THE UNIVERSITY OF MELBOURNE

School of Mathematics and Statistics

MAST20026 Real Analysis

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Problem Booklet

Contents

Problem Sheet 1: Logic, Sets, Numbers and Proofs

Logic and Notation

1. Write the following statements as a conditional statement in the form $p \implies q$
   (a) "A monkey is happy only if he is eating a banana."
   (b) "A snake will not bite you provided you don't step on its tail."
   (c) "A donkey laughs whenever he sees a mule."
   (d) "Happiness is a necessary condition for Wealth."
   (e) "Happiness is a sufficient condition for Wealth."

2. Indicate whether each statement is True or False.
   (a) Jupiter is a planet and Neptune is a moon.
   (b) Jupiter is a planet or Neptune is a moon.
   (c) Elvis was a woman or Cleopatra was a man.
   (d) *Harry Potter* was written by J.K. Rowling, or *Lord of the Rings* was written by J.R.R. Tolkein
   (e) If the capital of Egypt is Cairo, then apples can be used to make cider.
   (f) If Napoleon was born in Zimbabwe, then the eigenstates of the quantum harmonic oscillator are proportional to Hermite polynomials.
   (g) If the dodo is extinct, then pigs can fly!
   (h) It is not the case that if Luke Skywalker was a Jedi, then his father was not Darth Vader.

3. Translate the following into mathematical notation.
   (a) Six is not prime or eleven is not prime
   (b) The square of 10 is 50 and the cube of 5 is 12.
   (c) If 7 is an integer then 6 is not an integer.
   (d) If both 2 and 5 are prime then 2×5 is not prime.

   Which of these are true, which are false, and which are neither?

4. Construct truth tables for the following statements.
   (a) $(p \wedge q) \vee (\sim p \wedge \sim q)$
   (b) $[\sim q \wedge (p \implies q)] \implies \sim p$
   (c) $[(p \vee q) \wedge r] \implies (p \wedge r)$

   One is equivalent to a simple binary operator: which one? One is a tautology: which one?

   For the last one, can you find a simpler equivalent statement?

5. Translate the following into mathematical notation.
   (a) All rational numbers are larger than 6.
   (b) There is a real-number solution to $x^2 + 3x - 7 = 0$.
   (c) There is a natural number whose cube is 8.
   (d) The set of all numbers that aren't multiples of 7.

   Which of these are true, which are false, and which are neither?

6. Translate the following mathematical statements into English.

   (a) $\forall a \in \mathbb{Q},\ a + 0 = a$

   (b) $\forall x \in \mathbb{R},\ x^2 > 1$

   (c) $\exists \delta$ such that $\forall x \in (m - \delta, m + \delta),\ f(m) \leq f(x)$

   (d) $\exists K \in \mathbb{R}$ such that $\forall s \in S \subseteq \mathbb{R},\ |s| \leq K$

7. The following two statements look similar, but say very different things. Which is true, and which is false?

   (a) $\exists b \in \mathbb{Z}$ such that $\forall a \in \mathbb{Z},\ a + b = 0$

   (b) $\forall a \in \mathbb{Z},\ \exists b \in \mathbb{Z}$ such that $a + b = 0$

8. Find the negation of

   (a) $\forall x \in R\ \ x^2 = 10$

   (b) $\exists y \in \mathbb{N}\ \ y < 0$

   (c) $\exists a \in \mathbb{N}\ \ \forall x \in \mathbb{R}\ \ ax = 4$

   (d) $\forall y \in \mathbb{Q}\ \ \exists x \in \mathbb{R}\ \ x/y = 30$

9. Verify the following:

   (a) $p \wedge q \equiv q \wedge p$

   (b) $\sim (p \implies q) \equiv (p \wedge \sim q)$

   What can you conclude about (i) $p \wedge q \iff q \wedge p$ and (ii) $\sim (p \implies q) \iff (p \wedge \sim q)$?

Sets

For questions 10 to 13, let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5, 7\}$ and $C = \{2, 3\}$.

10. Find the following sets:

    (a) $A \cup B$

    (b) $A \cap B$

    (c) $A \cup C$

    (d) $A \cap C$

    (e) $A \cup B \cup C$

    (f) $A \cap B \cap C$

11. Find the following sets:

    (a) $B \times C$

    (b) $A \times \emptyset$

12. Which of the following statements are true?

    (a) $3 \subseteq B$

    (b) $\emptyset \subseteq A$

    (c) $7 \in B \cup C$

    (d) $C \subseteq B$

    (e) $C = B \cup C$

    (f) $C \in A$

    (g) $\forall a \in A,\ a \leq 4$

    (h) $\forall b \in B,\ \exists c \in C,\ b - c \geq 0$

13. Calculate the following:

    (a) $\displaystyle\sum_{a \in A} a^2$

    (b) $\displaystyle\prod_{b \in B} (b - 2)$

14. Write the following as a single interval or set:

(a) $(-5, 2] \cup (-4, 3)$

(b) $[-6, 12] \cup (2, 12)$

(c) $(-\pi, \pi] \cap (-1, 4]$

(d) $(-\infty, 1) \cap (1, \infty)$

(e) $\displaystyle\bigcup_{n\in\mathbb{N}} \left[\frac{1}{n}, 1\right]$

(f) $\displaystyle\bigcap_{n\in\mathbb{N}} \left[\frac{1}{n}, 1\right]$

15. Let A, B and C be sets. Let $\mathcal{U} = A \cup B \cup C$ be the universal set. Prove the following theorems.

(a) $A \subseteq ((A \cap B) \cup (A \cap (\mathcal{U} \backslash B)))$

(b) $A \subseteq B \implies (C \cap A) \subseteq (C \cap B)$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) $A \subseteq B$ if and only if $A \cup B \subseteq B$

Note: Hints are available in the answers. Some proofs are easier to do using Proof by Cases - wait until this method has been covered in lectures (or read ahead).

16. *The Cantor Set*

We are going to construct an unusual set made famous by Georg Cantor.

We begin with the set $C_0 = [0, 1]$

(a) Draw this set on the real number line. What is its length?

(b) Construct C_1 by removing the middle third of C_0 as an open interval. In other words, remove $(\frac{1}{3}, \frac{2}{3})$. Write C_1 as the union of two intervals. Draw C_1 on the real line, and determine its length.

(c) Construct C_2 by removing the middle third of each segment in C_1. C_2 should have 4 segments. Write C_2 as a union of 4 intervals. Draw it on the real line, and determine its length.

(d) By now you probably understand the pattern. Draw a few more: C_3, C_4, C_5 etc.

(e) The Cantor Set C is what remains after you repeat this process *ad infinitum*. Find several points that are in the set. Find several points that aren't in the set.

(f) The Cantor Set is a simple example of a *fractal*. It exhibits self-similairty: meaning if you zoom in on sections, you see the overal shape repeating itself. Type "fractal cauliflower" into Google to see another example of a fractal.

Proof by Counterexample

17. Use a counterexample to show that the following statements are false.

(a) If the product of two integers is even then both of those integers are even.

(b) For all real numbers, if $x^2 = y^2$ then $x = y$.

(c) $A \cup (B \cap C) = (A \cap B) \cup C$

(d) Let $a \in \mathbb{Z}$. If a divided by 7 gives remainder 4, then $5a$ divided by 7 gives remainder 4.

Direct Proofs

18. Carefully prove the following results for integers.

(a) The product of an even integer with an odd integer is even.

(b) The sum of an even integer and an odd integer is odd.

(c) The cube of an odd integer is odd.

(d) If k is odd, $k^2 - 1$ is divisible by 4.

Numbers

19. For each equation below, list the sets of numbers ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$) in which there exists a solution.

(a) $x^2 - 8 = 0$
(b) $x + 8 = 0$
(c) $4x = 8$
(d) $x^2 + 8 = 0$
(e) $3x + 8 = 0$
(f) $0x = 8$

20. What property separates the following sets? In other words, what vital property does one set have that the other doesn't?

(a) $\mathbb{N}$ and $\mathbb{Z}$
(b) $\mathbb{Z}$ and $\mathbb{Q}$
(c) $\mathbb{Q}$ and $\mathbb{R}$

21. Let x, y, $z \in \mathbb{R}$. Using the axioms of the Real numbers, prove the following "obvious" results. Be very careful that you only use the axioms. Each step should use exactly one axiom, and you should indicate which one.

(a) $x + z = y + z \implies x = y$
(b) $0x = 0$
(c) $-(-x) = x$
(d) if $x < y$ and $z < 0$ then $xz > yz$

22. Let x, y, $z \in \mathbb{R}$. Using the axioms of the real numbers, or results proved in lectures, tutorials or previous questions prove the following results. You should indicate which axioms you used.

(a) $(-x)y = -(xy)$
(b) If $x \neq 0$ then $(x^{-1})^{-1} = x$
(c) $(x \cdot z = y \cdot z) \wedge (z \neq 0) \implies (x = y)$
(d) $x \neq 0 \implies x^2 > 0$
(e) $0 < 1$

Proof by Contrapositive

23. Prove the following theorems using the contrapositive.

(a) Let $n \in \mathbb{Z}$. Prove that if n^4 is even, then n is even.
(b) Let $n \in \mathbb{Z}$. Prove that if n^3 is odd, then n is odd.
(c) For all $m, n \in \mathbb{Z}$, if $m \cdot n$ is odd then m and n are odd.

Hint: To show $p \implies q$, show $\sim q \implies \sim p$.

Proof by Contradiction

24. Consider the following theorem:

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size. If $\mathbf{AB} = \mathbf{0}$ then at least one of $\mathbf{A}$ and $\mathbf{B}$ is singular.[1]

(a) Begin a proof by contradiction by assuming that both A and B are invertible.
(b) Starting with $\mathbf{AB} = \mathbf{0}$, use the assumption you just made to arrive at a contradiction. Hint: remember that $\mathbf{AA}^{-1} = \mathbf{I}$.

25. Consider the following theorem:

Let $p, q \in \mathbb{Z}$ with $p, q > 0$. If $pq = 1$ then $p = q = 1$

(a) Begin a proof by contradiction by assuming that at least one of p and q is not equal to 1.
(b) Starting with $pq = 1$, arrive at a contradiction

[1] Recall that *singular* means non-invertible

26. Consider following equation:
$$x^2 - n^2y^2 = 1$$
where $n \in \mathbb{N}$ is fixed.
Show that it has no positive integer solutions for x, y using a proof by contradiction in the following steps.
   (a) Begin by assuming there exists a positive integer solution.
   (b) Factorise the left hand side, and conclude that both factors are integers.
   (c) Using the result of question 25, conclude both factors equal 1.
   (d) Attempt to solve the two equations you have just developed and arrive at a contradiction.

27. *The Adventures of π-casso, the Mathematical Artist*
Your friend π-casso has invited you over to see his new artwork. Before he unveils it, he tells you that it is a 3×3 grid of squares, each painted either red or blue. He goes on to say he used a special rule to paint it.

THE SPECIAL RULE:
Every square has either 2 or 4 blue neighbours.

   (a) Suppose he tells you the centre square is red. Prove that the remaining squares are all blue using a contradiction argument. Hint: you will need to consider 2 cases.
   (b) Suppose he tells you the centre square is blue. Prove that at least one of the remaining squares is red using a contradiction argument.
   (c) How many possible paintings are there that satisfy the special rule?
   (d) *Extension* Consider paintings of 4×4 grids that use the same special rule. What are the possiblities?

Proof by Cases aka Proof by Exhaustion

28. Prove the following theorems by dividing in two or more cases.
   (a) Let $n \in \mathbb{Z}$. Prove that if n is not divisible by 3, then n^2 is not divisible by 3.
   (b) For all $m, n \in \mathbb{Z}$, if m and n are either both odd or both even then $m + n$ is even.

The Rational Numbers

29. *Constructing the Rational Numbers*

In this question we will build the rational numbers using the integers. Remember since the integers have no innate concept of "division", our definitions cannot use it.

We will use $(a, b) \in \mathbb{Z} \times \mathbb{Z} \backslash \{0\}$ to represent $\dfrac{a}{b} \in \mathbb{Q}$. Next, we will *define* what equality, addition, and multiplication mean to mimic what we know about rational numbers.
   (a) When we say "$\dfrac{a}{b} = \dfrac{c}{d}$", what do we mean? Make sure you rewrite it to avoid division, since integers have no concept of division.
   (b) Now, *define* $(a, b) = (c, d)$ to match.
   (c) What is $\dfrac{a}{b} + \dfrac{c}{d}$ equal to? Use this to *define* $(a, b) + (c, d)$.
   (d) What is $\dfrac{a}{b}\dfrac{c}{d}$ equal to? Use this to *define* $(a, b)(c, d)$.

Warning! Don't treat the ordered pairs as vectors! Same objects, but different interpretation.

Irrational numbers

30. Prove that the following are irrational:

(a) $\sqrt{3}$ (b) $\sqrt{15}+\sqrt{5}$ (c) $\log_2 7$

(d) The sum of a rational number and an irrational number.

Hint: For part (c) you may need THE FUNDAMENTAL THEOREM OF ARITHMETIC: All natural numbers greater than 1 have a unique prime factorisation.

Mathematical Induction

31. Prove by induction that each formula is true for every natural number n.

(a) $2+7+12+\cdots+(5n-3)=\frac{1}{2}n(5n-1)$

(b) $1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+\cdots+n\cdot 2^{n-1}=1+(n-1)2^n$

(c) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$

(d) $\dfrac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots$
$+ab^{n-2}+b^{n-1} \quad (a\neq b)$

Rewrite these equations using summation notation.

32. Prove by induction that the following statements are true for every natural number n:

(a) 3 is a factor of n^3-n+3;

(b) 9 is a factor of $10^{n+1}+3\cdot 10^n+5$;

(c) 4 is a factor of 5^n-1;

(d) $x-y$ is a factor of x^n-y^n;

(e) $7^{2n}-48n-1$ is divisible by 2304.

33. Write the following inequalities in summation notation and then prove them for all $n\in\mathbb{N}$, using summation notation throughout your proof. If you find this difficult, first try the proofs using the more informal notation.

(a) $1^3+2^3+\cdots+(n-1)^3<\frac{1}{4}n^4<1^3+2^3+\cdots+n^3$, for $n\geq 2$;

(b) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\geq\sqrt{n}$.

34. In each case try to find $n_0\in\mathbb{N}$ such that the inequality appears to hold for $n\geq n_0$. If you think you have found such an n_0, give a proof by induction that the inequality holds for $n\geq n_0$. If you think that no such n_0 exists, try to prove that it cannot exist.

(a) $1+2n\leq 3^n$

(b) $n!>2^n$

(c) $n(n+1)\geq(2n-1)^2$

(d) $n!>2n^3$

35. If $\mathbf{A}$ is a square matrix and $\mathbf{1}$ is the identity matrix, we define $\mathbf{A}^n$ by

$$\mathbf{A}^0 = \mathbf{1} \quad \text{and} \quad \mathbf{A}^{n+1} = \mathbf{A}^n\mathbf{A},\ n = 0, 1, 2, \ldots,$$

which is a more precise and careful way of saying that

$$\mathbf{A}^n = \underbrace{\mathbf{A}\,\mathbf{A}\cdots\mathbf{A}}_{n \text{ factors}}.$$

Prove by mathematical induction that for all natural numbers n,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.$$

36. Show for $n \in \mathbb{N}$ that

$$\frac{d^n}{dx^n}\log(1+x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n}$$

37. Given $a > -1$, prove *Bernoulli's Inequality*

$$(1+a)^n \geq 1 + na \quad \forall n \in \mathbb{N}$$

38. Let $I_n = \displaystyle\int_0^1 x^n\sqrt{1-x}\,dx$. Prove that

$$I_n = \frac{4^{n+1}n!(n+1)!}{(2n+3)!} \quad \text{for } n = 0, 1, 2, 3, \ldots$$

Inequalities

This subject will involve heavy use of inequalities. The following exercises will help you practice and become comfortable manipulating them.

39. Let $a, b, c \in \mathbb{R}$ and let $a < b$. Which of the following statements are always true? Which are sometimes true? Which are never true?

(a) $a + 1 < b + 1$

(b) $a + c < b + c$

(c) $5a < 5b$

(d) $ac < bc$

(e) $\dfrac{1}{a} < \dfrac{1}{b}$

(f) $\dfrac{1}{c+a} < \dfrac{1}{c}$

(g) $c < c + a$

(h) $-a < -b$

40. Give the solutions to the folowing inequalities in terms of intervals

(a) $|1 + 2x| \leq 4$

(b) $|x + 2| \geq 5$

(c) $|x - 5| < |x + 1|$

(d) $|x - 2| < 3 \vee |x + 1| < 1$

(e) $|x - 2| < 3 \wedge |x + 1| < 1$

41. Use the triangle inequality or other properties of inequalities to find a bound for $|f|$ on the stated interval.

    (a) $f(x) = \dfrac{2x^2+1}{x+3}, \quad |x| < 1$

    (b) $f(x) = \dfrac{x^3+3x+1}{10-x^3}, \quad |x+1| < 2$

42. Prove that $\forall a, b \in \mathbb{R}$ that $|a-b| \geq ||a| - |b||$.

43. Let $a, b \in \mathbb{R}$. If $0 < \epsilon < \min\{|a|, |b|\}$ show that

$$\left|\frac{a+\varepsilon}{b+\varepsilon}\right| \leq \frac{|a|+\varepsilon}{|b|-\varepsilon}$$

44. Let $a, b \geq 0$, $p > 1$ and $q = p/(p-1)$. We will prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

    (a) Consider the cases where either $a = 0$ or $b = 0$: the inequality is trivially true for these.

    (b) Treat a as a real variable and define

$$f(x) = \frac{x^p}{p} + \frac{b^q}{q} - bx, \quad x > 0$$

    Show that f has a minimum at $x = b^{\frac{1}{p-1}}$ using calculus techniques.

    (c) Find the value of the function at this point, and conclude $f(x) \geq 0$.

    (d) Finish off the proof.

45. Use a similar method to the one used in question 44 to prove the following inequalities:

    (a) $1 + x \leq \exp x$

    (b) $\log x \geq \dfrac{x-1}{x}$ for all $x > 0$

46. For any $x, y \in \mathbb{R}$, prove that

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

Hint: Show first that $f(u) = u/(1+u)$ is increasing for $u \geq 0$.

