# MAST30013 Assignment 3 2021 Michael Le LaTex

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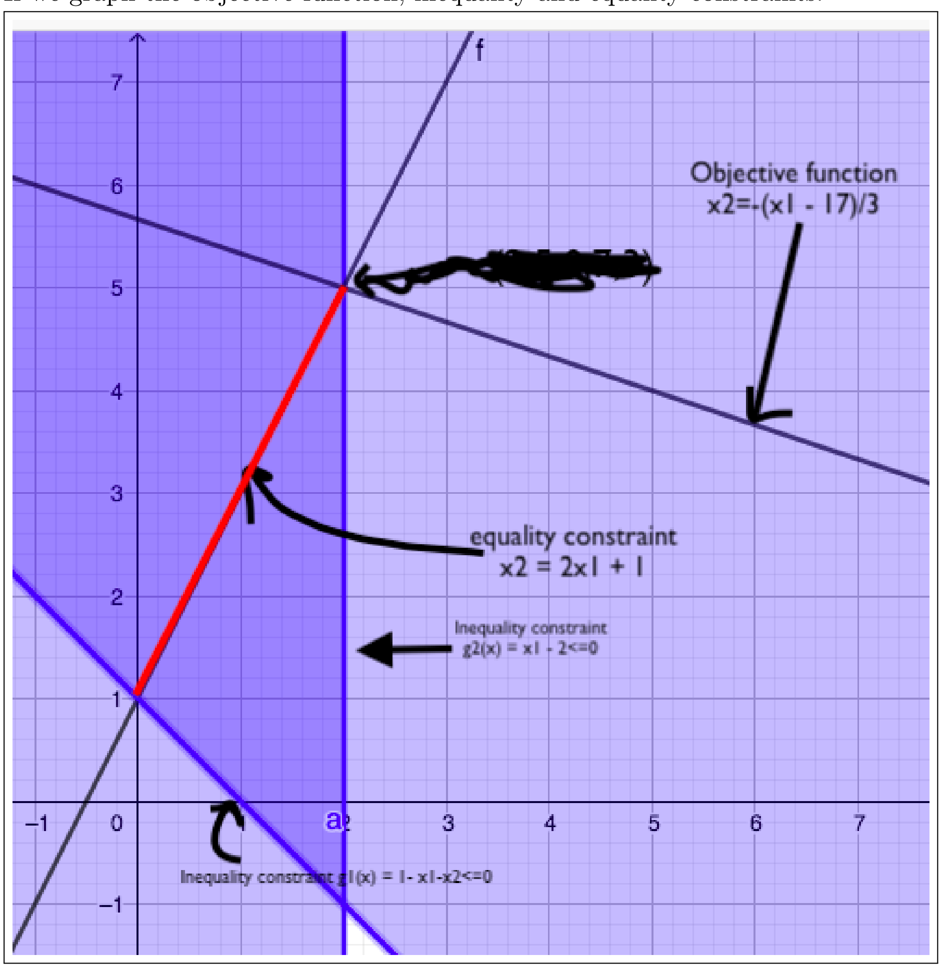
***Question* 1 :**

Part a:

If we graph the objective function, inequality and equality constraints.



We could see that the line $x_2 = 2x_1 + 1$ is optimal within the domain x $\in$ [0,2]. Assume we know from DMOR where we are changing the f(x) value, (i.e. the y value).

Part b:

In standard form,

min $-x_1$—-3x$_2$

s.t. $g_1(x) =$ 1 - $x_1$ - $x_2 \leq 0$

$g_2(x) = x_1$ - 2 $\leq 0$

h(x) $= x_2$ - 2$x_1$ - 1 $= 0$

Part c:

Finding the KKT Conditions:

The Lagrangian,

$$L(x, \lambda_i, \eta_j) = f(x) + \sum_{i=1}^{\infty} \lambda_i g_i(x) + \sum_{j=1}^{\infty} \eta_j h_j(x)$$

$$= -x_1 - 3x_2 + \lambda_1(1 - x_1 - x_2) + \lambda_2(-2 + x_1) + \eta(x_2 - 2x_1 - 1)$$

Take the derivative with respect to $x_1$ and $x_2$ respectively for each row.

$$\nabla L_x(x, \lambda_i, \eta_j) = \begin{bmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 - \lambda_1 + \lambda_2 - 2\eta \\ -3 + \eta - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$KKT_a$ conditions:,

