STM4PSD – Workshop 1 Solutions

1. (a) Continuous
   (b) Discrete
   (c) Discrete
   (d) Continuous
   (e) Discrete
   (f) Continuous

2. (a) $\binom{10}{9} = \frac{10!}{9!\,1!} = \frac{10\times \cancel{9}\times \cancel{8}\times\ldots\times \cancel{2}\times \cancel{1}}{\cancel{9}\times \cancel{8}\times \cancel{7}\times\ldots\times \cancel{2}\times \cancel{1}} = 10.$
   (b) $\binom{10}{1} = \frac{10!}{1!\,9!} = \frac{10\times \cancel{9}\times \cancel{8}\times\ldots\times \cancel{2}\times \cancel{1}}{\cancel{9}\times \cancel{8}\times \cancel{7}\times\ldots\times \cancel{2}\times \cancel{1}} = 10.$
   (c) $\binom{9}{4} = \frac{9!}{4!\,5!} = \frac{9\times 8\times 7\times 6\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}}{4\times 3\times 2\times 1\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times \cancel{1}} = \frac{9\times 8\times 7\times 6}{4\times 6} = 9\times 2\times 7 = 126$
   (d) $\binom{11}{8} = \frac{11!}{8!\,3!} = \frac{11\times 10\times 9\times \cancel{8}\times \cancel{7}\times\ldots\times \cancel{2}\times \cancel{1}}{\cancel{8}\times \cancel{7}\times\ldots\times \cancel{2}\times \cancel{1}\times 3\times 2\times 1} = \frac{11\times 10\times 9}{3\times 2} = 11\times 5\times 3 = 165$

3. Since there are 23 seats to choose from, and 18 of them will be chosen, the total number of seating arrangements is

$$\binom{23}{18} = \frac{23!}{18!\,(23-18)!} = \frac{23!}{18!\,5!} = \frac{23\times 22\times 21\times 20\times 19}{5\times 4\times 3\times 2\times 1} = 33649$$

4. (a)

x	1	2	3	4
$P(X=x)$	1/4	1/4	1/4	1/4

   (b) i. $P(X=3) = \frac{1}{4}$
   ii. $P(X\neq 3) = 1 - P(X=3) = \frac{3}{4}$
   iii. $P(X\leqslant 3) = P(X=1)+P(X=2)+P(X=3) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{3}{4}$
   iv. $P(X<3) = P(X=1)+P(X=2) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$
   v. $P(X\geqslant 2) = P(X=2)+P(X=3)+P(X=4) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{3}{4}$
   (c) $E(X) = \frac{1}{4}\times 1 + \frac{1}{4}\times 2 + \frac{1}{4}\times 3 + \frac{1}{4}\times 4 = \frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4} = \frac{10}{4} = \frac{5}{2}.$
   (d) $\mathrm{Var}(X) = \frac{1}{4}(1-\frac{5}{2})^2 + \frac{1}{4}(2-\frac{5}{2})^2 + \frac{1}{4}(3-\frac{5}{2})^2 + \frac{1}{4}(4-\frac{5}{2})^2 = 1.25$
   $\mathrm{SD}(X) = \sqrt{1.25} = 1.12$ (to 2 decimal places).

5. (a)
$$\begin{aligned}\Omega_Y = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$
   (b) Your intuition may vary, so there is no one correct answer here.
   (c) $A = \{(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}$
   $B = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$
   $C = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}$
   Since A has 10 elements, it follows that $P(A) = \frac{10}{36} = \frac{5}{18}$.
   Since B has 6 elements, it follows that $P(B) = \frac{6}{36} = \frac{1}{6}$.
   Since C has 9 elements, it follows that $P(C) = \frac{9}{36} = \frac{1}{4}$.
   From this, we can see that A is most likely, and B is least likely.

(d) i. $A \cup B = \{(1,1),(2,2),(3,3),(3,6),(4,4),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}$
Since $A \cup B$ has 14 elements, it follows that $P(A \cup B) = \frac{14}{36}$

ii. $A \cap B = \{(5,5),(6,6)\}$.
Since $A \cap B$ has 2 elements, it follows that $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$.

iii. $B \cap C = \{(1,1),(3,3),(5,5)\}$
Since $B \cap C$ has 3 elements, it follows that $P(B \cap C) = \frac{3}{36} = \frac{1}{12}$

iv. $(A \cup B) \cap C = \{(1,1),(3,3),(5,5)\}$
Observe that this is the same as $B \cap C$, so it follows that $P\big((A \cup B) \cap C\big) = P(B \cap C) = \frac{1}{12}$.

