2015 past exam solutions

Stochastic Modelling (University of Melbourne)

Semester 2 Assessment, 2015

School of Mathematics and Statistics

MAST30001 Stochastic Modelling

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam.

This paper consists of 10 pages (including this page)

Authorised materials:

- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You may remove this question paper at the conclusion of the examination.
- This paper has **7 questions**. Attempt as many questions, or parts of questions, as you can. The number of marks allocated to each question is shown in the brackets after the question statement. There are **80 total marks** available for this examination. A table of **normal distribution probabilities** can be found at the end of the exam. Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination.

1. (a) Analyse the state space $S = \{1, 2, 3, 4\}$ for each of the three Markov chains given by the following transition matrices. That is, write down the communication classes and their periods, label each class as essential or not, and as transient or positive recurrent or null recurrent.

   i.
$$\begin{pmatrix} 1/6 & 5/6 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   ii.
$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 3/5 & 2/5 & 0 \end{pmatrix}.$$

   iii.
$$\begin{pmatrix} 1/6 & 1/3 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix}.$$

   (b) For the Markov chain given by the transition matrix in part iii above, discuss the long run behaviour of the chain including deriving long run probabilities.

   (c) For the Markov chain given by the transition matrix in part iii above, find the expected number of steps taken for the chain to first reach state 3 given the chain starts at state 1.

   (d) For the Markov chain given by the transition matrix in part iii above, find the expected number of steps taken for the chain to first return to state 2 given the chain starts at state 2.

[15 marks]

Ans.

(a) Since all of these Markov chains are finite, any essential communicating classes are positive recurrent and non-essential classes are transient.

i. There are three communicating classes: $\{1, 2\}, \{4\}, \{3\}$. The first two are essential and the last non-essential. All classes are aperiodic due to the presence of loops.

ii. The chain is irreducible with period 2.

iii. The chain has two communicating classes: $\{1, 2, 3\}, \{4\}$, the first essential and the second not. Both classes are aperiodic because of loops.

(b) Regardless of where the chain starts it ends up in the essential communicating class and stays forever. The long run probabilities are given by the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3)$ satisfying

$$\pi \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix} = \pi.$$

Solving shows $\pi = (2/5, 4/15, 1/3)$.

(c) We perform a first step analysis. Let e_i be the expected time to reach state 3 given the chain starts at state i. Then first step analysis implies

$$\begin{aligned} e_1 &= 1 + \frac{e_1}{6} + \frac{e_2}{3}, \\ e_2 &= 1 + \frac{e_2}{2}, \end{aligned}$$

and so $e_1 = 2$.

(d) Because we're starting in the essential class, $E[T(2)|X_0 = 2] = 1/\pi_2 = 15/4$.

2. A continuous time Markov chain $(X_t)_{t\geq 0}$ has generator

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

The states are labelled $\{1, 2, 3\}$. Use the Kolmogorov forward equation to find $p_{1,j}(t) = P(X_t = j|X_0 = 1)$ for $j = 1, 2, 3$.

[6 marks]

Ans.

The Kolmogorov forward equation for the transition matrix $P(t)$ is

$$P'(t) = P(t)A.$$

The first column of this matrix relation is

$$\begin{aligned} p'_{11}(t) &= -2p_{11}(t) + p_{13}(t), \\ p'_{12}(t) &= p_{11}(t) - p_{12}(t) + p_{13}(t), \qquad (1) \\ p'_{13}(t) &= p_{11}(t) + p_{12}(t) - 2p_{13}(t). \qquad (2) \end{aligned}$$

Using that $p_{13}(t) = 1 - p_{12}(t) - p_{11}(t)$ in (1) yields

$$p'_{12}(t) = 1 - 2p_{12}(t),$$

and solving using integrating factors and the boundary condition $p_{12}(0) = 0$ yields

$$p_{12}(t) = \frac{1}{2}(1 - e^{-2t}).$$

Similarly we can use that $p_{12}(t) = 1 - p_{13}(t) - p_{11}(t)$ in (2) to find

$$p'_{13}(t) = 1 - 3p_{13}(t),$$

and solving using integrating factors and the boundary condition $p_{13}(0) = 0$ yields

$$p_{13}(t) = \frac{1}{3}(1 - e^{-3t}).$$

Now using $p_{11}(t) = 1 - p_{13}(t) - p_{12}(t)$ we find

$$p_{11}(t) = \frac{1}{6} + \frac{e^{-3t}}{3} + \frac{e^{-2t}}{2}.$$

3. A Markov chain $(X_n)_{n\geq 0}$ on $\{0,1,2,\ldots\}$ has transition probabilities for $i=0,1,2,\ldots$,

$$p_{i,i+1} = 1 - p_{i,0} = e^{-(i+1)^{-\alpha}},$$

where we only consider $\alpha \geq 1$. This chain is irreducible. For which values of α is the chain transient? Null recurrent? Positive recurrent?

