MAST30001 Stochastic Modelling

Assignment 1

Please complete the Plagiarism Declaration Form (available through the LMS) before submitting this assignment.

Don't forget to staple your solutions, and to print your name, student ID, tutorial time and day, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 7 September, 2018 by 4pm** in the appropriate assignment box at the north end of Peter Hall Building (near Wilson Lab).

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. A discrete time Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ has the following transition matrix.

$$P = \begin{pmatrix} 0 & 3/10 & 7/10 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 \\ 1/4 & 1/8 & 0 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) Write down the communication classes of the chain.

(b) Find the period of each communicating class.

(c) Determine which classes are essential.

(d) Classify each essential communicating class as transient or positive recurrent or null recurrent.

(e) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

(f) Find the expected number of steps taken for the chain to first reach state 3, given the chain starts at state 4.

Ans.

(a) The communicating classes of the chain are $S_1 := \{1, 2, 3\}$ and $S_2 := \{4, 5\}$.

(b) S_1 is aperiodic (loop) and S_2 has period 2.

(c) S_1 is the only essential communicating class.

(d) S_1 is positive recurrent because it is finite and essential.

(e) Because there is one essential communicating class, the chain will eventually end up in it. Because the class is positive recurrent and aperiodic, it is ergodic, with long run probabilities given by the unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, 0, 0)$ satisfying $\pi P = \pi$ and $\pi_1 + \pi_2 + \pi_3 = 1$. Solving these equations yields

$$\pi = \left(\frac{20}{111}, \frac{40}{111} \frac{51}{111} \right).$$

(f) For $j \in S, j \neq 3$, let e_j be the expected amount of time to reach state 3 given the chain starts in state j. We want to find e_4. First step analysis implies

$$
\begin{aligned}
e_5 &= 1 + e_4, \\
e_4 &= 1 + \frac{5}{8}e_5 + \frac{1}{4}e_1 + \frac{1}{8}e_2, \\
e_1 &= 1 + \frac{3}{10}e_2 \\
e_2 &= 1 + \frac{1}{2}e_1,
\end{aligned}
$$

and solving gives $e_4 = 101/17$.

2. Let $0 < \alpha < 1/2$. A Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ has transition probabilities given by: for $|i| = 0, 2, 3, 4, \ldots$,

$$p_{i,i+1} = p_{i,i-1} = \alpha, \quad p_{i,0} = 1 - 2\alpha,$$

and

$$
\begin{aligned}
p_{1,2} &= \alpha, \quad p_{1,0} = 1 - \alpha, \\
p_{-1,-2} &= \alpha, \quad p_{-1,0} = 1 - \alpha.
\end{aligned}
$$

Note that the chain is irreducible.

(a) For which values of α is the chain transient? Null recurrent? Positive recurrent?

(b) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

(c) For $j \in \mathbb{Z}$, define $T(j) = \inf\{n \geq 1 : X_n = j\}$. Find an expression in terms of α for $\mathbb{E}[T(j)|X_0 = j]$.

Ans.

(a) Since the chain is irreducible, and aperiodic (loop at state 0), we can determine positive recurrence by whether or not $\pi P = \pi$ has a probability vector solution π (necessarily unique), where P is the transition matrix of the chain. These equations are for $i \neq 0$,

$$\pi_i = \alpha \pi_{i-1} + \alpha \pi_{i+1},$$

and

$$\pi_0 = \alpha \pi_1 + \alpha \pi_{-1} + (1 - 2\alpha).$$

By symmetry in the state space about zero, it's clear that any stationary distribution solution will have $\pi_i = \pi_{-i}$. Thus we solve the the system of equations for $i \geq 1$:

