Decision Making
Part 7: Utility Theory

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Topics in this part

- Lotteries and utilities: simple lotteries, compound lotteries, von Neumann-Morgenstern Axioms, von Neumann-Morgenstern Theorem, expected utility criterion

- Decision tables and utility

- Attitude towards risk: expected value, certainty equivalent, risk premium, attitude towards risk

References:

S. French, Decision theory: an introduction to the mathematics of rationality, Ellis Horwood, 1986, Sections 5.1–5.3, 5.5

W. L. Winston, Operations Research, Section 13.2

Lotteries and utilities

<u>Lotteries</u>

Example. Choose one of the following gambles to play.

Gamble A: Toss a fair coin. If it is heads, you win \$10; if it is tails, you lose \$4 (i.e. with probability $1/2$ you win \$10 and with probability $1/2$ you lose \$4).

Gamble B: Throw a fair die. If you get an even number, you win \$100; if you get an odd number, you lose \$50 (i.e. with probability $1/2$ you win \$100 and with probability $1/2$ you lose \$50).

These are examples of lotteries. Which gamble do you prefer to play?

Expected monetary payoff of A $= (1/2) \cdot 10 + (1/2) \cdot (-4) = 3$

Expected monetary payoff of B $= (1/2) \cdot 100 + (1/2) \cdot (-50) = 25$

Many people may choose A to avoid a big loss (\$50).

However, for some rich people, \$50 is not a big amount, and since B has a good probability to win a large amount (\$100), they may choose B.

The decision relies on the utility of the monetary payoff. Roughly, a utility function of a decision maker is a function representing his preference order.

Example. (cont.) If the utility function of a decision maker gives

$$u(100) = 1, u(10) = 0.2, u(-4) = -0.1, u(-50) = -1,$$

then

$$\mathbf{E}(\text{utility of A}) = (1/2) \cdot u(10) + (1/2) \cdot u(-4) = 0.05$$

$$\mathbf{E}(\text{utility of B}) = (1/2) \cdot u(100) + (1/2) \cdot u(-50) = 0$$

Since $\mathbf{E}(\text{utility of } A) > \mathbf{E}(\text{utility of } B)$, this decision maker may choose A.

We will study cases where a decision maker's preferred lottery can be determined by using the so-called "expected utility criterion".

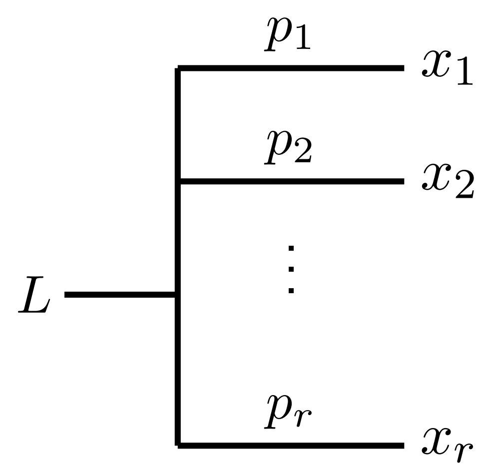
Definition 1. A simple lottery is a game which has a finite number of rewards whose probabilities are known to the player before the game.

Suppose a simple lottery has r rewards $x_1, x_2, \ldots, x_r$ which occur with probabilities $p_1, p_2, \ldots, p_r$ respectively, where each $p_i \geq 0$ and $\sum_{i=1}^{r} p_i = 1$. Then we denote this simple lottery by

$$L = \langle p_1, x_1; p_2, x_2; \ldots; p_r, x_r \rangle \,.$$

In this notation we allow some of the rewards x_i to be repeated.

A lottery is often represented by a tree in which each edge stands for a possible outcome. The numbers along the edges indicate the probability that the corresponding outcome will occur. The lottery $L = \langle p_1, x_1; p_2, x_2; \ldots; p_r, x_r \rangle$ can thus be represented as follows.



Example 1. (simple lottery)

Pay \$20 to buy a ticket for playing roulette. If it shows number i $(0 \leq i \leq 36)$, you get i dollars. This game is the simple lottery:

$$\left\langle \frac{1}{37}, -20; \frac{1}{37}, -19; \frac{1}{37}, -18; \ldots; \frac{1}{37}, 16 \right\rangle$$

Example 2. (simple lotteries: investments)

Suppose that in the next one year you want to invest \$100,000 in exactly one of the following.

A (shares): With probabilities $0.1, 0.3, 0.25, 0.35$ the market conditions next year will be very good, good, average, bad, respectively, and the corresponding return rate is 20%, 12%, 5%, -8%, respectively.

B (investment bank account): With probabilities 0.4 and 0.6 the interest rate next year will be 5% and 6.5%, respectively.

C (real estate): With probabilities $0.3, 0.4, 0.3$ the real estate market will be very strong, strong, weak, respectively, and the corresponding return rate is 10%, 7%, 3%, respectively.

Which investment strategy do you choose?

