MAST30001 Stochastic Modelling

$\boxed{\textbf{Assignment 2}}$

Please complete the Plagiarism Declaration Form on LMS.

Don't forget to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **3pm, Friday Oct 20** in the appropriate assignment box at the north end of Peter Hall building (near Wilson Lab).

Marks may be lost where answers are not clear and concise (or where lacking in explanation).

1. A CTMC $(X_t)_{t\geq 0}$ with state space $\mathcal{S} = \{1,2,3,4\}$ has the following generator

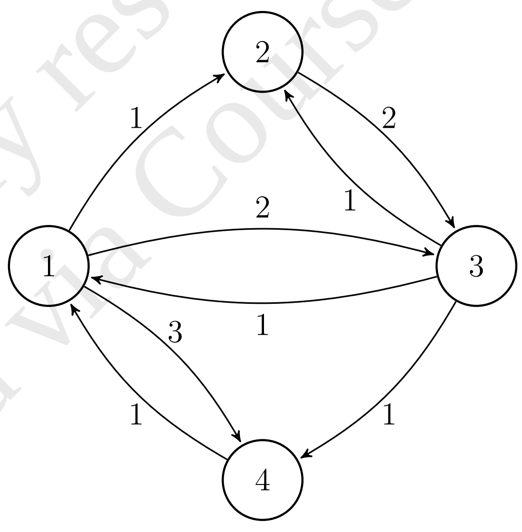
$$\begin{pmatrix} q_{1,1} & 1 & 2 & 3 \\ 0 & q_{2,2} & 2 & 0 \\ 1 & 1 & q_{3,3} & 1 \\ 1 & 0 & 0 & q_{4,4} \end{pmatrix}.$$

[8 Marks]

(a) What are the values of $q_{i,i}$ for $i = 1,2,3,4$?

$q_{1,1} = -6$, $q_{2,2} = -2$, $q_{3,3} = -3$, $q_{4,4} = -1$ **[1/2 Marks]**

(b) Draw the transition diagram for this chain.



[1/2 Marks]

(c) Is this chain reducible?

No, every state communicates with every state **[1 Marks]**

(d) Find the 1 step transition matrix for the jump chain.

$$P = \begin{pmatrix} 0 & 1/6 & 2/6 & 3/6 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

[1 Marks]

(e) Given that $X_t = 4$, find the distribution of $T^* = \inf\{s > t : X_s \neq 4\}$.
$T^* = t + Y$ where $Y \sim$exp(1). **[1 Marks]**

(f) Let $T(4) = \inf\{t \geq 0 : X_t = 4\}$. Find $\mathbb{E}[T(4)|X_0 = i]$.
This is the expected hitting time of 4, which is equal to 0 if $i = 4$ and otherwise it can be found by solving

$$
\begin{aligned}
m_{1,4} &= \frac{1}{6} + \frac{1}{6}m_{2,4} + \frac{2}{6}m_{3,4} \\
m_{2,4} &= \frac{1}{2} + m_{3,4} \\
m_{3,4} &= \frac{1}{3} + \frac{1}{3}m_{1,4} + \frac{1}{3}m_{2,4}
\end{aligned}
$$

Solving gives

$$m_{1,4} = \frac{5}{6}, \qquad m_{2,4} = \frac{10}{6}, \qquad m_{3,4} = \frac{7}{6}.$$

[1 Marks]

(g) Let $T' = \inf\{t > T(4) : X_t \neq 4\}$. Find the distribution of $X_{T'}$.
$\mathbb{P}(X_{T'} = 1) = 1$ **[1 Marks]**

(h) Starting from the uniform initial distribution, find the limiting proportion of time spent in state i for each i.
This is an irreducible finite state CTMC so the limiting proportion of time spent in i is π_i. The full balance equations are

$$
\begin{aligned}
6\pi_1 &= \pi_4 + \pi_3 \\
2\pi_2 &= \pi_1 + \pi_3 \\
3\pi_3 &= 2\pi_2 + 2\pi_1 \\
\pi_4 &= 3\pi_1 + \pi_3
\end{aligned}
$$

Solving gives $\pi = (4/33, 5/33, 6/33, 18/33)$ **[1 Marks]**

(i) Find the limiting distribution for this chain if $\mathbb{P}(X_0 = 1) = 1$.
This is an irreducible finite-state CTMC, so the limiting disribution is the stationary distribution $\pi = (4/33, 5/33, 6/33, 18/33)$ **[1 Marks]**

2. The Joker escaped from Batman since you handed in assignment 1. Now Batman chases the Joker around a regular polyhedron with $k \geq 3$ corners (the corners are labelled $0, 1, 2, \dots, k-1$ clockwise around the polyhedron). At each corner of the polyhedron there is a traffic light. Unfortunately, the traffic lights have been hijacked by the Riddler, who lets them turn green only for an instant according to the following rule: Let $(N_t)_{t\geq 0}$ be a Poisson process of rate λ. At each jump time T_j of this process the Riddler chooses a corner $i \in \{0, \dots, k-1\}$ uniformly at random and turns the traffic light at that corner green for an instant. Both Batman and the Joker are responsible drivers, so they move clockwise to the next corner i' if and only if they are at the corner i when the traffic light at i turns green. (Note that $i' = 0$ if $i = k-1$ and otherwise $i' = i + 1$). Batman catches the Joker as soon as they are at the same corner of the polyhedron (but the Riddler continues his control of the lights regardless). **[6 Marks]**

(a) What is the distribution of the number of green lights that occur at corner i up to time t?
Poisson$(\lambda t/k)$ **[1 Marks]**

(b) Find $\mathbb{P}(\cap_{r=1}^{m}\{N_r = 2r\})$.
This is the same as $\mathbb{P}(\cap_{r=1}^{m}\{N_r - N_{r-1} = 2\})$ *which is equal to*

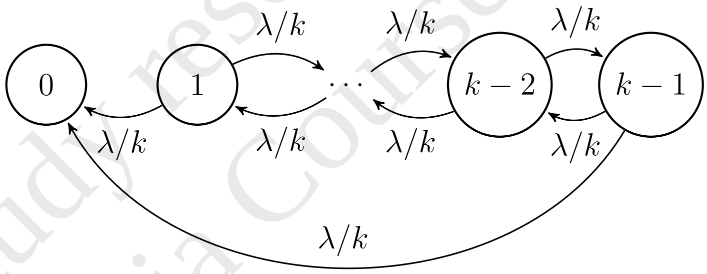
$$\left(\frac{e^{-\lambda}\lambda^2}{2!}\right)^m.$$

[1 Marks]

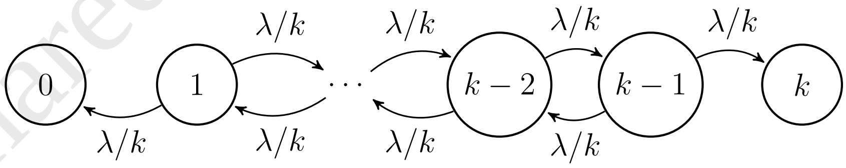
(c) Find the probability that all of the green lights up to time t have been at corner 0, given that there have been 2 green lights at corner 0 by time t.
This is the probability that there are no green lights anywhere else by time t which is $e^{-\lambda t\frac{k-1}{k}}$. **[1 Marks]**

(d) Suppose that Batman starts at corner i and the Joker starts at corner $j \neq i$. Find the probability that neither of them have moved by time t.
$(e^{-\lambda t/k})^2$ **[1 Marks]**

(e) Suppose that Batman starts at corner 0 and the Joker starts at corner i. Find the expected time until Batman catches the Joker.
Let C_t *denote the clockwise distance from Batman to the Joker at time t. Then* $(C_t)_{t\geq 0}$ *is a CTMC, taking values in* $\mathcal{S} = \{0, 1, \ldots, k-1\}$, *and we are looking for* $m_{i,0}$. *The transition diagram is*



This is the same as $m_{i,\{0,k\}}$ *for the following chain*



All of the rates here are the same, so the expected time to reach $\{0, k\}$ *is the same as for the symmetric Gambler's ruin expected hitting time problem (in discrete time - see a previous tutorial) multiplied by* $\frac{k}{2\lambda}$ *(the expected time for the CTMC to make a jump). Thus the answer is* $i(k-i)\frac{k}{2\lambda}$. **[2 Marks]**

3. At an infuriating airport, passengers arrive as a Poisson process of rate λ, and an airport official rolls a fair die for each passenger to decide which of 6 independent (FCFS) $M/M/1$ queues s/he will be sent to. Suppose that each server serves at rate $\mu > \lambda/6$. **[6 Marks]**

(a) Let $X_t^{(i)}$ denote the number of customers in queue i (including any customer being served). Find the stationary distribution for the Markov chain $(X_t^{(1)}, \dots, X_t^{(6)})$.
According to Poisson thinning, the queues are completely independent of each other, with each one behaving as an $M/M/1$ queue with parameters $\lambda/6$ and μ. Therefore the stationary distribution is just the product of the marginal stationary distributions. In other words, with $\rho = \lambda/(6\mu)$,

$$\pi_{n_1,\dots,n_6} = \prod_{i=1}^{6}(1-\rho)\rho^{n_i}.$$

[1 Marks]

(b) Find the expected time spent in this system by a passenger at stationarity. Will this be smaller or larger than the expected time a customer spends in an $M/M/6$ queue (with the same arrival and service rates, and also at stationarity)?
The expected time spent in this system by a passenger is $d = \frac{1}{\mu-\lambda/6}$.
This will be larger than the expected time spent in an $M/M/6$ queue because servers are underutilized in our system (there are times when there are idle servers and customers waiting for service). **[1 Marks]**

(c) Find the long run proportion of time that there are exactly k idle servers.
For a given queue, the proportion of time that it is idle is $1-\rho$. The queues are independent of each other, so the proportion of time that exactly k of them are idle is

$$\binom{6}{k}(1-\rho)^k\rho^{6-k}.$$

[1 Marks]

Suppose that passenger B is the next passenger joining the system after passenger A, and that immediately before passenger A enters the system, the system was stationary. Let Δ denote the exit time of B from the system minus the exit time of A from the system.

