SCHOOL OF MATHEMATICS AND STATISTICS

MAST30013 Techniques in Operations Research Semester 1, 2021

Assignment 3- SOLUTION

2. (a) The Lagrange function

$$L(x,\lambda) = (x_1-2)^4/4 + x_2^4 + 4 + \lambda_1(x_1 - x_2 - 8) + \lambda_2(-x_1 + x_2^2 + 4).$$ **1**

KKTa:

$$\nabla_x L(x^*,\lambda^*) = 0 \Rightarrow \begin{bmatrix} (x_1-2)^3 + \lambda_1 - \lambda_2 \\ 4x_2^3 - \lambda_1 + 2\lambda_2 x_2 \end{bmatrix} = 0$$ **2**

KKTb:

$$\begin{aligned} &x_1^* - x_2^* - 8 \le 0, && -x_1^* + {x_2^*}^2 + 4 \le 0, \\ &\lambda_1^* \ge 0, && \lambda_2^* \ge 0, \\ &\lambda_1^*(x_1^* - x_2^* - 8) = 0, && \lambda_2^*(-x_1^* + {x_2^*}^2 + 4) = 0, \end{aligned}$$ **6**

i. $\lambda_1^* = \lambda_2^* = 0, \Rightarrow (x_1^*-2)^3 = 0$ and $x_2^* = 0$. $-x_1^* + {x_2^*}^2 + 4 = 2 \not\le 0$, which is not allowed by KKTb. **1**

ii. $\lambda_1^* = 0, \lambda_2^* > 0, \Rightarrow -x_1^* + {x_2^*}^2 + 4 = 0, 4{x_2^*}^3 + 2\lambda_2^* x_2^* = 0, (x_1^* - 2)^3 = \lambda_2^*$. Note $4{x_2^*}^3 + 2\lambda_2^* x_2^* = 0$ gives one solution when $\lambda_2^* > 0$, that is $x_2^* = 0$. Then $x_1^* = 4$ and $\lambda_2^* = 8$. **2**

iii. $\lambda_1^* > 0, \lambda_2^* = 0, \Rightarrow x_1^* - x_2^* - 8 = 0, 4{x_2^*}^3 - \lambda_1^* = 0, (x_1^* - 2)^3 + \lambda_1 = 0$. The three equalities give $4{x_2^*}^3 = -(x_2^* + 6)^3$, for which root(s) must be negative. Then $\lambda_1^* = 4{x_2^*}^3 < 0$, which is not allowed by KKTb. **1**

iv. $\lambda_1^* > 0,\ \lambda_2^* > 0, \Rightarrow x_1^* - x_2^* - 8 = 0$ and $-x_1^* + {x_2^*}^2 + 4 = 0$, $\Rightarrow {x_2^*}^2 - x_2^* - 4 = 0$. Two roots $x_2^* = (1 \pm \sqrt{17})/2$ and $x_1^* = (17 \pm \sqrt{17})/2$. Both lead to negative λ. So they are not KKT points. **1**

So one KKT point is found, that is $x^* = (4,0)$ with multiplier $\lambda^* = (0,8)$.

(b) The active constraint $g_2(x) := -x_1 + x_2^2 + 4 \leq 0$ is not affine. **2**

$$\nabla g_2(x^*) = \begin{bmatrix} -1 \\ 2x_2^* \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

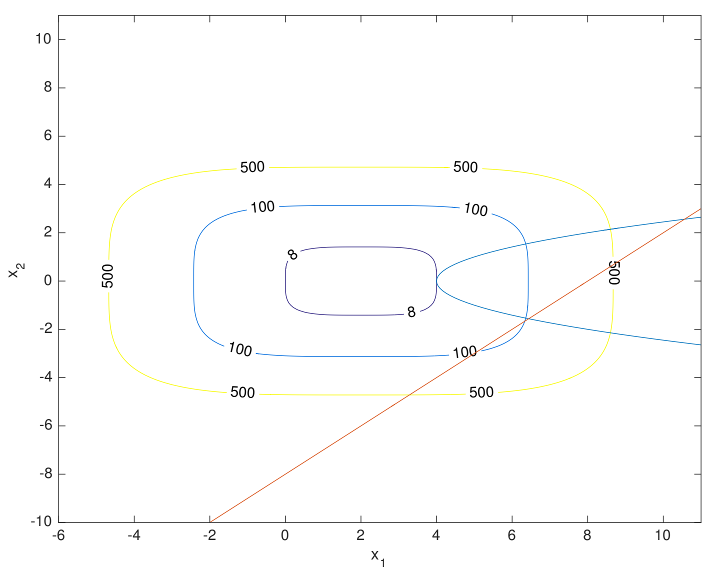
The active gradient is linearly independent, and thus the constraint qualification is satisfied.

