MAT4MDS

Quiz 2, Solution

Q1

$$A = \begin{bmatrix} -3 & -5 \\ 2 & -4 \end{bmatrix} \quad B = \begin{bmatrix} -3 & -3 \\ 5 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 3 & -3 \\ 5 & 5 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$$

- $2A - 3B = \begin{bmatrix} -6 & -10 \\ 4 & -8 \end{bmatrix} - \begin{bmatrix} -9 & -9 \\ 15 & -9 \end{bmatrix} = \begin{bmatrix} -6+9 & -10+9 \\ 4-15 & -8+9 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -11 & 1 \end{bmatrix}$

- $CD = \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & -3 \\ 5 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 2-20 & 3-20 & -3+0 \\ 6+10 & 9+10 & -9+0 \end{bmatrix}$

  $= \begin{bmatrix} -18 & -17 & -3 \\ 16 & 19 & -9 \end{bmatrix}$

- $\det C = 1 \times 2 - 3 \times (-4) = 2 + 12 = 14$

- $\det (D^T C) = ?$

  $\underbrace{D^T}_{\underline{3} \times 2} \underbrace{C}_{2 \times \underline{2}}$ is of the size $3 \times 2 \Rightarrow$

  $\det (D^T C)$ is not possible to calculate, as $D^T C$ is not a square matrix

- $(E)^{-1} = \dfrac{1}{\underbrace{-12-4}_{\det E}} \begin{bmatrix} -6 & -1 \\ -4 & +2 \end{bmatrix} = \dfrac{1}{-16} \begin{bmatrix} -6 & -1 \\ -4 & 2 \end{bmatrix}$

  $= \begin{bmatrix} \frac{6}{16} & \frac{1}{16} \\ \frac{4}{16} & -\frac{2}{16} \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{3}{8} & \frac{1}{16} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix}}}$

Q2

$$A = \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix}$$

- Determine the largest and smallest eigenvalues of A.

λ is egenvalue of A if $\underbrace{\det(A - \lambda I) = 0}_{\text{Charact. equation of } A}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 6-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} =$$

$$= (6-\lambda)(5-\lambda) - 9 = 30 - 5\lambda - 6\lambda + \lambda^2 - 9$$

$$= \lambda^2 - 11\lambda - 21$$

$$\det(A - \lambda I) = 0 \iff \lambda^2 - 11\lambda - 21 = 0$$

$$\lambda = \frac{11 \pm \sqrt{121 - 84}}{2} = \frac{11 \pm \sqrt{37}}{2}$$

$\underbrace{\lambda \approx 2.46}_{\text{smallest eigen value}}$ $\underbrace{\lambda \approx 8.54}_{\text{largest eigenvalue}}$

(Both eigenvalues are roundet to 2nd dec. place)

- $B = \begin{bmatrix} 8 & 0 & -1 \\ 0 & 7 & 0 \\ 7 & 0 & 0 \end{bmatrix}$, $u = \begin{bmatrix} 10 \\ -4 \\ 10 \end{bmatrix}$

Is u eigenvenctor of B ?

$$Bu = \begin{bmatrix} 8 & 0 & -1 \\ 0 & 7 & 0 \\ 7 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 10 \\ -4 \\ 10 \end{bmatrix}}_{= u} = \begin{bmatrix} 80 + 0 - 10 \\ 0 - 28 + 0 \\ 70 - 0 - 0 \end{bmatrix} = \begin{bmatrix} 70 \\ -28 \\ 70 \end{bmatrix} = 7 \underbrace{\begin{bmatrix} 10 \\ -4 \\ 10 \end{bmatrix}}_{= u} = 7u$$

Since $Bu = 7u \Rightarrow u$ is an eigenvector of B and $\lambda = 7$ is the associated eigenvalue

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Q3. • $A = \begin{bmatrix} x & 7 \\ 9 & 2x \end{bmatrix}$

Find x, $x > 0$ so that A is not invertable

A is not invertable if det A = 0

$\det A = x \times 2x - 9 \times 7 = 2x^2 - 63$

$\det A = 0 \iff 2x^2 = 63$

$x^2 = \frac{63}{2}$

$x = \pm\sqrt{\frac{63}{2}}$

We are looking for $x > 0$, so $x = \sqrt{\frac{63}{2}} \approx 5.61$ rounded to the second decimal place

• Given $A = \begin{bmatrix} 5 & 7 \\ 0 & -11 \end{bmatrix}$

$v = \begin{bmatrix} 1 \\ x \end{bmatrix}$

Find such x, that v is eigenvector of A an $\lambda = -11$ the assosiated eigenvalue.

