Your assignment is marked out of 10. Not every question has been marked; you should self-assess the unmarked questions. A completeness mark of 2 is added for making a serious attempt at all questions.

MATLAB, row operations

1.
```
function M = pivot(A,r,c)
%This function takes a matrix and pivots around row r and column c
    if A(r,c)==0
        fprintf('I can not pivot about 0\n');
        M=1;
        return
    end
    M=A;
    M(r,:)=M(r,:)/A(r,c);
    s=size(A);
    for i=[1:r-1 r+1:s(1)]
        M(i,:)=M(i,:)-M(i,c)*M(r,:);
    end
end
```

Canonical form, basic solutions.

2. **2 marks.** The second equation in system (a) does not have a basic variable. The systems (b) and (c) are in canonical form. For (b) the basic variables are x_4, x_3, x_6 and the basic solution is $(0, 0, 1, 3, 0, 2)$. For (c) the basic variables are x_5, x_3, x_1 and the basic solution is $(1, 0, 3, 0, 2)$.

3. (a) **1 mark.** The code

```
M=sym([1 2 3 4 5 6; 1 0 1 0 1 0; 5 4 3 2 1 0])
M1=pivot(M,1,2)
M2=pivot(M1,2,5)
M3=pivot(M2,3,4)
```

produces the following matrices:

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}, \qquad M1 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & -3 & -6 & -9 & -12 \end{bmatrix},$$

$$M2 = \begin{bmatrix} -2 & 1 & -1 & 2 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 12 & 0 & 6 & -6 & 0 & -12 \end{bmatrix}, \qquad M3 = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 0 & -1 & 1 & 0 & 2 \end{bmatrix}.$$

(b) **1 mark.** No, it is not uniquely given. Any matrix with the same rows, but in a different order will do. For example, swapping rows 2 and 3,

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & -1 \\ -2 & 0 & -1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

4. (a) **2 marks.** We maximise $-z$ and introduce slack variables x_3, x_4:

$$\begin{aligned} \text{maximise} \quad & -z = 2x_1 - 4x_2 \\ \text{subject to} \quad & x_1 - 2x_2 + x_3 = 2 \\ & 2x_1 + 3x_2 + x_4 = 12 \\ & \mathbf{x} \geq 0. \end{aligned}$$

The symplex algorithm gives:

$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & 1 & 0 & 2 \\ 2 & 3 & 0 & 1 & 12 \\ -2 & 4 & 0 & 0 & 0 \end{array}\right] \equiv \left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 2 \\ 0 & 7 & -2 & 1 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{array}\right].$$

This gives the minimal solution $z = -4$ at $(2, 0)$. As there is a zero in the bottom row in the non-basic column x_2, we can find another solution:

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 2 \\ 0 & \boxed{7} & -2 & 1 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{array}\right] \equiv \left[\begin{array}{ccccc} 1 & 0 & \frac{3}{7} & \frac{2}{7} & \frac{30}{7} \\ 0 & 1 & -\frac{2}{7} & \frac{1}{7} & \frac{8}{7} \\ 0 & 0 & 2 & 0 & 4 \end{array}\right].$$

This gives the second solution $z = -4$ at $(30/7, 8/7)$. The complete set of minimisers is the set of all convex combinations

$$\{t(2, 0) + (1 - t)(30/7, 8/7),\ t \in [0, 1]\},$$

which is called the **convex hull** of the points $(2, 0)$ and $(30/7, 8/7)$.

(b) We introduce slack variables x_3, x_4 and artificial variable x_5. We first solve phase 1:

$$\begin{aligned} \text{maximise} \quad & w = -x_5 = x_1 + x_2 - x_4 - 3 \\ \text{subject to} \quad & x_1 + 4x_2 + x_3 = 12 \\ & x_1 + x_2 - x_4 + x_5 = 3 \\ & \mathbf{x} \geq 0. \end{aligned}$$

The symplex algorithm gives:

$$\begin{pmatrix} 1 & 4 & 1 & 0 & 0 & 12 \\ \boxed{1} & 1 & 0 & -1 & 1 & 3 \\ -1 & -1 & 0 & 1 & 0 & -3 \end{pmatrix} \equiv \begin{pmatrix} 0 & 3 & 1 & 1 & -1 & 9 \\ 1 & 1 & 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This gives the optimal solution $w = -x_5 = 0$ at $(3, 0, 9, 0)$. We continue phase 2, by deleting the artificial column and reinstating the original objective row:

$$\begin{aligned} \begin{pmatrix} 0 & 3 & 1 & 1 & 9 \\ \boxed{1} & 1 & 0 & -1 & 3 \\ -1 & -2 & 0 & 0 & 0 \end{pmatrix} &\equiv \begin{pmatrix} 0 & \boxed{3} & 1 & 1 & 9 \\ 1 & 1 & 0 & -1 & 3 \\ 0 & -1 & 0 & -1 & 3 \end{pmatrix} \\ &\equiv \begin{pmatrix} 0 & 1 & \frac{1}{3} & \boxed{\frac{1}{3}} & 3 \\ 1 & 0 & -\frac{1}{3} & -\frac{4}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 6 \end{pmatrix} \\ &\equiv \begin{pmatrix} 0 & 3 & 1 & 1 & 9 \\ 1 & 4 & 1 & 0 & 12 \\ 0 & 2 & 1 & 0 & 12 \end{pmatrix} \end{aligned}$$

