MAST30022 Decision Making
2021
Tutorial 10

1. **(PS8-4)** Consider the network shown in Figure 1 below.

   (a) Is the labelling given on the network a proper labelling? If it is not proper, find a proper labelling by using the method of counting the number of predecessors of each vertex.

   (b) Based on the proper labelling you found in (a), find **all** Pareto minimal paths from vertex u to vertex v.

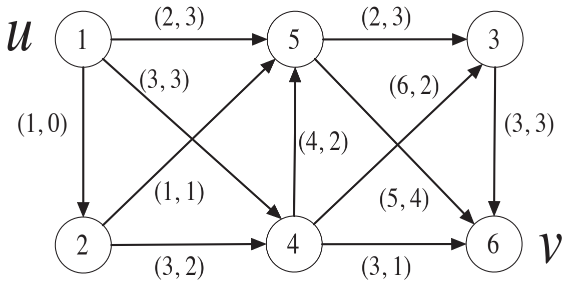

Figure 1: PS8-4

Solution

(a) The given labelling is not proper since, for example, for the arc (4, 3) the label 4 of the tail is larger than the label 3 of the head.

The numbers of predecessors of the nodes can be counted easily. They are: $0, 3, 4$ (top row from left to right), $1, 2, 5$ (bottom row from left to right), see Figure 2. Arranging these numbers in non-decreasing order we get a proper labelling, which is shown in Figure 3.

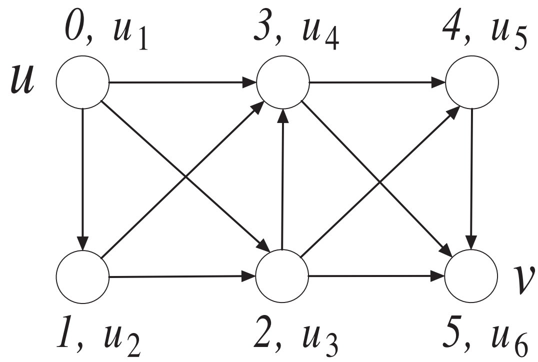

Figure 2: PS8-4

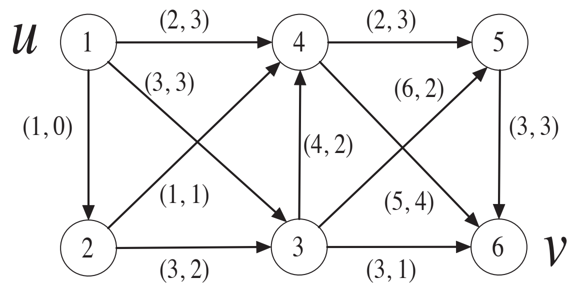

Figure 3: PS8-4

(b) Functional equations:

$$f(1) = \{(0,0)\}$$

$$\begin{aligned}
f(2) &= P\text{-}\min\left(\bigcup_{j\in P(2)} \left(f(j) + (c^1_{j2}, c^2_{j2})\right)\right) \\
&= P\text{-}\min\left(\bigcup_{j\in \{1\}} \left(f(j) + (c^1_{j2}, c^2_{j2})\right)\right) \\
&= P\text{-}\min(f(1) + (c^1_{12}, c^2_{12})) \\
&= P\text{-}\min(\{(0,0) + (1,0)\}) \\
&= P\text{-}\min(\{(1,0)\}) \\
&= \{\ \underbrace{(1,0)}_{\text{path }1-2}\ \}, \qquad P^*(2) = \{1\}
\end{aligned}$$

$$\begin{aligned}
f(3) &= P\text{-}\min\left(\bigcup_{j\in \{1,2\}} \left(f(j) + (c^1_{j3}, c^2_{j3})\right)\right) \\
&= P\text{-}\min[(f(1) + (c^1_{13}, c^2_{13})) \cup (f(2) + (c^1_{23}, c^2_{23}))] \\
&= P\text{-}\min[\{(0,0) + (3,3)\} \cup \{(1,0) + (3,2)\}] \\
&= P\text{-}\min(\{(3,3), (4,2)\}) \\
&= \{\ \underbrace{(3,3)}_{\text{path }1-3;}\ ,\ \underbrace{(4,2)}_{\text{path }1-2-3}\ \}, \qquad P^*(3) = \{1,2\}
\end{aligned}$$

