1. Let $\mathbf{f} = \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 - x_2 + x_3(x_2 - 1) \\ x_1(x_2 + 1) - x_3 \end{pmatrix}$$

(i) [2 marks] Determine the set of non-regular points of $\mathbf{f}$.

(ii) [1 mark] Determine which level-set of $\mathbf{f}$ contains the set of non-regular points from (i).

(iii) [3 marks] Find a basis for, and state the dimension of, the tangent space $T\mathcal{F}(\mathbf{p})$ at $\mathbf{p} = (2, 1, 3)$.

(iv) [2 marks] The curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^3$, given by

$$\boldsymbol{\gamma}(t) = \left(\frac{2}{t}, t, 1 + \frac{2}{t}\right)^T$$

is a curve in $\mathcal{F}$. Find t' such that $\mathbf{p} = (2, 1, 3) = \boldsymbol{\gamma}(t')$ and decide, with reasons, whether $D\boldsymbol{\gamma}(t')$ is contained in $T\mathcal{F}(\mathbf{p})$.

2. Consider the set-constraint problem:

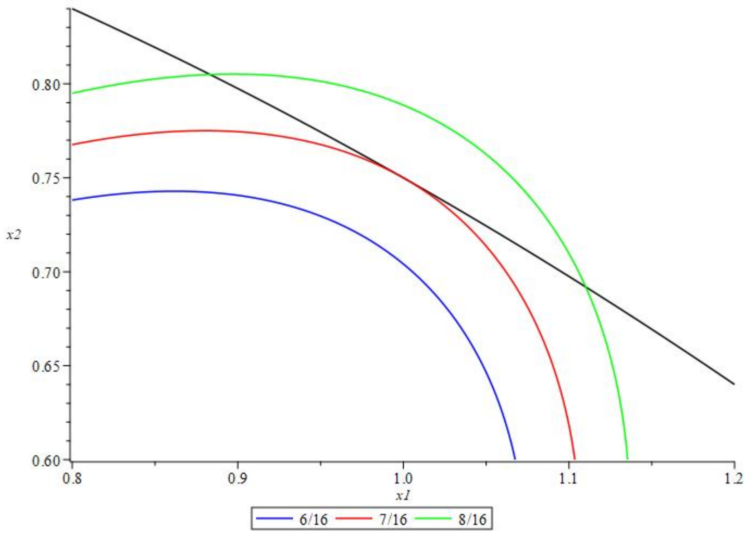
$$\begin{array}{ll} \text{maximize} & x_1^3 + 3x_2^2 - 3x_1x_2 \\ \text{subject to} & \mathbf{x} \in \Omega = \{\mathbf{x} : \ x_1 \geq 0, \ x_2 \geq 0, \ \text{and } 4x_2 \leq (x_1 + 2)(2 - x_1)\} \end{array}$$

(i) [5 marks] Determine and draw in one diagram: the tangent line to the curve $(x_1 + 2)(2 - x_1) - 4x_2 = 0$ at the point $\mathbf{p} = (1, 3/4)^T$, a normal vector at $\mathbf{p}$ and the feasible set Ω.

(ii) [2 marks] Describe the set of feasible directions at $\mathbf{p}$ using the normal vector you found in (i). State whether $(2, -1)^T$ is feasible.

(iii) [2 marks] Is $\mathbf{p}$ a possible maximiser according to the FONC? Justify your answer.

(iv) [1 mark] The plot below contains three level sets of the objective function $x_1^3 + 3x_2^2 - 3x_1x_2 = a/16$, with $a = 6, 7, 8$, as well as the boundary of the constraint set. The point $\mathbf{p} = (1, 3/4)$ is contained in the 7/16-level set.

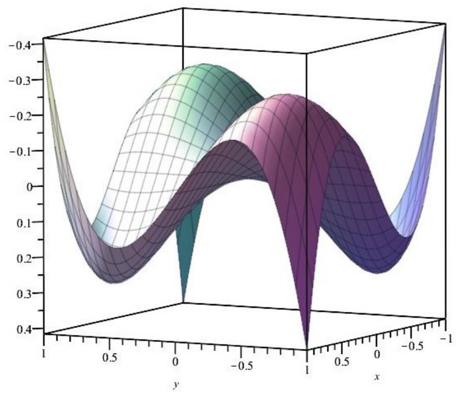


State whether the point $\mathbf{p}$ is a local maximizer. No reasons required.

3. [7 marks] Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$, given by

$$\mathbf{f}(\mathbf{x}) = x_1 x_2 \cos(x_1^2 + x_2^2).$$

Its graph



shows several maxima and minima. In particular, there is a minimum close to $\mathbf{q} = (0.7, 0.7)$ (this is the only thing that is relevant about the graph). Perform one step of Newton's method to find a point which is closer to the minimum. Verify that it is closer to the minimum.

4. [5 marks] Solve the LP problem

$$\begin{aligned} \text{Maximise} \quad & z = x_1 + x_2 \\ \text{Subject to} \quad & \tfrac{1}{2}x_1 - x_2 \leq -1 \\ & \tfrac{1}{2}x_1 + x_2 \leq 4 \\ & 3x_1 + x_2 \geq 5 \\ & 2x_1 - x_2 \leq 2 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

by sketching the feasible region and drawing some level sets of the objective function. State the maximum and the corner at which the maximum occurs.

5. [4 marks] After having applied the simplex algorithm to an LP problem, we have arrived at the following (augmented) matrix

$$\left(\begin{array}{cccc|c} 2 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

where the 3rd and 4th columns correspond to slack variables and the bottom row represents the objective function. Write down the set of all optimal solutions.

6. Consider the LP problem

$$\begin{aligned} \text{maximize} \quad & z = x_1 + 5x_2 \\ \text{subject to} \quad & 5x_1 + x_2 \geq 5 \\ & x_2 \leq 3 \\ & -x_1 + x_2 \geq 1 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

(i) [3 marks] By introducing slack variables and artificial variables as appropriate, and using the 2-phase method, write down the augmented matrix for the first phase.

Indicate the entry to pivot on in this matrix.

(ii) [3 marks] Solve the first phase using the simplex method (this requires two steps), and write down the basic solution it provides.

Note: You do not have to solve the second phase.

7. Consider the nonlinear problem

Minimise $\quad f(\mathbf{x}) = x_1^2 + x_2^3 - 2x_1x_2$

Subject to $\quad g(\mathbf{x}) = x_1^2 + x_2^2 \leq 1$

(i) [1 mark] Write down the relevant KKT condition.

(ii) [3 marks] Decide, with reasons, whether there is a minimum in the interior of the feasible region.

(iii) [2 marks] Assuming that $\mu \neq 0$, solve the KKT-condition using the MATLAB routine `vpasolve`. How many points are possibly minimisers?

(iv) [3 marks] For the possible minimisers identified in (iii), does the SOSC imply they are strict local minimisers? Give reasons.