Linear Statistical Models Exam 2017

Linear Statistical Models (University of Melbourne)

Student ID

Semester 1 Assessment, 2017

School of Mathematics and Statistics

MAST30025 Linear Statistical Models

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 8 pages (including this page)

Authorised materials:

- Scientific calculators are permitted, but not graphical calculators.
- Two A4 double-sided handwritten sheets of notes.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- The total number of marks available is 90.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

This paper may be held in the Baillieu Library

Thi **oom**

Question 1 (9 marks)

(a) Show that if A is a symmetric and idempotent matrix, then $r(A) = tr(A)$.

(b) Show, without using the above result, that if X is an $n \times p$ matrix with $n > p$, then $r(X(X^TX)^cX^T) = r(X)$.

(c) Let $B\mathbf{x} = \mathbf{g}$ be a consistent linear system. Show that $\mathbf{x} = B^c\mathbf{g}$ is a solution to this system.

Question 2 (13 marks) Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \sim MVN\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right).$$

You are given the following R calculations.

```
> V <- matrix(c(2,1,1,3),2,2)
> P <- eigen(V)$vectors
> P%*%diag(sqrt(eigen(V)$values))%*%t(P)

          [,1]      [,2]
[1,] 1.3763819 0.3249197
[2,] 0.3249197 1.7013016

> P%*%diag(1/sqrt(eigen(V)$values))%*%t(P)

           [,1]       [,2]
[1,]  0.7608452 -0.1453085
[2,] -0.1453085  0.6155367
```

(a) Find two independent standard normal random variables which are linear combinations of $\mathbf{y}$ elements (and constants).

(b) Calculate $E[y_1^2 - 2y_1y_2]$.

(c) Find all quadratic forms in $\mathbf{y}$ which have a non-central χ^2 distribution with 2 degrees of freedom.

(d) Show that $4y_1^2 - 4y_1y_2 + y_2^2$ is independent of $y_1^2 + 6y_1y_2 + 9y_2^2$.

Question 3 (14 marks) The following data is a sample of 5 random countries. For each country we measure the following variables:

- `logPPgdp`: The logarithm of the 2001 gross domestic product per person in US dollars;
- `logFertility`: The logarithm of the birth rate per 1000 females in the year 2000;
- `Purban`: The percentage of the population which lives in an urban area.

Name	logFertility	logPPgdp	Purban
Moldova	0.23	5.8	41
Netherlands	0.38	10.1	90
Estonia	0.14	8.3	69
Uganda	1.36	5.5	15
Hungary	0.13	8.6	65

We wish to predict `logFertility` using `logPPgdp` and `Purban`, using a linear model $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ *with no constant (intercept) term.*

You are given the following R calculations.

```
> qt(0.975,1:5)

[1] 12.706205  4.302653  3.182446  2.776445  2.570582

> qf(0.95,1,1:5)

[1] 161.447639  18.512821  10.127964   7.708647   6.607891

> qf(0.95,2,1:5)

[1] 199.500000  19.000000   9.552094   6.944272   5.786135
```

(a) Calculate the least squares estimates of $\boldsymbol{\beta}$.

(b) Calculate and interpret a 95% confidence interval for the parameter associated with `Purban`. You are given the sample variance $s^2 = 0.0887$.

(c) Test for model relevance at a 5% significance level.

(d) Croatia has a gross domestic product of \$4500 per person, and 58% of its population lives in an urban area. Calculate a 95% prediction interval for its birth rate.

Question 4 (13 marks) Consider a full rank linear model $\mathbf{y} = X\boldsymbol{\beta}+\boldsymbol{\varepsilon}$. We wish to derive the formula for a prediction interval for the sum of the responses of two independent future observations with predictors $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. This uses the unbiased point estimator $(\mathbf{x}_1 + \mathbf{x}_2)^T\mathbf{b}$, where $\mathbf{b}$ is the least squares estimator of $\boldsymbol{\beta}$.

(a) Calculate the variance of the prediction error of this estimator.

(b) Show that this estimator is normally distributed.

(c) Show that this estimator is independent of SS_{Res}.