$$
\begin{aligned}
f(4) &= P\text{-}\min\left(\bigcup_{j\in\{1,2,3\}}\left(f(j)+(c^1_{j4},c^2_{j4})\right)\right)\\
&= P\text{-}\min[(f(1)+(c^1_{14},c^2_{14}))\cup(f(2)+(c^1_{24},c^2_{24}))\\
&\quad \cup(f(3)+(c^1_{34},c^2_{34}))]\\
&= P\text{-}\min(\{(0,0)+(2,3)\}\cup\{(1,0)+(1,1)\}\\
&\quad \cup\{(3,3)+(4,2),(4,2)+(4,2)\})\\
&= P\text{-}\min(\{(2,3),(2,1),(7,5),(8,4)\})\\
&= \{\underbrace{(2,1)}_{\text{path }1-2-4}\},\qquad P^*(4)=\{2\}
\end{aligned}
$$

$$
\begin{aligned}
f(5) &= P\text{-}\min\left(\bigcup_{j\in\{3,4\}}\left(f(j)+(c^1_{j5},c^2_{j5})\right)\right)\\
&= P\text{-}\min[(f(3)+(c^1_{35},c^2_{35}))\cup(f(4)+(c^1_{45},c^2_{45}))]\\
&= P\text{-}\min[\{(3,3)+(6,2),(4,2)+(6,2)\}\cup\{(2,1)+(2,3)\}]\\
&= P\text{-}\min(\{(9,5),(10,4),(4,4)\})\\
&= \{\underbrace{(4,4)}_{\text{path }1-2-4-5}\},\qquad P^*(5)=\{4\}
\end{aligned}
$$

$$
\begin{aligned}
f(6) &= P\text{-}\min\left(\bigcup_{j\in\{3,4,5\}}\left(f(j)+(c^1_{j6},c^2_{j6})\right)\right)\\
&= P\text{-}\min[(f(3)+(c^1_{36},c^2_{36}))\cup(f(4)+(c^1_{46},c^2_{46}))\\
&\quad \cup(f(5)+(c^1_{56},c^2_{56}))]\\
&= P\text{-}\min[\{(3,3)+(3,1),(4,2)+(3,1)\}\cup\{(2,1)+(5,4)\}\\
&\quad \cup\{(4,4)+(3,3)\}]\\
&= P\text{-}\min\{(6,4),(7,3),(7,5),(7,7)\}\\
&= \{\underbrace{(6,4)}_{\text{path }1-3-6;}\ ,\ \underbrace{(7,3)}_{\text{path }1-2-3-6}\},\qquad P^*(6)=\{3\}
\end{aligned}
$$

The information above is summarised in Figure 4. From the figure, the Pareto minimal distances from node 1 to node 6 are: $(6,4),(7,3)$. Since

$$
\begin{aligned}
(6,4)&=(3,1)+(3,3),\ \text{where } (3,1)=(c^1_{36},c^2_{36}),(3,3)\in f(3)\\
(3,3)&=(3,3)+(0,0),\ \text{where } (3,3)=(c^1_{13},c^2_{13}),(0,0)\in f(1)
\end{aligned}
$$

one can see that the Pareto minimal path for $(6,4)$ is 1-3-6. Similarly, the Pareto minimal path for $(7,3)$ is 1-2-3-6.

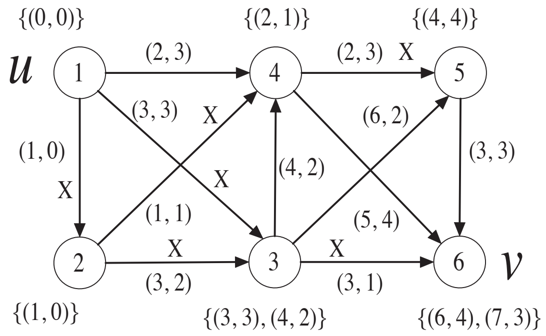

Figure 4: PS8-4

2. **(PS8-8)** J. R. Carrington has \$4 million to invest in three oil well sites. The amount of revenue earned from site i ($i = 1, 2, 3$) depends on the amount of money invested in site i as shown in the table below. Assuming that the amount invested in a site must be an exact multiple of \$1 million, use dynamic programming to determine an investment policy that will maximize the revenue that J. R. Carrington will earn from his three oil wells.

		Revenue (\$ Million)		
		Site 1	Site 2	Site 3
	0	4	3	3
Amount Invested	1	7	6	7
(\$ Million)	2	8	10	8
	3	9	12	13
	4	11	14	15

(Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004.)