Solution: v, $\lambda = -11$ must satisfy the following condition:

$Av = \lambda v \iff \begin{bmatrix} 5 & 7 \\ 0 & -11 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = -11 \begin{bmatrix} 1 \\ x \end{bmatrix}$

$\iff \begin{matrix} 5 + 7x = -11 \\ -11x = -11x \end{matrix} \quad (*)$

The secon equation in $(*)$ is valid for any x, so x must satisfy: $5 + 7x = -11$

$\Rightarrow 7x = -16$

$\boxed{x = -\frac{16}{7}}$

MAT4MDS, Quiz 2, Solution

Q4

- $A = \begin{bmatrix} a, & 5 \\ -5 & b \end{bmatrix}$. It is given that $\lambda_1 = 5$, $\lambda_2 = 5$ are eigenvalues of A

Using trace and det. properties find $a > 0$, b.

If λ_1, λ_2 are eigenvalues of A, (2×2) then: $\lambda_1 + \lambda_2 = tr(A)$

$$\lambda_1 \times \lambda = \det A$$

$$\Leftrightarrow \quad 5+5 = \underbrace{a+b}_{Tr(A)}$$

$$5 \times 5 = \underbrace{ab + 25}_{\det A}$$

$$\Leftrightarrow \quad a+b = 10$$

$$ab + 25 = 25$$

$$\Leftrightarrow \quad a+b = 10$$

$$ab = 0$$

$a \neq 0 \Rightarrow \boxed{b = 0}$ and $\boxed{a = 10}$
(a must be positive)

- It is known $v = \begin{bmatrix} 6 \\ 6 \\ -10 \end{bmatrix}$ is eigenvector of A with $\lambda = -1$; find $A^3 v$.

$$Av = -\begin{bmatrix} 6 \\ 6 \\ -10 \end{bmatrix}$$

$$A^3 v = A \times A \times A \times v = A\,A(\underbrace{Av}_{\lambda v})$$

$$= AA \times \lambda v = \lambda \times A \times (Av)$$

$$= \lambda^2 \underbrace{Av}_{\lambda v} = \lambda^3 v = (-1)^3 \begin{bmatrix} 6 \\ 6 \\ -10 \end{bmatrix} = -\begin{bmatrix} 6 \\ 6 \\ -10 \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \\ 10 \end{bmatrix}$$

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Q4 part 3

A is 3×3 matrix

$\lambda_1 = 5$, $\lambda_2 = -2$ two eigen values.

A is not invertable, $\Rightarrow$ $\det A = 0$

Find λ_3.

From det. properties:

$$\lambda_1 \times \lambda_2 \times \lambda_3 = \underbrace{\det A}_{=0}$$

$$\Rightarrow 5 \times -2 \times \lambda_3 = 0$$

$$\Rightarrow -10 \lambda_3 = 0$$

$$\Rightarrow \boxed{\lambda_3 = 0}$$

MATH4MDS Quiz 2

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Q5

The following Data are given:

Date	1/03	8/03	15/03	22/03	29/03
Number of cases	90	403	1808	8103	36316

Assume $N = N_0 e^{an}$, where

N is # of cases, n = number of weeks passed after March 1.

- n is given by

n	0	1	2	3	4

- $\ln N$ is given by

$\ln N$	$\ln 90 \approx 4.5$	$\ln 403 \approx 6.0$	$\ln 1808 \approx 7.5$	$\ln 8103 \approx 9.0$	$\ln 36316 \approx 10.5$

- $\ln N = \alpha n + \beta$

where α, β are solution of $A\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = B$.

