MAST30013 – Techniques in Operations Research

Semester 1, 2021

Tutorial 10 Solutions

1. (a) The log barrier penalty function is

$$P_k(\boldsymbol{x}) \;=\; \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 + x_2 - \frac{1}{k}\log(-x_1) - \frac{1}{k}\log(-x_2)\,.$$

(b)

$$\boldsymbol{\nabla} P_k(\boldsymbol{x}) \;=\; \begin{pmatrix} x_1 - 1 - \dfrac{1}{kx_1} \\ x_2 + 1 - \dfrac{1}{kx_2} \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here, $kx_1^2 - kx_1 - 1 = 0 \implies x_1^k = \dfrac{k \pm \sqrt{k^2+4k}}{2k}$, and $kx_2^2 + kx_2 - 1 = 0 \implies x_2^k = \dfrac{-k \pm \sqrt{k^2+4k}}{2k}$. As $x_1^k \leq 0$ and $x_2^k \leq 0$,

$$\begin{aligned} \boldsymbol{x}^k \;&=\; \left(\frac{k - \sqrt{k^2+4k}}{2k}\,,\quad \frac{-k - \sqrt{k^2+4k}}{2k} \right)^T \\ &=\; \left(\frac{1 - \sqrt{1+4/k}}{2}\,,\quad \frac{-1 - \sqrt{1+4/k}}{2} \right)^T. \end{aligned}$$

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(c) Now,

$$\begin{aligned} \boldsymbol{x}^* \;&=\; \left(\lim_{k\to\infty} \frac{1 - \sqrt{1+4/k}}{2}\,,\quad \lim_{k\to\infty} \frac{-1 - \sqrt{1+4/k}}{2} \right)^T \\ &=\; (\,0\,,\;\; -1\,)^T. \end{aligned}$$

(d) For the KKT multipliers $\boldsymbol{\lambda}^*$ we have

$$\begin{aligned} \boldsymbol{\lambda}^* \;&=\; \left(\lim_{k\to\infty} \frac{-1}{kg_1(\boldsymbol{x}^k)}\,,\quad \lim_{k\to\infty} \frac{-1}{kg_2(\boldsymbol{x}^k)} \right)^T \\ &=\; \left(\lim_{k\to\infty} \frac{-1}{k\dfrac{1 - \sqrt{1+4/k}}{2}}\,,\quad \lim_{k\to\infty} \frac{-1}{k\dfrac{-1 - \sqrt{1+4/k}}{2}} \right)^T \\ &=\; \left(\lim_{k\to\infty} \frac{-1}{\dfrac{k - \sqrt{k^2+4k}}{2}}\,,\quad \lim_{k\to\infty} \frac{-1}{\dfrac{-k - \sqrt{k^2+4k}}{2}} \right)^T \\ &=\; \left(\lim_{k\to\infty} \frac{-2}{k - \sqrt{k^2+4k}}\,,\quad \lim_{k\to\infty} \frac{-2}{-k - \sqrt{k^2+4k}} \right)^T \\ &=\; (\,1\,,\;\; 0\,)^T. \end{aligned}$$

2. (a) The Lagrangian is

$$L((x_1,x_2),\lambda,\eta) \;=\; x_1^2+2x_2^2+\lambda(x_1^2+x_2^2-1)+\eta(x_1+x_2-1).$$

The KKT conditions are:

KKTa: $2x_1+2\lambda x_1+\eta=0, \quad 4x_2+2\lambda x_2+\eta=0.$

KKTb: $x_1^2+x_2^2-1\le 0, \quad \lambda\ge 0, \quad \lambda(x_1^2+x_2^2-1)=0.$

KKTc: $x_1+x_2-1=0.$

If $\lambda=0$, then from KKTa and KKTc, $\eta=-2x_1=-4x_2=4x_1-4 \Longrightarrow x_1=\frac{2}{3}$ and $x_2=\frac{1}{3}$, and $\eta=-\frac{4}{3}$.
Thus, the KKT point is $((x_1^*,x_2^*),\lambda^*,\eta^*)=\left(\left(\frac{2}{3},\frac{1}{3}\right),0,-\frac{4}{3}\right)$. We have that

$$\nabla^2 L((x_1,x_2),\lambda,\eta) \;=\; \begin{pmatrix} 2+2\lambda & 0 \\ 0 & 4+2\lambda \end{pmatrix}.$$

Now,

$$\nabla^2 L\left(\left(\tfrac{2}{3},\tfrac{1}{3}\right),0,-\tfrac{4}{3}\right) \;=\; \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix},$$

which is positive definite. Therefore, $\left(\left(\frac{2}{3},\frac{1}{3}\right),0,-\frac{4}{3}\right)$ is a local (and hence a global - the nonlinear program is convex) minimum.