Example (cont.) Formulate each of the options as a simple lottery:

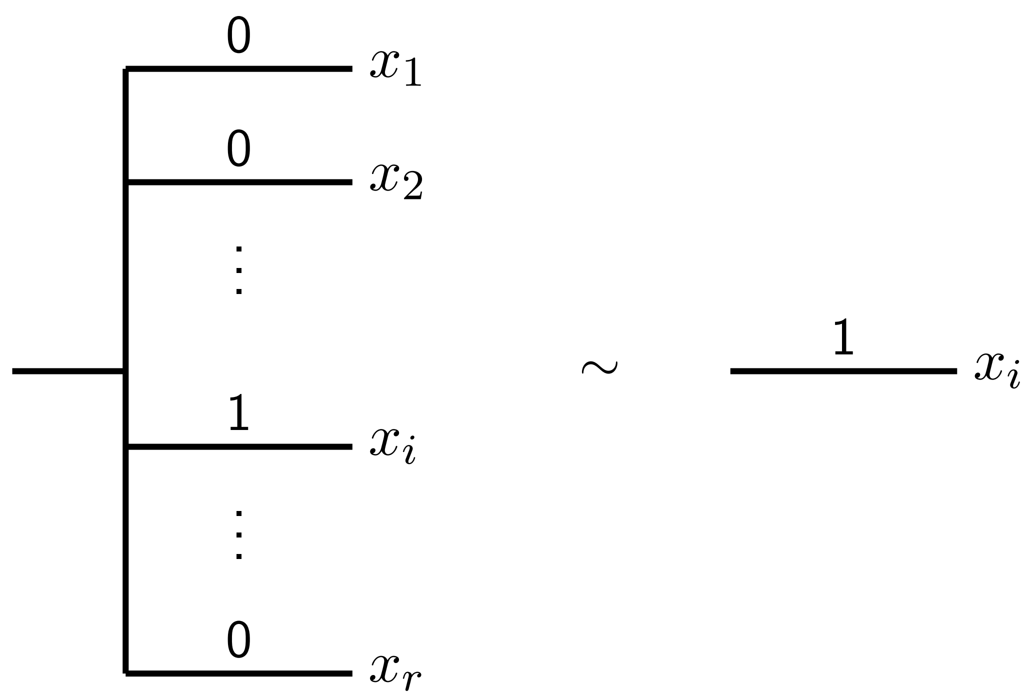
This problem involves choosing the most preferable lottery from a given set of lotteries.

A number of decision making problems can be formulated as the problem of choosing the most preferable lottery from a given set of lotteries.

We assume that the decision maker is indifferent between a reward and its corresponding simple lottery:

$$x_i \sim \langle 0, x_1; \ldots; 1, x_i; \ldots; 0, x_r \rangle.$$

By suppressing 0 probabilities, we may write $x_i \sim \langle 1, x_i \rangle$. That is, a reward can be identified with a simple lottery.



Definition 2. In a compound lottery some or all outcomes may be entries into further lotteries. If with probability q_i the player wins an entry into lottery L_i, $1 \leq i \leq s$, where each $q_i \geq 0$ and $\sum_{i=1}^{s} q_i = 1$, then we represent this compound lottery by

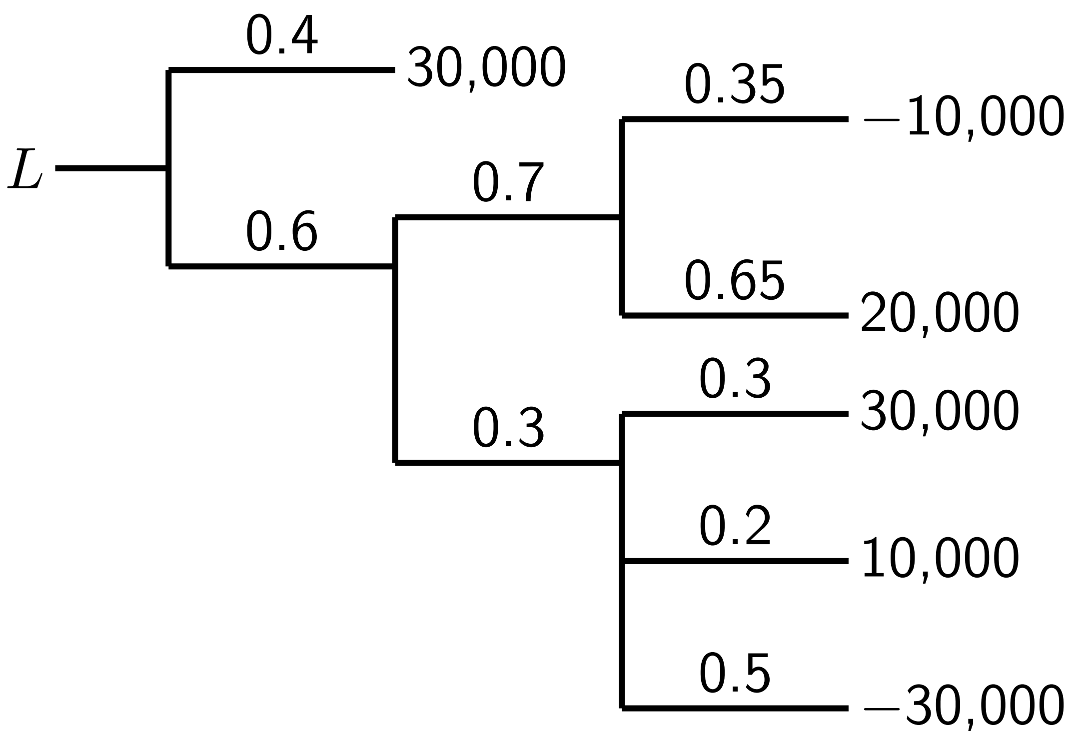
$$\langle q_1, L_1; q_2, L_2; \ldots; q_s, L_s \rangle.$$

In this compound lottery, some of the L_i may be compound lotteries, which themselves may be compound, and so on.