Supremum and Infimum

47. Find the supremum and infinum of the set S (where $S \subseteq \mathbb{R}$), if they exist, and if they do, explain whether the supremum or infimum is an element of S.

    (a) $S = \{x : x^2 \leq 9\}$

    (b) $S = \{x : |x-2| < 3\}$

    (c) $S = \{x : |2x+1| < 5\}$

    (d) $S = \{x : |x-2| < 3 \wedge |x+1| < 1\}$

    (e) $S = \{x : |x+2| \leq 2 \vee |x| > 1\}$

    (f) $S = \{x \in \mathbb{Q} : x^2 \leq 7\}$

48. If $S \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ we define $c + S = \{c + x : x \in S\}$ and $cS = \{cx : x \in S\}$. If S is bounded, prove the following:

    (a) $c + S$ and cS are bounded

    (b) $\sup(c+S) = c + \sup(S)$

    (c) $\sup(cS) = c\sup(S)$ if $c \geq 0$

    (d) $\sup(cS) = c\inf(S)$ if $c \leq 0$

49. Let $A \subseteq \mathbb{R}$. Prove that the supremum of A is unique if it exists.

    Hint: A useful method for showing some quantity is unique is to assume there are two, and prove they must be equal. In this case, assume that A has two suprema, s_1 and s_2, and show that $s_1 = s_2$.

Functions

50. Classify each function as injective, surjective, bijective or none. Formal proof is not needed. You may need to do a little research to answer some of these.
    (a) Let W be the set of all English words. Let $f : W \to \mathbb{N}$ be defined by $f(w) =$ the number of letters in the word w.
    (b) Let A be the set of all countries on Earth, and let B be the set of all cities on Earth. Let $f : A \to B$ be defined by $f(a) =$ the capital city of country a.
    (c) Let P be the set of all people on Earth, and let C be the set of all countries. Let $f : P \to C$ be defined by $f(p) =$ the country of birth of person p.
    (d) Let A be the alphabet, and let Ph be the words in the phoentic alphabet. Let $f : Ph \to A$ be defined by $f(p) =$ the letter of the alphabet that the phonetic word p represents.

51. Consider again the function of question 50a). Let S be the set of words in Shakespeare's famous line, "Now is the winter of our discontent".
    (a) What is $f(S)$, the image of S under f?
    (b) List seven elements of $f^{-1}(\{1, 2, 3\})$, the pre-image of $\{1, 2, 3\}$ under f.

52. Let $f : A \to B$ and $g : B \to C$ be functions. Prove the following:
    (a) If f and g are injective, then $g \circ f$ is also injective.
    (b) If f and g are surjective, then $g \circ f$ is also surjective.
    (c) If f and g are bijective, then $g \circ f$ is also bijective.

53. Find a bijection between the sets $[0, 1]$ and $[1, 2]$. Don't forget to prove it!

54. Find a bijection between the sets $(0, 1)$ and $(1, \infty)$. Again, a proof is needed.

Cardinality

55. We say two sets have the same *cardinality* if there exists a bijection between them.

    Consider the function $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(z) = \begin{cases} -2z & z < 0 \\ 2z + 1 & z \geq 0 \end{cases}$$

    By showing that f is a bijection, prove that $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality.

Problem Sheet 2: Sequences

Standard Sequences: Throughout this problem sheet, and your tutorial sheets, certain limits will be marked with the symbol ♣. These limits are special limits that will be used for later problems.

ϵ–N Definition of Convergence

All the questions in this section will require you to use the ϵ-N definition of convergence.

56. Use ϵ-N definition of convergence to prove the following.

 (a) $\lim_{n\to\infty} \frac{1}{n+2} = 0$ (b) $\lim_{n\to\infty} (n+4)^{-2} = 0$ (c) $\lim_{n\to\infty} \frac{n^2-1}{2n^2+4} = \frac{1}{2}$.

57. For the following sequences, guess the limit, then verify the result using an ϵ–N argument.

 (a) $\frac{n}{n^2+1}$ (b) $\frac{2n}{n+1}$ (c) $\frac{3n+1}{2n+5}$ (d) $\frac{n^2-1}{2n^2+3}$

58. Guess the limits of the two following special sequences and prove your result using ϵ-N arguments.

 (a) ♣ $a_n = c$ where $c \in \mathbb{R}$ (b) ♣ $b_n = \frac{1}{n^p}$, where $p > 0$

59. Prove that, if the sequence (u_n) converges, then $\lim_{n\to\infty} 2u_n = 2\lim_{n\to\infty} u_n$.

60. Prove that, if the sequence (u_n) converges, then $\lim_{n\to\infty} ku_n = k\lim_{n\to\infty} u_n$ for $k \in \mathbb{R}$.

61. Prove the following theorem: If a_n is a bounded sequence, then $\lim_{n\to\infty} \frac{a_n}{n} = 0$

62. Prove the following theorem, which is a generalisation of question 61.

 Let a_n and b_n be sequences. If a_n is bounded, and $\lim_{n\to\infty} b_n = 0$ then $\lim_{n\to\infty} a_nb_n = 0$

63. Prove the Sandwich Theorem:

 Let a_n, b_n and c_n be sequences. If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

64. The following sequences are divergent. Describe the nature of the divergence. Calculating some terms and drawing a sketch may help. Use the negation of the ϵ-M definition of convergence to prove that each sequence diverges.

 (a) $e_n = \frac{n+1}{\sqrt{n}}$ (b) $g_n = 1 + (-1)^{n+1}$

65. Let $a_n = \frac{n^p}{a^n}$ where $p > 0$ and $a > 1$.

 (a) Show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{a}$.

 (b) Let $r \in \mathbb{R}$ such that $\frac{1}{a} < r < 1$. Put $\epsilon = r - \frac{1}{a}$. Conclude that

 $$\exists K \in \mathbb{N}\ \forall n \in \mathbb{N}\ \ n > K \implies \left|\frac{a_{n+1}}{a_n} - \frac{1}{a}\right| < r - \frac{1}{a}.$$

 (c) Show that $a_{n+1} < ra_n$ for all $n > K$.

 (d) Note that $a_n > 0$. Show that $a_{K+n} < r^{n-1}a_{K+1}$ for all $n \in \mathbb{N}$, $n > 1$.

 (e) ♣ Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^p}{a^n} = 0$.

Special types of sequences

66. Determine whether or not the sequence is increasing, decreasing or not monotonic. Also determine whether each sequence is bounded or unbounded.

(a) $\dfrac{1}{3n+5}$ (b) $3 + \dfrac{(-1)^n}{n}$ (c) $\dfrac{n-2}{n+2}$ (d) $\left(\dfrac{n}{25}\right)^{1/3}$ (e) $\dfrac{2^{2n}}{n!}$

Using the sandwich rule

You are not expected to use ϵ-N, but rather the limit theorems that you have learned so far. Each step in your calculation must be backed up with references to the sandwich rule and special (♣) limits.

67. Do questions 61 and 62 again, but use the sandwich rule.

68. Use the sandwich rule to find the following limits

(a) $a_n = (1 + (-1)^n)\dfrac{1}{n}$ (c) $c_n = \dfrac{\sin n}{n^3}$ (e) $e_n = \dfrac{2^n}{n!}$

(b) $b_n = \dfrac{\sin^2 n}{n^3}$ (d) ♣ $d_n = \dfrac{n!}{n^n}$ (f) ♣ $f_n = \dfrac{a^n}{n!}$ where $a > 0$

(g) What can you conclude about $\lim\limits_{n\to\infty} g_n$ where $g_n = \dfrac{a^n}{n^n}$?

For f_n you may find it useful to notice that there is some number $K \in \mathbb{N}$ where $n > a$ for all $n \geq K$.

Subsequences

Definition A *subsequence* is a sequence that can be derived from another sequence by deleting some elements without changing the order of the remaining elements. (For example, the prime numbers are a subsequence of the natural numbers.)

69. Let (a_n) be a convergent sequence with limit L. Prove that if $\{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ is a subsequence of (a_n), then $\lim\limits_{k\to\infty} a_{n_k} = L$.

70. Let (a_n) be a sequence and let (b_n) and (c_n) be two distinct subsequences of (a_n). Show that if $\lim\limits_{n\to\infty} b_n = L_1$ and $\lim\limits_{n\to\infty} c_n = L_2$ where $L_1 \neq L_2$ then (a_n) diverges.

Hint: use the result from Question 69.

71. For each of the following sequences, find all r such that a subsequence converges to r. Make a conclusion about the convergence or otherwise of the sequence.

(a) $a_n = (-1)^n \dfrac{n}{n+1}$ (b) $b_n = \sin\left(\dfrac{n\pi}{2}\right) \cdot \dfrac{n}{n+1}$ (c) $c_n = (-1)^{n+1} \dfrac{n^2}{n^3+5}$

72. *Challenging!* Can you find a sequence $\{a_n\}$ so that for *every* real number $r \in [0, 1]$, there is a subsequence of $\{a_n\}$ which converges to r?

Recursive Sequences

73. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

(a) By induction or otherwise, show that $\{a_n\}$ is increasing and bounded above by 3. Deduce that $\lim\limits_{n\to\infty} a_n$ exists.

(b) Find $\lim\limits_{n\to\infty} a_n$. (You may assume that $\lim\limits_{n\to\infty} \sqrt{f_n} = \sqrt{f_n}$ if $\lim\limits_{n\to\infty} f_n$ exists and is ≥ 0.)

74. Show that the sequence defined by

$$a_1 = 2 \qquad\qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and that $a_n < 3$ for all $n \in \mathbb{N}$. Deduce that $\{a_n\}$ is convergent and find its limit.

75. Explain why you know that the sequence defined by $a_1 = 3$ and $a_n = \frac{1}{2}(a_{n-1} + 5)$ if $n = 2, 3, 4, \ldots$ is convergent, and evaluate its limit.

76. ♣ Prove that $a^n \to 0$ as $n \to \infty$ for $|a| < 1$ without use of logarithms by using properties of the sequence $u_n = |a|^n$ and its subsequence u_{2n}. [Hint: $u_{2n} = u_n^2$.]

77. ♣ Suppose that $c > 1$.
   (a) Prove that $c^{1/n} > 1$, and that $c^{1/n}$ is decreasing.
   (b) Use the observation that $c^{2/n} = (c^{1/n})^2$ to prove that $c^{1/n} \to 1$ as $n \to \infty$. Deduce that for all $c > 0$, $c^{1/n} \to 1$ as $n \to \infty$. Use a similar argument to prove that $c^{1/n} \to 1$ as $n \to \infty$ for $c < 1$.

78. ♣ Challenge extension to question 77. Use a similar argument to prove that $n^{1/n} \to 1$ as $n \to \infty$. [2]

79. *Newton's Method.*[3]
   For a given function f, Newton's method approximates a solution to the equation $f(x) = 0$, by starting with an initial guess x_1 and recursively defining

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for each } n \geq 1.$$

   Consider applying Newton's method to the function

$$f(x) = x^2 - 2 = 0$$

   and using $x_1 = 2$ as the initial approximation.
   (a) Verify that $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$. Use this to calculate x_2 and x_3.
   (b) Show that if the limit $\lim\limits_{n\to\infty} x_n = L$ exists, then it must satisfy $L^2 = 2$.
   (c) Show, by induction on n, that $\sqrt{2} < x_n \leq 2$, for all n.
   (d) Show that $x_{n+1} < x_n$ for all n.
   (e) Deduce that the sequence $\{x_n\}$ has a limit, and that $\lim\limits_{n\to\infty} x_n = \sqrt{2}$.

Algebra of Limits

80. Prove that if $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} a_n = B$ where $A, B \in \mathbb{R}$ then $\lim\limits_{n\to\infty} a_n - b_n = A - B$.

81. Prove the following theorem:
   Let a_n be a sequence. If a_n converges to $L \neq 0$ then

$$\exists N \text{ such that } n > N \Rightarrow |a_n| > \frac{|L|}{2}.$$

82. Prove the limit theorem for reciprocals of sequences:
   Let a_n be a sequences. If $\lim\limits_{n\to\infty} a_n = L \neq 0$ then

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}.$$

   Hint: You will need your result to question 81.

83. Prove the limit theorem for quotients of sequences:
   Let a_n and b_n be sequences. If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b \neq 0$ then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

   Hint: This is very easy if you use one of the limit theorems you proved in tutorials, and the result of question 82.

[2] That $c^{1/n} \to 1$ and $n^{1/n} \to 1$ can be proved easily later by less clever means when we have proved additional results.

[3] This algorithm for numerical determination of square roots was known in Mesopotamia in 1500BC. With $x_1 = 2$ we find $x_2 = 3/2 = 1.5$, $x_3 = 17/12 \approx 1.417$, $x_4 = 577/408 \approx 1.141422$, cf $\sqrt{2} = 1.41421356\cdots$.

Cauchy Sequences

84. Use the definition of a Cauchy sequence to show that the sequence defined by $f_n = \dfrac{n+3}{2n-1}$ is Cauchy.

85. Suppose that (a_n) is a sequence such that $|a_{n+1} - a_n| \leq a\, r^{n-1}$ for all n, where $0 \leq r < 1$ and $a \geq 0$. Show that (a_n) is Cauchy.

    [Hint: For $m > n$ use the fact that $|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - a_{m-2} + \ldots + a_{n+2} - a_{n+1} + a_{n+1} - a_n|$ and the triangle inequality to find an upper bound for $|a_m - a_n|$. You are reminded that $\displaystyle\sum_{k=0}^{m-1} r^k = \frac{1-r^m}{1-r}$.]

86. Suppose that (a_n) is a sequence such that

    $$|a_{n+2} - a_{n+1}| \leq c\, |a_{n+1} - a_n| \text{ for all } n \in \mathbb{N},$$

    where $0 \leq c < 1$. (We then say that the sequence (a_n) is *contractive* with contraction factor c.)

    Use induction on k to prove that

    $$|a_{k+1} - a_k| \leq c^{k-1}\, |a_2 - a_1| \text{ for all } k \in \mathbb{N}.$$

    Deduce from Question 85 that (a_n) is Cauchy, hence convergent.

87. Let $\alpha \in (0, 1)$ and let (x_n) be defined recursively by $x_1 = \alpha$ and $x_{n+1} = (x_n^3 + 2)/7$ for all $n \geq 1$.

    (a) Show that (x_n) is contractive with contraction factor $\dfrac{3}{7}$ (as defined in Question 86).

    (b) Deduce that x_n converges and show that the limit x satisfies the equation $x^3 - 7x + 2 = 0$.

    (c) Use your calculator or computer to calculate the first 5 terms in the sequence, starting with $x_1 = 0.5$. Hence find a numerical approximation for the limit x.

Putting it together

While the emphasis in this course is on proving basic results, once proved you may use these results to find limits of sequences. In fact, when you study series later this semester, you will be expected to do this.

In the following questions you are not expected to use the ϵ-N definition, but rather the limit theorems that you have learned so far. Each step in your calculation must be backed up with references to the algebra (arithmetic) of limits, the sandwich rule and special (♣) limits. You may assume two additional limits. These will be proved later in semester when you have seen differentiability and Taylor series respectively.

♣ $\displaystyle\lim_{n\to\infty} (1 + \frac{a}{n})^n = e^a$ ♣ $\displaystyle\lim_{n\to\infty} \frac{\log n}{n^p} = 0$

You may find it useful to divide numerator and denominator by the term that gets biggest most quickly. The following hierarchy of terms may help you to identify the largest term:

$$1 \ll \log_e n \ll n^p \ll a^n \ll b^n \ll n! \ll n^n$$

where $p > 0$, $1 < a < b$ and $(a_n \ll b_n) \equiv (\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0)$.

88. Find the limits of the following sequences.

    (a) $\dfrac{n}{2^n}$

    (b) $\sqrt[n]{n^2}$

    (c) $\dfrac{\log n + 1}{n}$

    (d) $\dfrac{\log n}{\sqrt{n}}$

    (e) $\dfrac{(2n)^3}{3^n}$

    (f) $\left(\dfrac{n+3}{n}\right)^n$

    (g) $\left(\dfrac{n}{n+3}\right)^n$

    (h) $\dfrac{n^n}{(n+3)^{n+1}}$

    (i) $\dfrac{n^5}{2^n}$

    (j) $\dfrac{2^n}{(2n)!}$

    (k) $\sqrt[n]{2n+1}$

    (l) $\sqrt[n]{n^2+n}$

89. Find the limits for the convergent sequences below (ϵ–N arguments not required, but justify all conclusions). You may assume that if $\lim_{n\to\infty} f_n$ exists and is ≥ 0 then $\lim_{n\to\infty}\sqrt{f_n} = \sqrt{\lim_{n\to\infty} f_n}$.

(a) $\dfrac{n}{n+1}$

(b) $\dfrac{1}{\sqrt{n}}$

(c) $\sqrt{n+1}-\sqrt{n}$

(d) $\sqrt{n}(\sqrt{n+1}-\sqrt{n})$

(e) $\dfrac{n}{2n+3}$

(f) $\dfrac{n}{n+1}-\dfrac{n+1}{n}$

(g) $\dfrac{1-n}{n^3}$

(h) $\dfrac{3n-1}{2n+5}$

(i) $\dfrac{\sqrt{n}}{n+1}$

(j) $(1+(-1)^n)\dfrac{1}{n}$

90. Explain why the following limits do not exist. You should refer to result in Question 70 or to a theorem from class about bounded sequences and any other results used.

(a) $\sqrt{n}$

(b) $\dfrac{(-1)^n 2n+1}{n+1}$

(c) $\dfrac{n!}{10^{6n}}$

(d) $\dfrac{n\cos n\pi + 4n}{3n}$

91. Determine whether the sequences for which the nth element of a_n is given below converge or diverge. Full explanations are required, including standard sequences used, and reference to results from class or previous problems.