-1 -$\lambda_1$ + $\lambda_2$ - 2$\eta$ $= 0$ ***Equation 1***

-3 -$\lambda_1$ + $\eta$ $= 0$ ***Equation 2***

$KKT_b$ conditions states:,

$g_i(x^*) \leq 0$, $\lambda_i^* \geq 0$, then $\lambda_i g_i(x^*) = 0$

$\lambda_1(1 - x_1 - x_2) = 0$ ***Equation 3***

$\lambda_2(-2 + x_1) = 0$ ***Equation 4***

Lastly the $KKT_c$ condition:

$x_2 - 2x_1 - 1 = 0$ ***Equation 5***

Part d:

***Case 1 :*** $\boldsymbol{\lambda_1 > 0}$ **,** $\boldsymbol{\lambda_2 > 0}$

From $KKT_b$ using ***Equation 3***

and solving $x_2$ from ***Equation* 4**.
$x_1 = 2$,
Substitute into ***Equation* 3**
$(1 - 2 - x_2) = 0$
$x_2 =$ -1
But, substituting into $x_1$ and $x_2$ into the $KKT_c$ in ***Equation* 5** condition gives us,
-1-2(2)-1 $=$ -6 $\neq = 0$
This is a contradiction, violates $KKT_c$

***Case* 2 : $\boldsymbol{\lambda_1 = 0}$ , $\boldsymbol{\lambda_2 > 0}$**
From $KKT_a$
Substituting $\lambda_1 = 0$ into ***Equation* 1**
and ***Equation* 2**
-1 -0 $+ \lambda_2$ - $2\eta = 0$ ***Equation* 1**
-3 -0 $+ \eta = 0$ ***Equation* 2**
$\eta = 3$
Substitute $\eta = 3$ into ***Equation* 1**
-1 -0 $+ \lambda_2$ - 2x3$=$ 0
-1 -0 $+ \lambda_2$ - 6 $=$ 0
$\lambda_2 = 7$
Now were now solving $x_1$ and $x_2$ from $KKT_b$ and $KKT_c$ conditions:
$x_1 = 2$ ***Equation* 4**
Substitute $x_1 = 2$ into ***Equation* 5**
$x_2 - 2x2 - 1 = 0$
$x_2 - 4 - 1 = 0$
$x_2$= 5
($x^*$, $\lambda^*$) $=$ (2,5,0,7,3)

***Case* 3 : $\boldsymbol{\lambda_1 > 0}$ , $\boldsymbol{\lambda_2 = 0}$**
$(1 - x_1 - x_2) = 0$ ***Equation* 3**
$x_2 - 2x_1 - 1 = 0$ ***Equation* 5**
Solving these simultaneous from ***Equation* 3**
and ***Equation* 5**
gives us,
$x_1 = 0$ and $x_2 = 1$.
Now we need to solve for ***Equation* 1**
and ***Equation* 2**.
-1 -$\lambda_1$ $+$ 0 - $2\eta = 0$ ***Equation* 1**
-3 -$\lambda_1$ $+ \eta = 0$ ***Equation* 2**

solving the simultaneous equations gives us,
$\eta = \frac{2}{3}$ and $\lambda_1 = \frac{-7}{3}$
However we stated that $\lambda_1 > 0$,
This violates $KKT_b$

$\boldsymbol{Case\ 4 : \lambda_1 = 0\ , \lambda_2 = 0}$
from $\boldsymbol{Equation\ 5}$
$x_2 = 2x_1 + 1$
where $x_1$ and $x_2$ are taken as arbitrary values
What about $KKT_a$?
-1-2$\eta = 0$ $\boldsymbol{Equation\ 1}$
and, -3+$\eta = 0$
$\eta$ has two different values $\frac{-1}{2}$ and 3.
Both values do not equal each other, violates $KKT_a$
Overall, there is only 1 KKT point (2,5,0,7,3) that should be tested for optimally.

