Regular points

Given a function $\mathbf{h}\colon \mathbb{R}^n \to \mathbb{R}^m$, and an assocated level set

$$\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{h}(\mathbf{x}) = \mathbf{c}\},$$

a point $\mathbf{x} \in \mathcal{H}$ is regular if the set

$$\{\nabla h_1(\mathbf{p}), \nabla h_2(\mathbf{p}), \ldots, \nabla h_n(\mathbf{p})\}$$

is linearly independent.

1. For each of the functions $h_i\colon \mathbb{R}^2 \to \mathbb{R}$ below, sketch the level set that contains the point $\begin{pmatrix}1 & 1\end{pmatrix}^T$, and decide whether $\begin{pmatrix}1 & 1\end{pmatrix}^T$ is a regular point.
   (a) $h_1(x_1, x_2) = x_1 - x_2$
   (b) $h_2(x_1, x_2) = x_1^2 - x_2^2$
   (c) $h_3(x_1, x_2) = (x_1 - x_2)^2$

The method of Lagrange

Given $f\colon \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{h}\colon \mathbb{R}^n \to \mathbb{R}^k$, the Lagrangian of f and $\mathbf{h}$ is the function $\mathcal{L}$ defined by

$$\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T h(\mathbf{x}).$$

Lagrange's multiplier theorem then states that if $\mathbf{x}^*$ is a minimiser of $f(\mathbf{x})$ subject to $\mathbf{h}(\mathbf{x}) = \mathbf{0}^T$ and $\mathbf{x}^*$ is a regular point of $\mathbf{h}$, then there exists $\boldsymbol{\lambda}$ such that $D\mathcal{L}(\mathbf{x}^*; \boldsymbol{\lambda}) = \mathbf{0}^T$, where the derivatives are with respect to $\mathbf{x}$ and $\boldsymbol{\lambda}$; i.e.,

$$D\mathcal{L}(\mathbf{x}; \lambda) = \begin{pmatrix} \frac{\partial \mathcal{L}}{\partial x_1} & \cdots & \frac{\partial \mathcal{L}}{\partial x_n} & \frac{\partial \mathcal{L}}{\partial \lambda_1} & \cdots & \frac{\partial \mathcal{L}}{\partial \lambda_k} \end{pmatrix}$$

Thus, to find candidates for local minimisers, state the Lagrangian of f and $\mathbf{h}$ and then solve $D\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda}) = \mathbf{0}^T$ for $\mathbf{x}$ and $\boldsymbol{\lambda}$. The SOSC can then be used to deduce the status of those points:

- If $D\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda}) = \mathbf{0}^T$ and $D^2\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda})$ is positive definite on $T\mathcal{H}(\mathbf{x}^*)$, then $\mathbf{x}^*$ is a strict local minimiser.
- If $D\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda}) = \mathbf{0}^T$ and $D^2\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda})$ is negative definite on $T\mathcal{H}(\mathbf{x}^*)$, then $\mathbf{x}^*$ is a strict local maximiser.

For this condition, the derivatives are with respect to $\mathbf{x}$ only; i.e.,

$$D^2\mathcal{L}(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \cdots & \frac{\partial \mathcal{L}}{\partial x_n \partial x_1} \\ \vdots & & \vdots \\ \frac{\partial^2 \mathcal{L}}{\partial x_1 x_n} & \cdots & \frac{\partial \mathcal{L}}{\partial x_n^2} \end{pmatrix} (\mathbf{x})$$

2. Consider the nonlinear optimisation problem

$$\begin{aligned} \text{minimise} \quad & f(\mathbf{x}) = x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3 \\ \text{subject to} \quad & h_1(\mathbf{x}) = x_1 + 2x_2 = 3, \\ & h_2(\mathbf{x}) = 4x_1 + 5x_3 = 6. \end{aligned}$$

   Let $\mathbf{h}(\mathbf{x}) = \begin{pmatrix} h_1(\mathbf{x}) - 3 & h_2(\mathbf{x}) - 6 \end{pmatrix}^T$ and let $\mathcal{H}$ denote the feasible set.
   (a) Write down $\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda})$, the Lagrangian of f and $\mathbf{h}$.
   (b) Find the unique point $\mathbf{x} = \mathbf{x}^*$ in the feasible set for which $D\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda}) = \mathbf{0}^T$.
   (c) Show that $\mathbf{x}^*$ is regular.
   (d) Determine the tangent space of $\mathcal{H}$ at $\mathbf{x}^*$.
   (e) Show that $D^2\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda})$ is positive definite on the tangent space $T\mathcal{H}(\mathbf{x}^*)$. What do you conclude?

3. Consider the nonlinear optimisation problem

$$\begin{aligned} \text{maximise} \quad & f(\mathbf{x}) = 4x_1 + x_2^2 \\ \text{subject to} \quad & h(\mathbf{x}) = x_1^2 + x_2^2 = 9. \end{aligned}$$

   Find, and classify, all local extremisers.