(d) Derive a t-distributed quantity based on this estimator and state its degrees of freedom.

(e) Thus write down a formula for a $100(1-\alpha)\%$ prediction interval for the sum of two responses.

Question 5 (14 marks) Consider the general linear model, $\mathbf{y} = X\boldsymbol{\beta}+\boldsymbol{\varepsilon}$. This model may be of full or less than full rank.

(a) State two methods that can be used to fit this model to data and compare them.

(b) Explain the difference between a confidence interval and a prediction interval.

(c) Define Akaike's information criterion and explain why it is useful as a goodness-of-fit measure for model selection.

(d) Give an advantage and a disadvantage of stepwise selection over forward selection.

(e) Define interaction between two categorical predictors.

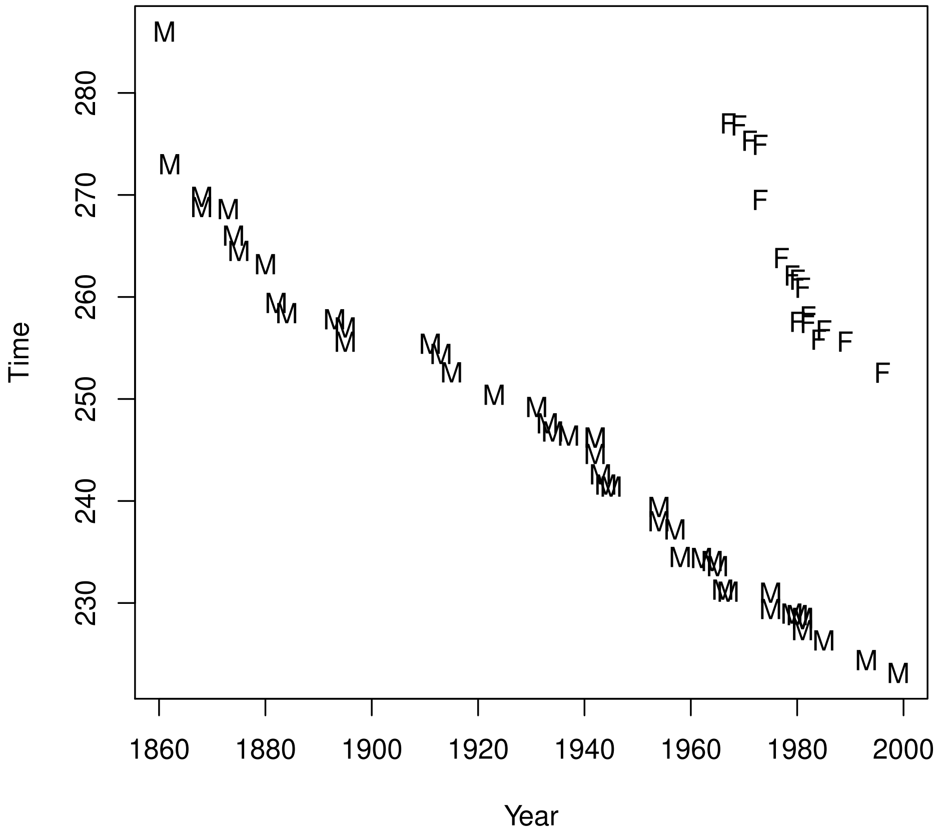
(f) List two principles of experimental design.

(g) Explain what a Latin square is and its use in experimental design.

