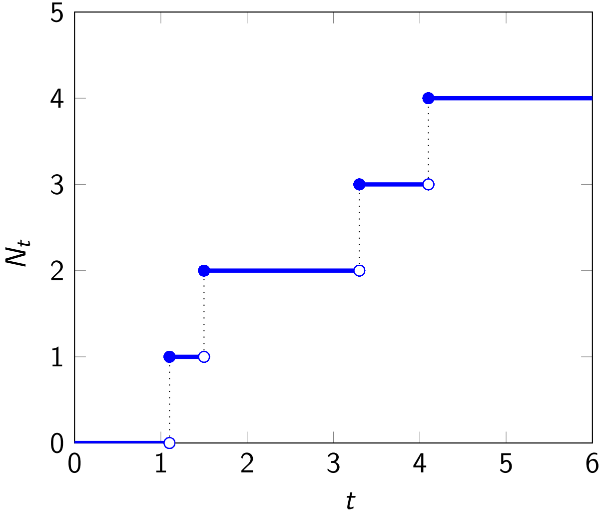
Renewal theory

Renewal process

A renewal process $(N_t)_{t \geq 0}$ is a counting process for which the times $\tau_j \geq 0$ between successive events, called renewals, are i.i.d. non-negative-real-valued random variables with an arbitrary common distribution function F.

- We assume $F(0) < 1$.
- The mean of τ_1 is $\mu > 0$ (which may or may not be finite).
- A Poisson process is a renewal process where the τ_i have an exponential distribution.
- A renewal process that is not a Poisson process is not Markovian.

A picture of N_t



$T_n = \sum_{i=1}^{n} \tau_i$ is time of nth jump.

Counting vs waiting representations

When we looked at the Poisson process, we saw that we could use a counting process description in terms of the number N_t of points in the interval $[0, t]$ or a waiting time description in terms of the time T_n until the nth event. This carries over to the study of renewal processes. Specifically

- $\{N_t \geq n\} = \{T_n \leq t\}$
- $\{N_t < n\} = \{T_n > t\}$
- $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$
- $T_{N_t} \leq t < T_{N_t+1}$.

Example

Light bulbs have a lifetime that has distribution function F. If a light bulb burns out, it is immediately replaced. Let N_t be the number of bulbs that have failed by time t. Then N_t is a renewal process.

N_t goes to ∞ as $t \to \infty$

For any fixed n,

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}(N_t > n) &= \lim_{t\to\infty} \mathbb{P}(T_n < t) \\ &= 1.\end{aligned}$$

This implies that with probability 1, $N_t \to \infty$ as $t \to \infty$.

Questions

- Can there be an explosion (that is an infinite number of renewals in a finite time)?
- What is the distribution of N_t?
- What is the average renewal rate? That is, at which rate does $N_t \to \infty$?

Explosion?

For any fixed $t < \infty$, $\mathbb{P}(N_t = \infty) = 0$. To see this, write

$$\begin{aligned}\mathbb{P}(N_t = \infty) &= \lim_{n\to\infty} \mathbb{P}(N_t \geq n) = \lim_{n\to\infty} \mathbb{P}(T_n \leq t) \\ &= \lim_{n\to\infty} \mathbb{P}(\sum_{i=1}^{n} \tau_i \leq t) \\ &= \lim_{n\to\infty} \mathbb{P}(e^{-\sum_{i=1}^{n}\tau_i} \geq e^{-t}).\end{aligned}$$

Using Markov's inequality ($\mathbb{P}(X \geq a) \leq \mathbb{E}[X]/a$ for $X \geq 0$ and $a > 0$) we have

$$\mathbb{P}(e^{-\sum_{i=1}^{n}\tau_i} \geq e^{-t}) \leq e^t \mathbb{E}[e^{-\sum_{i=1}^{n}\tau_i}] = e^t \left(\mathbb{E}[e^{-\tau_1}]\right)^n,$$

which goes to 0 as $n \to \infty$ since $\mathbb{E}[e^{-\tau_1}] < 1$ (why?)

