MAST30025: Linear Statistical Models

Refer way back from the first week of lectures!

Week 8 Lab

tao and beta repectively

i = 1,2 and j = 1,2,3

1. Consider a dataset where each sample has a response and two factors, which have 2 and 3 possible levels respectively. We take 2 samples from each possible combination of factor levels. We model this with a less than full rank model with one parameter for the overall mean, and one parameter for each level of each factor which adjusts the overall mean additively. Write down the linear model in both equation and matrix form.

2. Let
$$A = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}.$$

   (a) Show that $r(A) = 2$. ->Can I do the sample rref method in Linear Algebra?!

   (b) Find two distinct conditional inverses for A.

3. Show that $A = A(A^TA)^cA^TA$. You may use the result that if $A^TA = 0$, then $A = 0$. (*Hint: Consider the matrix* $A - A(A^TA)^cA^TA$.)

4. It is known that toxic material was dumped into a river that flows into a large salt-water commercial fishing area. We are interested in the amount of toxic material (in parts per million) found in oysters harvested at three different locations in this area. A study is conducted and the following data obtained:

Site 1	Site 2	Site 3
15	19	22
26	15	26

   (a) Write down the linear model in matrix form.

   (b) Write down the normal equations.

   (c) Find a conditional inverse for X^TX.

   (d) Find a solution for the normal equations.

5. In a manufacturing plant, filters are used to remove pollutants. We are interested in comparing the lifespan of 5 different types of filters. Six filters of each type are tested, and the time to failure in hours is given in the dataset (on the website) `filters` (in `csv` format).

   (a) Use the `read.csv` function to read the data. Then convert the `type` component into a factor.

   (b) Construct X and $\mathbf{y}$ matrices for this linear model.

   (c) Using the algorithm given in the lecture slides, find a conditional inverse for X^TX.

   (d) Use `ginv` to find another conditional inverse for X^TX.

   (e) Verify that $X(X^TX)^cX^T$ is the same for your two conditional inverses.

   (f) Find two solutions for the normal equations.

   (g) Express one of your solutions in terms of the other.

   (h) Write down a form for all solutions to the normal equations.

6. Show that $A^TA = 0 \Rightarrow A = 0$.

   Hence show that $AB = AC \Leftrightarrow A^TAB = A^TAC$.

PROBLEM 1

$$y_{ijk} = \mu + \overbrace{\tau_i + \beta_j}^{\text{2 way anova}} + \varepsilon_{ijk}$$

with $i = 1,2$ (2 levels)

$j = 1,2,3$ (3 levels)

$k = 1,2$

Matrix form

$$\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \end{bmatrix} = \begin{array}{c} \begin{array}{cccccc} \mu & \tau_1 & \tau_2 & \beta_1 & \beta_2 & \beta_3 \end{array} \\ \left[\begin{array}{c|cc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array} + \begin{bmatrix} \varepsilon_{111} \\ \varepsilon_{112} \\ \varepsilon_{121} \\ \varepsilon_{121} \\ \varepsilon_{131} \\ \varepsilon_{132} \\ \varepsilon_{211} \\ \varepsilon_{212} \\ \varepsilon_{221} \\ \varepsilon_{222} \\ \varepsilon_{231} \\ \varepsilon_{232} \end{bmatrix}$$

PROBLEM 2

$$A = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}$$

(a) To show rank, convert to RE form,

$$\sim \begin{bmatrix} 1 & 2 & 5 & 2 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & 2 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Rank $= 2$

(b) Take minor of rank $r(A) \times r(A) = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

$$M^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

$\Rightarrow$ New matrix $=$
A^c

$$\begin{bmatrix} 7 & -3 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Take bottom left & we got $M = \begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$

$$M^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -7/3 \\ 0 & 1 \end{bmatrix}$$

$$A^c = \begin{bmatrix} 0 & 1/3 & -7/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

