1 a) $\log P(Y=y) = y \log \lambda - \nu \log y! - \log Z(\lambda)$

$$= \frac{y \log \lambda^{1/\nu} - \log Z(\lambda)^{1/\nu}}{1/\nu} - \frac{\log y!}{1/\nu}$$

$$= \frac{y\theta - b(\theta)}{\phi} + c(y, \phi)$$

where $\theta = \log \lambda^{1/\nu}$ ✓

$\phi = 1/\nu \qquad a(\phi) = \phi$ ✓

$b(\theta) = \log Z(e^{\nu\theta})^{1/\nu}$

$c(y, \phi) = -\log y! / \phi$

Can just take $\theta = \log \lambda$, $\phi = 1$

b) ϕ is known so can get the difference between scaled deviances, which has an asymptotic χ^2 dist. ✓✓

c) Overdispersion is when variance is larger than it should be (for a given model).
Can use quasi-poisson or quasi-binomial ✓✓

$$\log L(p) = y \log p + (m-y)\log(1-p) + \log\binom{m}{y}$$

2 a) $L(p) = \binom{m}{y} p^y (1-p)^{m-y}$

$$J(p) = -\frac{\partial^2}{\partial p^2} \log L(p) = -\frac{\partial^2}{\partial p^2}\left(y \log p + (m-y)\log(1-p)\right)$$

$$= -\frac{\partial}{\partial p}\left(\frac{y}{p} - \frac{m-y}{1-p}\right)$$

$$= \frac{y}{p^2} + \frac{m-y}{(1-p)^2}$$ ✓

$$I(p) = \mathbb{E}\, J(p) = \frac{m}{p} + \frac{m}{1-p} = \frac{m}{p(1-p)}$$ ✓

b) $$p(n+1) = p(n) - \frac{f'(p(n))}{f''(p(n))}$$ ✓

$$= p(n) + \frac{y/p(n) - (m-y)/(1-p(n))}{y/p(n)^2 + (m-y)/(1-p(n))^2}$$

$$= p + \frac{y p(1-p)^2 - (m-y)p^2(1-p)}{y(1-p)^2 + (m-y)p^2}$$

$$p(1) = \tfrac{1}{2} + \frac{4 \times \frac{1}{8} - 6 \times \frac{1}{8}}{4 \times \frac{1}{4} + 6 \times \frac{1}{4}}$$

$$= \tfrac{1}{2} - \frac{2}{20} = \frac{4}{10}.$$ ✓

$$p(2) = \frac{4}{10} + 0 = \frac{4}{10}$$ ✓

c) $$p(n+1) = p(n) + f'(p(n)) / I(p(n))$$ ✓

$$= p + \frac{y/p - (m-y)/(1-p)}{m/p + m/(1-p)}$$

$$p(1) = \tfrac{1}{2} - 4/40 = 4/10$$ ✓

$$p(2) = 4/10 + 0 = 4/10$$

d) $$\hat{p} \approx N(p, I(p)^{-1}) = N(p, p(1-p)/m)$$ ✓✓

3 a) $y_i \sim bin(n_i, p)$ ✓

$logit(p) = \alpha + \beta \text{ dose} = \log \frac{p}{1-p}$

$\left(probit(p) = \alpha + \beta \text{ dose} \right.$

$\left(p = \Phi(\alpha + \beta \text{ dose} \right.$ ✓

b) AIC too close to really matter, but smaller for probit so prefer it. ✓

c) n_i large enough that resid deviance $\approx \chi^2_4$

95% critical value for 4 d.f. is $9.49 > 2.62$

so model is adequate ✓

$$T = -2\sum\left(y_i \log \hat{y}_i / y_i + (n_i - y_i) \log \frac{n_i - \hat{y}_i}{n_i - y_i} \right)$$ ✓

$\sim \chi^2_{n-p} = \chi^2_4$ ✓

observed value 2.62 ✓

d) $d_i = -2\left(y_i \log \hat{y}_i / y_i + (n_i - y_i) \log \frac{n_i - \hat{y}_i}{n_i - y_i} \right)$ ✓

residuals OK ✓

discrete responses mean banding in Pearson resid. ✓

e) point estimate is 0.815. For a 95% CI ✓

use $logit(1.4684 \pm 2 \times 0.4838)$

$= logit(0.5008, 2.436)$

$= \left(\frac{1}{1+e^{-0.5008}}, \frac{1}{1+e^{-2.436}} \right) = (0.623, .920)$ ✓

f) Estimate ϕ by $X^2 / n-p$

If $\phi \gg 1$ then overdispersed ✓

Can get overdispersion in binary models if p not constant or trials dependent ✓

3 g) Observation 2 could be skewing results a little, but expect P(tumor | dose = 0) > 0. Want this to be underlying tumor rate for population, which we could estimate independently.