[6 marks]

Ans.

Since the chain is irreducible, we only need to check transience, pos/null recurrence at a single state, in this case state 0. If $T(0) = \min\{n \geq 1 : X_n = 0\}$, then the chain is recurrent if $P(T(0) < \infty | X_0 = 0) = 1$ and positive recurrent if additionally $E[T(0)|X_0 = 0] < \infty$. But we can compute exactly

$$P(T(0) > k | X_0 = 0) = p_{0,1}p_{1,2}\cdots p_{k,k+1} = \prod_{i=0}^{k} e^{-(i+1)^{-\alpha}} = \exp\left\{-\sum_{i=0}^{k}(i+1)^{-\alpha}\right\}.$$

Taking the limit, we find

$$P(T(0) = \infty) = \lim_{k\to\infty} P(T(0) > k|X_0 = 0) = \exp\left\{-\lim_{k\to\infty}\sum_{i=0}^{k}(i+1)^{-\alpha}\right\}.$$

So for $\alpha > 1$, $P(T(0) = \infty) > 0$ in which case the chain is <u>transient</u> and for $\alpha = 1$, $P(T(0) = \infty) = 0$, in which case the chain is <u>recurrent</u>.

To determine pos/null recurrence, in the case $\alpha = 1$, we need to compute $E[T(0)|X_0 = 0]$.

$$E[T(0)|X_0 = 0] = \sum_{k\geq 0} P(T(0) > k|X_0 = 0) = \sum_{k\geq 0} \exp\left\{-\sum_{i=0}^{k}(i+1)^{-1}\right\}.$$

To evaluate the sum, first note

$$\sum_{i=0}^{k}(i+1)^{-1} \leq 1 + \int_1^{k+2} x^{-1}dx = 1 + \log(k+2),$$

and so

$$\sum_{k\geq 0} \exp\left\{-\sum_{i=0}^{k}(i+1)^{-1}\right\} \geq e^{-1}\sum_{k\geq 0}(k+2)^{-1} = \infty$$

and so in this case the chain is <u>null recurrent</u>.

4. A petrol station in the country is at the intersection of two roads, one running north-south and the other east-west. Cars drive from the north-south road according to a Poisson process with rate 3 per hour and from the east-west road according to an independent Poisson process with rate 5 per hour. Cars that drive by stop at the petrol station independently, with probability 1/10 if they are coming from the north-south road and with probability 1/20 if they are coming from the east-west road.

   (a) What is the chance that exactly five cars from the north-south road drive by the petrol station between 9am and 11am?

   (b) What is the chance that exactly five cars from the north-south road stop at the petrol station between 9am and 11am?

   (c) What is the expected amount of time between when the petrol station opens at 7am and the first car from either road drives by?

   (d) What is the expected amount of time between when the petrol station opens at 7am and the first car from either road stops?

   (e) Given exactly five cars from the north-south road have stopped at the petrol station between 9am and 11am, what is the chance that exactly two cars from the north-south road have driven by and not stopped in that time period?

   (f) Given exactly five cars from the north-south road have stopped at the petrol station between 9am and 11am, what is the chance that exactly two cars in total (that is, from either road) have driven by and not stopped in that time period?

   (g) Given that exactly five cars from the north-south road have stopped at the petrol station between 9am and 11am, what is the chance that exactly two of those five arrived between 9am and 9:30am?

   (h) Given that exactly five cars from the north-south road have stopped at the petrol station between 9am and 11am, what is the chance that exactly two cars in total (that is, from either road) arrived between 9am and 9:30am?

[15 marks]

Ans.

Let N_t (W_t) be the number of cars that have driven by on the north-south (east-west) road t hours after 9am and M_t (V_t) the number that stop at the station. Then thinning says that M_t, $N_t - M_t$, V_t, $W_t - V_t$ are independent Poisson processes with rates 0.3, 2.7, 0.25, 4.75. Also note that superposition of any of these processes is again a Poisson process and the relevant rates add.

(a) N_2 is Poisson with mean $2 \cdot 3 = 6$. $P(N_2 = 5) = e^{-6}6^5/5!$.

