SCHOOL OF MATHEMATICS AND STATISTICS

MAST30022 Decision Making **Semester 2, 2021**

Assignment 1 Solutions

1. (a)

$$
\begin{aligned}
X_{\mathrm{I}} &= \{T_1T_2, T_1B_2, B_1T_2, B_1B_2\} \\
X_{\mathrm{II}} &= \{t_1t_2, t_1b_2, b_1t_2, b_1b_2\}
\end{aligned}
$$

(b) There are eight different plays of the game.

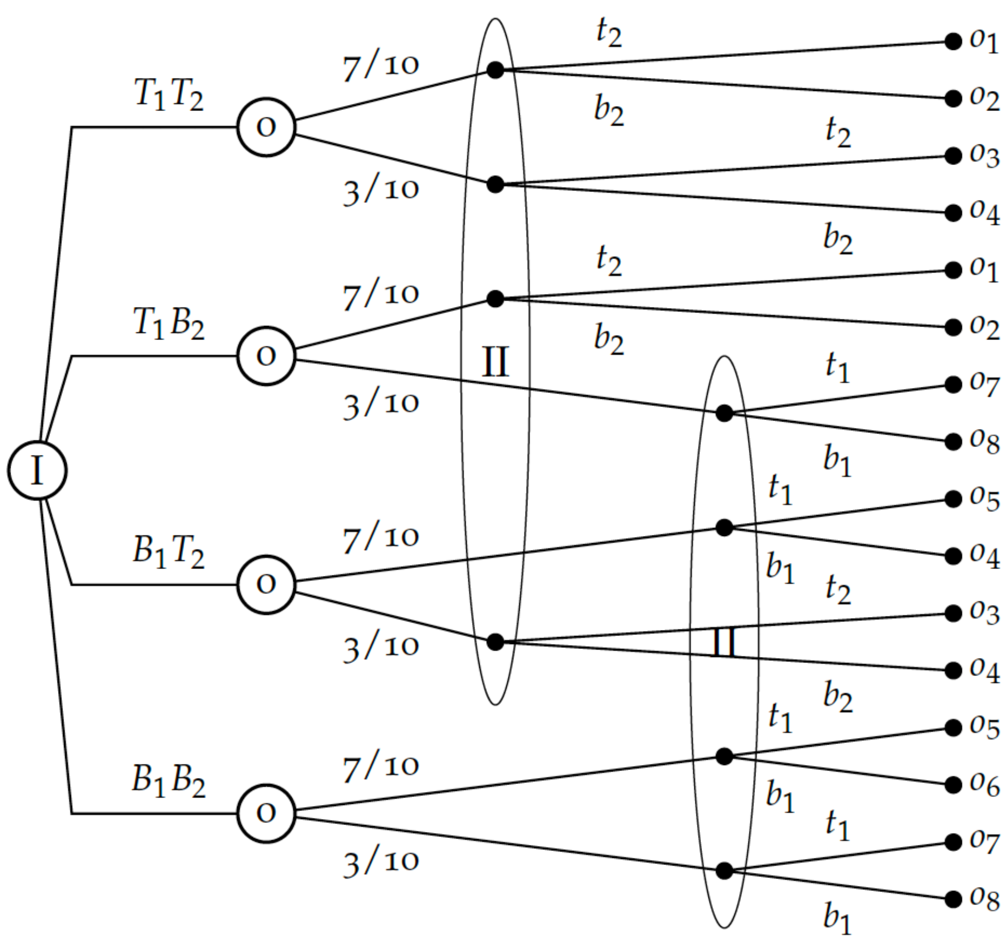
They are $T_1t_2, T_1b_2, B_1t_1, B_1b_1, T_2t_2, T_2b_2, B_2t_1$, and B_2b_1.

Alternatively, there are eight leaves.

(c) Player II does not know the result of the chance move, at each of her information sets.

On the other hand, she knows whether Player I selected the top action or whether he selected the bottom action.