We assume that all compound lotteries in this part are finitely compounded lotteries, that is, the ultimate reward is obtained after a finite number of random steps.

Example 3. (compound lottery)

Let $L = \langle 0.4, 30{,}000; 0.6, L_1\rangle$, with $L_1 = \langle 0.7, L_2; 0.3, L_3\rangle$ and $L_2 = \langle 0.35, -10{,}000; 0.65, 20{,}000\rangle$, $L_3 = \langle 0.3, 30{,}000; 0.2, 10{,}000; 0.5, -30{,}000\rangle$. Then L is a "3-level" compound lottery. A graphical representation of L is given in the picture below.



von Neumann-Morgenstern Axioms

Problem 1. Suppose $\mathcal{L}$ is a set of simple and finitely compounded lotteries. Suppose X is the set of all possible rewards. Let A be the union of X, $\mathcal{L}$, and a set of certain additional lotteries that will become clear later.

How should the decision maker choose the most preferable lottery or reward from A?

We will see that if the decision maker accepts a number of "rational" axioms, then the most preferable lottery or reward is determined by the "expected utility criterion", i.e. the larger the expected utility, the better the lottery for the decision maker.

Denote by $x_{\min}$ and $x_{\max}$ the minimum and maximum rewards in X respectively.

von Neumann-Morgenstern Axioms

Axiom 1. (Weak order) The decision maker's preference over A is a weak order (i.e. it is transitive and comparable).

Axiom 2. (Non-triviality) $x_{\max} > x_{\min}$.

If $x_{\max} = x_{\min}$, then all rewards in X are equal and so all the lotteries in $\mathcal{L}$ have the same expected utility. In this case the problem is trivial.

Axiom 3. (Reduction of compound lotteries) Let

$$L = \langle q_1, L_1; q_2, L_2; \ldots; q_s, L_s \rangle$$

be a compound lottery. Suppose each

$$L_j = \langle p_{j1}, x_1; p_{j2}, x_2; \ldots; p_{jr}, x_r \rangle$$

is a simple lottery. Then

$$L \sim L'$$

where

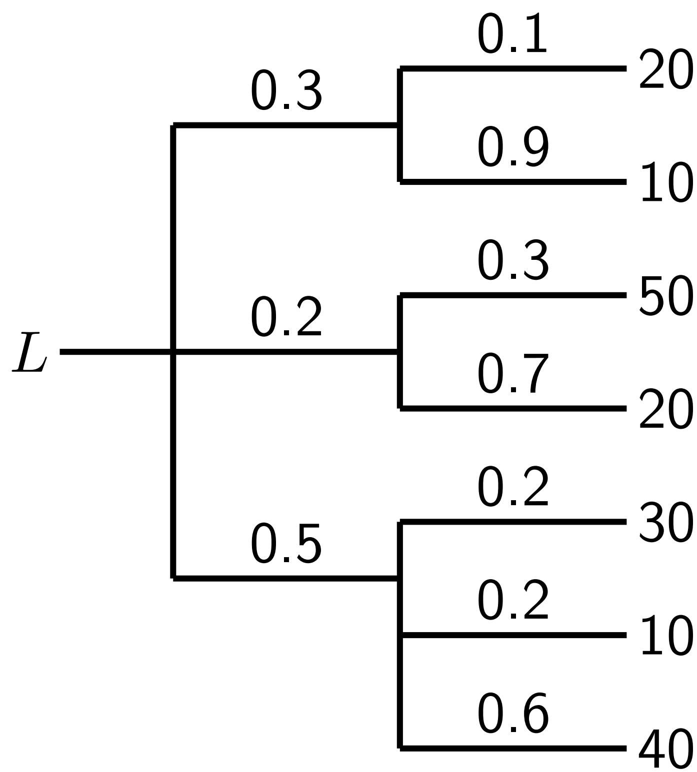
$$L' = \langle p_1, x_1; p_2, x_2; \ldots; p_r, x_r \rangle$$

with

$$p_i = q_1 p_{1i} + q_2 p_{2i} + \cdots + q_s p_{si}, \text{ for } i = 1, 2, \ldots, r.$$

By applying this axiom repeatedly, any finitely compounded lottery can be "reduced" to a simple lottery.

Example 4. (reduction of a finitely compounded lottery to a simple lottery)
Consider the lottery L as depicted below.



Determine a simple lottery L' such that $L \sim L'$.

Example (cont.)

Example 5. (reduction of a finitely compounded lottery to a simple lottery)
Let $L = \langle 0.4, \text{30,000}; 0.6, L_1 \rangle$, with $L_1 = \langle 0.7, L_2; 0.3, L_3 \rangle$ and
$L_2 = \langle 0.35, -\text{10,000}; 0.65, \text{20,000} \rangle$,
$L_3 = \langle 0.3, \text{20,000}; 0.2, \text{10,000}; 0.5, -\text{20,000} \rangle$.