(b) M_2 is Poisson with mean $2 \cdot 0.3 = 0.6$. $P(M_2 = 5) = e^{-0.6}0.6^5/5!$.

(c) This is the time of the first arrival in the superposition process $N_t + W_t$ which is a Poisson process with rate 8 cars per hour. The first arrival is exponential rate 8 with mean 1/8 hours.

(d) This is the time of the arrival in $M_t + V_t$ which is exponential with mean 20/11 hours.

(e) We want $P(N_2 - M_2 = 2 | M_2 = 5)$. But $N_2 - M_2$ and M_2 are independent so $P(N_2 - M_2 = 2 | M_2 = 5) = P(N_2 - M_2 = 2) = e^{-5.4}5.4^2/2$.

(f) We want $P(N_2 - M_2 + W_2 - V_2 = 2|M_2 = 5)$. But $N_2 - M_2$, $W_2 - V_2$, and M_2 are all independent and moreover $N_2 - M_2 + W_2 - V_2$ is Poisson distributed with mean 14.9. Thus

$$P(N_2 - M_2 + W_2 - V_2 = 2|M_2 = 5) = e^{-14.9}(14.9)^2/2.$$

(g) We want $P(M_{1/2} = 2|M_2 = 5)$. Given five cars stopped during the two hour time period, the distribution of times the cars stopped are iid uniform over the two hour period. Each car has a 1/4 chance of having stopped in the first half hour and the five cars behave independently so the number that stop in the first half hour is binomial with parameters 5 and 1/4 and so the chance that this number is two is

$$P(M_{1/2} = 2|M_2 = 5) = \binom{5}{2}(1/4)^2(3/4)^3.$$

(h) We want $P(M_{1/2} + V_{1/2} = 2|M_2 = 5)$. $V_{1/2}$ is independent of M_t, so we have, using the description from Part (g),

$$\begin{aligned}
P(M_{1/2} + V_{1/2} = 2|M_2 = 5) &= \sum_{i=0}^{2} P(M_{1/2} = i|M_2 = 5)P(V_{1/2} = 2 - i) \\
&= \sum_{i=0}^{2} P(M_{1/2} = i|M_2 = 5)P(V_{1/2} = 2 - i) \\
&= \sum_{i=0}^{2} \binom{5}{i}(1/4)^i(3/4)^{5-i}\frac{e^{-1/8}(1/8)^{2-i}}{(2-i)!}.
\end{aligned}$$

5. At a certain tram stop, trams run from 5am to 1am. We model the times of arrival of trams starting from 5am as a renewal process with inter-arrival times, in minutes, uniform on the interval $(0, 20)$.

   (a) Compute the mean and variance of the inter-arrival distribution.

   (b) On average, about how many trams arrive at the stop between 5am and 5pm?

   (c) Give an interval around your estimate from (b) that will have a 95% chance of covering the true number of trams that arrive at the stop over the course of the 12 hours.

   (d) If you arrive at the tram stop at 10pm, what would you estimate to be the mean and variance of the time until the next tram arrives?

[11 marks]

Ans.

(a) The inter-renewal time τ is distributed as uniform $(0, 20)$. Thus

$$E[\tau] = 10, \quad Var(\tau) = 20^2/12 = 100/3 = 33.333.$$

(b) $N_t/t \to 1/E[\tau] = 1/10$ as $t \to \infty$, and so in the first $12 \times 60 = 720$ minutes of the day we expect about $720/10 = 72$ trams.

(c) The renewal CLT says that $N_t \approx \mathrm{N}(t/\mu, t\sigma^2/\mu^3)$ and so for $t = 720$, we expect with there is a 95% chance the number of trams that have arrived is in the interval

$$\left(72 - (1.96)\sqrt{720 \cdot 100/(3 \cdot 10^3)}, 72 + (1.96)\sqrt{720 \cdot 100/(3 \cdot 10^3)}\right) \approx 72 \pm (1.96)\sqrt{24}.$$

(d) Defining the renewal times $T_k = \sum_{i=1}^{k} \tau_i$, we know that for large t and $Y_t := T_{N_t+1} - t$ roughly has density $(1 - F(t))/\mu$ for $0 < t < 20$. In this case this density on this support is

$$\frac{1 - t/20}{10}, \quad 0 < t < 20.$$

and now its only a matter of computing the first and second moments to find $E[Y_t] \approx 20/3$ and $Var(Y_t) \approx 200/9$.

