Vector Calculus - Lecture Notes

Topic 1 - Functions of Several Variables

1.01 Limits

The function $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ has the **limit** L as (*x, y*) approaches (x_0, y_0):

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if when (*x, y*) approaches (x_0, y_0) along any path in the domain D, *f(x, y)* gets close to L. Note that:

1. The limit can exist if *f* is undefined at (x_0, y_0)
2. The limit L must be finite. If the function approaches infinity, then limit does not exist.

Suppose that c, L, M are all real constants, and:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$
$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

Then:

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y) + g(x,y)] = L + M$$
$$\lim_{(x,y)\to(x_0,y_0)} [cf(x,y)] = cL$$
$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\,g(x,y)] = LM$$
$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L}{M} \text{ if } M \neq 0$$

Example 1

$$Evaluate \lim_{(x,y)\to(0,1)} \frac{x+3}{5xy - y^3}$$

$$\lim_{(x,y)\to(0,1)} \frac{x+3}{5xy - y^3} = \frac{0+3}{0-1} = -3$$

Example 2

$$Evaluate \lim_{(x,y)\to(2,1)} \frac{x^2 - 3xy + 2y^2}{x - 2y}$$

$$\begin{aligned}
\lim_{(x,y)\to(2,1)} \frac{x^2 - 3xy + 2y^2}{x - 2y} &= \lim_{(x,y)\to(2,1)} \frac{(x-2y)(x-y)}{x-2y} \\
&= \lim_{(x,y)\to(2,1)} (x-y) \\
&= 2 - 1 \\
&= 1
\end{aligned}$$

$$So \lim_{(x,y)\to(2,1)} \frac{x^2 - 3xy + 2y^2}{x - 2y} = 1$$

Note that if cannot apply any 'tricks' to evaluate limit, then must examine limit along different paths approaching the particular point.

Example 3

$$Evaluate \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

$$If\ approach\ (0,0)\ along\ x=0\ (y\ axis):$$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} &= \lim_{y\to 0} \frac{0}{0+y^2} \\ &= \lim_{y\to 0} 0 \\ &= 0\end{aligned}$$

$$If\ approach\ (0,0)\ along\ y=0\ (x\ axis):$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+0} = 1$$

Since the limiting values are different when approach (0,0) along $x = 0$ and $y = 0$, then the limit does not exist. Note that the path test can only be used to show that the limit does not exist since there are infinite amount of paths that could be used.

Example 4

$$Evaluate \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$$If\ approach\ (0,0)\ along\ x=0\ (y\ axis):$$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} &= \lim_{y\to 0} \frac{0}{0+y^2} \\ &= \lim_{y\to 0} 0 \\ &= 0\end{aligned}$$

$$If\ approach\ (0,0)\ along\ y=0\ (x\ axis):$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2+0} = 0$$

$$If\ approach\ (0,0)\ along\ y=kx,\ then:$$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} &= \lim_{x\to 0} \frac{kx^2}{x^2+k^2x^2} \\ &= \lim_{x\to 0} \frac{k}{1+k^2} \\ &= \frac{k}{1+k^2}\end{aligned}$$

Since limiting values when approach (0, 0) along $y = kx$ depend on k, then the limit does not exist.

Example 5

$$\textit{Evaluate} \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}}$$

$$\textit{If approach } (0,0) \textit{ along } x = 0:$$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} &= \lim_{y\to 0} \frac{0}{\sqrt{0+y^2}} \\ &= \lim_{y\to 0} 0 \\ &= 0\end{aligned}$$

$$\textit{If approach } (0,0) \textit{ along } y = kx:$$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} &= \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+k^2x^2}} \\ &= \lim_{x\to 0} \frac{|x|}{\sqrt{1+k^2}} \\ &= 0\end{aligned}$$

However, one cannot conclude that the limit is 0 since there are other paths to (0, 0); e.g. parabolas, cubics.

To do so, the **Sandwich Theorem** can be used, which states that:

Suppose for (x, y) near (x_0, y_0) that f, g, and h are continuous and $g(x,y) \le f(x,y) \le h(x,y)$, then

$$\text{if} \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L \text{ and } \lim_{(x,y)\to(x_0,y_0)} h(x,y) = L,$$

$$\text{then} \lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Hence, need to prove the limit is 0 by using the Sandwich Theorem:

$$0 \le \frac{x^2}{\sqrt{x^2+y^2}} \le \frac{x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\textit{So } 0 \le \frac{x^2}{\sqrt{x^2+y^2}} \le \sqrt{x^2+y^2}$$

$$\textit{Since} \lim_{(x,y)\to(0,0)} 0 = 0 \textit{ and} \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$$

$$\textit{by continuity of square root functions, then:}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0 \textit{ by Sandwich Theorem}$$

1.02 Continuity

f is **continuous** at a point (x, y) if:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

The following function types are continuous at every point in their domains:

1. Polynomials
2. Trigonometric functions
3. Exponentials
4. Logarithms
5. Root function
6. Hyperbolic functions

If f and g are continuous at a point and c is a real constant, then the following functions are continuous at that point:

1. $f+g$
2. cf
3. fg
4. $\frac{f}{g}$ if $g(x_0,y_0)\neq 0$
5. $h\circ f$

   where h is continuous at $z=f(x_0,y_0)$.
   [Recall $(h\circ f)(x,y)=h(f(x,y))$.]

Example 1

Where is f, given below, continuous?

$$f(x,y)=\log(1-xy)$$

$Given\ Theorem\ 1; 1-xy\ is\ continuous\ for\ (x,y)\in\mathbb{R}^2$

$$\log(z)\ is\ continuous\ if\ z>0$$

$Given\ Theorem\ 2\ (no.5):$

$$Let\ z=1-xy$$

$$Then\ 1-xy>0$$

$\log(1-xy)\ is\ a\ composition\ of\ continuous\ functions\ so\ is\ continuous$

$$if\ 1-xy>0, so\ xy<1$$

$Hence\ f\ is\ continuous\ if:$

$$A=\{(x,y)\in\mathbb{R}^2\backslash xy\geq 1\}$$

1.03 Differentiability

f is **differentiable** at a point if:

(i) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at (x_0, y_0)

(ii) a tangent plane

$$f(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y-y_0)$$

exists at (x_0, y_0) and is a "good approximation" to $f(x,y)$ at (x_0, y_0).

Note that (ii) is often hard to prove, so we use the following result instead.

Theorem: If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continuous at (x_0, y_0) then f is differentiable at (x_0, y_0).

A function is C^N if all of its N-th order partial derivatives exist and are continuous. For example, if is C^1 if its first order partial derivatives exist and are continuous. Hence, all C^1 functions are differentiable.

Note that if a function is C^N, then the order of differentiation is not important. Moreover, if a function is C^N, it is automatically C^1, C^2,…, C^{N-1}.

Example 1

Where is the following function f C^1?

$$f(x,y) = (x^2+y^2)^{3/4}$$

$$f_x = \frac{3x}{2}(x^2+y^2)^{-1/4} \; for \; (x,y) \neq (0,0)$$

$At\,(0,0):$

$$f_x = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$

$$f_x = \lim_{h\to 0}\frac{(h^2)^{3/4}-0}{h}$$

$$f_x = \lim_{h\to 0}\sqrt{h} = 0$$

$$f_y = \frac{3y}{2}(x^2+y^2)^{-1/4}$$

$By\ symmetry,\ f_y = 0$

Hence the first order partial derivatives exist everywhere. When (*x, y*) do not equal (0, 0), then the first order partial derivatives consist of compositions and quotients of polynomials and root functions, so is continuous.

If *f(x,y)* = (0,0) then:

$$\lim_{(x,y)\to(0,0)} f_x = 0$$

$$Using\ Sandwich\ Theorem:$$

$$\frac{3x}{2}\left(x^2+y^2\right)^{-1/4} \le \frac{3|x|}{2}\left(x^2+y^2\right)^{-1/4} = \frac{3\left(x^2\right)^{1/4}\left(x^2\right)^{1/4}}{2\left(x^2+y^2\right)^{1/4}} \le \frac{3\left(x^2\right)^{1/4}}{2}$$

$$So - \frac{3\left(x^2\right)^{1/4}}{2} \le \frac{3x}{2}\left(x^2+y^2\right)^{-1/4} \le \frac{3\left(x^2\right)^{1/4}}{2}$$

$$\lim_{(x,y)\to(0,0)} \pm\frac{3\left(x^2\right)^{1/4}}{2} = 0$$

$$So \lim_{(x,y)\to(0,0)} f_x = 0, and\ is\ hence\ continuous.$$

$$f_y\ is\ also\ continuous\ (symmetry)\ so\ f\ is\ C^1.$$

1.04 Chain Rule for Functions of Several Variables

For change of variables in multiple integrals, we need the Jacobian; the determinant of the Jacobi Matrix.

$$\text{Jacobian} = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

Suppose that the Jacobi does not equal 0. Then the linear equations relating the changes in coordinates can be inverted and:

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Taking determinants, one gets:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}\,\frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

If *f* is a differentiable function, then the derivative is an m x n matrix given by:

$$\underline{\underline{D}}f = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Example 1

Find the derivative matrix for the function *g* defined by:

$$g(x,y) = (x^2+1, y^2)$$

$$\boldsymbol{D}g = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

If *f* and *g* are differentiable functions and the composition is defined, then:

$$\underline{\underline{D}}(f \circ g) = \underline{\underline{D}}f\ \underline{\underline{D}}g$$

↑

matrix multiplication

Example 2

Find the derivative of *f(g[f(x,y)])* at (1,0) given that:

$$f(x,y) = (x^2, 2x+y, y^3)$$

$$g(u,v,w) = (u^2+2w, u-v^2)$$

$$\boldsymbol{D}f = \begin{bmatrix} 2x & 0 \\ 2 & 1 \\ 0 & 3y^2 \end{bmatrix}$$

$$\boldsymbol{D}g = \begin{bmatrix} 2u & 0 & 2 \\ 1 & -2v & 0 \end{bmatrix}$$

$$At\,(x,y) = (1,0), (u,v,w) = f(1,0) = (1,2,0)$$

$$g(1,2,0) = (1,-3)$$

$$So\ \boldsymbol{D}f(g(f(x,y,))) = \begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 0 & 27 \end{bmatrix}\begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}$$

1.05 Taylor Polynomials

If f is of order C^n, we can approximate $f(x,y)$ by a polynomial of order n around (a, b). Approximate $f(x,y)$ by a linear function at (a, b):

$$f(x,y) \approx p_1(x,y)$$

$$= f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y-b)$$

$$= f(a,b) + (x-a, y-b).\underbrace{\left.\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right|_{(a,b)}}_{\nabla f(a,b)}$$

$$p_1(x,y) = f(a,b) + \left[(x-a, y-b).\underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}_{\nabla}\right] f|_{(a,b)}$$

Let $\boldsymbol{x} = (x, y)$ and $\boldsymbol{a} = $ (a, b). In 'operator notation' we have:

$$p_1(\boldsymbol{x}) = f(\boldsymbol{a}) + \underbrace{[(\boldsymbol{x}-\boldsymbol{a}).\nabla]}_{\text{operator}} f|_{\boldsymbol{a}}$$

In general, the n-th order **Taylor polynomial** for f near $x = a$ is:

$$\boxed{p_n(\boldsymbol{x}) = \sum_{k=0}^{n} \frac{1}{k!}[(\boldsymbol{x}-\boldsymbol{a}).\nabla]^k f|_{\boldsymbol{a}}}$$

Note that the remainder is given by:

$$R_n(\boldsymbol{x}, \boldsymbol{a}) = \frac{1}{(n+1)!}[(\boldsymbol{x}-\boldsymbol{a}).\nabla]^{n+1} f|_{(\boldsymbol{a}+\xi(\boldsymbol{x}-\boldsymbol{a}))}$$

where $0 < \xi < 1$.

Example 1

Find the second order Taylor polynomial for *f(x, y)* near (1, 0).

$$f(x,y) = e^{xy}$$
$$f_x = ye^{xy}$$
$$f_y = xe^{xy}$$
$$f_{xx} = y^2 e^{xy}$$
$$f_{xy} = f_{yx} = e^{xy} + xye^{xy}$$
$$f_{yy} = x^2 e^{xy}$$

$At\ (1,0)$:

$$f = 1, f_x = 0, f_y = 1, f_{xx} = 0, f_{xy} = f_{yx} = 1, f_{yy} = 1$$

$$So\ P_2(x,y) = 1 + y + \frac{1}{2}[2(x-1)y + y^2]$$
$$= 1 + xy + \frac{y^2}{2}$$

$To\ approximate\ e^{0.11}$:

$$Need\ xy = 0.11\ and\ near\ (1,0).\ Use\ x = 1.1\ and\ y = 0.1$$
$$e^{0.11} = 1 + (1.1)(0.1) + \frac{1}{2}(0.1)^2$$
$$e^{0.11} = 1.115$$

1.06 Extrema

The **critical points** of a function of several variables *f* occur when either:

(i) $\nabla f = \underset{\sim}{0}$ OR

(ii) ∇f does not exist.

