School of Mathematics and Statistics
MAST30030
Applied Mathematical Modelling

Problem Sheet 1
One-Dimensional Dynamical Systems

Question 1

Consider the dynamical system $\dot{x} = \sin x$.

(a) Find all fixed points of this system.
(b) At what values of x does the flow have its greatest velocity?
(c) Find the flow's acceleration $\ddot{x}$ as a function of x.
(d) When does the flow have maximum positive acceleration?

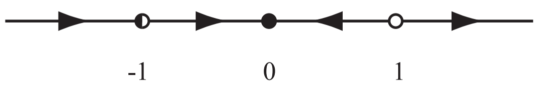
Question 2

For the following dynamical systems, sketch the vector field on the real line, locate all fixed points, discuss their stability, and sketch $x(t)$ - do not explicitly solve the dynamical systems, but perform these calculations graphically.

(a) $\dot{x} = 4x^2 - 16$
(b) $\dot{x} = 1 - x^{14}$
(c) $\dot{x} = x - x^3$
(d) $\dot{x} = e^{-x} \sin x$
(e) $\dot{x} = 1 + (1/2) \cos x$
(f) $\dot{x} = 1 - 2 \cos x$

Question 3

Consider the phase portrait below for some unknown dynamical system $\dot{x} = f(x)$. Graph a possible form for $f(x)$.



Question 4

Use a linear stability analysis to determine the stability of fixed points for the following dynamical systems. Compare this to the graphical procedure you used in Question 2. In some problems, the graphical procedure is essential in determining stability - discuss why.

(a) $\dot{x} = x(1 - x)$
(b) $\dot{x} = x(1 - x)(2 - x)$
(c) $\dot{x} = \tan x$
(d) $\dot{x} = x^2(6 - x)$
(e) $\dot{x} = \ln x$

Question 5

A simple harmonic oscillator $\ddot{x} = -ax$ oscillates in one-dimension along the x-axis. Does this contradict the statement that one-dimensional systems can not oscillate? Explain your answer.

Two-Dimensional Linear Dynamical Systems

Question 6

Consider the dynamical system

$$\dot{x} = ax$$
$$\dot{y} = -y$$

where $a < -1$.

(a) Find explicit solutions to $x(t)$ and $y(t)$. Hint: The system is decoupled, so this can be performed trivially.
(b) Eliminate the independent variable t in (a), and hence express y as a function of x.
(c) Using the result in (b), sketch the phase portrait for this system.
(d) What direction do trajectories follow as $t \to \infty$?

Question 7

Consider the dynamical system

$$\dot{x} = 4x - y$$
$$\dot{y} = 2x + y$$

(a) Write the system in matrix form.
(b) Determine the corresponding eigenvalues and eigenvectors.
(c) Find the general solution to this system.
(d) Classify the fixed points of this system.
(e) Solve the system subject to the initial condition $(x_0, y_0) = (3, 4)$.

Question 8

Consider the dynamical system

$$\dot{x} = x - y$$
$$\dot{y} = x + y$$

(a) Write the system in matrix form, calculate the corresponding eigenvalues and show they are complex conjugates.
(b) Calculate the corresponding eigenvectors.
(c) Find the general solution to this system, and write it in terms of real functions and coefficients.

Question 9

Classify the fixed points of the following dynamical systems and plot their phase portraits. In some case(s), the phase portrait can not be determined using standard stability analysis. Explain why.

(a) $\dot{x} = 3x - 4y$, $\dot{y} = x - y$
(b) $\dot{x} = 5x + 2y$, $\dot{y} = -17x - 5y$
(c) $\dot{x} = -3x + 4y$, $\dot{y} = -2x + 3y$
(d) $\dot{x} = 4x - 3y$, $\dot{y} = 8x - 6y$