THE UNIVERSITY OF MELBOURNE

SCHOOL OF MATHEMATICS AND STATISTICS

SEMESTER 1, 2015

MAST30030 APPLIED MATHEMATICAL MODELLING

Exam duration - Three hours

Reading time - 15 minutes

This paper has 7 pages, including this cover sheet.

Instructions to Invigilators:

Initially, students are to receive a 14 page script book.

Authorised Materials:

No materials are authorised.

No calculators or computers are permitted.

No written or printed material may be brought into the examination room.

Instructions to Students:

There are 4 questions on this examination paper.

All questions may be attempted.

All questions are of equal value.

The maximum number of marks available to students in the exam paper is 100.

Formula sheets are attached to this paper on Pages 6 and 7.

Question 1

(a) We wish to examine the stability of the following dynamical system

$$\dot{x} = x^2 + 2x - 3$$

i. Find the fixed points of this system.
ii. Plot the vector field for this system and hence determine the stability of its fixed points.
iii. Without implementing a linear stability analysis directly, explain how the graph of $\dot{x}$ vs. x can be used to infer the outcome of such an analysis at each fixed point [without using the vector field in (ii)]. Explain your answer.

(b) A general one-dimensional dynamical system has the form

$$\dot{x} = f(x)$$

i. Conduct a linear stability analysis on this system and hence derive the following general condition for a fixed point to be locally stable:

$$f'(\bar{x}) < 0$$

ii. If a particular fixed point is stable, do neighbouring phase points ever reach the fixed point? What is the rate at which neighbouring phase points approach it? Does the result change for an unstable fixed point? Explain your answers.
iii. What does this analysis indicate about the stability of a fixed point $\bar{x}$ when $f'(\bar{x}) = 0$? Explain your answer.
iv. Extend the analysis to handle the case in (iii), i.e., $f'(\bar{x}) = 0$. Explain the motivation behind your derivation. Hint: Do not solve the resulting differential equation.
v. Using your result in (iv), under what condition is the fixed point half-stable? Explain your answer.

(c) Consider the following two-dimensional dynamical system

$$\frac{dx}{dt} = y + x^2$$
$$\frac{dy}{dt} = x + y^2$$

i. Find the fixed points and nullclines of this system.
ii. Calculate and draw the local phase portraits for each fixed point.
iii. Using the information in (i) and (ii), draw the global phase portrait.

Question 2

(a) The method of characteristics can be used to solve first-order partial differential equations of the form

$$\frac{\partial f}{\partial t} + g(x,t)\frac{\partial f}{\partial x} = s(x,t)$$

Outline the basic steps involved in applying the method to convert such a partial differential equation into a system of ordinary differential equations.

(b) Traffic moving along a road obeys the conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

where t and x are scaled time and position, respectively, ρ is the scaled car density distribution (maximum density is one), $J = \rho v$ is the scaled flux, and v is the scaled car speed (maximum of one). Measurements show that $v = 1 - \rho$. Initially, at time $t = 0$, the car density is

$$\rho(x,0) = \begin{cases} \exp(x) & : x < 0 \\ 1 & : x \geq 0 \end{cases}$$

i. Calculate the initial car velocity distribution as a function of position, and plot this distribution. Using this information only, how you would expect the car dynamics to evolve in time? Explain your answer.

ii. Convert the above conservation law into two equivalent ordinary differential equations using the method outlined in (a).

iii. Identity all the key features of the problem, including any fans or shocks, and sketch the characteristics on a space-time diagram.

iv. Accounting for all of the above details, calculate the time taken for a car initially at position $x = -2$ (at time $t = 0$) to reach the position $x = 0$. How does this compare with your expectation in (i)? Explain your answer.

v. Without repeating the calculation, how would you expect the density distribution ρ to evolve in time if its initial value was halved for all x? Explain your answer.

Question 3

(a) The principles of conservation of mass, linear momentum and angular momentum are used to derive the governing equations for fluid flow. Give the equation that each principle produces without derivation and without specifying the constitutive equation.

(b) Define what is meant by a Lagrangian and an Eulerian description of fluid flow. For a moving boundary at the interface between two fluids, which description would you expect to be most appropriate? Explain you answer.

(c) In a certain flow, the fluid density, ρ, has the Eulerian form $\rho^E(\boldsymbol{r},t)$ and the Lagrangian form $\rho^L(\boldsymbol{R},t)$, where t is time. What do $\boldsymbol{r}$ and $\boldsymbol{R}$ refer to? Derive the formula for the material derivative

$$\frac{D\rho}{Dt}$$

in terms of $\rho^E(\boldsymbol{r},t)$ and the Eulerian velocity field $\boldsymbol{u}^E(\boldsymbol{r},t)$.

(d) A certain *irrotational* flow possesses *curved* streamlines. Explain why this is not a contradiction in terms. Give an example of the opposite case where the flow is *rotational* but possesses *straight* streamlines.

(e) Give the boundary conditions for

i. The interface between a viscous fluid and a moving solid boundary;

ii. A flat fluid-fluid interface between two viscous fluids.

