$ but negative for $(x_1, x_2) = (1, 2)$, which makes A indefinite on the set $\{\mathbf{x} : x_3 - x_1 + 2x_2 = 0\}$. Note, one can parametrise the subspace differently, $A(\mathbf{x}) = 2x_1(x_1 + x_3) = 4(x_2 + x_3)(2x_2 + x_3)$.

5. Conditions for minimisers, an unconstrained problem.

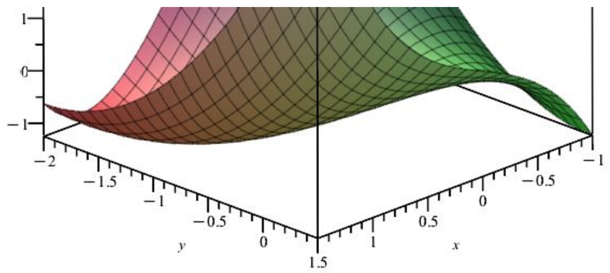
(a) $\nabla f = (3x_1^2 + 2x_2, 2x_1 + 2x_2 + 1)^T$ and $D^2 f = \begin{bmatrix} 6x_1 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) We need to solve the system $3x_1^2 + 2x_2 = 0, 2x_1 + 2x_2 + 1 = 0$. This gives $x_2 = -x_1 - 1/2$, and $3x_1^2 - 2x_1 - 1 = (3x_1 + 1)(x_1 - 1) = 0$, so all solutions are: $\mathbf{p} = (1, -3/2)$, $\mathbf{p} = (-1/3, -1/6)$. 1/2

(c) For $\mathbf{p} = (1, -3/2)$ we $D^2 f(\mathbf{p}) = \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}$. The principal minors are $\Delta_1 = 2$, $\Delta_2 = 8$. The matrix is positive definite, so $(1, -3/2)$ is a minimum. 1/2

For $\mathbf{p} = (-1/3, -1/6)$ we $D^2 f(\mathbf{p}) = \begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix}$. The principal minors are $\Delta_1 = -2$, $\Delta_2 = -8$. The matrix is indefinite, so $(-1/3, -1/6)$ is a saddle-point. 1/2

Note, this is consistent with the following plot of the function



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(d) The values of the function at the points $(1, -3/2)$, $(-1/3, -1/6)$ are $-5/4$, $-7/108$ respectively. A **global minimiser** of f over Ω is a feasible vector $\mathbf{x}^*$ for which the value of the function is the smallest possible, i.e.,

1/2

$$(\forall \mathbf{x} \in \Omega)\ f(\mathbf{x}) \geq f(\mathbf{x}^*).$$

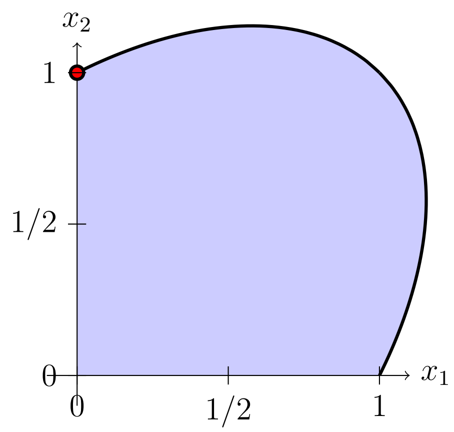
Since $f(-2, 1) = -10 < \min(-5/4, -7/108)$ the points are not global minimisers. Similarly, a **global maximiser** of f over Ω is a feasible vector $\mathbf{x}^*$ for which the value of the function is the largest possible, i.e.,

$$(\forall \mathbf{x} \in \Omega)\ f(\mathbf{x}) \leq f(\mathbf{x}^*).$$

Because e.g. $f(0, 1) = 2 > \max(-5/4, -7/108)$, the points are not global maximisers.

6. *Conditions for minimisers, a constrained problem.*

(a) The feasible region is:



(b) Differentiating $x_1^2 - x_1x_2 + x_2^2 - 1 = 0$ with respect to x_1 we find $2x_1 - x_2 - x_1\frac{\mathrm{dx}_2}{\mathrm{dx}_1} + 2x_2\frac{\mathrm{dx}_2}{\mathrm{dx}_1} = 0$, so that the slope of the tangent line to this constraint, at $\mathbf{x}^* = (0, 1)$, is

$$\frac{\mathrm{dx}_2}{\mathrm{dx}_1} = \frac{x_2 - 2x_1}{2x_2 - x_1}\bigg|_{\mathbf{x}^*=(0,1)} = \frac{1}{2}.$$

Therefore the set of feasible directions is $\{\mathbf{d} : d_1 \geq 0, d_2 \ < \frac{1}{2}d_1\}$.

(c) (i) The set of feasible directions of length 1 is not closed we cannot use Theorem 5 (FOSC)

(ii) We have that $\nabla(\mathbf{x}^*)^T\mathbf{d} = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}d_1 \\ d_2\end{pmatrix} = d_1 - d_2 \geq d_1 - \frac{1}{2}d_1 = \frac{1}{2}d_1 \geq 0$. So the FONC is not not satisfied.

(iii) As the FONC is satisfied, $\mathbf{x}^*$ is possibly a minimiser.

(d) Yes, Conjecture 1 (FOSC) would (if it were true) imply that $\mathbf{x}^*$ *is* a minimiser.