Topic 1: Random fields

This topic introduces the first class of models, which is used to represent raster data. In particular, we consider

- Revision of basic definitions of **random variables**.
- Revision of basic definitions and properties of their **expectations and variances**.
- **Weakly stationary time series**.
- Introduction to **random fields**.
- **Positive definiteness**.
- Properties of **positive definite** functions.

RANDOM VARIABLES.

A **random variable** is a variable that takes a numerical value for each possible outcome of a statistical experiment. For simplicity we denote a random variable $X(\omega)$ by X.

If X is a random variable, then we cannot predict its value with certainty, but can assign probabilities to events such as $\{X = 1\}$ and $\{X > 2\}$ etc.

Discrete random variables

A random variable X is called **discrete** if all of its possible values can be written down in a list. The probability distribution of a discrete random variable X is a list of the possible values that X can take (put in increasing order), together with the probabilities that X takes each of these possible values.

Expected value of X

The **expected value** of X, denoted $E(X)$, is defined to be

$$\boxed{E(X) = \sum_x x P(X = x).}$$

This is a weighted average of the possible values of X where the weights are the corresponding probabilities. $E(X)$ is a measure of the **centre** of the probability distribution.

Continuous Random Variables

A random variable that can take on a continuum of possible values is called a **continuous** random variable.

We describe the probability properties of a continuous random variable Y by a function $f(y)$ called the **probability density function (pdf)**. Probabilities are given by the relevant area under the probability density function curve.

Expected value of a continuous random variable Y

Suppose that Y is a continuous random variable.
The expected value of Y, denoted $E(Y)$, is defined mathematically as

$$\boxed{\int_{-\infty}^{\infty} y f(y) dy.}$$

Variance of X

Suppose that X is a random variable. The variance of X, denoted $\mathrm{Var}(X)$, is defined as

$$\boxed{\mathrm{Var}(X) = E\left[(X - E(X))^2\right].}$$

The standard deviation is defined to be $\sqrt{\mathrm{Var}(X)}$.

Properties of $E(X)$ and $\mathrm{Var}(X)$

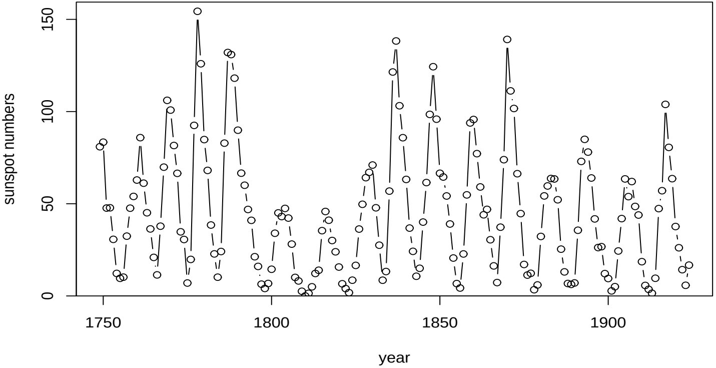
Suppose that X and Y are random variables. Also, suppose that a and b are numbers. Then

- $E(a) = a$
- $E(bX) = bE(X)$
- $E(X + Y) = E(X) + E(Y)$
- $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$.

Time Series

Observations of random variables over time typically display dependence. It is this dependence that we model by using time series models.

We use the notation $\{X_t(\omega) : t \in T\}$ (for simplicity X_t) to denote a time series, where T (usually $\mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$) is the index set.



Weakly stationary time series

Definition 1

A time series $\{X_t\}$ is said to be **weakly stationary** *if*

- $E(X_t)$ *does not depend on* t.
- *For every integer* s, $E\big(X_t X_{t-s}\big)$ *does not depend on* t.

For a weakly stationary time series $\{X_t\}$ we define:

- $m = E(X_t)$ (for simplicity we often use $m = 0$)
- The autocovariance function γ by

$$\gamma(s) = E\big(X_t X_{t-s}\big) \quad \text{for all integer } s.$$

- The autocorrelation function ρ by

$$\rho(s) = \frac{\gamma(s)}{\gamma(0)} \quad \text{for all integer } s.$$

Random Fields

A **random field** is a collection of random variables indexed by an index taking values in some multidimensional space, on a surface, etc. A time series is a particular case of random fields, when T is a one-dimensional space.