6. In a certain computer system, jobs arrive according to a Poisson process with rate λ. There are two servers that process jobs, Server A works at exponential rate μ_A and Server B at exponential rate $\mu_B < \mu_A$. Since Server A is faster than Server B, the system works as follows. When there is one job in the system, Server A processes it, and if there is more than one job, both servers process separate jobs. If there are exactly two jobs in the system and Server A finishes its job before the arrival of an additional job, then Server B instantly passes its job to Server A to process. When both servers are busy jobs queue in an infinite buffer.

   (a) Model the number of jobs in the system (including those being worked on) as a continuous time Markov chain $(X_t)_{t\geq 0}$, and write down its generator.

   (b) Determine for what parameters λ, μ_A, μ_B, the Markov chain is ergodic and for these values write down the steady state distribution.

   Assume for the rest of the problem that the parameters satisfy the constraints of part (b) so that a steady state exists.

   (c) What is the average number of jobs in the system?

   (d) What is the average number of jobs waiting in the queue (that is, in the system but not in service)?

   (e) What is the average amount of time an arriving job waits for service?

   (f) What is the average amount of time an arriving job is in the system?

   (g) What proportion of time is Server B idle?

[15 marks]

Ans.

(a) If $(X_t)_{t\geq 0}$ is a continuous time birth-death process with rates

$$\lambda_i = \lambda, \quad i \geq 0, \qquad \mu_1 = \mu_A, \qquad \mu_i = \mu_A + \mu_B, \qquad i \geq 2.$$

This is because the arrival rate is from the Poisson process and so is the same no matter what the state of the system, and if there are at least two jobs in the system, then both servers are working with combined rate $\mu_A + \mu_B$, and if there is exactly one job in the system, then Server A is always the server processing it.

(b) Since this is a birth-death chain, it is ergodic if and only if

$$\sum_{j\geq 0} \prod_{i=0}^{j-1} \frac{\lambda_i}{\mu_{i+1}} = 1 + \frac{\lambda}{\mu_A} \left(\sum_{j\geq 1} \left(\frac{\lambda}{\mu_A + \mu_B} \right)^{j-1} \right) < \infty,$$

and so the chain is ergodic if $\lambda < \mu_A + \mu_B$. In this case the steady state distribution is π with

$$\begin{aligned}
\pi_0 &= \left[1 + \frac{\lambda}{\mu_A} \left(\sum_{j\geq 1} \left(\frac{\lambda}{\mu_A + \mu_B} \right)^{j-1} \right) \right]^{-1} \\
&= \left(1 + \frac{\lambda(\mu_A + \mu_B)}{\mu_A(\mu_A + \mu_B - \lambda)} \right)^{-1}, \\
\pi_i &= \pi_0 \left(\frac{\lambda}{\mu_A} \right) \left(\frac{\lambda}{\mu_A + \mu_B} \right)^{i-1}, \qquad i \geq 1.
\end{aligned}$$

(c) The average number of jobs in the system is the expected value agains π:

$$\sum_{j\geq 1} j\pi_j = \pi_0\left(\frac{\lambda}{\mu_A}\right)\sum_{j\geq 1} j\left(\frac{\lambda}{\mu_A+\mu_B}\right)^{j-1} = \pi_0\left(\frac{\lambda(\mu_A+\mu_B)^2}{\mu_A(\mu_A+\mu_B-\lambda)^2}\right).$$

(d) The number of jobs waiting in the queue at time t is $\max\{X_t - 2, 0\}$. So the average number is

$$\begin{aligned}\sum_{j\geq 3}(j-2)\pi_j &= \sum_{j\geq 1} j\pi_j - 2\pi_2 - \pi_1 - 2(1-\pi_0-\pi_1-\pi_2) \\ &= \pi_0\left(\frac{\lambda(\mu_A+\mu_B)^2}{\mu_A(\mu_A+\mu_B-\lambda)^2} + \frac{\lambda}{\mu_A}\right) - 2(1-\pi_0).\end{aligned}$$

(e) By Little's law, the expected waiting time is the expected queue length divided by λ, given in Part (d).

(f) Again by Little's law, the expected time in the system is the expected number of jobs in the system divided by λ, given in Part (c).

(g) Server B is idle when there are zero or one jobs in the system: $\pi_0 + \pi_1$.

7. Let $(B_t)_{t\geq 0}$ and $(\widetilde{B}_t)_{t\geq 0}$ be two independent standard Brownian motions, let $-1 < \rho < 1$, and for $t \geq 0$ set $W_t = \rho B_t + \sqrt{1-\rho^2}\widetilde{B}_t$.

