Q1. $A_0, \ldots, A_m$

$$P\left(\bigcap_{j=0}^{i} A_j\right) > 0 \quad , \quad i = 0, \ldots, m-1$$

show:

$$P\left(\bigcap_{j=0}^{m} A_j\right) = P(A_0) \prod_{i=1}^{m} P\left(A_i \,\middle|\, \bigcap_{j=0}^{i-1} A_j\right).$$

$$= P(A_0) \prod_{i=1}^{m} \frac{P\left(\bigcap_{k=0}^{i} A_k\right)}{P\left(\bigcap_{j=0}^{i-1} A_j\right)}$$

$$= \cancel{P(A_0)} \frac{\prod_{i=1}^{m} \cancel{P\left(\bigcap_{k=0}^{i} A_k\right)}}{\prod_{i=0}^{m-1} P\left(\bigcap_{j=0}^{i} A_j\right)}.$$

$$P\left(\bigcap_{k=0}^{m} A_k\right)$$

Q5. $N \sim Geo(p)$

$$p(N=n) = (1-p)^n p \, , \quad n = 0, 1, 2, \ldots$$

$$E(s^N) = \frac{p}{1-s(1-p)}.$$

$X \mid N = n$:

$$\boxed{\begin{aligned} \alpha &= n+1 \\ \beta &= 1 \end{aligned}} \quad \theta = \frac{1}{\beta}$$

$$f_{X|N}(x|n) = \frac{x^n e^{-x}}{n!} \, , \quad x > 0.$$

(a) $E(\underbrace{e^{tx}}|N=n)$

Gamma(α, β)

$$= \int_0^\infty e^{+tx} \frac{x^{n+1-1} e^{-x}}{n!} dx$$

$$\boxed{\frac{\beta^\alpha}{\Gamma(\alpha)}} x^{\alpha-1} e^{-\beta x}$$

$$= \int_0^\infty \frac{\boxed{e^{-(1-t)x} x^{n+1-1}}}{n!} dx$$

β' α'

$$\boxed{\Gamma(n+1) = n!}$$

$$= \frac{1}{\cancel{n!}} \cdot \frac{\cancel{n!}}{(1-t)^{\cancel{\alpha}+1}}$$

$$= (1-t)^{-(N+1)}. \qquad t<1. \checkmark$$

(b) mgf of X. wrt N.

$$\rightarrow E(e^{tX}) = E[E[e^{tX}|N]]$$

(a)

$$= E[(1-t)^{-(N+1)}]$$

pgf of N

$s = (1-t)^{-1}$

$$= (1-t)^{-1} E((1-t)^{-N}).$$

$$\boxed{\frac{p}{1-s(1-p)}}$$

$$= (1-t)^{-1} \cdot \frac{p}{1-(1-t)^{-1}(1-p)}$$

$$\boxed{s = \frac{1}{(1-t)}}.$$

$\vdots$

$$= \boxed{\frac{p}{p-t}}, \quad \boxed{t<p}$$

$X \sim \exp(p)$.

(c) kth mom of X : $E(X^k)$

mgf of X.

$$EX^k = \frac{d^k}{dt^k} E(e^{tX}) \Big|_{t=0}.$$

ex. $M_x^{(k)}(t) = \dfrac{p\,k!}{(p-t)^{k+1}}$ ✓

(d) cond pmf $N | X = x$

$$P(N=n \mid X=x)$$

$$= \frac{f_{X,N}(x,n)}{f_X(x)}$$ ← joint dist'n of X, N. ← dist'n of X.

$$= \frac{f_{X|N}(x) \cdot P(N=n)}{f_X(x)}$$ — density exp(p).

$$= \frac{x^n e^{-x}/n! \times (1-p)^n \cancel{p}}{\cancel{p} e^{-px}}$$

$$= \frac{(x(1-p))^n e^{-(1-p)x}}{n!}.$$ ✓

$N \mid X = x \sim \mathrm{Poi}(x(1-p))$ $\quad \frac{x^n e^{-x}}{n!} \underline{\underline{dx}}$

Q6. win D wp 0.493

$\ominus D \quad 0.507$

(a) mean $= D \cdot 0.493 - D \cdot 0.507 = M_D$.

(b) variance $= E[(D - M_D)^2]$.

$= (D - M_D)^2 \cdot 0.493 + 0.507(-D - M_D)^2$

winnings.

(c) Y : bet \$100 in one attempt.

W : \$1 100

- mean, var of Y

$D = 100$.

$E(Y) = M_{100} = \underset{\overset{\shortparallel}{D}}{100}(0.493 - 0.507)$.

Var : plug in $D = 100$ in part (b).

