MAST30022 Decision Making
2021
Tutorial 8

1. **(PS6-2)** Find what each of the decision methods: Wald's Maximin, Hurwicz's Maximax, Savage's Minimax Regret, and Laplace's Criterion, would tell a company manager to do in the following decision situation. The manager has no information about what the economy will be like 3 years from now when the payoff will come, and so he/she may suppose that each state may occur equally likely. The figures in the table are profit to company in $ million. (Adapted from P. D. Straffin, "Game Theory and Strategy".)

		Economy			
		Way up	Slightly up	Slightly down	Way down
	Hold steady	3	2	2	0
Manager	Expand slightly	4	2	0	0
	Expand greatly	6	2	0	-2
	Diversity	1	1	2	2

Table 1: Question 2, Problem Set 6

Solution

	Way up	Slightly up	Slightly down	Way down	s_i	o_i	$(\sum_j v_{ij})/4$
Steady	3	2	2	0	0	3	7/4*
Slightly	4	2	0	0	0	4	3/2
Greatly	6	2	0	-2	-2	6*	3/2
Diversity	1	1	2	2	1*	2	1

From this table the Wald's Maximin, Hurwicz's Maximax, and Laplace's Criteria will pick the actions 'diversity', 'expand greatly', and 'hold steady', respectively.

The regret matrix is

	Way up	Slightly up	Slightly down	Way down	Maximum Regret
Steady	3	0	0	2	3
Slightly	2	0	2	2	2*
Greatly	0	0	2	4	4
Diversity	5	1	0	0	5

Since the minimum of the maximum regrets is attained by the second row, the Savage's Minimax Regret Criterion will pick the action 'expand slightly'.

2. **(PS6-3)** Pizza King and Nobel Greek are two competing restaurants. Each must determine the price they will charge for each pizza sold. Pizza King believes that Nobel Greek's price is a random variable D having the following probability distribution: $\mathbf{Pr}(D = \$6) = 1/4$, $\mathbf{Pr}(D = \$8) = 1/2$, $\mathbf{Pr}(D = \$10) = 1/4$. If Pizza King charges a price p_{PK} and Noble Greek charges a price p_{NG}, Pizza King will sell $100 + 25(p_{NG} - p_{PK})$ pizzas. It costs \$4 to make a pizza. Pizza King is considering charging \$5, \$6, \$7, \$8, or \$9 for a pizza. Use each of the four decision criteria (Wald's Maximin, Hurwicz's Maximax, Savage's Minimax Regret, and Laplace) to determine the price that Pizza King should charge. (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004)

Solution

Let $N = 100 + 25(p_{NG} - p_{PK})$. Pizza King can earn a profit of $\$(p_{PK} - 4)N$ when it charges a price p_{PK} and Noble Greek charges a price p_{NG}. Based on this one can work out the decision table:

	Nobel Greek's Price		
PK's Price	\$6	\$8	\$10
\$5	$125 \times 1 = 125$	$175 \times 1 = 175$	$225 \times 1 = 225$
\$6	$100 \times 2 = 200$	$150 \times 2 = 300$	$200 \times 2 = 400$
\$7	$75 \times 3 = 225$	$125 \times 3 = 375$	$175 \times 3 = 525$
\$8	$50 \times 4 = 200$	$100 \times 4 = 400$	$150 \times 4 = 600$
\$9	$25 \times 5 = 125$	$75 \times 5 = 375$	$125 \times 5 = 625$

Maximin criterion: The secured profits associated with the actions "\$5, \$6, \$7, \$8, \$9" are \$125, \$200, \$225, \$200, and \$125, respectively. Thus, if Pizza King uses the maximin criterion, it should charge \$7, and by doing so it will earn a profit of at least \$225.

Maximax criterion: The maximum profits associated with the actions "\$5, \$6, \$7, \$8, \$9" are \$225, \$400, \$350, \$600, and \$625, respectively. Thus, if Pizza King uses the maximax criterion, it should charge \$9, and by doing so it will earn a profit of at most \$625.

Minimax regret criterion: The regret matrix is shown below.

	Nobel Greek's Price			
PK's Price	\$6	\$8	\$10	Maximum Regret
\$5	100	225	400	400
\$6	25	100	225	225
\$7	0	25	100	100
\$8	25	0	25	25
\$9	100	25	0	100

Pizza King should charge \$8 in order to minimize maximum regret.

Laplace criterion: For $i = 5, \ldots, 9$, the expected profit when $p_{PK} = \$i$ is

$$\begin{aligned} \mathbf{E}(\$i) &= [100 + 25(6 - i)](i - 4)/4 + [100 + 25(8 - i)](i - 4)/2 + \\ &\quad [100 + 25(10 - i)](i - 4)/4 \\ &= [100 + 25(8 - i)](i - 4). \end{aligned}$$

So $\mathbf{E}(\$5) = 175$, $\mathbf{E}(\$6) = 300$, $\mathbf{E}(\$7) = 375$, $\mathbf{E}(\$8) = 400$, $\mathbf{E}(\$9) = 375$. Pizza King should charge \$8 for a pizza, and by doing so it will earn an expected profit of \$400.