If in the first step you chose x_1 as departing variable instead of x_3, the second matrix would be different but your final answer should be the same. The maximum of 12 is attained at $(12, 0)$

(c) We combine the first two inequalities into one equality, multiply the third inequality by -1, and introduce a slack variable to get

$$
\begin{array}{rl}
\text{maximise} & z = x_1 + 3x_2 + 2x_3 \\
\text{subject to} & x_1 + 2x_2 + x_3 = 1 \\
 & -2x_1 + x_2 + x_4 = 2 \\
 & \mathbf{x} \geq \mathbf{0}.
\end{array}
$$

The symplex algorithm gives:

$$
\begin{pmatrix} 1 & \boxed{2} & 1 & 0 & 1 \\ -2 & 1 & 0 & 1 & 2 \\ -1 & -3 & -2 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{2} & 1 & \boxed{\frac{1}{2}} & 0 & \frac{1}{2} \\ -\frac{5}{2} & 0 & -\frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix}
$$
$$
\equiv \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ -2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}
$$

This gives the optimal solution $z = 2$ at $(0, 0, 1)$.

5. (a) We multiply the first equation by -1, introduce slack variables x_3, x_4, x_5, x_6 and artificial variables x_7, x_8 and then maximise $w = -x_7 - x_8$. The MATLAB code

```
M=[ -sym(1)/2      1     -1     0      0     0     1     0       1;
    sym(1)/2       1      0     1      0     0     0     0       4;
        3      1      0     0     -1     0     0     1      5;
        2     -1      0     0      0     1     0     0      2;
    -sym(5)/2     -2      1     0      1     0     0     0      -6];
M=pivot(M,4,1);
M=pivot(M,3,2);
M=pivot(M,1,6)
```

solves the first phase

$$
\begin{bmatrix} 0 & 0 & -\frac{10}{7} & 0 & \frac{3}{7} & 1 & \frac{10}{7} & -\frac{3}{7} & \frac{9}{7} \\ 0 & 0 & \frac{5}{7} & 1 & \frac{2}{7} & 0 & -\frac{5}{7} & -\frac{2}{7} & \frac{13}{7} \\ 0 & 1 & -\frac{6}{7} & 0 & -\frac{1}{7} & 0 & \frac{6}{7} & \frac{1}{7} & \frac{11}{7} \\ 1 & 0 & \frac{2}{7} & 0 & -\frac{2}{7} & 0 & -\frac{2}{7} & \frac{2}{7} & \frac{8}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.
$$

For the second phase we apply the simplex algorithm to the matrix `M=[M(1:4,[1:6 9]); -1 -1 0 0 0 0 0]`, that is

```
M=pivot(M,4,1)
M=pivot(M,3,2)
M=pivot(M,2,3)
M=pivot(M,2,5)
```

gives us

$$
\begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 1 & 5 \\ 0 & 0 & 1 & \frac{3}{5} & 0 & -\frac{2}{5} & \frac{3}{5} \\ 0 & 1 & 0 & \frac{4}{5} & 0 & -\frac{1}{5} & \frac{14}{5} \\ 1 & 0 & 0 & \frac{2}{5} & 0 & \frac{2}{5} & \frac{12}{5} \\ 0 & 0 & 0 & \frac{6}{5} & 0 & \frac{1}{5} & \frac{26}{5} \end{bmatrix},
$$

whose basic solution is (12/5,14/5) with maximum 26/5.

(b) **2 marks.** For this problem the first phase is the same as in (a) and then for the second phase we apply the simplex algorithm to the matrix `M=[M(1:4,[1:6 9]); 1 1 0 0 0 0 0]`, that is

```
M=pivot(M,4,1)
M=pivot(M,3,2)
```

gives us

$$\begin{bmatrix} 0 & 0 & -\frac{10}{7} & 0 & \frac{3}{7} & 1 & \frac{9}{7} \\ 0 & 0 & \frac{5}{7} & 1 & \frac{2}{7} & 0 & \frac{13}{7} \\ 0 & 1 & -\frac{6}{7} & 0 & -\frac{1}{7} & 0 & \frac{11}{7} \\ 1 & 0 & \frac{2}{7} & 0 & -\frac{2}{7} & 0 & \frac{8}{7} \\ 0 & 0 & \frac{4}{7} & 0 & \frac{3}{7} & 0 & -\frac{19}{7} \end{bmatrix},$$

whose basic solution is (8/7,11/7) with maximum -19/7, and so the minimum we were after is 19/7.

One can check their answer using the solutions of the first assignment: