SCHOOL OF MATHEMATICS AND STATISTICS

MAST30022 Decision Making **Semester 2, 2021**

Assignment 4 Solutions

1. (a) We first use Axiom 3 (reduction of compound lotteries) to find simple lotteries L_1' and L_3' such that $L_1 \sim L_1'$ and $L_3 \sim L_3'$:

$$\begin{aligned} L_1' &= \langle 0.5 \times 0.5, -2; 0.5 \times 0.75, 0; 0.5 \times 0.25 + 0.5 \times 0.4, 3; 0.5 \times 0.1, 10 \rangle \\ &= \langle 0.25, -2; 0.375, 0; 0.325, 3; 0.05, 10 \rangle. \end{aligned}$$

$$\begin{aligned} L_3' &= \langle 0.8 \times 0.4 + 0.1, -5; 0.1, -2; 0.8 \times 0.6, 10 \rangle \\ &= \langle 0.42, -5; 0.1, -2; 0.48, 10 \rangle. \end{aligned}$$

To determine Paul's preferences over $\mathcal{L}$, we use the expected utility criterion for the simple lotteries L_1', L_2, and L_3':

$$\begin{aligned} \mathbb{E}(U \text{ of } L_1') &= 0.25 \times \sqrt{3} + 0.375 \times \sqrt{5} + 0.325 \times \sqrt{8} + 0.05 \times \sqrt{15} \approx 2.3844 \\ \mathbb{E}(U \text{ of } L_2) &= 0.25 \times \sqrt{0} + 0.55 \times \sqrt{5} + 0.2 \times \sqrt{15} \approx 2.0044 \\ \mathbb{E}(U \text{ of } L_3') &= 0.42 \times \sqrt{0} + 0.1 \times \sqrt{3} + 0.48 \times \sqrt{15} \approx 2.0322. \end{aligned}$$

Paul holds $L_1' \succ L_3' \succ L_2$, and because of transitivity of '$\succ$' and '$\sim$' (Axiom 1), he also holds $L_1 \succ L_3 \succ L_2$.

(b) $RP(L_1) = EV(L_1) - CE(L_1) = EV(L_1') - CE(L_1')$. We have

$$EV(L_1') = 0.25 \times (-2) + 0.375 \times 0 + 0.325 \times 3 + 0.05 \times 10 = 0.975$$

and

$$\sqrt{CE(L_1') + 5} = \mathbb{E}(U \text{ of } L_1') = 2.3844 \implies CE(L_1') = (2.3844)^2 - 5 = 0.685$$

$$\implies RP(L_1) = 0.975 - 0.685 = 0.290$$

(c) Note that it is it not sufficient to check that $RP(L_i) > 0$ for $i = 1, 2, 3$ to conclude that Paul is risk-averse (but it is a good indicator). It should formally be checked for *every* non-degenerate lottery L.

We use the properties of the utility function: since $u''(x) = -\frac{1}{4}(x+5)^{-3/2} < 0$ for all $x \in (-5, \infty)$, it follows that u is strictly concave. Therefore, Paul is risk-averse.

2. (a) Let M1 and M2 denote Machine 1 and 2, respectively, and let AMC denote "annual maintenance cost". The decision tree is represented in Figure 1. Decision vertices are represented by a □, and event vertices are represented by a ○.