(a) $\dfrac{4n^2-2n+\cos n}{3n^2+7n+6}$

(b) $\sqrt[n]{10n}$

(c) $\left(1-\dfrac{6}{n}\right)^n$

(d) $\dfrac{3^n+2^n}{n!+3^n}$

(e) $\dfrac{1-n!}{4^n}$

(f) $1+\dfrac{(-1)^n}{n}$

(g) $\sin(\pi n)$

(h) $\cos(\pi n)$

(i) $\sqrt[n]{n^3}$

(j) $\left(1-\dfrac{1}{n^2}\right)^n$

(k) $\dfrac{\log n + n^3}{3n^3}$

(l) $\dfrac{\log n + 1}{n^{1/n}}$

(m) $\sqrt[n]{n+2n^3}$

(n) $n^{(-1)^n}$

(o) $\dfrac{\log 3n}{\log 4n}$

(p) $\dfrac{2-\sin n}{n^5}$

(q) $\dfrac{\log n + 5n^2}{2n^2+100}$

(r) $\dfrac{3^n+n!}{100^n+n^7}$

(s) $(2^n+1)^{1/n}$

(t) $\cos\dfrac{n\pi}{2}$

Problem Sheet 3: Limits of Functions and Continuity

Before you start these, you may wish to refresh your skills with inequalities; in particular the Triangle Inequality, which will be your best friend.

Limit Points

92. Find all the limit points of the following sets:

(a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(b) $\left\{\frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N}\right\}$

(c) $\left\{(-1)^n \left(1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$

(d) $\mathbb{Z}$

(e) $\mathbb{Q}$

Limits of Real-Valued Functions

93. Prove by ϵ–δ arguments that as $x \to 3$:

(a) $5x \to 15$

(b) $3/x \to 1$

(c) $x^4 \to 81$.

94. In each of the following, guess the limit and then use the ϵ–δ definition to prove that your guess is correct. (If there is no limit, explain clearly why this is so.)

(a) $\lim_{x\to 4} \left(\frac{1}{2}x - 3\right)$

(b) $\lim_{x\to 0} \frac{1}{1+x}$

(c) $\lim_{x\to 4} \frac{1}{1+x^2}$

(d) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

(e) $\lim_{x\to 9} \frac{x+1}{x^2+1}$

(f) $\lim_{x\to\infty} x\sin x$

95. Prove using ϵ-δ arguments that

(a) $\lim_{x\to 0} x^4 = 0$;

(b) for $a \neq 0$, $\lim_{x\to a} x^4 = a^4$.

Hint In part (b) it may help to commence with the restriction that $|x - a| < |a|/2$.

96. ♣ Prove the following limits using the ϵ-δ definition for limits at ∞. You may assume these in future problems.

(a) $\lim_{x\to\infty} c = c$; $c \in \mathbb{R}$

(b) $\lim_{x\to\infty} \frac{1}{x^p} = 0$; $p > 0$.

97. Prove the limit laws:

If $f \to \alpha$ and $g \to \beta$ as $x \to l$ where $\alpha, \beta \in \mathbb{R}$ then

(a) $\lim_{x\to l} f(x) + g(x) = \alpha + \beta$

(b) $\lim_{x\to l} f(x) \cdot g(x) = \alpha\beta$

(c) $\lim_{x\to l} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta}$, $\beta \neq 0$.

98. Decide if the limit exists. If it does, evaluate the limit and justify each step by referring to the algebra of limits and/or sandwich rule as appropriate. If it does not, explain with reference to results from class/tutorials.

(a) $\lim_{x\to-1} \frac{2x+1}{3-4x}$

(b) $\lim_{x\to-1} \frac{x^2-3x-4}{2x^2+x-1}$

(c) $\lim_{x\to 9} \frac{2x-17}{x-9}$

(d) $\lim_{x\to\infty} \sin x$

(e) $\lim_{x\to\infty} x$

(f) $\lim_{x\to\infty} \frac{\sin x}{x}$

(g) $\lim_{x\to 2} \left(\frac{x^2}{x-2} - \frac{4}{x-2}\right)$

(h) $\lim_{x\to 0} \frac{1}{x}$

99. You will prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ later in this sheet. Assuming this result and that $\lim_{x\to 0} \cos x = 1$ find the following limits:

(a) $\lim_{x\to 0} \frac{\sin 2x}{x}$

(b) $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

(c) $\lim_{x\to 0} \frac{\sin^2 x}{x}$

(d) $\lim_{x\to 0} \frac{\tan^2 x + 2x}{x + x^2}$

Continuity

100. Using ϵ-δ arguments prove that if two functions f, g are continuous at $x = a$, then so are

(a) $f - g$

(b) $f + 2g$

101. At which points is $(x^2 + 1)/(x^4 + x^2 - 2)$ continuous? Give reasons (ϵ–δ arguments not required).

102. Find the limit of the following functions at the given point. Redefine the function so that it is continuous at that point. You may assume that the function $f(x) = \sqrt{x}$ is continuous for all $x > 0$.

(a) $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

(b) $\lim_{x\to -4} \frac{1/4 + 1/x}{4 + x}$.

103. Is the function f is continuous on the interval $[-1, 1]$?

(a) $f(x) = |x|$

(b) $f(x) = \begin{cases} (1-x)^{-1}, & x \neq 1; \\ 1, & x = 1. \end{cases}$

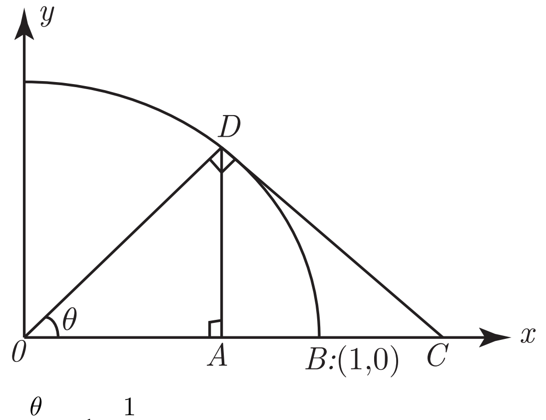
(c) $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}$

(d) $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0; \\ 0, & x = 0. \end{cases}$

(e) $f(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0; \\ 0, & x = 0. \end{cases}$

(f) $f(x) = \tan(\frac{\pi}{2} - x)$.

104. (a) Use the following diagram to find the areas of the triangles OAD and OCD, and the area of the part of the unit circle OBD



(b) Hence show that $\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$.

(c) Find $\lim_{\theta\to 0^+} \frac{\theta}{\sin\theta}$ for $\theta > 0$.

(d) What is $\lim_{\theta \to 0^-} \dfrac{\theta}{\sin\theta}$? Why?

What is $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta}$?

105. Sketch the function $F(x)$ defined by

$$F(x) = \lim_{n\to\infty} \frac{x^{2n}\sin(\pi x/2) + x^2}{x^{2n}+1}.$$

106. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y)\ \forall x, y \in \mathbb{R}$. If f is continuous at $x = 0$, show that it is continuous at all $x = a \in \mathbb{R}$. Can you think of a function that has this property?

107. Let $f : (0,\infty) \to \mathbb{R}$ satisfy $f(xy) = f(x) + f(y)\ \forall x, y \in (0,\infty)$. If f is continuous at $x = 1$, show that it is continuous at all $x = a \in (0,\infty)$. Can you think of a function that has this property?

108. Let $f : I \to \mathbb{R}$ be continuous on the interval I. Show that the function g defined by $g(x) = |f(x)|$ is continuous on I. Is the converse true?

109. Let $f : I \to \mathbb{R}$ with $f(x) \geq 0\ \ \forall x \in I$. If f is continuous at $x = a$, prove that g given by $g(x) = \sqrt{f(x)}$ is continuous at $x = a$ as well.

Intermediate Value Theorem

110. (a) If $f(x) = x^3 - 5x^2 + 7x - 9$, prove that there is a real number c such that $f(c) = 100$.

(b) Show that equation $x^5 - 3x^4 - 2x^3 - x + 1 = 0$ has at least one solution between 0 and 1.

(c) Show that equation $x + \sin x = 1$ has at least one solution in the interval $[0, \pi/6]$.

(d) Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that $\exists c \in [0,1]$ such that $f(c) = c$.

(e) Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(1) = 0$. Prove that $\exists c \in [0, \frac{1}{2})$ such that $f(c + \frac{1}{2}) = f(c)$.

Problem Sheet 4: Differentiability

Differentiability

111. Here are some exercises to try if you feel your differentiation skills are rusty.

Use appropriate rules to find derivatives of the following functions. Simplify wherever possible.

(a) $x^2\sqrt{1-x^3}$

(b) $x^3e^{x^2}$

(c) $\log \dfrac{x}{\sqrt{x^2+1}}$

(d) $\dfrac{x^2}{\sqrt[3]{1-x}}$

(e) $\dfrac{x^2-1}{(x+1)^2}$

(f) $\dfrac{\sin 3x}{\sin^3 x}$

112. Find the derivatives of the following functions, all defined with domain $\mathbb{R}$. If at any point the derivative does not exist, explain clearly why it doesn't and report the values of the left and right derivatives if these exist.

(a) $f(x) = \dfrac{x}{1+|x|}$

(b) $f(x) = \sin(|x|)$

(c) $f(x) = \begin{cases} x^4 & x \geq 0, \\ \sin^4 x, & x < 0; \end{cases}$

(d) $f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ \sin x, & x \notin \mathbb{Q}; \end{cases}$

(e) $f(x) = \lfloor x \rfloor = \begin{cases} \text{the largest integer less} \\ \text{than or equal to } x. \end{cases}$

113. Decide whether the following functions are continuous and/or differentiable at $x = 0$.

(a) $\begin{cases} -x^2, & x \leq 0 \\ x, & x > 0 \end{cases}$

(b) $\begin{cases} -x^2, & x \leq 0 \\ x^3, & x > 0 \end{cases}$

114. Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x) + f(y)$ for all $x, y \in (0, \infty)$.

(a) If f is differentiable at $x = 1$, show that f is differentiable on $(0, \infty)$ and $f'(x) = f'(1)/x$.

(b) Show that f is in fact infinitely differentiable.

115. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

(a) If f is differentiable at $x = 0$, show that f is differentiable on $\mathbb{R}$ and $f'(x) = f'(0)f(x)$.

(b) Show that f is in fact infinitely differentiable.

116. ♣ The proof that $e^a = \lim_{n\to\infty} (1 + a/n)^n$, $a \neq 0$. (The result is easy for $a = 0$.)

We first define e^x as follows: Let $f : \mathbb{R} \to \mathbb{R}$ be the function that has the properties in Sheet 3 Question 106 and Sheet 4 Question 115 where $f(0) = 1$ and $f'(0) = 1$ be called the *exponential function* denoted by $f(x) = e^x$. Then $f'(x) = e^x$.

(a) Let $y = \log x$, be the inverse function for e^x, i.e. $y = \log x \iff x = e^y$. Differentiate implicitly to prove that $(\log x)' = \dfrac{1}{x}$.

(b) Show that $\lim_{h\to 0} \dfrac{\log(1 + h)}{h} = 1$.

(c) Put $h = \dfrac{a}{n}$ to prove that $\lim_{n\to\infty} \dfrac{n}{a} \log(1 + a/n) = 1$ and hence show that $e^a = \lim_{n\to\infty} (1 + a/n)^n$.

Optional Exercise: Note that you have assumed that if $\lim_{x\to 0} f(x) = L$, then putting $n = \dfrac{\alpha}{x}$ gives $\lim_{n\to\infty} f(\alpha/n) = L$. Prove this.

117. Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$. Prove that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and is equal to $f'(c)$. Does the existence of $\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$ imply that f differentiable at c?

118. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere except possibly at $x = a$. If $\lim_{x \to a} f'(x)$ exists, does this guarantee that $f'(a)$ exists? If so, prove it. If not, find a counterexample.

119. Prove the *Straddle Lemma*:

Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. $\forall \epsilon > 0$, $\exists \delta > 0$ such that $c - \delta < u \le c \le v < c + \delta$ implies

$$|f(v) - f(u) - (v - u)f'(c)| \le \epsilon |v - u|$$

Hint: Subtract and add the term $f(c) - cf'(c)$ inside the left hand side and use the triangle inequality.

120. Consider the following theorem that you will be familiar with from high school:

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The maximum of f occurs either at an endpoint, at a point where f is not differentiable, or at a point $c \in [a, b]$ where $f'(c) = 0$.

To prove the theorem we will assume the maximum doesn't occur at an endpoint or a point where f is not differentiable, and show that it must occur at a point where the derivative is zero.

(a) Sketch three example functions to illustrate the theorem.

(b) Before we start the proof, explain why f attains a maximum.

(c) Let f achieve its maximum at $x = m$. Assume $m \ne a$ and $m \ne b$, and that f is differentiable at $x = m$. Then prove that the maximum must occur at a stationary point: i.e. $f'(m) = 0$.

Mean Value Theorem

121. This question test your understanding of the mean value theorem.

(a) Draw a picture of a function to illustrate what the mean value theorem says.

(b) The MVT requires the function to be continuous on $[a, b]$. Draw a function which fails this condition so that the conclusion of the MVT doesn't hold.

(c) The MVT requires the function to be differentiable on (a, b). Draw a function which fails this condition so that the conclusion of the MVT doesn't hold.

122. Use the mean value theorem on the following functions, using the interval $[-1, 4]$. Remember to confirm the conditions of the MVT before you use it. Then, find the value of c that the theorem says must exist.

(a) $f(x) = x^2$

(b) $g(x) = (x + 1)^a$ where $a \ge 1$

(c) $h(x) = |x - 1|$

(d) $j(x) = \log(2x + 4)$

123. Use the mean value theorem to prove the following inequalities. You will need to figure out what f to use, as well as what inverval $[a, b]$ will be needed.

(a) $|\cos x - \cos y| \le |x - y| \quad \forall x, y \in \mathbb{R}$

(b) $|\log x - \log y| \le \frac{1}{2}|x - y| \quad \forall x, y \in [2, \infty)$

(c) $\sqrt{1 + x} < 1 + \frac{1}{2}x \quad \forall x > 0$

(d) $e^x > 1 + x \quad \forall x > 0$

124. Using the MVT on the function $f : [36, 40] \to \mathbb{R}$ given by $f(x) = \sqrt{(x)}$, find upper and lower bounds for $\sqrt{40}$. This shows that the mean value theorem can be used to approximate irrational numbers (although not necessarily very well).

125. Prove the following theorem:

   Let f be continuous on $[a, b]$ and differentiable on (a, b). If $\forall x \in [a, b] \;\; f'(x) = 0$, then f is constant on $[a, b]$.

126. Recall that a function is *increasing* on an interval I if $\forall x, y \in I \;\; x < y \implies f(x) < f(y)$. Prove the following theorem:

   Let f be differentiable on the interval I. If $\forall x \in I \;\; f'(x) > 0$, then f is increasing on I.

   State and prove the corresponding theorem for a decreasing function.

127. Let $f : [a, b) \to \mathbb{R}$ be continuous on $[a, b)$ and differentiable on (a, b) and let the limit

$$\lim_{x \to a^+} f'(x) = L$$

   exist. Prove that the right derivative $f'_+(a)$ exists and is equal to L.

Problem Sheet 5: Integration

The Riemann Integral

128. Let $f : [-2, 4] \to \mathbb{R}$ be defined by $f(x) = -x + 3$. Let $P = \{-2, -1, 1, 4\}$ be a partition.
    (a) Find the upper sum $U(f, P)$.
    (b) Find the lower sum $L(f, P)$.
    (c) Find $\displaystyle\int_{-2}^{4} f\, dx$ using the Fundamental Theorem of Calculus.
    (d) Do your answers satisfy the correct inequality?
    (e) Make a refinement of P and recalculate the lower and upper sums. Do all your numbers still line up?

129. For each of the following functions defined on [-1,1], decide if it is integrable or not. Full proofs are not required, but a brief discussion of why or why not is needed.
    (a) $f(x) = x^7 - 5x^3$
    (b) $g(x) = \sin(e^x - 12)$
    (c) $h(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ -x^2 & x \notin \mathbb{Q} \end{cases}$
    (d) $p(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$
    (e) $q(x) = \begin{cases} x^2 + 2 & x < \frac{1}{3} \\ 9 - \log x & x \geq \dfrac{1}{3} \end{cases}$
    (f) $r(x) = \begin{cases} 17 & x = -\frac{1}{2} \\ \dfrac{1}{3x^2 + 5} & x \neq -\frac{1}{2} \end{cases}$

130. Let $f : [-1, 1] \to \mathbb{R}$ be given by
$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$
By finding a partition that gives $U(f, P) - L(f, P) < \epsilon$, show that f is integrable.
Hint: Pick a partition $P = \{-1, 0, a, 1\}$ and choose a so that you end up with $U(f, P) - L(f, P) < \epsilon$

131. Let $f : [0, 1] \to \mathbb{R}$ be given by
$$f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \end{cases}$$
By finding a partition that gives $U(f, P) - L(f, P) < \epsilon$, show that f is integrable.
Hint: Try a similar approach to the previous question.

132. Prove that if f is increasing on $[a, b]$ then f is integrable.
    (a) First, argue that f must be bounded on $[a, b]$.
    (b) Given $\epsilon > 0$, justify the existence of $k \in \mathbb{R}$ such that
$$0 < k < \frac{\epsilon}{f(b) - f(a)}$$
    (c) Let P be a partition of $[a, b]$ such that
$$x_i - x_{i-1} < k$$
for all i. Find the upper and lower sums of f.
    (d) Finally, show that $U(f, P) - L(f, P) < \epsilon$ and then conclude that f is integrable.
    (e) *Extra:* Prove the corresponding theorem for decreasing functions.

133. Let $f : [a, b] \to \mathbb{R}$ be integrable and let $k \in \mathbb{R}$. Prove that kf is integrable and

$$\int_a^b kf\ dx = k\int_a^b f\ dx$$

134. Let f and g be integrable on $[a, b]$ and let $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that

$$\int_a^b f\ dx \leq \int_a^b g\ dx$$

135. We will prove the Mean Value Theorem for Integrals.

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $\exists c \in [a, b]$ such that

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

(a) First, how do we know f is integrable?

(b) Argue that f has a maximum and a minimum on $[a, b]$. Call these M and m respectively.

(c) Use the very simple partition $P = \{a, b\}$ to create lower and upper sums, and relate them to

$$I = \int_a^b f(x)\ dx$$

(d) Finally, use the intermediate value theorem on f to finish the proof.

(e) *Extra:* If we let F be an antiderivative of f, then we know from the regular mean value theorem that $\exists c \in [a, b]$ such that

$$F(b) - F(a) = F'(c)(b-a)$$

Why is a proof using this approach invalid?