4) $\text{var}\,\hat{\beta} = (X^T \Sigma^{-1} X)^{-1}$

where

$$\Sigma = \text{diag}\left(g'(\hat{\mu}_i)^2\, v(\hat{\mu}_i)\, a(\phi) \right)_{i=1}^{n}$$

and $X = \begin{pmatrix} \underset{\sim}{x}_1 \\ \vdots \\ \underset{\sim}{x}_n \end{pmatrix}$ $\qquad \hat{\mu}_i = g^{-1}(\underset{\sim}{x}_i^T \underset{\sim}{\hat{\beta}})$

5 a) put $\gamma_{ij} = P(Y_i \le j)$

$g(\gamma_{ij}) = \theta_j - \underset{\sim}{x}_i^T \underset{\sim}{\beta}$ some θ_j, $\underset{\sim}{\beta}$

link function $g = F^{-1}$ where F is a cdf

b) $\underset{\sim}{x}_i^T \underset{\sim}{\beta} = 3 \times 0.4589 + 2 \times 0.2696 + 2.0816 + 3 \times 0.5635$
$= 5.688$

$g(\gamma_{i1}) = 5.9944 - 5.688 = 0.3064$
$g(\gamma_{i2}) = 7.3948 - 5.688 = 1.7068$

g is logit function

$\gamma_{i1} = \dfrac{1}{1+e^{-0.3064}}$ $\qquad \gamma_{i2} = \dfrac{1}{1+e^{-1.7068}}$

$= 0.5760$ $\qquad = 0.8464$

c) 650acc has the smallest t-value
size of effect also a consideration
but harder to judge without knowing
more about the predictor vars.

6. $\theta \sim \text{beta}(a, b)$ $\quad Y \sim \text{binom}(m, \theta)$

a) $p(\theta / y) \propto p(\theta, y) = p(y / \theta)\, p(\theta)$

$$\propto \binom{m}{y} \theta^{y} (1-\theta)^{m-y} \theta^{a-1} (1-\theta)^{b-1}$$

$$\propto \theta^{a+y-1} (1-\theta)^{b+(m-y)-1}$$

thus $\theta / Y = y \sim \text{beta}(a+y, b+m-y)$

posterior same form as prior, hence conjugate

b) Bayes estimate under squared error loss is the posterior mean

$$\frac{a+y}{a+b+m}$$

c) Jeffreys prior is $\propto \sqrt{I(\theta)}$

where $I(\theta)$ is the Fisher information

$$I(\theta) = -E \frac{d^2}{d\theta^2} \log L(\theta) = -E \frac{d^2}{d\theta^2} \log p(y/\theta)$$

$$= -E \frac{d^2}{d\theta^2} \left(y \log \theta + (m-y) \log (1-\theta) \right)$$

$$= -E \frac{d}{d\theta} \left(\frac{y}{\theta} - \frac{m-y}{1-\theta} \right)$$

$$= -E \left(\frac{-y}{\theta^2} - \frac{m-y}{(1-\theta)^2} \right)$$

$$= \frac{m}{\theta} + \frac{m}{1-\theta} = \frac{m}{\theta(1-\theta)}$$

$p(\theta) \propto \theta^{-1/2} (1-\theta)^{-1/2}$

thus $\theta \sim \text{beta}(1/2, 1/2)$

7

7 a) sim1

runif(m) < p returns a vector of T/F with $Pr(T) = p$

sum(runif(m) < p) counts the number of T's in the vector

Thus the value returned $\sim bin(m, p)$ ✓

sim2 We have for $X \sim bin(m, p)$

$$p_x = \mathbb{P}(X = x) = p(x)$$
$$F_x = \mathbb{P}(X \leq x) = F(x)$$ ✓

since $\mathbb{P}(X = x) = \binom{m}{x} p^x (1-p)^{m-x}$

$$= \frac{m-x+1}{x} \frac{p}{(1-p)} \binom{m}{x-1} p^{x-1} (1-p)^{m-x+1}$$

$$= \frac{m-x+1}{x} \frac{p}{1-p} \mathbb{P}(X = x-1)$$ ✓✓

thus loop exits when $F(x-1) < U \leq F(x)$

that is $\mathbb{P}(\text{return } x) = \mathbb{P}(F(x-1) < U \leq F(x))$

$$= F(x) - F(x-1) = p(x)$$ ✓

b) sim1 uses m, sim2 uses 1 uniform (uniforms are cheap, sim1 is much faster than sim2, so is preferable). ✓