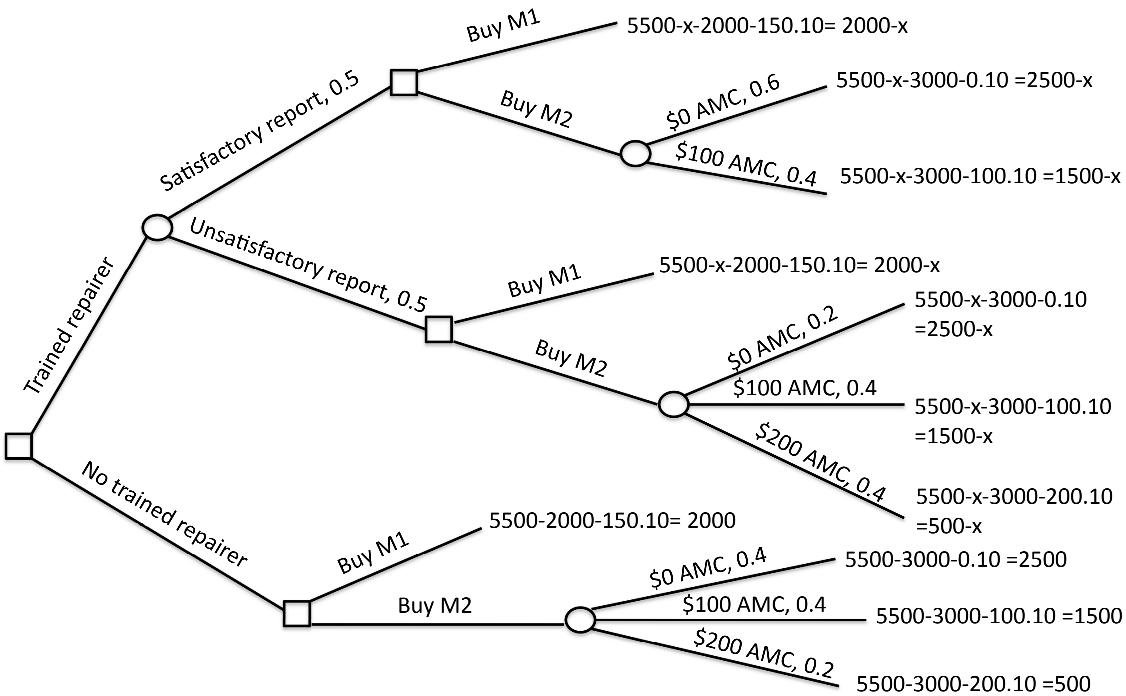

Figure 1: Question 4(a)

(b) When $x = \$0$, the optimal strategy for the department is to ask a trained repairer to evaluate the quality of Machine 2, and then to buy Machine 2 if the report is satisfactory, otherwise to buy Machine 1. The maximum expected amount of money left on the account if the department uses the optimal strategy is then \$2050 (see Figure 2). When $x = \$100$, the optimal strategy for the department is to not ask a trained repairer to evaluate the quality of Machine 2 and to buy Machine 1. The maximum expected amount of money left on the account if the department uses the optimal strategy is then \$2000 (see Figure 3).

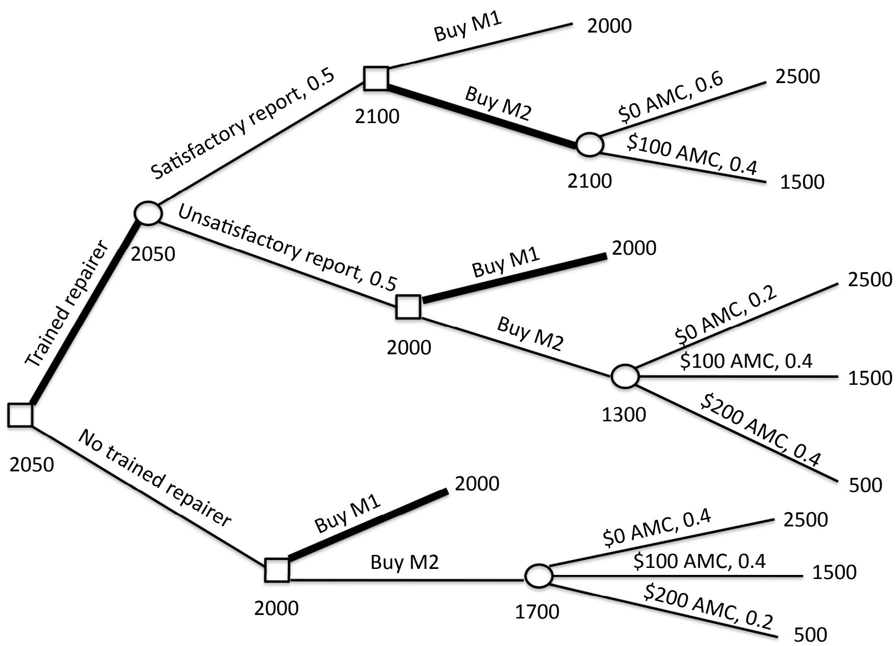

Figure 2: Question 4(b) with $x = 0$.

