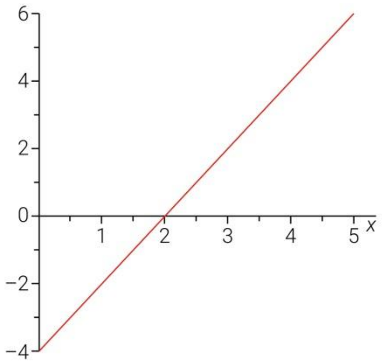
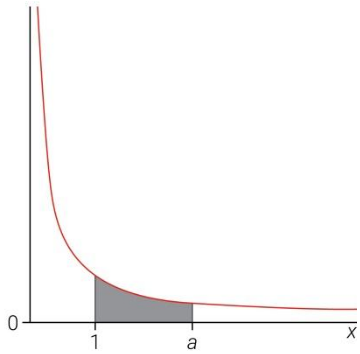
MAT4MDS — Practice 9 Worked Solutions

Model Answers to Practice 9

Diagrams for Questions 1 and 3:





Question 1.

(a) $\int_2^5 f(x)\,dx = \text{area of triangle} = \frac{1}{2}(5-2)f(5) = \frac{3}{2}\cdot 6 = 9$

(b) $\int_3^5 f(x)\,dx = \text{area of trapezium} = \frac{1}{2}(5-3)(f(5)+f(3)) = 6+2 = 8,$

(c) $\int_2^x f(t)\,dt = \text{area of triangle} = \frac{1}{2}(x-2)f(x) = (x-2)^2$

Question 2. The characteristic equation for K is

(a) $\int_1^2 f(x)\ dx = \int_{-1}^2 f(x)dx - \int_{-1}^1 f(x)\ dx = 6-2 = 4.$

(b) $\int_{-1}^0 f(x)\ dx = \int_{-1}^1 f(x)\ dx - \int_0^1 f(x)\ dx = 2-8 = -6.$

(c) $\int_0^2 f(x)\ dx = \int_0^1 f(x)\ dx + \int_1^2 f(x)\ dx = 8+4 = 12 \qquad \text{(using(a))}.$

Question 3.

(a) See above

(b) See above

(c) Using a basic antiderivative, since $\int_1^a \frac{1}{x}dx = \log_e(a) - \log_e(1)$, we must have $\log_e(a) = 1$, so that $a = e$.

Question 4.

(a) $\frac{d}{dx}(\frac{1}{2}x^2 + c) = \frac{1}{2}\frac{d}{dx}(x^2) + \frac{d}{dx}(c) = x + 0 = x$, as required.

(b) $\frac{d}{dx}(\frac{1}{2}e^{2x} + c) = \frac{1}{2}\frac{d}{dx}(e^{2x}) + \frac{d}{dx}(c) = \frac{1}{2}e^{2x}\cdot 2 + 0 = e^{2x}$, as required.

(c) $\frac{d}{dx}(\frac{2}{9}(3x+1)^{\frac{3}{2}} + c) = \frac{2}{9}\frac{d}{dx}[(3x+1)^{\frac{3}{2}}] + \frac{d}{dx}(c) = \frac{2}{9}\times\frac{3}{2}(3x+1)^{\frac{1}{2}}(3) + 0 = (3x+1)^{\frac{1}{2}}$, as required.

Question 5.

(a) $\int (t^2 + 3t + t^{-1})\ dt = \int t^2 dt + 3\int t\ \ dt + \int t^{-1}\ dt = \frac{1}{3}t^3 + \frac{3}{2}t^2 + \log_e(t) + c$, where $c \in \mathbb{R}$.

(b) $\int (5\log_e(x) + 2e^x)\ dx = 5\int \log_e(x)\ dx + 2\int e^x\ dx = 5(x\log_e(x) - x) + 2e^x + c$, where $c \in \mathbb{R}$.

(c) $\int 3p^{0.2}\ dp = 3\int p^{0.2}\ dp = \frac{3}{1.2}p^{1.2} = 2.5p^{1.2} + c$, where $c \in \mathbb{R}$.

(d) $\int dx = \int 1\ dx = x + c$, where $c \in \mathbb{R}$.

Question 6.

(a) $\int_{-1}^{1} x^2(x+2)\ dx = \int_{-1}^{1} (x^3 + 2x^2)\ dx = \left[\frac{1}{4}x^4 + \frac{2}{3}x^3\right]_{-1}^{1} = \frac{1}{4} + \frac{2}{3} - \frac{1}{4} + \frac{2}{3} = \frac{4}{3}$.