(c) The Hessian is **3**

$$\nabla_{xx}^2 L(x^*, \lambda^*) = \begin{bmatrix} 3(x_1^* - 2)^2 & 0 \\ 0 & 12x_2^{*2} + 2\lambda_2^* \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 16 \end{bmatrix}$$

$\nabla_{xx}^2 L(x^*, \lambda^*)$ is positive-definite for all $d \in \mathbb{R}^2$. Thus, by the 2nd order sufficiency condition, $x^* = (4, 0)$ is a local minimum of the NLP.

(d) There is only one active constraint, which is $g_2(x)$. There is a unique local minimum, which is the intersect between g_2 and the level curve at 8. **2**



6

Figure 1: Red line: $x_1 - x_2 - 2 = 0$, blue curve: $x_1 - x_2^2 - 4 = 0$.

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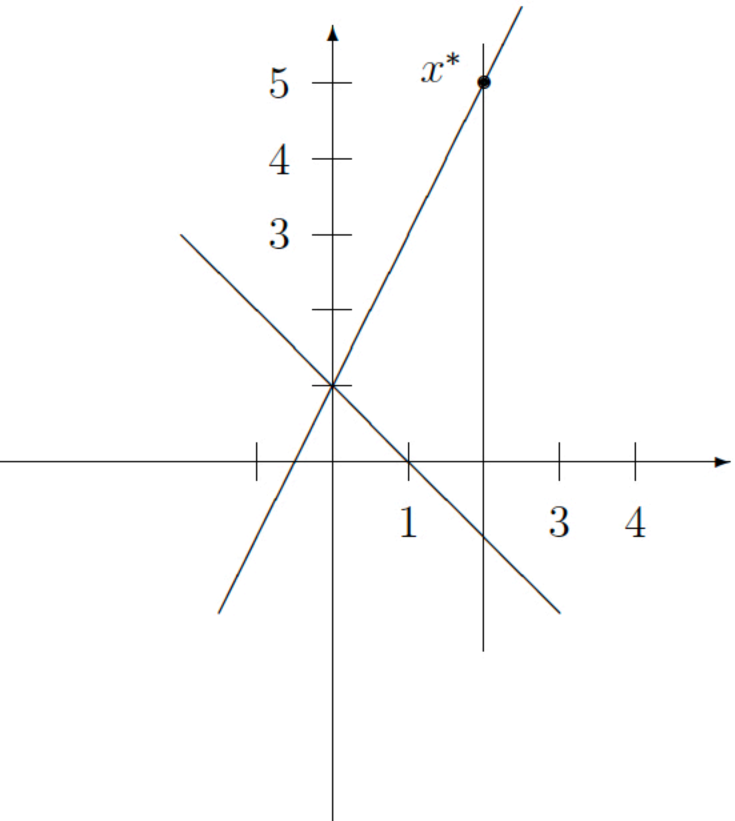
x = -10:0.1:11;
y = -10:0.1:11;
[X,Y] = meshgrid(x,y);
Z = (X-2).^4./4+Y.^4+4;
figure(1)
[c,h]=contour(X,Y,Z,[8, 100,500]);
clabel(c,h);
xlabel('x_1');
ylabel('x_2');
hold on
x1 = y.^2+4;
plot(x1,y);
x1 = y+8;
plot(x1,y);
hold off
xlim([-6 11]);
```

(e) The Hessian of the Lagrangian is positive semidefinite on the constraint set. Therefore, the objective function is convex on the constraint set. **1**

a) Solution: The feasible region is the region bounded by the three lines in Figure 2. Minimizing $-x$ is equivalent to maximizing x so the solution we seek is the corner, $x^* = (2, 5)$, and the optimal value is $-2 - 3(5) = -17$.



(b) Set up the problem in standard form.

Solution:

$$\begin{array}{rl} \min & -x_1 - 3x_2 \\ \text{subject to} & h(x) = x_2 - 2x_1 - 1 = 0 \\ & g_1(x) = 1 - x_1 - x_2 \leq 0 \\ & g_1(x) = x_1 - 2 \leq 0 \end{array}$$

(c) State the KKT conditions that the solution will have to satisfy. Make sure that you have as many conditions as variables.

Solution: We have five unknowns, $x_1^*, x_2^*, \lambda^*, \mu_1^*$, and μ_2^*. With

$$f(x) = \begin{bmatrix} -1 & -3 \end{bmatrix} x,$$

the conditions are

$$\begin{aligned} \mu^* &\geq 0, \\ Df(x^*) + \lambda^{*T} Dh(x^*) + \mu^* Dg(x^*) &= 0, \\ \mu_1^* g_1(x^*) = \mu^*(1 - x_1 - x_2) &= 0, & (1) \\ \mu_2^* g_2(x^*) = \mu_2^* x_1 &= 2, & (2) \\ h(x^*) = x_2 - 2x_1 - 1 &= 0, & (3) \\ g_1(x^*) = 1 - x_1 - x_2 &\leq 0, \end{aligned}$$

and

$$g_2(x^*) = x_1 - 2 \leq 0.$$

The transpose of the condition on the derivatives is