(d) $W = \sum_{i=1}^{100} X_i$

$\rightarrow X_i = \begin{cases} +1 & \text{wp. } 0.493 \\ -1 & 0.507 \end{cases}$

(a), (b), $D = 1$

$E(X_i)$, $var(X_i)$ ✓

$$EW = \sum_{i=1}^{100} E(X_i)$$

$$Var(X_i) = \sum_{i=1}^{100} Var(X_i)$$

(e) $P(Y>0)$

$Y>0 \iff Y=D$

$P(Y>0) = 0.493$.

$P(W>0)$ $\boxed{CLT}$.

Q7. X : # heads in n fair coin tosses.

$Y = n - X$. $\leftarrow$

$X \sim Bin(n, \frac{1}{2})$.

$cov(X, W \pm Y)$
$= cov(X, W) \pm cov(X, Y)$

(a) $Cov(X, Y)$

$= Cov(X, n-X)$

$= cov(X, n) - \underbrace{cov(X, X)}$

$= 0 - Var(X)$

$= -Var(X)$

$= -n/4$ $\quad = -\frac{1}{2}\cdot\frac{1}{2}(n)$. $\quad Bin(n, \frac{1}{2})$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = -1.$$

$Y \sim \mathrm{Bin}(n, \frac{1}{2})$.

$V(Y) = V(X)$

$X|N \sim \mathrm{Bin}(N, \frac{1}{2})$

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tosses random N.

$EN = \mu \qquad \mathrm{Var}(N) = \sigma^2$

(b) $EX = \underset{\text{wrt } N}{E}(E(X|N)) = E(\frac{1}{2}N)$.

$= \frac{\mu}{2}$.

(c) $\mathrm{Var}(X) = V(E(X|N)) + E(V(X|N))$

$= V(\frac{N}{2}) + E(\frac{1}{4} \cdot N)$

$= \frac{1}{4}V(N) + \frac{1}{4}E(N)$

$= \frac{1}{4}(\mu + \sigma^2)$.

(c) $\frac{\sigma^2 + \mu}{4}$

(d) $\mathrm{cov}(X,Y)$

$= \mathrm{cov}(X, N - X)$

$=$ Var(X)

$= \mathrm{cov}(X,N) - \mathrm{cov}(X,X) = \frac{\sigma^2 - \mu}{4}$.

$\mu/2 \qquad \mu$

$$\mathrm{cov}(X,N) = E(XN) - E(X)EN.$$

, wrt N

$$E(E(XN|N)) = E(N\,E(X|N))$$
$$= E\left(N\cdot\frac{N}{2}\right)$$
$$= \frac{1}{2}EN^2$$
$$= \frac{1}{2}(\sigma^2+\mu^2).$$

use (d)

(e) $\mathrm{corr}(X,Y)$

$$= \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} = \frac{\sigma^2-\mu}{\sigma^2+\mu}.$$

use (c), $Y|N \sim \mathrm{Bin}(N,\frac{1}{2})$.

Q2. $X_1, X_2, \dots, X_n$ indep.

$X_1 \sim \exp(\lambda)$.

$X = \min(X_1, \dots, X_n)$.

$\cdot\ P(X > x) = P(X_1 > x, \dots, X_n > x).$

$$= \prod_{i=1}^{n} P(X_i > x)$$

$$= [P(X_1 > x)]^n$$

$$= (e^{-\lambda x})^n$$

$$= e^{-\underbrace{\lambda n}x}$$ cdf, differentiate to get density.

$X \sim \exp(\lambda n)$.

Q3. B : Bob wins.

B_i : Bob wins at round i

$B = \bigcup_{i\geq 1} B_i \leftarrow$ disjoint

$$\underline{P(B)} = \sum_{i\geq 1} \underline{P(B_i)}$$

$$= \frac{5}{36} \sum_{i\geq 1} \left(\frac{31}{36} \times \frac{5}{6}\right)^{i-1}$$ geom. sum

N_j : no one wins at round j.

$$\underline{P(B_i)} = P(B_i \cap \underline{N_{i-1} \cap \cdots \cap N_1}).$$

$$= P(B_i | N_{i-1} \cap \cdots \cap N_1) \times$$

$$P(N_{i-1} | \underline{N_{i-2} \cap \cdots \cap N_1}) \times \ldots$$

$$\ldots \times P(N_2 | N_1) P(N_1).$$

$$= P(B_i | N_{i-1} \cap \ldots \cap N_1) \times$$

$$\left(P(\underline{\text{no one wins at round } 1}) \right)^{i-1}$$

Bob gets 6.

$$= \boxed{\frac{5}{36} \times \left(\frac{31}{36} \times \frac{5}{6} \right)^{i-1}}$$

bob does not get 6.

drawing no 7

$$= \frac{5}{5 + 31/6} < \frac{1}{2}$$