La Trobe University
Semester 1 Examination Period 2019

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Student ID: ☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐ Seat Number: ☐ ☐ ☐

Subject code:	MAT4MDS	Paper number:	1 (of 1)
Reading time:	15	minutes	
Writing time:	120	minutes	
Number of pages:	20 (including 4 page Fact Sheet)	(including cover sheet)	

Campus:

☐ Albury-Wodonga ☐ Bendigo ☒ Bundoora ☐ City ☐ Mildura ☐ Shepparton

Allowable materials:

Number: Description:

Instructions to candidates:

1. This exam paper consists of 55 marks.
2. Write your answers in the spaces provided using blue or black pen. If you need extra space, continue your answer on an `extra space' page.
3. Attempt all questions. Show all of your working unless instructed otherwise.
4. If you cannot do part of a question, you should still attempt later parts; information given in the question may enable you to answer them correctly.
5. Calculators may NOT be used in the exam.

Question 1. *Total: 7 marks*

Consider the matrix

$$M = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 64 & 0 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of M. *4 marks*

(b) *3 marks*

(i) Is M of full rank? Why or why not?

(ii) Is M invertible? Why or why not?

Question 2. *Total: 10 marks*

Four portfolios of similar stocks were observed in 2001 and 2002. Their percentage returns in both years are given in the following table – profits are positive, losses are negative. It is plausible that there is an underlying linear relationship between the returns given by $y = \alpha x + \beta$ where x is the return in 2001, and y is the return in 2002.

Return 2001 (%)	-2	-1	0	3
Return 2002 (%)	1	-3	5	9

(a) Fill in the entries in the matrix-vector equation $AX = b$ which would correspond to a single line passing through all 4 points. *2 marks*

$$A = \begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix}, \quad X = \begin{bmatrix} \\ \end{bmatrix}, \quad b = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

(b) Explain why the equation $AX = b$ does not have a solution. *1 mark*

(c) Write out the modified 2×2 matrix-vector equation which uniquely determines the coefficients of the linear least squares line of best fit, $y = \alpha x + \beta$. *2 marks*

(d) Solve the system you found in the previous part for α and β, and hence write down the linear least squares line of best fit. *1 mark*

(e) What is the best estimate of the return in 2002 of a similar portfolio of stocks which increased by 2% in 2001. *1 mark*

(f) Briefly and carefully explain what is meant by linear least squares line of best fit: illustrate your answer with a sketch. *2 marks*

(g) Suppose there is a portfolio of stocks which increased by 1% in 2002. Explain briefly why the linear least squares regression line you obtained above is not appropriate to use to estimate the return in 2001. *1 mark*

Use this page if you need more space. Label your answers with the question number.