Distribution of N_t

$$
\begin{aligned}
\mathbb{P}(N_t = n) &= \mathbb{P}(T_n \le t < T_{n+1}) \\
&= \mathbb{P}(T_n \le t) - \mathbb{P}(T_n \le t, T_{n+1} \le t) \\
&= \mathbb{P}(T_n \le t) - \mathbb{P}(T_{n+1} \le t) \\
&= F_{T_n}(t) - F_{T_{n+1}}(t)
\end{aligned}
$$

where F_n is the distribution function of T_n.
There are very few F for which this is easy to evaluate (can you name one?)

Growth of N_t

Above, we saw that $T_{N_t} \leq t < T_{N_t+1}$. It follows that

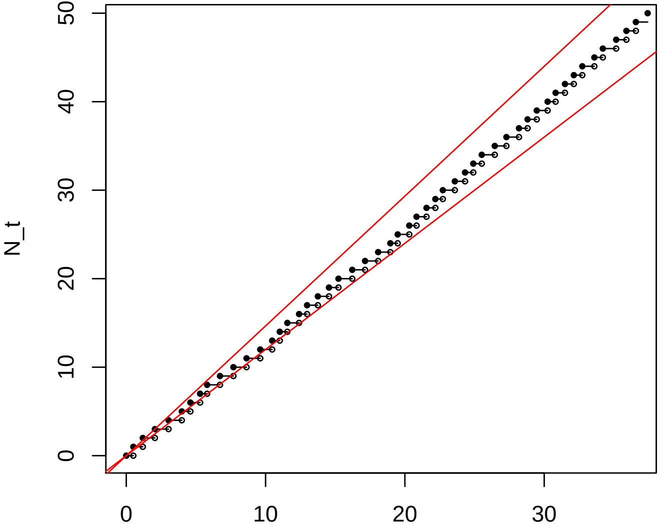
$$\frac{N_t}{N_t+1} \cdot \frac{N_t+1}{T_{N_t+1}} = \frac{N_t}{T_{N_t+1}} < \frac{N_t}{t} \leq \frac{N_t}{T_{N_t}}$$

Since $N_t \to \infty$ as $t \to \infty$, the Law of Large Numbers tells us that, with probability one, both the first and last terms approach μ^{-1}. Therefore, with probability one,

$$\lim_{t\to\infty} \frac{N_t}{t} = \mu^{-1},$$

and we see that, for large t, if $\mu < \infty$ then N_t grows like t/μ.

The LLN for N_t



Example

Jenny uses her smart phone a lot, so she carries a powerful portable charger with her. Whenever her phone gives her a low energy warning she immediately charges her phone for 30 minutes and that charge lasts for a $U(30, 60)$ (minutes, independent of previous charges) amount of time before she receives a warning. On average how many times per hour does Jenny charge her phone?

- $\mu = \mathbb{E}[\tau_1] = (45 + 30)/60$, so the rate is

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} = \frac{60}{75} = \frac{4}{5} \text{ per hour}$$

$M/G/1/1$ queue

The arrival process is Poisson with parameter λ.

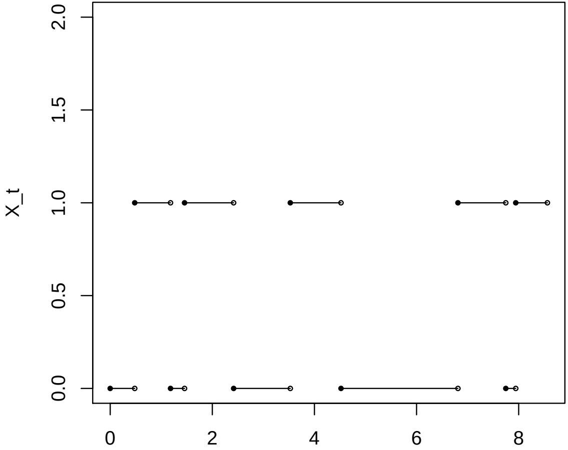
There is no "queue": when an arriving customer finds the server busy, s/he does not enter. Service times are independent and identically-distributed with distribution function G (with mean μ_G).

- What is the rate at which customers actually enter the system?
- What proportion of potential customers actually receive service?

$M/G/1/1$ queue

Let N_t be the number of customers who have been admitted by t. Then the times between successive entries of customers are made up of:

- a service time, and then
- a waiting time from the end of service until the next arrival.