3. **(PS6-4)** Consider decision making with risk in which probabilities $\mathbf{Pr}(\theta_j)$, $j = 1, 2, \ldots, n$, are associated with the states. Consider the *expected utility rule* which chooses a_k to maximize

$$V_k = \sum_{j=1}^{n} \mathbf{Pr}(\theta_j) v_{kj}.$$

Show that this rule satisfies the following six Axioms: complete ranking, independence of labelling, independence of value scale, strong domination, independence of irrelevant alternatives, and independence of addition of a constant to a column. (Adapted from "Decision Theory", S. French, 1986)

Solution

- **Complete ranking**: Any two actions a_i and a_j are comparable since either $V_i \geq V_j$, or $V_i \leq V_j$, or both. Transitivity is satisfied since if a_i, a_j, and a_k are three actions such that $V_i \leq V_j$ and $V_j \leq V_k$, then $V_i \leq V_k$. We also have reflexivity since, for any action a_i, $V_i \leq V_i$. Antisymmetry is satisfied since $V_i \leq V_j$ and $V_j \leq V_i$ imply $V_i = V_j$.
- **Independence of labelling**: If π is a permutation of actions and τ is a permutation of states, let (v'_{ij}) be the decision table whose $\pi(i)$-th row is the i-th row of (v_{ij}) and whose $\tau(j)$-th column is the j-th column of (v_{ij}). Now observe that $V'_{\pi(i)} = V_i$, so the decision maker's prefered action should not depend on the actions and states labelling.
- **Independence of value scale**: If $v'_{ij} = av_{ij} + b,\ \ 1 \leq i \leq m, 1 \leq j \leq n$ for some $a > 0$ and b, then $V'_i = aV_i + b$, and therefore $V_i > V_k \Longleftrightarrow V'_i > V'_k$.
- **Strong Domination**: If $v_{ij} > v_{kj}$ for all j then

$$\begin{aligned} V_i &= \sum_{j=1}^{n} \mathbf{Pr}(\theta_j) v_{ij} \\ &> \sum_{j=1}^{n} \mathbf{Pr}(\theta_j) v_{kj} \\ &= V_k. \end{aligned}$$

- **Independence of irrelevant alternatives**: V_i only depends on row i and not on other rows, so holding $V_i > V_k$ for some i and k does not change if you add an extra row.
- **Independence of addition of a constant to a column**: Let (v'_{ij}) be constructed from (v_{ij}) by adding a constant c to every entry in column ℓ and

keeping all other entries unchanged, that is, $v'_{i\ell} = v_{i\ell} + c$ for all i, and $v'_{ij} = v_{ij}$ for all i and $j \neq \ell$. Then

$$\begin{aligned} V'_i &= \sum_{j=1}^{n} \mathbf{Pr}(\theta_j) v'_{ij} \\ &= \sum_{j \neq \ell} \mathbf{Pr}(\theta_j) v_{ij} + \mathbf{Pr}(\theta_\ell)(v_{i\ell} + c) \\ &= V_i + \mathbf{Pr}(\theta_\ell) c. \end{aligned}$$

Thus $V_i > V_k \iff V'_i > V'_k$.

4. **(PS6-5)** Consider the decision table of Milnor (as discussed in lectures):

		States			
		θ_1	θ_2	θ_3	θ_4
	a_1	2	2	0	1
Actions	a_2	1	1	1	1
	a_3	0	4	0	0
	a_4	1	3	0	0

Suppose that the situation is one under risk. Show that there are values of the probabilities $\mathbf{Pr}(\theta_1), \mathbf{Pr}(\theta_2), \mathbf{Pr}(\theta_3)$, and $\mathbf{Pr}(\theta_4)$ such that a_1 is optimal under the expected utility rule (see Question 3 above). Similarly, show that there are other sets of probabilities such that a_2 and a_3 are optimal under the expected utility rule. However, show that there is no set of probabilities such that a_4 is optimal under this rule. (Adapted from "Decision Theory", S. French, 1986)

Solution

Let $p_i = \mathbf{Pr}(\theta_i)$, $i = 1, 2, 3, 4$.

- Action a_1 is optimal under the expected utility rule if and only if

  $$V_1 \geq V_2, \quad V_1 \geq V_3, \quad V_1 \geq V_4,$$

  that is,

  $$2p_1 + 2p_2 + p_4 \geq 1, \quad 2p_1 + 2p_2 + p_4 \geq 4p_2, \quad 2p_1 + 2p_2 + p_4 \geq p_1 + 3p_2.$$

  These inequalities are satisfied by choosing, for instance, $p_1 = 1/2$, $p_2 = 1/4$, $p_3 = 0$, and $p_4 = 1/4$.
- Action a_2 is optimal under the expected utility rule if and only if

  $$V_2 \geq V_1, \quad V_2 \geq V_3, \quad V_2 \geq V_4,$$

  that is,

  $$1 \geq 2p_1 + 2p_2 + p_4, \quad 1 \geq 4p_2, \quad 1 \geq p_1 + 3p_2.$$

  These inequalities are satisfied by choosing, for instance, $p_1 = p_2 = p_4 = 0$, and $p_3 = 1$.

