MAST30001 Stochastic Modelling – 2015

$\boxed{\textbf{Assignment 1}}$

Please complete and sign the Plagiarism Declaration Form (available from the LMS or the department's webpage), which covers all work submitted in this subject. The declaration should be attached to the front of your first assignment.

Don't forget to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 11 September, 2015 by 4pm** in the appropriate assignment box at the north end of Richard Berry Building.

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. A switch is either in the *on* or *off* position. At the discrete times T_1, $T_1 + T_2, \ldots$ the switch is toggled from the position it is in to the other position (so if the switch is *off* at time $T_1 - 1$, then it is *on* at time T_1), where $T_1, T_2, \ldots$ are i.i.d. positive integer-valued random variables. Let $X_n = 1$ if at time n the switch is *on* and $X_n = 2$ if the switch is *off* at time n.

   (a) Assume that T_1 has the geometric distribution with parameter $0 < p < 1$; that is, for $k = 1, 2, \ldots$, $\mathbb{P}(T_1 = k) = (1-p)^k p$.

   i. Explain in at most a few sentences why $(X_n)_{n\geq 0}$ is a Markov chain.

   ii. Find the transition matrix of $(X_n)_{n\geq 0}$, analyse its state space and discuss its long run behaviour (including deriving long run probabilities where appropriate).

   (b) If now $\mathbb{P}(T_1 = 1) = 1 - \mathbb{P}(T_1 = 2) = q$, where $0 < q < 1$, explain why $(X_n)_{n\geq 0}$ is *not* a Markov chain.

   Ans.

   (a)i. Since the T_i are geometric, we can alternatively describe the chain by saying that at each step we toss independent p-headed coins and at a given step if the coin comes up heads then we change the switch position and otherwise leave it in the same position. Thus given we're in state 1 (2), the chance of switching to state 2 (1) is p, and by independence of the coin tosses, the chance of switching does not depend further on the past history of the chain.

   (a)ii. From the previous part, we find

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}.$$

   Since $0 < p < 1$, the chain is irreducible, aperiodic (loops), and since the state space is finite it is positive recurrent. Thus the chain is ergodic with long run distribution π (i.e., $\lim_{n\to\infty} p_{i,j}^{(n)} = \pi_j$) which is stationary, satisfying

$$\pi P = \pi.$$

   So $\pi = (1/2, 1/2)$.

(b) For this distribution,

$$\mathbb{P}(X_2 = 1 | X_1 = 1, X_0 = 2) = (1 - q)$$

while

$$\mathbb{P}(X_2 = 1 | X_1 = 1, X_0 = 1) = 0.$$

If the process was Markov, then these two quantities would be equal. The issue is that the age distribution T_i does not have the memoryless property.

2. A Markov chain $(X_n)_{n \geq 0}$ on $\{0, 1, 2, \ldots\}$ has transition probabilities for $i = 0, 1, 2, \ldots$,

$$p_{i,i+1} = 1 - p_{i,0} = \left(\frac{i+1}{i+2} \right)^\alpha,$$

where $\alpha > 0$. This chain is irreducible.

(a) For which values of α is the chain transient? Null recurrent? Positive recurrent?

(b) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

(c) If $T(i) = \min\{n \geq 1 : X_n = i\}$, find $E[T(i) | X_0 = i]$ for $i = 0, 1, \ldots$.

(d) If $X_0 = 0$, what is the chance the chain reaches state 3 before it returns to state 0?

Ans.

(a) Since the chain is irreducible, we only need to check transience, pos/null recurrence at a single state, in this case state 0. If $T(0) = \min\{n \geq 1 : X_n = 0\}$, then the chain is recurrent if $\mathbb{P}(T(0) < \infty | X_0 = 0) = 1$ and positive recurrent if additionally $\mathbb{E}[T(0) | X_0 = 0] < \infty$. But we can compute exactly

$$\mathbb{P}(T(0) = k | X_0 = 0) = p_{0,1} p_{1,2} \cdots p_{k-1,0} = \frac{1}{k^\alpha} \left(1 - \left(\frac{k}{k+1} \right)^\alpha \right).$$

Thus

$$\begin{aligned} \mathbb{P}(T(0) < \infty | X_0 = 0) &= \sum_{k=1}^{\infty} \mathbb{P}(T(0) = k | X_0 = 0) \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} \right) \\ &= \lim_{n \to \infty} \left(1 - \frac{1}{n^\alpha} \right) = 1, \end{aligned}$$

so the chain is <u>recurrent</u> for all values of α.

To determine pos/null recurrence, we need to compute $\mathbb{E}[T(0) | X_0 = 0]$. First note that

$$\mathbb{P}(T(0) \geq k | X_0 = 0) = k^{-\alpha},$$

and then

$$\mathbb{E}[T(0) | X_0 = 0] = \sum_{k \geq 1} \mathbb{P}(T(0) \geq k | X_0 = 0) = \sum_{k \geq 1} k^{-\alpha},$$

which is finite if $\alpha > 1$ and infinite if $\alpha \leq 1$. So the chain is positive recurrent for $\alpha > 1$ and null recurrent if $0 < \alpha \leq 1$.

(b) The chain is irreducible and aperiodic (loop at zero), and so it's ergodic when it's positive recurrent ($\alpha > 1$).

If $\alpha > 1$, then the the long-run probabilities are given by the stationary distribution π which solve the recurrence for $i \geq 1$,

$$\pi_{i-1} p_{i-1,i} = \pi_i,$$

which has solution

$$\pi_i = \pi_0 p_{0,1} p_{1,2} \cdots p_{i-1,i} = \pi_0 (i+1)^{-\alpha}.$$

Now using that $\sum_{k=0}^{\infty} \pi_k = 1$, we have for $i = 0, 1, \ldots$,

$$\pi_i = (i+1)^{-\alpha} / \zeta(\alpha),$$

where $\zeta(\alpha) = \sum_{k \geq 1} k^{-\alpha}$.

If $0 < \alpha \leq 1$, then then the chain is null recurrent and so the long run probabilities tend to zero.

(c) We know that $\mathbb{E}[T(i)|X_0 = 0] = 1/\pi$, where π are the long run probabilities from Part (b). Thus if $\alpha > 1$, then

$$\mathbb{E}[T(i)|X_0 = 0] = \zeta(\alpha)(i+1)^{\alpha},$$

and these expectations are infinite for $0 < \alpha \leq 1$.

(d) If $X_0 = 0$ then the chain can only reach state 3 before state 0 via the path $0 \mapsto 1 \mapsto 2 \mapsto 3$ which is taken with probability $p_{0,1} p_{1,2} p_{2,3} = 4^{-\alpha}$.