MAST30022 Decision Making
2021
Tutorial 6

1. **(PS4-1)** Consider the following 3-person game, taken from "Games, Theory and Applications" by L. C. Thomas, Ellis Horwood, 1984.

   Country 1 has oil which it can use to run its transport system at a profit of \$$a$ per barrel. Country 2 wants to buy the oil to use in its manufacturing industry, where it gives a profit of \$$b$ per barrel. Country 3 wants it for food manufacturing where the profit is \$$c$ per barrel. Assume $a < b \leq c$.

   (a) Justify to yourself that an appropriate characteristic function for this situation is
   $$v(\emptyset) = 0,\ v(\{1\}) = a,\ v(\{2\}) = v(\{3\}) = 0$$
   $$v(\{1,2\}) = b,\ v(\{1,3\}) = c,\ v(\{2,3\}) = 0,\ v(\{1,2,3\}) = c.$$

   (b) Verify that this characteristic function is superadditive.

   In parts (c)–(e) let $a = 10$, $b = 16$ and $c = 18$.

   (c) State, with reasons, which of the following allocations are imputations.
   $$\boldsymbol{x}^1 = (12,2,4),\ \boldsymbol{x}^2 = (8,6,4),\ \boldsymbol{x}^3 = (10,3,4),\ \boldsymbol{x}^4 = (13,3,2),\ \boldsymbol{x}^5 = (17,0,1).$$
   For each imputation state, with reasons, whether it is a core element or not.

   (d) Find the core of this game. Interpret your result in terms of coalitions formed and who pays whom what.

   (e) Calculate the Shapley value of this game. Is the Shapley value a core element of this game?

Solution

(a) Clearly $v(\emptyset) = 0$. If Country 1 is on their own they use the oil for transport and earn \$$a$ per barrel, so $v(\{1\}) = a$. Countries 2 and 3 have no oil to sell so $v(\{2\}) = v(\{3\}) = 0$. If countries 1 and 2 cooperate they collectively earn \$$b$ per barrel selling it for manufacturing, so $v(\{1,2\}) = b$. If countries 1 and 3 cooperate they collectively earn \$$c$ per barrel selling it for food production, so $v(\{1,3\}) = c$. Countries 2 and 3 together have no oil to sell so $v(\{2,3\}) = 0$. If all countries cooperate they are best selling all the oil for food production, rather than for manufacturing, at \$$c$ per barrel, so $v(\{1,2,3\}) = c$.

(b) A TU-game (N, v) is superadditive if
$$v(S \cup T) \geq v(S) + v(T) \tag{1}$$
for any disjoint coalitions $S, T \in 2^N$.

If S or T is empty, then inequality (1) is trivial. Hence v is superadditive if and only if the following inequalities hold

$$\begin{aligned}
v(\{1\}) + v(\{2\}) \leq v(\{1,2\}) &\iff a \leq b \\
v(\{1\}) + v(\{3\}) \leq v(\{1,3\}) &\iff a \leq c \\
v(\{2\}) + v(\{3\}) \leq v(\{2,3\}) &\iff 0 \leq 0 \\
v(\{1\}) + v(\{2,3\}) \leq v(N) &\iff a \leq c \\
v(\{2\}) + v(\{1,3\}) \leq v(N) &\iff c \leq c \\
v(\{3\}) + v(\{1,2\}) \leq v(N) &\iff b \leq c.
\end{aligned}$$

All the inequalities are true and we conclude that v is superadditive.

(c) For an imputation $\boldsymbol{x} = (x_1, x_2, x_3) \in I(v)$ we require $x_1+x_2+x_3 = v(N) = 18$, and $x_1 \geq v(\{1\}) = 10$, $x_2 \geq v(\{2\}) = 0$, and $x_3 \geq v(\{3\}) = 0$. So $\boldsymbol{x}^3$ does not satisfy the first condition and $\boldsymbol{x}^2$ does not satisfy the second condition. Therefore, $\boldsymbol{x}^1$, $\boldsymbol{x}^4$, and $\boldsymbol{x}^5$ are imputations.

For an imputation $\boldsymbol{x} = (x_1, x_2, x_3) \in C(v)$ we require in addition to the conditions above, $x_1 + x_2 \geq v(\{1,2\}) = 16$, $x_1 + x_3 \geq v(\{1,3\}) = 18$, and $x_2 + x_3 \geq v(\{2,3\}) = 0$. Only $\boldsymbol{x}^5$ satisfies all conditions and therefore is in the core.

