1. [2+2+2+2=8 marks] Let $f = \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = \frac{e^{2x_1x_2}}{(x_1-2)(x_2+1)},$$

which has partial derivatives

$$\frac{\partial f}{\partial x_1} = \frac{e^{2x_1x_2}\,(2x_1x_2 - 4x_2 - 1)}{(x_1-2)^2\,(x_2+1)}, \qquad \frac{\partial f}{\partial x_2} = \frac{e^{2x_1x_2}\,(2x_1x_2 + 2x_1 - 1)}{(x_1-2)\,(x_2+1)^2},$$

and let $\mathbf{p} = (1, -\frac{1}{2})^T \in \mathbb{R}^2$.

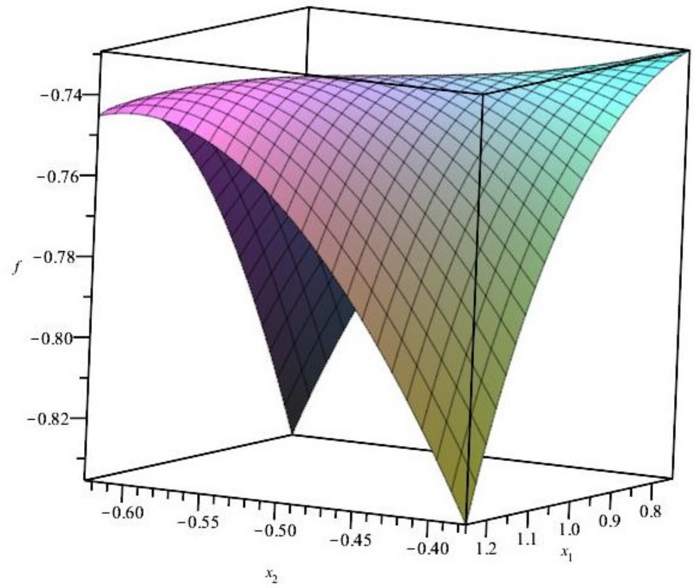
(a) Show that the point $\mathbf{p}$ is stationary. Is the FONC satisfied?

(b) You are given that the Hessian at $\mathbf{p}$ is

$$H = -\frac{2}{e}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Is the SONC satisfied? Is the SOSC satisfied?

(c) Looking at the graph below, the point seems to be on an edge.



What is the direction $\mathbf{d}$ of the edge at $\mathbf{p}$? Choose $\mathbf{d} = (a, b)^T$ with a, b integers with no common factor (other than 1) and $a > 0$.

(d) Substituting $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ in $f(\mathbf{x})$ we obtain

$$f(t) = -\frac{2\,e^{-(2t+1)^2}}{(2t-1)^2}$$

which has derivatives

$$f'(t) = \frac{32\,e^{-(1+2t)^2}t^2}{(-1+2t)^3}$$

$$f''(t) = -\frac{64\,e^{-(1+2t)^2}t\,(8t^3 - t + 1)}{(-1+2t)^4}$$

$$f'''(t) = \frac{64\,e^{-(1+2t)^2}\,(128t^6 - 48t^4 + 48t^3 + 1)}{(-1+2t)^5}.$$

Write down the third order Taylor series for $f(t)$ about $t = 0$. Can you use the Taylor series to classify the stationary point $\mathbf{p}$? If yes, classify the point. If no, explain why not.

2. [4+3+2+1=10 marks] In this question we consider the point $\mathbf{p} = (0,1)^T$ and the set-constraint problem:

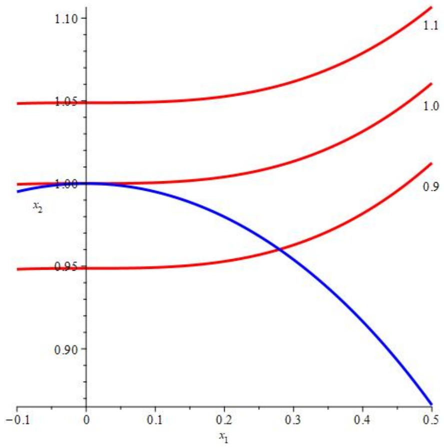
$$\begin{array}{ll} \text{maximize} & x_2^2 - x_1^3 \\ \text{subject to} & \mathbf{x} \in \Omega = \{\mathbf{x} :\ x_1 \geq 0,\ x_2 \geq 0,\ \text{and } x_1^2 + x_2^2 \leq 1\}. \end{array}$$

(a) Determine and draw in one diagram: the gradient of the objective function at $\mathbf{p}$, normal vectors to the active constraints at $\mathbf{p}$, and the feasible set Ω.