Thereby $A = \begin{bmatrix} \sum n_i^2 & \sum n_i \\ \sum n_i & \underset{\uparrow}{n} \end{bmatrix} = \begin{bmatrix} 0^2+1^2+2^2+3^2+4^2 & 0+1+2+3+4 \\ 0+1+2+3+4 & 5 \end{bmatrix}$

tot. number of data

$\Rightarrow$ • $A = \begin{bmatrix} 30 & 10 \\ 10 & 5 \end{bmatrix}$;

• $B = \begin{bmatrix} \sum n_i \ln N_i \\ \sum \ln N_i \end{bmatrix} = \begin{bmatrix} 0 \times 4.5 + 1 \times 6 + 2 \times 7.5 + 3 \times 9 + 4 \times 10.5 \\ 4.5 + 6 + 7.5 + 9 + 10.5 \end{bmatrix}$

$= \begin{bmatrix} 90 \\ 37.5 \end{bmatrix}$; • $A^{-1} = \dfrac{1}{150-100}\begin{bmatrix} 5 & -10 \\ -10 & 30 \end{bmatrix} = \dfrac{1}{50}\begin{bmatrix} 5 & -10 \\ -10 & 30 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

Q5 (continue)

- $$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}}_{A^{-1}} \times \begin{bmatrix} 90 \\ \underset{\frac{75}{2}}{\overset{\parallel}{37.5}} \end{bmatrix} = \begin{bmatrix} 9 - \frac{15}{2} \\ -18 + \frac{45}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ +\frac{9}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1.5 \\ 4.5 \end{bmatrix} \Rightarrow \ln N = 1.5n + 4.5$$

- Determ. sum of square of residuals, i.e.

$\cancel{\alpha, \beta}$: $\sum (\ln N_i - \widehat{\ln N_i})^2 = ?$ estimation with lin. model

n	$\ln N_i$	$\widehat{\ln N_i}$
0	4.5	$1.5 \times 0 + 4.5 = 4.5$
1	6	$1.5 \times 1 + 4.5 = 6$
2	7.5	$1.5 \times 2 + 4.5 = 7.5$
3	9	$1.5 \times 3 + 4.5 = 9$
4	10.5	$1.5 \times 4 + 4.5 = 10.5$

$$\Rightarrow \sum_i (\ln N_i - \ln \widehat{N_i})^2 = (4.5 - 4.5)^2 + (6-6)^2 + (7.5 - 7.5)^2 + (9-9)^2 + (10.5 - 10.5)^2 = 0$$

- Find model for N from $\ln N = 1.5n + 4.5$

$\ln N = 1.5n + 4.5$

$$\Leftrightarrow N = e^{1.5n + 4.5} = e^{1.5n} \times e^{4.5} \approx 90 \times e^{1.5n}$$

$$\Rightarrow \boxed{N_0 = 90, \ a = 1.5 \ \text{for} \ N = N_0 e^{an}}$$

- April 5 means $n = 5 \Rightarrow N \approx 90 \times e^{1.5 \times 5} \approx 162724$

Q6. AX gives the final percentage in each portfolio categorie

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \text{score vector for 3 assesments}$$

Skills Categories: writing, presentation, analytical

$$\begin{array}{r} \\ Wr. \\ AX = Pr. \\ Anal \end{array} \underbrace{\begin{array}{c} \text{Test1 Test2 Test3} \\ \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \end{array}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2x_1 + 0.4x_2 + 0.4x_3 \\ 0.2x_1 + 0.3x_2 + 0.5x_3 \\ 0.5x_1 + 0.2x_2 + 0.3x_3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.64 \\ 0.65 \\ 0.48 \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.64 \\ 0.65 \\ 0.48 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 0.64 \\ 0.65 \\ 0.48 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.2 & 0.3 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}^{-1} \begin{bmatrix} 0.64 \\ 0.65 \\ 0.48 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.7 \\ 0.8 \end{bmatrix}$$