(d) Find the expected value of Δ given that passenger B joins the same queue as passenger A.
Let N be a random variable with $\mathbb{P}(N=n) = (1-\rho)\rho^n$ for $n = 0, 1, 2, \dots$ (this is a Geometric random variable starting from 0). Let passenger A enter the system at time 0, when the initial distribution was stationary. Without loss of generality we may assume that A is directed to queue 1.
Let T_n denote the time it takes for n passengers to be served (so, T_{N+1} is the time it takes for $N+1$ passengers to be served, where N is random as above), and let Y be an independent exp(μ) random variable, and Z be an independent exp(λ) random variable (think of Y as being the service time of passenger B and Z the time it takes passenger B to arrive). The time it takes for passenger A to be served is T_{N+1} which has an exponential($\mu(1-\rho)$) distribution (recall our results about a geometric sum of exponential random variables from a tutorial).

Conditional on passenger B joining the same queue as A, the time at which passenger B exits the system is

$$(T_{N+1}+Y)\mathbb{1}_{\{Z<T_{N+1}\}}+(T_{N+1}+Y+Z-T_{N+1})\mathbb{1}_{\{Z>T_{N+1}\}} = T_{N+1}+Y+(Z-T_{N+1})\mathbb{1}_{\{Z>T_{N+1}\}}.$$

Thus the difference in this case is

$$\Delta = Y + (Z - T_{N+1})\mathbb{1}_{\{Z>T_{N+1}\}}.$$

This has expectation

$$\frac{1}{\mu} + \mathbb{E}[Z - T_{N+1}|Z > T_{N+1}]\mathbb{P}(Z > T_{N+1}) = \frac{1}{\mu} + \frac{1}{\lambda}\mathbb{P}(Z > T_{N+1}).$$

Note that Z and T_{N+1} are independent exponential random variables with parameters λ and $\mu(1-\rho)$. Therefore

$$\gamma := \mathbb{P}(Z > T_{N+1}) = \frac{\mu(1-\rho)}{\mu(1-\rho)+\lambda}.$$

It follows that the conditional expectation of Δ given that B joins the same queue as A is

$$\frac{1}{\mu} + \frac{1}{\lambda}\gamma = \frac{1}{\mu} + \frac{1}{\lambda}\cdot\frac{\mu(1-\rho)}{\mu(1-\rho)+\lambda}$$

[1 Marks]

(e) Find the (unconditional) expected value of Δ

Conditional on B joining a different queue, the service time of B can be written in the form

$$\begin{aligned}&(S_{N'}+Y)\mathbb{1}_{\{Z<S_{N'}\}} + (S_{N'}+Y+(Z-S_{N'}))\mathbb{1}_{\{Z>S_{N'}\}}\\ &= (S_{N'}+Y) + (Z-S_{N'})\mathbb{1}_{\{Z>S_{N'}\}},\end{aligned}$$

where the time $S_{N'}$ represents the time to serve the N' customers that were in the queue eventually joined by passenger B at time 0. Now $(S_{N'}+Y) \sim T_{N+1}$ so the expectations are the same. Thus the expected value of Δ conditional on joining different queues is

$$\mathbb{E}[(Z-S_{N'})\mathbb{1}_{\{Z>S_{N'}\}}] = \frac{1}{\lambda}\mathbb{P}(Z > S_{N'}).$$

Here,

$$\begin{aligned}\mathbb{P}(Z > S_{N'}) &= \mathbb{P}(Z > S_{N'}|N'=0)\mathbb{P}(N'=0) + \mathbb{P}(Z > S_{N'}|N'>0)\mathbb{P}(N'>0)\\ &= \mathbb{P}(N'=0) + \mathbb{P}(Z > T_{N+1})\mathbb{P}(N'>0),\end{aligned}$$

since the conditional distribution of N' given that $N' > 0$ is bigger than 0 is the same as the distribution of $N'+1$, which is the same as the distribution of $N+1$ etc. Since $\mathbb{P}(N'=0) = 1-\rho$ we get that the conditional expectation of Δ given that B joins a different queue is

$$\frac{1}{\lambda}[1-\rho+\gamma\rho] = \frac{1}{\lambda}[1-\rho+\frac{\mu(1-\rho)}{\mu(1-\rho)+\lambda}\rho].$$

Finally,

$$\begin{aligned}\mathbb{E}[\Delta] &= \frac{1}{6}\left[\frac{1}{\mu}+\frac{1}{\lambda}\gamma\right]+\frac{5}{6}\frac{1}{\lambda}[1-\rho+\gamma\rho] \\ &= \frac{1}{6}\cdot\left[\frac{1}{\mu}+\frac{1}{\lambda}\cdot\frac{\mu(1-\rho)}{\mu(1-\rho)+\lambda}\right]+\frac{5}{6}\cdot\frac{1}{\lambda}[1-\rho+\frac{\mu(1-\rho)}{\mu(1-\rho)+\lambda}\rho]\end{aligned}$$

[2 Marks]