(b) Describe the set of feasible directions at $\mathbf{p}$ using the normal vectors you found in (a). State whether $(1,0)^T$ is feasible.

(c) Is the FONC satisfied at $\mathbf{p}$? Justify your answer.

(d) The c-level sets with $c \in \{0.9, 1, 1.1\}$, and one of the active constraints, are given below.

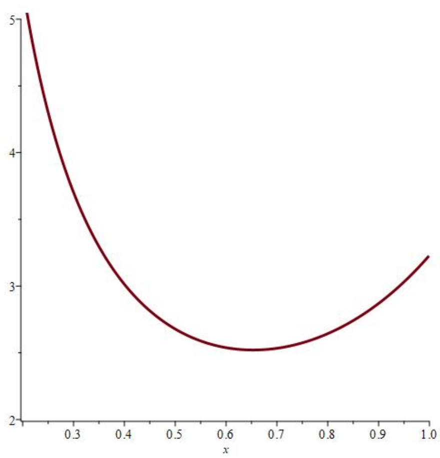


State whether the point $\mathbf{p}$ is a local maximiser. No reasons required.

3. [2+2+2=6 marks] Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \frac{e^{x^2}}{\sin(x)}.$$

Its graph



shows there is a minimum between $a = .6$ and $c = .7$. Let $\rho = \frac{3-\sqrt{5}}{2} \approx 0.381966$. Note that if your calculator is set to degrees you should multiply x by $\frac{360}{2\pi} \approx 57.2957$.

(a) Use the point $b = a + \rho(c-a) = 0.638197$ to prove that there is a minimum between a and c.

(b) Let $d = b + (1 - 2\rho)(c - a) = 0.661803$. Find out whether the minimum is in the interval (a, d) or (b, c).

(c) Update a, b, c and determine a new d. Once more, find out whether the minimum is in the interval (a, d) or (b, c).

4. [5 marks] Solve the LP problem

$$\begin{aligned} \text{Maximise} \quad & z = x_1 - x_2 \\ \text{Subject to} \quad & 2x_1 + x_2 \geq 2 \\ & 3x_1 + 2x_2 \leq 6 \\ & x_1 - 2x_2 \leq 0 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

by sketching the feasible region and drawing some level sets of the objective function. State the maximum and the corner at which the maximum occurs.

5. [2+4+3=9 marks] Consider the ILP problem

$$\begin{aligned} \text{maximize} \quad & z = 3x_2 - x_1 \\ \text{subject to} \quad & x_2 + 2x_1 \geq 8 \\ & 3x_2 - 2x_1 \leq 3 \\ & \mathbf{x} \geq \mathbf{0} \\ & \mathbf{x} \in \mathbb{Z}^2. \end{aligned}$$

(a) The simplex algorithm was applied to the corresponding LP problem and it found the canonical matrix of the optimal solution:

$$\begin{pmatrix} 1 & 0 & \frac{3}{8} & -\frac{1}{8} & \frac{21}{8} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & \frac{3}{8} & \frac{7}{8} & \frac{45}{8} \end{pmatrix}.$$

Introduce the Gomory cut which corresponds to the first row, and write it in standard form.

(b) Introduce an artificial variable x_6, write down the objective function for $w = -x_6$ (which should be maximised), give the augmented matrix for the phase 1 problem, and solve it.

(c) Reintroduce the original objective and solve the phase 2 problem. Do you find an integer solution?

6. [3+3+3+3=12 marks]

(a) Find the points on the parabola $y = 1 - x^2$ which are local extremisers with respect to the squared distance function $x^2 + y^2$.

(b) For each of the points you found in (a), determine the tangent space to the parabola.

(c) Using the Hessian of the Lagrangian, classify the points you found in (a).

(d) Solve the nonlinear optimisation problem with inequality constraint

$$\begin{aligned} \text{extremise} \quad & x^2 + y^2 \\ \text{subject to} \quad & y \leq 1 - x^2. \end{aligned}$$