Solution

Define

- stage i = investing money in site i, $i = 1, 2, 3$;
- x_i = state at stage i = the amount of money available to invest in sites $i, \ldots, 3$, where x_i is an integer smaller than or equal to 4;
- y_i = the amount of money actually invested in site i ($y_i \le x_i$). Denote the revenue of investing y_i money in site i by $r_i(y_i)$;
- action at stage i: invest y_i;
- $f_i(x_i)$ = the maximum revenue that can be obtained from investment in sites $i, \ldots, 3$, if x_i million dollars are available at stage i.

Then

$$f_3(x_3) = \max_{y_3 \in \{0,1,\ldots,x_3\}} \{r_3(y_3)\} = r_3(x_3), \quad \text{and } y_3^*(x_3) = x_3,$$

since $r_3(\cdot)$ is an increasing function, it is optimal to invest all money available in site 3 at stage 3.

For $i = 1, 2$ we have

$$f_i(x_i) = \max_{y_i \in \{0,1,\ldots,x_i\}} \{r_i(y_i) + f_{i+1}(x_i - y_i)\}.$$

Stage 3 computations:

x_3	$f_3(x_3) = r_3(x_3)$	$y_3^*(x_3) = x_3$
0	3	0
1	7	1
2	8	2
3	13	3
4	15	4

Stage 2 computations:

	$r_2(y_2) + f_3(x_2 - y_2)$					Optimum solution	
x_2	$y_2 = 0$	$y_2 = 1$	$y_2 = 2$	$y_2 = 3$	$y_2 = 4$	$f_2(x_2)$	$y_2^*(x_2)$
0	6	–	–	–	–	6	0
1	10	9	–	–	–	10	0
2	11	13	13	–	–	13	1 or 2
3	16	14	17	15	–	17	2
4	18	19	18	19	17	19	1 or 3

Stage 1 computations (note that we know that $x_1 = 4$):

	$r_1(y_1) + f_2(4 - y_1)$					Optimum solution	
x_1	$y_1 = 0$	$y_1 = 1$	$y_1 = 2$	$y_1 = 3$	$y_1 = 4$	$f_1(x_1)$	$y_1^*(x_1)$
4	23	24	21	19	17	24	1

Optimal strategy: invest $y_1^*(4) = 1$ million dollar in site 1, $y_2^*(4-1) = y_2^*(3) = 2$ million dollars in site 2, and $y_3^*(3-2) = y_3^*(1) = 1$ million dollar in site 3. The total return is 24 million dollars.

3. **(PS8-9)** (The knapsack problem)
You have k types of items, and you want to bring some of them to an aircraft. Each item of type i has value v_i and weight w_i. The airline's weight limit is w. How should you pack your belongings such that the total value of the packed items is maximised subject to the weight constraint?

Solve the knapsack problem with 3 types of items such that

$$w_1 = 4,\ w_2 = 3,\ w_3 = 5,$$

$$v_1 = 11,\ v_2 = 7,\ v_3 = 12,$$

and $w = 10$. (Winston, Section 18.4, knapsack problem.)

Solution

Assume you pack y_i items of type i, $i = 1, 2, 3$. Then the problem is

$$\max \sum_{i=1}^{3} y_i v_i$$

$$\text{s.t.} \quad \sum_{i=1}^{3} y_i w_i \le w$$

$$y_1, y_2, y_3 \ge 0 \text{ are integers.}$$

This is a special case of the resource allocation problem for which

$$g_i(y_i) = y_i w_i,\ \ r_i(y_i) = y_i v_i,\ \ i = 1, 2, 3.$$

In this problem we have 3 stages, each corresponding to one type of item. The state x_i at stage i is the weight limit for items of types $i, \ldots, 3$. The possible actions at stage i are "pack y_i items of type i" where y_i is an integer such that $y_i w_i \le x_i$. Let

$$f_i(x_i) = \text{max. total value with weight limit } x_i \text{ and types } i, i+1, \ldots, 3.$$

Then the DP equation is

$$f_i(x_i) = \max_{y_i}\{y_i v_i + f_{i+1}(x_i - y_i w_i) : y_i \ge 0 \text{ an integer s.t. } y_i w_i \le x_i\}.$$

Beginning with $f_3(\cdot)$ and $y_3^*(\cdot)$ and working backward, we compute $f_2(\cdot), y_2^*(\cdot)$ and $f_1(\cdot), y_1^*(\cdot)$ sequentially.