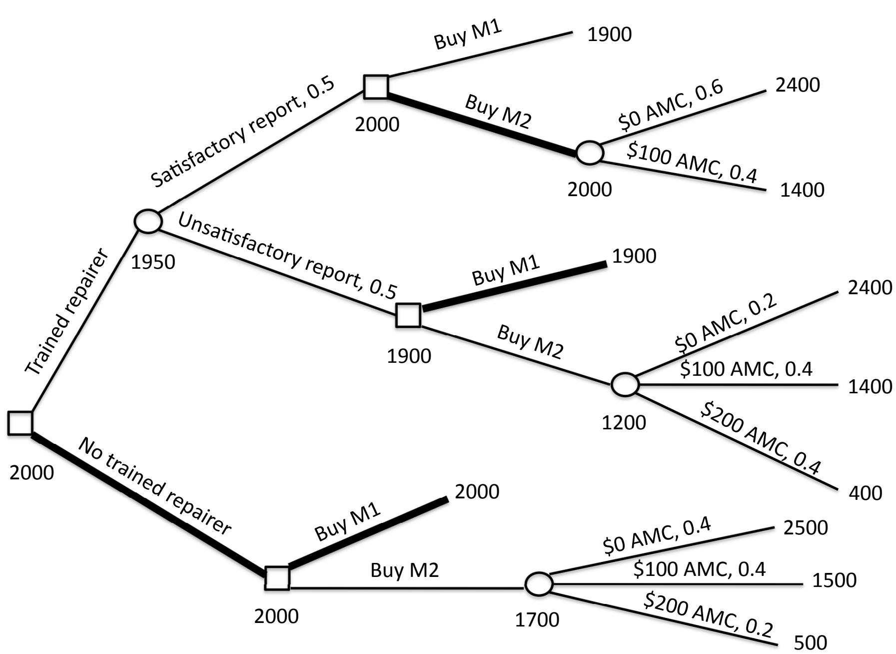

Figure 3: Question 2(b) with $x = \$100$.

(c) If $x = \$0$, and the department wants to maximise the expected utility of the money left on the account, then the optimal strategy for the department is to buy Machine 1 (the department is indifferent between asking or not asking a trained repairer to evaluate the quality of Machine 2) (see Figure 4).

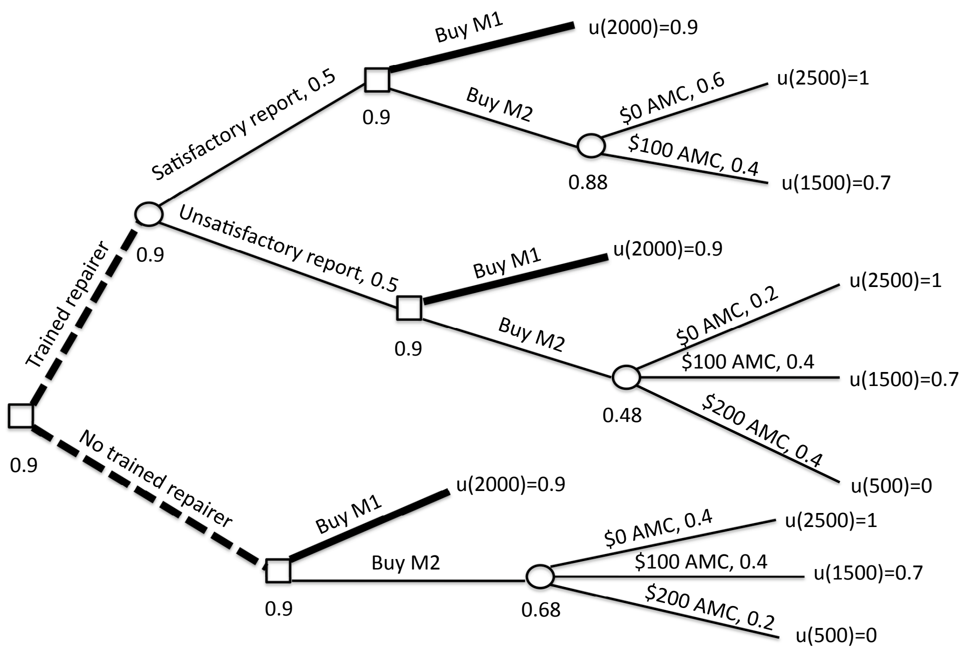

Figure 4: Question 2(c)

