Question 1.

(a) $f(x) = x^2 - 2x + 3 \implies f'(x) = 2x - 2$ (using the sum rule).

(b) $g(x) = (x^2+1)^3 \implies g'(x) = 3(x^2+1)^2 \cdot 2x = 6x(x^2+1)^2$, by the chain rule.

(c) $f(x) = x^2e^x \implies f'(x) = 2xe^x + x^2e^x$, by the product rule.

(d)

$$\begin{aligned} y = x^3\ln(x) \implies \frac{dy}{dx} &= 3x^2\ln(x) + x^3 \cdot \frac{1}{x} \qquad \text{product rule} \\ &= 3x^2\ln(x) + x^2 = x^2(3\ln(x)+1) \end{aligned}$$

(e) Let $y = \frac{u}{v}$ where $u = 2x-3$ and $v = 3x+1$. Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 3$. Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{2(3x+1) - 3(2x-3)}{(3x+1)^2} \qquad \text{quotient rule} \\ &= \frac{6x+2-6x+9}{(3x+1)^2} = \frac{11}{(3x+1)^2} \end{aligned}$$

[Note that you could have divided first to get $y = \frac{2}{3} - \frac{11}{3}(3x+1)^{-1}$ and used the sum, constant and chain rules.]

(f) $f(x) = (3x+2)\ln(3x+2) \implies f'(x) = 3\ln(3x+2) + (3x+2)\cdot\dfrac{1}{3x+2}\cdot 3 = 3\ln(3x+2)+3.$

(g) Note that we can avoid the quotient rule by simplifying: $y = \dfrac{x^2+\sqrt{x}}{x} = x + x^{-\frac{1}{2}}$.

It follows that $\dfrac{dy}{dx} = 1 - \frac{1}{2}x^{-\frac{3}{2}}$.

Question 2.

(a)

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \implies f'(f^{-1}(x))(f^{-1}(x))' &= 1 \quad \text{using the chain rule} \\ \implies (f^{-1}(x))' &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

(b) $g(x) = \log_e(x)$ is the inverse of the function $h(x) = e^x$. Then

$$g'(x) = \frac{1}{h'(g(x))} = \frac{1}{e^{\log_e(x)}} = \frac{1}{x}.$$

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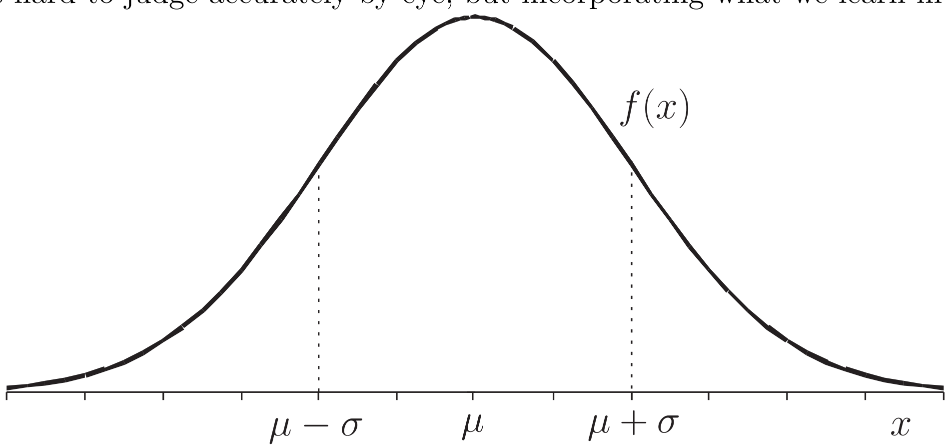
Question 3. Consider $y = f(g(h(x)))$. Write $k(x) = g(h(x))$, so that $y = f(k(x))$. Then $k'(x) = g'(h(x))h'(x)$ by the standard chain rule. Now

$$\begin{aligned} \frac{dy}{dx} &= [f(k(x))]' \\ &= f'(k(x))k'(x) \qquad \text{by the chain rule} \\ &= f'(k(x))[g'(h(x))h'(x)] \\ &= f'(g(h(x)))g'(h(x))h'(x) \end{aligned}$$

Question 4.