$$\pi_i = \alpha \pi_{i-1} + \alpha \pi_{i+1},$$

with

$$\pi_0 = 2\alpha \pi_1 + (1 - 2\alpha).$$

Using the approach from lectures, we try solutions of the form, for $j \geq 1$,

$$\pi_j = A u_1^j + B u_2^j,$$

where u_1, u_2 satisfy the characteristic equation

$$u = \alpha + \alpha u^2.$$

These solutions are

$$u_1 = \frac{1+\sqrt{1-4\alpha^2}}{2\alpha}; \quad u_2 = \frac{1-\sqrt{1-4\alpha^2}}{2\alpha};$$

note they are distinct as $0 < \alpha < 1/2$. The first solution u_1 decreases in α and equals 1 when $\alpha = 1/2$, and so $u_1 > 1$ for $0 < \alpha < 1/2$. Similarly, $0 < u_2 < 1$, and so no matter the choice of B, if $A \neq 0$, then $Au_1^j + Bu_2^j \to \pm\infty$ as $j \to \infty$. Thus in order for π to be a probability vector, we must have $A = 0$.
Thus, for $j \geq 1$, we let $\pi_j = Bu_2^j$ for some $B > 0$ and use the boundary condition to establish

$$\pi_0 = 2\alpha B u_2 + (1 - 2\alpha).$$

(After finding B, we should check that this is non-negative.) To find B, we use that the probabilities have to sum to one:

$$1 = \pi_0 + 2\sum_{i=1}^{\infty} \pi_i = 2\alpha B u_2 + (1-2\alpha) + 2B\frac{u_2}{1-u_2};$$

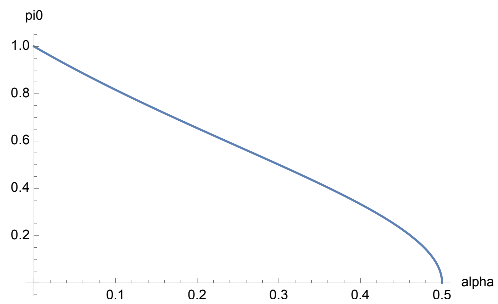
or

$$B = \frac{\alpha}{\alpha u_2(1-u_2) + u_2}(1-u_2),$$

and thus our candidate solution is

$$\pi_i = \frac{\alpha}{\alpha(1-u_2)+1} u_2^{|i|-1}(1-u_2), \quad i \neq 0$$
$$\pi_0 = \frac{2\alpha^2(1-u_2)}{\alpha(1-u_2)+1} + 1 - 2\alpha.$$

Regardless of the rigour of the steps above, it's easy to verify that this π satisfies the $\pi P = \pi$ and that it is a probability vector (note that $\pi_0 > 0$; see plot below), and therefore the chain is positive recurrent for all $\alpha \in (0, 1/2)$.



(b) Since the chain is irreducible, aperiodic, and positive recurrent, it is ergodic, with long run probabilities given by the stationary distribution π from (a).

(c) Since the Markov chain is ergodic with stationary distribution π from (a), $\mathbb{E}[T(j)|X_0 = j] = 1/\pi_j$.

In case you're interested here is R code to simulate X_n:

```
a<-1/4 ## alpha
n<-100 ## number of steps to run the chain
N<-1000 ## number of iterations of Xn
X<-c(1:N) ## c[i] is value of Xn on ith interation

for (i in 1:N){
  x<-0
  for (j in 1:n){
   u<-runif(1)
    if (u<a){
      x<-x+1
    } else
      if ({a<u} && {u<2*a})
        {x<-x-1}
   else {x<-0}
      }

  X[i]<-x
}

hist(X)

sum(X==0)/N
```

And here is Mathematica code for the stationary distribution:

```
p[i_, a_] := a/(a (1 - (1 - Sqrt[1 - 4 a^2])/(2 a)) + 1) ((1 - Sqrt[1 - 4
    a^2])/(2 a))^(Abs[i] - 1) (1 - (1 - Sqrt[1 - 4 a^2])/(2 a))

 p0[a_] := 2 a^2 (1 - (1 - Sqrt[1 - 4 a^2])/(2 a))/(a (1 - (1 - Sqrt[1 - 4
      a^2])/(2 a)) + 1) + 1 - 2 a
```