Assignment 4 Questions

Probability (University of Melbourne)

MAST20004 Probability

Assignment Four: Questions [Due 4:00 pm Monday 28/10]

There are 5 problems in total, of which 3 randomly chosen ones will be marked. You are expected to submit answers to all questions, otherwise a mark penalty will apply. Calculations and reasoning must be given in order to obtain full credit.

Problem 1. (1) Let $X_1, \cdots, X_n$ be independent exponential random variables with parameter α. Compute the moment generating function of $Y = X_1 + \cdots + X_n$, and hence show that $Y \stackrel{d}{=} \gamma(n, \alpha)$.
(2) Let X be a standard normal random variable. What is the moment generating function of X^2?
(3) Let $X_1, \cdots, X_n$ be independent standard normal random variables. Compute the moment generating function of $X_1^2 + \cdots + X_n^2$, and hence identify it with a gamma distribution.

Problem 2. (1) Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables, each following the exponential distribution with parameter α. Let N be another independent random variable following a Poisson distribution with parameter λ. What is the moment generating function of $X_1 + \cdots + X_N$?
(2) Let $U_1, U_2, \cdots$ be a sequence of independent random variables uniformly distributed over $[0, 1]$. Let N be another independent random variable whose probability mass function is given by

$$\mathbb{P}(N = n) = \frac{1}{(e-1)n!}, \quad n = 1, 2, \cdots.$$

Define $X = \max(U_1, \cdots, U_N)$.
(2-i) Find the cumulative distribution function and probability density function of $\max(U_1, U_2, U_3)$.
(2-ii) What is the moment generating function of X?

(2-iii) Let R be another independent random variable whose probability mass function is given by

$$\mathbb{P}(R = r) = (e-1)e^{-r}, \quad r = 1, 2, \cdots.$$

What is the distribution of $R - X$?

Problem 3. Let $X_1, X_2, \cdots$ be an independent sequence of Bernoulli random variables with parameter $p = 1/2$. Let $S_n = X_1 + \cdots + X_n$.
(i) By computing the moment generating function of S_n, show that $S_n \stackrel{d}{=} \mathrm{Bi}(n, 1/2)$.
(ii) By using moment generating functions, show that

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}[S_n]}} \xrightarrow{d} N(0,1).$$

Problem 4. The amount of milk consumed by a child per day in a kindergarten follows certain distribution with mean 300 ml and standard deviation 40 ml. Assume that the milk demand of different child is independent. Suppose that there are 49 children in the kindergarten, and the kindergarten orders 15 litres of fresh milk from a farm every day.
(i) On a particular day, what is the probability that the milk supplied by the kindergarten will not meet the demand of the children?
(ii) Suppose that the amount of milk consumed by the children is independent from day to day. What is the probability the kindergarten will not meet the milk demand of the children for at least 1 of the next 5 days?
(iii) During the next year, what is the probability that the kindergarten will not meet the milk demand of the children for more than 40 days?

Problem 5. A frog is jumping among the vertices of a triangle. If the frog is at the upper vertex, in the next move it is equally likely to remain at its current position or jumps down to one of the two lower vertices. If the frog is at one of the two lower vertices, in the next move it is twice as likely to jump to the other lower vertex as to the upper one but it will not remain at its current position. Suppose that initially the frog is located at the upper vertex. Let X_n $(n \geqslant 0)$ denote the position of the frog after the n-th move.
(i) Write down the transition matrix of the Markov chain $\{X_n : n \geqslant 0\}$ and draw the corresponding state space diagram.
(ii) What is the probability that, after 2019 moves the frog returns to its initial position (i.e. the upper vertex)?
(iii) Compute the equilibrium distribution of $\{X_n : n \geqslant 0\}$.