PROBLEM 3

Claim : $A = A(A^TA)^c A^TA$

Proof: Using lemma given $X^TX = 0 \Rightarrow X = 0$

we can try to let $X = A - A(A^TA)^c A^TA$.

$$\Rightarrow X^TX = \left[A - A(A^TA)^c A^TA\right]^T \left[A - A(A^TA)^c A^TA\right]$$

$$= A^TA - A^T\underbrace{A(A^TA)^c A^TA}_{A^TA} - (A^TA)\underbrace{(A^TA)^c (A^TA)}_{A^TA}$$

$$+ \underbrace{A^TA(A^TA)^c A^TA}_{A^TA}\underbrace{(A^TA)^c A^TA}_{A^TA}$$

$$= 0$$

$\Rightarrow$ we have that $A = A(A^TA)^c A^TA$ ∎

PROBLEM 4

(a) $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$. where $i = 1,2,3$
$j = 1,2$

$$\Rightarrow \underbrace{\begin{bmatrix} 15 \\ 26 \\ 19 \\ 15 \\ 22 \\ 26 \end{bmatrix}}_{y} = \underbrace{\begin{array}{c} \begin{array}{cccc} \mu & \tau_1 & \tau_2 & \tau_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}}_{X} \underbrace{\begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}}_{\beta} + \underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{31} \\ \varepsilon_{32} \end{bmatrix}}_{\varepsilon}$$

(b) The normal equation is given by

$$(X^T X) b = X^T y$$

- Calculate $X^T X =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 26 \\ 19 \\ 15 \\ 22 \\ 26 \end{bmatrix}$$

$$= \begin{bmatrix} 123 \\ 41 \\ 34 \\ 48 \end{bmatrix}$$

c) $\begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 123 \\ 41 \\ 34 \\ 40 \end{bmatrix}$

(c) Rank of X^TX is given by calculating the RE form

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} \\ 3R_2 - R_1 \\ 3R_3 - R_1 \\ 3R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 6 & 2 & 2 & 2 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 4 & -2 \\ 0 & -2 & -2 & 4 \end{bmatrix} \begin{array}{l} \\ \\ 2R_3 + R_2 \\ 2R_4 + R_2 \end{array}$$

$$\sim \begin{bmatrix} 6 & 2 & 2 & 2 \\ 0 & 4 & -2 & -2 \\ 0 & 0 & 6 & -6 \\ 0 & 0 & -6 & 6 \end{bmatrix} \qquad \underline{\underline{\text{Rank} = 3}}$$

* Take $M = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$$M^{-1} = \left[\begin{array}{ccc|ccc} 6 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ 3R_2 - R_1 \\ 3R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 6 & 2 & 2 & 1 & 0 & 0 \\ 0 & 4 & -2 & -1 & 3 & 0 \\ 0 & -2 & 4 & -1 & 0 & 3 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 + R_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 6 & 2 & 2 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 3 & 0 \\ 0 & 0 & 6 & -3 & 3 & 6 \end{array}\right]$$

$$= \ldots$$

$$(X^T X)^c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

(b) $b = (X^T X)^c (X^T y) = \begin{bmatrix} 0 \\ 20.5 \\ 17 \\ 24 \end{bmatrix}$

PROBLEM 6

Proof $A^T A = 0 \Rightarrow A = 0$

we have $Ax = 0 \Leftrightarrow A^T A x = 0$

Thus $A^T A = 0$

$\Rightarrow Ax = 0 \quad \forall x$

$\Rightarrow A = 0$ (only possibilities)

Claim: $AB = AC \Leftrightarrow A^T A B = A^T A C$

$(\Rightarrow)$ Assume $AB = AC$.

$(A^T A B - A^T A C)^T (A^T A B - A^T A C)$

$$= B^T A^T A A^T A B - B^T A^T A A^T A C - C^T A^T A A^T A B$$

$$+ C^T A^T A A^T A C$$

$$= C^T A^T A A^T A C - C^T A^T A A^T A C$$

$$= 0$$

(Proved)

* Similar for the other way