(d) From the conditions for the core given in part (c) we have that $x_1 \geq 10$, $x_2 \geq 0$, $x_3 \geq 0$, $x_1 + x_2 = 18 - x_3 \geq 16 \implies x_3 \leq 2$, $x_1 + x_3 = 18 - x_2 \geq 18 \implies x_2 \leq 0 \implies x_2 = 0$, and $x_2 + x_3 = 18 - x_1 \geq 0 \implies x_1 \leq 18$. Now, $0 \leq x_3 \leq 2 \implies 0 \leq 18 - x_1 \leq 2 \implies 16 \leq x_1 \leq 18$. Therefore

$$C(v) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 | 16 \leq x_1 \leq 18, x_2 = 0, x_3 = 18 - x_1 \right\}.$$

In conclusion, there is no benefit to Country 1 to trade with Country 2, so Country 2 gets nothing. In trading with Country 3, Country 1 can earn from Country 3 anything between \$16 per barrel and \$18 per barrel, so Country 3 will earn anything between \$0 and \$2 per barrel.

(e) We first write v as a linear combination of unanimity games $v = \sum_{T \in 2^N \backslash \emptyset} c_T u_T$ with $c_T \in \mathbb{R}$ for all $T \in 2^N \backslash \emptyset$. In order to compute the coefficients c_T in the linear combination, we use the fact that

$$v(S) = \sum_{T \subseteq S} c_T \qquad \text{for all } S \in 2^N,$$

and we use this relation starting with coalitions S of size $|S| = 1$, then $|S| = 2$, $\ldots$, until $|S| = n$ (for $S = N$).

We then have

$$
\begin{aligned}
10 &= v(\{1\}) = c_1 \Longrightarrow c_1 = 10 \\
0 &= v(\{2\}) = c_2 \Longrightarrow c_2 = 0 \\
0 &= v(\{3\}) = c_3 \Longrightarrow c_3 = 0 \\
16 &= v(\{1,2\}) = c_1 + c_2 + c_{12} \Longrightarrow c_{12} = 6 \\
18 &= v(\{1,3\}) = c_1 + c_3 + c_{13} \Longrightarrow c_{13} = 8 \\
0 &= v(\{2,3\}) = c_2 + c_3 + c_{23} \Longrightarrow c_{23} = 0 \\
18 &= v(\{1,2,3\}) = c_1 + c_2 + c_3 + c_{12} + c_{13} + c_{23} + c_{123} \Longrightarrow c_{123} = -6,
\end{aligned}
$$

and therefore

$$v = 10u_{\{1\}} + 6u_{\{1,2\}} + 8u_{\{1,3\}} - 6u_{\{1,2,3\}}.$$

Then the Shapley value $\mathbf{\Phi}(v)$ is given by $\Phi_i(v) = \displaystyle\sum_{T \in 2^N; i \in T} \frac{c_T}{|T|}$ for all $i \in N$, and we obtain

$$
\begin{aligned}
\Phi(v) &= (10,0,0) + (3,3,0) + (4,0,4) + (-2,-2,-2) \\
&= (15,1,2)\,.
\end{aligned}
$$

The Shapley value is not in the core.

2. **(PS4-2)** Consider the 3-person TU-game with characteristic function as below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	a	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

(a) Find the maximum a such that the characteristic function is superadditive.

(b) With this maximum value of a, show that

$$\boldsymbol{p} = \left(\tfrac{1}{4}, \tfrac{1}{8}, \tfrac{5}{8}\right), \ \boldsymbol{q} = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right), \ \boldsymbol{r} = \left(\tfrac{5}{16}, \tfrac{1}{8}, \tfrac{9}{16}\right)$$

are imputations. Which of these imputations are core elements?

(c) Draw the core of the game with a as found in part (a) and find its extreme points.

(d) Find the Shapley value when $a = \frac{1}{2}$.

