MAST30001 Stochastic Modelling

Assignment 2

Please complete the Plagiarism Declaration Form on LMS.

Don't forget to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **3pm, Friday Oct 20** in the appropriate assignment box at the north end of Peter Hall building (near Wilson Lab).

Marks may be lost where answers are not clear and concise (or where lacking in explanation).

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1. A CTMC $(X_t)_{t\geq 0}$ with state space $\mathcal{S} = \{1, 2, 3, 4\}$ has the following generator

$$\begin{pmatrix} q_{1,1} & 1 & 2 & 3 \\ 0 & q_{2,2} & 2 & 0 \\ 1 & 1 & q_{3,3} & 1 \\ 1 & 0 & 0 & q_{4,4} \end{pmatrix}.$$

[8 Marks]

(a) What are the values of $q_{i,i}$ for $i = 1, 2, 3, 4$?

(b) Draw the transition diagram for this chain.

(c) Is this chain reducible?

(d) Find the 1 step transition matrix for the jump chain.

(e) Given that $X_t = 4$, find the distribution of $T^* = \inf\{s > t : X_s \neq 4\}$.

(f) Let $T(4) = \inf\{t \geq 0 : X_t = 4\}$. Find $\mathbb{E}[T(4)|X_0 = i]$.

(g) Let $T' = \inf\{t > T(4) : X_t \neq 4\}$. Find the distribution of $X_{T'}$.

(h) Starting from the uniform initial distribution, find the limiting proportion of time spent in state i for each i.

(i) Find the limiting distribution for this chain if $\mathbb{P}(X_0 = 1) = 1$.

2. The Joker escaped from Batman since you handed in assignment 1. Now Batman chases the Joker around a regular polyhedron with $k \geq 3$ corners (the corners are labelled $0, 1, 2, \ldots, k-1$ clockwise around the polyhedron). At each corner of the polyhedron there is a traffic light. Unfortunately, the traffic lights have been hijacked by the Riddler, who lets them turn green only for an instant according to the following rule: Let $(N_t)_{t\geq 0}$ be a Poisson process of rate λ. At each jump time T_j of this process the Riddler chooses a corner $i \in \{0, \ldots, k-1\}$ uniformly at random and turns the traffic light at that corner green for an instant. Both Batman and the Joker are responsible drivers, so they move clockwise to the next corner i' if and only if they are at the corner i when the traffic light at i turns green. (Note that $i' = 0$ if $i = k-1$ and otherwise $i' = i + 1$). Batman catches the Joker as soon as they are at the same corner of the polyhedron (but the Riddler continues his control of the lights regardless). **[6 Marks]**

(a) What is the distribution of the number of green lights that occur at corner i up to time t?

(b) Find $\mathbb{P}(\cap_{r=1}^{m}\{N_r = 2r\})$.

(c) Find the probability that all of the green lights up to time t have been at corner 0, given that there have been 2 green lights at corner 0 by time t.

(d) Suppose that Batman starts at corner i and the Joker starts at corner $j \neq i$. Find the probability that neither of them have moved by time t.

(e) Suppose that Batman starts at corner 0 and the Joker starts at corner i. Find the expected time until Batman catches the Joker.

3. At an infuriating airport, passengers arrive as a Poisson process of rate λ, and an airport official rolls a fair die for each passenger to decide which of 6 independent (FCFS) $M/M/1$ queues s/he will be sent to. Suppose that each server serves at rate $\mu > \lambda/6$. **[6 Marks]**

(a) Let $X_t^{(i)}$ denote the number of customers in queue i (including any customer being served). Find the stationary distribution for the Markov chain $(X_t^{(1)}, \ldots, X_t^{(6)})$.

(b) Find the expected time spent in this system by a passenger at stationarity. Will this be smaller or larger than the expected time a customer spends in an $M/M/6$ queue (with the same arrival and service rates, and also at stationarity)?

(c) Find the long run proportion of time that there are exactly k idle servers.

Suppose that passenger B is the next passenger joining the system after passenger A, and that immediately before passenger A enters the system, the system was stationary. Let Δ denote the exit time of B from the system minus the exit time of A from the system.

(d) Find the expected value of Δ given that passenger B joins the same queue as passenger A.

(e) Find the (unconditional) expected value of Δ