In this workshop, you will use MATLAB to solve the Support Vector Machine problem from the text, namely to find the optimal separating hyperplane for the (training) data

$$\mathbf{p}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \ \mathbf{p}_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \ \mathbf{p}_3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ \mathbf{p}_4 = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

with labels $\mathbf{l} = \begin{pmatrix} +1 & +1 & -1 & -1 \end{pmatrix}^T$. The nonlinear optimisation problem to solve is

$$\begin{aligned} \text{minimise} \quad & f(\mathbf{w}, b) = \tfrac{1}{2}\|\mathbf{w}\|^2 \\ \text{subject to} \quad & \mathbf{g}(\mathbf{w}, b) \leqslant \mathbf{0}, \end{aligned}$$

where $g_i(\mathbf{w}, b) = 1 - l_i(\mathbf{w}^T\mathbf{p}_i + b)$. The Lagrangian is given by

$$\mathcal{L} = f(\mathbf{w}, b) + \boldsymbol{\mu}^T\mathbf{g}(\mathbf{w}, b).$$

Proceed as follows:

1. Show that $Df(\mathbf{w}, b) = \begin{pmatrix} \mathbf{w} & 0 \end{pmatrix}$.
2. Run the following code in MATLAB to create a symbolic 2×1 vector (array) `w` and a symbolic variable `b` that are assumed to be real numbers (see `syms` documentation) and a vector `Df` representing $Df(\mathbf{w}, b)$.
```
syms w [2 1] real
syms b real
Df = [w' 0];
```
3. Create a matrix `p` that has $\mathbf{p}_i$ as its ith column, and a column vector `l` for the labels.
4. Show that
$$D\mathbf{g}(\mathbf{w}, b) = \begin{pmatrix} -l_1\mathbf{p}_1^T & -l_1 \\ -l_2\mathbf{p}_2^T & -l_2 \\ -l_3\mathbf{p}_3^T & -l_3 \\ -l_4\mathbf{p}_4^T & -l_4 \end{pmatrix}.$$
5. Using the matrices `p` and `l`, create a matrix `Dg` representing $D\mathbf{g}(\mathbf{w}, b)$.
6. Create a symbolic 4×1 real-valued vector `mu`.
7. Create a vector `DL` that represents $D\mathcal{L}(\mathbf{w}, b; \boldsymbol{\mu})$.
8. Create a vector `g` whose i-th row is $g_i(\mathbf{w}, b)$ and a row vector `mug` whose i-th element equals $\mu_i g_i(\mathbf{w}, b)$.
9. Create a vector `KKT` by concatenating `DL` and `mug`.
10. Define `vars = [w' b mu']` and run `sol = solve(KKT, vars)`. The solution is a structure that can be easily substituted into an expression by typing `subs(expr,sol)`. Verify that the solutions are valid by running `subs(KKT, sol)`.
11. Run `subs(mu', sol)` and observe which rows have non-negative values of μ.
12. Run `subs(g', sol)` and observe which rows correspond to feasible solutions.
13. Explain what the variables `r1` and `r2` contain after running this code:
```
r1 = all(subs(g', sol) <= 0, 2);
r2 = all(subs(mu', sol) >= 0, 2);
```
14. Use the MATLAB function `find` and the elementwise `&` operator to determine the rows corresponding to possible solutions according to the KKT theorem.
15. Based on the previous answers, determine the optimal separating hyperplane of the data points.
16. Plot the data points and the optimal separating hyperplane.
17. Use the optimal separating hyperplane, and the MATLAB function `sign` to label the new points
$$\mathbf{q}_1 = \begin{pmatrix} 1/2 \\ 3 \end{pmatrix}, \ \mathbf{q}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \ \mathbf{q}_3 = \begin{pmatrix} 7/2 \\ 2 \end{pmatrix}.$$