Solution

(a) A TU-game (N, v) is superadditive if

$$v(S \cup T) \geq v(S) + v(T) \tag{2}$$

for any disjoint coalitions $S, T \in 2^N$. If S or T is empty, then inequality (2) is trivial. Hence v is superadditive if and only if the following inequalities hold

$$\begin{aligned}
v(\{1\}) + v(\{2\}) \leq v(\{1,2\}) &\Longleftrightarrow 0 \leq \tfrac{1}{4} \\
v(\{1\}) + v(\{3\}) \leq v(\{1,3\}) &\Longleftrightarrow a \leq \tfrac{1}{2} \\
v(\{2\}) + v(\{3\}) \leq v(\{2,3\}) &\Longleftrightarrow a \leq \tfrac{3}{4} \\
v(\{1\}) + v(\{2,3\}) \leq v(N) &\Longleftrightarrow \tfrac{3}{4} \leq 1 \\
v(\{2\}) + v(\{1,3\}) \leq v(N) &\Longleftrightarrow \tfrac{1}{2} \leq 1 \\
v(\{3\}) + v(\{1,2\}) \leq v(N) &\Longleftrightarrow a + \tfrac{1}{4} \leq 1.
\end{aligned}$$

We conclude that v is superadditive if $a \leq \frac{1}{2}$ and the maximum a such that v is superadditive is $a = \frac{1}{2}$.

(b) A vector $\boldsymbol{x}$ is an imputation if

(i) $\sum_{i=1}^{3} x_i = v(N) = 1$, and

(ii) $x_i \geq v(\{i\})$ for $i = 1, 2, 3$.

It is easily checked that these conditions hold for each of the three given vectors.

Furthermore, $\boldsymbol{x} \in C(v)$ if

(i) $\sum_{i=1}^{3} x_i = v(N) = 1$, and

(ii) $\sum_{i \in S} x_i \geq v(S)$ for all $S \in 2^N$.

By checking the conditions in (ii) one by one, we see that $\boldsymbol{p} \in C(v)$. Furthermore, $\boldsymbol{q} \notin C(v)$ since $q_2 + q_3 = \frac{1}{2} < \frac{3}{4}$ and $\boldsymbol{r} \notin C(v)$ since $r_2 + r_3 = \frac{11}{16} < \frac{3}{4}$.

(c) Let $\boldsymbol{x} \in C(v)$, then

$$\begin{aligned}
x_1 + x_2 + x_3 &= 1 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
x_3 &\geq \tfrac{1}{2} \\
x_1 + x_2 &\geq \tfrac{1}{4} \\
x_1 + x_3 &\geq \tfrac{1}{2} \\
x_2 + x_3 &\geq \tfrac{3}{4}.
\end{aligned}$$

Using the first equality, the last three inequalities can be rewritten as $x_3 \leq \frac{3}{4}$, $x_2 \leq \frac{1}{2}$, and $x_1 \leq \frac{1}{4}$, respectively. The core is depicted in Figure 1 (note that the inequality $x_2 \leq \frac{1}{2}$ holds for all imputations).

The extreme points of $C(v)$ are $\left(\frac{1}{4}, 0, \frac{3}{4}\right)$, $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, and $\left(0, \frac{1}{4}, \frac{3}{4}\right)$.

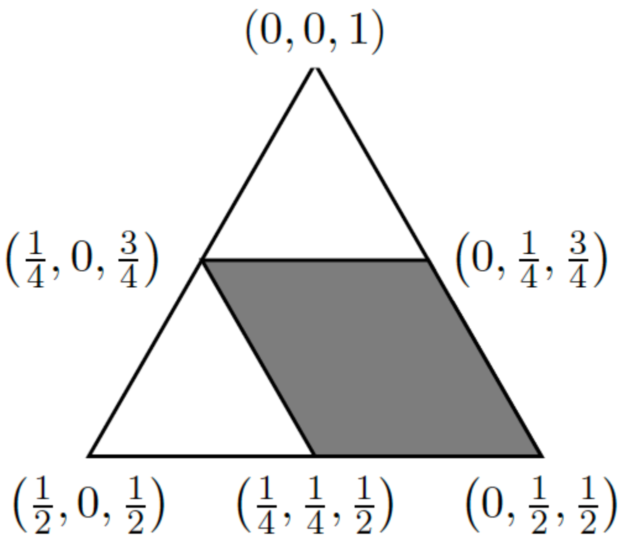

Figure 1: Core of the game for PS4-2.