Question 6 (15 marks) Data was collected on the world record times for the one-mile run. For males, the records are from the period 1861–2003, and for females, from the period 1967–2003. This data is analysed below.

```
> mile <- read.csv('mile.csv', header=T)
> mile$Gender <- factor(mile$Gender)
> plot(Time ~ Year, data = mile, pch=as.character(Gender))
> imodel <- lm(Time ~ (Year + Gender)^2, data = mile)
> summary(imodel)

Call:
lm(formula = Time ~ (Year + Gender)^2, data = mile)

Residuals:
    Min      1Q  Median      3Q     Max 
-5.4512 -1.6160 -0.1137  1.1784 13.7265 

Coefficients:
                  Estimate Std. Error t value Pr(>|t|)    
(Intercept)      2309.4247   202.0583  11.429  < 2e-16 ***
Year               -1.0337     0.1021 -10.126 1.95e-14 ***
GenderMale      -1355.6778   203.1441  -6.673 1.03e-08 ***
Year:GenderMale     0.6675     0.1027   6.502 2.00e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.989 on 58 degrees of freedom
Multiple R-squared:  0.9663,        Adjusted R-squared:  0.9645 
F-statistic: 553.8 on 3 and 58 DF,  p-value: < 2.2e-16

> amodel <- lm(Time ~ Year + Gender, data = mile)
> summary(amodel)

Call:
lm(formula = Time ~ Year + Gender, data = mile)

Residuals:
    Min      1Q  Median      3Q     Max 
-5.9071 -2.0988 -0.1141  1.2002 13.1863 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept) 1003.00334   27.84691   36.02   <2e-16 ***
Year          -0.37364    0.01406  -26.57   <2e-16 ***
GenderMale   -34.85078    1.30099  -26.79   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 3.896 on 59 degrees of freedom
```

```
Multiple R-squared:  0.9417,       Adjusted R-squared:  0.9397
F-statistic: 476.3 on 2 and 59 DF,  p-value: < 2.2e-16

> anova(amodel, imodel)

Analysis of Variance Table

Model 1: Time ˜ Year + Gender
Model 2: Time ˜ (Year + Gender)ˆ2
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1     59 895.62
2     58 518.03  1    377.59 42.276 2.001e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

> linearHypothesis(imodel, c(0,1,0,1), -0.3)

Linear hypothesis test

Hypothesis:
Year  + Year:GenderMale = - 0.3

Model 1: restricted model
Model 2: Time ˜ (Year + Gender)ˆ2

  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1     59 850.63
2     58 518.03  1     332.6 37.238 9.236e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

> qt(c(0.95,0.975,0.99,0.995), df=58)

[1] 1.671553 2.001717 2.392377 2.663287

> qt(c(0.95,0.975,0.99,0.995), df=59)

[1] 1.671093 2.000995 2.391229 2.661759
```



(a) Do you think that this data satisfies the linear model assumptions? Explain.

(b) What are the covariates used in the `imodel` object (in the context of the study)?

(c) What is being tested in the `anova` function call? What do you conclude from the results?

(d) Write down the final fitted models for the male and female records.

(e) Calculate a point estimate for the year when the female world record will equal the male world record. Do you expect this estimate to be accurate? Why or why not?

(f) Calculate a 95% confidence interval for the amount by which the gap between the male and female world records narrow every year.

(g) What is the hypothesis being tested in the `linearHypothesis` function call? What do you conclude from the output?

Question 7 (12 marks)

(a) Randomisation eliminates confounding from both known and unknown factors. Explain why it is additionally advantageous to use blocking for some confounding factors.

(b) A study is to be conducted to evaluate the effect of a drug on brain function. The evaluation consists of measuring the response of a particular part of the brain using an MRI scan. The drug is prescribed in doses of 1, 2 and 5 milligrams. Funding allows only 24 observations to be taken in the current study.

Explain how control might be used in a design of this experiment.

(c) For the scenario above, explain how replication might be used in a design of this experiment.

(d) For a complete block design with b blocks and k treatments, the reduced design matrix $X_{2|1}$ satisfies

$$X_{2|1}^T X_{2|1} = b\left[I_k - \frac{1}{k}J_k\right],$$

where I_k is the $k \times k$ identity matrix and J_k is the $k \times k$ matrix with all elements equal to 1. Show that

$$(X_{2|1}^T X_{2|1})^c = \frac{1}{b} I_k.$$

(e) For the above complete block design, show that if a quantity $\mathbf{t}^T\boldsymbol{\tau}$ involving only the treatment parameters is estimable, then it must be a treatment contrast. (That is, $\mathbf{t}^T\mathbf{1} = 0$.)

End of Exam—Total Available Marks = 90.