136. Consider the following sum:

$$\frac{\pi}{2n}\sin\left(\frac{\pi}{2n}\right) + \frac{\pi}{2n}\sin\left(\frac{2\pi}{2n}\right) + \cdots + \frac{\pi}{2n}\sin\left(\frac{(n-1)\pi}{2n}\right)$$

Interpret this expression as a Riemann Sum, and find the limit as $n \to \infty$ by using a definite integral. You might want to begin by trying to identify a term which could be the interval width.

The Fundamental Theorem of Calculus

137. Prove the integration by parts formula

$$\int_a^b f'(x)g(x)dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)dx$$

explaining sufficient conditions on f and g that ensure its validity.

138. *Riemann-Lebesgue Lemma*

Consider $f : [a, b] \to \mathbb{R}$ such that f' is continuous. Prove that

$$\lim_{\lambda\to\infty}\int_a^b f(x)\cos(\lambda x)\ dx = 0$$

This result is very important in the theory of *Fourier Analysis*. It is true for very general f, though it is significantly harder to prove in the more general setting.

139. Prove that if f is continuous on $[0, b]$ then

$$\int_0^b f(x)(b-x)\,dx = \int_0^b \left(\int_0^x f(t)\,dt\right) dx$$

140. Let $f : [a, b] \to \mathbb{R}$ be continuous and nonnegative and let $\int_a^b f(x)\,dx = 0$.

(a) Prove that $f(x) = 0$ for all $x \in [a, b]$

(b) Can the requirement of continuity of f be dropped? If so, prove it. If not, find a counterexample.

141. *The Trapezoidal Rule*

Let f be C^2, i.e. twice differentiable on $[a, b]$ with $f''(x)$ continuous on $[a, b]$. Let

$$I = \int_a^b (b-x)(x-a)f''(x)\,dx.$$

(a) By integrating by parts, prove that

$$\int_a^b f(x)\,dx = \frac{(b-a)[f(a)+f(b)]}{2} - \frac{I}{2}$$

Interpret the first term on the right-hand side as the area of a trapezium.

(b) We can think of this result as saying the integral is equal to the area of a trapezoid plus an "error term" $E = -I/2$. Use the mean value theorem for integrals to bound the error, giving

$$|E| \leq \frac{(b-a)^3}{2} M \qquad \text{where } M = \sup\Big\{|f''(x)| : x \in (a, b)\Big\}$$

The trapezoid is a simple way to approximate the value of integrals that you can't evaluate exactly. In practice we generally break up the interval $[a, b]$ into many smaller intervals: this reduces the total error quite a lot because of the $(b-a)^3$ term.

Improper Integration

142. Which of the following integrals are proper Riemann integrals, and which are improper?

(a) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

(b) $\displaystyle\int_0^{\pi/2} \tan x\,dx$

(c) $\displaystyle\int_1^2 \frac{1}{3x-1}\,dx$

(d) $\displaystyle\int_0^2 \frac{1}{3x-1}\,dx$

(e) $\displaystyle\int_{-\infty}^\infty e^{-x^4}\,dx$

(f) $\displaystyle\int_{-1}^1 \frac{\sin x}{x^2}\,dx$

(g) $\displaystyle\int_0^1 t^{1/2}e^{e^t}\,dt$

(h) $\displaystyle\int_0^1 \frac{dt}{t-\frac{1}{2}}$

(i) $\displaystyle\int_0^{100} e^{\lfloor t\rfloor}dt$ where $\lfloor t\rfloor$ is the *floor function.*

143. Evaluate the following improper integrals, or show they are divergent.

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (d) $\displaystyle\int_0^\infty xe^{-2x}\,dx$ (g) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int_{-1}^1 \frac{1}{x^{2/3}}\,dx$ (e) $\displaystyle\int_1^\infty \frac{\log x}{x}\,dx$ (h) $\displaystyle\int_0^5 \frac{1}{\sqrt{5-x}}\,dx$

(c) $\displaystyle\int_{-\infty}^2 \frac{1}{3-x}\,dx$ (f) $\displaystyle\int_1^\infty \frac{\log x}{x^2}\,dx$ (i) $\displaystyle\int_0^\infty \frac{1}{(x+1)(x+2)}\,dx$

Note that you may assume that $\lim_{x\to\infty} x^n e^{-x} = 0$ for $n > 0$.

Theorem: Comparison Test *Let f and g be continuous functions on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$.*

- *If $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ converges.*
- *If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges.*

144. Determine whether the following integrals converge or diverge using the comparison test.

(a) $\displaystyle\int_1^\infty \frac{1}{1+x^6}\,dx$ (c) $\displaystyle\int_0^\infty e^{-x^4}\,dx$ (e) $\displaystyle\int_1^\infty \frac{1}{x+x^2}\,dx$

(b) $\displaystyle\int_\pi^\infty \frac{2+\cos x}{x}\,dx$ (d) $\displaystyle\int_1^\infty \left(\frac{1}{x}+\frac{1}{x^2}\right)dx$ (f) $\displaystyle\int_2^\infty \frac{1}{\log x}\,dx$

145. *The Gamma Function*

We define the Gamma Function $\Gamma : (0,\infty) \to \mathbb{R}$ by the following integral:

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$$

Use induction to show that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$.

Notice how the Gamma Function is defined for all values between the natural numbers as well, so we can think of it as being a *continuous extension* of the factorial function.

146. *The Beta Function* (Challenging!!)

For which u, $v \in \mathbb{R}$ does the Beta Function

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt$$

exist? By use of the change of variable $t = x/(1+x)$ prove that with the same restrictions on u and v,

$$B(u,v) = \int_0^\infty \frac{x^{u-1}dx}{(1+x)^{u+v}}$$

Problem Sheet 6: Series

Series

147. For each of the following series calculate the first five partial sums and make a *guess* as to whether the series converges. (NOTE: It is perfectly acceptable if your guesses are incorrect.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(\log n)^2}{n^2}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

148. Let $a_n = \dfrac{2n}{3n+1}$.

(a) Determine whether $\{a_n\}$ is convergent.

(b) Is $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent?

149. The following series are telescoping (although it may not appear so at first glance). For each series, determine if it is convergent or divergent, and calculate its value if it converges. (Hint: partial fractions may be useful in some parts.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \log \frac{n+1}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

150. The following series can all be written as geometric series, perhaps after a little algebra. Determine the value of each. You will need to use the theorem:

THEOREM: *The geometric series* $\displaystyle\sum_{n=0}^{\infty} r^n$ *converges to* $\dfrac{1}{1-r}$ *if* $|r| < 1$. *Otherwise it diverges.*

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+2}}{6^{2n+1}}$

(c) $\displaystyle\sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^n$

(d) $\displaystyle\sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{2}\right) \frac{2^n}{7^{n-1}}$

151. For this question, refer back to the Cantor Set from problem sheet 1, question 16.

We will calculate the "length" of the Cantor Set.

(a) What is the length of C_0?

(b) What are the lengths of C_1 and C_2?

(c) To figure out the final length of C, we will figure out the length of all the segments removed.

(d) Create a sequence a_n= the number of segments removed at step n. Clearly $a_1 = 1$ and $a_2 = 2$. Find a few more terms, then create a formula for the nth term.

(e) Create another sequence b_n= the length of each segment removed. Clearly $b_1 = \frac{1}{3}$ and $b_2 = \frac{1}{9}$. Find a few more terms, then create a formule for the nth term.

(f) The total lengh removed each step is therefore $a_n b_n$ and the total length removed is the sum

$$\sum_{n=1}^{\infty} a_n b_n$$

Determine the value of this sum, and hence the length of the Cantor Set C. Does the result seem unusual?

Convergence Tests

152. Use comparison tests to determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n4^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+3^n}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^2}$

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\sqrt[3]{n^2-2}}$

Extra: in part 152f, are there any special values of x for which you can evaluate the series exactly?

153. Use the ratio test to determine the convergence or divergence of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

154. Use the modified integral test to determine whether the following series converge or diverge. The modified integral test is:

THEOREM: *Let (a_n) be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$, $n \in \mathbb{N}$. Then the series $\displaystyle\sum_{n=N}^{\infty} a_n$ and the integral $\displaystyle\int_N^{\infty} f(x)\,dx$ either both converge or both diverge.*

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\log n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\log n\log(\log n)}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+9}$

155. Determine whether the following alternating series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2n}{4n^2-3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2n}{4n-3}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$ (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1+3^n}{1+4^n}$

156. Determine whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos n}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+7n}{2^n(n^2+1)}$

(c) $\displaystyle\sum_{n=2}^{\infty} \left(\sqrt[n]{n+3} - \sqrt[n-1]{n+2}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{(n+1)^2}{n(n+2)}\right)$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{2n+1}}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt[3]{n^2+2}}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$,

(j) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+4n}{n^4+200}$

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}$

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

(n) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{2^n}$,

(o) $\displaystyle\sum_{n=1}^{\infty} \frac{(\log n)^2}{n^2}$,

(p) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ (Warning: this is *not* a p-series!).

157. For each of the series in question 155, determine if it converges absolutely, converges conditionally, or diverges.

158. Prove that for a convergent alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, $a_n > 0$, that $\left|\sum_{n=k}^{\infty}(-1)^{n+1}a_n\right| \leq a_k$.

159. How many terms of the following alternating series do we need to add to find the sum to the indicated accuracy?

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$ error < 0.001

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$ error < 0.01

160. Prove the following theorem:

The Limit Comparison Test *Let a_n and b_n be positive sequences.*

1. If $\lim_{n\to\infty}\frac{a_n}{b_n} = c > 0$, then $\sum_{n=1}^{\infty}a_n$ converges if and only if $\sum_{n=1}^{\infty}b_n$ converges.

2. If $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

3. If $\lim_{n\to\infty}\frac{a_n}{b_n}$ diverges to infinity and $\sum_{n=1}^{\infty}b_n$ diverges, then $\sum_{n=1}^{\infty}a_n$ diverges.

161. Is it true that the convergence of $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ implies the convergence of $\sum_{n=1}^{\infty}(a_n+b_n)$? If it is true, prove it. If it is not true, find a counterexample.

162. Is it true that the convergence of $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ implies the convergence of $\sum_{n=1}^{\infty}a_nb_n$? If it is true, prove it. If it is not true, find a counterexample.

163. Prove that absolute convergence implies convergence. Your first step should be to convert that last sentence into mathematical language.

Power Series

164. Find the interval of convergence and radius of convergence for each of the series below.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n^2}$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2n+1}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{n^2}{2^n}x^n$

(d) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x+6}{3}\right)^n$

(e) $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!}$

(f) $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n}{n}x^{n+1}$

165. Use integration and differentiation of power series to find the sum of the following series. The geometric series might be a good starting point. Keep track of the radius of convergence through each step.

(a) $\displaystyle\sum_{n=1}^{\infty}nx^{n-1}$

(b) $\displaystyle\sum_{n=1}^{\infty}nx^n$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$

(d) $\displaystyle\sum_{n=2}^{\infty}n(n-1)x^n$

166. (a) ♣ Determine if the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges. What can we conclude about the convergence of $\int_{x=1}^{\infty} \frac{x}{e^x}\, dx$? What do you deduce about $\lim_{x\to\infty} \frac{x}{e^x}$? Use a similar argument to show that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

(b) ♣ Find the radius of convergence for the series $\sum_{n=1}^{\infty} n^p x^n$. What do you conclude about the $\lim_{n\to\infty} n^p x^n$?

Taylor and Maclaurin Series

167. Find the Maclaurin series for the given function. Find also the interval $|x| < r$ on which it converges, using the ratio test or otherwise.

(a) $\sin(2x)$

(b) $\cos x$

(c) $\frac{1}{1+x}$

(d) $\sinh x$

(e) $\log(2x+1)$

(f) $(1+x)^{-2}$

168. Find the Taylor series for the function f about the indicated point a.

(a) $f(x) = \sin x; \quad a = \frac{1}{4}\pi$

(b) $f(x) = \cos x; \quad a = \frac{1}{3}\pi$

(c) $f(x) = 1/x; \quad a = 2$

(d) $f(x) = e^x; \quad a = -3$

169. Find a series representation for the following.

(a) e^{2x} in powers of $x+1$.

(b) $\log x$ in powers of $x-1$.

170. Find the Maclaurin series for the given functions by manipulating simpler Taylor series such as the geometric and exponential series.

(a) $\sin\left(\theta^2\right)$

(b) $x \sin 3x$

(c) $\cos^2(x)$

(d) $\frac{t}{1+t}$

(e) $\frac{z}{e^{2z}}$

Hint for (c): Use $\cos(2x) = 2\cos^2(x) - 1$.

171. (a) Show that for $\alpha \notin \mathbb{N}$ the binomial series for $(1+x)^\alpha$

$$1 + \sum_{n=1}^{\infty} \frac{\alpha \cdot (\alpha-1) \cdot (\alpha-2) \cdots (\alpha-(n-1))}{n!} x^n$$

has radius of convergence $R = 1$. You may assume that the series converges to $(1+x)^\alpha$ within the radius of convergence.

(b) Using your answer to (a), find the Maclaurin series for $\sqrt{1+x}$

(c) Hence find the Maclaurin series for $\frac{1}{\sqrt{1-x^2}}$

(d) From(c) find the Maclaurin series for $\arcsin x$.

(e) Hence find a power series expression for $\frac{\pi}{6}$.

172. Use Taylor series to evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{1-\frac{1}{2}x^2-\cos x}{x^4}$ (b) $\displaystyle\lim_{x\to 0} \frac{x-\arctan x}{x^3}$ (c) $\displaystyle\lim_{x\to 0} \frac{x}{e^x}$

173. Consider the function $f : \mathbb{R}\backslash\{-1\} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{1+x}$$

(a) Find the nth order Taylor polynomial about $x = 0$, and the corresponding remainder term.

(b) For $0 < x < 1$ show that the remainder vanishes as $n \to \infty$

(c) Using Taylor's theorem, conclude that $f(x)$ is equal to its Taylor series about $x = 0$ for $0 < x < 1$.

174. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = e^x$$

(a) Find the nth order Taylor polynomial about $x = 0$, and the corresponding remainder term.

(b) Show that the remainder vanishes as $n \to \infty$.

(c) Using Taylor's theorem, conclude that f is equal to its Taylor series about $x = 0$ for all $x \in \mathbb{R}$.

175. Integration of power series.

(a) Find the Maclaurin series for $\sin(x^2)$, using the Maclaurin series for $\sin(x)$.

(b) Find the Maclaurin series for $f(t) = \displaystyle\int_0^t \sin(x^2)\,dx$ by integrating the previous series.

(c) Estimate the integral $\displaystyle\int_0^1 \sin(x^2)\,dx$ to 3 decimal places. Use the result of question 158 to justify your result.

176. Estimate $\displaystyle\int_0^1 \left(\frac{\sin t}{t}\right) dt$ using Taylor polynomials for $\sin t$ about $t = 0$ of

(a) degree 3 (b) degree 5.

Use the remainder in Taylor's theorem to estimate the error in these approximations.

177. Here we give an example of function f which has a convergent Taylor series, but with the Taylor series *not* equal to the function! Define

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0\,, \\ 0 & x = 0\,. \end{cases}$$

(a) Show that f is continuous at 0.

(b) Show that for any $m \in \mathbb{N}$

$$\lim_{x \to 0} \frac{f(x)}{x^m} = 0\,.$$

(c) Show that

$$f'(x) = \begin{cases} \frac{2}{x^3}\, e^{-\frac{1}{x^2}} & x \neq 0\,, \\ 0 & x = 0\,. \end{cases}$$

(d) Use induction to show that for every $n \in \mathbb{N}$ there is a rational function R such that

$$f^{(n)}(x) = \begin{cases} R(x)\, e^{-\frac{1}{x^2}} & x \neq 0\,, \\ 0 & x = 0\,. \end{cases}$$

(e) What is the Maclaurin series for $f(x)$? How good an approximation is it?

178. Here we prove that e is irrational.

(a) Show that for any $n \in \mathbb{N}$, we have

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n$$

where $R_n = \dfrac{1}{(n+1)!} + \dfrac{1}{(n+2)!} + \ldots$.

(b) Begin a proof by contradiction by assuming $e \in \mathbb{Q}$. Find m such that $R = m!R_m \in \mathbb{N}$.

(c) Find an upper bound for R that is a geometric series and sum this and work out why it is less than 1 so cannot be a natural number. Conclude that e is irrational.

Problem Sheet 7: Fourier Series

Note: This topic is particularly relevant for students planning to do Physics, Engineering and Applied Mathematics. If studying in these areas, you are recommended to keep a copy of these questions for future reference.

Background for these questions

Fourier series are introduced here using concepts that you have studied in Linear Algebra.

Let $L^2([-\pi,\pi])$ be the vector space of functions $\{f : [-\pi,\pi] \text{ to } \mathbb{R}\}$ such that f^2 is Riemann integrable. Assume that $L^2([-\pi,\pi])$ is equipped with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

The set of vectors,

$$\mathcal{B} = \left\{\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \dots\right\},$$

forms an orthonormal set in $L^2([-\pi,\pi])$.

Then the Fourier series for $f(x)$ is the projection of $f(x)$ onto $\mathcal{B}$ given by

$$\langle f, \frac{1}{\sqrt{2\pi}}\rangle \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\langle f, \frac{\cos nx}{\sqrt{\pi}}\rangle\frac{\cos nx}{\sqrt{\pi}} + \sum_{n=1}^{\infty}\langle f, \frac{\sin nx}{\sqrt{\pi}}\rangle\frac{\sin nx}{\sqrt{\pi}} \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \sin nx \;+ b_n \cos nx\Big)$$

where a_n and b_n, the *Fourier coefficients* for f, are defined by:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \;\; n\in\mathbb{N} \text{ and } \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \;\; n\in\mathbb{N}$$

Notes:

(1) These coefficients make sense even if f is not defined on a finite number of points in $[\pi,\pi]$.

(2) The coefficients are valid for functions that are 2π-periodic, that is, functions $f : \mathbb{R}\to\mathbb{R}$ such that $f(x) = f(x+2\pi)$ for all $x\in\mathbb{R}$.