Reduce L to a simple lottery.

Axiom 4. (Substitutability) Let $b, c \in A$ be such that the decision maker holds $b \sim c$. Let

$$L = \langle \ldots; q, b; \ldots \rangle \in A$$

be any lottery, simple or compound, and let

$$L' = \langle \ldots; q, c; \ldots \rangle \in A$$

be obtained from L by replacing b by c but keeping all other terms unchanged. Then the decision maker holds

$$L \sim L'.$$

Consider

$$\langle p, x_{\max}; 1-p, x_{\min}\rangle.$$

This is a simple lottery under which one has probability p to win the largest reward $x_{\max}$ and the remaining probability to win the smallest reward $x_{\min}$.

Axiom 5. (The reference experiment) For any p with $0 \le p \le 1$,

$$\langle p, x_{\max}; 1-p, x_{\min}\rangle \in A.$$

Axiom 6. (Monotonicity)

$$\langle p, x_{\max}; 1-p, x_{\min}\rangle \succeq \langle p', x_{\max}; 1-p', x_{\min}\rangle \Longleftrightarrow p \ge p'.$$

In other words, if two simple lotteries have $x_{\max}$ and $x_{\min}$ as the only rewards, then the decision maker prefers the lottery with higher probability of obtaining $x_{\max}$.

Axiom 7. (Continuity) For any $x_i \in X$, there exists u_i with $0 \leq u_i \leq 1$ such that the decision maker holds

$$x_i \sim \langle u_i, x_{\max}; 1 - u_i, x_{\min} \rangle.$$

This number u_i determined by x_i is the utility of x_i. That is, $u_i = u(x_i)$ for the utility function $u(x)$ whose existence will be proved soon.

Definition 3. Let $L = \langle p_1, x_1; \ldots; p_r, x_r \rangle$ be a simple lottery. Define the expected utility of L as

$$\mathbf{E}(U \text{ of } L) = \sum_{i=1}^{r} p_i u(x_i).$$

Theorem 1. (von Neumann and Morgenstern 1950's) If the decision maker's preferences over A satisfy all the seven axioms above, then there exists a utility function $u(x)$ on X such that

$$x_i \succeq x_j \Longleftrightarrow u(x_i) \geq u(x_j) \text{ for any } x_i, x_j \in X$$

and, for any simple lotteries $L = \langle p_1, x_1; \ldots; p_r, x_r \rangle \in A$ and $L' = \langle p_1', x_1; \ldots; p_r', x_r \rangle \in A$,

$$L \succeq L' \Longleftrightarrow \mathbf{E}(U \text{ of } L) \geq \mathbf{E}(U \text{ of } L').$$

Expected utility of compound lotteries

If the decision maker accepts the axioms above, then any finitely compounded lottery is equivalent to a simple lottery. Hence we can compute the expected utility of any lottery, simple or compound.

Utility function

From the axiom of continuity, we have

Definition 4. The utility of x is the number u with $0 \le u \le 1$ such that

$$x \sim \langle u, x_{\max}; 1 - u, x_{\min} \rangle.$$

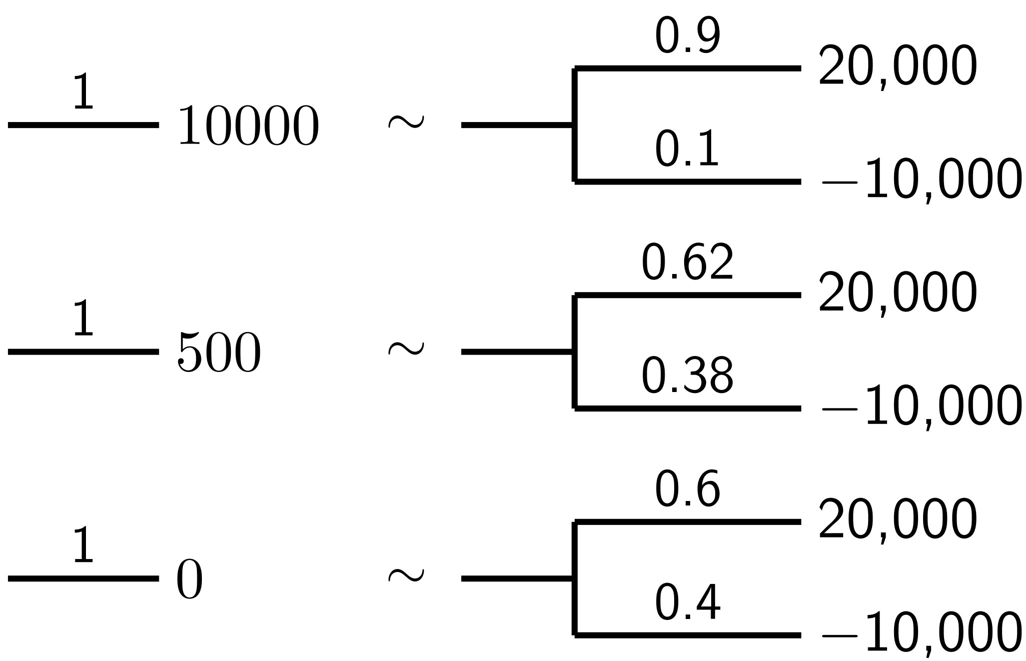
This can be used to determine the utility of any reward.