Near the critical point (a, b), the second order Taylor polynomial reduces to:

$$p_2(x,y) = f(a,b) + \frac{1}{2}\Big[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big]$$

since $\nabla f(a,b) = \underset{\sim}{0}$.

$$\Rightarrow p_2(x,y) = f(a,b) + \frac{1}{2}\begin{bmatrix} x-a & y-b \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\Bigg|_{(a,b)}}_{\underline{\underline{H}}(a,b) = \text{Hessian matrix}} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

If f is C^2 then $f_{xy} = f_{yx}$ and so:

$$\det \underline{\underline{H}} = f_{xx} f_{yy} - (f_{xy})^2$$

The **Hessian** function determines whether the critical points are maxima, minima or saddle points. If $(x, y) = (a, b)$ is a critical point then:

- if $\det \underline{\underline{H}}(a,b) > 0$ then

  (a) if $f_{xx}(a,b) < 0 \Rightarrow$ MAXIMUM at (a,b)

  (b) if $f_{xx}(a,b) > 0 \Rightarrow$ MINIMUM at (a,b)

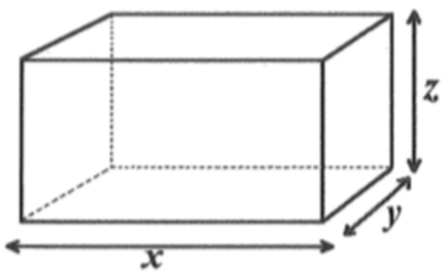
- If $\det \underline{\underline{H}}(a,b) < 0 \Rightarrow$ SADDLE POINT at (a,b)

If the Hessian = 0 then the test is inconclusive. One then needs to use higher derivatives in Taylor expansion.

Constrained extrema involves finding extrema of a function of several variables subject to one or more constraints.

Example 1

An open rectangular box is to be made with fixed volume of 4m^3. What dimensions should the box have to minimise the amount of material used to make it?



$$SA = xy + 2yz + 2xz$$
$$V = xyz = 4$$
$$Need\ to\ \min imise\ f(x,y,z) = xy + 2yz + 2xz$$
$$Given\ z = \frac{4}{xy}\ then:$$
$$F(x,y) = xy + \frac{8}{x} + \frac{8}{y}$$
$$Critical\ po\operatorname{int}s\ satisfy\ \nabla f = \mathbf{0}:$$
$$So\ \nabla f = (y - \frac{8}{x^2}, x - \frac{8}{y^2}) = (0,0)$$

$$So\ y = \frac{8}{x^2}$$

$$x = \frac{8}{y^2}$$

$$Then\ y = \frac{8}{\left(\frac{8}{y^2}\right)^2} = \frac{8}{64/y^4}$$

$$So\ y = \frac{y^4}{8}$$

$$y^4 - 8y = 0$$

$$y\left(y^3 - 8\right) = 0$$

$$So\ y = 0, y^3 = 8, then\ y = 2$$

$$But\ y \neq 0, so\ y = 2$$

$$Then\ x = 8/4 = 2$$

$$So\ critical\ po\operatorname{int}\ of\ F\ at\ (2,2)$$

$$Now\ F_{xx} = \frac{16}{x^3}$$

$$F_{yy} = \frac{16}{y^3}$$

$$F_{xy} = 1$$

$$So\ \det(H) = \frac{16}{x^3}.\frac{16}{y^3} - 1$$

$$At\ (2,2), \det(H) = 4 - 1 = 3 > 0$$

$$F_{xx}(2,2) = 16/8 = 2$$

$$So\ \min imum\ \sin ce\ F_{xx} > 0\ at\ (2,2)$$

$$At\ (x,y) = (2,2), z = 4/xy = 1$$

Hence the box has dimensions 2m x 2m x 1m.

1.07 Lagrange Multipliers

If **a** is an extrema of *f* subject to the constraint g(***x***) = 0, then there exists any real number such that:

$$\nabla f(\boldsymbol{a}) = \lambda \nabla g(\boldsymbol{a}).$$

i.e. $\nabla f(\boldsymbol{a})$ and $\nabla g(\boldsymbol{a})$ are parallel.

Note that lambda is called the **Lagrange multiplier**. Geometrically:

- $\nabla f(\boldsymbol{a})$ is normal to a level set of f.
- $\nabla g(\boldsymbol{a})$ is normal to $g(\boldsymbol{x}) = 0$.

So the constraint $g(\mathbf{x}) = 0$ will be tangent to a level set of f at the critical points. The guidelines for extrema include:

1. If constraint is closed and bounded, there exists a maximum and minimum.
2. If there are only 2 critical points, one is a maximum and one is a minimum.
3. If constraint is open or unbounded then maxima and minima need not exist.

If **a** is an extrema of f subject to the constraints $g_1(\mathbf{x}) = 0$ and $g_2(\mathbf{x}) = 0$, there exists Lagrange multipliers such that:

$$\nabla f(\boldsymbol{a}) = \lambda_1 \nabla g_1(\boldsymbol{a}) + \lambda_2 \nabla g_2(\boldsymbol{a})$$

Example 1

Find the extrema of f below subject to the conditions.

$$f(x,y,z) = x + y + z$$
$$x^2 + y^2 = 2$$
$$x + z = 1$$
$$g_1(x,y,z) = x^2 + y^2 - 2 = 0$$
$$g_2(x,y,z) = x + z - 1 = 0$$
$$Let\ \nabla \boldsymbol{f} = \lambda_1 \nabla \boldsymbol{g_1} + \lambda_2 \nabla \boldsymbol{g_2}$$
$$So\ (1,1,1) = \lambda_1 (2x, 2y, 0) + \lambda_2 (1,0,1)$$
$$Equate\ components:$$
$$1 = 2\lambda_1 x + \lambda_2$$
$$1 = 2\lambda_1 y$$
$$1 = \lambda_2$$
$$So\ 2x\lambda_1 = 0\ implies\ x = 0\ or\ \lambda_1 = 0$$
$$But\ \lambda_1 = 0\ contradicts\ 1 = 2\lambda_1 y,\ so\ x = 0$$
$$Also,\ x = 0 \Longrightarrow y^2 = 2,\ y = \pm\sqrt{2}$$
$$x = 0 \Longrightarrow z = 1$$
$$Critical\ po\operatorname{int}s\ at\ (0,\sqrt{2},1)\ or\ (0,-\sqrt{2},1)$$

The intersection of the constraints is an ellipse which is closed and bounded. Hence, maxima and minima must exist.

$$f(0,\sqrt{2},1) = 1 + \sqrt{2}$$
$$f(0,-\sqrt{2},1) = 1 - \sqrt{2}$$
$$Since\ 1 + \sqrt{2} > 1 - \sqrt{2},\ local \max imum\ at\ (1,\sqrt{2},0)\ and$$
$$local \min imum\ at\ (1,-\sqrt{2},0).$$

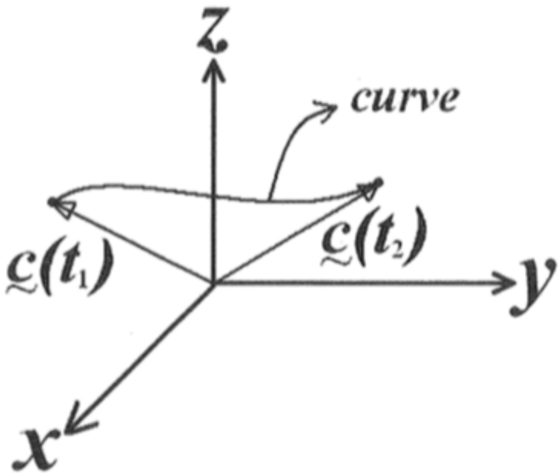
Topic 2 - Space Curves and Vector Fields

2.01 Parametric Paths

The **position** at time t of a particle moving in space is given by the path **c** where:

$$\mathbf{c}(t) = (x(t), y(t), z(t))$$

c parametrises the curve C, which is traced out by **c**(t) as t varies.



Let **c** be a differentiable path. Then the **velocity** and **speed** is given by:

$$\mathbf{v}(t) = \frac{d\mathbf{c}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

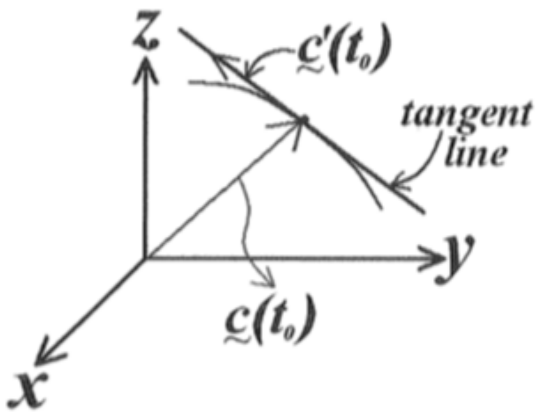
$$\text{speed} = |\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Note that the direction of **v**(t) is tangent to the path **c**. The **acceleration** is hence given by:

$$\mathbf{a}(t) = \frac{d^2\mathbf{c}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$$

The equation of the **tangent line** to **c** at a certain value of t is:

$$\ell(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0)$$



Let **b, c** be differentiable paths. Then the following differentiation rules apply:

$$(1)\quad \frac{d}{dt}[\mathbf{b}+\mathbf{c}] = \frac{d\mathbf{b}}{dt}+\frac{d\mathbf{c}}{dt}$$

$$(2)\quad \frac{d}{dt}[\mathbf{b}\cdot\mathbf{c}] = \frac{d\mathbf{b}}{dt}\cdot\mathbf{c}+\mathbf{b}\cdot\frac{d\mathbf{c}}{dt}$$

$$(3)\quad \frac{d}{dt}[\mathbf{b}\times\mathbf{c}] = \frac{d\mathbf{b}}{dt}\times\mathbf{c}+\mathbf{b}\times\frac{d\mathbf{c}}{dt}$$

The length s of a path **c** is called the **arclength**, and is given by:

$$\mathbf{c}(t) = (x(t), y(t), z(t))$$

for $a \leq t \leq b$, is

$$s = \int_a^b \left|\frac{d\mathbf{c}}{dt}\right| dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\, dt$$

We can parametrise a path in terms of the arclength s by defining the following function $s(t)$, which is the length from a point P to any point on the path (note that this is a dummy variable).

$$\boxed{s(t) = \int_a^t \left|\frac{d\mathbf{c}}{d\tau}\right| d\tau}$$

T(t) is the unit **tangent vector** to the path **c** at the point **c**(t), and is given by:

$$\text{Now } \frac{d\mathbf{c}}{ds} = \frac{d\mathbf{c}}{dt}\frac{dt}{ds} = \frac{\frac{d\mathbf{c}}{dt}}{\frac{ds}{dt}}$$

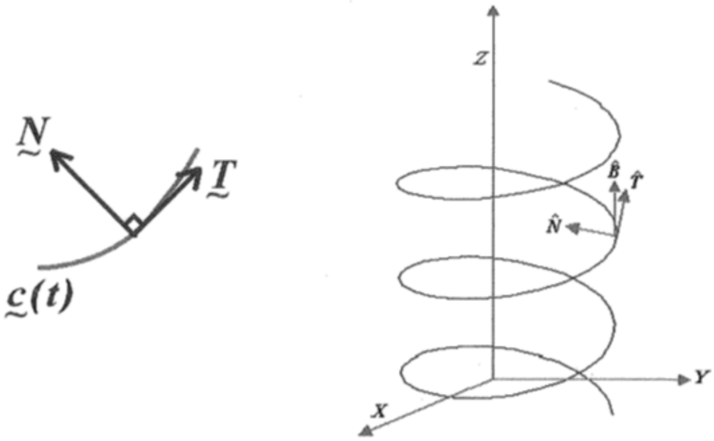
$$\text{Since } \boxed{\frac{ds}{dt} = \left|\frac{d\mathbf{c}}{dt}\right| = \text{speed}}$$

$$\Rightarrow \boxed{\frac{d\mathbf{c}}{ds} = \frac{\frac{d\mathbf{c}}{dt}}{\left|\frac{d\mathbf{c}}{dt}\right|} = \boldsymbol{T}(t)}$$

The below vector is a unit normal vector to the path **c** at the point **c**(t); note that **N**(t) is called the **principal normal**.

$$\boxed{N(t) = \frac{\dfrac{dT}{dt}}{\left|\dfrac{dT}{dt}\right|}}$$

Geometrically, this can be visualised as:



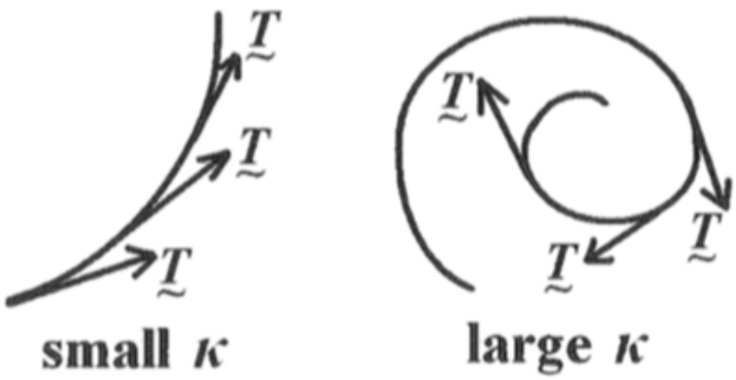
A third unit vector perpendicular to both **T**(t) and **N**(t) is the **binormal vector** **B**(t), given by:

$$\boxed{B(t) = T(t) \times N(t)}$$

Hence, if **c**(t) lies in the page, then **B**(t) points out of the page. The vectors **T, N** and **B** form a right hand set of axes moving along the curve **c**(t) with time.

