COPY

The University of Melbourne
Semester 2 Assessment 2011

Department of Mathematics and Statistics

# MAST10007 Linear Algebra

**Reading Time:** 15 minutes
**Writing Time:** 3 hours

**This paper has:** 7 pages

**Identical Examination Papers:** None
**Common Content Papers:** None

**Authorized Materials:**

No materials are authorized.

Calculators and mathematical tables are not permitted.

Candidates are reminded that no written or printed material related to this subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

**Instructions to Invigilators:**

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination.

**Instructions to Students:**

This examination consists of 12 questions.

The total number of marks is 120.

All questions may be attempted. All answers should be appropriately justified.

— BEGINNING OF EXAMINATION QUESTIONS —

1. Let

$$A = \begin{bmatrix} 0 & 3 & -2 \\ 4 & -5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1 & -2 \end{bmatrix}.$$

Evaluate, if possible:

(a) $AB$

(b) $CD$

(c) $3B^2$

(d) $(AC)^T + D$

[4 marks]

2. (a) Consider the following linear system:

$$\begin{array}{rcrcrcrl} 3x_1 & + & 2x_2 & + & x_3 & - & 4x_4 = & 4 \\ x_1 & + & 3x_2 & + & 5x_3 & + & x_4 = & -1 \\ 2x_1 & + & x_2 & + & 2x_3 & + & 3x_4 = & 0 \end{array}$$

(i) Write down the augmented matrix corresponding to the linear system.

(ii) Reduce the matrix to row-echelon form.

(iii) Use the row-echelon form to give all solutions to the linear system.

(b) Determine the values of $k$ (if any) for which the following linear system has:

(i) a unique solution,

(ii) no solution,

(iii) infinitely many solutions.

$$\begin{array}{rcrcrcl} x & + & ky & + & & = & 1 \\ kx & + & 9y & + & & = & 3 \\ 2x & + & 2ky & + & kz & = & 4 \end{array}$$

[12 marks]

3. (a) Consider the matrix

$$M = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

(i) Calculate its determinant $\det(M)$.

(ii) Find $M^{-1}$ (if it exists).

(iii) Suppose that $N$ is a $3 \times 3$ matrix with $\det(3N) = 54$. Calculate $\det(M^{-1}N^T)$.

(b) Let $A$ and $B$ be $3 \times 3$ matrices, such that the reduced row-echelon form of both matrices is the identity matrix. Show that the reduced row-echelon form of $AB$ is the identity matrix.

[12 marks]

4. Let $\Pi$ be the plane in $\mathbb{R}^3$ given by the following vector equation:

$$(x, y, x) = (2, 1, 1) + s(1, 2, 2) + t(-1, 0, 2), \qquad s, t \in \mathbb{R}.$$

(a) Find a Cartesian equation for $\Pi$.

(b) Find a vector equation of the line of intersection of $\Pi$ and the plane $2x+y+4z = 6$.

[8 marks]

5. For each of the following, decide whether or not the given set $S$ is a subspace of the vector space $V$. Justify your answers by either appealing to appropriate theorems, or providing a counter-example.

(a) $V = \mathcal{P}_2$ (all real polynomials of degree at most 2) and

$$S = \left\{ ax^2 + bx + c \in \mathcal{P}_2 \,:\, a + b + c = 0 \right\}$$

(b) $V = M_{2,2}$ (all real $2 \times 2$ matrices) and

$$S = \left\{ A \in M_{2,2} \,:\, A^2 = A \right\}$$

[6 marks]

6. Let

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 & 6 \\ 2 & 4 & 2 & 1 & 7 \\ 3 & 5 & 4 & 1 & 8 \\ 4 & 5 & 7 & -2 & 4 \\ 1 & -3 & 6 & 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $B$ is the reduced row-echelon form of the matrix $A$. Using this information, or otherwise, answer the following, giving reasons for your answers.

(a) What is the rank of $A$?

(b) Write down a basis for the column space of $A$.

(c) Write down (or calculate) the dimension of the row space of $A$.

(d) Are the rows of $A$ linearly independent? Explain your answer.

(e) Write down a basis for the row space of $A$.

(f) Do the vectors $(1,2,3,4,1), (3,4,5,5,-3), (0,2,4,7,6), (1,1,1,-2,3), (6,7,8,4,-4)$ span $\mathbb{R}^5$? Explain your answer.

(g) Write $(0,2,4,7,6)$ as a linear combination of $(1,2,3,4,1)$ and $(3,4,5,5,-3)$.

