School of Mathematics and Statistics
MAST30030
Applied Mathematical Modelling

Assignment 3. Due: May 31, 3pm

This assignment counts for 15% of the marks for this subject.

Question 1

At time $t \geq 0$, the velocity field $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ is given by

$$u(x, y, t) = \frac{x}{2+t}; \; v(x, y, t) = -\frac{y}{1+t^2}.$$

i. Find the streamlines in parametric form.

ii. Using the result in (i) find the curves giving the shape of the streamlines in Cartesian form.

iii. Give the expression for the streamline passing through the point, $(x, y) = (1, 1)$, for all time $t \geq 0$. What form does it take at $t = 0$, $t \to \infty$?

iv. Streaklines can be measured by taking snapshots of dye injected continuously into a single point in the flow, i.e., a streakline is the locus of material points originating from a specified spatial point. Calculate the streakline emanating from the point $(x, y) = (1, 1)$, for $t \geq 0$. Express your answer in Cartesian form and specify its domain.

v. Plot the curves in (iii) and (iv) for a number of discrete times in $0 \leq t \leq 10$ over the x-domain for the streaklines. Discuss the origin of any similarities or differences you observe between these curves.

Question 2

i. The generalised Lorentz reciprocal theorem for two Stokes flows (which may or may not be subject to body forces) in the same spatial domain, with velocity and stress tensor $(\mathbf{u}, \mathbf{T})$ and $(\mathbf{u}', \mathbf{T}')$, respectively, is

$$\int_S \mathbf{n} \cdot (\mathbf{u}' \cdot \mathbf{T} - \mathbf{u} \cdot \mathbf{T}') \, dS = \int_V \mathbf{u} \cdot (\nabla \cdot \mathbf{T}') - \mathbf{u}' \cdot (\nabla \cdot \mathbf{T}) \, dV$$

where $\mathbf{n}$ is the unit vector into the fluid domain, V, and S is the surface of this domain. Use Cartesian tensor methods to prove this identity. Hint: You may need to use the constitutive equation for an incompressible viscous fluid and the continuity equation.

ii. The hydrodynamic force acting on a sphere of radius a moving with steady velocity, $U\mathbf{k}$, in a (continuum) Stokes flow in a stationary fluid (no body force) is $\mathbf{F}' = -F'_{\text{drag}}\mathbf{k}$, where the drag force, $F'_{\text{drag}} = 6\pi\mu a U$, μ is the fluid's shear viscosity and $\mathbf{k}$ is the Cartesian basis vector. Its corresponding velocity field is defined by the Stokes streamfunction,

$$\Psi'(r, \theta) = \frac{U}{4}\left(3ar - \frac{a^3}{r}\right)\sin^2\theta,$$

where r and θ are the radial and azimuthal coordinates.

(a) Calculate the corresponding velocity field, $\mathbf{u}'$, for this Stokes flow.

(b) The no-slip condition does not hold when the sphere is small and the flow violates the continuum hypothesis. Consider the case where the mean free path of the gas, λ, is small (but finite) relative to the sphere radius, a, i.e., the near-continuum limit where the 'Knudsen number', $\mathrm{Kn} \equiv \lambda/a \ll 1$. In this limit, the gas velocity, $\mathbf{u}$, at the sphere's surface obeys the 'Navier slip condition':

$$\mathbf{u} = \mathbf{U}_s + 2\lambda\, \mathbf{n} \times ([\mathbf{e} \cdot \mathbf{n}] \times \mathbf{n}),$$

where $\mathbf{U}_s$ is the velocity of the sphere's surface, $\mathbf{n}$ is the outward unit normal to the surface and $\mathbf{e}$ is the rate-of-strain tensor of the gas. The Stokes equation still applies in this limit.

Approximate the rate-of-strain tensor, $\mathbf{e}$, on the right-hand side by the corresponding result for a continuum Stokes flow, whose velocity field is $\mathbf{u}'$. Hence, calculate the the hydrodynamic force, $\mathbf{F} = -F_{\mathrm{drag}}\mathbf{k}$, experienced by a sphere moving with steady velocity $U\mathbf{k}$ (no body force) in this zero inertia non-continuum limit. i.e., determine F_{drag}.

NOTE: Do not directly solve the resulting Stokes equation, but instead, apply the generalised Lorentz reciprocal theorem in (i). Explain why the bounding surface of the fluid domain far from the spheres does not contribute to the required force.

(c) How do non-continuum effects modify the drag force? Do you recover the expected drag force in the limit, $\mathrm{Kn} \to 0$? Explain your answers.

Question 3

Consider a circular cylinder of radius a whose central axis is stationary. The cylinder is surrounded by a fluid that is moving with a uniform steady velocity $U\hat{\mathbf{x}}$ far from the cylinder; the cylinder axis is in the z-direction. The cylinder is also spinning on its axis with constant angular velocity, $\Omega\,\hat{\mathbf{z}}$, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are the usual Cartesian unit vectors.

We wish to model this 2D flow using an inviscid approximation. This is achieved by first calculating the rotating steady flow of an inviscid fluid that is stationary far from the cylinder and whose angular velocity at the cylinder's surface is $\Omega\,\hat{\mathbf{z}}$. The velocity field resulting solely from the impinging uniform flow, $U\hat{\mathbf{x}}$, is then superposed onto the rotating flow to give the complete flow.

i. Calculate the velocity potential, velocity field and streamfunction for this complete flow, using the above approximation.

ii. Hence calculate the pressure and vector force acting on the cylinder. Express the force as a function of the *circulation* $\Gamma = -2\pi a^2\Omega$.

iii. What are the drag and lift on the cylinder? How are these related to the cylinder's direction of rotation?

iv. Non-dimensionalise the streamfunction and express it in terms of the dimensionless angular velocity of the cylinder $\bar{\Omega} \equiv \Omega a/U$. Explain the physical significance of $\bar{\Omega}$. Hence plot the streamlines of the flow for a range of different values of $\bar{\Omega}$ (using any numerical package, e.g., Matlab, Mathematica, WolframAlpha etc). Connect the shape of the streamlines to the vector force acting on the cylinder, and thus provide a physical explanation for the underlying force.

v. What common occurrence (particularly in sport) does this flow (and force) represent? Hint: The same phenomenon occurs for rotating spheres. Does the above solution give you any insight into the physics underlying this common occurrence? Explain your answer.