MAST30022 Decision Making
2021
Tutorial Solutions 1

1. **(PS1-2)** Prove that in a rooted tree any vertex other than the root has exactly one entering edge.

   Solution

   Suppose the vertex v, which is not the root r, has no entering edges. Then there is no unique path from r to v (there is no path), so v must have at least one entering edge. Suppose v has more than one entering edge. Then there will be more than one path from r to v, so the path from r to v will not be not unique. Since v is an arbitrary vertex, every vertex other than the root has exactly one entering edge.

2. **(PS1-4)** Show that any rooted tree with at least two vertices contains a non-terminal vertex such that all of its children are terminal vertices.

   Solution

   Consider a rooted tree (T, r). For each terminal vertex v calculate the length of the path from r to v. Choose v^* to be such that the length of the path from r to v^* is maximal. Then v^* is a leaf. Let u^* be the parent of v^*. Then u^* is a non-terminal vertex. If w is a child of u^*, then w is a leaf since we have chosen v^* so that the length of the path from r to v^* is maximal and the path from r to u^* is unique. Hence, all children of u^* are leaves.

3. **(PS1-5)** The game of Fingers is played as follows: Two players, Alice and Bob, simultaneously hold up either one or two fingers. Alice wins in case of a match (the same number), and Bob wins in case of a non-match. The amount won is the number of fingers held up by the winner. It is paid to the winner by the loser.

   (a) Describe the strategies for both players.

   (b) Write down the normal form (as a pair of payoff matrices).

   (c) Verify that there is no equilibrium pair of (pure) strategies.

   (Adapted from "Introduction to Game Theory", P. Morris, 1994)

   Solution

   (a) The strategies of Alice are: $a_1 =$ "1" (holding one finger), $a_2 =$ "2" (holding two fingers). The strategies of Bob are $A_1 =$ "1" (holding one finger), $A_2 =$ "2" (holding two fingers).

(b) The payoff bi-matrix of the game is

		Bob	
		A_1	A_2
	a_1	$(1,-1)$	$(-2,2)$
Alice	a_2	$(-1,1)$	$(2,-2)$

(c) To find any Nash equilibria look for the maximum first entry down a column and the maximum second entry across a row in the same payoff pair. There is no such payoff pair, so there are no Nash equilibria.

4. **(PS1-6)** The normal form of a three-person game is given in the following table. Player I has three strategies (1, 2, 3), and Players II and III each have two strategies (1, 2). Find the two equilibrium 3-tuples of strategies.

Strategy triples	Payoff vectors
(1,1,1)	$(0,-1,0)$
(1,1,2)	$(0,-2,0)$
(1,2,1)	$(3,0,-1)$
(1,2,2)	$(1,-1,-1)$
(2,1,1)	$(0,0,0)$
(2,1,2)	$(0,0,-1)$
(2,2,1)	$(-1,1,1)$
(2,2,2)	$(2,1,-1)$
(3,1,1)	$(0,0,2)$
(3,1,2)	$(0,-1,1)$
(3,2,1)	$(1,-2,1)$
(3,2,2)	$(1,1,-1)$

Table 1: Question 6, Problem Set 1

Solution

Consider the strategy triple $(1,1,1)$. We have $u_1(1,1,1)=0$, $u_2(1,1,1)=-1$, and $u_3(1,1,1)=0$. Now deviating from this strategy gives $u_1(2,1,1)=0$, $u_1(3,1,1)=0$, $u_2(1,2,1)=0$, and $u_3(1,1,2)=0$. Since $u_2(1,2,1)\not\leq u_2(1,1,1)$, $(1,1,1)$ is not an equilibrium strategy.

Consider the strategy triple $(1,2,1)$. We have $u_1(1,2,1)=3$, $u_2(1,2,1)=0$, and $u_3(1,2,1)=-1$. Now deviating from this strategy gives $u_1(2,2,1)=-1$, $u_1(3,2,1)=1$, $u_2(1,1,1)=-1$, and $u_3(1,2,2)=-1$. Since $u_1(2,2,1)\leq u_1(1,2,1)$, $u_1(3,2,1)\leq$

$u_1(1,2,1)$, $u_2(1,1,1) \leq u_1(1,2,1)$, and $u_3(1,2,2) \leq u_1(1,2,1)$, $(1,2,1)$ is an equilibrium strategy.

Continuing in this manner it can be seen that $(3,1,1)$ is the only other equilibrium strategy.

5. **(PS1-8)** Two nuclear powers are engaged in an arms race in which each power stockpiles nuclear weapons. An issue is the rationality of such a strategy on the part of both powers.

   Let us examine a stylized version. Assume Country 1 moves in the first stage and may choose between nuclear weapons N or non-proliferation NP. Country 2 in stage 2 of the game observes the choice of country 1 and chooses between N and NP. A representative game tree of the situation is shown in Figure 1.