Expected utility criterion

Corollary 1. If the decision maker's preferences over A satisfy the seven axioms above, then his most preferable lottery in A is the one with maximum expected utility.

This rule is called the expected utility criterion.

Example 6. (expected utility)
Let $\mathcal{L} = \{L_1, L_2, L_3, L_4\}$, with $L_1 = \langle 1, \text{10,000} \rangle$, $L_2 = \langle 0.5, \text{20,000}; 0.5, 0 \rangle$, $L_3 = \langle 1, 0 \rangle$ and $L_4 = \langle 0.02, -\text{10,000}; 0.98, 500 \rangle$. Then $X = \{-\text{10,000}, 0, 500, \text{10,000}, \text{20,000}\}$. Suppose that the decision maker holds the equivalence relations as depicted below.



Determine for each $x \in X$ the utility $u(x)$ of x. Use this information to determine the expected utility of L_i, $i = 1, 2, 3, 4$ and the decision maker's preferences over $\mathcal{L}$ based on the expected utility criterion.

Example (cont.)

Note that $X = \{20{,}000, 10{,}000, 500, 0, -10{,}000\}$, $x_{\max} = 20{,}000$ and $x_{\min} = -10{,}000$. The utility of x is the number $u(x)$ with $0 \leq u(x) \leq 1$, such that

$$x \sim \langle u(x), 20{,}000; 1 - u(x), -10{,}000\rangle.$$

From the given equivalence relations we derive that

$$\begin{aligned} u(10{,}000) &= 0.9, \\ u(500) &= 0.62, \\ u(0) &= 0.6. \end{aligned}$$

Furthermore, (by definition),

$$\begin{aligned} u(20{,}000) &= 1, \\ u(-10{,}000) &= 0. \end{aligned}$$

Example (cont.)

From the information

- $L_1 = \langle 1, 10000\rangle$, $L_2 = \langle 0.5, \text{20,000}; 0.5, 0\rangle$, $L_3 = \langle 1, 0\rangle$ and $L_4 = \langle 0.02, -\text{10,000}; 0.98, 500\rangle$;
- $u(\text{10,000}) = 0.9$, $u(500) = 0.62$, $u(0) = 0.6$, $u(\text{20,000}) = 1$, $u(-\text{10,000}) = 0$;

we can calculate the expected utilities of the lotteries as follows

$$\mathbf{E}(U \text{ of } L_1) = 1 \times u(\text{10,000}) = 1 \times 0.9 = 0.9$$

$$\mathbf{E}(U \text{ of } L_2) = 0.5u(\text{20,000}) + 0.5u(0) = 0.5 \times 1 + 0.5 \times 0.6 = 0.8$$

$$\mathbf{E}(U \text{ of } L_3) = 1 \times u(0) = 1 \times 0.6 = 0.6$$

$$\mathbf{E}(U \text{ of } L_4) = 0.02u(-\text{10,000}) + 0.98u(500) = 0.02 \times 0 + 0.98 \times 0.62 = 0.6076.$$

We have that $\mathbf{E}(U \text{ of } L_1) > \mathbf{E}(U \text{ of } L_2) > \mathbf{E}(U \text{ of } L_4) > \mathbf{E}(U \text{ of } L_3)$. Therefore the preference of this decision maker with respect to $\mathcal{L}$ is given by

$$L_1 \succ L_2 \succ L_4 \succ L_3.$$

Remark In practice it will often be the case that it is assumed that a decision maker's preferences over A satisfy all the seven axioms of von Neumann-Morgenstern and in addition the preferences of this decision maker are represented by a strictly increasing utility function $\bar{u}(x) : [x_{\min}, x_{\max}] \to \mathbb{R}$ such that

$$x_i \succeq x_j \iff \bar{u}(x_i) \geq \bar{u}(x_j), \text{ for any } x_i, x_j \in [x_{\min}, x_{\max}].$$

The utility function $u(x) : [x_{\min}, x_{\max}] \to [0, 1]$ as described in the theorem of von Neumann-Morgenstern can be derived from $\bar{u}(x)$ by a linear transformation

$$u(x) = a\bar{u}(x) + b,$$

where a and b are such that $u(x_{\max}) = 1$ and $u(x_{\min}) = 0$. Since $\bar{u}(x)$ is strictly increasing, we have that $a > 0$.

Let $L = \langle p_1, x_1; \ldots; p_r, x_r \rangle \in A$ and $L' = \langle p'_1, x_1; \ldots; p'_r, x_r \rangle \in A$. The above observation implies that

$$\sum_{i=1}^{r} p_i u(x_i) \geq \sum_{i=1}^{r} p'_i u(x_i) \Longleftrightarrow \sum_{i=1}^{r} p_i \bar{u}(x_i) \geq \sum_{i=1}^{r} p'_i \bar{u}(x_i).$$

Which, in combination with the result of von Neumann-Morgenstern, gives us

$$L \succeq L' \Longleftrightarrow \sum_{i=1}^{r} p_i \bar{u}(x_i) \geq \sum_{i=1}^{r} p'_i \bar{u}(x_i).$$

Hence, if a strictly increasing utility function is given, we can directly derive the preferences of the decision maker over a set of lotteries by applying the expected utility criterion based on this utility function. It is not necessary to explicitly determine the utility function which has its values in the interval $[0, 1]$.

