MAT4MDS — Practice 2 Worked Solutions

Model answers for Practice 2

Questions 1 to 4 concern $F\colon [a,\infty) \to \mathbb{R} \quad F(x) = 1 - \left(\frac{a}{x}\right)^b$.

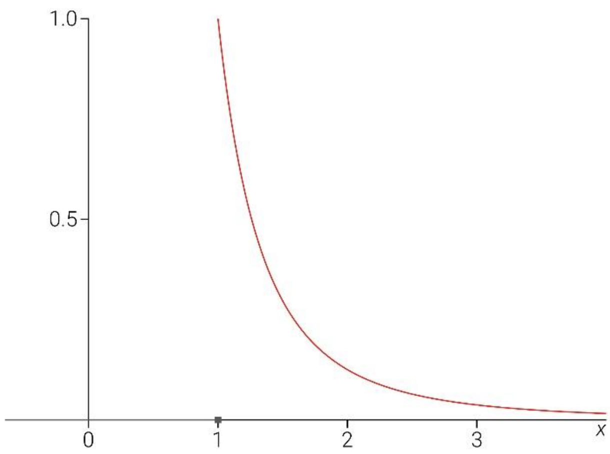
Question 1.

(a) $F(a) = 1 - 1^b = 0$

(b) As x gets very large, x^{-b} gets small, so that $F(x) \to 1$.

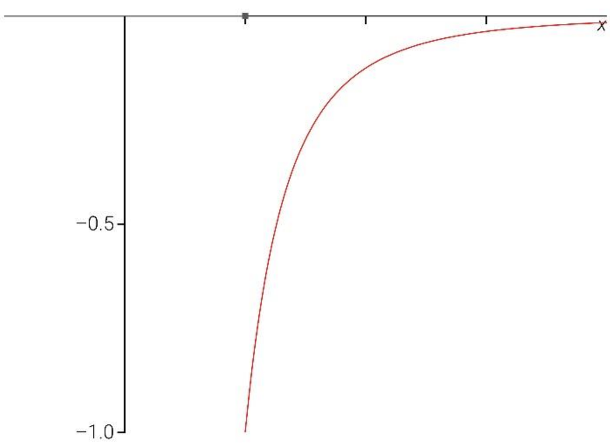
Question 2.

(a) In the graph of x^{-b}, the exponent b determines the shape of the curve as it decreases.

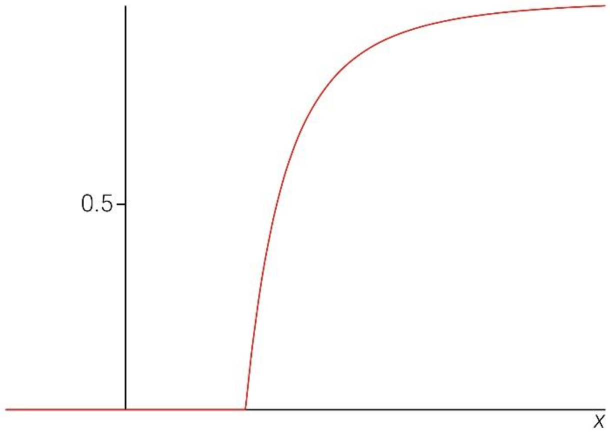


(b) The parameter a causes the graph to be scaled horizontally.

(c) The graph of $-\left(\frac{x}{a}\right)^{-b}$ is



(d) Finally, shift the graph up by one unit to obtain $F(x)$.



Question 3.

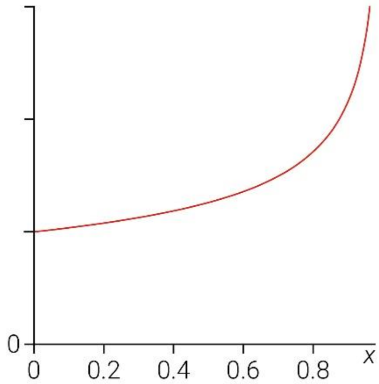
(a) If $f(x) = \frac{a}{x}$, let $g(x) = 1 - x^b$. Then $g(f(x)) = 1 - \left(\frac{a}{x}\right)^b = F(x)$.

(b) If $h(x) = x^b$, let $k(x) = 1 - \frac{a^b}{x}$. Then, $k(h(x)) = 1 - \frac{a^b}{x^b} = F(x)$.

Question 4.

(a) F^{-1} will exist because each value on the vertical axis can be traced back to one input value. The input values of F^{-1} are $[0,1)$. The output values for F^{-1} are $[a, \infty)$.

(b) By reflecting in the line $y = x$ we sketch the graph of F^{-1}.