   (a) Use the axioms of Brownian motion to show that $(W_t)_{t\geq 0}$ is a standard Brownian motion.

   (b) Find $P(W_2 \leq 0 | W_1 = 1)$.

   (c) Find $P(W_1 \leq 0 | W_2 = 1)$.

   For the remaining parts of the problem, set $\rho = \sqrt{2}/3$.

   (d) Find $P(W_2 \leq -1 | B_1 = 0)$.

   (e) Find $P(W_1 \leq 0 | B_2 = -1)$.

[12 marks]

Ans.

(a) We need to show that $(W_t)_{t\geq 0}$ has independent increments and that its marginal distributions W_t are normal with mean zero and variance t.

Independent increments are inherited from the independent increments of B and $\widetilde{B}$. That is, for $0 \leq s_1 < t_1 \leq \cdots \leq s_k < t_k$, $W_{t_i} - W_{s_i} = \rho(B_{t_i} - B_{s_i}) + \sqrt{1-\rho^2}(\widetilde{B}_{t_i} - \widetilde{B}_{s_i})$ and $B_{t_i} - B_{s_i}$ is independent of $\widetilde{B}$ and any other of the increments of B appearing in other W increments and a similar statement holds for $\widetilde{B}_{t_i} - \widetilde{B}_{s_i}$. Thus the $W_{t_i} - W_{s_i}$ are functions of variables that are independent for different i and so independent increments holds.

For the marginal distribution, since W_t is a linear function of independent normals, it is normal it has the right mean and variance:

$$\begin{aligned}E[W_t] &= E[\rho B_t + \sqrt{1-\rho^2}\widetilde{B}_t] = \rho E B_t + \sqrt{1-\rho^2}E\widetilde{B}_t = 0, \\ Var(W_t) &= \rho^2 Var(B_t) + (1-\rho^2)Var(\widetilde{B}_t) = t.\end{aligned}$$

(b) Note that $(W_t)_{t\geq 0}$ is just a standard Brownian motion. By the independent increments property,

$$P(W_2 \leq 0 | W_1 = 1) = P(W_2 - W_1 \leq -1 | W_1 = 1) = P(W_2 - W_1 \leq -1) = 1 - 0.8413,$$

since $W_2 - W_1$ is standard normal and we use the table.

(c) $(W_1, W_2/\sqrt{2})$ is standard bivariate normal with correlation $1/\sqrt{2}$. Thus we can write

$$W_1 = \frac{1}{2}W_2 + \frac{1}{\sqrt{2}}Z,$$

where Z is standard normal independent of W_2. Thus we have

$$P(W_1 \leq 0 | W_2 = 1) = P\left(\frac{1}{2} + \frac{1}{\sqrt{2}}Z \leq 0\right) = P\left(Z \leq \frac{-1}{\sqrt{2}}\right) \approx P(Z \leq -.707) = 1 - 0.71.$$

(d) We can write $W_2 = \rho(B_2 - B_1) + \rho B_1 + \sqrt{1-\rho^2}\widetilde{B}_2$. $B_2 - B_1$ is independent of B_1 so given $B_1 = 0$, W_2 is a linear combination of independent normals having mean zero and variance $\rho^2 + 2(1-\rho^2) = 2 - \rho^2$. Thus for Z standard normal,

$$P(W_2 \leq -1 | B_1 = 0) = P\left(Z \leq \frac{-1}{\sqrt{2-\rho^2}}\right) = P\left(Z \leq -\frac{3}{4}\right) = 1 - .7734.$$

(e) As in part (c) we can write

$$B_1 = \frac{1}{2}B_2 + \frac{1}{\sqrt{2}}Z,$$

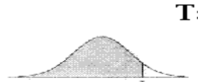
where Z is standard normal independent of B_2. Thus conditional on $B_2 = -1$, we can write

$$W_1 = \rho\left(\frac{1}{\sqrt{2}}Z - \frac{1}{2}\right) + \sqrt{1-\rho^2}\widetilde{B}_1,$$

which is normal with mean $-\rho/2$ and variance $\rho^2/2 + (1-\rho^2) = 1 - \rho^2/2$. Thus

$$\begin{aligned} P(W_1 \leq 0 | B_2 = -1) &= P\left(Z \leq \frac{\rho/2}{\sqrt{1-\rho^2/2}}\right) = P\left(Z \leq \frac{1/\sqrt{18}}{\sqrt{1-1/9}}\right) \\ &= P\left(Z \leq \frac{1}{4}\right) = 0.5987. \end{aligned}$$

Tables of the Normal Distribution



Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015

End of Exam