6. (a) Here we want the possible *sums* of the two rolls.
Hence, $\Omega_Z = \{2,3,4,5,6,7,8,9,10,11,12\}$

(b) To see how this table was obtained, consider a few examples:

- There is only 1 way out of 36 to obtain a sum of 2 (from $(1,1)$). So $P(Z=2) = \frac{1}{36}$.
- There are 2 ways out of 36 to obtain a sum of 3 (from $(1,2)$ and $(2,1)$). So $P(Z=3) = \frac{2}{36} = \frac{1}{18}$.
- There are 5 ways out of 36 to obtain a sum of 6 (from $(1,5),(2,4),(3,3),(4,2)$ and $(5,1)$). So $P(Z=6) = \frac{5}{36}$.

x	2	3	4	5	6	7	8	9	10	11	12
$P(Z=x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

(c) $P(A) = P(Z \geqslant 9) = P(Z=9) + P(Z=10) + P(Z=11) + P(Z=12) = \frac{1}{9} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} = \frac{5}{18}$.

(d) $$E(Z) = \frac{2}{36} + \frac{3}{18} + \frac{4}{12} + \frac{5}{9} + \frac{30}{36} + \frac{7}{36} + \frac{40}{36} + \frac{9}{9} + \frac{10}{12} + \frac{11}{18} + \frac{12}{36} = 7$$
$$\begin{aligned}\mathrm{Var}(Z) &= \tfrac{1}{36}(2-7)^2 + \tfrac{1}{18}(3-7)^2 + \tfrac{1}{12}(4-7)^2 + \tfrac{1}{9}(5-7)^2 + \tfrac{5}{36}(6-7)^2 + \tfrac{1}{6}(7-7)^2 \\ &\quad + \tfrac{5}{36}(8-7)^2 + \tfrac{1}{9}(9-7)^2 + \tfrac{1}{12}(10-7)^2 + \tfrac{1}{18}(11-7)^2 + \tfrac{1}{36}(12-7)^2 = 5.833\end{aligned}$$
$$\mathrm{SD}(Z) = \sqrt{5.833} = 2.415$$

7. (a) i. Let X denote the number of days the server does not crash.

ii. Then $X \sim \mathrm{Geo}(0.003)$.
(We are treating a crash as a success, so that we can count the number of days before a crash).

iii. The expected number of days before a crash is $E(X)$.

(b) First approach:

i. Let X denote the number of bones fed to the wolf before it is tamed.

ii. Then $X \sim \mathrm{Geo}(1/3)$.

iii. Since we have 6 bones, we want to determine $P(X < 6)$.
(It is not $P(X \leqslant 6)$, because that would imply the sixth bone was a failure, and then we have run out of bones.)

Second approach:

i. Suppose we feed every bone to the wolf; let Y denote the number of successful attempts.

ii. Then $Y \sim \mathrm{Bin}(6, 1/3)$.

iii. The probability of having a successful attempt is then $P(X \geqslant 1) = 1 - P(X = 0)$.

(c) i. Let X denote the number of bones fed to the wolf before it is tamed.

ii. Then $X \sim \mathrm{Geo}(1/3)$.

iii. The average number of bones needed is $E(X) + 1$.
(We have to add one, because the successful bone is not counted by X).

(d) The probability that a day has an accident is $\frac{20}{365}$ (ignoring leap years).

i. Let X denote the number of days with an accident in one year.

ii. Then $X \sim \mathrm{Bin}(365, \frac{20}{365})$.

iii. The probability that there are fewer than 10 accidents is $P(X < 10)$.

(e) i. Let X denote the number of spins which did not result in a prize, before winning three prizes.

ii. Then $X \sim \mathrm{NB}(3, 0.1)$.

iii. The number of spins to win 3 prizes is then $E(X) + 3$.
(Add 3 because X does not count the successful attempts.)

(f) i. Let X denote the number of successful spins.
  ii. Then $X \sim \mathrm{Bin}(5, 0.1)$.
  iii. The probability of winning at least one prize is $P(X \geqslant 1)$.

(g) i. Let X denote the number of dented cans.
  ii. Then $X \sim \mathrm{Bin}(1250, 0.01)$.
  iii. The probability that at most 20 cans are dented is $P(X \leqslant 20)$.

8. Let $X \sim \mathrm{Bin}(10, 0.3)$

(a) Note that $\binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10\times9\times8}{3\times2} = 10 \times 3 \times 4 = 120$. So,

$$P(X = 3) = \binom{10}{3} \times 0.3^3 \times (1 - 0.3)^{10-3} = 120 \times 0.3^3 \times 0.7^7 \approx 0.267$$

(b)
$$\begin{aligned} P(X \leqslant 1) &= P(X = 0) + P(X = 1) \\ &= \binom{10}{0} 0.3^0 0.7^{10} + \binom{10}{1} 0.3^1 \times 0.7^{10-1} \\ &= 1 \times 1 \times 0.7^{10} + 10 \times 0.3 \times 0.7^9 \\ &\approx 0.149 \end{aligned}$$