(a) This is hard to judge accurately by eye, but incorporating what we learn in (d):



(b)

$$\begin{aligned} f(x) &= \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2} \\ \Longrightarrow \quad f'(x) &= \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2} \\ \Longrightarrow \quad f''(x) &= \frac{1}{\sigma^3\sqrt{2\pi}} \left[-1 + \frac{(x-\mu)^2}{\sigma^2}\right] e^{-(x-\mu)^2/2\sigma^2} \end{aligned}$$

(c) $f'(x) = 0$ when $x = \mu$. We note that $f''(\mu) < 0$. This is both a local and global maximum.

(d) Using (b), $f''(x) = 0$ when

$$-1 + \frac{(x-\mu)^2}{\sigma^2} = 0 \implies (x-\mu)^2 = \sigma^2 \implies x = \mu \pm \sigma.$$

So the density function changes curvature twice, at points equally spaced on either side of the mode.

Question 5.

(i) (a) $f'(x) = 2x - 2 \implies f''(x) = 2.$

(b) $g'(x) = 6x(x^2+1)^2 \implies g''(x) = 6(x^2+1)^2 + 24x^2(x^2+1) = 6(x^2+1)(5x^2+1).$

(c) $f'(x) = (2x + x^2)e^x \implies f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(2 + 4x + x^2).$

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(d)

$$\begin{aligned} \frac{dy}{dx} &= x^2(3\ln(x)+1) \\ \implies \frac{d^2y}{dx^2} &= 2x(3\ln(x)+1) + \frac{3x^2}{x} \\ &= x(6\ln(x)+5) \end{aligned}$$

(e)

$$\begin{aligned} \frac{dy}{dx} &= \frac{11}{(3x+1)^2} \\ \implies \frac{d^2y}{dx^2} &= \frac{11\cdot 3\cdot(-2)}{(3x+1)^3} = \frac{-66}{(3x+1)^3} \end{aligned}$$

(f) $f'(x) = 3\ln(3x+2)+3 \implies f''(x) = \dfrac{9}{(3x+2)}$.

(g) $\dfrac{dy}{dx} = 1 - \frac{1}{2}x^{-\frac{3}{2}}$, so that $\dfrac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{5}{2}}$

(ii) (a) This function has constant positive curvature.

(b) $g(x)$ has positive curvature on its whole domain.

(c) The curvature changes at $x = -2 \pm \sqrt{2}$.

(d) This function is defined for $x \in (0, \infty)$. It changes curvature at $x = e^{-\frac{5}{6}}$.

(e) This function is not defined at $x = -\frac{1}{3}$, which is a vertical asymptote. The curvature is positive to the left of this line, and negative to the right of this line.

(f) This function is defined on $(-\frac{2}{3}, \infty)$. Thus it has positive curvature on all of its domain.

(g) This function is defined on $(0, \infty)$. Thus it has positive curvature on all of its domain.

(iii) (a) $f(x) = x^2 - 2x + 3$ has a stationary point at $x = 1$.

(b) $g(x) = (x^2+1)^3$ has a stationary point at $x = 0$.

(c) $f(x) = x^2e^x$ has two turning points, one at $x = 0$, the other at $x = -2$.

(d) For $y = x^3\log_e(x)$, $x \in (0, \infty)$, the only stationary point is at $x = e^{-\frac{1}{3}}$.

(e) The graph of $y = \dfrac{2x-3}{3x+1}$ does not have any stationary points.

(f) $f(x) = (3x+2)\ln(3x+2)$, $x \in \left(-\frac{2}{3}, \infty\right)$ has a stationary point where $\ln(3x+2) = -1$, so that $x = \frac{1}{3}(e^{-1} - 2)$.

(g) $\frac{dy}{dx} = 0$ when $x = 4^{-\frac{1}{3}}$.

(iv) (a) The stationary point at $x = 1$ is a minimum, as $f''(1) = 2 > 0$.

(b) The stationary point at $x = 0$ is a minimum, as $g''(0) = 6 > 0$.

(c) The stationary point at $x = 0$ is a minimum as $f''(0) = 2$. The stationary point at $x = -2$ is a maximum, as $f''(-2) = -2e^{-2}$.

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(d) The stationary point at $x = e^{-\frac{1}{3}}$ is a minimum, because $\frac{d^2y}{dx^2} = 3e^{-\frac{1}{3}} > 0$.

(f) The stationary point is a minimum. (There is positive curvature on the whole domain.)

(g) The stationary point is a minimum. (There is positive curvature on the whole domain.)

(v) The graphs (with key features) are (in order):