MAST30013 – Techniques in Operations Research

Semester 1, 2021

Tutorial 9 Solutions

1. (a) The penalty function is

$$P_k(\boldsymbol{x}) \;=\; \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 - x_1 + x_2 + \tfrac{k}{2}[(x_1)_+]^2 + \tfrac{k}{2}[(x_2)_+]^2.$$

(b)

$$\boldsymbol{\nabla} P_k(\boldsymbol{x}) \;=\; \begin{pmatrix} x_1 - 1 + k(x_1)_+ \\ x_2 + 1 + k(x_2)_+ \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If $x_1 \leq 0$, we have $x_1 - 1 = 0 \Longrightarrow x_1 = 1$, which is a contradiction. If $x_2 \geq 0$, we have $x_2 + 1 + kx_2 = 0 \Longrightarrow x_2^k = -1/(k+1)$, which is a contradiction . Therefore, $x_1 > 0$ and $x_2 < 0$ for any stationary points of $P_k(\boldsymbol{x})$.

(c) Now, for $x_1 > 0$ and $x_2 < 0$,

$$\boldsymbol{\nabla} P_k(\boldsymbol{x}) \;=\; \begin{pmatrix} x_1 - 1 + kx_1 \\ x_2 + 1 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives $x_1^k = 1/(k+1)$ and $x_2^k = -1$. Now, $x_1^* = \lim\limits_{k\to\infty} x_1^k = 0$, so the limit point is $\boldsymbol{x}^* = (0, -1)^T$.

(d) For $\left(x_1^k, x_2^k\right) = (1/(k+1), -1)$,

$$\boldsymbol{\lambda}^k \;=\; \begin{pmatrix} k\left(\dfrac{1}{1+k}\right)_+ \\ k(-1)_+ \end{pmatrix} \;=\; \begin{pmatrix} \dfrac{k}{1+k} \\ 0 \end{pmatrix} \;\to\; \boldsymbol{\lambda}^* \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2. (a) The penalty function is

$$P_k(\boldsymbol{x}) \;=\; \tfrac{1}{4}x_1^4 - \tfrac{1}{2}x_1^2 + x_2^2 + \tfrac{k}{2}[(-x_1)_+]^2 + \tfrac{k}{2}[(2-x_2)_+]^2.$$

(b)

$$\boldsymbol{\nabla} P_k(\boldsymbol{x}) \;=\; \begin{pmatrix} x_1^3 - x_1 - k(-x_1)_+ \\ 2x_2 - k(2-x_2)_+ \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If $x_1 < 0$, we have, $x_1^3 - x_1 + kx_1 = 0 \Longrightarrow x_1 = 0, \pm\sqrt{1-k}$. Here, we preclude 0 and $x_1^k = \sqrt{1-k}$ as they are nonnegative. $x_1^k = \sqrt{1-k}$ is also precluded as $k \leq 1$ for x_1^k to be defined and we need to let $k \to \infty$. Also, If $x_2 \geq 2$, we have $2x_2 = 0 \Longrightarrow x_2 = 0$, which is a contradiction. Therefore, $x_1 \geq 0$ and $x_2 < 2$ for any stationary points of $P_k(\boldsymbol{x})$.

(c) Now, for $x_1 \geq 0$ and $x_2 < 2$,

$$\boldsymbol{\nabla} P_k(\boldsymbol{x}) \;=\; \begin{pmatrix} x_1^3 - x_1 \\ 2x_2 - k(2-x_2)_+ \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives $x_1^k = 0, 1$ (we preclude the other solution $x_1^k = -1$ as it is negative), and $x_2^k = 2k/(k+2)$. Now, $x_2^* = \lim_{k\to\infty} x_2^k = 2$, so the cluster (limit) points are $\boldsymbol{x}^* = (0,2)^T$ and $(1,2)^T$.

(d) For $\left(x_1^k, x_2^k\right) = (0, 2k/(k+2))$,

$$\boldsymbol{\lambda}^k \;=\; \begin{pmatrix} k(0)_+ \\ k\left(2 - \dfrac{2k}{k+2}\right)_+ \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ \dfrac{4k}{k+2} \end{pmatrix} \;\to\; \boldsymbol{\lambda}^* \;=\; \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

For $\left(x_1^k, x_2^k\right) = (1, 2k/(k+2))$,

$$\boldsymbol{\lambda}^k \;=\; \begin{pmatrix} k(-1)_+ \\ k\left(2 - \dfrac{2k}{k+2}\right)_+ \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ \dfrac{4k}{k+2} \end{pmatrix} \;\to\; \boldsymbol{\lambda}^* \;=\; \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$