MAT4MDS — Practice 6

Model Answers to Practice 6

Question 1.

(a) $f(x) = x^2 - 2x + 3 \Rightarrow f'(x) = 2x - 2$ (using the sum rule).

(b) $g(x) = (x^2+1)^3 \Rightarrow g'(x) = 3(x^2+1)^2 \cdot 2x = 6x(x^2+1)^2$, by the chain rule.

(c) $f(x) = x^2e^x \Rightarrow f'(x) = 2xe^x + x^2e^x$, by the product rule.

(d)

$$\begin{aligned} y = x^3\ln(x) \Rightarrow \frac{dy}{dx} &= 3x^2\ln(x) + x^3 \cdot \frac{1}{x} \qquad \text{product rule} \\ &= 3x^2\ln(x) + x^2 = x^2(3\ln(x)+1) \end{aligned}$$

(e) Let $y = \frac{u}{v}$ where $u = 2x-3$ and $v = 3x+1$. Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 3$. Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{2(3x+1)-3(2x-3)}{(3x+1)^2} \qquad \text{quotient rule} \\ &= \frac{6x+2-6x+9}{(3x+1)^2} = \frac{11}{(3x+1)^2} \end{aligned}$$

[Note that you could have divided first to get $y = \frac{2}{3} - \frac{11}{3}(3x+1)^{-1}$ *and used the sum, constant and chain rules.]*

(f) $f(x) = (3x+2)\ln(3x+2) \Rightarrow f'(x) = 3\ln(3x+2) + (3x+2)\cdot\frac{1}{3x+2}\cdot 3 = 3\ln(3x+2) + 3$

(g) Note that we can avoid the quotient rule by simplifying: $y = \frac{x^2+\sqrt{x}}{x} = x + x^{-\frac{1}{2}}$.

It follows that $\frac{dy}{dx} = 1 - \frac{1}{2}x^{-\frac{3}{2}}$.

Question 2.

(a)

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \Rightarrow f'(f^{-1}(x))(f^{-1}(x))' &= 1 \quad \text{usingthechainrule} \\ \Rightarrow (f^{-1}(x))' &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

(b) $g(x) = \log_e(x)$ is the inverse of the function $h(x) = e^x$. Then

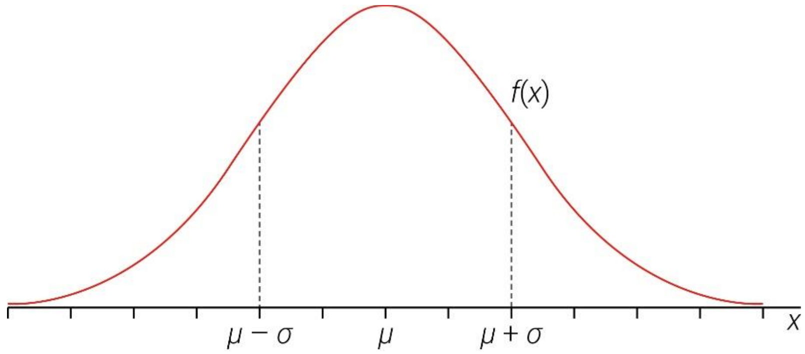
$$g'(x) = \frac{1}{h'(g(x))} = \frac{1}{e^{\log_e(x)}} = \frac{1}{x}.$$

Question 3. Consider $y = f(g(h(x)))$. Write $k(x) = g(h(x))$, so that $y = f(k(x))$. Then $k'(x) = g'(h(x))h'(x)$ by the standard chain rule. Now

$$\begin{aligned} \frac{dy}{dx} &= [f(k(x))]' \\ &= f'(k(x))k'(x) \qquad \text{by the chain rule} \\ &= f'(k(x))[g'(h(x))h'(x)] \\ &= f'(g(h(x)))g'(h(x))h'(x) \end{aligned}$$

Question 4.

(a) This is hard to judge accurately by eye, but incorporating what we learn in (d):



(b)

$$
\begin{aligned}
f(x) &= \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2} \\
\Rightarrow f'(x) &= \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2} \\
\Rightarrow\ f''(x) &= \frac{1}{\sigma^3\sqrt{2\pi}}\left[-1+\frac{(x-\mu)^2}{\sigma^2}\right] e^{-(x-\mu)^2/2\sigma^2}
\end{aligned}
$$

(c) $f'(x) = 0$ when $x = \mu$. We note that $f''(\mu) < 0$. This is both a local and global maximum.