   According to this game tree, country 2 likes the option N whether country 1 chooses NP or N. If country 1 chooses NP, then country 2 by choosing N guarantees for itself a very powerful position. If country 1 chooses N, then country 2 would like to choose N as this allows a credible deterrence against a possible nuclear attack by country 1.

   Give the normal form of this game. Find all its Nash equilibria or otherwise prove that it has no Nash equilibrium.

   (Adapted from "Game and Decision Making", C. D. Aliprantis and S. K. Chakrabarti, 2000)

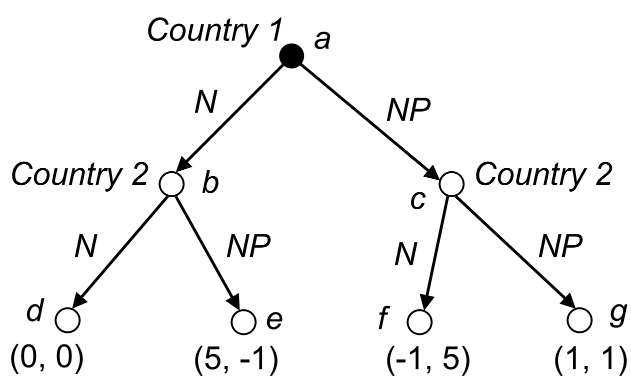


Figure 1: Question 8, Problem Set 1

Solution

Note that the game is one of perfect information because Country 2 responds to what Country 1 does, so there exists at least one Nash equilibrium. Let (N,N) denote the strategy Country 2 takes if they choose N if Country 1 chooses N, and N if Country 1 chooses NP, and so on.

The game in normal form is

		Country 2			
		(N,N)	(N,NP)	(NP,N)	(NP,NP)
	N	$(0,0)$	$(0,0)$	$(5,-1)$	$(5,-1)$
Country 1	NP	$(-1,5)$	$(1,1)$	$(-1,5)$	$(1,1)$

To find any Nash equilibria look for the maximum first entry down a column and the maximum second entry across a row in the same payoff pair. Here (N, (N,N)) is a Nash equilibrium, with corresponding payoff pair $(0,0)$. Note the best solution for both players is either (NP, (N,NP)) or (NP, (NP,NP)), each with corresponding payoff pair $(1,1)$, but neither is a Nash equilibrium.

6. **(PS1-9)** Represent the following game in extensive form, and then give its normal form using a payoff matrix.

   Anne has an Ace and a Queen. Bea has a King and a Joker (J). The rank of the cards is $A > K > Q$, but the Joker is peculiar as will be seen.

   Each antes (places an initial bet before any move is made) a penny to the pot. The players then select one of their cards and simultaneously reveal their selections. If Bea selects the King, the highest card chosen wins the pot and the game ends. If Bea selects the Joker and Ann the Queen, they split the pot and the game ends. If Bea selects the Joker and Ann the Ace, then Ann may either resign (in which case Bea gets the pot) or demand a replay. If Ann demands a replay they each ante another penny to the pot and play again. This time if Bea selects the Joker and Ann the Ace, Bea gets the pot.

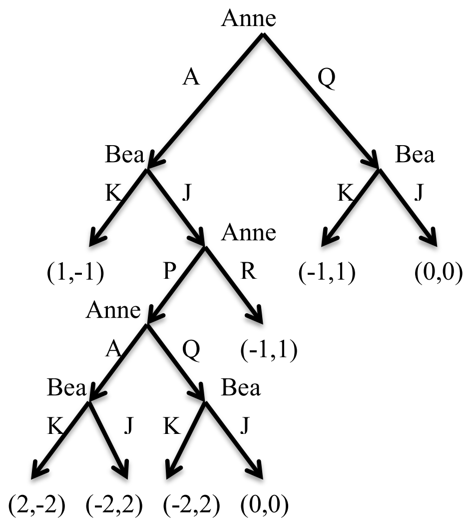
   (Adapted from "Game Theory: Mathematical Models of Conflict", A. J. Jones, 2000)

Solution

Below is the tree for the game in extensive form. P indicates that Anne plays again and R indicates that she resigns.

Let the strategy sets for Anne and Bea be

$$\begin{aligned} X_1 &= \{Q, AR, APA, APQ\} \\ X_2 &= \{K, JK, JJ\}, \end{aligned}$$



respectively. Understand that the strategies AR, APA, and APQ for Anne mean that she plays nothing should Bea play the King on her first move. Also understand that the strategies JK and JJ for Bea mean she only plays the Joker once should Anne play the Queen on her first move. The payoff bi-matrix of the game is

		Bea		
		K	JK	JJ
	Q	$(-1,1)$	$(0,0)$	$(0,0)$
Anne	AR	$(1,-1)$	$(-1,1)$	$(-1,1)$
	APA	$(1,-1)$	$(2,-2)$	$(-2,2)$
	APQ	$(1,-1)$	$(-2,2)$	$(0,0)$

Note that there are no Nash equilibria.