(f) Using Cartesian tensor methods, prove the following vector identities

i. $\nabla \times \nabla F = \mathbf{0}$

ii. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d}))\mathbf{c} - (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))\mathbf{d}$

The following relationship may be useful in proving at least one of these identities

$$\varepsilon_{ijk}\,\varepsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

You may also use the properties of the scalar triple product without proof.

Question 4

(a) Two-dimensional incompressible flow is often expressed in terms of a streamfunction ψ.

i. Prove that the streamlines are given by the level sets of ψ.

ii. Apart from the feature listed in (i), list two other advantages of using the streamfunction.

(b) A model viscous flow around a cylinder of radius a in a uniform stream of velocity $U\hat{\mathbf{x}}$ is given by the streamfunction in cylindrical coordinates

$$\psi(\sigma,\phi) = U\sigma\left\{1-\left(\frac{a}{\sigma}\right)^2\right\}\sin\phi - B\log\sigma$$

where U is the (uniform) velocity upstream from the cylinder and B is a constant.

i. Calculate the velocity field for this flow, and thus explain what the constant B represents physically.

ii. Show that the flow is irrotational. Using this property of irrotationality, explain how you can find the drag force acting on the cylinder and give its value. Hint: There is no need to explicitly calculate the stress tensor from the streamfunction.

iii. Can the preceding analysis in (ii) be used to find the lift on the cylinder? Without doing a calculation, give the direction of the lift. Explain your answer.

(c) Consider the following velocity field

$$\boldsymbol{u} = w(x,y,z,t)\,\hat{\boldsymbol{z}}$$

i. To which class of flow does this velocity field belong?

ii. Starting from the Navier-Stokes equations, derive the following governing equation for w when the flow is incompressible

$$\rho\frac{\partial w}{\partial t} = G(t) + \mu\tilde{\nabla}^2 w$$

where $-G(t)$ is the pressure gradient in the z-direction and $\tilde{\nabla}^2$ is the two-dimensional Laplacian operator with respect to the x and y-coordinates. You may ignore body forces.

iii. What special feature of this class of flow facilitates analytical solutions?

iv. Hence show that the velocity field for the steady incompressible flow through an annular pipe of inner radius a and outer radius $2a$, due to an applied constant pressure gradient $-G$ in the z-direction, is

$$w = \frac{Ga^2}{4\mu}\left(1-\left(\frac{\sigma}{a}\right)^2 + \frac{3}{\ln 2}\ln\left(\frac{\sigma}{a}\right)\right)$$

END OF EXAMINATION

Vector Identities

$\nabla \cdot (\phi\,\mathbf{q}) = (\nabla\phi)\cdot\mathbf{q} + \phi\nabla\cdot\mathbf{q}$ $\qquad$ $\nabla \times (\phi\,\mathbf{q}) = (\nabla\phi)\times\mathbf{q} + \phi\nabla\times\mathbf{q}$

$\nabla\cdot(\nabla\times\mathbf{q}) = 0 \qquad \nabla\times(\nabla\phi) = \mathbf{0}$ $\qquad$ $\nabla\times(\nabla\times\mathbf{q}) = \nabla(\nabla\cdot\mathbf{q}) - \nabla^2\mathbf{q}$

$\nabla\times(\mathbf{p}\times\mathbf{q}) = \mathbf{p}(\nabla\cdot\mathbf{q}) - \mathbf{q}(\nabla\cdot\mathbf{p}) + (\mathbf{q}\cdot\nabla)\mathbf{p} - (\mathbf{p}\cdot\nabla)\mathbf{q}$

$\nabla(\mathbf{p}\cdot\mathbf{q}) = (\mathbf{p}\cdot\nabla)\mathbf{q} + (\mathbf{q}\cdot\nabla)\mathbf{p} + \mathbf{p}\times(\nabla\times\mathbf{q}) + \mathbf{q}\times(\nabla\times\mathbf{p})$

Polar Coordinates

For *cylindrical polar coordinates* σ, φ, z, with z measured along the axis of the cylinder, σ the distance from the axis of the cylinder, and φ the azimuthal angle:

$$\nabla f = \hat{\boldsymbol{\sigma}}\,\frac{\partial f}{\partial\sigma} + \hat{\boldsymbol{\varphi}}\,\frac{1}{\sigma}\frac{\partial f}{\partial\varphi} + \hat{\mathbf{z}}\,\frac{\partial f}{\partial z} \qquad \nabla\cdot(u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}) = \frac{1}{\sigma}\frac{\partial}{\partial\sigma}(\sigma u) + \frac{1}{\sigma}\frac{\partial v}{\partial\varphi} + \frac{\partial w}{\partial z}$$

$$\nabla\times(u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}) = \left\{\frac{1}{\sigma}\frac{\partial w}{\partial\varphi} - \frac{\partial v}{\partial z}\right\}\hat{\boldsymbol{\sigma}} + \left\{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial\sigma}\right\}\hat{\boldsymbol{\varphi}} + \left\{\frac{1}{\sigma}\frac{\partial}{\partial\sigma}(\sigma v) - \frac{1}{\sigma}\frac{\partial u}{\partial\varphi}\right\}\hat{\mathbf{z}}$$