(d) Using (b), $f''(x) = 0$ when

$$-1 + \frac{(x-\mu)^2}{\sigma^2} = 0(x-\mu)^2 = \sigma^2 x = \mu \pm \sigma.$$

So the density function changes curvature twice, at points equally spaced on either side of the mode.

(e) See the graph above (in (a))

Question 5. The following data gives the population of India in the census years since 1951.

(i) (a) $f'(x) = 2x - 2 \Rightarrow f''(x) = 2$

(b) $g'(x) = 6x(x^2+1)^2 \Rightarrow g''(x) = 6(x^2+1)^2 + 24x^2(x^2+1) = 6(x^2+1)(5x^2+1)$.

(c) $f'(x) = (2x + x^2)e^x \Rightarrow f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 4x + x^2)$

(d)

$$
\begin{aligned}
\frac{dy}{dx} &= x^2(3\ln(x) + 1) \\
\Rightarrow \frac{d^2y}{dx^2} &= 2x(3\ln(x) + 1) + \frac{3x^2}{x} \\
&= x(6\ln(x) + 5)
\end{aligned}
$$

(e)

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{11}{(3x+1)^2} \\
\Rightarrow \frac{d^2y}{dx^2} &= \frac{11\cdot 3\cdot(-2)}{(3x+1)^3} = \frac{-66}{(3x+1)^3}
\end{aligned}
$$

(f) $f'(x) = 3\ln(3x+2) + 3f''(x) = \frac{9}{(3x+2)}$

(g) $\frac{dy}{dx} = 1 - \frac{1}{2}x^{-\frac{3}{2}}$, so that $\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{5}{2}}$

(ii) (a) This function has constant positive curvature.

(b) $g(x)$ has positive curvature on its whole domain.

(c) The curvature changes at $x = -2 \pm \sqrt{2}$.

(d) This function is defined for $x \in (0, \infty)$. It changes curvature at $x = e^{-\frac{5}{6}}$.

(e) This function is not defined at $x = -\frac{1}{3}$, which is a vertical asymptote. The curvature is positive to the left of this line, and negative to the right of this line.

(f) This function is defined on $(-\frac{2}{3}, \infty)$. Thus it has positive curvature on all of its domain.

(g) This function is defined on $(0, \infty)$. Thus it has positive curvature on all of its domain.

(iii) (a) $f(x) = x^2 - 2x + 3$ has a stationary point at $x = 1$.

(b) $g(x) = (x^2 + 1)^3$ has a stationary point at $x = 0$.

(c) $f(x) = x^2 e^x$ has two turning points, one at $x = 0$, the other at $x = -2$.

(d) For $y = x^3 \log_e(x)$, $x \in (0, \infty)$, the only stationary point is at $x = e^{-\frac{1}{3}}$.

(e) The graph of $y = \frac{2x-3}{3x+1}$ does not have any stationary points.

(f) $f(x) = (3x + 2)\ln(3x + 2)$, $x \in \left(-\frac{2}{3}, \infty\right)$ has a stationary point where $\ln(3x + 2) = -1$, so that $x = \frac{1}{3}(e^{-1} - 2)$.

(g) $\frac{dy}{dx} = 0$ when $x = 4^{-\frac{1}{3}}$.

(iv) (a) The stationary point at $x = 1$ is a minimum, as $f''(1) = 2 > 0$.

(b) The stationary point at $x = 0$ is a minimum, as $g''(0) = 6 > 0$.

(c) The stationary point at $x = 0$ is a minimum as $f''(0) = 2$. The stationary point at $x = -2$ is a maximum, as $f''(-2) = -2e^{-2}$.

(d) The stationary point at $x = e^{-\frac{1}{3}}$ is a minimum, because $\frac{d^2y}{dx^2} = 3e^{-\frac{1}{3}} > 0$.

(e) The stationary point is a minimum. (There is positive curvature on the whole domain.)

(f) The stationary point is a minimum. (There is positive curvature on the whole domain.)

(g) The stationary point is a minimum. (There is positive curvature on the whole domain.)

(v) The graphs (with key features) are (in order):