(c) $P(X \geqslant 2) = 1 - P(X < 2) = 1 - P(X \leqslant 1) = 1 - 0.149 = 0.851$

(d)
$$\begin{aligned} P(X > 8) &= P(X = 9) + P(X = 10) \\ &= \binom{10}{9} 0.3^9 0.7^1 + \binom{10}{10} 0.3^{10} 0.7^0 \\ &= 10 \times 0.3^9 \times 0.7 + 1 \times 0.3^{10} \times 1 \\ &\approx 0.0001 \end{aligned}$$

(e) $P(X \leqslant 9) = 1 - P(X > 9) = 1 - P(X = 10) = 1 - \binom{10}{10} 0.3^{10} 0.7^0 = 1 - 0.3^{10} \approx 0.999$

(f) $P(2 \leqslant X \leqslant 9) = P(X \leqslant 9) - P(X < 2)$
We already know $P(X \leqslant 9)$ from the previous part, and $P(X < 2) = P(X \leqslant 1)$, which we also already know.
So $P(2 \leqslant X \leqslant 9) = 0.999 - 0.149 = 0.85$

9. Let $Y \sim \mathrm{NB}(3, 0.15)$.

(a)
$$\begin{aligned} P(Y = 3) = \binom{3+3-1}{3} 0.15^3 (1 - 0.15)^3 &= \binom{5}{3} 0.15^3 0.85^3 \\ &= \frac{5!}{3!\,2!} 0.15^3 0.85^3 \\ &= 10 \times 0.15^3 \times 0.85^3 \approx 0.021 \end{aligned}$$

(b)
$$\begin{aligned} P(Y \leqslant 1) &= P(Y = 0) + P(Y = 1) \\ &= \binom{0+3-1}{0} 0.15^3 (1 - 0.15)^0 + \binom{1+3-1}{1} 0.15^3 (1 - 0.15)^1 \\ &= \binom{2}{0} \times 0.15^3 \times 1 + \binom{3}{1} 0.15^3 \times 0.85 \\ &= 1 \times 0.15^3 + 3 \times 0.15^3 \times 0.85 \\ &\approx 0.012 \end{aligned}$$

(c) $P(Y \geqslant 2) = 1 - P(Y < 2) = 1 - P(Y \leqslant 1) = 1 - 0.012 = 0.988$

(d)
$$\begin{aligned} P(1 < Y \leqslant 3) &= P(Y = 2) + P(Y = 3) \\ &= \binom{2+3-1}{2} 0.15^3 (1 - 0.15)^2 + 0.021 \quad \text{(using (a))} \\ &= \binom{4}{2} 0.15^3 0.85^2 + 0.021 \\ &= 6 \times 0.15^3 \times 0.85^2 + 0.021 \\ &\approx 0.035 \end{aligned}$$

10. (a) For $X \sim \mathrm{Geo}(0.003)$, we have $E(X) = \frac{1-0.003}{0.003} \approx 332$ days.

(b) Approach 1:

For $X \sim \mathrm{Geo}(1/3)$, we have

$$\begin{aligned} P(X < 6) &= P(X=0) + P(X=1) + \ldots + P(X=5) \\ &= (1-\tfrac{1}{3})^0 \times \tfrac{1}{3} + (1-\tfrac{1}{3})^1 \times \tfrac{1}{3} + (1-\tfrac{1}{3})^2 \times \tfrac{1}{3} + (1-\tfrac{1}{3})^3 \times \tfrac{1}{3} + (1-\tfrac{1}{3})^4 \times \tfrac{1}{3} + (1-\tfrac{1}{3})^5 \times \tfrac{1}{3} \\ &\approx 0.912 \end{aligned}$$

Approach 2:

For $Y \sim \mathrm{Bin}(6, \frac{1}{3})$, we have

$$\begin{aligned} P(X \geqslant 1) = 1 - P(X=0) &= 1 - \binom{6}{0} \left(\tfrac{1}{3}\right)^0 \times \left(1 - \tfrac{1}{3}\right)^6 \\ &= 1 - 1 \times 1 \times \left(\tfrac{2}{3}\right)^6 \\ &\approx 0.912 \end{aligned}$$

(c) The average number of bones is given by $E(X)+1$, for $X \sim \mathrm{Geo}(1/3)$. For $X \sim \mathrm{Geo}(1/3)$, we have $E(X) = \frac{1-1/3}{1/3} = 3 \times \frac{2}{3} = 2$.

So the average number of bones required is $E(X) + 1 = 3$.

(e) For $X \sim \mathrm{NB}(3, 0.1)$, we have

$$E(X) = \frac{3(1-0.1)}{0.1} = \frac{3 \times 0.9}{0.1} = 27$$

So the average number of spins is $E(X) + 3 = 30$.

(f) For $X \sim \mathrm{Bin}(5, 0.1)$, we have

$$P(X \geqslant 1) = 1 - P(X=0) = 1 - \binom{5}{0} 0.1^0 (1-0.1)^5 = 1 - 1 \times 1 \times 0.9^5 = 0.40951$$