Theorem (Fourier Series) *Every piecewise differentiable function f on $[-\pi,\pi]$ with $f(\pi) = f(-\pi)$ is given by its Fourier series.*

That is, the Fourier series for f converges to $f(x)$ for all $x\in[-\pi,\pi]$. We then say

$$f(x) \;=\; a_0 + \sum_{n=1}^{\infty}\Big(a_n\cos nx \;+ b_n \sin nx\Big).$$

General Intervals

Note that the Fourier series for $f : [-L, L] \to \mathbb{R}$ may be found by replacing x with $\pi x/L$. That is, the Fourier series is

$$A_0 + \sum_{n=1}^{\infty}\Big(A_n\cos\Big(\frac{n\pi x}{L}\Big) \;+ B_n\sin\Big(\frac{n\pi x}{L}\Big)\Big),$$

where A_n and B_n, the *Fourier coefficients* for f, are defined by:

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx, \;\; n\in\mathbb{N}, \qquad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx, \;\; n\in\mathbb{N}$$

Please turn over for questions ...

179. You may assume the following:

- $\sin nx \sin mx = \frac{1}{2}(\cos((n-m)x) - \cos((n+m)x))$
- $\cos nx \cos mx = \frac{1}{2}(\cos((n-m)x) + \cos((n+m)x))$
- $\cos nx \sin mx = \frac{1}{2}(\sin((m-n)x) + \sin((m+n)x))$

This is part of the proof that $\mathcal{B}$ is an orthonormal set.

(a) Show that $\langle \cos nx, \sin mx \rangle = 0$ for all $m, n \in \mathbb{N}$.

(b) Show that $\langle \sin nx, \sin mx \rangle = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$.

180. (a) Consider the square wave, $f : \mathbb{R}\backslash\{n\pi\} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

with $f(x) = f(x+2\pi)$. Sketch $f(x)$ and find the Fourier series for $f(x)$.

(b) Consider the saw tooth wave, $f : \mathbb{R}\backslash\{n\pi\} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ t, & 0 < x < \pi \end{cases}$$

with $f(x) = f(x+2\pi)$.

Sketch $f(x)$ for $-3\pi \leq x \leq 3\pi$ and find the Fourier series for $f(x)$.

181. (a) Let $f : [-\pi, \pi] \to \mathbb{R}$ be an even function with integrable f^2. (Even means $\forall x \;\; f(-x) = f(x)$.) Show that

$$\forall n \in \mathbb{N} \;\; b_n = 0.$$

(b) Let $f : [-\pi, \pi] \to \mathbb{R}$ be an odd function with integrable f^2. (Odd means $\forall x \;\; f(-x) = -f(x)$.) Show that

$$\forall n \in \mathbb{N} \;\; a_n = 0.$$

182. Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x \leq 0 \\ x(\pi - x) & \text{if } 0 \leq x \leq \pi \end{cases}.$$

(a) Sketch the function $f(x)$.

(b) Show that the Fourier series for f is

$$f(x) = \frac{\pi^2}{12} - \sum_{n=1}^{\infty} \frac{2}{(2n)^2} \cos(2nx) + \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)^3} \sin((2n-1)x)$$

(c) Substitute $x = 0$ into the Fourier series to find $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

183. Let $f : [0, 2\pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ x - 2\pi, & \pi < x \leq 2\pi \end{cases}.$$

(a) Find the Fourier series associated with f.

(b) For what values does the Fourier series converge to f?

(c) Although $f'(x) = 1$, show that the series obtained by term by term differentiation of the Fourier series in (a) diverges where the Fourier series converge to f.

Problem Sheet 1: Logic, Sets, Numbers and Proofs

Logic and Notation

1. (a) "A monkey is happy" $\implies$ "The monkey is eating a banana"
   (b) "You don't step on the tail of a snake" $\implies$ "The snake will not bite you"
   (c) "A donkey sees a mule" $\implies$ "The donkey laughs at him"
   (d) Wealth $\implies$ Happiness
   (e) Happiness $\implies$ Wealth

2. (a) False: Neptune is a planet, not a moon.
   (b) True: Jupiter *is* a planet.
   (c) False: both parts are false.
   (d) True: at least one of the parts is true (in this case, both are true).
   (e) True
   (f) True: Napoleon was born in Corsica, France. The second part happens to be true, although it doesn't matter in this case.
   (g) False: Dodos *are* unfortunately extinct (hunted to extinction by man), but pigs cannot fly.
   (h) True

3. Let $\mathcal{P}$ = the set of prime numbers.
   (a) $(\sim(6 \in \mathcal{P})) \vee (\sim(11 \in \mathcal{P}))$
   (b) $(10^2 = 50) \wedge (5^3 = 12)$
   (c) $(7 \in \mathbb{Z}) \implies (\sim(6 \in \mathbb{Z}))$.
   (d) $((2 \in \mathcal{P}) \wedge (5 \in \mathcal{P})) \implies (\sim(2 \times 5 \in \mathcal{P}))$

4. (a) The truth table for $(p \wedge q) \vee (\sim p \wedge \sim q)$ is:

p	q	$(p \wedge q) \vee (\sim p \wedge \sim q)$
T	T	T
T	F	F
F	T	F
F	F	T

This one is equivalent to the biconditional operator.

   (b) The truth table for $[\sim q \wedge (p \implies q)] \implies \sim p$ is:

p	q	$[\sim q \wedge (p \implies q)] \implies \sim p$
T	T	T
T	F	T
F	T	T
F	F	T

This one is a tautology

(c) The truth table for $[(p \vee q) \wedge r] \implies (p \wedge r)$ is:

p	q	r	$[(p \vee q) \wedge r] \implies (p \wedge r)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

This one is equivalent to $\sim (\sim p \wedge q \wedge r)$

5. (a) $\forall r \in \mathbb{Q}, r > 6$ This is false
   (b) $\exists x \in \mathbb{R}$ such that $x^2 + 3x - 7 = 0$ This is true
   (c) $\exists n \in \mathbb{N}$ such that $n^3 = 8$ This is true
   (d) $\{n \in \mathbb{N} : \nexists p \in \mathbb{N}$ such that $n = 7p\}$ Just an object: neither true nor false

6. There are of course many answers.
   (a) "Adding zero to any rational number doesn't change the number."
   (b) "The square of any real number is greater than one."
   (c) "m is a local minimum of f"
   (d) "S is bounded"

7. The second statement is the true one.

8. (a) $\exists x \in R \;\; x^2 \neq 10$
   (b) $\forall y \in \mathbb{N} \;\; y \geq 0$
   (c) $\forall a \in \mathbb{N} \;\; \exists x \in \mathbb{R} \;\; ax \neq 4$
   (d) $\exists y \in \mathbb{Q} \;\; \forall x \in \mathbb{R} \;\; x/y \neq 30$

9. (a)

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	T	F	F
T	F	F	F
T	F	F	F

As the truth tables are the same, the two statements $p \wedge q$ and $q \wedge p$ are the same.

(b)

$\sim (p \implies q)$	$p \wedge \sim q$
F	F
T	T
F	F
F	F

As the truth tables are the same, the two statements $p \sim (p \implies q)$ and $p \wedge \sim q$ are the same.

Both (i) and (ii) are tautologies.

Sets

10. (a) $\{1, 2, 3, 4, 5, 7\}$
    (b) $\{1, 3\}$
    (c) $\{1, 2, 3, 4\} = A$
    (d) $\{2, 3\} = C$
    (e) $\{1, 2, 3, 4, 5, 7\} = A \cup B$
    (f) $\{3\}$

11. (a) $\{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3), (7, 2), (7, 3)\}$ (b) $\emptyset$

12. (a) False: 3 is not a set, so it can't be a subset of anything. However, we could say 3 is an element of B ($3 \in B$).
    (b) True: the empty set is a subset of all sets.
    (c) True
    (d) False: 2 is in C, but not in B.
    (e) False: B contains numbers which aren't in C.
    (f) False: A does not contain sets of numbers, so no set is an element of A. However, we could say that $C \subseteq A$.
    (g) True
    (h) False: The number 1 is in B, and there is no number in C you can subtract from it that gives a result that is non-negative.

13. (a) 30 (b) -15

14. The intervals are:
    (a) $(-5, 3)$
    (b) $[-6, 12]$
    (c) $(-1, \pi]$
    (d) $\emptyset$
    (e) $(0, 1]$
    (f) $\{1\}$

15. General hints:
    - To prove that $A = B$ you usually need to prove $A \subseteq B$ and $B \subseteq A$.
    - To prove that $A \subseteq B$, start with $x \in A$ as a premise and try to prove that $x \in B$. Use the Deduction Principle and definition of subset to conclude $A \subseteq B$.
    - One way to prove set theorems is to rewrite the theorem in logic form and then prove the logic form eg. $A \subseteq B$ is true iff $\forall x \in \mathcal{U} \quad [(x \in A) \implies (x \in B)]$ is true.

    Hints for specific subquestions:
    (a) Use the axiom $x \in B \lor \sim x \in B$ and then proof by cases.
    (c) You may need to use the logical equivalence $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
    (d) If you have a premise $x \in A \lor x \in B$ then try proof by cases.

16. (a) The length is 1.
    (b) $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
    (c) $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

Proof by Counterexample

17. (a) Consider $3 \times 4 = 12$ which is even, but 3 is odd, so the statement is false.
    (b) Put $x = 1$ and $y = -1$. Then $x^2 = y^2$, but $x \neq y$.
    (c) Let $A = \{1\}$, $B = \emptyset$, $C = \{2\}$. Then $A \cup (B \cap C) = A \neq C = (A \cap B) \cup C$.
    (d) Put $a = 11$, so $a = 7 + 4$ and the remainder is 4. The $5a = 55 = 7 \times 7 + 6$ and the remainder is $6 \neq 4$.

Direct Proofs

18. These proofs will all follow the same basic pattern. Identify the premise and express it in mathematical language; identify the conclusion and express this in mathematical notation; connect the two with a reasoned argument. For example in (a):

Claim: The product of an even integer with an odd integer is even.

Proof:

Let	a be an even integer			Premise
$\Rightarrow$	$\exists n \in \mathbb{Z}\ \ a$	$=$	$2n$	Definition of even
Let	b be an odd integer			Premise
$\Rightarrow$	$\exists m \in \mathbb{Z}\ \ b$	$=$	$2m+1$	Definition of odd
Therefore	ab	$=$	$(2n)(2m+1)$	
		$=$	$4mn + 2n$	
		$=$	$2(2mn + n)$	
		$=$	$2k$	where$k = 2mn + n$
	$k \in \mathbb{Z}$			$\mathbb{Z}$ closed under multiplication and addition
Thus	$\exists k \in \mathbb{Z}\ \ ab$	$=$	$2k$	
$\Rightarrow$	ab is even.			Definition of even and Conclusion

■

Numbers

19. The sets that go with each equation are:

(a) $\mathbb{R}$
(b) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$,
(c) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
(d) none
(e) $\mathbb{Q}, \mathbb{R}$
(f) None

20. (a) The integers have additive inverses, but the natural numbers do not. In particular, this lets us define *subtraction* of integers.
    (b) The rational numbers (besides zero) have multiplicative inverses, but the integers do not (except 1 and -1). In particular, this lets us define *division* of rational numbers.
    (c) The real numbers satisfy the Supremum Axiom, but the rational numbers do not. In particular, this will let us define *convergence* later in the subject.

21. We give part (a) as an example plus hints for the other parts.

(a) **Claim:** For all $x, y, z \in \mathbb{R}$, if $x + z = y + z$ then $x = y$.

Proof:

	$x, y, z \in \mathbb{R}$			Premise
	$x + z$	$=$	$y + z$	Premise
Let	$w \in \mathbb{R}$ such that $z + w$	$=$	0	w exists by Additive Inverse
Then	$(x + z) + w$	$=$	$(y + z) + w$	Transitivity
$\Rightarrow$	$x + (z + w)$	$=$	$y + (z + w)$	Associativity
$\Rightarrow$	$x + 0$	$=$	$y + 0$	Additive Inverse
$\Rightarrow$	x	$=$	y	Additive Identity
Hence	$(x + z = y + z)$	$\implies$	$(x = y)$	■

(b) Hint: First prove that $0 + 0 = 0$. Then, use that fact with $0x$.

If you pay attention, you may be able to use the result of (a) to save time.

(c) Hint: Start with $x + (-x) = 0$ and add $-(-x)$ to both sides.

(d) Hint: Argue that $-z > 0$, then multiply both sides of $x < y$ by $-z$ using MO.

22. Hints:

(a) The key here is to realise what the theorem is saying. It says

"The additive inverse of xy is equal to $(-x)y$"

So, we need to show that the sum of these is equal to zero: i.e. $xy + (-x)y = 0$. You might need to use the fact that additive inverses are unique at some point in your proof.

(b) Again the key here is to decipher what the theorem is saying. Finish this interpretation of the theorem: "The multiplicative inverse of x^{-1} is..."

(c) Start with $(-x) + (-(-x)) = 0$.

(d) Use the multiplicative inverse of z.

(e) Split into two cases and use MO2 and part (d) of Q21.

(f) Use previous part.

Proof by Contrapositive

23. Here is a proof of part (a).

Claim: If n^4 is even, then n is even.

Proof: Assume n is odd. Premise

$\Rightarrow$	$\exists m \in \mathbb{Z}\ n$	$=$	$2m + 1$	Definition of odd
$\Rightarrow$	n^4	$=$	$(2m + 1)^4$	
		$=$	$32m^4 + 32m^3 + 24m^2 + 8m + 1$	
		$=$	$2(16m^4 + 16m^3 + 12m^2 + 4m) + 1$	
		$=$	$2k + 1$	where $k = 16m^4 + 16m^3 + 12m^2 + 4m$
	$k \in \mathbb{Z}$			$\mathbb{Z}$ closed under addition and multiplication
Hence	n^4 is odd			Definition of odd

We conclude that $\sim (n \text{ is even}) \implies \sim (n^4 \text{ is even})$.

By the contrapositive if n^4 is even, then n is even. ■

Proof by Contradiction

24. Mulitply both sides by the inverses of both matrices. Remember though that matrix multiplication is *not* commutative, so the order does matter.

25. See your tutor if you need a hint.

26. See your tutor if you need a hint.

27. There are 3 paintings which satisfy the special rule.

Proof by Cases aka Proof by Exhaustion

28. Here are some hints:

    (a) Show the result for the cases $n = 3k + 1$ and $n = 3k + 2$. See the lecture solution of the square of an odd integer is odd and use the same idea for each case separately.

    (b) There are two obvious cases here, m and n both odd or both even.

The Rational Numbers

29. (a) We mean $ad = bc$

    (b) Definition: $(a, b) = (c, d)$ means $ad = bc$

    (c) Definition: $(a, b) + (c, d) = (ad + bc, bd)$

    (d) Definition: $(a, b)(c, d) = (ac, bd)$

Irrational numbers

30. (a) Mimic the proof of the irrationality of $\sqrt{2}$

    (b) Assume it is rational, so that $\sqrt{15} + \sqrt{5} = \dfrac{p}{q}$. Then, following the standard procedure for removing square roots, square both sides. Next simplify and find an expression for $\sqrt{3}$ as a rational number. Finally use part (a).

    (c) Use the same procedure: assume it is rational, and then remove the logarithm (how do you remove logarithms?)

(d) **Claim:** For all $x, r \in \mathbb{R}$, if $x \notin \mathbb{Q}$ and $r \in\ Q$ then $x + r$ is irrational.

Proof: Let

$$
\begin{array}{lrll}
\text{Let} & x, r \in \mathbb{R} & & \text{Premise} \\
& x \notin \mathbb{Q} & & \text{Premise} \\
& r \in\ Q & & \text{Premise} \\
\Rightarrow & \exists a, b \in \mathbb{Z}\ \ b \neq 0\ \ r & = \dfrac{a}{b} & \text{Definition of rational number} \\
& x + r \in \mathbb{Q} & & \text{Contradiction premise} \\
\Rightarrow & \exists p, q \in \mathbb{Z}\ \ q \neq 0\ \ x + r & = \dfrac{p}{q} & \text{Definition of rational number} \\
\Rightarrow & x + \dfrac{a}{b} & = \dfrac{p}{q} & \\
& & = \dfrac{p}{q} - \dfrac{a}{b} & \\
& & = \dfrac{pb - aq}{bq} & \\
& pb - aq \in \mathbb{Z} & & \mathbb{Z} \text{ closed under addition and multiplication} \\
& bq \in \mathbb{Z} & & \mathbb{Z} \text{ closed under mulitplication} \\
\text{Thus} & x \in \mathbb{Q} & & \text{Definition of rational number}
\end{array}
$$

This contradicts the fact that x is irrational.

Therefore, $x + r$ is irrational. ■

Mathematical Induction

31. Mimic the proofs from lectures and tutorials.

The equations, rewritten using summation notation, are:

(a) $\displaystyle\sum_{j=1}^{n}(5j - 3) = \tfrac{1}{2}n(5n - 1)$

(b) $\displaystyle\sum_{m=1}^{n} m2^{m-1} = 1 + (n - 1)2^n$

(c) $\displaystyle\sum_{\kappa=1}^{n} \frac{1}{\kappa(\kappa + 1)} = \frac{n}{n + 1}$

(d) $\displaystyle\frac{a^n - b^n}{a - b} = \sum_{i=1}^{n} a^{n-i}b^{i-1}$

32. Here is a proof of part (a).

Claim: 3 is a factor of $n^3 - n + 3 \quad \forall n \in \mathbb{N}$

Proof: Let $P(n)$ be the statement "$\exists m \in \mathbb{N}$ such that $n^3 - n + 3 = 3m$"

(1) $1^3 - 1 + 3 = 3 \times 1$.

$\Rightarrow\ \ \exists k \in \mathbb{N}\ \ 1^3 - 1 + 3 = 3k$.

$\Rightarrow P(1)$.

(2) Assume $\exists k \in \mathbb{N}\ \ P(k)$. Inductive premise

$\Rightarrow\ \ \exists p \in \mathbb{N}\ \ k^3 - k + 3 = 3p$.