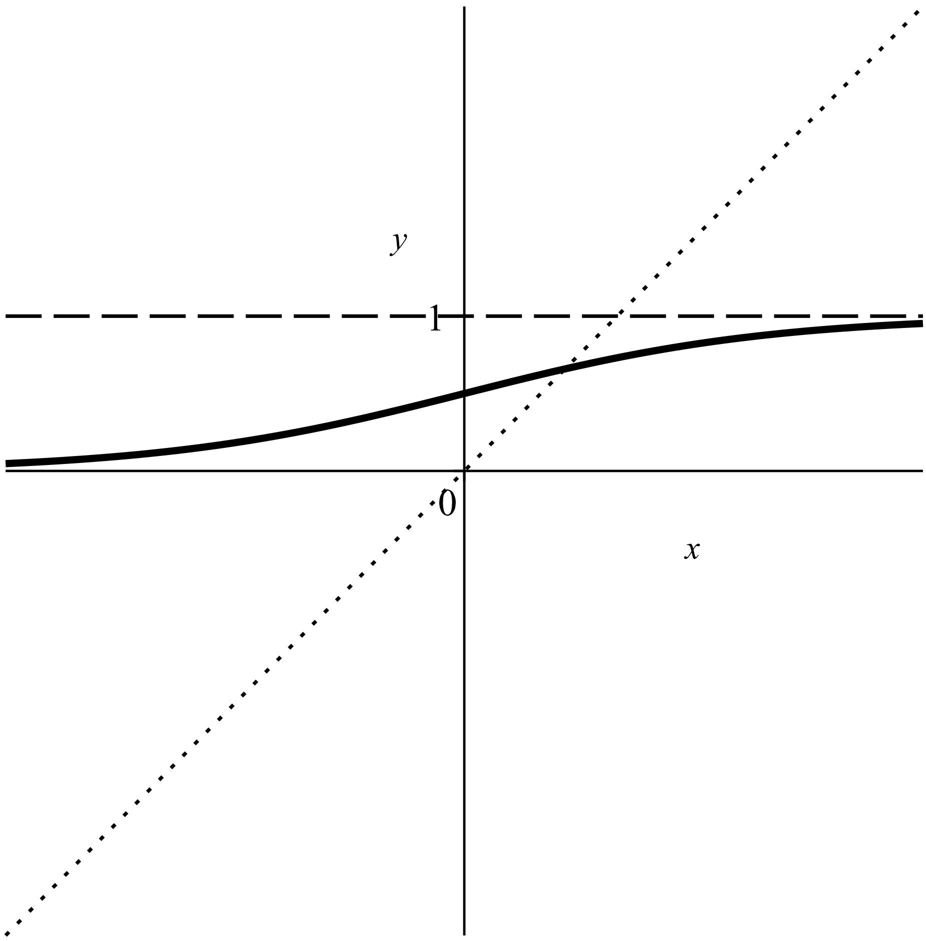
Question 3. *Total: 7 marks*

Let

$$f(x) = \frac{e^x}{1 + e^x}$$

(a) The graph of f is shown on the axes below (solid line). Also indicated is the line $y = x$ (dotted) and an asymptote of f (dashed).

On the same axes, sketch the graph of f^{-1}. *3 marks*



(b) Find the rule for the inverse function f^{-1}, where *3 marks*

$$f(x) = \frac{e^x}{1 + e^x}$$

(c) State the domain (allowed input values) of f^{-1}. *1 mark*

Question 4. *Total: 5 marks*

(a) Using an appropriate substitution, find an antiderivative of $\dfrac{x}{x^2+1}$. *2 marks*

(b) By an appropriate method, find an antiderivative of $\dfrac{x^2}{x+1}$ *3 marks*

Question 5. *Total: 15 marks*

Consider the function f (which is a probability density function) with rule

$$f(y) = \begin{cases} \dfrac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

in which α and β are both positive parameters.

(a) *3 marks*

(i) Is α a shape parameter or a scale parameter? Explain.

(ii) Is β a shape parameter or a scale parameter? Explain.

(b) (i) Calculate *2 marks*

$$\int_{-\infty}^{\infty} y f(y) dy$$

(ii) Calculate *2 marks*

$$\int_{-\infty}^{\infty} y^2 f(y) dy$$

(iii) Using your answers to (i) and (ii), show that the variance associated with this probability distribution is $\dfrac{\alpha}{\beta^2}$. *3 marks*

Use this page if you need more space. Label your answers with the question number.

(c) It can be shown that

$$f'(y) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, y^{\alpha-2} e^{-\beta y} \left[\alpha - 1 - \beta y\right]$$

so that there is a stationary point at $y = \dfrac{\alpha - 1}{\beta}$.

(Note: This information is being given to you, and you do not have to show it.)

Using the extended product rule, and the second derivative test, show that this point is a maximum (for $\alpha > 1$). *5 marks*

Question 6. *Total: 11 marks*

Consider the function of two variables

$$f(x,y) = xe^{-y^2} + x^2y.$$

(a) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ *2 marks*

(b) Find $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ *3 marks*

(c) Hence find the second order Taylor polynomial for $f(x,y)$ near $(1,0)$. *4 marks*

(d) Find *2 marks*

$$\int_0^6 f(x,y)\ dx$$

$* * * *$ **End of Questions** $* * * *$

Use this page if you need more space. Label your answers with the question number.

Matrices

- For each order $(m \times n)$, the matrix of ones $J_{m\times n}$ is the matrix in which every entry is 1. The square $(n \times n)$ matrix of ones is usually written as J_n.
- The centering matrix $C_n := I_n - \frac{1}{n}J_n$
- When it exists, the inverse of an $n \times n$ matrix A is the unique matrix, denoted by A^{-1}, with the property
$$AA^{-1} = I_n,$$
where I_n is the $n \times n$ identity matrix.
- Multiplicative Property of Determinants: $\det(AB) = \det(A)\det(B)$.
- Associated with each eigenvalue λ of a square matrix A, there is an eigenvector X, which is a non-zero column vector such that $AX = \lambda X$.
- For a 2×2 matrix A, the characteristic equation of A is
$$\lambda^2 - \text{trace}(A)\lambda + \det(A) = 0.$$
where trace(A) is the sum of the diagonal entries of A.
- The least squares solution to the system $AX = b$ is given by the solution to $A^TAX = A^Tb$.