Decision table and utility

Decision making with risk

Consider a decision table:

Consequences		States			
		θ_1	θ_2	$\dots$	θ_n
	a_1	x_{11}	x_{12}	$\dots$	x_{1n}
	a_2	x_{21}	x_{22}	$\dots$	x_{2n}
Actions	$\vdots$	$\vdots$			$\vdots$
	a_m	x_{m1}	x_{m2}	$\dots$	x_{mn}
		$\mathbf{Pr}(\theta_1)$	$\mathbf{Pr}(\theta_2)$	$\dots$	$\mathbf{Pr}(\theta_n)$
		Probabilities of states			

This is an example of decision making with risk. The decision maker does not know the true state for certain before he makes his decision. However, he can quantify his uncertainty by a probability distribution on the set of states.

Main question: which action is preferred by the decision maker?

For each $i = 1, 2, \ldots, m$, let

$$L_i = \langle \mathbf{Pr}(\theta_1), x_{i1}; \mathbf{Pr}(\theta_2), x_{i2}; \ldots; \mathbf{Pr}(\theta_n), x_{in} \rangle.$$

It may be reasonable to assume that a rational decision maker holds

$$a_i \sim L_i$$

for each i.

The set of possible rewards is

$$X = \{x_{ij} : i = 1, \ldots, m, j = 1, \ldots, n\}.$$

In L_i above the rewards $x_{i1}, \ldots, x_{in}$ need not be distinct.

If, for example, $x_{i1} = x_{i2}$, then it may be reasonable to assume a decision maker holds

$$\begin{aligned} L_i &= \langle \mathbf{Pr}(\theta_1), x_{i1}; \mathbf{Pr}(\theta_2), x_{i2}; \ldots; \mathbf{Pr}(\theta_n), x_{in} \rangle \\ &\sim \langle \mathbf{Pr}(\theta_1) + \mathbf{Pr}(\theta_2), x_{i1}; 0, x_{i2}; \ldots; \mathbf{Pr}(\theta_n), x_{in} \rangle. \end{aligned}$$

Denote

$$\mathcal{L} = \{L_i : i = 1, \ldots, m\}.$$

Axiom 8. (Equivalence of situations of uncertainty) The decision maker considers his choice in the decision table as equivalent to the choice between lotteries in $\mathcal{L}$. In particular,

$$a_i \succeq a_k \Longleftrightarrow L_i \succeq L_k, \;\; i, k = 1, \ldots, m.$$

Theorem 2. If a decision maker accepts the axiom above as well as the seven von Neumann-Morgenstern axioms, then he holds

$$a_i \succeq a_k \Longleftrightarrow \mathbf{E}(U \text{ of } L_i) \geq \mathbf{E}(U \text{ of } L_k).$$

In particular, his most preferable alternative is determined by the expected utility criterion (that is, he would choose a_k with largest $\mathbf{E}(U \text{ of } L_k)$).

Example 7. (W.L. Winston, decision making with risk)

David's utility function for his asset position x is given by $u(x) = x^{\frac{1}{2}}$. Currently, David's assets consist of \$10,000 in cash and a \$90,000 home. During a given year, there is 0.001 chance that David's home will be destroyed by fire or other causes. How much would David be willing to pay for an insurance policy that would replace his home if it were destroyed?

To solve this problem: provide a decision table that describes this situation, formulate the lotteries that corresponds to the actions, and use the expected utility criterion to answer the question.

Example (cont.)

Attitude towards risk

Expected value

The attitude of a decision maker towards risk is implied in his utility function. To explain this let us give two definitions first.

Definition 5. Let

$$L = \langle p_1, x_1; p_2, x_2; \ldots; p_r, x_r \rangle.$$

be a simple lottery. The expected value of L is defined as

$$EV(L) = \sum_{i=1}^{r} p_i x_i.$$

Note that $EV(L)$ is different from the expected utility $\mathbf{E}(U \text{ of } L) = \sum_{i=1}^{r} p_i u(x_i)$ of L.

We may think of x_i's as monetary rewards. Then $EV(L)$ is the expected *monetary* reward of L, and $\mathbf{E}(U \text{ of } L)$ is the expected *utility* of the reward of L.

Certainty equivalent

Definition 6. Let $u(x)$ be the utility function of a decision maker. The certainty equivalent of a simple lottery L, denoted by $CE(L)$, is defined as the value x^* such that the decision maker is indifferent between playing L and receiving x^* for certain. That is, $CE(L)$ is the value x^* such that

$$u(x^*) = \mathbf{E}(U \text{ of } L).$$

Given a simple lottery $L = \langle p_1, x_1; p_2, x_2; \ldots; p_r, x_r \rangle$ and the decision maker's utility function u, we can use the following method to find $CE(L)$:

- compute $\mathbf{E}(U \text{ of } L) = \sum_{i=1}^{r} p_i u(x_i)$;
- solve for x the equation $u(x) = \mathbf{E}(U \text{ of } L)$;
- the solution x^* to this equation gives us $CE(L)$.