(h) Find a basis for the solution space of $A$.

[10 marks]

7. (a) Determine whether or not the given set $S$ is a spanning set for the given vector space $V$. If the set is a spanning set, find a subset that is a basis for $V$.

(i)

$$S = \left\{ \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$$

$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \quad \text{(all real symmetric } 2 \times 2 \text{ matrices)}$$

(ii)

$$S = \{-1 + x - x^2, 3 - 3x + 3x^2, 4 - x + 3x^2, 7 - x + 5x^2, 2 + x + x^2\}$$

$$V = \mathcal{P}_2 \quad \text{(all real polynomials of degree at most 2)}$$

(b) Let $V$ and $W$ be a vector spaces and $T : V \to W$ a linear transformation. Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a subset of $V$. Show that if $\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_k)\}$ is linearly independent, then $S$ is linearly independent.

[11 marks]

8. (a) Let $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations of the plane where $S$ is the rotation by $90°$ anticlockwise about the origin and $T$ is the reflection in the line $y = x$.
   (i) Find the standard matrices for $S$, $T$, and $S \circ T$.
   (ii) Give a single geometric transformation with the same effect as $S \circ T$.
   (b) Let $\mathcal{P}_2$ denote the vector space of all real polynomials of degree at most 2.
   (i) Show that the function $T : \mathcal{P}_2 \to \mathbb{R}$ defined by

$$T(p) = \int_0^1 p(x)\, dx$$

   is a linear transformation. (You may use standard properties of integration.)
   (ii) Find bases for the image and kernel of $T$.

[12 marks]

9. Consider the following three bases for $\mathbb{R}^2$:

$$\mathcal{S} = \{(1,0),(0,1)\} \qquad \mathcal{B} = \{(3,1),(1,1)\} \qquad \mathcal{C} = \{(1,-1),(1,3)\}$$

   (a) (i) Write down the transition matrix $P_{\mathcal{S},\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{S}$.
   (ii) Find the transition matrix $P_{\mathcal{C},\mathcal{S}}$ from $\mathcal{S}$ to $\mathcal{C}$.
   (iii) Find the transition matrix $P_{\mathcal{C},\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.
   (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by:

$$T(x,y) = (-x + 4y, -x + 2y)$$

   (i) Find the matrix $[T]_{\mathcal{S}}$ of the transformation $T$ with respect to the basis $\mathcal{S}$.
   (ii) Find the matrix $[T]_{\mathcal{C},\mathcal{B}}$ of the transformation $T$ with respect to bases $\mathcal{B}$ for the domain and $\mathcal{C}$ for the codomain.
   (iii) Given that $\boldsymbol{v} \in \mathbb{R}^2$ satisfies $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, find $[T(\boldsymbol{v})]_{\mathcal{C}}$.

[10 marks]

10. (a) For each of the following formulae either prove that it defines an inner product on $\mathbb{R}^2$, or prove that it does not.

   (i) $\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2$

   (ii) $\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + 2x_1y_2 + 3x_2y_2$

   (b) Let $W$ be the subspace of $\mathbb{R}^4$ having basis:

$$\{(1,1,1,1), (2,1,0,1), (3,-2,5,-2)\}$$

   Apply the Gram-Schmidt procedure to the above basis to find an orthonormal basis for $W$ using the dot product as inner product.

[14 marks]

11. (a) For each of the following matrices decide whether or not the matrix is diagonalizable. You should justify your answers.

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

   (b) Consider the matrix

$$C = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}.$$

   (i) Find all eigenvalues of the matrix $C$ and a basis for each eigenspace.

   (ii) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $C = PDP^{-1}$.

   (iii) Use your results from (ii) to find a formula for $C^n$ valid for each integer $n \geq 1$.

   (iv) Describe the limiting behaviour of $C^n$ as $n \to \infty$.

[14 marks]

12. (a) Verify that the vectors

$$\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are *eigenvectors* of the matrix

$$A = \begin{bmatrix} 7 & 24 & 0 \\ 24 & -7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and find their associated *eigenvalues.*

(b) Write down an invertible matrix $P$ and a diagonal matrix $D$ so that

$$D = P^{-1}AP.$$

(c) Write down an *orthogonal* matrix $Q$ so that

$$D = Q^T AQ$$

(where $D$ is the diagonal matrix above).

(d) Write down $Q^{-1}$.

[7 marks]

— END OF EXAMINATION QUESTIONS —