(d) We first write v as a linear combination of unanimity games $v = \sum_{T \in 2^N \setminus \emptyset} c_T u_T$ with $c_T \in \mathbb{R}$ for all $T \in 2^N \setminus \emptyset$. In order to compute the coefficients c_T in the linear combination, we use the fact that

$$v(S) = \sum_{T \subseteq S} c_T \qquad \text{for all } S \in 2^N,$$

and we use this relation starting with coalitions S of size $|S| = 1$, then $|S| = 2$, $\ldots$, until $|S| = n$ (for $S = N$). We then have

$$\begin{aligned}
0 &= v(\{1\}) = c_1 \Longrightarrow c_1 = 0 \\
0 &= v(\{2\}) = c_2 \Longrightarrow c_2 = 0 \\
\tfrac{1}{2} &= v(\{3\}) = c_3 \Longrightarrow c_3 = \tfrac{1}{2} \\
\tfrac{1}{4} &= v(\{1,2\}) = c_1 + c_2 + c_{12} \Longrightarrow c_{12} = \tfrac{1}{4} \\
\tfrac{1}{2} &= v(\{1,3\}) = c_1 + c_3 + c_{13} \Longrightarrow c_{13} = 0 \\
\tfrac{3}{4} &= v(\{2,3\}) = c_2 + c_3 + c_{23} \Longrightarrow c_{23} = \tfrac{1}{4} \\
1 &= v(\{1,2,3\}) = c_1 + c_2 + c_3 + c_{12} + c_{13} + c_{23} + c_{123} \Longrightarrow c_{123} = 0,
\end{aligned}$$

and therefore

$$v = \tfrac{1}{2} u_{\{3\}} + \tfrac{1}{4} u_{\{1,2\}} + \tfrac{1}{4} u_{\{2,3\}}.$$

Then the Shapley value $\mathbf{\Phi}(v)$ is given by $\Phi_i(v) = \sum_{T \in 2^N; i \in T} \frac{c_T}{|T|}$ for all $i \in N$ and we obtain

$$\begin{aligned}
\Phi(v) &= \left(0, 0, \tfrac{1}{2}\right) + \left(\tfrac{1}{8}, \tfrac{1}{8}, 0\right) + \left(0, \tfrac{1}{8}, \tfrac{1}{8}\right) \\
&= \left(\tfrac{1}{8}, \tfrac{1}{4}, \tfrac{5}{8}\right).
\end{aligned}$$

Or, alternatively, the Shapley value can be calculated using marginal vectors.

σ	$m^\sigma(v)$
(1 2 3)	$\left(0, \frac{1}{4}, \frac{3}{4}\right)$
(1 3 2)	$\left(0, \frac{1}{2}, \frac{1}{2}\right)$
(2 1 3)	$\left(\frac{1}{4}, 0, \frac{3}{4}\right)$
(2 3 1)	$\left(\frac{1}{4}, 0, \frac{3}{4}\right)$
(3 1 2)	$\left(0, \frac{1}{2}, \frac{1}{2}\right)$
(3 2 1)	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

The Shapley value is determined by the average of all marginal vectors

$$\Phi(v) = \tfrac{1}{6}\left(\tfrac{3}{4}, \tfrac{6}{4}, \tfrac{15}{4}\right) = \left(\tfrac{1}{8}, \tfrac{1}{4}, \tfrac{5}{8}\right),$$

as before. Note that the Shapley value is in the core.

3. **(PS4-6)** Let $N = \{1, 2, 3\}$ and let $v \in \mathrm{TU}^N$ as described in the table below, with $r, t \in \mathbb{R}$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	$t+5$	5	7	$2t$	$3t+4$	14	$2t+r$

Furthermore, it is given that $(x_1, x_2, x_3) = (2t, 10, 9) \in C(v)$. Determine t and r.

Solution

If $\boldsymbol{x} \in C(v)$, then

$$\begin{aligned} x_1 + x_2 + x_3 &= v(N) \\ x_1 &\geq v(\{1\}) \\ x_2 &\geq v(\{2\}) \\ x_3 &\geq v(\{3\}) \\ x_1 + x_2 &\geq v(\{1,2\}) \\ x_1 + x_3 &\geq v(\{1,3\}) \\ x_2 + x_3 &\geq v(\{2,3\}). \end{aligned}$$

From the first equality it follows that

$$2t + 19 = 2t + r \Longrightarrow r = 19.$$

Working out the other inequalities, yields

$$\begin{aligned} 2t &\geq t + 5, \\ 10 &\geq 5, \\ 9 &\geq 7, \\ 2t + 10 &\geq 2t, \\ 2t + 9 &\geq 3t + 4, \\ 19 &\geq 14. \end{aligned}$$

From the first inequality we conclude that $t \geq 5$ and from the fifth inequality it follows that $t \leq 5$. All other inequalities are true for any value of t. Therefore $t = 5$.

