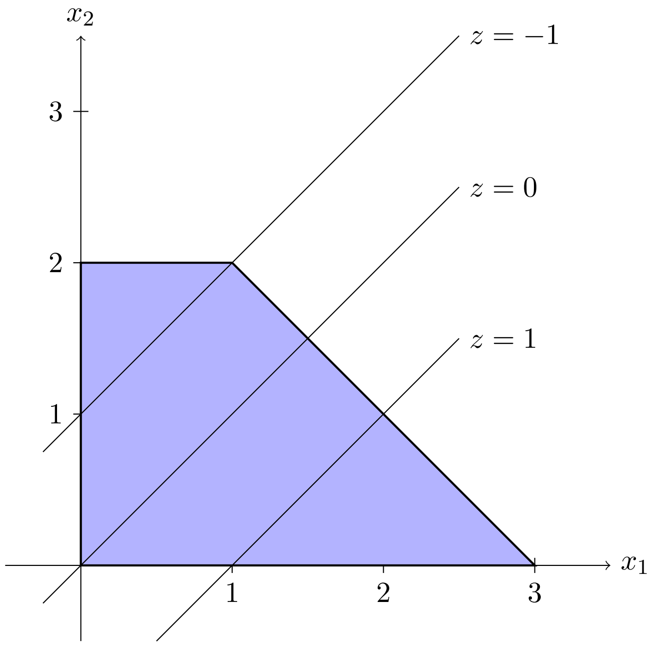
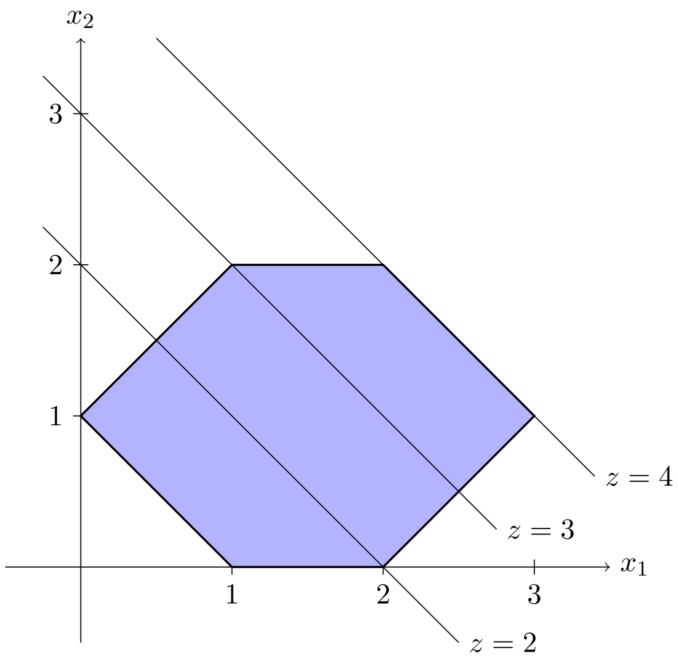
1. (a)



The optimal solution is attained at the corner point $\mathbf{x}^T = \begin{pmatrix} 3 & 0 \end{pmatrix}$, with $z = 3$.
If the question was instead to minimise z, then the solution would be $\mathbf{x}^T = \begin{pmatrix} 0 & 2 \end{pmatrix}$.

(b)

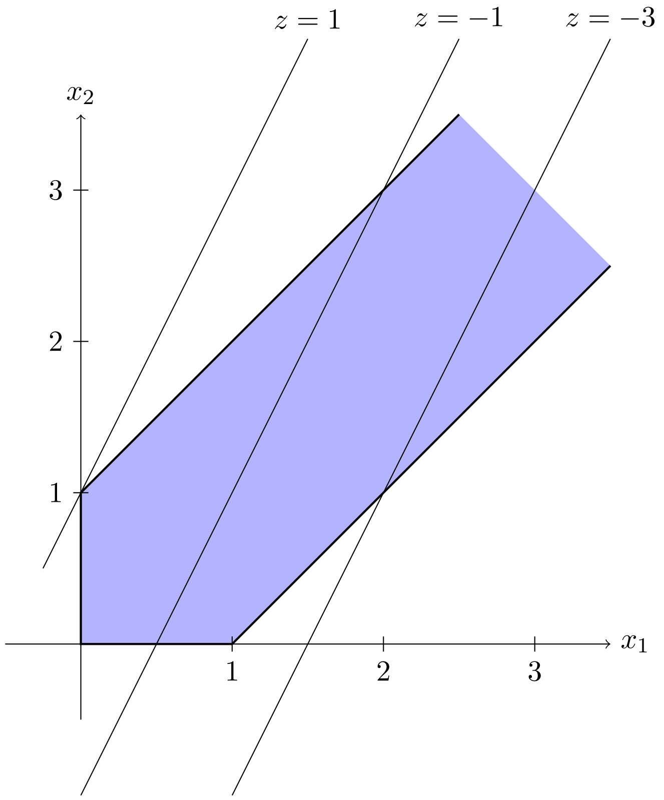


The optimal solution is attained at the two corner points $\begin{pmatrix} 3 & 1 \end{pmatrix}^T$ and $\begin{pmatrix} 2 & 2 \end{pmatrix}^T$. All points on the line segment joining them are also optimal, so the solutions are all points in the set

$$\left\{ \begin{pmatrix} 3t + 2(1-t) \\ t + 2(1-t) \end{pmatrix} \mid t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} t + 2 \\ 2 - t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

The value of the objective function is $z = 4$.
If the question was instead to minimise z, then the solution would be the line segment joining $\begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 1 \end{pmatrix}^T$.