Calculus

function	$f(x)$	x^r $(r \neq 0)$	constant	e^x	$\log_e(x)$
derivative	$f'(x)$	rx^{r-1}	0	e^x	$\frac{1}{x}$

- **The constant rule**

  If $y = cu = cg(x)$ where $c \in \mathbb{R}$ then $\quad \dfrac{dy}{dx} = c\dfrac{du}{dx} = cg'(x).$

- **The sum rule**

  If $y = u + v = g(x) + h(x)$ then $\quad \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx} = g'(x) + h'(x).$

- **The product rule**

  If $y = u{\cdot}v = g(x){\cdot}h(x)$ then $\quad \dfrac{dy}{dx} = \dfrac{du}{dx}\,v + u\,\dfrac{dv}{dx} = g'(x)\,h(x) + g(x)\,h'(x)$

- **The extended product rule**

  If $y = f(x){\cdot}g(x){\cdot}h(x)$ then $\quad \dfrac{dy}{dx} = f'(x)g(x)h(x) + f(x)g'(x)\,h(x) + f(x)g(x)\,h'(x)$

- **The chain rule**

  If $y = f(u) = f(g(x))$ and $u = g(x)$ then $\quad \dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx} = f'(g(x))\,g'(x).$

- **The quotient rule**

  If $y = \dfrac{u}{v} = \dfrac{g(x)}{h(x)}$ where $h(x) \neq 0$ then $\dfrac{dy}{dx} = \dfrac{\frac{du}{dx}\,v - u\,\frac{dv}{dx}}{v^2} = \dfrac{g'(x)\,h(x) - g(x)\,h'(x)}{[h(x)]^2}$.

- **The Second Derivative test** Suppose that $f'(x_0) = 0$.

  - If $f''(x_0) > 0$, then x_0 is a local minimum point.
  - If $f''(x_0) < 0$, then x_0 is a local maximum point.
  - If $f''(x_0) = 0$, the test is inconclusive. (This point may be a local maximum, a local minimum or a point of inflection.)

- The nth **Taylor polynomial** to f about a is the function $T_nf : \mathbb{R} \to \mathbb{R}$ where

  $$(T_nf)(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

  where $f^{(n)}(x) = \dfrac{d^n}{dx^n}(f(x))$ is the n^{th} derivative of f. T_1f is a linear approximation to f.

- **Taylor's Theorem.** Let f be a function which has an $(n+1)^{th}$ derivative defined on an interval I containing 0. If there is a positive number M such that $-M \leqslant f^{(n+1)}(x) \leqslant M$ for all $x \in I$ then

  $$\big|(E_nf)(x)\big| \;\leqslant\; \frac{M|x|^{n+1}}{(n+1)!} \qquad \text{for } x \in I.$$

  Here $(E_nf)(x) = f(x) - (T_nf)(x)$ is the error which arises when T_nf is used as an approximation to f.

Table of Common Antiderivatives.

$f(x)$	Anti-derivative $F(x)$	Comments
x^k	$\dfrac{1}{k+1}x^{k+1}$	$k \neq -1$, $x > 0$.
e^{ax}	$\frac{1}{a}e^{ax}$	
$\dfrac{1}{x}$	$\log_e(x)$	$x > 0$
$\log_e(x)$	$x\log_e(x) - x$	$x > 0$

- **Sum/difference property:** $\int(f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$
- **Constant multiple property:** $\alpha f(x)\ dx = \alpha \int f(x)\ dx$ for all $\alpha \in \mathbb{R}$.
- For all $a, b \in \mathbb{R}$ with $a \neq 0$: If $\int f(x)\ dx = F(x)$ then $\int f(ax+b)\ dx = \frac{1}{a}F(ax+b)$
- **Substitution rule:** For suitable functions f and g we have

$$\int_a^b f(u)\frac{du}{dx}\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

where $u = g(x)$.

- **Integration by Parts**

$$\int_a^b u\frac{dv}{dx}\ dx = uv|_a^b - \int_a^b v\frac{du}{dx}\ dx$$

- The **cumulative distribution function** F is an anti-derivative of the **probability density function** f for continuous data. That is:

$$P(X \leqslant x) = F(x) = \int_{-\infty}^{x} f(t)dt$$

The **mean** value is given by

$$\int_{-\infty}^{\infty} xf(x)dx$$

- **The trapezoidal rule:** The integral on $[a, b]$ of the function f can be approximated by

$$\frac{(b-a)}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

where $x_k = a + k\frac{(b-a)}{n}, \quad k = 0, 1, \ldots n$

Special functions and functions of two variables:

- **The Gamma Function:**
$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$$
It has the properties:
$$\Gamma(x+1) = x\Gamma(x) \qquad\qquad \Gamma(n+1) = n! \text{ for } n \in \mathbb{N}$$
Special values:
$$\Gamma(1) = 1 \qquad\qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

- **The Beta Function**
$$B(p,q) = \int_0^1 y^{p-1}(1-y)^{q-1}dy = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

- **The second order Taylor polynomial** of a function of two variables $f(x,y)$ about (a,b):
$$\begin{aligned} T^2_{(a,b)}f(x,y) =& f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &+ \frac{1}{2}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right\}. \end{aligned}$$