We now prove that $P(k+1)$ is true:

$$\begin{aligned}
(k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 2k + 3 \\
&= (k^3 - k + 3) + 3(k^2 + 3k) \\
&= 3p + 3(k^2 + k) && \text{By inductive premise} \\
&= 3(p + k^2 + k) \\
&= 3m && \text{where } m = p + k^2 + k \\
m &\in \mathbb{N} && \text{As } \mathbb{N} \text{ closed under multiplication and addition}
\end{aligned}$$

Thus $\exists m \in \mathbb{N} \ \ (k+1)^3 - (k+1) + 3 = 3m$

$\Rightarrow \ \ P(k+1)$

Hence, $P(k) \implies P(k+1)$

Therefore, by Mathematical Induction, $P(n)$ is true $\forall n \in \mathbb{N}$. ■

The other parts of this question are similar, except (d). Recall that when we talk about factors of polynomials, we mean something different to factors of integers.

33. See your tutor if you need a hint.

34. Here is the inductive step for (a).

    Assume $\exists k \in \mathbb{N}$ such that $1 + 2k \leq 3^k$.

    Claim: $1 + 2(k+1) \leq 3^{k+1}$.

    Proof: First note that $1 + 2(k+1) = 3 + 2k$

$$\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&\geq 3(1 + 2k) && \text{from inductive premise} \\
&= 3 + 6k \\
&= 3 + 2k + 4k \\
&\geq 3 + 2k && \text{for } k \geq 0 \\
&= 1 + 2(k+1)
\end{aligned}$$

    Therefore $1 + 2(k+1) \leq 3^{k+1}$ ■

35. See your tutor if you need a hint.

36. See your tutor if you need a hint.

37. See your tutor if you need a hint.

38. This one is a little tricky, algebra-wise.

    In the inductive step, you will need to evalute the integral

$$I_{k+1} = \int_0^1 x^{k+1}\sqrt{1-x}\,dx$$

    Use integration by parts, letting

$$u = x^{k+1} \text{ and } v' = \sqrt{1-x}$$

    You will need to do a little bit of manipulation, but eventually you will get I_k and I_{k+1} appearing again on the right hand side.

Inequalities

39.

(a) True

(b) True

(c) True

(d) True only if $c > 0$. What if $c = 0$ or $c < 0$?

(e) Sometimes true (Hint: $a < 0 < b$ vs $0 < a < b$)

(f) Sometimes true

(g) True only if $a > 0$.

(h) False. What should it be?

40. (a) $x \in [-\frac{5}{2}, \frac{3}{2}]$

(b) $x \in (-\infty, -7] \cup [3, \infty)$

(c) $x \in (2, \infty)$

(d) $x \in (-2, 5)$

(e) $x \in (-1, 0)$

41. There are many correct answers here, but these answers are ones you might reasonably get:

(a) $\frac{3}{2}$

(b) $\frac{37}{9}$

42. We need to consider two cases

- Case 1: Show $|a| - |b| \leq |a - b|$

  Hint: Put $x = a - b$ and $y = b$, then use $|x + y| \leq |x| + |y|$.

- Case 2: Show $|a| - |b| \leq |a - b|$

Then use $|c| = \max\{c, -c\}$.

43. This looks harder than it really is. Use the triangle inequality and it's variations, but watch out for a possible divide-by-zero issue.

44. (a) Simply plug in $a = 0$. The right hand side will be positive. Repeat for $b = 0$.

(b) Remember to check the second derivative to make sure it's is a minimum, and not a maximum.

(c) Plus in $x = b^{\frac{1}{p-1}}$ and $q = p/(p-1)$ and use a bit of algebra to simplify.

(d) Replace x with a and, with a small amount of rearranging, voila!

45. For part (a), let $f(x) = \exp x - x - 1$. Show it has a minimum, and that the function is equal to zero at this minimum. Try a similar approach for part (b).

46. Since f is increasing, and $|x + y| \leq |x| + |y|$, then we know $f(|x + y|) \leq f(|x| + |y|)$. The rest falls out with a little bit of manipulation.

Supremum and Infimum

47. (a) $\sup S = 3$ and $\inf S = -3$. Both are in S.

(b) $\sup S = 5$ and $\inf S = -1$. Neither is in S.

(c) $\sup S = 2$ and $\inf S = -3$. Neither is in S.

(d) $\sup S = 0$ and $\inf S = -1$. Neither is in S.

(e) No supremum or infinum.

(f) $\sup S = \sqrt{7}$ and $\inf S = -\sqrt{7}$. Neither is in S.

48. Here is a proof of the first half of part (a).

Claim: If S is bounded, then $c + S$ is bounded.

Proof:	Assume S is bounded.	Premise
$\Rightarrow$	$\exists K \in \mathbb{R}\ \forall x \in S\ \ |x| \leq K$.	Definition of bounded
	$|x + c| \leq |x| + |c|$.	Triangle inequality
$\Rightarrow$	$|x + c| \leq |x| + |c| \leq K + |c|$.	
$\Rightarrow$	$c + S$ is bounded.	Definition of bounded ∎

You should be able to mimic this proof to show that cS is bounded.

49. The key to this proof is to write what you know. For instance, s_1 is a supremum, so we know two things:
   - s_1 is an upper bound.
   - if K is any upper bound, then $s_1 \leq K$

   If you write out the equivalent facts for s_2 you should be able to figure out some inequalities relating s_1 and s_2

Functions

50. (a) f is neither injective nor surjective.
   (b) f is injective since no city is the capital of two countries. But it is not surjective, since not every city is a capital of some country.
   (c) f is not injective because many people are born in the same country. However, f is surjective because every country has had at least one person born in it.
   (d) f is Bravo India Juliet Echo Charlie Tango India Victor Echo!

51. (a) $f(S) = \{2, 3, 6, 10\}$
   (b) There are a very large number of elements of $f^{-1}(\{1, 2, 3\})$. A few of them are “a”, “bun”, “is”, “cat”, “dog”, “no” ...

52. (a) **Claim:** If $f : A \to B$ and $g : B \to C$ are injective then $g \circ f$ is injective.

Assume	$f : A \to B$ is injective.	Premise	
$\Rightarrow$	$\forall\ a_1, a_2 \in A\ \ (f(a_1) = f(a_2) \Rightarrow a_1 = a_2)$	Definition of injective	(1)
Assume	$g : B \to C$ is surjective.	Premise	
$\Rightarrow$	$\forall\ b_1, b_2 \in B\ \ (f(b_1) = f(b_2) \Rightarrow b_1 = b_2)$	Definition of injective	(2)
Let	$a_1, a_2 \in A$ such that $g \circ f(a_1) = g \circ f(a_2)$	Premise	
$\Rightarrow$	$g(f(a_1)) = g(f(a_2))$	Definition of composition	
$\Rightarrow$	$f(a_1) = f(a_2)$	from(2)	
$\Rightarrow$	$a_1 = a_2$	from(1)	
Hence	$\big(g \circ f(a_1) = g \circ f(a_2)\big) \Rightarrow (a_1 = a_2)$		
Therefore	$g \circ f$ is injective.	Definition of injective	

Thus If $f : A \to B$ and $g : B \to C$ are injective then $g \circ f$ is injective. ∎

(b) We do the first couple of steps of the proof for you:

Assume	$f : A \to B$ is surjective.	Premise	
$\Rightarrow$	$\forall\ b \in B\ \exists a \in A\ \ f(a) = b$	Definition of surjective	(1)
Assume	$g : B \to C$ is surjective.	Premise	
$\Rightarrow$	$\forall\ c \in C\ \exists b \in B\ \ g(b) = c$	Definition of surjective	(2)
Let	$c \in C$	Premise	

$\vdots$

(c) This follows immediately from the previous two results.

53. We need a bijection that goes from $[0, 1]$ to $[3, 2]$. A linear function will do nicely: let $f(x) = ax + b$ and choose a and b so that $f(0) = 3$ and $f(1) = 2$

54. Clearly a linear function cannot be the bijection we need (why not?). Instead, try to think of a function f so that $f(x)$ is small when x is large, and vice versa. A very simple one will do.

Cardinality

55. We need to prove that f is injective and surjective.

To show that f is injective, pick two integers a and b, start with $f(a) = f(b)$, and show that $a = b$.

To show that f is surjective, pick a natural number m and find an integer z such that $f(z) = m$. You will need to consider even and odd m separately.

Problem Sheet 2: Sequences

ϵ–N Definition of Convergence

56. Hints:

    (a) $\dfrac{1}{n+2} < \dfrac{1}{n} < \epsilon$; Choose $M > \dfrac{1}{\epsilon}$

    (b) $\dfrac{1}{(n+4)^2} < \dfrac{1}{n} < \epsilon$; Choose $M > \dfrac{1}{\epsilon}$

    (c) $|a_n - \dfrac{1}{2}| = \dfrac{3}{2n^2+4} < \dfrac{3}{2n} < \epsilon$; Choose $M > \dfrac{3}{2\epsilon}$.

57. Limits:

    (a) 0

    (b) 2

    (c) $\dfrac{3}{2}$

    (d) $\dfrac{1}{2}$

58. Possible choices for M:

    (a) $M = 1$.

    (b) $M > \dfrac{1}{\epsilon^{1/p}}$

59. Let $\lim_{n\to\infty} u_n = L$. Use the fact that $\forall \epsilon > 0\ \exists M \in \mathbb{N}\ \forall n \in \mathbb{N}\ n > M \implies |u_n - L| < \dfrac{\epsilon}{2}$

60. Use a similar method to the previous question.

61. Use fact that $\exists K \in \mathbb{R}\ |a_n| < K$. Then pick $M > \dfrac{K}{\epsilon}$.

62. Use fact that $\exists K \in \mathbb{R}\ |a_n| < K$ and the fact that $\forall \epsilon > 0\ \exists M \in \mathbb{N}\ \forall n \in \mathbb{N}\ n > M \implies |b_n - 0| < \dfrac{\epsilon}{K}$.

63. Use $\forall \epsilon > 0\ \exists M_1 \in \mathbb{N}\ \forall n \in \mathbb{N}\ n > M_1 \implies |a_n - L| < \epsilon$ and
    $\forall \epsilon > 0\ \exists M_2 \in \mathbb{N}\ \forall n \in \mathbb{N}\ n > M_2 \implies |c_n - L| < \epsilon$.
    Choose $M = \max\{M_1, M_2\}$ and show that $L - \epsilon < a_n \le b_n \le c_n < L + \epsilon$ for $n > M$.

64. (a) Diverges to infinity.

    (b) Oscillates between 0 and 1 so consider $L < \frac{1}{2}$ and $L \ge \frac{1}{2}$.

65. (e) Let $n > K$ and solve $a_n < r^{n-K} a_K < \epsilon$ to get a suitable M. You might need to use log.

Special types of sequences

66. (a) Decreasing, bounded

    (b) Not monotonic, bounded

    (c) Increasing, bounded

    (d) Increasing, unbounded

    (e) Not monotonic, bounded

Using the sandwich rule

67. As $\exists K \in \mathbb{R}\ |a_n| < K$ we obtain $-\dfrac{K}{n} \le \dfrac{a_n}{n} \le \dfrac{K}{n}$ and $-K|b_n| \le a_n b_n \le K|b_n|$.

68. (a) $L = 0;\ \ 0 \le a_n \le \frac{1}{n}$

    (b) $L = 0;\ \ 0 \le a_n \le \frac{1}{n^3}$

    (c) $L = 0;\ \ -\frac{1}{n^3} \le a_n \le \frac{1}{n^3}$

    (d) $L = 0;\ \ 0 \le a_n \le \frac{1}{n}$

    (e) $L = 0;\ \ 0 \le a_n \le \frac{2^4}{4!} \cdot \frac{2}{n},\ n \ge 5$

    (f) $L = 0$

    (g) $L = 0$ as $\lim_{n\to\infty} \dfrac{a^n}{n!}\dfrac{n!}{n^n} = 0 \times 0$ by algebra of limits

Subsequences

69. Use the fact that if a_n is the mth term in the subsequence then $m < n$.
70. This follows from the contrapositive of Question 69.
71. (a) $-1,\ 1;$ divergent. (b) $-1,\ 0,\ 1;$ divergent. (c) $0;$ convergent.
72. For you to try!

Recursive Sequences

73. 2
74. $\dfrac{3+\sqrt{5}}{2}$
75. $\dfrac{5}{2}$
76. Show monotonic decreasing and bounded below by 0.
77. For $c > 1$ show bounded below by 1 and monotonic decreasing. For bounded, put $x = c^{1/n}$ and use a contrapositive proof, i.e. show $x \leq 1 \implies x^n = c \leq 1$. To show decreasing you may find it useful to show that $f(x) = e^{(\log c)/x} = c^{1/x}$ is decreasing for $c > 1$.

    For $c < 1$ show bounded above by 1 and monotonic increasing.
78. Use the same approach as in the previous problem.
79. Proof required for (a), (b), (c), (d) and (e).

Algebra of Limits

80. Either use the method in lectures for showing that $\lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = A + B$ or substitute $-b_n$ for b_n in this result.
81. Let $\epsilon = \dfrac{|L|}{2}$ and use $\big||a_n| - |L|\big| \leq |a_n - L| < \epsilon$.
82. Proof required.
83. Use the product rule from algebra of limits.

Cauchy Sequences

84. One possible M is from solving $|f_n - f_m| < \dfrac{7m+7n}{mn} = \dfrac{7}{n} + \dfrac{7}{m} < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2}$. Then pick $M > \dfrac{14}{\epsilon}$.
85. Proof required.
86. Proof required.
87. (a) Proof required. (b) Proof required. (c) Limit is approximately 0.289

Putting it together

88. (a) 0 (e) 0 (i) 0

    (b) 1 (f) e^3 (j) 0

    (c) 0 (g) e^{-3} (k) 1 (Use sandwich rule)

    (d) 0 (h) 0 (l) 1 (Use sandwich rule)

89. (a) 1

(b) 0

(c) 0 (Diff. of 2 squares.)

(d) $\dfrac{1}{2}$

(e) $\dfrac{1}{2}$

(f) 0

(g) 0

(h) $\frac{3}{2}$

(i) 0

(j) 0 (Use sandwich rule)

Sample Solutions: to give some idea of how to set out your answers.

(e)
$$\begin{aligned}\lim_{n\to\infty}\frac{n}{2n+3} &= \lim_{n\to\infty}\frac{1}{2+3/n}\\ &= \frac{1}{2+3\times 0} \qquad \text{Using the algebra of limits and } \lim_{n\to\infty}\frac{1}{n^p}=0\\ &= \frac{1}{2}\end{aligned}$$

(j)
$$\begin{aligned}\text{For all } n\in\mathbb{N}, \quad \frac{1-1}{n} &\le \frac{1+(-1)^n}{n} \le \frac{1+1}{n}\\ \text{so} \quad 0 &\le \frac{1+(-1)^n}{n} \le \frac{2}{n}\end{aligned}$$

Now $\lim\limits_{n\to\infty} 0 = 0$ and $\lim\limits_{n\to\infty}\dfrac{2}{n} = 2\times 0 = 0$ using the algebra of limits and standard limits.

Hence $\lim\limits_{n\to\infty}\dfrac{1+(-1)^n}{n} = 0$ by the sandwich rule.

90. (a) Unbounded: Let $K>0$. Choose $n>K^2$ to get $\sqrt{n}>K$.

(b) Use odd and even terms to get subsequences with limits -2 and $+2$.

(c) Unbounded: for $K>0$, as $\dfrac{10^{6n}}{n!}$ converges to 0 there exists n such that $\left|\dfrac{10^{6n}}{n!}-0\right|<\dfrac{1}{K}$; hence $\dfrac{n!}{10^{6n}}>K$.

(d) Use odd and even terms to get subsequences with limits 1 and $\frac{5}{3}$ respectively.

91. (a) $\dfrac{4}{3}$ (Sandwich rule)

(b) 1

(c) e^{-6}

(d) 0

(e) Dvgt; unbounded

(f) 1 (Sandwich rule)

(g) 0

(h) Dvgt; $a_{2n-1}\to -1$; $a_{2n}\to 1$

(i) 1

(j) 1

(k) $\dfrac{1}{3}$

(l) Dvgt; unbounded

(m) 1 (Sandwich rule)

(n) Dvgt: subsequence unbounded

(o) 1; Use $\log kn = \log k + \log n$

(p) 0 (Sandwich rule)

(q) $\dfrac{5}{2}$

(r) Dvgt; unbounded.

(s) 2 (Sandwich rule)

(t) Dvgt; look at 3 subsequences

Problem Sheet 3: Limits of Functions and Continuity

Limit Points

92. (a) 0
(b) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$
(c) $1, -1$
(d) None
(e) $\mathbb{R}$

Limits of Real-Valued Functions

93. Possible choices for δ:

(a) $\frac{\epsilon}{5}$
(b) $\min\{1, 2\epsilon\}$
(c) $\min\{\frac{\epsilon}{175}, 1\}$.

Proof for (a): Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{5}$.

$$\begin{array}{lrcl} \text{Assume} & 0 < |x-3| & < & \delta \\ \text{Then} & |5x-15| & = & 5|x-3| \\ & & < & 5\frac{\epsilon}{5} \\ & & = & \epsilon. \end{array}$$

Therefore $0 < |x-3| < \delta$ implies $|5x-15| < \epsilon$.

94. The limits are:

(a) -1
(b) 1
(c) $\frac{1}{17}$
(d) 2
(e) $\frac{5}{41}$
(f) Diverges.

For (c), use $\left|\frac{1}{1+x^2} - \frac{1}{17}\right| = \frac{|x-4||x+4|}{17(1+x^2)} < |x-4||x+4| < 9|x-4|$ for $|x-4| < 1$ to find $\delta = \min\{\frac{\epsilon}{9}, 1\}$.

For part (f), the proof is:

Let $L \in \mathbb{R}$. Choose $\epsilon = \frac{|L|}{2}$. Let $\delta > 0$. Choose $x = 2n\pi + \frac{\pi}{2}$ such that $n \in \mathbb{N}$ and $2n\pi > \max\{\frac{3|L|}{2}, \delta\}$.

Then $x > \delta$ and $|f(x) - L| = |2n\pi - L| \geq |2n\pi| - |L| > \frac{3|L|}{2} - |L| = \epsilon$ by a version of the triangle inequality.

So $\forall L \in \mathbb{R}\ \exists \epsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R}\ \ (x > \delta \text{ and } f(x) - L > \epsilon)$.

