MAST20004 Probability

Tutorial Set 10

1. If $X \stackrel{d}{=} R(0, \frac{\pi}{2})$ and $Z = \sin X$, find $V(Z)$ and compare this with the approximate value calculated using $V(\psi(X)) \approx \psi'(\mu)^2 V(X)$.

2. Let $N \geq 0$ be an integer-valued random variable with $\mathbb{E}[N] = a, V(N) = b^2$ and $X_1, X_2, \cdots$ be independent random variables, also independent of N, with $\mathbb{E}[X_j] = \mu$ and $V(X_j) = \sigma^2$. Using conditional expectations, compute $\mathrm{Cov}(S_N, N)$, where $S_N = \sum_{j=1}^N X_j$.

3. Let the random variable X have the probability generating function

$$P_X(z) = c + 0.1(1+z)^3 + 0.3z^5.$$

   (a) Find the constant c.
   (b) Give the distribution of X.
   (c) Use the $P_X(z)$ to compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.
   (d) Compute the probability generating function of $Y = X + 2$.

4. Let $X \stackrel{d}{=} \mathrm{R}(0,1)$ and $Y \stackrel{d}{=} \mathrm{R}(1,3)$ be independent random variables.
   (a) Compute the moment generating function $M_X(t)$ of X.
   (b) Compute the moment generating function of Y.
   (c) Compute the moment generating function of $Z = X - 2Y + 2$.
   (d) Use the moment generating function $M_X(t)$ to verify that $\mathbb{E}[X] = 1/2$ and $V(X) = 1/12$.

5. Let $Y_\lambda \stackrel{d}{=} \mathrm{Pn}(\lambda)$.
   (a) Write down the moment generating function of Y_λ.
   (b) Compute the moment generating function of $Z_\lambda = (Y_\lambda - \lambda)/\sqrt{\lambda}$.
   (c) Using part (b) show that $Z_\lambda \xrightarrow{d} N(0,1)$ as $\lambda \to \infty$.

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Computer Lab 10

In this lab you

- simulate the total amount T claimed from an insurance company in one day and compare your simulation estimates against the theoretical values of $\mathbb{E}[T]$ and $V(T)$.
- investigate the accuracy of the approximation formulae for the mean and variance of a function of a random variable.

Exercise A - Simulation of insurance company total claims

Suitably modified, the **incomplete** Matlab m-file **Lab10ExA.m** will simulate the total amount claimed from an insurance company in one day. You will need to add a few lines to the program to generate the required distributions. **Lab10ExA.m** produces estimates for $\mathbb{E}[T]$ and $V(T)$ and also plots the empirical pdf for T.

Let the number of claims in one day be $N \stackrel{d}{=} \text{Pn}(10)$ and $X_1, X_2, \cdots$ be random variables representing claim amounts. We assume that N and $X_1, X_2, \cdots$ are independent, with $X_i \stackrel{d}{=} X$ (for all i) for some claim size X. Then $T = \sum_{i=1}^{N} X_i$ is the sum of a (random) number of random variables and represents the total amount claimed in one day.

1. We start with the assumption that $X_i \stackrel{d}{=} \exp(\lambda)$. Using the appropriate formulae from lectures calculate the theoretical values for $\mathbb{E}[T]$ and $V(T)$.
2. Open **Lab10ExA.m** in the m-file editor and add the code required to generate the claims. Run the program for a couple of different values of λ and compare your theoretical answers with the simulation estimates. Also comment on the shape of the empirical pdf for T.
3. Repeat this exercise for a claim distribution $X \stackrel{d}{=} R(10, 20)$.

Exercise B - Approximations for mean and variance of functions

Let X be a random variable with $\mathbb{E}[X] = \mu$ and $V(X) = \sigma^2$, and let $\psi(X)$ be a transformation of X. As we have seen, it is **not true** in general that the mean and the variance of the transformation $\psi(X)$ are equal simply to the transformations of the mean and variance of X, respectively (an important exception is $\psi(x) = ax + b$ for which $\mathbb{E}[\psi(X)] = \psi(\mathbb{E}[X])$). Often it is difficult to find the exact values of $\mathbb{E}[\psi(X)]$ and $V(\psi(X))$ (due to the fact that the integrals or sums are complicated). In lectures (refer to slides 446–448) we derived the following approximation formulae for the mean and the variance of $\psi(X)$:

$$\begin{aligned} \mathbb{E}[\psi(X)] &\approx \psi(\mu) + \frac{1}{2}\psi''(\mu)\sigma^2, \\ V(\psi(X)) &\approx \psi'(\mu)^2\sigma^2. \end{aligned} \tag{1}$$

These relations are based on the Taylor series approximations of the form

$$\psi(X) \approx \psi(\mu) + \psi'(\mu)(X - \mu) + \frac{1}{2}\psi''(\mu)(X - \mu)^2. \tag{2}$$

To help you understand these formulae and test how well they work, in this lab we will apply them and verify the results using the m-files **Lab10ExB1.m** and **Lab10ExB2.m**. **Lab10ExB1.m** plots $\psi(x)$ and its Taylor series approximations over a specified domain. **Lab10ExB2.m** simulates 'nreps' observations on $\psi(X)$ to estimate the mean and variance.

1. This first example examines the transformation that we looked at in questions 3 and 4 of tutorial 9. Let $X \stackrel{d}{=} R(0, 1)$ and $Y = \sqrt{X}$ (so that $X = \psi(X)$ with $\psi(x) = \sqrt{x}$).

   (a) Write down the approximating functions $l(x)$ and $q(x)$ of $\sqrt{x}$, given by the first two and three terms on the right-hand side of (2) respectively. (Note: $l(x)$ is the tangent line approximation and $q(x)$ the quadratic approximation). By adding the appropriate code to **Lab10ExB1.m**, plot $\psi(x)$, $l(x)$ and $q(x)$ on the same graph, over an appropriate domain. Do you expect both approximations to be good?

   (b) Add the appropriate code to **Lab10ExB2.m** to produce observations on $\psi(X)$. Compare the simulation estimates with your approximations.

   (c) How do your simulation results compare to the exact values of $\mathbb{E}[Y]$ and $V(Y)$?

   (d) Repeat this exercise for $X \stackrel{d}{=} R(1, 2)$. Do the approximations work better or worse? Explain.

2. Let $V = e^{-1/X}$, where $X \stackrel{d}{=} N(5, 4)$. In this case, no exact values are readily available. Complete parts (a) and (b) of section 1 for this random variable.