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{1}{\sigma}\frac{\partial}{\partial\sigma}\left(\sigma\frac{\partial f}{\partial\sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

For *spherical polar coordinates* r, θ, φ, with r the distance from the origin, θ the colatitudinal angle and φ the azimuthal angle:

$$\nabla f = \hat{\mathbf{r}}\,\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{\boldsymbol{\varphi}}\,\frac{1}{r\sin\theta}\,\frac{\partial f}{\partial\varphi}$$

$$\nabla\cdot(u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta\, v) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi}$$

$$\nabla\times(u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}}) = \left\{\frac{\partial}{\partial\theta}(w\sin\theta) - \frac{\partial v}{\partial\varphi}\right\}\frac{\hat{\mathbf{r}}}{r\sin\theta} + \left\{\frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi} - \frac{\partial}{\partial r}(rw)\right\}\frac{\hat{\boldsymbol{\theta}}}{r} + \left\{\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial\theta}\right\}\frac{\hat{\boldsymbol{\varphi}}}{r}$$

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\,\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Stream Functions and Potentials

If $\nabla\times\mathbf{q} = \mathbf{0}$ in a simply-connected domain, then $\mathbf{q} = \nabla\phi$.

If $\nabla\cdot\mathbf{q} = 0$, then $\mathbf{q} = \nabla\times\mathbf{A}$.

In two-dimensional flow, if $\nabla\cdot\mathbf{q} = 0$, then $\mathbf{q} = \nabla\times(\psi\,\hat{\mathbf{k}})$, where $\hat{\mathbf{k}}$ is the unit basis vector associated with the z direction, and ψ is independent of z.

In axisymmetric three-dimensional flow, if $\nabla\cdot\mathbf{q} = 0$, then with Λ and χ independent of the azimuthal angle φ,

$$\mathbf{q} = \nabla\times\left(\frac{\Lambda}{\sigma}\,\hat{\boldsymbol{\varphi}}\right) + \chi\,\hat{\boldsymbol{\varphi}} = \left\{\frac{1}{r^2\sin\theta}\,\frac{\partial\Lambda}{\partial\theta}\right\}\hat{\mathbf{r}} - \left\{\frac{1}{r\sin\theta}\,\frac{\partial\Lambda}{\partial r}\right\}\hat{\boldsymbol{\theta}} + \chi\,\hat{\boldsymbol{\varphi}}.$$

The rate-of-strain tensor

The rate-of-strain tensor $\mathbf{e}$ is related to the velocity field $\mathbf{q}$ by the equation $\mathbf{e} = \frac{1}{2}\{\nabla\mathbf{q} + (\nabla\mathbf{q})^{\mathrm{T}}\}$, where $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of the tensor $\mathbf{A}$.

For *cylindrical polar coordinates* σ, φ, z, if $\mathbf{q} = u\hat{\boldsymbol{\sigma}} + v\hat{\boldsymbol{\varphi}} + w\hat{\mathbf{z}}$, the components of $\mathbf{e}$ are

$$
\begin{aligned}
e_{\sigma\sigma} &= \frac{\partial u}{\partial \sigma} \\
e_{\varphi\varphi} &= \frac{1}{\sigma}\frac{\partial v}{\partial \varphi} + \frac{u}{\sigma} \\
e_{zz} &= \frac{\partial w}{\partial z} \\
e_{\sigma\varphi} &= e_{\varphi\sigma} = \frac{\sigma}{2}\frac{\partial}{\partial \sigma}\left(\frac{v}{\sigma}\right) + \frac{1}{2\sigma}\frac{\partial u}{\partial \varphi} \\
e_{\sigma z} &= e_{z\sigma} = \frac{1}{2}\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial w}{\partial \sigma} \\
e_{z\varphi} &= e_{\varphi z} = \frac{1}{2\sigma}\frac{\partial w}{\partial \varphi} + \frac{1}{2}\frac{\partial v}{\partial z}
\end{aligned}
$$

For *spherical polar coordinates* r, θ, φ, if $\mathbf{q} = u\hat{\mathbf{r}} + v\hat{\boldsymbol{\theta}} + w\hat{\boldsymbol{\varphi}}$, the components of $\mathbf{e}$ are

$$
\begin{aligned}
e_{rr} &= \frac{\partial u}{\partial r} \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \\
e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{u}{r} + \frac{v\cot\theta}{r} \\
e_{r\theta} &= e_{\theta r} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{2r}\frac{\partial u}{\partial \theta} \\
e_{r\varphi} &= e_{\varphi r} = \frac{1}{2r\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{w}{r}\right) \\
e_{\theta\varphi} &= e_{\varphi\theta} = \frac{\sin\theta}{2r}\frac{\partial}{\partial \theta}\left(\frac{w}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial v}{\partial \varphi}
\end{aligned}
$$

These formulae may also be used to deduce the polar coordinate representations of the linear strain tensor in the linear theory of elasticity.