MAST30027: Modern Applied Statistics

Assignment 2

Due: 1:00pm Friday 28th August (week 5)

This assignment is worth 3 1/3% of your total mark.

1. Show that the gamma distribution is an exponential family.

   Obtain the canonical link and the variance function.

   Solution: The gamma distribution with shape $\nu > 0$ and rate $\lambda > 0$ has log density

$$\begin{aligned}\log f(x;\nu,\lambda) &= (\nu-1)\log(x) - \lambda x + \nu\log(\lambda) - \log(\Gamma(\nu)) \\ &= \frac{x(-\lambda/\nu) + \log(\lambda/\nu)}{1/\nu} - \nu\log(1/\nu) + (\nu-1)\log(x) - \log(\Gamma(\nu))\end{aligned}$$

   Put $\theta = -\lambda/\nu$ and $\phi = 1/\nu$ then we have

$$\log f(x;\nu,\lambda) = \frac{x\theta - \log(-1/\theta)}{\phi} - \frac{\log(\phi)}{\phi} + \left(\frac{1}{\phi} - 1\right)\log(x) - \log(\Gamma(1/\phi))$$

   This is in the form of an exponential family, with

$$\begin{aligned}b(\theta) &= \log(-1/\theta) \\ a(\phi) &= \phi \\ c(x,\phi) &= \frac{-\log(\phi) + (1-\phi)\log(x) - \phi\log(\Gamma(1/\phi))}{\phi}\end{aligned}$$

   Note that with this parameterisation we have $\theta < 0$ and $\phi > 0$.

   For the canonical link g we have $g(\mu) = \theta$. Here $\mu = \nu/\lambda = -1/\theta$, so $g(x) = -1/x$. (Note that in practice people tend to use the inverse link $x \mapsto 1/x$ rather than $x \mapsto -1/x$, because it is convenient to keep things positive.) The variance is $\nu/\lambda^2 = \phi\mu^2 = a(\phi)v(\mu)$. That is, the variance function is $v(\mu) = \mu^2$.

2. Prove that if a random variable X has density

$$f(x;\theta,\phi) = \exp\left[\frac{x\theta - b(\theta)}{a(\phi)} + c(x,\phi)\right]$$

   Then

$$\mathbb{E}X = b'(\theta) \text{ and } \operatorname{Var} X = b''(\theta)a(\phi).$$

   Hint: show that for any likelihood L we have

$$\begin{aligned}\mathbb{E}\frac{\partial \log L}{\partial\theta} &= 0 \\ \mathbb{E}\frac{\partial^2 \log L}{\partial\theta^2} &= -\mathbb{E}\left(\frac{\partial \log L}{\partial\theta}\right)^2.\end{aligned}$$

Solution: first note that for any likelihood L we have

$$\begin{aligned}
\mathbb{E}\frac{\partial \log L}{\partial \theta} &= \int \frac{\partial \log L(\theta;x)}{\partial \theta} L(\theta;x)dx \\
&= \int \frac{1}{L(\theta;x)} \frac{\partial L(\theta;x)}{\partial \theta} L(\theta;x)dx \\
&= \int \frac{\partial L(\theta;x)}{\partial \theta} dx \\
&= \frac{\partial}{\partial \theta} \int L(\theta;x)dx \\
&= \frac{\partial}{\partial \theta} 1 \quad = \quad 0.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{E}\frac{\partial^2 \log L}{\partial \theta^2} &= \mathbb{E}\frac{\partial}{\partial \theta}\left(\frac{1}{L(\theta;x)} L'(\theta;x)\right) \\
&= \mathbb{E} - \frac{L'(\theta;x)^2}{L(\theta;x)^2} + \frac{L''(\theta;x)}{L(\theta;x)} \\
&= -\mathbb{E}\left(\frac{\partial \log L}{\partial \theta}\right)^2 + \int \frac{L''(\theta;x)}{L(\theta;x)} L(\theta;x)dx \\
&= -\mathbb{E}\left(\frac{\partial \log L}{\partial \theta}\right)^2 + \frac{\partial^2}{\partial \theta^2} \int L(\theta;x)dx \\
&= -\mathbb{E}\left(\frac{\partial \log L}{\partial \theta}\right)^2.
\end{aligned}$$

Applying the first result to f we have

$$\begin{aligned}
0 &= \mathbb{E}\frac{\partial}{\partial \theta}\left[\frac{X\theta - b(\theta)}{a(\phi)} + c(X,\phi)\right] \\
&= \mathbb{E}\frac{X - b'(\theta)}{a(\phi)}
\end{aligned}$$

whence $\mathbb{E}X = b'(\theta)$.

Applying the second result we have

$$\begin{aligned}
-\mathbb{E}\left(\frac{X - b'(\theta)}{a(\phi)}\right)^2 &= \mathbb{E}\frac{\partial^2 \log L}{\partial \theta^2} \\
-\frac{\operatorname{Var} X}{a(\phi)^2} &= \mathbb{E}\frac{\partial}{\partial \theta}\frac{X - b'(\theta)}{a(\phi)} \\
&= \mathbb{E}\frac{-b''(\theta)}{a(\phi)} \quad = \quad -\frac{b''(\theta)}{a(\phi)}
\end{aligned}$$

whence $\operatorname{Var} X = b''(\theta)a(\phi)$, as required.