We will use the same notation $X_{\mathbf{t}}$ for random fields as for time series. But you should remember that the index $\mathbf{t}$ can be multidimensional.

However, because of randomness, $X_{\mathbf{t}}$ is a function of two variables, i.e. $X_{\mathbf{t}} = X(\mathbf{t}, \omega)$. For a fixed $\omega \in \Omega$, the function $X(\mathbf{t}, \omega)$ is a non-random function of $\mathbf{t}$. This deterministic function is usually called a **sample path (sample function)** or a **realization.** This is what we usually observe in our experiments or datasets.

For simplicity we denote $X(\mathbf{t}, \omega)$ by $X_{\mathbf{t}}$.

Expectation and Covariance

The **expectation** of a random field equals

$$m(\mathbf{t}) = E\{X_{\mathbf{t}}\}.$$

The **(auto-) covariance function** is defined by

$$C(\mathbf{t},\mathbf{s}) \;=\; Cov\{X_{\mathbf{t}}, X_{\mathbf{s}}\} = E\{X_{\mathbf{t}}X_{\mathbf{s}}\} - m(\mathbf{t})m(\mathbf{s}),$$

whereas the *variance* is $\sigma^2(\mathbf{t}) = C(\mathbf{t},\mathbf{t})$. The **(auto-)correlation function** of a random field equals

$$\rho(\mathbf{t},\mathbf{s}) = Corr\{X_{\mathbf{t}}, X_{\mathbf{s}}\} = \frac{C(\mathbf{t},\mathbf{s})}{\sigma(\mathbf{t})\sigma(\mathbf{s})}.$$

The covariance/correlation function describes spatial dependencies between observations at locations $(\mathbf{t},\mathbf{s})$.

- The expectation $m(\mathbf{t})$ represents a spatial trend in data. This trend can be modelled by any function.

  However, in applications it is common to use some parametric trend, which is a function that can be controlled by a small number of parameters. To fit such trend one just needs to estimate these unknown values of parameters. Popular trends are: constant, polynomials, exponential, logit, etc.

- To select an appropriate covariance function is much more challenging problem. Not every function can be used as a covariance function.

  There is a well-known result that the class of covariance functions coincides with the class of positive definitive functions.

Definition 2

A function of two variables $B(\cdot,\cdot)$ is positive definite if

$$\boxed{\sum_{k=1}^{n}\sum_{l=1}^{n} c_k \bar{c}_l B(\mathbf{t}_k, \mathbf{t}_l) \geq 0}$$

for any positive integer n, sequences $\mathbf{t}_k \in T$ and $c_k \in \mathbb{C}$ ($\bar{c}_k$ denotes a complex conjugate of c_k).

The concept of positive definiteness is fundamental in many areas of statistics, data sciences and mathematics.

We provide some simple properties of positive definite functions that are used to construct new covariance functions from the known ones.

Properties of $\mathcal{P}_T$

Let $\mathcal{P}_T$ be the class of positive functions on T.

(1) $B(t,s) \in \mathcal{P}_T, \alpha \geq 0 \Rightarrow \alpha \cdot B(t,s) \in \mathcal{P}_T$.

(2) $B_1(t,s) \in \mathcal{P}_T, B_2(t,s) \in \mathcal{P}_T \Rightarrow B_1(t,s) + B_2(t,s) \in \mathcal{P}_T$.

(3) $\alpha_1 \geq 0, \ldots, \alpha_n \geq 0; B_1(t,s), \ldots, B_n(t,s) \in \mathcal{P}_T \Rightarrow$
$\sum\limits_{k=1}^{n} \alpha_k B_k(t,s) \in \mathcal{P}_T$.

(4) $B_1(t,s) \in \mathcal{P}_T, B_2(t,s) \in \mathcal{P}_T \Rightarrow B_1(t,s) \cdot B_2(t,s) \in \mathcal{P}_T$.

(5) $B_n(t,s) \in \mathcal{P}_T \Rightarrow \lim_{n\to\infty} B_n(t,s) \in \mathcal{P}_T$.