(c)



The optimal solution is attained at the corner point $\mathbf{x}^T = \begin{pmatrix}0 & 1\end{pmatrix}$, with $z = 1$.
If the question was instead to minimise z, then there would be no solutions.

2. (a) Canonical. (Note that the basis vector e_3 is the omitted z column.)
   The basic variables are x_2 and x_4 and the basic solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 15, 0, 35, 0)$.

   (b) Not canonical.

   (c) Not canonical. (The basis vector e_1 is missing).

   (d) Canonical.
   The basic variables are x_2, x_3 and x_4 and the basic solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 2, 44, 125, 0, 0)$.

3. (a) Introduce slack variables x_3 and x_4 to give:

$$\begin{aligned} \text{maximise} \quad & z = x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 3 \\ & x_2 + x_4 = 2 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

   The initial matrix is

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 \end{array}\right).$$

   This matrix is canonical; the basic variables are the slack variables (and z).

   (b) Introduce slack variables x_3, x_4, x_5, x_6 and x_7 to give:

$$\begin{aligned} \text{maximise} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 4 \\ & x_1 - x_2 + x_4 = 2 \\ & x_2 - x_1 + x_5 = 1 \\ & x_1 + x_2 - x_6 = 1 \\ & x_2 + x_7 = 2 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

Note carefully that the slack variable x_6 has a negative coefficient. The initial matrix is

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 4 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This matrix is not canonical; the basis vector e_4 is absent (due to the negative coefficient of x_6).

(c) Introduce slack variables x_3 and x_4 to give:

$$\begin{aligned} \text{maximise} \quad & z = x_2 - 2x_1 \\ \text{subject to} \quad & x_2 - x_1 + x_3 = 1 \\ & x_1 - x_2 + x_4 = 1 \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

The initial matrix is

$$\left(\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 1 \\ 2 & -1 & 0 & 0 & 0 \end{array}\right).$$

This matrix is canonical; the basic variables are the slack variables (and z).

4. (a) The basic solution $(0,0,0,0)$ has $x_1 = x_2 = 0$, giving $z = 0$.
The basic solution $(0,2,0,2)$ has $x_1 = 0$, $x_2 = 2$, giving $z = 2$.
The basic solution $(2,0,2,0)$ has $x_1 = 2$, $x_2 = 0$, giving $z = 2$.
The basic solution $(\frac{4}{3}, \frac{4}{3}, 0, 0)$ has $x_1 = x_2 = \frac{4}{3}$, giving $z = 4$.
The maximum value of the objective function is $z = 4$, so the maximiser is $\mathbf{x}^* = \begin{pmatrix}\frac{4}{3} & \frac{4}{3}\end{pmatrix}^T$.

(b) The minimum value of the objective function is $z = 0$, so the minimiser is $\mathbf{x}^* = \begin{pmatrix}0 & 0\end{pmatrix}^T$.

(c) The basic solution $(0,0,0,0)$ has $x_1 = x_2 = 0$, giving $z = 0$.
The basic solution $(0,2,0,2)$ has $x_1 = 0$, $x_2 = 2$, giving $z = -2$.
The basic solution $(2,0,2,0)$ has $x_1 = 2$, $x_2 = 0$, giving $z = 2$.
The basic solution $(\frac{4}{3}, \frac{4}{3}, 0, 0)$ has $x_1 = x_2 = \frac{4}{3}$, giving $z = 0$.
The maximiser is $\begin{pmatrix}2 & 0\end{pmatrix}^T$ and the minimiser is $\begin{pmatrix}0 & 2\end{pmatrix}^T$.

5. (a) To make x_1 basic and keep x_4 basic, we will pivot on the entry in row 1, column 1. This gives

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 3 \end{array}\right) \quad \begin{array}{l} \\ \\ R_3' = R_3 + R_1 \end{array}$$

(b) To make x_2 basic and keep x_3 basic, we will pivot on the entry in row 2, column 2. This gives

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 \end{array}\right) \equiv \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ -1 & 0 & 0 & -1 & -2 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 - R_2 \\ \\ R_3' = R_3 - R_2 \end{array}$$

Then, to make x_1 basic and keep x_2 basic, we will pivot on the entry in row 1, column 1.

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ -1 & 0 & 0 & -1 & -2 \end{array}\right) \equiv \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & -1 \end{array}\right) \quad \begin{array}{l} \\ \\ R_3' = R_3 + R_1 \end{array}$$

6.
```
q3 = [1 1 1 0 3; 0 1 0 1 2; -1 1 0 0 0];

% Transforming q3 so that x1 and x4 are basic:
m1 = q3;
m1(3,:) = m1(3,:) + m1(1,:);

% Transforming q3 so that x2 and x3 are basic:
m2 = q3;
m2([1 3],:) = m2([1 3],:) - m2(2,:);
% then make x1 and x2 basic
m2(3,:) = m2(3,:) + m2(1,:)
```