95. Possible values of δ are:

(a) $\sqrt[4]{\epsilon}$;

(b) Use $a - \frac{|a|}{2} < x < a + \frac{|a|}{2}$ to find $|x| < \frac{3}{2}|a|$.

Then use

$|x^4 - a^4| = |x-a||x^3 + ax^2 + a^2x + a^3| \leq |x-a|(|x|^3 + |a||x|^2 + a^2|x| + |a|^3)$
$< |x-a|(\frac{27}{8}|a|^3 + \frac{9}{4}|a|^3 + \frac{3}{2}|a|^3 + |a|^3) = \frac{65|a|^3}{8}|x-a|$.

Use this to choose $\delta = \min\left\{\frac{|a|}{2}, \frac{8\epsilon}{65|a|^3}\right\}$.

96. Possible choices for δ are:

 (a) $\delta = 1$

 (b) $\delta = \epsilon^{-1/p}$.

97. Proof required. (Hint: Use the techniques used in proving these for sequences.

98. The answers are:

 (a) $-\dfrac{1}{7}$;

 (b) $\dfrac{5}{3}$

 (c) Dvt; unbounded $(= 2 + \dfrac{1}{x-9})$

 (d) Dvt; (compare $\sin(2n\pi)$, $\sin((2n+1/2)\pi)$) **

 (e) Dvt (unbounded)

 (f) 0; (Sandwich: $-\dfrac{1}{|x|} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{|x|}$)

 (g) 4

 (h) Dvt; unbounded

** Note that this follows from the contrapositive of $\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} f(n) = L$ and $\lim_{n\to\infty} f(n) = L \implies \lim_{i\to\infty} f(n_i) = L$ where $f(n_i)$ is a subsequence of $f(n)$.

99. The limits are:

 (a) 2

 (b) $\dfrac{a}{b}$

 (c) 0

 (d) 2

Continuity

100. (a) Use $\forall\epsilon > 0\ \exists\delta_1$ s.t. $|x-a| < \delta_1 \implies |f(x) - f(a)| < \epsilon/2$

 and $\forall\epsilon > 0\ \exists\delta_2$ s.t. $|x-a| < \delta_2 \implies |g(x) - g(a)| < \epsilon/2$.

 Choose $\delta = \min\{\delta_1, \delta_2\}$ and use the triangle rule: $|f(x) - g(x) - f(a) + g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)|$.

 (b) Use the same idea as (a).

101. The function has domain $\mathbb{R}\backslash\{-1, 1\}$ and is continuous everywhere on its domain as it is the quotient of polynomials and the denominator is not equal to zero on the domain.

102. Redefine as

 (a) $f(x) = \begin{cases} \dfrac{\sqrt{x+2}-3}{x-7}, & x \neq 7 \\ \dfrac{1}{6}, & x = 7 \end{cases}$

 (b) $f(x) = \begin{cases} \dfrac{1/4 + 1/x}{4+x}, & x \neq -4 \\ -\dfrac{1}{16}, & x = -4 \end{cases}$

103. Is the function f is continuous on the interval $[-1, 1]$?

 (a) Yes

 (b) No

 (c) No

 (d) No

 (e) Yes

 (f) No

104. (c) As $\cos\theta$ is continuous at $\theta = 0$, $\lim_{\theta\to 0^+} \cos\theta = \lim_{\theta\to 0^+} \dfrac{1}{\cos\theta} = 1$ by the algebra of limits. Hence, by the sandwich rule $\lim_{\theta\to 0^+} \dfrac{\theta}{\sin\theta} = 1$.

 (d) The limit is 1. Note that $\lim_{\theta\to 0^-} \dfrac{\theta}{\sin\theta} = \lim_{\theta\to 0^+} \dfrac{\theta}{\sin\theta}$ because $\dfrac{\theta}{\sin\theta}$ is symmetric around the origin.

 Then $\lim_{\theta\to 0} \dfrac{\theta}{\sin\theta} = \lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ by the algebra of limits.

105. Firstly you need to evaluate the limit. Take care though, you will need to consider different cases. Try looking at $|x| < 1$, $|x| > 1$ and $|x| = 1$ separately.

106. First note that since $f(0+0) = f(0)f(0)$, we have either $f(0) = 0$ or $f(0) = 1$.

If $f(0) = 0$ it follows immediately that $f(x) = 0$ for all $x \in \mathbb{R}$ and hence the function is clearly continuous. So now consider the case when $f(0) = 1$.

Claim: f is continuous for all $x \in \mathbb{R}$.

Proof: Let $\epsilon > 0$ be fixed but arbitrary, and fix a point y.

Since we know f is continuous at $x = 0$, we know $\exists \delta_1$ such that

$|x - 0| < \delta_1 \implies |f(x) - 1| < \dfrac{\epsilon}{|f(y)|}$.

Now, choose $\delta = \delta_1$.

So, we take $|x - y| < \delta = \delta_1$ as our premise, which implies $|f(x-y) - 1| < \dfrac{\epsilon}{|f(y)|}$.

Now $|f(x) - f(y)| = |f(x-y+y) - f(y)| = |f(x-y)f(y) - f(y)| = |f(y)||f(x-y) - 1| < |f(y)|\dfrac{\epsilon}{|f(y)|} = \epsilon$

Therefore $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ and hence f is continuous everywhere.

107. Proof similar to previous question.

108. The converse is not true: Can you think of a function f that has $|f(x)| = |x| + 1$?

109. Here is an idea of the proof for the case $f(a) > 0$. The case for $f(a) = 0$ needs to be done separately.

$$
\begin{aligned}
|g(x) - g(a)| &= |\sqrt{f(x)} - \sqrt{f(a)}| \\
&= |\sqrt{f(x)} - \sqrt{f(a)}|\left|\frac{\sqrt{f(x)} + \sqrt{f(a)}}{\sqrt{f(x)} + \sqrt{f(a)}}\right| \\
&= \frac{|f(x) - f(a)|}{\sqrt{f(x)} + \sqrt{f(a)}} \\
&\leq \frac{|f(x) - f(a)|}{\sqrt{f(a)}}
\end{aligned}
$$

Now, since f is continuous at $x = a$, the numerator can be controlled to be less that $\sqrt{f(a)}\epsilon$.

Intermediate Value Theorem

110. (a) **Claim:** $\exists c \in \mathbb{R}$ such that $f(c) = 100$.

Proof: f is continuous for all $x \in \mathbb{R}$ since it is a polynomial.

Now $f(0) = -9 < 100$ and $f(10) = 561 > 100$.

Therefore, by the intermediate value theorem, $\exists c \in (0, 10)$ such that $f(c) = 100$.

(b) Define a function $f(x) = x^5 - 3x^4 - 2x^3 - x + 1$ and then find $f(0)$ and $f(1)$.

(c) Similar to the previous question.

(d) Try the function $g(x) = f(x) - x$.

(e) Create a new function that will be zero when $f(x+1) = f(x)$.

Problem Sheet 4: Differentiability

Differentiability

111. (a) $2x\sqrt{1-x^3} - \dfrac{3x^4}{2\sqrt{1-x^3}}$ (c) $\dfrac{1}{x} - \dfrac{x}{1+x^2}$ (e) $\dfrac{2}{(x+1)^2}$

(b) $3x^2e^{x^2} + 2x^4e^{x^2}$ (d) $\dfrac{1-\frac{5}{3}x^2}{(1-x)^{\frac{4}{3}}}$ (f) $\dfrac{-3\sin 2x}{\sin^4 x}$

112. (a) $f'(x) = \dfrac{1}{(1+|x|)^2}$ (Note: you need to check the case of $f'(0)$ separately.)

(b) $f'(x) = \begin{cases} -\cos(x) & x < 0 \\ \cos(x) & x > 0 \end{cases}$

f is not differentiable at the origin.

(c) $f'(x) = \begin{cases} 4x^4 & x \leq 0 \\ 4\sin^3 x\cos(x) & x > 0 \end{cases}$

(d) f is only differentiable at $x = \dfrac{\pi}{2} + n2\pi, n \in \mathbb{Z}$, where the derivative is 0.

(e) $f'(x) = 0$ for all $x \notin \mathbb{Z}$. At integer values of x, f is right differentiable, but not left differentiable.

113. (a) It is continuous at the origin, but not differentiable.

(b) It is continuous and differentiable at the origin.

114. The method used in the following question will help with this question

115. We prove part (a).

Claim: If f is differentiable at $x = 0$, then f is differentiable on $\mathbb{R}$ and $f'(x) = f'(0)f(x)$.

Proof: First note that $f(0+0) = f(0)f(0)$ implies that either $f(0) = 0$ or $f(0) = 1$.

In the case that $f(0) = 0$ we have that

$$f(x) = f(x+0) = f(x)f(0) = 0 \quad \forall x \in \mathbb{R}$$

So clearly in that case f is differentiable (constant function).

Consider now the case that $f(0) = 1$. Since f is differentiable at $x = 1$ we know that

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-1}{h}$$

exists.

Now we try to find the derivative at all points $x \in \mathbb{R}$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h} \\ &= f(x)\lim_{h\to 0}\frac{f(h)-1}{h} \\ &= f(x)f'(1) \end{aligned}$$

Therefore f is differentiable for all $x \in \mathbb{R}$ with derivative $f'(x) = f(x)f'(1)$. ■

116. Proof required.

117. Add and subtract $f(c)$ to the numerator, and break it into two pieces.

118. Consider the unit step function: $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

119. Here is a very rough outline of the main calculation in the proof.

The key is to try and make derivative-like terms appear since we know the function is differentiable at c. Differentiability at c means

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

exists. In other words, using the definition of limit, we can make the term

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right|$$

smaller than any number ϵ. With that in mind, we do some manipulation:

$$\begin{aligned}
&|f(v) - f(u) - (v-u)f'(c)| \\
&\quad = |\,(f(v) + vf'(c)) - (f(u) + uf'(c))\,| \\
&\quad = |\,(f(v) - f(c) + (v-c)f'(c)) \\
&\qquad\qquad - (f(u) - f(c) + (u-c)f'(c))\,| \\
&\quad \leq |f(v) - f(c) + (v-c)f'(c)| \\
&\qquad\qquad + |f(u) - f(c) + (u-c)f'(c)| \\
&\quad = |v-c| \left| \frac{f(v) - f(c)}{v - c} - f'(c) \right| \\
&\qquad\qquad + |u-c| \left| \frac{f(u) - f(c)}{u - c} - f'(c) \right|
\end{aligned}$$

Now that we have some derivative terms, we know we can control them, and all that's left to do is some fine details (such as picking δ)

120. For part (b), you need to quote a theorem from class. For part (c), here is a rough outline.

Since the maximum doesn't occur at the end points, we have that

$$f(m+h) \leq f(m)$$

Rearrange this and divide by h to get a derivative-like term. You will need to consider the case $h > 0$ and $h < 0$ separately, so you will be looking at left and right limits. If you can prove $f'(m) \geq 0$ and $f'(m) \leq 0$, then you should be able to finish off the proof from there.

Mean Value Theorem

121. Try making functions with sharp-points or jumps for parts (b) and (c)

122. For all of these, we know a value of c exists in the interval $(-1, 4)$ that has the following property.

    (a) $f'(c) = 3$

    (b) $g'(c) = 5^{\alpha - 1}$

    (c) Trick question! The MVT doesn't apply to this function because it is not differentiable at $x = 1$.

    (d) $j'(c) = (\log 7)/5$

123. (b) **Claim:** $|\log x - \log y| \leq \frac{1}{2}|x-y| \qquad \forall x, y \in [2, \infty)$

Proof: Without loss of generality, let $x > y \geq 2$. The case $y = x$ is trivial.

We try to identify parts that look like the mean value theorem. So, let $f(x) = \log x$.

f is continuous and differentiable everywhere on its domain. So we can apply the mean value theorem on the interval $[y, x]$.

Hence $\exists c \in (y, x)$ such that

$$\begin{aligned} \frac{f(x)-f(y)}{x-y} &= f'(c) \\ \Rightarrow \frac{\log x - \log y}{x-y} &= \frac{1}{c} \\ \Rightarrow \frac{|\log x - \log y|}{|x-y|} &= \frac{1}{c} \\ &< \frac{1}{2} \text{ since } c > y \geq 2 \\ \Rightarrow |\log x - \log y| < \tfrac{1}{2}|x-y| \end{aligned}$$

Combining with the $x = y$ case we get

$|\log x - \log y| \leq \frac{1}{2}|x-y|$ ■

(d) **Claim:** $e^x > 1 + x \qquad \forall x > 0$

Proof: This time it's harder to see the mean-value-form, until we recall that $e^0 = 1$. Hence, let $f(x) = e^x$ which is continuous and differentiable everwhere. Now we can apply the MVT on the interval $[0, x]$.

Therefore $\exists c \in (0, x)$ such that

$$\begin{aligned} \frac{f(x)-f(y)}{x-y} &= f'(c) \\ \Rightarrow \frac{e^x - e^0}{x} &= e^c \\ &> 1 \text{ since } c > 0 \\ \Rightarrow e^x - 1 &> x \\ e^x &> 1 + x \end{aligned}$$

And we are done! ■

124. Since you know $36 < c < 40$, you should be able to find bounds for $f'(c)$ as well.

125. This proof is actually very simple. Let $x \in (a, b]$ and apply the MVT to f on the interval $[a, x]$. The result falls right out.

126. All you need to do is apply the MVT on an appropriate interval.

127. Use the Mean Value Theorem.

Problem Sheet 5: Integration

The Riemann Integral

128. (a) $\sup\{f(x) : x \in [-2,-1]\} = 5$, $\sup\{f(x) : x \in [-1,1]\} = 4$, $\sup\{f(x) : x \in [1,4]\} = 2$

$$\therefore U(f,P) = 5(-1+2) + 4(1+1) + 2(4-1) = 19$$

(b) $L(f,P) = 5$

(c) $\displaystyle\int_{-2}^{4} f\,dx = 12$

(d) $5 \leq 12 \leq 19$

(e) This will depend on the refinement you choose.

129. (a) f is continuous, so it is integrable.

(b) g is continuous, so it is integrable.

(c) h is not integrable because the upper sums will always be governed by x^2, while the lower sums will always be governed by $-x^2$. This is similar to Dirichlet's Function.

(d) p is not integrable because it is unbounded: the definition of the Riemann integral requires the function be bounded.

(e) q is integrable because it is piecewise continuous and bounded.

(f) r is integrable because it is piecewise continuous and bounded.

130. For the partition to be sensible, we need $0 < a < 1$. With the suggested partition, we get $U(f,P) = 1$ and $L(f,P) = 1 - a$. So we have

$$U(f,P) - L(f,P) = 1 - (1-a) = a$$

Now simply choose a value of a that will make that expression less than ϵ and then you can write up the proof.

131. Try the partition $P = 0, \frac{1}{2}, a, 1$ where $\frac{1}{2} < a < 1$. Drawing a sketch will make this a lot easier.

So we have $U(f,P) = a - \frac{1}{2}$ and $L(f,P) = 0$. Hence

$$U(f,P) - L(f,P) = a - \tfrac{1}{2}$$

Now simply choose a value of a that will make that expression less than ϵ and then you can write up the proof.

132. We give some hints on how to do each step.

(a) f must be bounded because that is a requirement of the definition of integrability.

(b) Since $\dfrac{\epsilon}{f(b) - f(a)} > 0$ we can use the density of real numbers to prove the existence of k.

(c) Note that since f is increasing, we know that $\sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i)$ and similarly for the infimum.

(d) For this step, use the fact $x_i - x_{i-1} < k$ to bound the expression.

133. It is easiest if you consider the case $k > 0$ first. You will need to show that for any partition P,

$$U(kf,P) = kU(f,P) \text{ and } L(kf,P) = kL(f,P)$$

You may also need a result from question 48 in the "Logic, Sets, Numbers and Proof" problem sheet.

The case $k = 0$ is very straightforward.

The $k < 0$ is a little more complex. What will $U(kf,P)$ equal?

134. For this proof, you will need to argue that

$$L(f) \leq L(g) \text{ and } U(f) \leq U(g)$$

135. We will give hints for each step.

(a) Because f is continuous.

(b) You need to argue that a continuous function on a closed interval achieves a maximum and minimum value. Can you give an example of a continuous function on an open interval which *doesn't* achieve a maximum and a minimum?

(c) $U(f, P) = M(b-a)$ and $L(f, P) = m(b-a)$ which gives us

$$m(b-a) \leq I \leq M(b-a)$$

(d) We now know that

$$m \leq \frac{I}{b-a} \leq M$$

The intermediate value theorem will allow you to conclude that $\exists c \in [a, b]$ such that

$$f'(c) = \frac{I}{b-a}$$

(e) You actually need the theorem we just proved (Mean Value Theorem for Integrals) in order to prove the Fundamental Theorem of Calculus!

136. Make a partition

$$P = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$$

and let $x_i = \dfrac{i\pi}{2n}$. Then you need to interpret the expression as a Riemann sum of some function f.

The Fundamental Theorem of Calculus

137. The integration by parts formula is just the integral of the product rule. Give it a try.

138. Use integration by parts followed by the triangle inequality. Use the fact that, as $\dfrac{f'(x)\sin(\lambda x)}{\lambda}$ is continuous, from the MVT for integrals we know that there exists $c \in [a, b]$ such that

$$\int_a^b \frac{f'(x)\sin(\lambda x)}{\lambda}\, dx = (b-a)\frac{f'(c)\sin(\lambda c)}{\lambda}.$$

Finally use the sandwich rule.

139. The Fundamental Theorem of Calculus tells us that since f is continuous, then

$$F(x) = \int_0^x f(t)\ dt$$

satisfies $F'(x) = f(x)$. So, use Integration By Parts on

$$\int_0^b f(x)(b-x)\ dx$$

140. (a) Use the Fundamental Theorem of Calculus and argue that $F(x) = \int_0^x f(t)\,dt$ must be non-decreasing. (What sign is its derivative?) (b) You cannot drop the condition that f is continuous on $[a, b]$.

141. You will need to use integration by parts twice on I.

Note: It is possible to improve the error bound to $|E| \leq \frac{(b-a)^3}{12} M$ using a different approach. See a calculus textbook for details if interested.

Improper Integration

142.