3. (a) Decision table

Decision alternative	"Bull" market ($)	"Bear" market ($)	s_i	o_i	$\bar{v}_i$
Company X	5000	-2000	-2000	5000	1500
Company Y	1500	500	500	1500	1000
Probability of occurence	0.6	0.4			

Regret table

Decision alternative	"Bull" market ($)	"Bear" market ($)	ρ_i
Company X	0	2500	2500
Company Y	3500	0	3500

Wald, Hurwicz, Savage, and Laplace say to invest in Company Y, X, X, and X, respectively.

(b) $L_X = \langle 0.6, 5000; 0.4, -2000\rangle$ and $L_Y = \langle 0.6, 1500; 0.4, 500\rangle$.

$\mathbb{E}(U, \text{of } L_X) = 0.6 \times u(5000) + 0.4 \times u(-2000) = 0.6 \times 1 + 0.4 \times 0 = 0.6$ and $\mathbb{E}(U \text{ of } L_Y) = 0.6 \times u(1500) + 0.4 \times u(500) = 0.6 \times 0.5 + 0.4 \times 0.2 = 0.38$.

Since $\mathbb{E}(U \text{ of } L_X) > \mathbb{E}(U \text{ of } L_Y)$, we have $L_X \succ L_Y$. So invest in Company X.

(c) Invest in the stock market and choose Company X. See the decision tree in Figure 5.

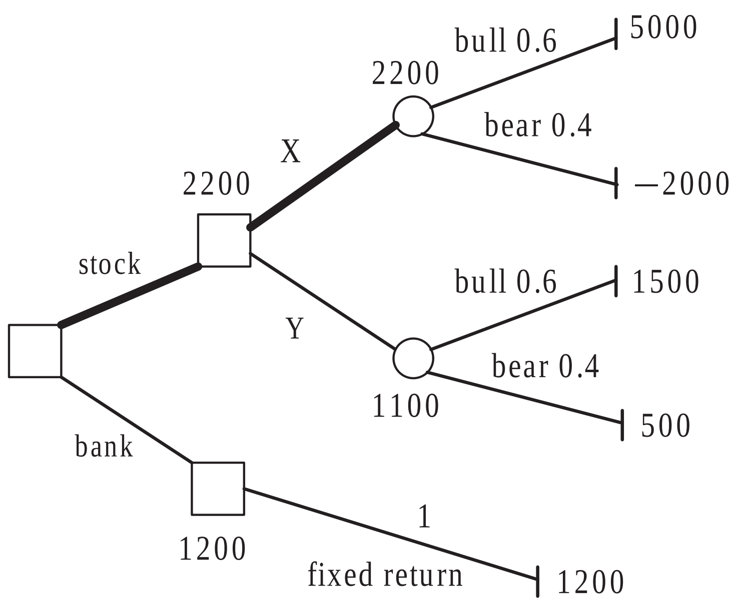

Figure 5: Question 3(c)

4. (a) Let $p(v)$ denote the number of vertices u such that there exists a path from u to v. Then $p(A) = 0$, $p(B) = 2$, $p(C) = 1$, $p(D) = 4$, $p(E) = 3$, and $p(F) = 5$. From this we can derive the proper labelling as depicted in Figure 6.

