TUTORIAL 3 (week 4)

The Simplex Method

Input: A linear program in standard form
Output: An optimal solution

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Construct canonical form and obtain a basic feasible solution
while There are negative reduced costs do
    Select entering variable with most positive/negative reduced cost (Alysson's notation / slides notation).
    Select leaving variable using the ratio test.
    Change basis.
end while
```

1. Solve the following LP problem by the Simplex Method.

$$
\begin{aligned}
&\max z = x_1 + x_2 + x_3 \\
&2x_2 + x_3 \leq 1 \\
&2x_1 + x_2 + 2x_3 \leq 1 \\
&x_1 + 2x_2 \leq 1 \\
&x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Solutions:

We first convert the given problem to canonical form by introducing slack variables x_4, x_5, x_6:

$$
\begin{aligned}
&\max z = x_1 + x_2 + x_3 \\
&2x_2 + x_3 + x_4 = 1 \\
&2x_1 + x_2 + 2x_3 + x_5 = 1 \\
&x_1 + 2x_2 + x_6 = 1 \\
&x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

The initial tableau is:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	2	1	1	0	0	1
x_5	(2)	1	2	0	1	0	1
x_6	1	2	0	0	0	1	1
z	$\mathbf{-1}$	-1	-1	0	0	0	0

There are three negative entries in the z-row which are equal. We can choose any one of them for the pivot column. Let us choose the first one so that x_1 becomes the entering variable. The ratio test identifies x_5 as the leaving variable. So the pivot entry is in the first column and second row of the tableau. Pivoting on this entry we obtain the next tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	2	1	1	0	0	1
x_1	1	1/2	1	0	1/2	0	1/2
x_6	0	(3/2)	−1	0	−1/2	1	1/2
z	0	**−1/2**	0	0	1/2	0	1/2

There is only one negative entry in the z-row. So x_2 is the entering variable. The ratio test identifies x_6 as the leaving variable and $(i = 3, j = 2)$ as the pivot entry. Pivoting on this entry yields the next tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_4	0	0	(7/3)	1	2/3	−4/3	1/3
x_1	1	0	4/3	0	2/3	−1/3	1/3
x_2	0	1	−2/3	0	−1/3	2/3	1/3
z	0	0	**−1/3**	0	1/3	1/3	2/3

This tableau is still not optimal since there exists a negative entry in the z-row. Greedy Rule and Ratio Test identify x_3 as the entering variable and x_4 as the leaving variable. Pivoting on the $(i = 1, j = 3)$-entry, we obtain the next tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	0	1	3/7	2/7	−4/7	1/7
x_1	1	0	0	−4/7	2/7	3/7	1/7
x_2	0	1	0	2/7	−1/7	2/7	3/7
z	0	0	0	1/7	3/7	1/7	5/7

This tableau is optimal since there is no negative reduced costs (entries in the z-row). So we stop.

Report:

$(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*) = (1/7, 3/7, 1/7, 0, 0, 0)$ is an optimal solution to the problem in canonical form and the optimal objective value is $z^* = 5/7$.

$(x_1^*, x_2^*, x_3^*) = (1/7, 3/7, 1/7)$ is an optimal solution to the given LP problem (in standard form) with optimal value $z^* = 5/7$.

2. Solve the following LP problem, or identify that the problem is unbounded or infeasible, by using the Simplex Method.

$$
\begin{aligned}
&\max z = 2x_1 + 3x_2 \\
&x_1 - 2x_2 \leq 4 \\
&2x_1 + x_2 \leq 18 \\
&x_2 \leq 10 \\
&x_1, x_2 \geq 0
\end{aligned}
$$

Solutions: This problem is in standard form. Convert it to canonical form by introducing slack variables x_3, x_4, x_5.

$$
\begin{aligned}
&\max z = 2x_1 + 3x_2 \\
&x_1 - 2x_2 + x_3 = 4 \\
&2x_1 + x_2 + x_4 = 18 \\
&x_2 + x_5 = 10 \\
&x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

The initial tableau is:

	x_1	x_2	x_3	x_4	x_5	RHS
x_3	1	-2	1	0	0	4
x_4	2	1	0	1	0	18
x_5	0	(1)	0	0	1	10
z	-2	$\mathbf{-3}$	0	0	0	0

Since there are negative reduced costs, we choose the most negative of them. This means we select x_2 as the entering variable. The ratio test identifies x_5 as the leaving variable and

$(i = 3, j = 2)$ as the pivot entry. Pivoting on this entry yields:

	x_1	x_2	x_3	x_4	x_5	RHS
x_3	1	0	1	0	2	24
x_4	(2)	0	0	1	-1	8