The **curvature** at a point **c**(t) on a path **c** is the angular rate of change of the direction of **T** per unit change in distance along the path, given by:

$$\boxed{\kappa(t) = \left|\frac{dT}{ds}\right| = \frac{\left|\dfrac{dT}{dt}\right|}{\left|\dfrac{ds}{dt}\right|}}$$



The **torsion** measures how fast the path **c** is twisting out of the plane **T** and **N** at the point **c**(t):

$$\boxed{\frac{d\boldsymbol{B}}{ds} = -\tau(t)\boldsymbol{N}(t)}$$

So if the curve **c**(t) lies in a plane, then this will be equal to 0.

Example 1
Find the tangent vector, principal normal and binormal vector for the following path **c**(t).

$$\begin{aligned}
c(t) &= (5\cos 3t, 6t, 5\sin 3t) \\
c'(t) &= (-15\sin 3t, 6, 15\cos 3t) \\
|c'(t)| &= \sqrt{225\sin^2 3t + 36 + 225\cos^2 3t} \\
&= \sqrt{225+36} \\
&= 3\sqrt{29} \\
\boldsymbol{T}(t) &= \frac{c'(t)}{|c'(t)|} = \frac{1}{\sqrt{29}}(-5\sin 3t, 2, 5\cos 3t) \\
\boldsymbol{T}'(t) &= \frac{1}{\sqrt{29}}(-15\cos 3t, 0, -15\sin 3t) \\
&= \frac{-15}{\sqrt{29}}(\cos 3t, 0, \sin 3t) \\
So\,|\boldsymbol{T}'(t)| &= \frac{15}{\sqrt{29}} \\
Then\ \boldsymbol{N}(t) &= \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} = (-\cos 3t, 0, -\sin 3t) \\
\boldsymbol{B}(t) &= \boldsymbol{T}(t) \times \boldsymbol{N}(t) \\
&= \det\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{-5}{\sqrt{29}}\sin 35 & \frac{2}{\sqrt{29}} & \frac{5}{\sqrt{29}}\cos 3t \\ -\cos 3t & 0 & -\sin 3t \end{bmatrix} \\
&= \frac{1}{\sqrt{29}}(-2\sin 3t, -5, 2\cos 3t)
\end{aligned}$$

2.02 Vector Fields

A **vector field** is a function **F** such that:

$$\boldsymbol{F}(x, y) = u(x, y)\,\boldsymbol{i} + v(x, y)\,\boldsymbol{j}$$

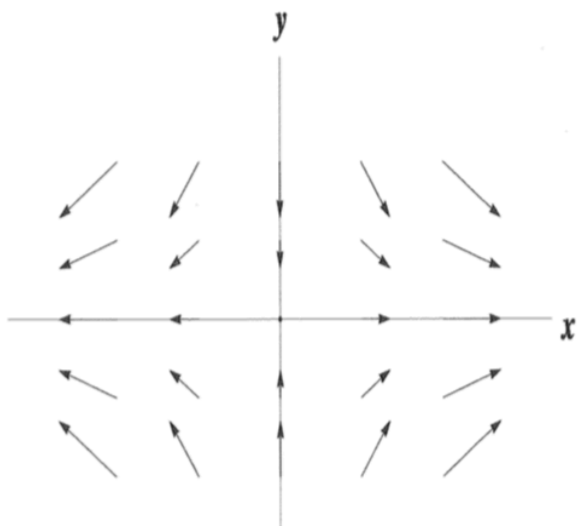
This has two independent variables *x*, *y*; **F**(*x*, *y*) is a vector (u, v) where each component is a scalar function of *x, y*. The vector field can be sketched by assigning to each point ***x*** a vector **F(*x*)** represented by an arrow whose tail is at ***x***. The vector field **F(*x*)** represents a physical vector quantity such as force or velocity at position ***x***.

Example 1

Sketch the following vector field (length of vectors not to scale).

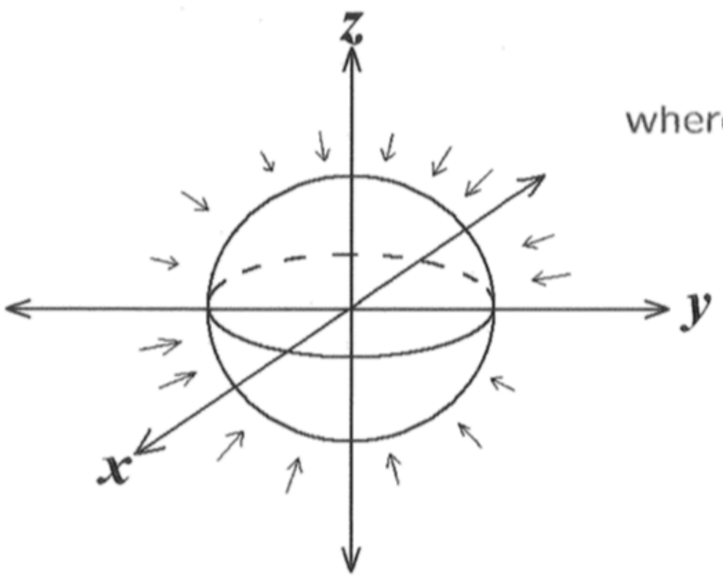
$$\boldsymbol{F}(x, y) = x\,\boldsymbol{i} - y\,\boldsymbol{j} = (x, -y).$$

(length of vectors not to scale)



Example 2

Sketch the vector field; this is the gravitational force of attraction of the earth (mass M) on a particle (mass *m*) at position (*x, y, z*).



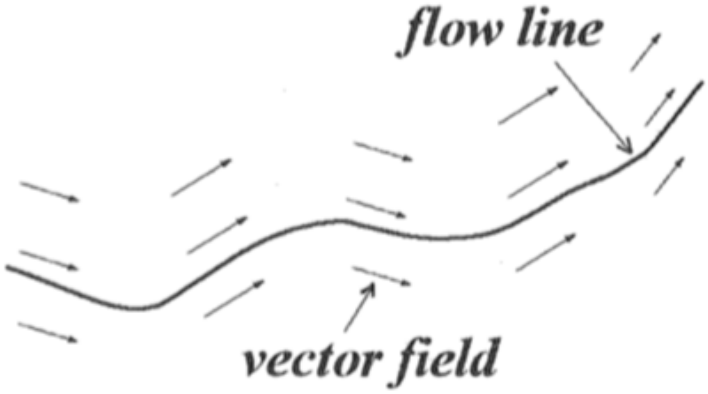
$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{GMm}{r^3}\,\boldsymbol{r}$$

where $\boldsymbol{r}(x, y, z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, $r = |\boldsymbol{r}|$.

A path **c** is a **flow line** (or streamline) of a vector field **F** if:

$$\mathbf{c}'(t) = \boldsymbol{F}[\mathbf{c}(t)]$$

Geometrically, a flow line is a curve whose tangent vector coincides with the vector field. For example, the flower of water through a pipe. The flow line represents the path followed by a particle suspended in the fluid.



Example 3
Determine the equation for the flow lines and hence sketch the flow lines of **F**(x, y) = (-y, x).

$$\text{If } \mathbf{c}(t) = (x(t), y(t)) \text{ is a flow line, then:}$$

$$\mathbf{c}'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (-y, x)$$

$$\text{So} -y = \frac{dx}{dt}$$

$$x = \frac{dy}{dt}$$

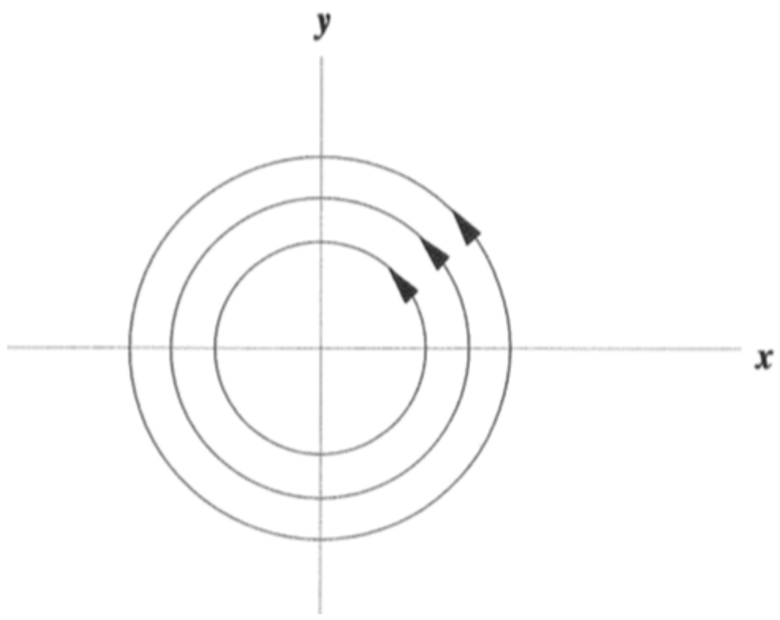
$$\text{Then } \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

$$x^2 + y^2 = D$$



2.03 Differentiation Operators

The del operator, known as **grad**, is defined as:

$$\text{In } \mathbb{R}^3: \ \nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

Note that if f is a C^1 scalar function then:

$$\nabla f = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}$$

Note that this implies that it is a vector field. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a C^1 vector field, the **divergence** of **F** is given by:

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note that div(**F**) is a scalar function.

Example 1

Find the divergence of **F.**

$$\boldsymbol{F}(x,y,z) = x^2 y\,\boldsymbol{i} + z\,\boldsymbol{j} + xyz\,\boldsymbol{k}$$

$$\begin{aligned} div(\boldsymbol{F}) &= \nabla . \boldsymbol{F} \\ &= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(xyz) \\ &= 2xy + 0 + xy \\ &= 3xy \end{aligned}$$

If **F** is the fluid velocity at position (x, y, z), then the divergence gives a measure of the net transport of fluid in/out of that point. If the divergence is greater than 0, then more fluid flows out than in, so the fluid is expanding. If the divergence is less than 0, then more fluid flows in than out, so the fluid is compressing. If the divergence is equal to 0, then the rate at which the fluid flows in equals the rate at which the fluid flows out. If the divergence is equal to 0, then **F** is called an **incompressible vector field**.

The **curl** of **F** is given by:

$$\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F}$$

$$= \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}.$$

Note that the curl **F** is a vector field.

Example 2
Find the curl of **F**.

$$\boldsymbol{F}(x,y,z) = x^2 y\,\boldsymbol{i} - 2xz\,\boldsymbol{j} + (x+y-z)\,\boldsymbol{k}$$

$$\begin{aligned} curl(\boldsymbol{F}) &= \nabla \times \boldsymbol{F} \\ &= (1+2x)\,\boldsymbol{i} - \boldsymbol{j} + (-x^2 - 2z)\,\boldsymbol{k} \end{aligned}$$

If **F** is the fluid velocity in a lake, and if one drops a twig into the lake, then the curl of **F** measures how quickly and in what orientation the twig rotates as it moves. If the twig does not rotate as it travels, then the curl of **F** is equal to **0** and is hence called **irrotational**. If the direction of the twig changes as it travels then the curl of **F** does not equal **0** and is called **rotational**. Note that if *f* is a scalar function the the divergence and curl of *f* are not defined; must be a vector field.