***Question* 2 :**

Part a:

min $\frac{(x_1-2)^4}{4}$ + $x_2^4$ + 4

s.t. $x_1$- $x_2$ $\leq$ 8

$x_1$- $x_2^2$ $\geq$ 4

Convert into standard form,

min $\frac{(x_1-2)^4}{4}$ + $x_2^4$ + 4

s.t. $g_1(x)$ = $x_1$- $x_2$ - 8 $\leq$ 0

$g_2(x)$ = 4-$x_1$+ $x_2^2$ $\leq$ 0

Compute the Lagrangian,

$$L(x,\lambda_i,\eta_j) = f(x) + \sum_{i=1}^{\infty} \lambda_i g_i(x) + \sum_{j=1}^{\infty} \eta_j h_j(x)$$

$$= -\frac{(x_1-2)^4}{4} + x_2^4 + 4 + \lambda_1(x_1 - x_2 - 8) + \lambda_2(4 - x_1 + x_2^2)$$

Computing the KKT Conditions:

$$\nabla L_x(x,\lambda_i,\eta_j) = \begin{bmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} (x_1-2)^3 + \lambda_1 - \lambda_2 \\ 4x_2^3 - \lambda_1 + 2\lambda_2 x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$KKT_a$ conditions:

***Equation* 1**

$$(x_1-2)^3 + \lambda_1 - \lambda_2 = 0$$

***Equation* 2**

$$4x_2^3 - \lambda_1 + 2\lambda_2 x_2 = 0$$

$KKT_b$ conditions:

$g_i(x^*) \leq 0$, $\lambda_i^* \geq 0$, then $\lambda_i g_i(x^*) = 0$

***Equation* 3**

$$\lambda_1(x_1 - x_2 - 8) = 0$$

***Equation* 4**

$$\lambda_2(4 - x_1 + x_2^2) = 0$$

***Case* $\boldsymbol{1 : \lambda_1 = 0 \ , \lambda_2 = 0}$**
From ***Equation* 1**
and ***Equation* 2**
$(x_1 - 2)^3 = 0$
$4x_2^3 = 0$
Solving both simultaneous equations gives us,
$x_1 = 2$ and $x_2 = 0$
gives us a solution (2,0,0,0).
But it fails under $g_2 x$ constraint, thus this point is infeasible.

***Case* $\boldsymbol{2 : \lambda_1 = 0 \ , \lambda_2 > 0}$**
Substitute for $\lambda_1 = 0$ into ***Equations* 1, 2, 3, 4**
which gives us three simultaneous equations.

$$(x_1 - 2)^3 + 0 - \lambda_2 = 0 \ (1)$$

$$4x_2^3 - 0 + 2\lambda_2 x_2 = 0 \ (2)$$

$$4 - x_1 + x_2^2 = 0 \ (3)$$

We first solve for $x_1$ in (3).
$x_1 = 4 + x_2^2$
In this case there are two scenarios to overlook if there exists an optimal point that satisfies all the KKT conditions.

<u>Scenario 1:</u>
Substitute $x_1$ into (1).
$(x_2^2 + 2)^3$ - $\lambda_2 = 0$
$\lambda_2 = (x_2^2 + 2)^3$
But, solving (2) for $\lambda_2$ gives us.
$\lambda_2 =$ -$2x_2^2$
In which $\lambda_2 > 0$, violates $KKT_b$.

<u>Scenario 2:</u>
Examining at (2) closely, we only care that $x_1$ and $x_2$ are real numbers.
Rearranging the equation take out $2x_2$ factor out shows us.
$2x_2$(x) ($2x_2^2$ + $\lambda_2$) = 0
One solution for $x_2$ is 0, $x_2 = 0$.
Substitute $x_2$ into (3) gives us,
4-$x_1$+ $0^2 = 0$,
$x_1 = 4$,
Substitute $x_1$ into (1) gives us,
$(4 - 2)^3$ + 0 - $\lambda_2 = 0$

$\lambda_2 = 8$
Therefore our solution is (4,0,0,8)
***Case* $\boldsymbol{3 : \lambda_1 > 0 \ , \lambda_2 = 0}$**
Substitute for $\lambda_2 = 0$ into ***Equations* $\boldsymbol{1,2,3,4}$**
which gives us three simultaneous equations.

$$x_1 - x_2 - 8 = 0 \ (1)$$

$$(x_1 - 2)^3 + \lambda_1 = 0 \ (2)$$

$$4x_2^3 - \lambda_1 = 0 \ (3)$$

Solving $x_1$ in (1)
$x_1 = x_2 + 8$
Substitute $x_1$ into (2).
$(x_2 + 6)^3 + \lambda_1 = 0$
Solving $\lambda_1$ in (3)
$\lambda_1 = 4x_2^3$
Substitute $\lambda_1$ into (2)
$(x_2 + 6)^3 + 4x_2^3 = 0$
Using the solver, we solve that $x_2 =$ -2.3189.
Substitute $x_2$ back into (1).
$x_1$ - (-2.3189) - 8 = 0
Solving $x_1$ gives us,
$x_1 = 5.6811$
$\lambda_1 = 4(x_2)^3 =$ -49.877 $< 0$
But $\lambda_1 > 0$, violates $KKT_b$