(d)



2. (a) Player III: $\tau_1\tau_2$

		Player II			
		t_1t_2	t_1b_2	b_1t_2	b_1b_2
	T	$(2,4,5)$	$(2,4,5)$	$(3,8,2)$	$(3,8,2)$
Player I	M	$(2,7,3)$	$(1,1,1)$	$(2,7,3)$	$(1,1,1)$
	B	$(2,4,8)$	$(2,4,8)$	$(2,4,8)$	$(2,4,8)$

Player III: $\tau_1\beta_2$

		Player II			
		t_1t_2	t_1b_2	b_1t_2	b_1b_2
	T	$(2,4,5)$	$(2,4,5)$	$(3,8,2)$	$(3,8,2)$
Player I	M	$(4,0,5)$	$(1,1,1)$	$(4,0,5)$	$(1,1,1)$
	B	$(2,4,8)$	$(2,4,8)$	$(2,4,8)$	$(2,4,8)$

Player III: $\beta_1\tau_2$

		Player II			
		t_1t_2	t_1b_2	b_1t_2	b_1b_2
	T	$(2,4,5)$	$(2,4,5)$	$(3,8,2)$	$(3,8,2)$
Player I	M	$(2,7,3)$	$(1,1,1)$	$(2,7,3)$	$(1,1,1)$
	B	$(27,9,3)$	$(27,9,3)$	$(27,9,3)$	$(27,9,3)$

Player III: $\beta_1\beta_2$

		Player II			
		t_1t_2	t_1b_2	b_1t_2	b_1b_2
	T	$(2,4,5)$	$(2,4,5)$	$(3,8,2)$	$(3,8,2)$
Player I	M	$(4,0,5)$	$(1,1,1)$	$(4,0,5)$	$(1,1,1)$
	B	$(27,9,3)$	$(27,9,3)$	$(27,9,3)$	$(27,9,3)$

(b) The Nash equilibria in pure strategies are $(T, b_1t_2, \tau_1\tau_2)$, $(T, b_1b_2, \tau_1\tau_2)$, $(B, t_1t_2, \tau_1\tau_2)$, $(B, t_1b_2, \tau_1\tau_2)$, $(T, b_1b_2, \tau_1\beta_2)$, and $(B, t_1b_2, \tau_1\beta_2)$.

3. $(\Longrightarrow)$

Suppose $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is in equilibrium. Then

$$\begin{aligned} v_1 &= \max_{\boldsymbol{x}\in X} \min_{\boldsymbol{y}\in Y} \boldsymbol{x}\boldsymbol{V}\boldsymbol{y}^T \\ &\geq \min_{\boldsymbol{y}\in Y} \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^T \\ &= s(\boldsymbol{x}^*) \\ &= \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^*. \end{aligned}$$

Also

$$\begin{aligned} v_2 &= \min_{\boldsymbol{y}\in Y} \max_{\boldsymbol{x}\in X} \boldsymbol{x}\boldsymbol{V}\boldsymbol{y}^T \\ &\leq \max_{\boldsymbol{x}\in X} \boldsymbol{x}\boldsymbol{V}\boldsymbol{y}^{*T} \\ &= S(\boldsymbol{y}^*) \\ &= \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^*. \end{aligned}$$

Therefore

$$v_2 \;\leq\; S(\boldsymbol{y}^*) \;=\; \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^* \;=\; s(\boldsymbol{x}^*) \;\leq\; v_1.$$

But, by the the fundamental theorem of matrix game theory, $v_1 = v_2$, and so

$$s(\boldsymbol{x}^*) \;=\; v_1 \;=\; v_2 \;=\; S(\boldsymbol{y}^*).$$

$(\Longleftarrow)$

Suppose $s(\boldsymbol{x}^*) = v_1 = v_2 = S(\boldsymbol{y}^*)$. Then

$$\begin{aligned} v_1 &= s(\boldsymbol{x}^*) \\ &= \min_{\boldsymbol{y}\in Y} \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^T \\ &\leq \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^{*T} \\ &\leq \max_{\boldsymbol{x}\in X} \boldsymbol{x}\boldsymbol{V}\boldsymbol{y}^{*T} \\ &= S(\boldsymbol{y}^*) \\ &= v_2. \end{aligned}$$

Since $v_1 = v_2$, all inequalities are equalities. Hence

$$\begin{aligned} \max_{\boldsymbol{x}\in X} \boldsymbol{x}\boldsymbol{V}\boldsymbol{y}^{*T} &= \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^* \\ &= \min_{\boldsymbol{y}\in Y} \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^T \end{aligned}$$

.

Therefore, for all $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in X \times Y$,

$$\boldsymbol{x}\boldsymbol{V}\boldsymbol{y}^* \;\leq\; \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^* \;\leq\; \boldsymbol{x}^*\boldsymbol{V}\boldsymbol{y}^{*T},$$

and $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is in equilibrium.