(a) Improper, infinite interval of integration.
(b) Improper, infinite discontinuity at $\pi/2$.
(c) Proper.
(d) Improper, infinite discontinuity at $\frac{1}{3}$.
(e) Improper, infinite interval of integration.
(f) Improper, infinite discontinuity at 0.
(g) Proper.
(h) Improper, infinite discontinuity at $\frac{1}{2}$
(i) Proper.

143.

(a) 2
(b) 6
(c) divergent
(d) $\frac{1}{4}$
(e) divergent
(f) 1
(g) $\frac{\pi}{2}$
(h) $2\sqrt{5}$
(i) $\log 2$

144.

(a) convergent
(b) divergent
(c) convergent
(d) divergent
(e) convergent
(f) divergent

145. For the inductive step, use integration by parts.

146. $u > 0$ and $v > 0$

Problem Sheet 6: Series

Series

147. (a) $1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{5125}{3600}$; converges
(b) $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}$; diverges
(c) diverges
(d) converges
(e) converges
(f) converges
(g) converges

148. a_n converges to $\dfrac{2}{3}$ which means the series $\displaystyle\sum_{n=1}^{\infty} a_n$ must diverge. Why?

149. (a) Converges to 1
(b) Diverges
(c) Converges to $\frac{1}{2}$

150. (a) Converges to 5
(b) Converges to $\dfrac{12}{17}$
(c) Converges to $\dfrac{1}{12}$
(d) This one is trickier. Note that $\cos\left(\frac{n\pi}{2}\right) = 0$ whenever n is odd, so we can delete all the odd terms and introduce $m = 2n$. If you do this the series becomes

$$\sum_{m=0}^{\infty} (-1)^m \frac{4^m}{7^{2m-1}} = \frac{7}{53}$$

151. (a) 1
(b) $\frac{2}{3}$ and $\frac{4}{9}$
(d) $a_n = 2^{n-1}$
(e) $b_n = \dfrac{1}{3^n}$
(c) The total length removed is

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$$

Hence the length of the Cantor set is zero!!!

Convergence Tests

152. Here is what you should compare to (or a multiple of that), and the conclusion.

(a) $\dfrac{1}{n^2}$, convergent.
(b) $\dfrac{1}{n}$, divergent.
(c) $\dfrac{1}{n^{3/2}}$, convergent.
(d) $\dfrac{1}{4^n}$, convergent.
(e) $\dfrac{1}{3^n}$, convergent.
(f) $\dfrac{1}{n^2}$, convergent.
(g) $\dfrac{1}{n^{2/3}}$, divergent.

153. In the following, $p = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

(a) $p = e^{-1} < 1$, convergent.
(b) $p = e^{-1} < 1$, convergent.
(c) $p = \frac{1}{2} < 1$, convergent.
(d) $p = 2 > 1$, divergent.

154. (a) Diverges
(b) Diverges
(c) Diverges
(d) Converges

155. (a) Converges
 (b) Diverges - note that you can't use the alternating series test to conclude this.
 (c) Diverges - note that you can't use the alternating series test to conclude this.
 (d) Converges

156. (a) Converges
 (b) Diverges
 (c) Converges
 (d) Converges to log(2)
 (e) Converges
 (f) Converges
 (g) Diverges
 (h) Diverges
 (i) Converges
 (j) Converges
 (k) Diverges
 (l) Converges
 (m) Diverges
 (n) Converges
 (o) Converges
 (p) Diverges

(o) Hints: Use $\lim_{n\to\infty} \frac{\log n}{n^{1/4}} = 0$ to conclude that $\frac{\log n}{n^{1/4}}$ is bounded so that $0 \leq \frac{\log n}{n^{1/4}} \leq K$ for some $K > 0$.

Then $\frac{(\log n)^2}{n^2} \leq K^2 \frac{1}{n^{3/2}}$ and then use the comparison test.

(p) Hints: Show $\lim_{n\to\infty} \frac{n}{n^{1+1/n}} = 1$. Let $\epsilon = \frac{1}{2}$.

Then $\exists M \in \mathbb{N}$ s.t. $n > M \implies \frac{n}{n^{1+1/n}} > 1 - \epsilon = \frac{1}{2}$. Then $\frac{1}{n^{1+1/n}} \geq \frac{1}{2n}$ for $n \geq M$. Use the comparison test.

157. (a) Converges conditionally
 (b) Diverges
 (c) Diverges
 (d) Converges absolutely

158. Hint: Use the fact that a_n is positive and decreasing. Consider the sequence defined by $b_1 = a_k, b_2 = a_k - a_{k+1}, b_3 = a_k - a_{k+1} + a_{k+2}, \ldots$ and show that this is decreasing.

159. To do this, you use the result proved in the previous question. So, you just need to find the first term in the series smaller than the desired accuracy. (a) 5 (b) 7

160. This one is simpler than it looks. Remember that if a sequence is convergent, then it is bounded. Use this fact on the sequence $\frac{a_n}{b_n}$ and apply the comparison test.

161. Yes this is true. Here is an outline of the argument.

Let $A_n = \sum_{i=1}^{n} a_i$ and $B_n = \sum_{i=1}^{n} b_i$.

Since both the series are convergent, we have $\lim_{n\to\infty} A_n = L$ and $\lim_{n\to\infty} B_n = M$.

Now consider the partial sum

$$\begin{aligned} S_n &= \sum_{i=1}^{n}(a_i + b_i) \\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \\ &= A_n + B_n \\ \text{Hence } \lim_{n\to\infty} S_n &= \lim_{n\to\infty}(A_n + B_n) \\ &= \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n \\ &= L + M \end{aligned}$$

Since the limit exists, the partial sums converge, and hence $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

162. This one is actually false!

To help you find a counterexample, choose

$$a_n = \frac{(-1)^n}{\sqrt{n}}$$

Now you pick b_n.

163. Here is a hint. Assume $\sum_{n=1}^{\infty} |a_n|$ converges, and consider the series $\sum_{n=1}^{\infty}(a_n + |a_n|)$. Is the second series a positive series? If so, try using the comparison test.

Power Series

In this section, $R =$ the radius of convergence

164. (a) $[-1, 1]$, $R = 1$
(b) $[-1, 1)$, $R = 1$
(c) $(-2, 2)$, $R = 2$
(d) $(-9, -3)$, $R = 3$
(e) $\mathbb{R}$, $R = \infty$
(f) $(-\frac{1}{3}, \frac{1}{3})$, $R = \frac{1}{3}$

165. (a) $\dfrac{1}{(1-x)^2}$
(b) $\dfrac{x}{(1-x)^2}$
(c) $-\log(1-x)$
(d) $\dfrac{2x^2}{(1-x)^3}$

166. (a) Series converges by ratio test; $\displaystyle\int_{x=1}^{\infty} \frac{x}{e^x}\,dx$ converges by the integral test; $\displaystyle\lim_{x\to\infty} \frac{x}{e^x} = 0$.
(b) ♣ $R = 1$; $\displaystyle\lim_{n\to\infty} n^p x^n = 0$ for $|x| < 1$.

Taylor and Maclaurin Series

167. (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$ converges for $x \in \mathbb{R}$
(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ converges for $x \in \mathbb{R}$
(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$ converges for $x \in (-1, 1)$
(d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ converges for $x \in \mathbb{R}$
(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(2x)^n}{n}$ converges for $x \in (-\frac{1}{2}, \frac{1}{2}]$
(f) $\displaystyle\sum_{n=0}^{\infty}(-1)^n (n+1) x^n$ converges for $x \in (-1, 1)$

168. (a) $\displaystyle\frac{1}{\sqrt{2}}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n (x-\frac{1}{4}\pi)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n (x-\frac{1}{4}\pi)^{2n+1}}{(2n+1)!}\right\}$
(b) $\displaystyle\frac{1}{2}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n (x-\frac{1}{3}\pi)^{2n}}{(2n)!} - \sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n (x-\frac{1}{3}\pi)^{2n+1}}{(2n+1)!}\right\}$
(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{2^{n+1}}$
(d) $\displaystyle e^{-3}\sum_{n=0}^{\infty} \frac{(x+3)^n}{n!}$

169. (a) $\displaystyle e^{2x} = e^{-2}\sum_{n=0}^{\infty} \frac{(x+1)^n 2^n}{n!}$
(b) $\displaystyle \log x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$

170. (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{\theta^{4n+2}}{(2n+1)!}$
(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} 3^{2n-1} \frac{x^{2n}}{(2n-1)!}$
(c) $\displaystyle\frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n 2^{2n} \frac{x^{2n}}{(2n)!}$
(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} t^n$
(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} z^{n+1}$

171. (a) Apply the ratio test as usual.

(b) To find the the Maclaurin series for $\sqrt{1+x}$, set $\alpha = \frac{1}{2}$

(c) Use the result from the previous question with $\alpha = -\frac{1}{2}$, and replace x with $-x^2$.

(d) Integrate the result you just obtained.

(e) Plug in an appropriate value of x to get

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 2^7} + \dots$$

172. (a) $-\dfrac{1}{24}$ (b) $\dfrac{1}{3}$ (c) 0

173. (a) $P_n(x) = \sum_{k=0}^{n} (-1)^k x^k$, $R_n(x) = \dfrac{(-1)^{n+1} x^{n+1}}{(1+c)^{n+2}}$

(b) With $0 < x < 1$ and $0 < c < x$ we have

$$\begin{aligned} |R_n(x)| &= \left| \frac{(-1)^{n+1} x^{n+1}}{(1+c)^{n+2}} \right| \\ &= \frac{|x|^{n+1}}{|1+c|^{n+2}} \\ &\leq \frac{|x|^{n+1}}{|1+0|^{n+2}} \\ &= |x|^{n+1} \to 0 \text{ as } n \to \infty \end{aligned}$$

(c) Finally, take the limit as $n \to \infty$ of

$$f(x) = P_n(x) + R_n(x)$$

174. (a) $P_n(x) = \sum_{k=0}^{n} \dfrac{x^k}{k!}$, $R_n(x) = e^c \dfrac{x^{n+1}}{(n+1)!}$

(b) As c is between 0 and x, $-|x| < c < |x|$.

So $e^c < e^|x|$ and $|R_n(x)| \leq e^|x| \dfrac{|x|^{n+1}}{(n+1)!}$.

As $\lim_{n\to\infty} \dfrac{|x|^{n+1}}{(n+1)!} = 0$, the result follows from the sandwich rule.

(c) Finally, take the limit as $n \to \infty$ of

$$f(x) = P_n(x) + R_n(x)$$

175. (a) $\sin(x^2) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{4n+2}$

(b) $\displaystyle\int_0^t \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!} t^{4n+3}$

(c) $\displaystyle\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \approx 0.31028$

Computer gives 0.310268. . .

176. (a) $17/18 \approx 0.9444$ (b) $1703/1800 \approx 0.94611$

177. This is a challenge question.

178. This is also a challenge question.

Problem Sheet 7: Fourier Series

179. (a)

$$\begin{aligned}
\langle \cos nx, \sin mx \rangle &= \int_{-\pi}^{\pi} \cos nx \sin mx \, dx \\
&= \int_{-\pi}^{\pi} \frac{1}{2} \left(\sin\left((m-n)x\right) + \sin\left((m+n)x\right) \right) \, dx \\
&= \begin{cases} \left[\frac{1}{2} \left(-\frac{\cos((n-m)x)}{n-m} - \frac{\cos((n+m)x)}{n+m} \right) \right]_{-\pi}^{\pi} & n \neq m \\ \left[\frac{1}{2} \left(-\frac{\cos((n+m)x)}{n+m} \right) \right]_{-\pi}^{\pi} & n = m \end{cases} \\
&= 0 \text{ as cos is an even function so that } \cos k\pi - \cos(-k\pi) = 0
\end{aligned}$$

(b)

$$\begin{aligned}
\langle \sin nx, \sin mx \rangle &= \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \\
&= \int_{-\pi}^{\pi} \frac{1}{2} \left(\cos((n-m)x) - \cos((n+m)x) \right) \, dx
\end{aligned}$$

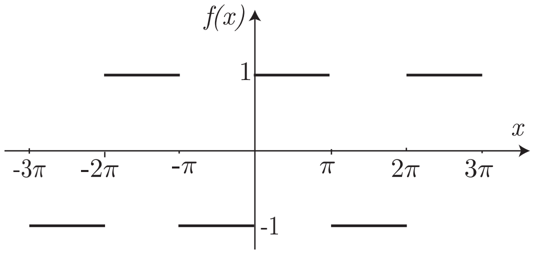
For $n = m$

$$\begin{aligned}
\langle \sin nx, \sin mx \rangle &= \int_{-\pi}^{\pi} \frac{1}{2} \left(1 - \cos(2nx) \right) \, dx \\
&= \left[\tfrac{1}{2} \left(x - \sin(2nx) \right) \right]_{-\pi}^{\pi} \\
&= \tfrac{1}{2} \left((\pi - 0) - (-\pi - 0) \right) = \pi.
\end{aligned}$$

For $n \neq m$

$$\begin{aligned}
\langle \sin nx, \sin mx \rangle &= \left[\tfrac{1}{2} \left(\frac{\sin((n-m)x)}{n-m} - \frac{\sin((n+m)x)}{n+m}) \right) \right]_{-\pi}^{\pi} \\
&= 0.
\end{aligned}$$

180. (a)



$$\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{0} -1 \, dx + \frac{1}{2\pi} \int_{0}^{\pi} 1 \, dx \\
&= \frac{1}{2\pi} \left[-x \right]_{-\pi}^{0} + \frac{1}{2\pi} \left[x \right]_{0}^{\pi} \\
&= \frac{1}{2\pi} \left[-\pi + \pi \right] \\
&= 0
\end{aligned}$$

For $n \geq 1$,

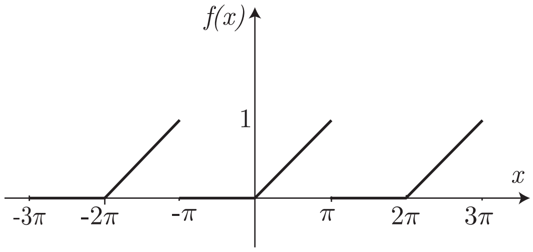
$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{0} -\cos(nx) \, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos(nx) \, dx \\
&= \frac{1}{\pi} \left[\frac{-\sin(nx)}{n} \right]_{-\pi}^{0} + \frac{1}{\pi} \left[\frac{\sin(nx)}{n} \right]_{0}^{\pi} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx \\
&= \frac{1}{\pi}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_{0}^{\pi} \\
&= \frac{1}{n\pi}\left(1-\cos(-n\pi)\right) + \frac{1}{n\pi}\left(-\cos(n\pi)+1\right) \\
&= \frac{2(1-\cos(n\pi))}{n\pi} \\
&= \frac{2(1-(-1)^n)}{n\pi} \\
&= \begin{cases} 0, & n \text{ even} \\ \dfrac{4}{n\pi}, & n \text{ odd} \end{cases}
\end{aligned}$$

The Fourier series is

$$\begin{aligned}
a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) &= \frac{4}{\pi}\sin x + \left(\frac{4}{3\pi}\right)\sin(3x) + \left(\frac{4}{5\pi}\right)\sin(5x) + \dots \\
&= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}
\end{aligned}$$

(b)



$a_0 = \frac{\pi}{4}$, $\quad a_n = \begin{cases} 0, & n \text{ even} \\ \dfrac{-2}{n^2\pi}, & n \text{ odd} \end{cases} \quad b_n = \frac{(-1)^{n+1}}{n}$

The Fourier series is:

$$\begin{aligned}
& a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) \\
= \quad & \frac{\pi}{4} - \sum_{n=1}^{\infty}\frac{2}{(2n-1)^2\pi}\cos((2n-1)x) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)
\end{aligned}$$

181. (a) For $n \in \mathbb{N}$,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Let $g_n(x) = f(x)\sin nx$. Now

$$g_n(-x) = f(-x)\sin(-nx) = f(x)(-\sin nx) = -g_n(x).$$

So g_n is an odd function. Hence

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} g_n(x)\,dx = 0.$$

(b) For $n \in \mathbb{N}$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

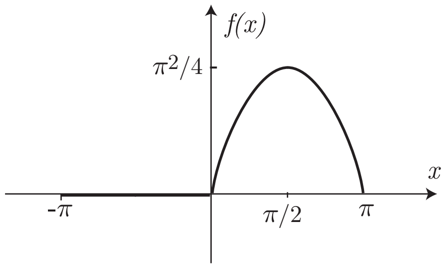
Let $h_n(x) = f(x)\cos nx$. Then

$$h_n(-x) = f(-x)\cos(-nx) = -f(x)\cos nx = -h_n(x).$$

So h_n is an odd function. Hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} h_n(x)\,dx = 0.$$

182. (a)



(b) $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{\pi^2}{12}$.

Using integration by parts twice we obtain: $a_n = \begin{cases} -\dfrac{2}{n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $\quad b_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{\pi n^3} & n \text{ odd} \end{cases}$

By the Fourier series theorem, as f is piecewise differentiable with $f(\pi) = f(-\pi)$, we obtain:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \sin nx \ + b_n \cos nx) = \frac{\pi^2}{12} - \sum_{n \text{ even}} \frac{2}{n^2}\cos nx + \sum_{n \text{ odd}} \frac{4}{\pi n^3}\sin nx$$

or

$$f(x) = \frac{\pi^2}{12} - \sum_{n=1}^{\infty}\frac{2}{(2n)^2}\cos(2nx) + \sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)^3}\sin((2n-1)x).$$

(c) Putting $x = 0$ gives

$$f(0) = \frac{\pi^2}{12} - \sum_{n=1}^{\infty}\frac{2}{(2n)^2}\cos(0) + \sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)^3}\sin(0)$$

or

$$0 = \frac{\pi^2}{12} - \sum_{n=1}^{\infty}\frac{2}{(2n)^2} = \frac{\pi^2}{12} - \sum_{n=1}^{\infty}\frac{1}{2n^2}.$$

Multiplying by 2 and rearranging gives:

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

183. (a) $f(x) = \displaystyle\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}\sin(nx)}{n}$

(b) From the Fourier series theorem the series converges to f for $0 \le x < \pi$ and $\pi < x \le 2\pi$.

(c) Proof required.