4. **(PS4-10)** The 3-person game of Couples is played as follows. Each player chooses one of the other two players; these choices are made simultaneously. If a couple forms (e.g. if Player I chooses Player II, and Player II chooses Player I), then each member of that couple receives a payoff of 1/2, while the person not in the couple receives -1. If no couple forms (e.g. if I chooses II, II chooses III, and III chooses I), then each receives a payoff of zero.

   (a) Using the technique of optimal security levels, determine the corresponding game in characteristic function form.

   (b) Prove that in this game
   $$v(S) + v(N \backslash S) = 0$$
   for all $S \in 2^N$.

   (c) Show that this game is essential (i.e. show that this game is not an additive game).

   (d) Show that this game has an empty core.

   (Adapted from "Games Theory: Mathematical Models of Conflict", A. J. Jones, 2000)

   Solution

   (a) Denote the strategy that Player 1 chooses Player 2 by a_2, and chooses Player 3 by a_3. Similarly, denote the strategies for Player 2 by b_1 and b_3, and for Player 3 by c_1 and c_2.

   The strategy triples and corresponding payoffs are tabulated below.

(a_2, b_1, c_1)	$\left(\frac{1}{2}, \frac{1}{2}, -1\right)$
(a_2, b_1, c_2)	$\left(\frac{1}{2}, \frac{1}{2}, -1\right)$
(a_2, b_3, c_1)	$(0, 0, 0)$
(a_2, b_3, c_2)	$\left(-1. \frac{1}{2}, \frac{1}{2}\right)$
(a_3, b_1, c_1)	$\left(\frac{1}{2}, -1, \frac{1}{2}\right)$
(a_3, b_1, c_2)	$(0, 0, 0)$
(a_3, b_3, c_1)	$\left(\frac{1}{2}, -1, \frac{1}{2}\right)$
(a_3, b_3, c_2)	$\left(-1. \frac{1}{2}, \frac{1}{2}\right)$

   Let $N = \{1, 2, 3\}$. First we have $v(\emptyset) = 0$. Now, if $S = \{1, 2\}$ and $N \backslash S = \{3\}$, the payoffs to S and $N \backslash S$ are given in the bi-matrix

$$
\begin{array}{cc|cc}
 & & \multicolumn{2}{c}{N\backslash S} \\
 & & c_1 & c_2 \\
\hline
 & (a_2, b_1) & (1, -1) & (1, -1) \\
S & (a_2, b_3) & (0, 0) & \left(-\frac{1}{2}, \frac{1}{2}\right) \\
 & (a_3, b_1) & \left(-\frac{1}{2}, \frac{1}{2}\right) & (0, 0) \\
 & (a_3, b_3) & \left(-\frac{1}{2}, \frac{1}{2}\right) & \left(-\frac{1}{2}, \frac{1}{2}\right)
\end{array}
$$

This is a 2-person zero-sum game with payoff (to S) matrix

$$
V = \begin{bmatrix} 1 & 1 \\ 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.
$$

Here v_{11} and v_{12} are saddles so $L = U = 1$. Thus $v(\{1,2\}) = 1$ and $v(\{3\}) = -1$. By symmetry we can calculate the security levels of the other coalitions, and v is given in the table below.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	-1	-1	-1	1	1	1	0

(b) From the table it is clear that $v(S)+v(N\backslash S) = 0$. This is because the game is a zero-sum game. Any coalition's payoff is just the negative of the corresponding counter coalition's payoff.

(c) v is not additive since $v(\{1\}) + v(\{2\}) + v(\{3\}) = -3 \neq v(\{1,2,3\}) = 0$.

(d) For $(x_1, x_2, x_3) \in C(v)$ we require

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
x_1 &\geq -1 \\
x_2 &\geq -1 \\
x_3 &\geq -1 \\
x_1 + x_2 &\geq 1 \\
x_1 + x_3 &\geq 1 \\
x_2 + x_3 &\geq 1.
\end{aligned}
$$

The equality and the fourth inequality imply that $x_3 \leq -1 \Longrightarrow x_3 = -1$, by the third inequality. By symmetry we also have $x_1 = x_2 = -1$ which contradicts the equality. Hence the core is empty.