***Case* $\boldsymbol{4 : \lambda_1 > 0 \ , \lambda_2 > 0}$**

$$(x_1 - 2)^3 + \lambda_1 - \lambda_2 = 0 \ (1)$$

$$4x_2^3 - \lambda_1 + 2\lambda_2 x_2 = 0 \ (2)$$

$$x_1 - x_2 - 8 = 0 \ (3)$$

$$4 - x_1 + x_2^2 = 0 \ (4)$$

Solving $x_1$ in (3) gives us,
$x_1 = x_2 + 8$
Substitute $x_1$ into (4)

4-($x_2$ + 8)+ $x_2^2 = 0$
4 - $x_2$ - 8 + $x_2^2 = 0$
$x_2^2$ - $x_2$ -4 = 0
Using the quadratic formula,
$x_2 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2*1}$
Gives us,
$x_2 = \frac{1 \pm \sqrt{17}}{2}$
In this case there are two scenarios to overlook if there exists an optimal point that satisfies all the KKT conditions.

<u>Scenario 1:</u>
$x_2 = \frac{1+\sqrt{17}}{2}$
Substituting $x_2$ into (3) gives us,
$x_1$- ($\frac{1+\sqrt{17}}{2}$) - 8 = 0
$x_1$= ($\frac{17+\sqrt{17}}{2}$)
Now we can substitute $x_1$ and $x_2$ into (1) and (2).
$((\frac{17+\sqrt{17}}{2}) - 2)^3 + \lambda_1$ - $\lambda_2 = 0$
$\lambda_1$ - $\lambda_2$ = -($\frac{17+\sqrt{17}}{2}) - 2)^3$
$\lambda_1$ - $\lambda_2$ = -($\frac{13+\sqrt{17}}{2}))^3$ (1)
$4(\frac{1+\sqrt{17}}{2})^3$ - $\lambda_1 + 2\lambda_2(\frac{17+\sqrt{17}}{2}) = 0$
- $\lambda_1 + 2\lambda_2(\frac{17+\sqrt{17}}{2})$ = -4$(\frac{1+\sqrt{17}}{2})^3$ (2)
solving $\lambda_1$ and $\lambda_2$.
gives us, $\lambda_1$ = -796.06 and $\lambda_2$ = -168.5
Which both lambda's violate the $KKT_b$ conditions.

<u>Scenario 2:</u>
$x_2 = \frac{1-\sqrt{17}}{2}$
Substituting $x_2$ into (3) gives us,
$x_1$- ($\frac{1-\sqrt{17}}{2}$) - 8 = 0
$x_1$= ($\frac{17-\sqrt{17}}{2}$)
Now we can substitute $x_1$ and $x_2$ into (1) and (2).
$((\frac{17-\sqrt{17}}{2}) - 2)^3 + \lambda_1$ - $\lambda_2 = 0$
$\lambda_1$ - $\lambda_2$ = -($\frac{17-\sqrt{17}}{2}) - 2)^3$
$\lambda_1$ - $\lambda_2$ = -($\frac{13-\sqrt{17}}{2}))^3$ (1)
$4(\frac{1-\sqrt{17}}{2})^3$ - $\lambda_1 + 2\lambda_2(\frac{1-\sqrt{17}}{2}) = 0$
- $\lambda_1 + 2\lambda_2(\frac{17-\sqrt{17}}{2})$ = -4$(\frac{1-\sqrt{17}}{2})^3$ (2)
solving $\lambda_1$ and $\lambda_2$.
gives us, $\lambda_1$ = -69.924 and $\lambda_2$ = 17.512
But $\lambda_1 > 0$, violates $KKT_b$.

Overall, the optimal point is (4,0,0,8).

Part b:

The constraint qualifications hold at 1 KKT point (4,0,0,8) since all two points are affine.

Part c:

We need to check the second order condition at the KKT point at (4,0,0,8).

