MAST30001 Stochastic Modelling – 2014

Assignment 1

Please complete and sign the Plagiarism Declaration Form (available from the LMS or the department's webpage), which covers all work submitted in this subject. The declaration should be attached to the front of your first assignment.

Please hand in your assignment directly to me. **Don't forget** to staple your solutions and to print your name, student ID, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 12 September, 2014 at 5:10pm (end of lecture).**

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness and style count.

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1. Let X_n be a Markov chain with transition matrix

$$P = \begin{pmatrix} 1/6 & 1/3 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

(a) Define

$$Y_n = \begin{cases} 1 & X_n \in \{1,2\}, \\ 2 & X_n = 3. \end{cases}$$

Is $(Y_n)_{n\geq 0}$ a Markov chain? If so, find its transition matrix.

Ans. The chain is Markov. We have to show that

$$P(Y_{n+1} = y_{n+1}|Y_n = y_n, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0) = P(Y_{n+1} = y_{n+1}|Y_n = y_n),$$

for all $y_j \in \{1,2\}$. Notice that $\{Y_n = 2\} = \{X_n = 3\}$ and $\{Y_n = 1\} = \{X_n \in \{1,2\}\}$, so for $y_n = 2$,

$$\begin{aligned} &P(Y_{n+1} = y_{n+1}|Y_n = 2, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0) \\ &\quad = P(X_{n+1} \in A_{n+1}|X_n = 3, X_{n-1} \in A_{n-1}, \ldots, X_0 \in A_0), \end{aligned}$$

where

$$A_j = \begin{cases} \{3\} & y_j = 2, \\ \{1,2\} & y_j = 1. \end{cases}$$

The Markov property for (X_n) implies

$$\begin{aligned} P(X_{n+1} \in A_{n+1}|X_n = 3, X_{n-1} \in A_{n-1}, \ldots, X_0 \in A_0) &= P(X_{n+1} \in A_{n+1}|X_n = 3) \\ &= P(Y_{n+1} = y_{n+1}|Y_n = 2), \end{aligned}$$

and so we now only need to check the Markov property for $y_n = 1$. First compute

$$\begin{aligned} P(Y_{n+1} = 2|Y_n = 1) &= \frac{P(X_{n+1} = 3, X_n \in \{1,2\})}{P(X_n \in \{1,2\})} \\ &= \frac{P(X_n = 1)p_{1,3} + P(X_n = 2)p_{2,3}}{P(X_n \in \{1,2\})} \\ &= \frac{1}{2}\frac{P(X_n = 1) + P(X_n = 2)}{P(X_n \in \{1,2\})} = \frac{1}{2}; \end{aligned}$$

the third inequality is because $p_{1,3} = p_{2,3} = 1/2$. On the other side Using the same definition of A_j and the definition of conditional expectation, we have

$$\begin{aligned}
&P(Y_{n+1} = 2|Y_n = 1, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0) \\
&\quad = P(X_{n+1} = 3|X_n \in \{1,2\}, X_{n-1} \in A_{n-1}, \ldots, X_0 \in A_0) \\
&\quad = \frac{\sum_{x_j \in A_j} P(X_0 = x_0)p_{x_0,x_1} \cdots p_{x_{n-1},1}p_{1,3} + \sum_{x_j \in A_j} P(X_0 = x_0)p_{x_0,x_1} \cdots p_{x_{n-1},2}p_{2,3}}{\sum_{x_j \in A_j} P(X_0 = x_0)p_{x_0,x_1} \cdots p_{x_{n-1},1} + \sum_{x_j \in A_j} P(X_0 = x_0)p_{x_0,x_1} \cdots p_{x_{n-1},2}} \\
&\quad = \frac{(1/2)\left(\sum_{x_j \in A_j} P(X_0 = x_0)p_{x_0,x_1} \cdots p_{x_{n-1},1} + \sum_{x_j \in A_j} P(X_0 = x_0)p_{x_0,x_1} \cdots p_{x_{n-1},2}\right)}{\sum_{x_j \in A_j} P(X_0 = x_0)p_{x_0,x_1} \cdots p_{x_{n-1},1} + \sum_{x_j \in A_j} P(X_0 = x_0)p_{x_0,x_1} \cdots p_{x_{n-1},2}} \\
&\quad = \frac{1}{2};
\end{aligned}$$

the third equality is because $p_{2,3} = p_{1,3} = 1/2$. So the Markov property holds in all cases (the case $Y_n = 1$ and $Y_{n+1} = 1$ follows by taking complements in the calculations above) and the transition matrix is

$$\begin{pmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{pmatrix}.$$

2. N balls are distributed among two urns, labelled A and B. At discrete time steps, an urn is chosen at random and a ball from that urn is moved to the other urn. The chance of choosing urn A at each step is $1-p$, where $0 < p < 1$, and if the chosen urn is empty, then no action is taken. Let X_n be the number of balls in urn A after n time steps.

   (a) Model $(X_n)_{n\geq 0}$ as a Markov chain and compute its transition probabilities.

   (b) Analyse the state space of the Markov chain and describe its long run behaviour.

   (c) If $T(i) = \min\{n \geq 1 : X_n = i\}$, find $E[T(i)|X_0 = i]$ for $0 \leq i \leq N$.

   (d) For $T(i)$ as above, compute $E[T(0)|X_0 = j]$ for $j = 0, 1, 2, 3$.