In general, $u(x)$ is a strictly increasing function, i.e. if $x > x'$, then $u(x) > u(x')$. So the equation above has a unique solution.

Example 8. (Expected value and certainty equivalent)

Let $L = \left\langle \frac{1}{3}, 100; \frac{1}{2}, 400, \frac{1}{6}, 900 \right\rangle$ and $u(x) = 3x^{\frac{1}{2}}$. Find the expected value and certainty equivalent of L. What do these values tell us about the attitude towards risk of this decision maker?

Risk premium

Definition 7. Let $u(x)$ be the utility function of a decision maker. The risk premium of L is defined as

$$RP(L) = EV(L) - CE(L).$$

A lottery is called non-degenerate if it has at least two outcomes.

Definition 8. A decision maker with utility function $u(x)$ is called

- risk-averse if $RP(L) > 0$ for **every** non-degenerate lottery L;
- risk-seeking if $RP(L) < 0$ for **every** non-degenerate lottery L;
- risk-neutral if $RP(L) = 0$ for **every** non-degenerate lottery L.

Example 9. (risk premium)

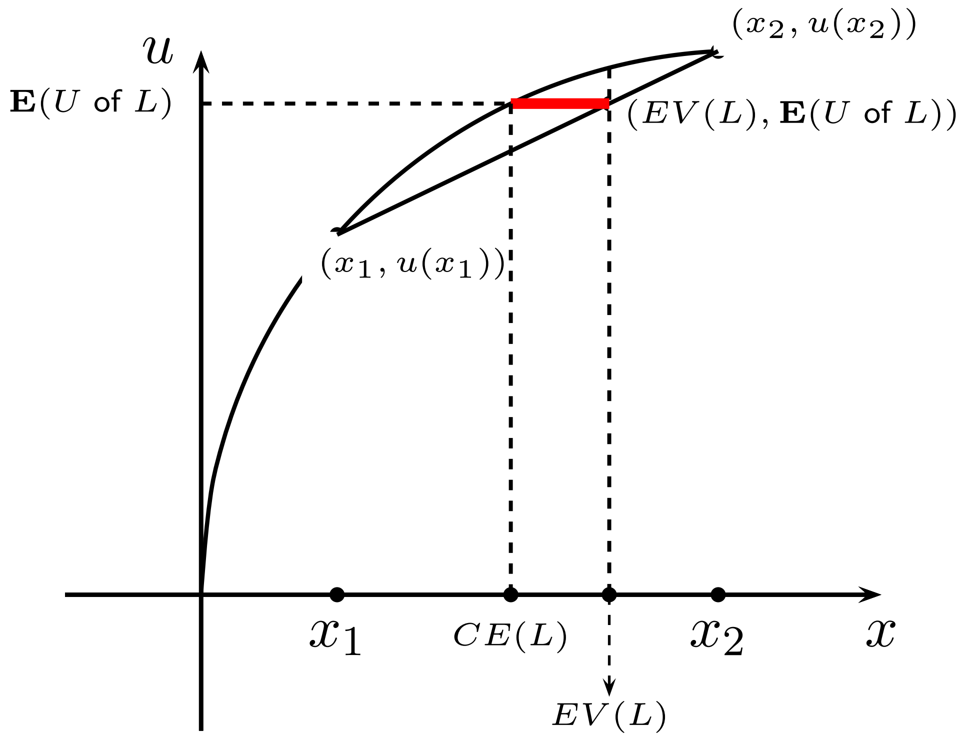
David's utility function for his asset position x is given by $u(x) = x^{\frac{1}{2}}$. Currently, David's assets consist of \$10,000 in cash and a \$90,000 home. During a given year, there is 0.001 chance that David's home will be destroyed by fire or other causes.

Let $L_2 = \langle 0.001, \text{10,000}; 0.999, \text{100,000} \rangle$, recall that this lottery corresponds to the action 'do not buy insurance'. Determine the risk premium for L_2.

Example 10. (graphical interpretation attitude towards risk)
Let $L = \langle p, x_1; (1-p), x_2 \rangle$, where $x_1 < x_2$. Then

$$\mathbf{E}(U \text{ of } L) = pu(x_1) + (1-p)u(x_2),$$
$$EV(L) = px_1 + (1-p)x_2.$$

The utility function of the decision maker is depicted in the figure below, as are the values of $\mathbf{E}(U \text{ of } L)$, $EV(L)$ and $CE(L)$.



$RP(L) = EV(L) - CE(L) > 0$, the decision maker is risk averse.