4. We first subtract $12/2 = 6$ from every entry which gives us the two-person zero-sum game

$$\boldsymbol{V} = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -4 & -1 \\ 2 & 3 & -4 \\ -1 & 0 & 0 \end{bmatrix}.$$

First we see if there are any saddle points. $L = \max\{-2, -4, -4, -1\} = -1$ and $U = \min\{2, 3, 1\} = 1$. Since $L < U$ there are no equilibrium strategies consisting of only pure strategies.

Next we see if any rows or columns are dominated. The 2nd row is dominated by the 1st (and the 4th) row. In the resultant matrix where the 2nd row is deleted, the 2nd column is dominated by the 1st column. After deleting the 2nd column we have

$$\hat{\boldsymbol{V}} = \begin{bmatrix} -2 & 1 \\ 2 & -4 \\ -1 & 0 \end{bmatrix}.$$

In order to use the linear programming method to solve a two-person zero-sum game, we need to add a constant to each entry of $\hat{\boldsymbol{V}}$ so that there is at least one row with strictly positive entries. Here $c = 2$ is sufficient to get

$$\boldsymbol{V}' = \begin{bmatrix} 0 & 3 \\ 4 & -2 \\ 1 & 2 \end{bmatrix}.$$

The linear program for Player 2 is

$$\begin{aligned} \max\ 1/v_2' &= y_1' + y_2' \\ \text{s.t.} & \\ 3y_2 &\leq 1 \\ 4y_1' - 2y_2 &\leq 1 \\ y_1' + 2y_2' &\leq 1 \\ y_1',\ y_2' &\geq 0 \end{aligned}$$

We now solve the linear program.

	$\boldsymbol{y_1'}$	y_2'	y_4'	y_5'	y_6'	RHS
y_4'	0	3	1	0	0	1
$\boldsymbol{y_5'}$	4^*	-2	0	1	0	1
y_6'	1	2	0	0	1	1
z	$\boldsymbol{-1}$	-1	0	0	0	0

	y_1'	$\boldsymbol{y_2'}$	y_4'	y_5'	y_6'	RHS
y_4'	0	3	1	0	0	1
y_1'	1	$-1/2$	0	$1/4$	0	$1/4$
$\boldsymbol{y_6'}$	0	$5/2^*$	0	$-1/4$	1	$3/4$
z	0	$\boldsymbol{-3/2}$	0	$1/4$	0	$1/4$

	y'_1	y'_2	y'_4	y'_5	y'_6	RHS
y'_4	0	0	1	3/10	−6/5	1/10
y'_1	1	0	0	1/5	1/5	2/5
y'_2	0	1	0	−1/10	2/5	3/10
z	0	0	0	1/10	3/5	7/10

Reading from the simplex algorithm tableau we have $1/v'_2 = \frac{7}{10}$, $(y'_1, y'_2) = \left(\frac{2}{5}, \frac{3}{10}\right)$, and $(x'_1, x'_2, x'_3) = \left(0, \frac{1}{10}, \frac{3}{5}\right)$. Therefore, $v'_2 = \frac{10}{7}$, $(\hat{y}_1, \hat{y}_2) = \frac{10}{7}\left(\frac{2}{5}, \frac{3}{10}\right) = \left(\frac{4}{7}, \frac{3}{7}\right)$, $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \frac{10}{7}\left(0, \frac{1}{10}, \frac{3}{5}\right) = \left(0, \frac{1}{7}, \frac{6}{7}\right)$, and $\hat{v} = v'_2 - 2 = -\frac{4}{7}$.
Since we deleted the 2nd row and the 2nd column from $\boldsymbol{V}$, the optimal strategy for the two-person zero-sum game is $\boldsymbol{x}^* = \left(0, 0, \frac{1}{7}, \frac{6}{7}\right), \boldsymbol{y}^* = \left(\frac{4}{7}, 0, \frac{3}{7}\right)$, where $v = -\frac{4}{7}$.

Thus, the optimal strategy for the two-person constant-sum game is $\boldsymbol{x}^* = \left(0, 0, \frac{1}{7}, \frac{6}{7}\right), \boldsymbol{y}^* = \left(\frac{4}{7}, 0, \frac{3}{7}\right)$, and $v_1 = -\frac{4}{7} + 6 = \frac{38}{7}$ and $v_2 = 12 - \frac{38}{7} = \frac{46}{7}$.