If $\lambda>0$, then from KKTb and KKTc, $x_1^2+x_2^2=1 \Longrightarrow x_1^2+(1-x_1)^2=1 \Longrightarrow 2x_1^2-2x_1=0$. Therefore, $(x_1,x_2)=(1,0)$ or $(0,1)$. In either case, $\eta=0$. However, from KKTa, if $x_1=1$ then $\lambda<0$, which is a contradiction. The same conclusion is reached if $x_2=1$.

Thus, the only KKT point is $((x_1^*,x_2^*),\lambda^*,\eta^*)=\left(\left(\frac{2}{3},\frac{1}{3}\right),0,-\frac{4}{3}\right)$.

(b) Now,

$$\begin{aligned} L(\boldsymbol{x}^*,\lambda,\eta) &= \tfrac{2}{3}-\tfrac{4}{9}\lambda \\ L(\boldsymbol{x}^*,\lambda^*,\eta^*) &= \tfrac{2}{3} \\ L(\boldsymbol{x},\lambda^*,\eta^*) &= x_1^2+2x_2^2-\tfrac{4}{3}(x_1+x_2-1) \\ &= \left(x_1-\tfrac{2}{3}\right)^2+2\left(x_2-\tfrac{1}{3}\right)^2+\tfrac{2}{3}. \end{aligned}$$

Clearly, for all feasible $\boldsymbol{x}$, $\lambda\ge 0$, and $\eta\in\mathbb{R}$,

$$L(\boldsymbol{x}^*,\lambda,\eta) \;\le\; L(\boldsymbol{x}^*,\lambda^*,\eta^*) \;\le\; L(\boldsymbol{x},\lambda^*,\eta^*).$$

3. (a) The log barrier penalty function is

$$P_k(\boldsymbol{x}) \;=\; \frac{1}{4}x_1^4 - \frac{1}{2}x_1^2 + x_2^2 - \frac{1}{k}\log(x_1) - \frac{1}{k}\log(x_2-2)\,.$$

(b)

$$\boldsymbol{\nabla} P_k(\boldsymbol{x}) \;=\; \begin{pmatrix} x_1^3 - x_1 - \dfrac{1}{kx_1} \\ 2x_2 - \dfrac{1}{k(x_2-2)} \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here, $kx_1^4 - kx_1^2 - 1 = 0 \Longrightarrow x_1^k = \pm\sqrt{\dfrac{k \pm \sqrt{k^2+4k}}{2k}}$, and $2kx_2^2 - 4kx_2 - 1 = 0 \Longrightarrow x_2^k = \dfrac{4k \pm \sqrt{16k^2+8k}}{4k}$. As $x_1^k \geq 0$ and $x_2^k \geq 2$,

$$\begin{aligned} \boldsymbol{x}^k \;&=\; \left(\sqrt{\frac{k+\sqrt{k^2+4k}}{2k}}\;,\;\; \frac{k+\sqrt{k^2+k/2}}{k} \right)^T \\ &=\; \left(\sqrt{\frac{1+\sqrt{1+4/k}}{2}}\;,\;\; 1+\sqrt{1+1/2k} \right)^T. \end{aligned}$$

(c) Now,

$$\begin{aligned} \boldsymbol{x}^* \;&=\; \left(\lim_{k\to\infty}\sqrt{\frac{k+\sqrt{k^2+4k}}{2k}}\;,\;\; \lim_{k\to\infty}\frac{k+\sqrt{k^2+k/2}}{k} \right)^T \\ &=\; \left(\lim_{k\to\infty}\sqrt{\frac{1+\sqrt{1+4/k}}{2}}\;,\;\; \lim_{k\to\infty}\left(1+\sqrt{1+1/2k}\right) \right)^T \\ &=\; (1,\; 2)^T. \end{aligned}$$

(d) For the KKT multipliers $\boldsymbol{\lambda}^*$ we have

$$\begin{aligned} \boldsymbol{\lambda}^* \;&=\; \left(\lim_{k\to\infty}\frac{-1}{kg_1(\boldsymbol{x}^k)}\;,\;\; \lim_{k\to\infty}\frac{-1}{kg_2(\boldsymbol{x}^k)} \right)^T \\ &=\; \left(\lim_{k\to\infty}\frac{-1}{k\left(-\sqrt{\dfrac{1+\sqrt{1+4/k}}{2}}\right)}\;,\;\; \lim_{k\to\infty}\frac{-1}{k\left(2-\left(1+\sqrt{1+1/2k}\right)\right)} \right)^T \\ &=\; \left(\lim_{k\to\infty}\frac{1}{\sqrt{\dfrac{k^2+\sqrt{k^4+4k^3}}{2}}}\;,\;\; \lim_{k\to\infty}\frac{-1}{k-\sqrt{k^2+k/2}} \right)^T \\ &=\; (\;0,\;\; 4\;)^T. \end{aligned}$$