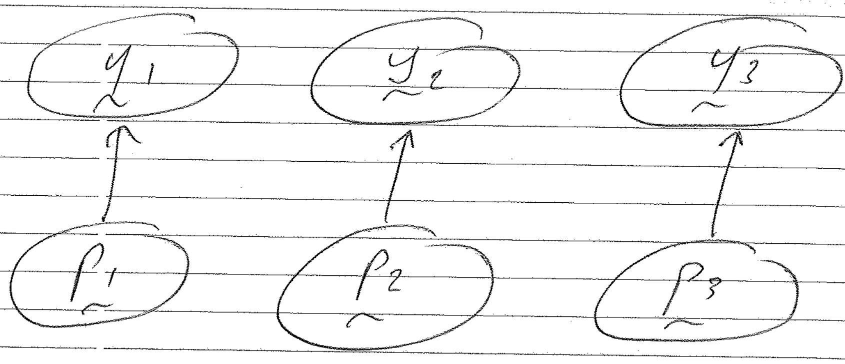
c) We have $\theta \sim U(0,1)$ $Y \sim bin(a+b, \theta)$ and algorithm returns

$$\theta \mid Y = a$$ ✓✓

$U(0,1) \equiv beta(1,1)$ beta is conjugate to binomial, and $\theta \mid Y = a \sim beta(a+1, b+1)$ ✓✓

10

8 a)



b) vague priors (prior is constant over domain)
an example of an informative prior would be

$$\underset{\sim}{p}_i \sim Dir(10, 10, 10)$$

c)

$$\underset{\sim}{y}_i \sim \text{multinomial}(m_i, \underset{\sim}{p}_i) \qquad i = 1, 2, 3$$

$$\underset{\sim}{p}_1 \leftarrow ((1-\delta)p + \delta, (1-\delta)q, 0)$$

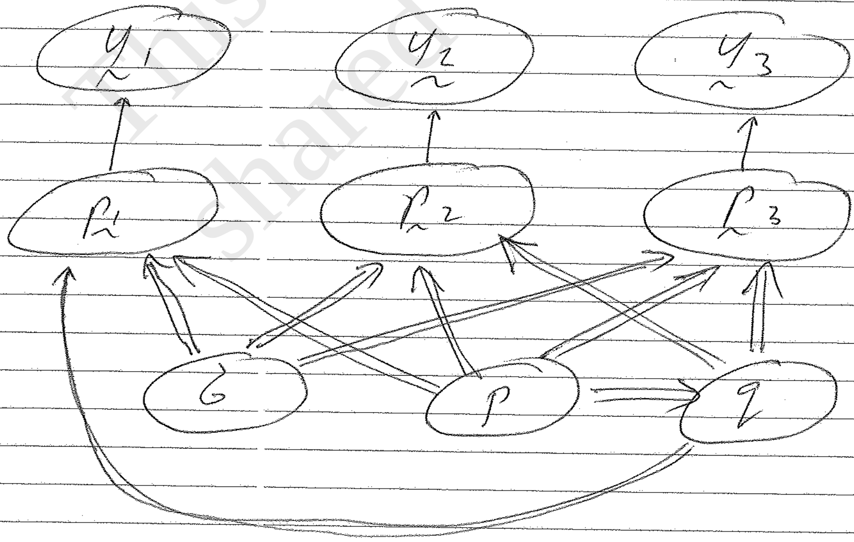
$$\underset{\sim}{p}_2 \leftarrow ((1-\delta)p/2 + \delta/4, 1/2, (1-\delta)q/2 + \delta/4)$$

$$\underset{\sim}{p}_3 \leftarrow (0, (1-\delta)p, (1-\delta)q + \delta)$$

$$\delta \sim U(0,1)$$

$$p \sim U(0,1)$$

$$q \leftarrow 1-p$$



8 d) $p(\delta, p) \propto$

$$((1-\delta)p+\delta)^{y_{11}}\,((1-\delta)q)^{y_{12}}$$
$$\times\,((1-\delta)p/2+\delta/4)^{y_{21}}\,((1-\delta)q/2+\delta/4)^{y_{23}}$$
$$\times\,((1-\delta)p)^{y_{32}}\,((1-\delta)q+\delta)^{y_{33}}$$

9

9 a) $p(u \mid v, w) \propto u^4(1-u-v-w)$

$$\propto \left(\frac{u}{1-v-w}\right)^4 \left(1-\frac{u}{1-v-w}\right)$$

for $u \in (0, 1-v-w)$

that is, for $\frac{u}{1-v-w} \in (0,1)$

thus $\frac{u}{1-v-w} \sim beta(5,2)$

$\equiv \; u \sim (1-v-w)\, beta(5,2)$

Similarly for $V \mid (u,W)$ & $W \mid (u,V)$

b) put $(u(0), v(0), w(0)) = (1/4, 1/4, 1/4)$ say

given $(u(n), v(n), w(n))$

$u(n+1) \sim beta(5,2) \times (1-v(n)-w(n))$

$v(n+1) \sim beta(4,2) \times (1-u(n+1)-w(n))$

$w(n+1) \sim beta(3,2) \times (1-u(n+1)-v(n+1))$

c) check for convergence using BGR
- multiple chains
- compare mean of between chain cred. int. to cred. int. of combined chain
- when both settle & ratio < 1.05 then OK

July						
M	T	W	T	F	S	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

August						
M	T	W	T	F	S	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

September						
M	T	W	T	F	S	S
30						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29

October						
M	T	W	T	F	S	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

November						
M	T	W	T	F	S	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	

December						
M	T	W	T	F	S	S
30	31					1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29

9