MAST20004 Probability

Tutorial Set 3

Tutorial problems:

1. The *newsvendor problem* is faced by retailers who sell perishable goods. (Newspapers could be the ultimate perishable good.) Newsvendors have to decide how many papers to order from their supplier every day. Assume that you are such a newsvendor and that historical data indicates that

   - with probability 0.25 you will have demand for 90 papers in a day,
   - with probability 0.5 you will have demand for 100 papers in a day, and
   - with probability 0.25 you will have demand for 120 papers in a day.

   On a given day, you decide to order 105 papers. Let X be the random variable equal to the number of unsold papers, and Y be the random variable equal to the number of customers that you have to turn away because you have run out of papers.

   (a) Write down the probability mass functions of X and Y.

   (b) Write down the expected values of X and Y.

   (c) Assume that the the supplier charges you \$0.70 for every paper that he sends to you and that you receive \$1.20 for every paper that you sell. Letting R be the amount of money that you make in a day, write down the probability mass function and expected value of R.

   (d) **Extension question:** How many papers should you order each day - justify your answer?

   Solution:

   (a) $S_X = \{15, 5, 0\}$, $p_X(15) = 0.25$, $p_X(5) = 0.5$, $p_X(0) = 0.25$.
   $S_Y = \{0, 15\}$, $p_Y(0) = 0.75$, $p_Y(15) = 0.25$.

   (b) $\mathbb{E}[X] = 15 \times 0.25 + 5 \times 0.5 = 6.25$
   $\mathbb{E}[Y] = 15 \times 0.25 = 3.75$.

   (c) If there is demand for 90 papers, you make $90 \times \$0.50 - 15 \times \$0.70 = \$34.50$.
   If there is demand for 100 papers, you make $100 \times \$0.50 - 5 \times \$0.70 = \$46.50$.
   If there is demand for 120 papers, you make $105 \times \$0.50 = \$52.50$.
   So $S_R = \{34.5, 46.5, 52.5\}$, $p_R(34.5) = 0.25$, $p_R(46.5) = 0.5$, $p_R(52.5) = 0.25$, and

   $$\mathbb{E}[R] = 34.5 \times 0.25 + 46.5 \times 0.5 + 52.5 \times 0.25 = \$45.00.$$

   (d) **Extension question:** The way to approach this is to let the number of papers that you order be a variable, say n, and then choose n so that $\mathbb{E}[R]$ is maximised. We follow reasoning similar to that above,

   (i) If $n \leq 90$, then you make (in dollars) $n \times 0.50$. This is maximised when $n = 90$, in which case you make \$45 with probability one.

   (ii) If $90 \leq n \leq 100$ then, if there is demand for 90 papers, you make (in dollars)

   $$90 \times 0.50 - (n - 90) \times 0.70 = 108 - 0.70n,$$

   if there is demand for 100 papers, you make $n \times 0.50$,
   if there is demand for 120 papers, you make $n \times 0.50$.
   So
   $$\mathbb{E}[R] = (108 - 0.70n) \times 0.25 + 0.50n \times 0.75 = 27 + 0.20n.$$

   This is maximised when $n = 100$, in which case $\mathbb{E}[R] = \$47.00$.

(iii) If $100 \le n \le 120$, then, if there is demand for 90 papers, you make (in dollars)

$$90 \times 0.50 - (n - 90) \times 0.70 = 108 - 0.70n,$$

if there is demand for 100 papers, you make

$$100 \times 0.50 - (n - 100) \times 0.70 = 120 - 0.70n,$$

if there is demand for 120 papers, you make $n \times 0.50$.
So

$$\mathbb{E}[R] = (108 - 0.70n) \times 0.25 + (120 - 0.70n) \times 0.5 + 0.50n \times 0.25 = 87 - 0.40n.$$

This is maximised when $n = 100$, in which case $\mathbb{E}[R] = \$47.00$.

(iv) You should clearly never order more than 120 papers.

So the answer is that you should order 100 papers, in which case your expected profit is \$47.00.

2. Five friends went to a picnic with a cooler, which contained three cans of strong and twelve cans of light beer. For the first round, they each took at random a can of beer from the cooler. Let X be the number of cans of light beer they drank in the first round.

   (a) Specify which possible values that X can take. Then give an explicit formula for the probability mass function $p_X(x)$.

   (b) Compute $p_X(3)$.