$M/G/1/1$ queue

The mean time between renewals is $\mu = \frac{1}{\lambda} + \mu_G$. So the rate at which customers actually enter the system is

$$\frac{1}{\mu} = \frac{1}{\frac{1}{\lambda} + \mu_G} = \frac{\lambda}{1 + \lambda\mu_G}.$$

Customers arrive at rate λ, and so the *proportion* that actually enter the system is

$$\frac{\text{entry rate}}{\text{arrival rate}} = \frac{\frac{\lambda}{1+\lambda\mu_G}}{\lambda} = \frac{1}{1 + \lambda\mu_G}.$$

If $\lambda = 10$ per hour and $\mu_G = 0.2$ hours, then on average only 1 out of 3 customers will actually get served!

$G/G/n/m$ queue

In the very general setting of the $G/G/n/m$ queue, the beginnings of busy periods (i.e. times at which a customer arrives to find the system empty) constitute renewal times.

The time of the first "renewal" will typically have a different distribution though

The Renewal Central Limit Theorem

If $\mathbb{E}[\tau_j] = \mu$, $\mathrm{Var}(\tau_j) = \sigma^2 < \infty$, then

$$\frac{N_t - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{d} \mathcal{N}(0,1) \text{ as } t \to \infty.$$

So for each x,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_t - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \leq x \right) = \Phi(x),$$

where Φ is the normal distribution function.

Residual lifetime

The residual lifetime R_t at time t is the amount of time until the next renewal time.

Since $T_{N_t} \leq t < T_{N_t+1}$, the residual lifetime at time t is $R_t = T_{N_t+1} - t > 0$.

If τ_i are exponential(δ) then what is the distribution of R_t?

Let F be the c.d.f. of τ_1. We will study the large t behaviour of R_t assuming that F is non-lattice (that is, it does not concentrate its mass at multiples of a fixed amount), and has finite mean μ.

Residual lifetime for large t

Theorem: If F is non-lattice with finite mean μ and $x \geq 0$ then

$$\lim_{t\to\infty} \mathbb{P}(R_t \leq x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy.$$

Recall that for $Z \geq 0$,

$$\mathbb{E}[Z] = \int_0^\infty (1 - F_Z(z))dz,$$

so $\frac{1-F(y)}{\mu}$, $y \geq 0$, is a probability density function.

Example

A computer receives packets of information whose sizes are uniformly distributed between 1 and 5 GB. It saves them on hard drives of total size 700GB, until the a hard drive is full.

- For the first file for which there is not enough space on a hard drive, find the approximate distribution and the mean of the length of the residual part that the hard drive does not have space for.

- Give a (symmetric) interval to which, the total number of saved files belongs with probability ≈ 0.95.

Example solution

Here, "time" is measured in GB! We are looking for the residual lifetime at time $t = 700$, where t is considered to be large (compared to the size of a packet).

- The limiting distribution of the residual part has density

$$\frac{1}{\mu}(1 - F(x)) = \begin{cases} \frac{1}{3} & \text{if } x \in [0, 1) \\ \frac{5-x}{12} & \text{if } x \in [1, 5]. \end{cases}$$

- The mean of the residual part is $31/18$, which is greater than half of the mean interval length, which is $3/2$.
- We have

$$\frac{N_t - \frac{t}{\mu}}{\sqrt{\frac{t}{\mu} \times \frac{\sigma^2}{\mu^2}}} \overset{d}{\approx} \mathcal{N}(0, 1).$$

With $t = 700$, $\mu = 3$, $\sigma^2 = 4/3$, the desired (symmetric) interval is $233.33 \pm 5.88 \times 1.96 = (221.81, 244.85)$.

Age

The age of the renewal process at time t is the time since the most recent renewal, i.e. $A_t = t - T_{N_t}$.

Theorem: If F is non-lattice with finite mean μ and $x \geq 0$ then

$$\lim_{t\to\infty} \mathbb{P}(A_t \leq x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy.$$

Age - some intuition

- Consider the process after it has been in operation for a long time.
- When we look backwards in time, the times between successive renewals are still independent and identically-distributed with distribution F.
- Looking backwards, the residual lifetime at t is exactly the age at t of the original process.

Example

Suppose $(N_t)_{t\geq 0}$ is a Poisson process with rate λ, find the distributions of R_t, A_t and (R_t, A_t) when t is large. What is the expected duration of the inter-event time $T_{N_t+1} - T_{N_t}$?

Renewal CLT - sketch proof

Recall that $T_n = \sum_{i=1}^{n} \tau_i$.