The active constraint are $g_1 = 4\text{-}x_1 + x_2^2 \leq 0$

The critical cone is,

$C(x^*, \lambda^*) = \{d \in R^2 : \nabla g_2 \ (4,0)^T d = 0\}$

$= \{d \in R^2 : (-1 \ 0) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0\}$

$= \{d \in R^2 : \ -d_1 = 0\}$

$= \{(d_1, d_2) \in R^2 : \ d_1 = 0\}$

The Hessian of the Lagrangian is,

$\nabla^2 L_{xx}(x, \lambda_i, \eta_j) = \begin{bmatrix} 3(x_1 - 2)^2 & 0 \\ 0 & 12(x_2)^2 + 12\lambda_2 \end{bmatrix}$

Substituting our point (4,0,0,8) into the Hessian.

$\nabla^2 L_{xx}((4,0),(0,8)) = \begin{bmatrix} 48 & 0 \\ 0 & 16 \end{bmatrix}$

Now for $d \in C(x^*, \lambda^*)$.

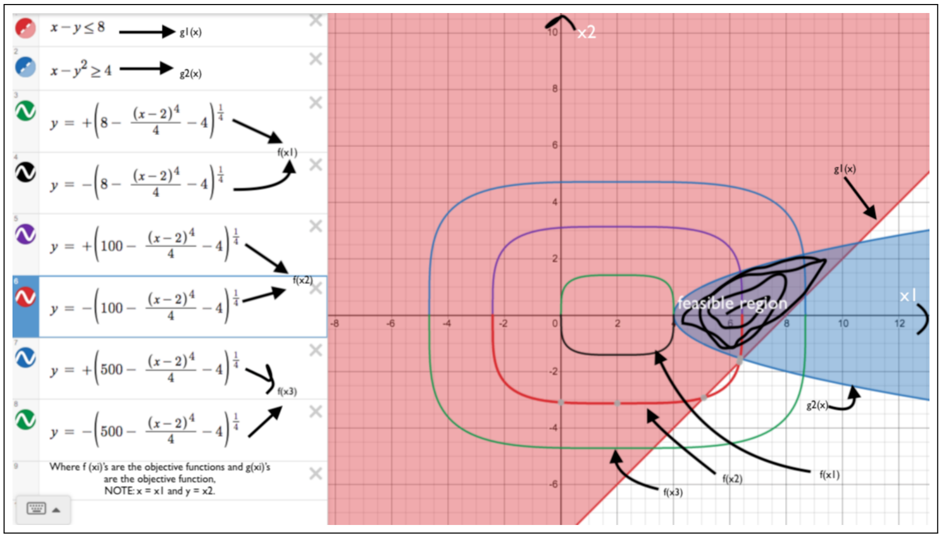
$\begin{bmatrix} 0 & d_2 \end{bmatrix} \begin{bmatrix} 48 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ d_2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 16d_2 \end{bmatrix} \begin{bmatrix} 0 \\ d_2 \end{bmatrix}$

$= 16d_2^2 > 0$

Thus, $\nabla^2 L_{xx}(x, \lambda_i, \eta_j)$ is positive definite on the critical cone. Therefore, $x^*$ = (4,0) is a local minimum. However, since the constraint set is closed and bounded, $x^*$ = (4,0) is a global minimum.

Part d:




At f(x) = 8, there is one mini-miser with 1 active constraint at point (4,0,0,8) where $\lambda_1$ = 0 and $\lambda_2 > 0$. Which we solved earlier that there is 1 optimal point in Part a.

Part e:

May change this later!

Following the Lemma (Convex Functions) or the Corollary (Convexity of quadratic function) from the lectures,

f(x) = $\frac{(x_1-2)^4}{4}$ + $x_2^4$ + 4

$\nabla f(x^*)$= $\begin{bmatrix} (\mathrm{x}_1 - 2)^3 \\ 4(\mathrm{x}_2)^3 \end{bmatrix}$

$\nabla^2 f(x^*)$= $\begin{bmatrix} 3(\mathrm{x}_1 - 2)^2 & 0 \\ 0 & 12(\mathrm{x}_2)^2 \end{bmatrix}$

Since it is an even function we can choose our arbitrary values for instance $x_1$ = 4 and $x_2$ = 0.

The Hessian of the objective function is symmetric and convex in the constraint set.

***END OF ASSIGNMENT.***