Stage 3 computation:

$$\begin{aligned} f_3(x_3) &= \max_{y_3 : y_3 w_3 \le x_3} \{y_3 v_3\} \\ &= \max_{y_3 : 5y_3 \le x_3} \{12 y_3\}. \end{aligned}$$

Here we only provide the details for $f_3(10)$, $f_3(7)$, and $f_3(6)$, the three values required in stage 2 computation. This is the case because we can only have a weight limit at stage 3 of 6 (one of Type 1 and none of Type 2 allocated), or 7 (none of Type 1 and one of Type 2 allocated), or 10 (none of Type 1 and none of Type 2 allocated). Also, note that we can have 0 weight limit left at stage 3 (one of Type 1 and two of Type 2 allocated) but $f_3(0) = 0$.

We thus have

$$f_3(10) = \max_{y_3:5y_3\le 10}\{12y_3\} = \max\{\underbrace{0}_{y_3=0}, \underbrace{12}_{y_3=1}, \underbrace{24}_{y_3=2}\} = 24,\ y_3^*(10) = 2$$

$$f_3(7) = \max_{y_3:5y_3\le 7}\{12y_3\} = \max\{\underbrace{0}_{y_3=0}, \underbrace{12}_{y_3=1},\} = 12,\ y_3^*(7) = 1$$

$$f_3(6) = \max_{y_3:5y_3\le 6}\{12y_3\} = \max\{\underbrace{0}_{y_3=0}, \underbrace{12}_{y_3=1},\} = 12,\ y_3^*(6) = 1$$

Stage 2 computation:

$$\begin{aligned} f_2(x_2) &= \max_{y_2:y_2w_2\le x_2}\{y_2v_2 + f_3(x_2 - y_2w_2)\} \\ &= \max_{y_2:3y_2\le x_2}\{7y_2 + f_3(x_2 - 3y_2)\}. \end{aligned}$$

Here we only provide the details for $f_2(10)$, $f_2(6)$, and $f_2(2)$, the three values required in stage 1 computation. This is the case because we can only have a weight limit at stage 2 of 2 (two of Type 1 allocated), or 6 (one of Type 1 allocated), or 10 (none of Type 1 allocated).

$$\begin{aligned} f_2(10) &= \max_{y_2:3y_2\le 10}\{7y_2 + f_3(10 - 3y_2)\} \\ &= \max\{\underbrace{f_3(10)}_{y_2=0}, \underbrace{7 + f_3(7)}_{y_2=1}, \underbrace{14 + f_3(4)}_{y_2=2}, \underbrace{21 + f_3(1)}_{y_2=3}\} \\ &= \max\{24, 19, 14, 21\} = 24, \\ y_2^*(10) &= 0 \end{aligned}$$

$$\begin{aligned} f_2(6) &= \max_{y_2:3y_2\le 6}\{7y_2 + f_3(6 - 3y_2)\} \\ &= \max\{\underbrace{f_3(6)}_{y_2=0}, \underbrace{7 + f_3(3)}_{y_2=1}, \underbrace{14 + f_3(0)}_{y_2=2}\} \\ &= \max\{12, 7, 14\} = 14 \\ y_2^*(6) &= 2 \end{aligned}$$

and finally, $f_2(2) = 0$ (items are too heavy), and $y_2^*(2) = 0$.

Stage 1 computation: we know that $x_1 = w = 10$, so

$$\begin{aligned} f_1(10) &= \max_{y_1:y_1w_1\le 10}\{y_1v_1 + f_2(10 - y_1w_1)\} \\ &= \max_{y_1:4y_1\le 10}\{11y_1 + f_2(10 - 4y_1)\} \\ &= \max\{\underbrace{f_2(10)}_{y_1=0}, \underbrace{11 + f_2(6)}_{y_1=1}, \underbrace{22 + f_2(2)}_{y_1=2}\} \\ &= \max\{24, 25, 22\} = 25 \\ y_1^*(10) &= 1. \end{aligned}$$

We have $f_1(10) = 25$ and $y_1^*(10) = 1$. Hence, we should include one Type 1 item in the knapsack. Then we have $10 - 4 = 6$ kilos left for Type 2 and Type 3 items, so we should include $y_2^*(6) = 2$ Type 2 items. Finally, we have $6 - 2 \cdot 3 = 0$ kilos left for Type 3 items, and we include $y_3^*(0) = 0$ Type 3 items.

In summary, the maximum value that can be gained from a 10 kilos knapsack is $f_1(10) = 25$. To obtain a value of 25, one Type 1 and two Type 2 items should be included.