x_2	0	1	0	0	1	10
z	$\mathbf{-2}$	0	0	0	3	30

There is only one negative reduced cost which is located in the first column. So we select x_1 as the entering variable. The ratio test gives the leaving variable x_4 and the pivot entry $(i = 2, j = 1)$. Pivoting on this entry yields the next tableau:

	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	0	1	$-1/2$	$5/2$	20
x_1	1	0	0	$1/2$	$-1/2$	4
x_2	0	1	0	0	1	10
z	0	0	0	1	2	38

Since all reduced costs are nonnegative, this is an optimal tableau. It gives the optimal solution $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (4, 10, 20, 0, 0)$ (to the problem in canonical form) and optimal value $z^* = 38$. So $(x_1^*, x_2^*) = (4, 10)$ is an optimal solution to the given problem (in standard form).

The fact that $x_4^* = x_5^* = 0$ indicates that $(x_1^*, x_2^*) = (4, 10)$ satisfies the last two functional constraints with equality.

3. Solve the following LP problem, or identify that the problem is unbounded or infeasible, by using the Simplex Method. Provide a graphical interpretation.

$$\max z = 5x_1 + 3x_2$$
$$-x_1 + x_2 \leq 4$$
$$x_1 - 2x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Solutions: Convert the problem to canonical form:

$$\max z = 5x_1 + 3x_2$$
$$-x_1 + x_2 + x_3 = 4$$
$$x_1 - 2x_2 + x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Initial tableau:

	x_1	x_2	x_3	x_4	RHS
x_3	-1	1	1	0	4
x_4	(1)	-2	0	1	6
z	$-\mathbf{5}$	-3	0	0	0

Next tableau:

	x_1	x_2	x_3	x_4	RHS
x_3	0	-1	1	1	10
x_1	1	-2	0	1	6
z	0	-13	0	5	30

There is a unique negative reduced cost, namely -13, which is in the second column. When we do the ratio test we find that all entries of the coefficient matrix in the second column are non-positive. So the ratio test fails. This means that the problem is unbounded, i.e. it has no optimal solution. The objective function can take arbitrarily large value over the feasible region.

Exercise: Convince yourself of this result by using the graphical method.

4. The tableau method consists in writing the problem in a table and doing simplex iterations by doing pivot operations in this table. The basic variables can be identified in the tableau by

finding the columns of an identity matrix. The reduced costs can be found in the row z for the columns of the non-basic variables.

	x_1	x_2	x_3	x_4	RHS
x_1	1	1/2	1/2	0	4
x_4	0	1/2	$-1/2$	1	1
z	0	0	1	0	8

(a) Identify the basic variables in the solution corresponding to the tableau above. What are their values ?

(b) Identify the non-basic variables and their reduced costs. Is this solution optimal for a maximisation problem ?

(c) Who will enter the basis and who will leave in the next iteration ?

Solutions: (a) The basic variables are indicated in the first column (x_1 and x_4) and correspond to the two columns of the identity matrix. Their values can be read directly from the last column in the tableay: $x_1 = 4$ and $x_4 = 1$.

(b) The non basic-variables are x_2 and x_3 and their reduced costs are $\hat{c}_2 = 0$ and $\hat{c}_3 = 1$. The solution of this maximisation problem is therefor not optimal.

(c) We have $A_B = I_{m\times m}$ (we will always keep this in the tableau method) and

$$A_{NB} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

.

We enter with variable x_3 (since it is the only one with a positive reduced cost) and therefore the vecyor y is given as:

$$y = A_B^{-1} a_3 = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

and since $A_B = I$, this can be read directly from the tableau.

Now, we see that the maximum value we can add to the current non-basic variable x_3 is given by $4/(1/2) = 8$ (from the row associated with basic variable x_1. The other row in the vector y is non-positive, so we disconsider it).

This gives us the conclusion that basic variable x_1 is going to leave the basis to give place to variable x_3 in our basis.