MAST30013 – Techniques in Operations Research

Semester 1, 2021

Tutorial 3 Solutions

1. An symmetric, real $n \times n$ matrix, $\boldsymbol{M}$, is *positive definite* if, for all nonzero vectors $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} > 0$. This is equivalent to having all its eigenvalues greater than 0. A matrix is *positive semi-definite* if $\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} \geq 0$. Similarly, we have a *negative definite* matrix if $\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} < 0$, and a *negative semi-definite* matrix if $\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} \leq 0$.

   (a) Solving

$$\begin{aligned} \det \begin{pmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{pmatrix} &= 0 \\ \implies (\lambda + 1)(\lambda - 3) &= 0, \end{aligned}$$

   gives eigenvalues $\lambda = -1, 3$. Since one is positive, the other negative, the matrix is neither positive definite nor negative definite.

   (b) Solving

$$\begin{aligned} \det \begin{pmatrix} \lambda - 7 & -\sqrt{3} \\ -\sqrt{3} & \lambda - 1 \end{pmatrix} &= 0 \\ \implies \lambda^2 - 8\lambda + 4 &= 0, \end{aligned}$$

   gives eigenvalues $\lambda = 4 \pm 2\sqrt{3}$. Since both are positive, the matrix is positive definite.

   (c) Solving

$$\begin{aligned} \det \begin{pmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{pmatrix} &= 0 \\ \implies \lambda(\lambda - 2) &= 0, \end{aligned}$$

   gives eigenvalues $\lambda = 0, 2$. Since one is 0, the other positive, the matrix is positive semi-definite (but not positive definite).

   (d) The eigenvalues of a diagonal matrix are the diagonal entries. These are all positive. Therefore this matrix is positive definite.

2. To find the stationary points solve $\boldsymbol{\nabla} f(\boldsymbol{x}) = \boldsymbol{0}$. The *second-order sufficiency condition* states that if $\boldsymbol{x}^*$ is a stationary point, $\boldsymbol{x}^*$ is a local minimum if $\boldsymbol{\nabla}^2 f(\boldsymbol{x}^*)$ is positive definite.

   (a) (i) $\boldsymbol{\nabla} f(x_1, x_2)^T = (1 + 2x_1 - 4x_2, 1 - 4x_1 + 2x_2) = (0, 0)$ gives the two equations

$$\begin{aligned} 2x_1 - 4x_2 &= -1 \\ -4x_1 + 2x_2 &= -1. \end{aligned}$$

   Solving them gives $\boldsymbol{x}^* = \left(\frac{1}{2}, \frac{1}{2}\right)^T$.

(ii) Now,

$$\nabla^2 f(x_1, x_2) = \nabla^2 f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{pmatrix} 2 & -4 \\ -4 & 2 \end{pmatrix}.$$

Solving

$$\det \begin{pmatrix} \lambda - 2 & -4 \\ -4 & \lambda - 2 \end{pmatrix} = 0$$
$$\implies (\lambda + 2)(\lambda - 6) = 0,$$

gives eigenvalues $\lambda = -2, 6$. Since one is positive, the other negative, the matrix is neither positive definite nor negative definite, and the second-order sufficient condition is not satisfied.

(iii) Since the second-order sufficient condition is not satisfied, $\boldsymbol{x}^* = \left(\frac{1}{2}, \frac{1}{2}\right)^T$ is not a minimum, it is a saddle point.

(b) (i) $\nabla f(x_1, x_2)^T = (2x_1 - 5x_2 - 25, -5x_1 + 4x_2^3 - 8) = (0, 0)$ gives the two equations

$$2x_1 - 5x_2 = 25$$
$$-5x_1 + 4x_2^3 = 8.$$

Rearranging the first equation gives $x_1 = \frac{1}{2}(5x_2 + 25)$, which when substituted into the second equation gives $8x_2^3 - 25x_2 - 141 = 0$. Using the hint we can factorise the left hand side to get $(x_2 - 3)(8x_2^2 + 24x_2 + 47) = 0$. The quadratic factor has no real roots since $b^2 - 4ac = -928 < 0$. Thus the only stationary point is $\boldsymbol{x}^* = (20, 3)^T$.

(ii) Now,

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 2 & -5 \\ -5 & 12x_2^2 \end{pmatrix},$$

therefore

$$\nabla^2 f(20, 3) = \begin{pmatrix} 2 & -5 \\ -5 & 108 \end{pmatrix}.$$

Solving

$$\det \begin{pmatrix} \lambda - 2 & -5 \\ -5 & \lambda - 108 \end{pmatrix} = 0$$
$$\implies \lambda^2 - 110\lambda + 191 = 0,$$

gives eigenvalues $\lambda = 55 \pm \sqrt{2834}$. Since both are positive the Hessian is positive definite, and the second-order sufficient condition is satisfied.

(iii) Since the second-order sufficient condition is satisfied, $\boldsymbol{x}^* = (20, 3)^T$ is a local minimum. In fact, it is a global minimum since it is the only stationary point.

3. For a descent direction $\boldsymbol{d} = (d_1, d_2)^T$ at $\boldsymbol{x} = (1, 2)^T$, we require $\boldsymbol{\nabla} f(1,2)^T \boldsymbol{d} < 0$.

(a)

$$\begin{aligned} \boldsymbol{\nabla} f(1,2)^T \boldsymbol{d} &= \begin{pmatrix} -5 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \\ &= -5d_1 + d_2 \ < \ 0. \end{aligned}$$

Thus the set of all possible descent directions is $\{(d_1, d_2) | -5d_1 + d_2 \ < \ 0\}$. The steepest descent direction is $-\boldsymbol{\nabla} f(1,2)^T = (5, -1)$.

(b)

$$\begin{aligned} \boldsymbol{\nabla} f(1,2)^T \boldsymbol{d} &= \begin{pmatrix} -33 & 19 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \\ &= -33d_1 + 19d_2 \ < \ 0. \end{aligned}$$

Thus the set of all possible descent directions is $\{(d_1, d_2) | -33d_1 + 19d_2 \ < \ 0\}$. The steepest descent direction is $-\boldsymbol{\nabla} f(1,2)^T = (33, -19)$.