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MAT4MDS **Practice 11**

GAMMA and BETA FUNCTIONS

The Gamma Function:

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$$

It has the properties:

$$\Gamma(x+1) = x\Gamma(x) \qquad \Gamma(n+1) = n! \text{ for } n \in \mathbb{N}$$

Special values:

$$\Gamma(1) = 1 \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Question 1.

(a) Find $\Gamma(\frac{5}{2})$. (b) Find $\Gamma(-\frac{1}{2})$. (c) Write $10{\cdot}8{\cdot}6{\cdot}4{\cdot}2$ in terms of the factorial function.

(d) Write the product $9{\cdot}7{\cdot}5{\cdot}3{\cdot}1$ in terms of the Gamma function.
(**Hint:** There are two different ways to do this. You need to re-write the product so that the factors differ from each other by 1. Then you can use $x\Gamma(x) = \Gamma(x+1)$ repeatedly.)

(e) Hence write the product $1{\cdot}3{\cdot}5\ldots(2k-1)(2k+1)$ in terms of the Gamma function.

Question 2. Using the substitution $t = u^2$, show that another way to represent $\Gamma(x)$ is

$$\Gamma(x) = 2\int_0^\infty e^{-u^2}u^{2x-1}du.$$

Question 3. Using the substitution $u = e^{-t}$, show that another way to represent $\Gamma(x)$ is

$$\Gamma(x) = \int_0^1 \left[\log(\tfrac{1}{u})\right]^{x-1} du.$$

The Beta Function

$$B(p,q) = \int_0^1 y^{p-1}(1-y)^{q-1}dy = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Question 4. Show, using properties of the Gamma Function, that

$$B(p,q+1) + B(p+1,q) = B(p,q).$$

P

The Beta Distribution is a two-parameter family of distributions that has probability density function

$$f(y) = \begin{cases} \dfrac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} & 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

α and β are both positive shape parameters.

Question 5. In Reading 11.2 it was shown that the mean of the Beta Distribution is $\dfrac{\alpha}{\alpha+\beta}$.

(a) Find $\displaystyle\int_{-\infty}^{\infty} y^2 f(y)dy$.

(b) Hence show that the variance is $\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

Question 6. The mode for a random variable occurs where its probability density function has a (global) maximum. Use differentiation to find the stationary point of $f(y)$ for the Beta Distribution. For what values of α and β is this a maximum?

Question 7. Show that with an appropriate substitution (which you must find)

$$B(p,q) = \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} du.$$

The Gamma Distribution is a two-parameter family of distributions that has probability density function

$$f(y) = \begin{cases} \dfrac{\beta^{\alpha} y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

α and β are both positive parameters.

Question 8. Find the mode of the Gamma Distribution. For what values of the parameters does this apply?

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The cumulative distribution function F (CDF) is an anti-derivative of the probability density function f (PDF) for continuous data. That is:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

Question 9. The following function is to be used as a PDF of a continuous random variable:

$$f(x) = \begin{cases} 2-2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Plot $f(x)$.

(b) Following the steps below, find the associated CDF.

(i) Find $F(x)$ for , $x < 0$

(ii) Find $F(x)$ for $0 \leq x \leq 1$

(iii) Find $F(x)$ for $x > 1$ $\left(\text{Hint: for } x > 1 \text{ we have } F(x) = F(1) + \int_1^x f(t)dt \right)$.

Question 10. Consider the function $f(x, y) = x^2 e^{2y+x} - \dfrac{x}{y}$. Integrate $f(x, y)$ with respect to y between $y = 1$ and $y = 8$. That is, calculate

$$\int_1^8 f(x, y)dy.$$

Two (or more) random variables are jointly continuous when they have a **joint cumulative distribution function** which is continuous in both variables:

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s, t)dt\ ds$$

The function $f(x, y)$ is the **joint probability density function**.

Question 11. Consider the joint probability density function

$$f(x, y) = \begin{cases} 6e^{-2x}e^{-3y} & x \geqslant 0, y \geqslant 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the associated joint cumulative distribution function $F(x, y)$.

Question 12. Check that for the function found in Question 7:

- As $x \to -\infty$ or $y \to -\infty$, $F(x, y) \to 0$.
- As $x \to \infty$ and $y \to \infty$, $F(x, y) \to 1$.