   (c) Compute $\mathbb{E}[X]$ and $V(X)$.

   Solution:

   (a) X can take the values 2, 3, 4, or 5. The probability that $X = x$ is calculated by observing that this happens when the friends choose $5 - x$ cans of strong beer from the 3 available and x cans of light beer from the 12 available. The number of ways that this can happen is $\binom{3}{5-x}\binom{12}{x}$. The total number of ways of choosing 5 cans from 15 is $\binom{15}{5}$. Thus, for $x = 2, \ldots, 5$, the probability mass function is

   $$p_X(x) = \mathbb{P}(X = x) = \frac{\binom{3}{5-x}\binom{12}{x}}{\binom{15}{5}}.$$

   (b)

   $$\begin{aligned} p_X(3) &= \frac{\binom{3}{2}\binom{12}{3}}{\binom{15}{5}} \\ &= 3 \times 220/3003 \\ &= 20/91 \end{aligned}$$

   (c) We also can work out that $p_X(2) = 2/91$, $p_X(4) = 45/91$, and $p_X(5) = 24/91$. So

   $$\mathbb{E}[X] = (2 \times 2 + 3 \times 20 + 4 \times 45 + 5 \times 24)/91 = 4$$

   and

   $$\mathbb{E}[X^2] = (4 \times 2 + 9 \times 20 + 16 \times 45 + 25 \times 24)/91 = 1508/91.$$

   Therefore $V(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 52/91 = 4/7$.

3. Let F be defined by

   $$F(x) = \begin{cases} 1 - (x+1)e^{-x}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

   (a) Verify that F is a distribution function.

   (b) Sketch the graph of F.

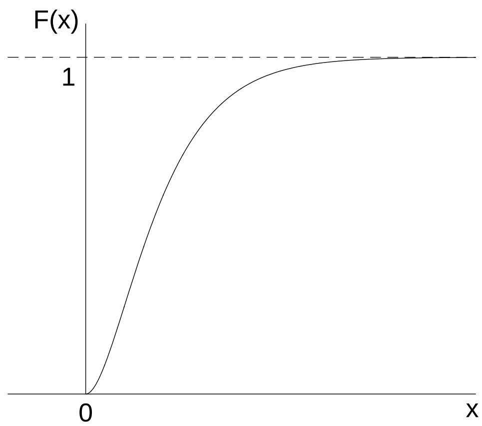
(c) If X is a random variable with this distribution function, write down the probability density function of X.

(d) Calculate the mean and variance of X.

Solution:

(a) $F(0) = 0$, $F(\infty) = 1$. Also $F'(x) = xe^{-x}$, which is positive for $x > 0$, so F is increasing. Therefore F is a distribution function.

(b)



(c) $f_X(x) = xe^{-x}$, $x > 0$.

(d) Integrating by parts we get

$$\begin{aligned}\mathbb{E}[X] &= \int_0^\infty x^2 e^{-x} dx \\ &= \left[-x^2 e^{-x}\right]_0^\infty + \int_0^\infty 2xe^{-x} dx \\ &= 0 + \left[-2xe^{-x}\right]_0^\infty + \int_0^\infty 2e^{-x} dx \\ &= 0 + 0 + 2.\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[X^2] &= \int_0^\infty x^3 e^{-x} dx \\ &= \left[-x^3 e^{-x}\right]_0^\infty + \int_0^\infty 3x^2 e^{-x} dx \\ &= 0 + 3 \times 2 \qquad\qquad \text{(from above)} \\ &= 6.\end{aligned}$$

Therefore $V(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 6 - 4 = 2$.

4. It is believed that the total loss caused by a fire at a factory is a random variable X (\$ million) having the probability density function $f_X(x) = cx^2$, $x \in [0, 2]$.

(a) Find the value of the constant c.

(b) Compute the distribution function $F_X(x)$.

(c) Find the 0.90 quantile of the distribution, that is the value u such that $F_X(u) = 0.90$.

(d) Calculate the mean and variance of X.

Solution:

(a) We need $\int_0^2 cx^2 dx = 1$. Therefore $c = 3/8$.

(b) For $x \in [0, 2]$, the probability density function is $f_X(x) = 3x^2/8$. Therefore the distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3/8 & \text{if } x \in [0, 2] \\ 1 & \text{if } x > 2. \end{cases}$$

(c) The 0.90 quantile is given by solving the equation $x^3/8 = 0.90$, so $x = (7.2)^{1/3}$.