- Action a_3 is optimal under the expected utility rule if and only if

$$V_3 \geq V_1, \quad V_3 \geq V_2, \quad V_3 \geq V_4,$$

that is,

$$4p_2 \geq 2p_1 + 2p_2 + p_4, \quad 4p_2 \geq 1, \quad 4p_2 \geq p_1 + 3p_2.$$

These inequalities are satisfied by choosing, for instance, $p_1 = 1/4$, $p_2 = 1/3$, $p_3 = 5/12$, and $p_4 = 0$.

- Action a_4 is optimal under the expected utility rule if and only if

$$V_4 \geq V_2, \quad V_4 \geq V_3, \quad V_4 \geq V_1,$$

that is,

$$p_1 + 3p_2 \geq 1, \quad p_1 + 3p_2 \geq 4p_2, \quad p_1 + 3p_2 \geq 2p_1 + 2p_2 + p_4.$$

$$\iff \qquad p_1 \geq 1 - 3p_2, \quad p_1 \geq p_2, \quad p_1 \leq p_2 - p_4.$$

The only way for these inequalities to be satisfied is to set $p_1 = p_2$ and $p_4 = 0$. But in that case, $V_4 = V_1 = V_3$, and the decision maker is indifferent between actions a_4, a_1, and a_3.

5. **(PS6-6)** Consider the decision table below (where x is a real number).

		States			
		θ_1	θ_2	θ_3	θ_4
	a_1	x	3	4	6
Actions	a_2	2	2	2	4
	a_3	3	2	1	9
	a_4	6	6	1	3

(a) Find which decision will be taken, as a function of x, according to: (i) Wald's Maximin criterion; (ii) Hurwicz's criterion (take $\alpha = 1/2$); (iii) Laplace's criterion; or (iv) Savage's Minimax Regret criterion.

(b) Find the range(s) of x for which all four criteria uniquely lead to the same choice.

(Adapted from "Decision Theory", S. French, 1986)

Solution

(a) The decision table

		States						
		θ_1	θ_2	θ_3	θ_4	s_i	$s_i/2 + o_i/2$	$\bar{v}_i$
	a_1	x	3	4	6	$\min\{x, 3\}$	$\min\{x, 3\}/2 + \max\{x, 6\}/2$	$(x + 13)/4$
Actions	a_2	2	2	2	4	2	3	2.5
	a_3	3	2	1	9	1	5	3.75
	a_4	6	6	1	3	1	3.5	4

The regret table

		States				
		θ_1	θ_2	θ_3	θ_4	ρ_i
	a_1	$\max\{x, 6\} - x$	3	0	3	$\max\{\max\{x, 6\} - x, 3\}$
Actions	a_2	$\max\{x, 6\} - 2$	4	2	5	$\max\{\max\{x, 6\} - 2, 5\}$
	a_3	$\max\{x, 6\} - 3$	4	3	0	$\max\{\max\{x, 6\} - 3, 4\}$
	a_4	$\max\{x, 6\} - 6$	0	3	6	$\max\{\max\{x, 6\} - 6, 6\}$

(i) Wald's maximin criterion: We need to choose an action that achieves $\max\{\min\{x, 3\}, 2\}$. If $x < 2$ choose a_2. If $x = 2$ choose a_1 or a_2. If $x > 2$ choose a_1.

(ii) Hurwicz's α-criterion. We need to choose an action that achieves $\max\{\min\{x/2, 1.5\} + \max\{x/2, 3\}, 5\}$. If $x < 7$ choose a_3. If $x = 7$ choose a_1 or a_3. If $x > 7$ choose a_1.

(iii) Laplace's criterion: We need to choose an action that achieves $\max\{(x + 13)/4, 4\}$. If $x < 3$ choose a_4. If $x = 3$ choose a_1 or a_4. If $x > 3$ choose a_1.

(iv) Savage's minimax regret criterion: From the regret table, for a_1, if $x < 3$ the maximum regret is $6-x$, otherwise it is 3; for a_2, if $x > 7$ the maximum regret is $x - 2$, otherwise it is 5; for a_3, if $x > 7$ the maximum regret is $x - 3$, otherwise it is 4; for a_4, if $x > 12$ the maximum regret is $x - 6$, otherwise it is 6.

So, if $x \geq 3$, the minimax regret of 3 is achieved by choosing a_1. If $2 < x < 3$, the minimax regret of $6 - x$ is achieved by choosing a_1. If $x = 2$, the minimax regret of 4 is achieved by choosing a_1 or a_3. If $x < 2$, the minimax regret of 4 is achieved by choosing a_3.

(b) If $x > 7$, then a_1 is chosen using all four criteria.