(b) $\int_{1}^{3} (x + x^{-1})\ dx = \left[\frac{1}{2}x^2 + \ln|x)\right]_{1}^{3} = \frac{9}{2} + \ln(3) - \frac{1}{2} - \ln(1) = 4 + \ln(3)$.

(c) $\int_{1}^{e} \big(2x - \ln(x)\big)\,dx = x^2 - (x\ln(x) - x)|_1^e = e^2 - e\ln(e) + e - (1 - \ln(1) + 1) = e^2 - 2$.

Question 7.

(a) (i) $\int (4t+1)^7\ dt = \frac{1}{4}.\frac{1}{8}(4t+1)^8 = \frac{1}{32}(4t+1)^8 + c$, where $c \in \mathbb{R}$.

(ii) $\int \frac{1}{(cx+d)^3}\ dx = \int (cx+d)^{-3}\ dx = \frac{1}{c}\cdot\left(-\frac{1}{2}\right)(cx+d)^{-2} = -\frac{1}{2c}(cx+d)^{-2} + a$, where $a \in \mathbb{R}$.

(iii) $\int_0^x e^{-3x+2}dx = -\frac{1}{3}[e^{-3x+2} - e^2] = \frac{e^2}{3}[1 - e^{-3x}]$

Note that, in (ii) above, c was already a constant in the expression so we used a as the constant of integration here.

(b) (i) $\int_1^2 (1+3x)^{-1}\ dx = \left[\frac{1}{3}\ln(1+3x)\right]_1^2 = \frac{1}{3}[\ln(7) - \ln(4)] = \frac{1}{3}\ln\left(\frac{7}{4}\right)$

(ii) $\int_{-2}^{2} \sqrt{2x+5}\ dx = \int_{-2}^{2} (2x+5)^{1/2}\ dx = \left[\frac{1}{2}\cdot\frac{2}{3}(2x+5)^{3/2}\right]_{-2}^{2} = \frac{1}{3}(9^{3/2} - 1^{3/2}) = \frac{1}{3}(3^3 - 1) = \frac{26}{3}$.

(iii) $\int_0^3 (x^2 + e^{-x})\ dx = \left[\frac{1}{3}x^3 - e^{-x}\right]_0^3 = 9 - e^{-3} + 1 = 10 - e^{-3}$.

Question 8.

(a) $\int_1^b x^{-5}\ dx = \left[-\frac{1}{4}x^{-4}\right]_1^b = \frac{1}{4}(1 - b^{-4})$. $\quad (b \in \mathbb{R}^+)$.

So, $\int_1^{\infty} x^{-5}\ dx = \lim_{b\to\infty} \int_1^b x^{-5}\ dx = \lim_{b\to\infty} \frac{1}{4}(1 - b^{-4}) = \frac{1}{4}$.

(b) $\int_0^b e^{-x}\ dx = [-e^{-x}]_0^b = 1 - e^{-b}$. $\quad (b \in \mathbb{R})$

So, $\int_0^{\infty} e^{-x}\ dx = \lim_{b\to\infty} \int_0^b e^{-x}\ dx = \lim_{b\to\infty}(1 - e^{-b}) = 1$.

(c) $\int_a^{-1} x^{-4}\ dx = \left[-\frac{1}{3}x^{-3}\right]_a^{-1} = \frac{1}{3}(1 + a^{-3})$. $\quad (a \in \mathbb{R}^-)$

So, $\int_{-\infty}^{-1} x^{-4}\ dx = \lim_{a\to-\infty} \frac{1}{3}(1 + a^{-3}) = \frac{1}{3}$.

Question 9.

(a) By the Fundamental Theorem of Calculus,

$$f(x) = \frac{d}{dx}\left[1 - \left(\frac{a}{x}\right)^b\right] = \frac{d}{dx}[-a^b x^{-b}] = a^b b x^{-b-1} = \frac{a^b b}{x^{b+1}}$$

(b)

$$\begin{aligned} \text{mean} &= \int_a^\infty x a^b b x^{-b-1}\ dx \\ &= a^b b \int_a^\infty x^{-b} dx \\ &= a^b b \left.\frac{x^{-b+1}}{-b+1}\right|_a^\infty \qquad \mathrm{b} \neq -1 \end{aligned}$$

So long as $-b + 1 < 0$, this has a well-defined limit. We obtain

$$\text{mean} = a^b b\left(0 - \frac{a^{-b+1}}{1-b}\right) = \frac{ab}{b-1} \quad b > 1$$

Question 10. By the Fundamental Theorem of Calculus,

$$f(x) = \frac{d}{dx}[1 - (1 - x^a)^b] = bax^{a-1}(1 - x^a)^{b-1}$$