(d) We have

$$\mathbb{E}[X] = \int_0^2 3x^3/8 dx = 3/2$$

and

$$\mathbb{E}[X^2] = \int_0^2 3x^4/8 dx = 96/40.$$

Therefore $\mathrm{Var}(X) = 96/40 - (3/2)^2 = 3/20$.

MAST20004 Probability

Computer Lab 3

This lab considers various aspects of the random experiment of throwing two fair dice. In particular we

- display the partitions of the sample space 'generated' by different random variables;
- calculate the theoretical distributions and then simulate the distributions for the sum, difference, absolute difference, minimum, and maximum of the numbers on the two dice;
- use a large number of trials to estimate the first and second moments for each of the above distributions.

MATLAB programming

In this lab you will need to do a very limited amount of programming for yourself — primarily adding a few important lines to the programs that have been provided on the server. To help you with these tasks:

- continue to work in groups of 2 or 3;
- follow the hints built into the comment lines in the programs themselves;
- remember to make use of the help system which is very extensive;
- try to do it yourself but not for too long — ask a tutor for help if you get stuck. At this stage of semester we expect students new to programming to need a lot of assistance.

Exercise A - Partitions of Ω generated by random variables

Any discrete random variable 'generates' a partition of the sample space. We hinted at this on lecture Slide 86, where we discussed the events $A_x = \{\omega : X(\omega) = x\}, \forall x \in S_X$. These events are both mutually exclusive and exhaustive and so form a partition of Ω.

Consider the experiment of rolling two fair dice. Let X be the number showing on the first die and Y be the number showing on the second. In this exercise you can define any discrete random variable R of your choosing by appropriately modifying the program **Lab3ExA.m**. The program will then plot the partition of Ω 'generated' by this random variable.

(a) Run program **Lab3ExA.m**. The program initially produces the partition formed by the random variable $R = X + Y$. Each possible value of R corresponds to a different colour on the plot. Verify that similarly coloured dots correspond to equivalent sums and use this diagram to write down the probability mass function for R.

(b) Modify the **Lab3ExA.m** program so that it produces partitions for the following random variables:

  (i) the difference $X - Y$;
  (ii) the absolute difference $|X - Y|$;
  (iii) the minimum value showing $\min(X, Y)$;
  (iv) the maximum value showing $\max(X, Y)$.

  Note the patterns that result for the different random variables. Use the resulting partitions to derive and write down the probability mass functions for all of these random variables.

(c) Type ‘open Lab3ExA.fig’ in the Command Window to open up a partition which has already been generated. In this case your task is to define a random variable which generates a partition exactly like the one in Lab3ExA.fig.

(d) (Extension task) Try to find a random variable which generates an interesting pattern in the sample space. Challenge your fellow students or your tutor (or your lecturer) to find a random variable which duplicates your pattern.

Exercise B - Simulations of various distributions

In this exercise we will simulate the distributions which you have theoretically derived in Exercise A. The relevant m-file is **Lab3ExB.m**, which is set up initially for the random variable $R = X + Y$. The program plots the theoretical probability mass function against the empirically generated one for ‘nreps’ repetitions of the experiment.

(a) Study the program to see how the theoretical probability mass function is defined and check it against your result from Exercise A. Run the program several times to assess the fit between the theoretical and estimated distributions.

(b) As you have input the theoretical probability mass function it is possible for the program to calculate the exact values for $\mathbb{E}[R]$ and $\mathbb{E}[R^2]$. This output is printed in the Command window. Note both values as you will need this information in Exercise C.

(c) Repeat steps 1 and 2 for a couple of other random variables from Exercise A. You will need to modify the program to input the correct theoretical probability mass function and to calculate the new random variable of interest.

Exercise C - Expected values

In this exercise we check the theoretical values for some expected values against empirical estimates based on a large number of trials of the experiment.

(a) Use your results from Exercise A to check a couple of the theoretical values of $\mathbb{E}[R]$ and $\mathbb{E}[R^2]$ produced by the Exercise B program. Make sure you know how these were computed.

(b) The m-file **Lab3ExC.m** produces estimates of these expectations, initially for the random variable $R = X + Y$. Run **Lab3ExC.m** and compare the theoretical values and empirical estimates.

(c) Modify **Lab3ExC.m** to run for your other selected random variables and again cross check the results.