Let $Z = \frac{T_n - n\mu}{\sqrt{n\sigma^2}} \overset{d}{\approx} \mathcal{N}(0,1)$. Then

$$\begin{aligned} \mathbb{P}(N_t \geq n) &= \mathbb{P}(T_n \leq t) \\ &\approx \mathbb{P}\left(Z \leq \frac{t - n\mu}{\sqrt{n\sigma^2}}\right) \\ &= \mathbb{P}\left(Z \geq \frac{n\mu - t}{\sqrt{n\sigma^2}}\right). \end{aligned}$$

Renewal CLT - sketch proof

Now, we choose $n(x)$ such that $\frac{n\mu - t}{\sqrt{n\sigma^2}} \approx x$. That is, we put $n(x) \approx \frac{t}{\mu} + x\sqrt{\frac{t}{\mu} \cdot \frac{\sigma^2}{\mu^2}}$.

Then, reversing the above argument, we have

$$\begin{aligned} \mathbb{P}(Z \geq x) &\approx \mathbb{P}(N_t \geq n(x)) \\ &\approx \mathbb{P}\left(\frac{N_t - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \geq x \right). \end{aligned}$$

Residual lifetime for large t - sketch of proof

Consider a period of n renewals. The proportion of time that the residual lifetime is greater than x is (by the Law of Large Numbers),

$$\begin{aligned}\frac{\sum_{i=1}^{n}(\tau_i - x)\mathbb{1}_{\{\tau_i > x\}}}{\sum_{i=1}^{n}\tau_i} &= \frac{\frac{1}{n}\sum_{i=1}^{n}(\tau_i - x)1_{[\tau_i > x]}}{\frac{1}{n}\sum_{i=1}^{n}\tau_i} \\ &\to \frac{\mathbb{E}\left[(\tau_1 - x)1_{[\tau_1 > x]}\right]}{\mathbb{E}[\tau_1]}.\end{aligned}$$

as n approaches infinity.

Residual lifetime for large t - sketch of proof

Under the stated conditions, it can also be shown that

$$\frac{\sum_{i=1}^{n}(\tau_i - x)1_{[\tau_i > x]}}{\sum_{i=1}^{n}\tau_i} \to \lim_{t\to\infty} \mathbb{P}(R_t > x).$$

Hence

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{P}(R_t > x) &= \frac{\mathbb{E}\left[(\tau_1 - x)1_{[\tau_1 > x]}\right]}{\mathbb{E}[\tau_1]} \\
&= \frac{1}{\mu}\int_0^\infty \mathbb{P}((\tau_1 - x)1_{[\tau_1 > x]} > y)dy \\
&= \frac{1}{\mu}\int_x^\infty \mathbb{P}(\tau_1 > u)du.
\end{aligned}$$

Age proof

For $x, y \geq 0$ the events $\{R_t > x, A_t > y\}$ and $\{R_{t-y} > x + y\}$ are equal so

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{P}(R_t > x, A_t > y) &= \lim_{t\to\infty} \mathbb{P}(R_{t-y} > x + y) \\
&= \frac{1}{\mu}\int_{x+y}^{\infty} [1 - F(z)]dz.
\end{aligned}$$

Setting $x = 0$ we get

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{P}(A_t \leq y) &= \frac{1}{\mu}\int_{0}^{y} [1 - F(z)]dz \\
&= \lim_{t\to\infty} \mathbb{P}(Y_t \leq y).
\end{aligned}$$

Example

For large t, find the joint probability density function of (R_t, A_t) in the computer packets example.

- First,

$$\mathbb{P}(A_t \le x, R_t \le y) = \mathbb{P}(A_t \le x) - \mathbb{P}(R_t > y) + \mathbb{P}(A_t > x, R_t > y),$$

so

$$\frac{\partial^2 \mathbb{P}(A_t \le x, R_t \le y)}{\partial x \partial y} = \frac{\partial^2 \mathbb{P}(A_t > x, R_t > y)}{\partial x \partial y}.$$

- When t is large, $\mathbb{P}(A_t > y, R_t > x) \approx \int_{x+y}^{\infty} \frac{1-F(z)}{\mu} dz$.
- Hence, the joint pdf is $1/12$ if $1 < x + y < 5$ and 0 otherwise.