(c) We have $F\colon [a,\infty) \to \mathbb{R}$ with $F(x) = 1 - \left(\frac{a}{x}\right)^b$. Let $y := F^{-1}(x)$. Then $F(F^{-1}(x)) = x$

$$\Rightarrow \quad F(y) = x$$

$$\Rightarrow \quad 1 - \left(\frac{a}{y}\right)^b = x \qquad \text{(definition of } F\text{)}$$

$$\Rightarrow \quad \left(\frac{a}{y}\right)^b = 1 - x$$

$$\Rightarrow \quad \frac{a}{y} = (1-x)^{\frac{1}{b}}$$

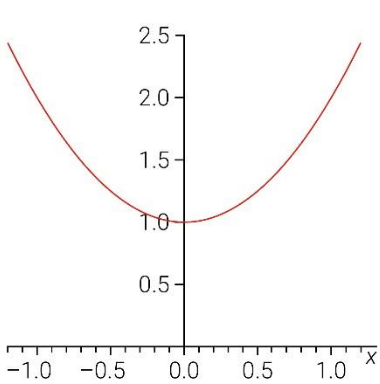
$$\Rightarrow \quad F^{-1}(x) = a(1-x)^{-\frac{1}{b}}$$

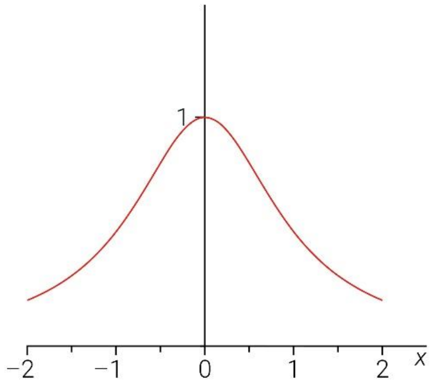
Hence $F^{-1}\colon [0,1) \to [a,\infty)$ $F^{-1}(x) = a(1-x)^{-\frac{1}{b}}$.

Question 5.

(a) If $k(x) = 1 + x^2$ and $h(x) = \frac{1}{x}$, then $f(x) = h(k(x))$.

(b) The graph on the left is $k(x) = 1 + x^2$. The graph on the right is $f(x)$.





(c) $f(x)$ does not have an inverse, because its graph is not one-to-one; that is, there is no unique input value corresponding to a particular output. The related restricted function g has an inverse, where $g\colon (-\infty, 0] \to \mathbb{R}$ $g(x) = \frac{1}{1+x^2}$.

(d) We have $g: (-\infty, 0] \to \mathbb{R}$ with $g(x) = \frac{1}{1+x^2}$. Let $y := g^{-1}(x)$. Then $g(g^{-1}(x)) = x$

$$\Rightarrow \quad g(y) = x$$

$$\Rightarrow \quad \frac{1}{1+y^2} = x \qquad \text{(definition of } g\text{)}$$

$$\Rightarrow \quad 1 + y^2 = \frac{1}{x}$$

$$\Rightarrow \quad y^2 = \frac{1}{x} - 1$$

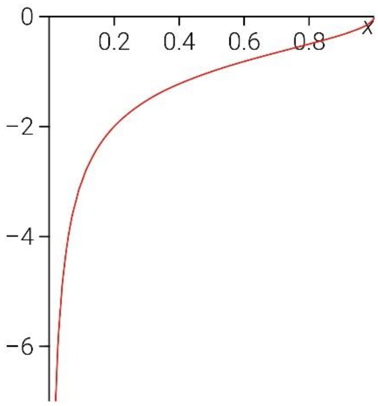
$$\Rightarrow \quad g^{-1}(x) = \pm\sqrt{\frac{1}{x} - 1}$$

$$\Rightarrow \quad g^{-1}(x) = -\sqrt{\frac{1}{x} - 1}$$

where we choose the negative sign because the outputs must lie in $(-\infty, 0]$.

Hence $g^{-1}: (0,1] \to (-\infty, 0]$ $g^{-1}(x) = -\sqrt{\frac{1}{x} - 1}$.

The graph of g^{-1} is a reflection of part of the graph of g, across the line $y = x$:



Question 6. We have $f: [0, \infty) \to \mathbb{R}$ where $f(x) = e^{-x^2}$. Let $y := f^{-1}(x)$. Then $f(f^{-1}(x)) = x$

$$\Rightarrow \quad f(y) = x$$

$$\Rightarrow \quad \mathrm{e}^{-y^2} = x \qquad \text{(definition of } f\text{)}$$

$$\Rightarrow \quad -y^2 = \log_e(x)$$

$$\Rightarrow \quad y^2 = -\log_e(x) = \log_e\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad g^{-1}(x) = \pm\sqrt{\log_e\left(\frac{1}{x}\right)}$$

$$\Rightarrow \quad g^{-1}(x) = +\sqrt{\log_e\left(\frac{1}{x}\right)}$$

where we choose the positive sign because the outputs must lie in $[0, \infty)$.

Hence $g^{-1}: (0,1] \to [0, \infty)$ $g^{-1}(x) = \sqrt{\log_e\left(\frac{1}{x}\right)}$.