   (e) Compute $P(T(N) > T(0)|X_0 = j)$ for each $0 \leq j \leq N$.

   (f) Now assume that N is even and the process is the same except now if the chosen urn has at least two balls in it, then two balls are moved from it to the other urn. If the chosen urn only has one ball in it, then it is moved to the other urn. Analyse the state space of the Markov chain and describe its long run behaviour.

   Ans.

   (a) If $X_n \neq N$ (that is, if urn B is not empty), then the chance of increasing the number of balls in urn A by one is the chance of choosing urn B, that is, equal to p. If $X_n = N$, then if urn B is chosen (again with probability p), then the ball distribution in the urns stays the same. Similarly, if $X_n > 0$, then the chance of decreasing the number of balls in urn A is $1-p$ and for $X_n = 0$, the chain holds with probability $1-p$. So the transition probabilities are for $j = 1, \ldots, N$:

$$p_{j,j+1} = 1 - p_{j,j-1} = p,$$

and

$$p_{0,1} = 1 - p_{0,0} = p \quad p_{N,N-1} = 1 - p_{N,N} = 1 - p.$$

(b) The chain is irreducible since all states can communicate with each other (since $p \neq 0, 1$). States 0 and N have a "loop" and so are aperiodic. Since periodicity is a class property, all states are aperiodic; the chain is aperiodic. The chain is positive recurrent because it is an irreducible Markov chain on a finite state space.

Since the chain is irreducible, aperiodic, and positive recurrent, it is ergodic. So if P is the transition matrix of the chain, then $\lim_{n\to\infty} P^n_{ij} = \pi_j$ for a probability distribution π and this distribution must be the unique stationary distribution of the chain restricted to the communicating class. [π_j is also the limiting proportion of time the chain spends in state j.]

So π solves $\pi P = \pi$ which implies that for $1 \leq i \leq N-1$,

$$\pi_i = p\pi_{i-1} + (1-p)\pi_{i+1},$$

and

$$\begin{aligned}\pi_0 &= (1-p)\pi_0 + (1-p)\pi_1,\\ \pi_N &= p\pi_{N-1} + \pi_N p.\end{aligned}$$

Also the entries of π must sum to one in order to be a probability distribution. We know from arguments above that these equations have a unique solution π so small cases lead us to guess the formula for $i = 1, \ldots, N$,

$$\pi_i = \pi_0 \left(\frac{p}{1-p}\right)^i$$

and

$$\pi_0 = \left(\sum_{i=0}^{N} (p/(1-p))^i\right)^{-1}.$$

And at this point it only must be checked that this formula satisfies the equations above (check this!).

(c) From lecture, we know that for an ergodic Markov chain, the expected return time to a state is one over the stationary probability:

$$E[T(i)|X_0 = i] = 1/\pi_i,$$

where π_i is computed in part (b).

(d) Let $e_j = E[T(0)|X_0 = j]$. First step analysis shows for $j = 2, \ldots, N-1$,

$$e_j = 1 + pe_{j+1} + (1-p)e_{j-1},$$

and

$$e_0 = (1-p) + p(1+e_1), \quad e_1 = (1-p) + p(1+e_2), \quad e_N = 1 + (1-p)e_{N-1} + pe_N.$$

Also note that Part (c) implies $e_0 = 1/\pi_0$ which determines the remaining e_j's. Using these equations, we can solve

$$e_1 = \frac{e_0 - 1}{p}, \quad e_2 = \frac{e_0 - 1 - p}{p^2}, \quad e_3 = \frac{e_0(1-p+p^2) - (1+p^2)}{p^3}.$$

(e) Let $f_j = P(T(N) > T(0)|X_0 = j)$. For $j \neq 0, N$, these are probabilities that were computed in the gambler's ruin problem in the lecture notes. Translating the results there ($M \to N$, $N \to 0$), if $p \neq 1/2$ and $j \neq 0, N$, then

$$f_j = \frac{\left(\frac{1-p}{p}\right)^j - \left(\frac{1-p}{p}\right)^N}{1 - \left(\frac{1-p}{p}\right)^N},$$

and for the same range of j, if $p = 1/2$,

$$f_j = \frac{N-j}{N}.$$

To compute f_0, f_N, first step analysis shows that for all p,

$$f_0 = (1-p) + pf_1, \quad f_N = (1-p)f_{N-1},$$

where the formulas for f_1 and f_{N-1} are those above.

(f) The transition probabilities are for $2 \leq j \leq N-2$,

$$p_{j,j+2} = 1 - p_{j,j-2} = p,$$

and the boundary probabilities are

$$\begin{aligned} p_{0,2} &= 1 - p_{0,0} = p, \\ p_{1,3} &= 1 - p_{1,0} = p, \\ p_{N,N} &= 1 - p_{N,N-2} = p, \\ p_{N-1,N} &= 1 - p_{N-1,N-3} = p. \end{aligned}$$

The chain is now reducible with the even numbers being an essential communicating class and the odd numbers a non-essential communicating class (since eventually all the balls are in either urn A or B and then since N is even, the number of balls in urn A stays even). The long term behaviour of the chain is that eventually it ends up in the even integers and stays there forever, with ergodic stationary distribution that follows exactly from the work of part (b),

$$\pi_{2j} = \left(\frac{p}{(1-p)}\right)^j \left(\sum_{i=0}^{N/2} (p/(1-p))^i\right)^{-1}.$$