Concave and convex functions

Definition 9. A function f over an interval $[a, b]$ is called strictly concave if, for any $x_1, x_2 \in [a, b]$,

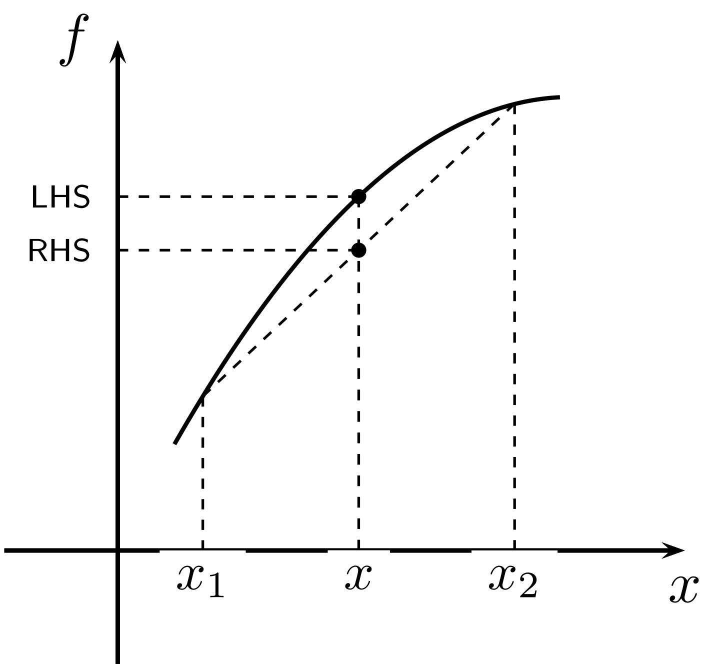
$$f\left(\frac{x_1 + x_2}{2}\right) > \frac{f(x_1) + f(x_2)}{2},$$

or equivalently if

$$f(\alpha x_1 + (1 - \alpha)x_2) > \alpha f(x_1) + (1 - \alpha)f(x_2),$$

for any $\alpha \in [0, 1]$.

A strictly concave function:



Definition 10. A function f is called strictly convex if, for any $x_1, x_2 \in [a, b]$,

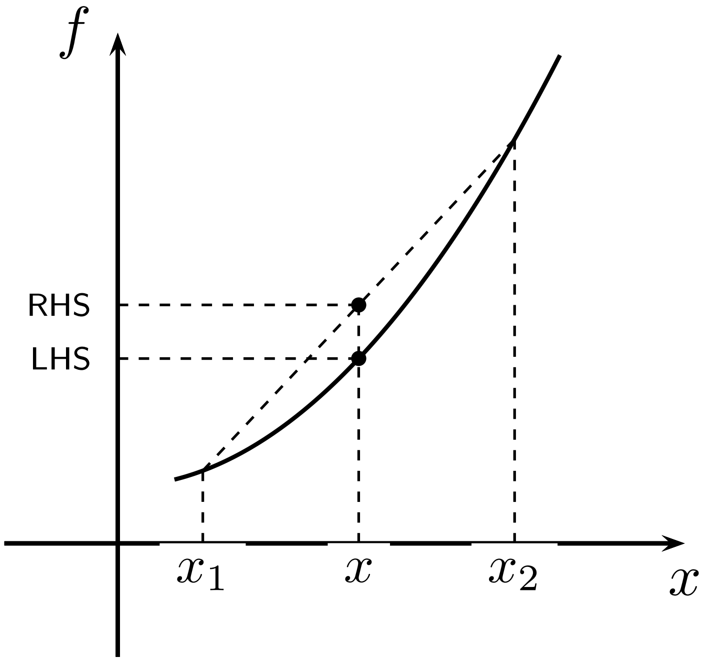
$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2},$$

or equivalently if

$$f(\alpha x_1 + (1 - \alpha)x_2) < \alpha f(x_1) + (1 - \alpha) f(x_2),$$

for any $\alpha \in [0, 1]$.

A strictly convex function:



Corollary 2. If f is strictly concave (convex, respectively), then for any $x_1, \ldots, x_n \in (a, b)$ and any $\alpha_1, \ldots, \alpha_n > 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) > \sum_{i=1}^{n} \alpha_i f(x_i)$$

$$\left(f\left(\sum_{i=1}^{n} \alpha_i x_i\right) < \sum_{i=1}^{n} \alpha_i f(x_i), \text{ respectively.}\right)$$

Finding decision maker's attitude towards risk

Theorem 3. A decision maker with utility function $u(x)$ is

- risk-averse if and only if $u(x)$ is strictly concave;
- risk-seeking if and only if $u(x)$ is strictly convex;
- risk-neutral if and only if $u(x)$ is a linear function.

Proof.

Proof. (cont.)

From calculus we have:

Theorem 4. If the second-order derivative f'' of a function f exists over (a, b), then

- it is strictly concave over $[a, b]$ iff $f''(x) < 0$ for any $x \in (a, b)$;
- it is strictly convex over $[a, b]$ iff $f''(x) > 0$ for any $x \in (a, b)$.

Corollary 3. If the decision maker's utility function $u(x)$ has the second-order derivative, then he is

- risk-averse if and only if $u''(x) < 0$ for any x;
- risk-seeking if and only if $u''(x) > 0$ for any x;
- risk-neutral if and only if $u''(x) = 0$ for any x.

Example 11. (attitude towards risk)

Determine the attitude of the decision maker towards risk (where $x \geq 0$) if

(i) $u(x) = \sqrt{x}$ (recall previous example of David's utility function for his asset position x);

(ii) $u(x) = x^2$;

(iii) $u(x) = 3x + 8$.