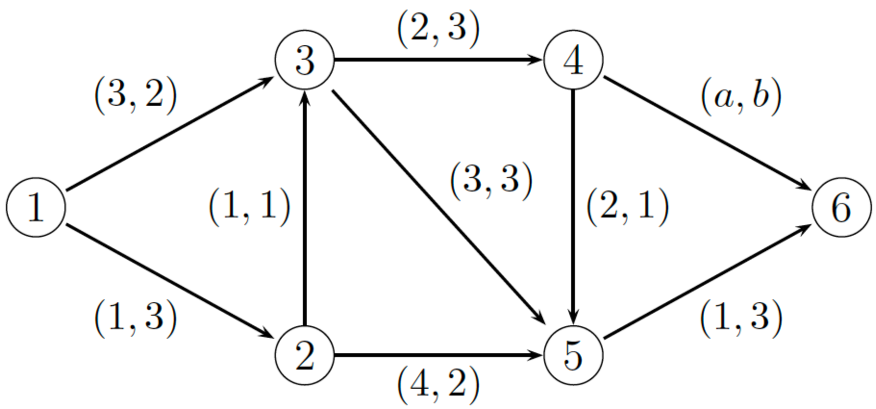

Figure 6: Question 4(a).

(b) We sequentially derive $f(1), \ldots, f(6)$.

$$f(1) = \{(0,0)\}$$

$$P(2) = \{1\}, \quad \begin{aligned} f(2) &= P_{\min}\big(\{f(1) + \boldsymbol{c}_{12}\}\big) \\ &= P_{\min}\big(\{(0,0) + (1,3)\}\big) \\ &= P_{\min}\big(\{(1,3)\}\big) \\ &= \{(1,3)\}. \end{aligned}$$

$$P(3) = \{1,2\}, \quad \begin{aligned} f(3) &= P_{\min}\big(\{f(1) + \boldsymbol{c}_{13}, f(2) + \boldsymbol{c}_{23}\}\big) \\ &= P_{\min}\big(\{(0,0) + (3,2), (1,3) + (1,1)\}\big) \\ &= P_{\min}\big(\{(3,2), (2,4)\}\big) \\ &= \{(3,2), (2,4)\}. \end{aligned}$$

$$P(4) = \{3\}, \quad \begin{aligned} f(4) &= P_{\min}\big(\{f(3) + \boldsymbol{c}_{34}\}\big) \\ &= P_{\min}\big(\{(3,2) + (2,3), (2,4) + (2,3)\}\big) \\ &= P_{\min}\big(\{(5,5), (4,7)\}\big) \\ &= \{(5,5), (4,7)\}. \end{aligned}$$

$$P(5) = \{2,3,4\}, \quad \begin{aligned} f(5) &= P_{\min}\big(\{f(2) + \boldsymbol{c}_{25}, f(3) + \boldsymbol{c}_{35}, f(4) + \boldsymbol{c}_{45}\}\big) \\ &= P_{\min}\big(\{(1,3) + (4,2), (3,2) + (3,3), (2,4) + (3,3) \\ &\qquad\quad (5,5) + (2,1), (4,7) + (2,1)\}\big) \\ &= P_{\min}\big(\{(5,5), (6,5), (5,7), (7,6), (6,8)\}\big) \\ &= \{(5,5)\}. \end{aligned}$$

$$P(6) = \{4,5\}, \quad \begin{aligned} f(6) &= P_{\min}\big(\{f(4) + \boldsymbol{c}_{46}, f(5) + \boldsymbol{c}_{56}\}\big) \\ &= P_{\min}\big(\{(5,5) + (a,b), (4,7) + (a,b), (5,5) + (1,3)\}\big) \\ &= P_{\min}\big(\{(5+a, 5+b), (4+a, 7+b), (6,8))\}\big). \end{aligned}$$

The points $(5+a, 5+b)$ and $(4+a, 7+b)$ are not comparable with respect to the Pareto order. Therefore, there is a unique Pareto minimal path between A and F if $(6,8) <_P (5+a, 5+b)$ and $(6,8) <_P (4+a, 7+b)$. This yields

$$\begin{aligned} &a \geq 1, b \geq 3, \quad \text{where at least one inequality is strict,} \quad \text{and} \\ &a \geq 2, b \geq 1, \quad \text{where at least one inequality is strict,} \end{aligned}$$

hence all values of a and b for which there is a unique Pareto minimal path between A and F are $a \geq 2$ and $b \geq 3$.

(c) From (b) we observe that

$$f(6) \quad = \quad L_{\min}\big(\{(4+a, 7+b), (6,8))\}\big).$$

Therefore, there are at least two lexicographic shortest paths if and only if $(4+a, 7+b) = (6,8)$, so $a = 2$ and $b = 1$.