$$\begin{bmatrix} -1 \\ -3 \end{bmatrix} + \lambda^* \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \mu_1^* \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \mu_2^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

which can be rewritten as the two scalar equations

$$\begin{aligned} -1 - 2\lambda^* - \mu_1^* + \mu_2^* &= 0 & (4) \\ -3 + \lambda^* - \mu_1^* &= 0. & (5) \end{aligned}$$

Thus we have five equations in five unknowns.

(d) Determine the candidate points that should be tested for optimality.

Solution: Because the equations are linear, the set of options will correspond to two pairs of options. From (1) we have either

$$\mu_1^* = 0 \quad \text{or} \quad x_1 + x_2 = 1$$

and from (2) either

$$\mu_2^* = 0 \quad \text{or} \quad x_1 = 2.$$

Case 1: If $\mu_1^* = 0$ then from the derivative constraints (4) and (5),

$$\begin{aligned} -1 - 2\lambda^* + \mu_2^* &= 0 \\ -3 + \lambda^* &= 0. \end{aligned}$$

Thus $\lambda^* = 3$ and $\mu_2^* = 7$, so by (2) $7(x_1 - 2) = 0$, which implies $x_2^* = 5$. The candidate augmented vector is then

$$\begin{bmatrix} \lambda & \mu_1 & \mu_2 & x_1 & x_2 \end{bmatrix}_1 = \begin{bmatrix} 3 & 0 & 7 & 2 & 5 \end{bmatrix}.$$

Case 2: If $\mu_1^* \neq 0$ then $x_1^* + x_2^* = 1$, and if $\mu_2^* = 0$ then from the derivative constraints (4) and (5),

$$\begin{aligned} -1 - 2\lambda^* - \mu_1^* &= 0 \\ -3 + \lambda^* - \mu_1^* &= 0. \end{aligned}$$

Subtracting the second equation from the first and solving for λ^* yields $\lambda^* = 2/3$. Then $\mu_1^* = 7/3$ and (1) and (3) together imply $x_1^* = 0$, in which case $x_2 = 1$. The resulting candidate augmented vector is

$$\begin{bmatrix} \lambda & \mu_1 & \mu_2 & x_1 & x_2 \end{bmatrix}_2 = \begin{bmatrix} 2/3 & 7/3 & 0 & 0 & 1 \end{bmatrix}.$$

Case 3: If $\mu_1^* \neq 0$ then $x_1^* + x_2^* = 1$, and if $\mu_2^* \neq 0$ then $x_1 = 2$. Then from (1) $x_2 = -1$. However, the equality constraint (3) is not satisfied, so this case does not occur.

Finally,

Case 4: $\mu_1^* = \mu_2^* = 0$. Then (4) and (5) are inconsistent, so this case cannot occur.

Thus we have two candidate solutions:

$$\begin{bmatrix} 3 \\ 0 \\ 7 \\ 2 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2/3 \\ 7/3 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$