The **Laplacian Operator** is given by:

$$\begin{aligned} \nabla^2 &= \nabla \cdot \nabla \\ &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

Note that if *f* is a C^2 scalar function then:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

If **F** is a C^2 vector field where u, v and w are scalar functions of (*x, y, z*) then:

$$\nabla^2 \boldsymbol{F} = \nabla^2 u\,\boldsymbol{i} + \nabla^2 v\,\boldsymbol{j} + \nabla^2 w\,\boldsymbol{k}$$

$$\text{where } \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \text{ etc.}$$

Note that this implies that the Laplacian Operator applied to **F** is a vector field. Some applications for the Laplacian Operator is the gravitational potential V of a mass *m* at (*x, y, z*) due to a mass M at (0, 0, 0), which satisfies Laplace's equation.

Example 1

Find the Laplacian operator of *f*.

$$f(x,y,z) = x^2y + xy^2 + yz^2$$

$$Now, \frac{\partial f}{\partial x} = 2xy + y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 2y$$

$$\frac{\partial f}{\partial y} = x^2 + 2xy + z^2$$

$$\frac{\partial^2 f}{\partial y^2} = 2x$$

$$\frac{\partial f}{\partial z} = 2yz$$

$$\frac{\partial^2 f}{\partial z^2} = 2y$$

$$So\ \nabla^2 f = 4y + 2x$$

2.04 Basic Identities of Vector Calculus

Let *f, g* be scalar functions and let **F, G** be vector fields. Then the following are all the basic identities of vector calculus:

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(\beta f) = \beta \nabla f \qquad (\beta \text{ constant})$
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \qquad \text{provided } g \neq 0$
5. $\nabla \cdot (\boldsymbol{F} + \boldsymbol{G}) = \nabla \cdot \boldsymbol{F} + \nabla \cdot \boldsymbol{G}$
6. $\nabla \times (\boldsymbol{F} + \boldsymbol{G}) = \nabla \times \boldsymbol{F} + \nabla \times \boldsymbol{G}$
7. $\nabla \cdot (f\boldsymbol{F}) = f\nabla \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \nabla f$
8. $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G})$
9. $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$
10. $\nabla \times (f\boldsymbol{F}) = f\nabla \times \boldsymbol{F} + \nabla f \times \boldsymbol{F}$
11. $\nabla \times (\nabla f) = \boldsymbol{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$
13. $\nabla \cdot (\nabla f \times \nabla g) = 0$
14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
15. $\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$

Note that all of these identities require *f, g,* **F, G** to be suitably differentiable, either C^1 or C^2.

For Identity 11, let **V** be a C^1 vector field. Hence, if the curl of **V** equals **0**, then **V** can be represented by the gradient of a scalar function, so:

$$\boldsymbol{V} = \nabla\phi.$$

Note that phi is unique up to an unknown constant C. We say that **V** is a gradient field and phi is the **scalar potential**.

Example 1

Given that **V** is a gradient field and irrotational, find the scalar potential.

$$\boldsymbol{V} = (2xy^2, 2x^2y + 2z^2y, 2y^2z)$$

$$So\,\boldsymbol{V} = \nabla\phi$$

$$(2xy^2, 2x^2y + 2z^2y, 2y^2z) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$Equating\ the\ components\ gives:$$

$$\frac{\partial\phi}{\partial x} = 2xy^2$$

$$\phi = \int 2xy^2\,dx$$

$$\phi = x^2y^2 + C(y,z)$$

$$\frac{\partial\phi}{\partial y} = 2x^2y + 2z^2y$$

$$\phi = \int 2x^2y + 2z^2y\,dy$$

$$\phi = x^2y^2 + y^2z^2 + D(x,z)$$

$$\frac{\partial\phi}{\partial z} = 2y^2z$$

$$\phi = \int 2y^2z\,dz$$

$$\phi = y^2z^2 + E(x,y)$$

$$So\,\phi(x,y,z) = x^2y^2 + z^2y^2 + A$$

Let **V** be a C^1 vector field. If the divergence of **V** is equal to 0 (incompressible), then **V** can be written as the curl of a vector field **F** given Identity 9, so:

$$\boldsymbol{V} = \nabla \times \boldsymbol{F}.$$

Note that **F** is not unique and we say that **F** is the **vector potential**.

Example 2

Express the vector field **V** as a curl of a vector field **F**.

$$\begin{aligned}
\boldsymbol{V} &= (x^2+1, z-2xy, y) \\
\boldsymbol{V} &= \nabla \times \boldsymbol{F} \\
\boldsymbol{F}(x,y,z) &= (F_1, F_2, F_3)
\end{aligned}$$

$$Equating\ components:$$

$$\begin{aligned}
x^2+1 &= \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\
z-2xy &= \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\
y &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\
Set\ F_3 &= 0 \\
So\ \frac{\partial F_2}{\partial z} &= -x^2-1 \\
F_2 &= \int -x^2-1\,dz \\
&= -x^2z - z + C(x,y) \\
And\ \frac{\partial F_1}{\partial z} &= z-2xy \\
F_1 &= \int z-2xy\,dz \\
F_1 &= \frac{z^2}{2} - 2xyz + D(x,y) \\
Also: \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= -2xz + \frac{\partial C}{\partial x} + 2xz - \frac{\partial D}{\partial y} \\
&= \frac{\partial C}{\partial x} - \frac{\partial D}{\partial y} \\
So\ y &= \frac{\partial C}{\partial x} - \frac{\partial D}{\partial y} \\
Choose\ \frac{\partial C}{\partial x} &= 0 \\
\frac{\partial D}{\partial y} &= -y \\
D &= \int -y\,dy \\
D &= \frac{-y^2}{2} + E(x) \\
Choose\ E(x) &= 0
\end{aligned}$$

$$So\ combining:$$

$$\begin{aligned}
F_1 &= \frac{z^2}{2} - 2xyz - \frac{y^2}{2} \\
F_2 &= -x^2z - z \\
F_3 &= 0
\end{aligned}$$

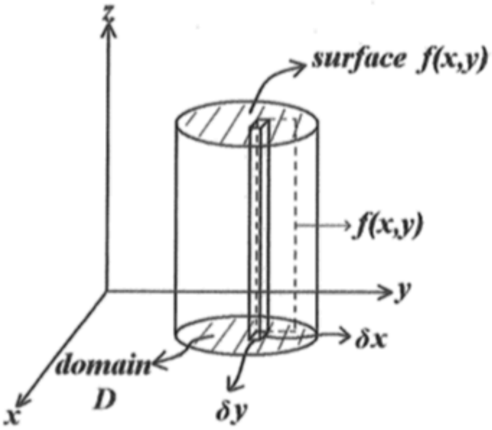
$$So\ one\ vector\ field\ is\ \boldsymbol{F} = \left(\frac{z^2}{2} - 2xyz - \frac{y^2}{2}, -x^2z - z, 0\right)$$

Topic 3 - Double and Triple Integrals

3.01 Double Integrals

If *f* is a continuous function over a domain D, we can evaluate the **double integral**:

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy.$$



$$\begin{aligned}\begin{matrix}\text{Volume}\\ \text{thin rod}\end{matrix} &= (\text{Area base})\ (\text{height})\\ &= \delta A\, f(x,y)\\ &= \delta x\,\delta y\, f(x,y)\end{aligned}$$

Hence:

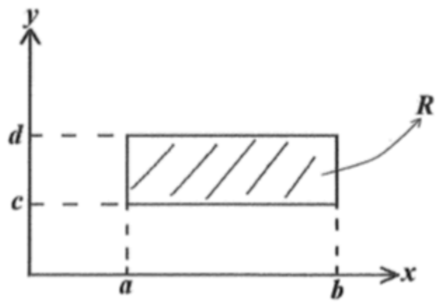
$$\begin{aligned}\iint_D f(x,y)\,dA &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,y_i)\delta A_i\\ &= \iint_D f(x,y)\,dx\,dy\end{aligned}$$

The double integral is the volume under the surface $z = f(x, y)$ that lies above the domain D in the *xy*-plane. Note that if $f(x, y) = 1$, then the double integral is the total area of domain D. That is:

$$\text{If } f(x,y) = 1 \text{ then}$$

$$\iint_D dA = \iint_D dx\,dy$$

R = [a, b] x [c, d] is a rectangular domain defined by a ≤ *x* ≤ b, c ≤ *y* ≤ d. That is:



$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

Note that the above formula means that one integrates with respect to *x* first and then integrate with respect to *y*.

Let *f* be a continuous function over the domain R = [a, b] x [c, d]. Then, **Fubini's Theorem** states that the order of integration is not important. That is:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example 1

Evaluate the following double integral.

$$\iint_R x^2 + y^2\,dxdy \text{ where } R = [-1,1]\times[0,1]$$

$$\int_0^1 \int_{-1}^1 x^2 + y^2\,dxdy = \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{-1}^1 dy$$

$$= \int_0^1 \frac{2}{3} + 2y^2\,dy$$

$$= \left[\frac{2}{3}y + \frac{2y^3}{3}\right]_0^1$$

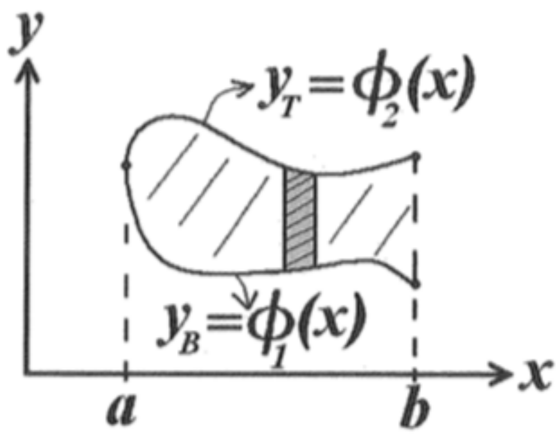
$$= \frac{4}{3}\,units^3$$

The domain determines the terminals and order of integration. Using vertical strips, one must always integrate with respect to *y* first.

$$\iint_D f(x,y)dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

Visually, these vertical strips are given as:

$$y_B \leq y \leq y_T \,, \quad a \leq x \leq b$$

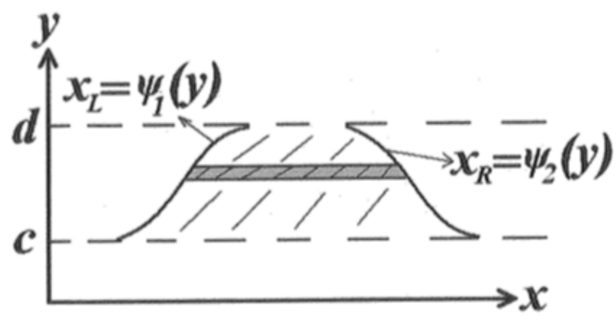


Alternatively, one can use horizontal strips where one must integrate with respect to x first. That is:

$$\iint_D f(x,y)dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\, dy$$

Visually, these horizontal strips are given as:

$$x_L \leq x \leq x_R \,, \quad c \leq y \leq d$$



Note that the order of integration is not important if the domain can be divided into horizontal and vertical strips and f is continuous in its domain (this is an extension to Fubini's Theorem).

Example 2

Find the area enclosed by the following functions and the y-axis for $x \geq 0$.

$$y = 3x^2$$

$$y = 4 - x^2$$

Use vertical strips:

$$3x^2 \leq y \leq 4 - x^2$$

$$0 \leq x \leq 1$$

Integrate with y first:

$$\text{Area of } D = \iint_D 1\, dA$$

$$= \int_0^1 \int_{3x^2}^{4-x^2} 1\, dy\, dx$$

$$= \int_0^1 [y]_{y=3x^2}^{y=4-x^2}\, dx$$

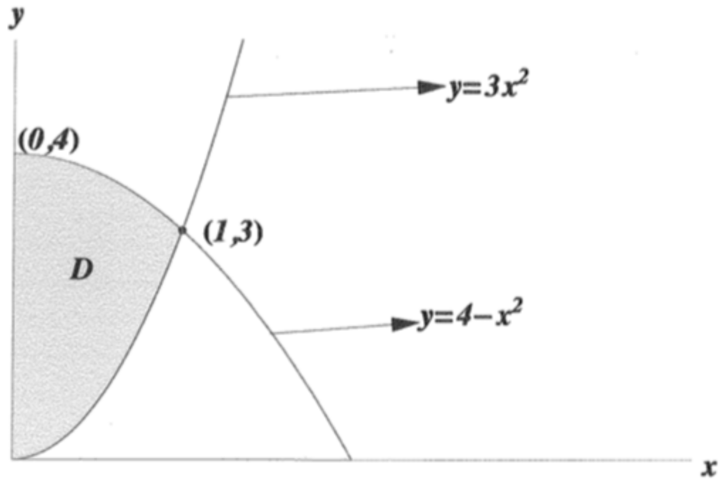
$$= \int_0^1 (4 - x^2 - 3x^2)\, dx$$

$$= \int_0^1 (4 - 4x^2)\, dx$$

$$= \left[4x - \frac{4x^3}{3}\right]_0^1$$

$$= \frac{8}{3}(units)^2$$

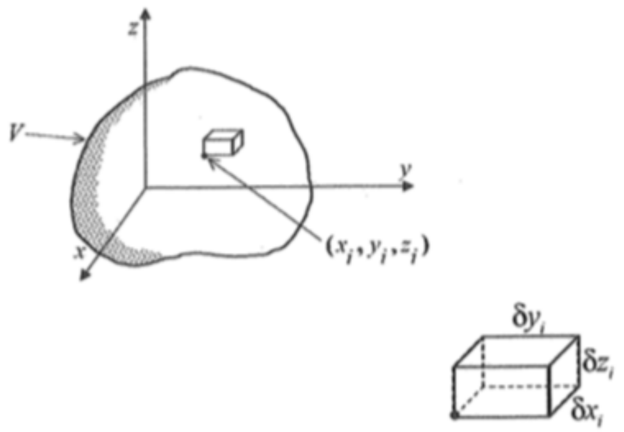
Visually this is the area D enclosed by the functions on the graph:



3.02 Triple Integrals

If f is a continuous function over a solid domain D, we can evaluate the **triple integral** as:

$$\iiint_D f(x,y,z)\, dV = \iiint_D f(x,y,z)\, dx\, dy\, dz$$



Let $\delta V = \delta z \delta y \delta x$ be a small subregion of V.
The triple integral of f over V is

$$\iiint_V f(x,y,z)\, dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \delta V_i$$
$$= \iiint_V f(x,y,z)\, dz\, dy\, dx$$

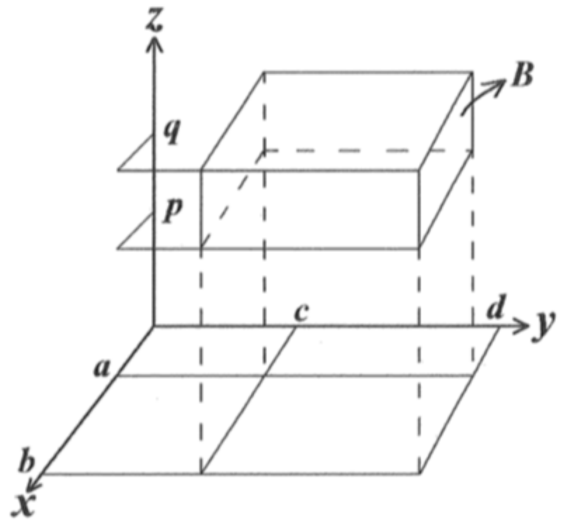
The physical meaning of the triple integral depends on *f*. For example, if *f* is the temperature at position (*x, y, z*), then the average temperature in D is given by:

$$\frac{\displaystyle\iiint_D f(x,y,z)\, dV}{\text{volume of } D}$$

The triple integral over a rectangular box domain B can be specified by:

B = [a, b] x [c, d] x [p, q]

Visually this is given by:



Let *f* be a continuous function over B = [a, b] x [c, d] x [p, q], then by Fubini's Theorem:

$$\iiint_B f(x,y,z)\, dV = \int_p^q \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

$$= \int_p^q \int_a^b \int_c^d f(x,y,z)\, dy\, dx\, dz$$

$$= \int_a^b \int_c^d \int_p^q f(x,y,z)\, dz\, dy\, dx$$

Example 1

Evaluate the following triple integral over B = [0, 1] x [-0.5, 0] x [0, 1/3].

$$\int_0^{1/3}\int_{-1/2}^{0}\int_0^1 (x+2y+3z)\,dx\,dy\,dz$$

$$=\int_0^{1/3}\int_{-1/2}^{0}\left[\frac{x^2}{2}+2xy+3xz\right]_{x=0}^{x=1} dy\,dz$$

$$=\int_0^{1/3}\int_{-1/2}^{0}(0.5+2y+3z)\,dy\,dz$$

$$=\int_0^{1/3}\left[\frac{y}{2}+y^2+3yz\right]_{y=-1/2}^{y=0} dz$$

$$=\int_0^{1/3}\frac{3z}{2}dz$$

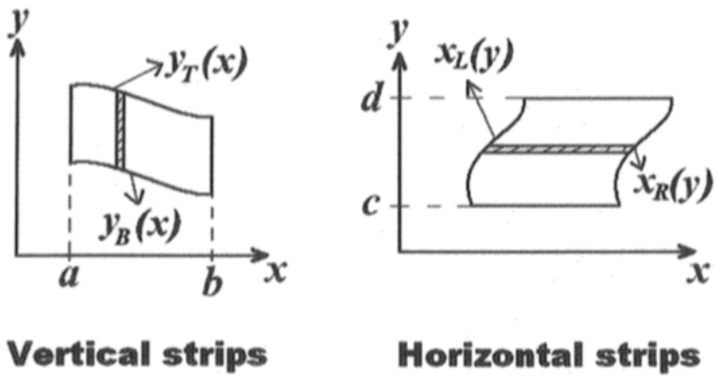
$$=\left[\frac{3z^2}{4}\right]_0^{1/3}$$

$$=\frac{1}{12}$$

The domain D is an **elementary region** if one variable is bounded by functions of two variables, the domains of these functions being described using horizontal or vertical strips. For example, if z is bounded by two functions of x and y, then:

$$f_1(x,y)\le z\le f_2(x,y)$$

The projection ("shadow") of D onto the xy plane is hence:



For example, if the region is oriented so the axis of symmetry is the x-axis, then let x be bounded by two functions of y and z and then project onto the yz plane.

Example 2

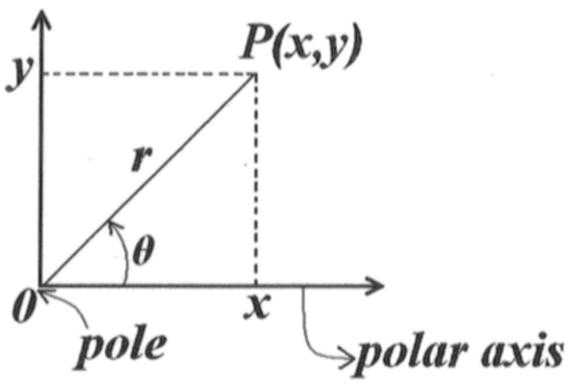
Evaluate the following triple integral where D is the solid tetrahedron with vertices at (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).

$$0 \le z \le 1 - x - y$$
$$0 \le y \le 1 - x$$
$$0 \le x \le 1$$

$$\int\int\int xy\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xy\, dz\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} [xyz]_{z=0}^{z=1-x-y}\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} (xy - x^2y - xy^2)\, dy\, dx$$
$$= \int_0^1 \left[\frac{xy^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3}\right]_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left(\frac{x}{6} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{6}\right) dx$$
$$= \frac{1}{120}$$

3.03 Orthogonal Curvilinear Coordinates

We can specify a point in the *xy* plane by Cartesian or **polar coordinates**.



- $r =$ length $\overrightarrow{OP} = \sqrt{x^2 + y^2}$

$$0 \le r < \infty$$

- $x = r\cos\theta$ and $y = r\sin\theta$
- $\theta =$ angle measured anticlockwise from positive x axis (polar axis) to $\overrightarrow{OP}$

Note that if r = 0, then theta is arbitrary. Hence, the pole is given by (0, ø). Also note that ø is not unique.

Example 1

Convert (x, y) = (-1, -1) to polar coordinates.

$$r = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \arctan(-1/-1)$$

$$\theta = \frac{5\pi}{4}$$

$$So\,(r,\theta) = (\sqrt{2}, \frac{5\pi}{4})$$

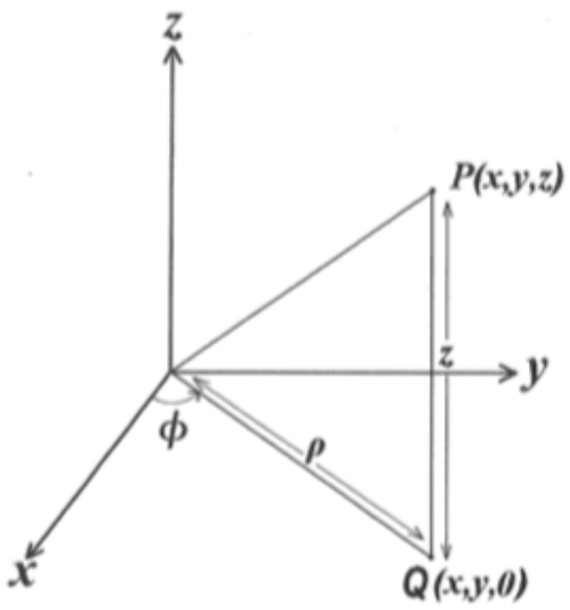
We define unit vectors in the direction of increasing coordinate. Hence:

$$\hat{\boldsymbol{r}} = (\cos\theta, \sin\theta)$$

$$\hat{\boldsymbol{\theta}} = (-\sin\theta, \cos\theta)$$

Hence, we can specify a point in space by Cartesian, cylindrical or spherical coordinates.

For **cylindrical coordinates**, it forms a right handed system and is defined as:



- $\rho =$ length $\overrightarrow{OQ} = \sqrt{x^2+y^2}$

$$0 \le \rho < \infty$$

- $\phi =$ angle measured anticlockwise from positive x axis to $\overrightarrow{OQ}$

$$0 \le \phi < 2\pi$$

Also:

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

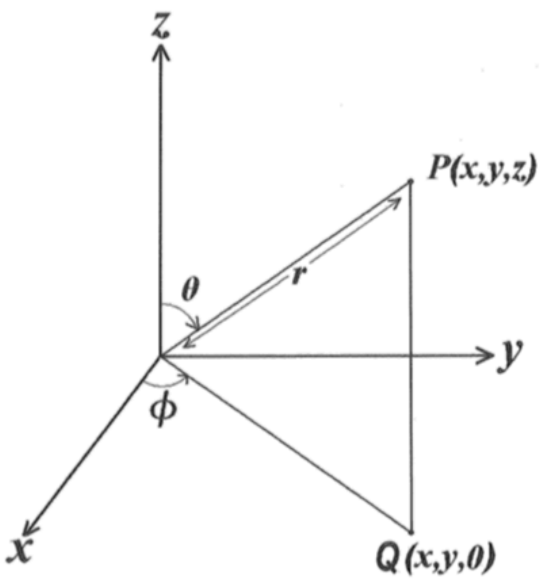
The unit vectors are given by:

$$\hat{\boldsymbol{\rho}} = (\cos \phi, \sin \phi)$$

$$\hat{\boldsymbol{\phi}} = (-\sin \phi, \cos \phi)$$

$$\hat{\boldsymbol{z}} = \boldsymbol{k}$$

The **spherical coordinates** are defined as:



- r = length of $\overrightarrow{OP} = \sqrt{x^2 + y^2 + z^2}$

$$0 \leq r < \infty$$

- ϕ (azimuthal angle) = angle measured anticlockwise from positive x axis to $\overrightarrow{OQ}$

$$0 \leq \phi < 2\pi$$

- θ (polar angle) = angle measured from positive z axis to $\overrightarrow{OP}$.

$$0 \leq \theta \leq \pi$$

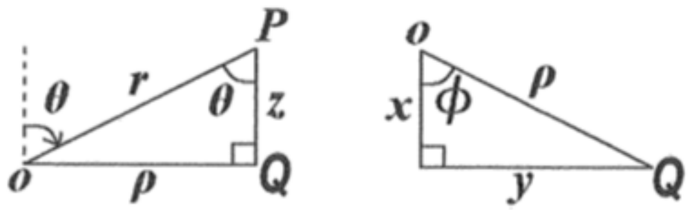
$$\theta = \arccos\left(\frac{z}{r}\right)$$

Also note that:

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$



$$\Rightarrow \rho = r\sin\theta \qquad \Rightarrow x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

The unit vectors in spherical coordinates are given by:

$$\hat{\boldsymbol{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\hat{\boldsymbol{\phi}} = (-\sin\phi, \cos\phi, 0)$$

$$\hat{\boldsymbol{\theta}} = (\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta)$$

Example 2

Convert the following cartesian coordinates into cylindrical and spherical coordinates.

$$(x, y, z) = (1, 1, \sqrt{3})$$

$$Cylindrical:$$

$$\rho = \sqrt{1+1} = \sqrt{2}$$

$$\phi = \arctan(1) = \frac{\pi}{4}$$

$$z = \sqrt{3}$$

$$So\,(\rho, \phi, z) = (\sqrt{2}, \frac{\pi}{4}, \sqrt{3})$$

$$Spherical:$$

$$r = \sqrt{1+1+3} = \sqrt{5}$$

$$\theta = \arccos\left(\frac{\sqrt{3}}{\sqrt{5}}\right) = 0.685\,radians$$

$$\phi = \frac{\pi}{4}$$

$$So\,(r, \theta, \phi) = (\sqrt{5}, 0.685, \frac{\pi}{4})$$

3.04 Change of Variable Theorems for Multiple Integrals

For 2 variables, let D and D* be elementary regions and T : D* —> D be C^1. If T is one-to-one and D = T(D*) (i.e. linear transformation from cartesian to spherical/cylindrical coordinates) then:

$$\iint_D f(x,y)\,dx\,dy$$

$$= \iint_{D^*} f\left[x(u,v), y(u,v)\right] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where the |Jacobian| is:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}\right|$$

Similarly for 3 variables, let D and D* be elementary regions and T : D* —> D be C^1. If T is one-to-one and D = T(D*), then:

$$\iiint_D f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_{D^*} f\left[x(u,v,w), y(u,v,w), z(u,v,w)\right] \cdot \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where the |Jacobian| is:

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{bmatrix}\right|$$

In polar coordinates, the Jacobian can be calculated as:

$$\text{Jacobian} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Example 1
Evaluate the following double integral.

$$\iint_D \log\left(x^2+y^2\right) dx\, dy$$

$$1 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}$$

$$So \log\left(x^2+y^2\right) = \log\left(r^2\right) = 2\log r$$

$$Jacobian = r$$

$$So \iint_D \log\left(x^2+y^2\right) dx\, dy = \int_1^2 \int_0^{\frac{\pi}{2}} 2r\log r\, d\theta\, dr$$

$$= \int_1^2 2r\log r[\theta]_0^{\frac{\pi}{2}}\, dr$$

$$= \pi \int_1^2 r\log r\, dr$$

$$= \pi\left[0.5r^2\log r - 0.5\int r\, dr\right]_1^2 (by\ parts)$$

$$= \pi\left(2\log 2 - 1 - (0.5\log 1 - 0.25)\right)$$

$$= \pi\left(2\log 2 - 0.75\right)$$

For cylindrical coordinates, the Jacobian is given by:

$$\text{Jacobian} = \det\begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{bmatrix}$$

$$= \det\begin{bmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

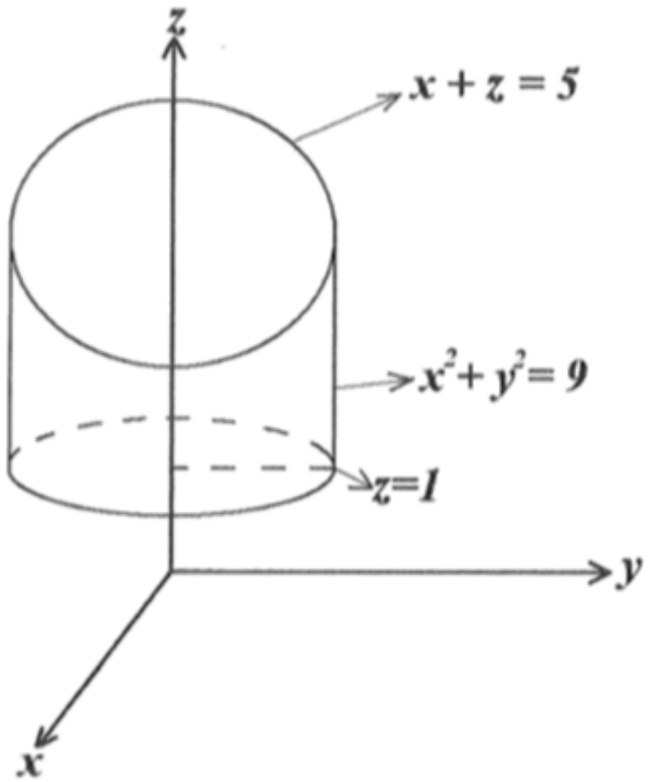
$$= \det\begin{bmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{bmatrix}$$

$$= \rho\cos^2\phi + \rho\sin^2\phi$$

$$= \rho$$

Example 2

Evaluate the following triple integral to find the volume of the solid below.



$Use\ cylindrical\ coordinates:$

$x = \rho \cos\phi$

$y = \rho \sin\phi$

$z = z$

$1 \le z \le 5 - \rho\cos\phi$

$0 \le \rho \le 3$

$0 \le \phi \le 2\pi$

$$
\begin{aligned}
V &= \int_0^3 \int_0^{2\pi} \int_1^{5-\rho\cos\phi} \rho \, dz \, d\phi \, d\rho \\
&= \int_0^3 \int_0^{2\pi} [pz]_1^{5-\rho\cos\phi} \, d\phi \, d\rho \\
&= \int_0^3 \int_0^{2\pi} (4p - p^2\cos\phi) \, d\phi \, d\rho \\
&= \int_0^3 [4p\phi - p^2\sin\phi]_0^{2\pi} \, dp \\
&= \int_0^3 8\pi p \, dp \\
&= [4\pi p^2]_0^3 \\
&= 36\pi \ units^3
\end{aligned}
$$

In spherical coordinates, the Jacobian is given by:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta \quad (r \ge 0)$$

$$\text{Jacobian} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix}$$

$$= \det\begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

$$= \cos\theta\left[r^2\cos\theta\sin\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right]$$

$$+ r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$$

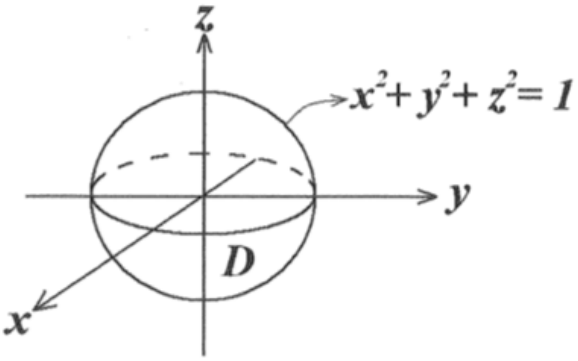
$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta \qquad (\sin\theta \geq 0)$$

Example 3

If D is the unit sphere centred at (0, 0, 0), evaluate:

$$\iiint_D \exp\left(\left[x^2+y^2+z^2\right]^{\frac{3}{2}}\right)\, dV$$



Use spherical coordinates:

$$0 \leq r \leq 1$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi \leq 2\pi$$
$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2 \sin\theta e^{r^3}\, dr\, d\theta\, d\phi$$

Let $u = r^3, u' = 3r^2$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{3}e^{r^3}\sin\theta\right]_{r=0}^{r=1} d\theta\, d\phi$$
$$= \frac{1}{3}(e-1)\int_0^{2\pi}\int_0^{\pi}\sin\theta\, d\theta\, d\phi$$
$$= \frac{1}{3}(e-1)\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi} d\phi$$
$$= \frac{1}{3}(e-1)\int_0^{2\pi} 2\, d\phi$$
$$= \frac{4\pi}{3}(e-1)$$

Note that if the terminals are constant and the integrand factories, then:

$$g(x,y,z) = g_1(x)g_2(y)g_3(z),$$

then

$$\int_a^b \int_c^d \int_e^f g(x,y,z)\,dx\,dy\,dz$$

$$= \left[\int_a^b g_3(z)\,dz\right]\left[\int_c^d g_2(y)\,dy\right]\left[\int_e^f g_1(x)\,dx\right].$$

3.04 Applications of Multiple Integrals

The first application of multiple integrals is using them to calculate average values. The formulae are given by:

- Average of f in D

$$\bar{f} = \frac{\iint\limits_D f(x,y)\,dx\,dy}{\iint\limits_D dx\,dy} \quad \leftarrow \text{ area of } D$$

- Average of f in D

$$\bar{f} = \frac{\iiint\limits_D f(x,y,z)\,dx\,dy\,dz}{\iiint\limits_D dx\,dy\,dz} \quad \leftarrow \text{ volume of } D$$

The second application is using multiple integrals to calculate the centre of mass. The centre of mass of a rigid body is that point at which the body can be supported so it will not experience any unbalanced torques that will cause it to rotate. The plate balances when supported at its centre of mass. For a 2D plate with mass per unit area μ, the centre of mass is given by the formulae below.

$$x_{cm} = \frac{\iint\limits_D x\,\mu(x,y)\,dx\,dy}{\text{mass}}$$

$$y_{cm} = \frac{\iint\limits_D y\,\mu(x,y)\,dx\,dy}{\text{mass}}$$

where

$$\text{mass of plate} = \iint\limits_D \mu(x,y)\,dx\,dy$$

For a 3D body with mass per unit volume μ, the centre of mass is given by:

$$x_{cm} = \frac{\iiint\limits_D x\,\mu(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

$$y_{cm} = \frac{\iiint\limits_D y\,\mu(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

$$z_{cm} = \frac{\iiint\limits_D z\,\mu(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

where

$$\text{mass of body} = \iiint\limits_D \mu(x,y,z)\,dx\,dy\,dz$$

The third application is to find the moment of inertia. It is denoted by I_n, which is the moment of inertia of a solid body about the *n* axis. It measures a body's response to spinning it about the *n* axis. As the moment of inertia increases, it becomes harder to spin the body about the *n* axis. The formula are given below.

$$I_x = \iiint\limits_D (y^2+z^2)\mu(x,y,z)\,dx\,dy\,dz$$

$$I_y = \iiint\limits_D (x^2+z^2)\mu(x,y,z)\,dx\,dy\,dz$$

$$I_z = \iiint\limits_D (x^2+y^2)\mu(x,y,z)\,dx\,dy\,dz$$

Topic 4 - Integrals Over Paths and Surfaces

4.01 Path Integrals

Let f be a continuous scalar function and **c** be a C^1 path where $\mathbf{c}(t) = (x(t), y(t), z(t))$. The **path integral** of f along **c** from $t = a$ to $t = b$ ($a \le t \le b$) is given by:

$$\int_{\mathbf{c}} f\, ds = \int_{\mathbf{c}} f(x, y, z)\, ds$$

$$= \int_a^b f\,[x(t), y(t), z(t)]\,\frac{ds}{dt}\,dt$$

$$= \int_a^b f\,[\mathbf{c}(t)]\,|\mathbf{c}'(t)|\;dt$$

Note that when $f = 1$, this path integral gives the arclength of **c**. Moreover, the path integral is independent of how **c** is parametrised.

The physical interpretation of the path integral depends on what f represents. For example, if f is the mass per unit length of a cable **c**, then the path integral is the total mass of the cable. Alternatively, if f is the charge per unit length of a cable **c**, then the path integral is the total charge of the cable.

Example 1

Let **c** be the helix given by $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le 2\pi$. Evaluate the following path integral.

$$\int_c (xy + z)\, ds$$

$$\mathbf{c}'(t) = (-\sin t, \cos t, 1)$$

$$So\ \frac{ds}{dt} = |\ \mathbf{c}'(t)\ | = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$Also\ xy + z = \cos t \sin t + t = \frac{1}{2}\sin 2t + t$$

$$\int_c (xy + z)\, ds = \sqrt{2}\int_0^{2\pi} (\frac{1}{2}\sin 2t + t)\, dt$$

$$= \sqrt{2}\Big[\frac{-1}{4}\cos 2t + \frac{1}{2}t^2\Big]_{t=0}^{t=2\pi}$$

$$= 2\sqrt{2}\,\pi^2$$

4.02 Line Integrals

Let **F** be a continuous vector field and **c** be a C^1 path where $\mathbf{c}(t) = (x(t), y(t), z(t))$. The **line integral** of **F** along **c** from $t = a$ to $t = b$ $(a \le t \le b)$ is:

$$\int_{\mathbf{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}\left[x(t), y(t), z(t)\right] \cdot \frac{d\boldsymbol{s}}{dt}\, dt$$

$$= \int_a^b \boldsymbol{F}\left[\mathbf{c}(t)\right] \cdot \mathbf{c}'(t)\, dt$$

- If $\boldsymbol{F} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$

$$\int_{\mathbf{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \left(u\frac{dx}{dt} + v\frac{dy}{dt} + w\frac{dz}{dt}\right) dt$$

$$= \int_{\mathbf{c}} \underbrace{u\, dx + v\, dy + w\, dz}_{\text{differential form}}$$

Note that the line integral is independent of how **c** is parametrised. If the traverse path **c** is in the opposite direction, then this will be the negative of the original line integral.

Let **F** be a force field (e.g. electric field) that acts on a particle moving along the path **c**. Then the line integral is the work done by **F** in moving the particle along **c**. If the line integral is less than 0, then **F** impedes the movement of the particle along **c**.

Example 1

Determine the work done by the force $\mathbf{F}(x, y) = (-y, 0)$ to move a particle around the following semicircle from (2, 0) to (-2, 0).

$$y = \sqrt{4 - x^2}$$

$$\boldsymbol{c}(t) = (2\cos t, 2\sin t) \quad 0 \le t \le \pi$$

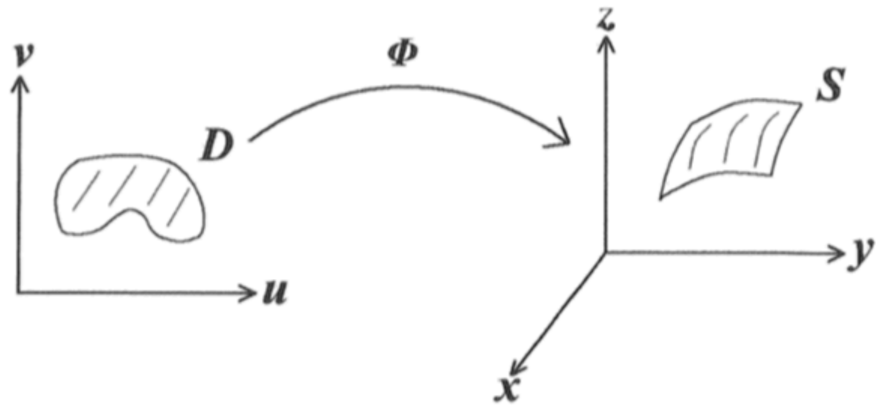
$$\boldsymbol{c}'(t) = (-2\sin t, 2\cos t)$$

$$\begin{aligned}
Work &= \int_c \boldsymbol{F} . d\boldsymbol{s} \\
&= \int_0^{\pi} (-2\sin t, 0) . (-2\sin t, 2\cos t)\, dt \\
&= \int_0^{\pi} 4\sin^2 t\, dt \\
&= \int_0^{\pi} 2(1 - \cos 2t)\, dt \\
&= 2\left[t - \frac{1}{2}\sin 2t\right]_{t=0}^{t=\pi} \\
&= 2\pi
\end{aligned}$$

4.03 Parametrised Surfaces

Some surfaces have the form $z = f(x, y)$. However, other surfaces cannot be written in this form, such as a torus. A parametrised surface is a function that maps a domain D to a surface S, namely:

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$



Note that if the transformation function phi is C^1, then S is a differentiable or C^1 surface.

Example 1

Identify the following surface given the parameterisations.

$$x = u\cos v$$

$$y = u\sin v$$

$$z = u$$

$$0 \leq v \leq 2\pi, u \geq 0$$

$$x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2$$

$$So\ z^2 = x^2 + y^2$$

$$z = \sqrt{x^2 + y^2}\ (\sin ce\ z = u \geq 0)$$

$$This\ is\ a\ cone.$$

Let S be a differentiable surface. Consider the curves on S given by:

$$\mathbf{c}_1(v) = \Phi(u_0, v) \quad \text{— } u \text{ is constant}$$

$$\mathbf{c}_2(u) = \Phi(u, v_0) \quad \text{— } v \text{ is constant}$$

Then the **tangent vector** to **c**(v) is given by:

$$T_v = \left.\frac{dc_1}{dv}\right|_{v=v_0}$$

$$= \left.\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\right|_{(u_0,v_0)}$$

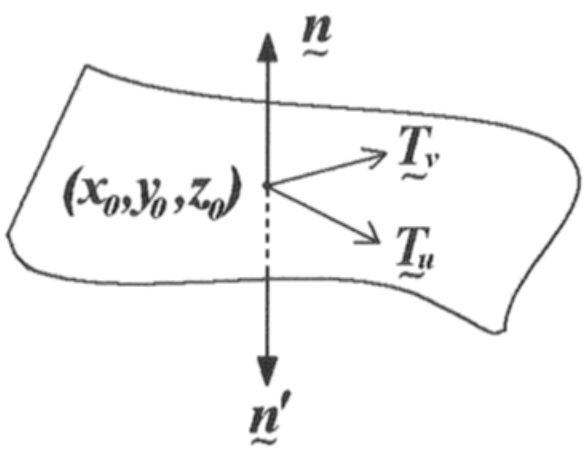
The tangent vector to **c**(u) is given by:

$$T_u = \left.\frac{dc_2}{du}\right|_{u=u_0}$$

$$= \left.\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\right|_{(u_0,v_0)}$$

Note that these tangent vectors change with position (x, y, z). They need not be orthogonal. There are two **normal vectors** to the surface, which are given by:

* $\boldsymbol{n} = \boldsymbol{T_u} \times \boldsymbol{T_v}$ <u>OR</u>
* $\boldsymbol{n}' = -\boldsymbol{n} = \boldsymbol{T_v} \times \boldsymbol{T_u}$



Note that **n** changes with position (x, y, z) and if $\mathbf{n} \neq \mathbf{0}$, then the surface is smooth.

If S is a smooth surface at (x, y, z), then the Cartesian equation of the **tangent plane** to S at (x, y, z) is:

$$(x - x_0, y - y_0, z - z_0) \cdot \boldsymbol{n}(u_0, v_0) = 0$$

Where **n** is the normal to S.

Let S be a smooth (except possibly at a finite number of points) parametrised surface. Then the area of the surface is given by:

$$\boxed{A(S) = \iint_S dS = \iint_D |\mathbf{T}_u \times \mathbf{T}_v| \, du \, dv}$$

Example 1
Find the surface area of the cone parametrised by the following equations.

$$\begin{aligned}
x &= u\cos v \\
y &= u\sin v \\
z &= u \\
\boldsymbol{n} &= (-u\cos v, -u\sin v, u) \\
So|\,\boldsymbol{n}\,| &= \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} \\
&= \sqrt{2u^2} \\
&= \sqrt{2}\,|\,u\,| \\
&= \sqrt{2}\,u \;\;(\sin ce\; u \geq 0) \\
SA &= \iint_S dS \\
&= \iint_D |\boldsymbol{n}\,|\,du\,dv \\
&= \int_0^{2\pi}\int_0^1 \sqrt{2}\,u\,du\,dv \\
&= \int_0^{2\pi} dv \int_0^1 \sqrt{2}\,u\,du \\
&= (2\pi)\,(\frac{\sqrt{2}}{2}) \\
&= \sqrt{2}\,\pi\,(units)^2
\end{aligned}$$

4.04 Integrals of Scalar Functions over Surfaces

Let f be a continuous function defined on a smooth, (except possibly at a finite number of points) parametrised surface S. We define:

$$\iint_S f\,dS = \iint_D f\,[\Phi(u,v)]\;\;|\mathbf{T}_u \times \mathbf{T}_v|\;du\,dv$$

where $\Phi : D \to S$ and

$$\Phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big).$$

Note that this integral is independent of the parametrised used for S. The area of surface is also a special case of this integral where $f = 1$. In terms of the physical interpretation, it depends on what f represents. For example, if f is the mass per unit area, then this integral is the total mass of the surface. If f is the charge per unit area, then this integral is the total charge on the surface.

Example 1

Evaluate the following surface integral, where S is the unit sphere centred at the origin.

$$x = \cos\phi \sin\theta$$
$$y = \sin\phi \sin\theta$$
$$z = \cos\theta$$
$$0 \le \theta \le \pi, 0 \le \phi \le 2\pi$$

$$\boldsymbol{T_\theta} \times \boldsymbol{T_\phi} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ -\sin\phi\sin\theta & \cos\phi\sin\theta & 0 \end{vmatrix}$$
$$= (\cos\phi\sin^2\theta, \sin\phi\sin^2\theta, \cos\theta\sin\theta)$$

$$So |\, \boldsymbol{T_\theta} \times \boldsymbol{T_\phi} \,| = \sqrt{\sin^4\theta\cos^2\phi + \sin^4\theta\sin^2\phi + \cos^2\theta\sin^2\theta}$$
$$= \sqrt{\sin^4\theta + \cos^2\theta\sin^2\theta}$$
$$= \sqrt{\sin^2\theta}$$
$$= \sin\theta \,(\sin ce\, 0 \le \theta \le \pi)$$

$$\iint_S z^2\, dS = \iint_D \cos^2\theta\sin\theta\, d\theta\, d\phi$$
$$= \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta\sin\theta\, d\theta$$
$$= 2\pi\left[\frac{-1}{3}\cos^3\theta\right]_{\theta=0}^{\theta=\pi}$$
$$= \frac{4\pi}{3}$$

Also note that if S can be written in the form $z = f(x, y)$ then:

$$\iint_S g\, dS$$

$$= \iint_D g[x, y, f(x,y)]\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dx\, dy$$

$$= \iint_D \frac{g[x, y, f(x,y)]}{|\hat{\boldsymbol{n}} \cdot \boldsymbol{k}\,|}\, dx\, dy$$

where $\hat{\boldsymbol{n}}$ is the unit normal to S.

4.05 Oriented Surfaces

An **oriented surface** is a two sided surface. At each point on S, there are two normal vectors. For example, a sphere is an oriented surface but a Möbius band is not an oriented surface because at each point there are two normal vectors, but the surface has only one side.

Let **F** be a continuous vector field defined on a smooth (except possibly at a finite number of points), orientable, parametrised surface S. We define:

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, dS$$
$$= \iint_D \boldsymbol{F} \cdot (\boldsymbol{T_u} \times \boldsymbol{T_v})\ du\, dv$$

where $\Phi : D \to S$ such that

$$\Phi(u,v) = \left(x(u,v), y(u,v), z(u,v)\right)$$

and $\hat{\boldsymbol{n}}$ is the unit outward normal to S.

If **F** is a vector field, then this integral is the **flux** of **F** across S. If **F** is the velocity of a fluid, then the flux is the net quantity of fluid to flow across the surface per unit time in the direction of the outward normal vector. Hence, the flux is the rate of fluid flow in the direction of the outward normal.

Example 1

Evaluate the following flux integral, where S is the unit sphere centred at the origin.

$$\begin{aligned}
&Let\ x = \cos\phi \sin\theta \\
&\quad y = \sin\phi \sin\theta \\
&\quad z = \cos\theta \\
&\quad 0 \le \theta \le \pi \\
&\quad 0 \le \phi \le 2\pi \\
&\quad \boldsymbol{n} = (\sin^2\theta \cos\phi, \sin^2\theta \sin\phi, \cos\theta \sin\theta)
\end{aligned}$$

$This\ is\ the\ outward\ normal\ \sin ce\ it\ has\ a\ positive\ k\ component\ for\ any\ value\ of\ \theta$

$when\ above\ the\ xy - plane.$

$Integrand\ is:$

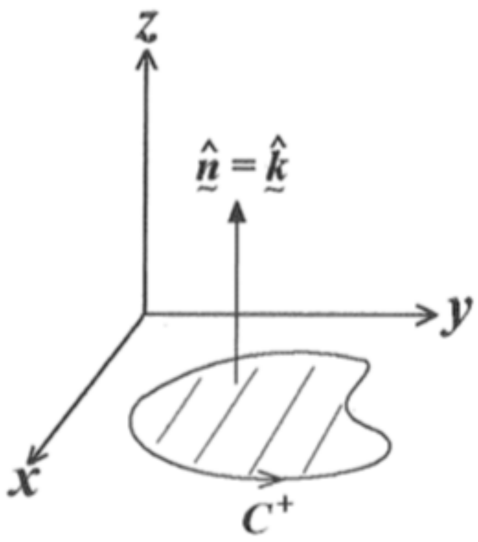
$$\begin{aligned}
I &= (x,y,z).(\sin^2\theta \cos\phi, \sin^2\theta \sin\phi, \cos\theta \sin\theta) \\
&= (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta).(\sin^2\theta \cos\phi, \sin^2\theta \sin\phi, \cos\theta \sin\theta) \\
&= \sin\theta
\end{aligned}$$

$$\begin{aligned}
So \iint_S \boldsymbol{F}.d\boldsymbol{S} &= \int_0^{\pi} \int_0^{2\pi} \sin\theta\, d\theta\, d\phi \\
&= 2\pi \int_0^{\pi} \sin\theta\, d\theta \\
&= 2\pi\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi} \\
&= 4\pi
\end{aligned}$$

Topic 5 - Integral Theorems

5.01 Oriented Closed Curves in *xy* Plane

Note that for an oriented closed curve in the *xy*-plane, it must have positive orientation (anticlockwise), where **n** is the normal to surface perpendicular to the *xy* plane.



Note that **n** and the orientation are related by the right-hand rule. If you walk around the boundary C in the positive orientation then region D will be on your left. It is also restricted to simple closed curves, i.e. non self intersecting.

Green's Theorem in the plane states:

$$\boxed{\int_{C=\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

Where D is a region in the *xy* plane bounded by a simple closed curve C with position orientation (anticlockwise). **F** is a C^1 vector field on D, and D is composed of regions of both vertical and horizontal strips. The vector form of Green's Theorem is given by:

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_D (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{k}\; dx\,dy$$

Example 1

Verify Green's Theorem for the following region bounded by the following curves.

$$\boldsymbol{F} = (xy^2, x + y)$$

$$y = x^2, y = x \,(x, y \geq 0)$$

$$Let \; P = xy^2, Q = x + y$$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx\,dy = \iint_D (1 - 2xy)\,dx\,dy$$

$$U\sin g \; vertical \; strips:$$

$$x^2 \leq y \leq x, 0 \leq x \leq 1$$

$$So \iint_D (1 - 2xy)\,dx\,dy = \int_0^1 \int_{x^2}^{x} (1 - 2xy)\,dy\,dx$$

$$= \int_0^1 (x - x^3 - x^2 + x^5)\,dx$$

$$= \frac{1}{12}$$

If C is a simple closed curve that bounds a region D, then the area of D is given by:

$$\boxed{\text{Area of } D = \frac{1}{2}\int_{C=\partial D} x\,dy - y\,dx}$$

5.02 Divergence Theorem in the Plane

The **Divergence Theorem** in the plane is given by:

$$\boxed{\int_{C=\partial D} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\,ds = \iint_D \boldsymbol{\nabla} \cdot \boldsymbol{F}\,dx\,dy}$$

Where D is a region in the *xy* plane bounded by a simple closed curve C with position orientation (anticlockwise). **F** is a C^1 vector field on D, and D is composed of regions of both vertical and horizontal strips. Moreover, **n** is the unit outward normal to ∂D in the *xy* plane.

Example 2

Let C be the triangle with vertices (0, 0), (1,1), (1.5, 0) traversed anticlockwise.

$$Evaluate \int_C \boldsymbol{F}.\hat{\boldsymbol{n}}\, ds$$

$$\boldsymbol{F}(x,y) = (2x^2y - 3x + \sin 5y, 5y - 2xy^2 - \cos^3 4x)$$

$$Using\ Divergence\ Theorem:$$

$$\begin{aligned}\nabla.\boldsymbol{F} &= \frac{\partial}{\partial x}(2x^2y - 3x + \sin 5y) + \frac{\partial}{\partial y}(5y - 2xy^2 - \cos^3 4x)\\ &= 4xy - 3 + 5 - 4xy\\ &= 2\end{aligned}$$

$$\begin{aligned}\int_C \boldsymbol{F}.\hat{\boldsymbol{n}}\, ds &= \iint_D \nabla.\boldsymbol{F}\, dx\, dy\\ &= \iint_D 2\, dx\, dy\\ &= 2\iint_D 1\, dx\, dy\\ &= 2 \times 0.5 \times 1.5 \times 1\\ &= 1.5\end{aligned}$$

5.03 Stokes' Theorem

Stokes' Theorem is given by:

$$\boxed{\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s}}$$

Where S is an open, oriented surface parametrised by the C^2 mapping. ∂S is the oriented closed boundary of S, **F** is a C^1 vector field on S. S and ∂S are oriented so that **n** is the unit outward normal to S. The orientation of ∂S and **n** are related by the right hand rule. Think of S as a fishing net and ∂S as its rim.

Walk along the boundary ∂S with the normal as your upright direction. You are moving in the positive direction if the surface S is on your left. Note that Green's Theorem is a special case of Stokes' Theorem where S and ∂S are confined to the *xy*-plane.

Example 1

Use Stokes' theorem to evaluate the following surface integral.

$$z = 9 - x^2 - y^2 \,, z \geq 0$$

$$\boldsymbol{F} = (2z - y, x + z, 3x - 2y)$$

$$Evaluate\ u\sin g \int_{\partial S} \boldsymbol{F} \,.\, d\boldsymbol{s}$$

$$Let\ x = 3\cos t, y = 3\sin t, z = 0, 0 \leq t \leq 2\pi$$

$$x' = -3\sin t, y' = 3\cos t, z' = 0$$

$$\int_{\partial S} \boldsymbol{F} \,.\, d\boldsymbol{s} = \int_0^{2\pi} (-3\sin t, 3\cos t, 9\cos t - 6\sin t) \,.\, (-3\sin t, 3\cos t, 0)\, dt$$

$$= \int_0^{2\pi} 9\sin^2 t + 9\cos^2 t\, dt$$

$$= \int_0^{2\pi} 9\, dt$$

$$= 18\pi$$

Note that for any two surfaces that have the same boundary C, Stokes' theorem implies that:

$$\boxed{\begin{aligned} \iint_{S_1} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} &= \iint_{S_2} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} \\ &= \int_C \boldsymbol{F} \cdot d\boldsymbol{s} \end{aligned}}$$

Hence, one should apply Stokes' theorem and use the simplest surface with consistent orientation of C and **n**. Also note that if the surface is closed, then Stokes' theorem implies that:

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = 0$$

5.04 Conservative Fields

Let **F** be a C^1 vector field defined on R^2 or R^3. The following conditions are all equivalent:

1. For any oriented simple closed curve C, then:

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

2. For any two oriented simple curves with the same end points, then:

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$$

This is also called **path independence**.

3. **F** = ∇ø for some scalar function ø (also called a **gradient field**).

4. ∇ x **F** = **0** (also called an **irrotational field**).

A vector field satisfying one (and hence all) of the four conditions is called a **conservative vector field**. If **F** is conservative, then the work integral depends only on the endpoints of C.

Example 1

Let **F** = (x, y). Evaluate the work integral along the parabola $y = x^2$ from (0, 0) to (1, 1).

$\boldsymbol{F}$ *is* C^1 *on* $\mathbb{R}^2$ *and* $\nabla \times \boldsymbol{F} = \boldsymbol{0}$, *so* $\boldsymbol{F}$ *is a conservative vector field.*

$$So\ \boldsymbol{F} = \nabla ø$$

$$That\ is, (x,y) = \left(\frac{\partial ø}{\partial x}, \frac{\partial ø}{\partial y}\right)$$

Equating components:

$$\frac{\partial ø}{\partial x} = x$$

$$ø = \frac{x^2}{2} + C(y)$$

$$Also, \frac{\partial ø}{\partial y} = y$$

$$ø = \frac{1}{2}y^2 + D(x)$$

$$So\ ø(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + C$$

$$Now, \int_C \boldsymbol{F} . d\boldsymbol{s} = ø(1,1) - ø(0,0)$$

$$= (0.5 + 0.5 + C) - (0 + 0 + C)$$

$$= 1$$

5.05 Gauss' Divergence Theorem

Gauss' Divergence Theorem is given by:

$$\boxed{\iiint_{\Omega} \nabla \cdot \boldsymbol{F}\, dV = \iint_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{S}}$$

Where Ω is a closed solid region in R^3, ∂Ω is the oriented closed surface that bounds Ω, **F** is a C^1 vector field on Ω, and orientation is defined by unit outward normal **n** to ∂Ω. For example, a solid hemisphere (which has two surfaces).

Example 1

Use Gauss' Divergence theorem to find the following surface integral:

$$\iint_S (x^3, y^3) \,.\, d\boldsymbol{S}$$

$$\textit{where } S \textit{ is the closed cylinder } x^2 + y^2 = 4, z = 0, z = 5.$$

$$U\sin g\ Gauss'\ Theorem:$$

$$\begin{aligned}\iint_S (x^3, y^3) \,.\, d\boldsymbol{S} &= \iint\int_V \nabla \,.\, \boldsymbol{F}\, dV \\ &= \iint\int_V 3x^2 + 3y^2\, dV\end{aligned}$$

$$U\sin g\ cylindrical\ coordinates:$$

$$0 \le p \le 2, 0 \le \phi \le 2\pi, 0 \le z \le 5$$

$$\begin{aligned}So \iint\int_V 3x^2 + 3y^2\, dV &= \iint\int_V 3p^3\, dp\, d\phi\, dz \\ &= \int_0^2 \int_0^{2\pi} \int_0^5 3p^3\, dz\, d\phi\, dp \\ &= 120\pi\end{aligned}$$

If **F** = (*x, y, z*) and Ω is a region tow which Gauss' theorem applies, then:

$$\boxed{\text{volume of } \Omega = \frac{1}{3} \iint_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{S}}$$

Topic 6 - General Curvilinear Coordinates

6.01 Curvilinear Coordinates Systems

For each point P with Cartesian coordinates (x, y, z), associate a unique set of **curvilinear coordinates** where:

- $$x = f_1(u_1, u_2, u_3),$$
  $$y = f_2(u_1, u_2, u_3),$$
  $$z = f_3(u_1, u_2, u_3).$$

- $$u_1 = g_1(x, y, z),$$
  $$u_2 = g_2(x, y, z),$$
  $$u_3 = g_3(x, y, z).$$

Let $\mathbf{r} = (x, y, z)$ be the position vector of P. A tangent vector at P for u_2 and u_3 constant is given by the following, where a unit tangent vector in the direction of u_1 increasing is:

$$\boldsymbol{e}_1 = \frac{\dfrac{\partial \boldsymbol{r}}{\partial u_1}}{\left|\dfrac{\partial \boldsymbol{r}}{\partial u_1}\right|}$$

$$\Rightarrow \quad \frac{\partial \boldsymbol{r}}{\partial u_1} = \left|\frac{\partial \boldsymbol{r}}{\partial u_1}\right| \boldsymbol{e}_1 = h_1\, \boldsymbol{e}_1$$

h_1 is called a **scale factor**. The curvilinear coordinates system is orthogonal if the dot product of the unit tangent vector is equal to 0.

Example 1

Define cylindrical coordinates; find the scale factors and unit tangent vectors.

$$x = p\cos\phi, y = p\sin\phi, z = z$$

$$0 \le \phi \le 2\pi, p \ge 0, z \in \mathbb{R}$$

$$\frac{\partial \boldsymbol{r}}{\partial p} = (\cos\phi, \sin\phi, 0)$$

$$h_p = \left|\frac{\partial r}{\partial p}\right| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

$$So\ \boldsymbol{e}_p = (\cos\phi, \sin\phi, 0)$$

$$\frac{\partial \boldsymbol{r}}{\partial \phi} = (-p\sin\phi, p\cos\phi, 0)$$

$$h_\phi = \left|\frac{\partial r}{\partial \phi}\right| = \sqrt{p^2\cos^2\phi + p^2\sin^2\phi} = p$$

$$So\ \boldsymbol{e}_\phi = (-\sin\phi, \cos\phi, 0)$$

$$\frac{\partial \boldsymbol{r}}{\partial z} = (0, 0, 1)$$

$$h_z = 1$$

$$\boldsymbol{e}_z = (0, 0, 1)$$

Using the chain rule, the tangent to **r** in any coordinate system is:

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial u_1}\frac{du_1}{dt} + \frac{\partial \boldsymbol{r}}{\partial u_2}\frac{du_2}{dt} + \frac{\partial \boldsymbol{r}}{\partial u_3}\frac{du_3}{dt}$$

$$\Rightarrow \frac{d\boldsymbol{r}}{dt} = h_1\boldsymbol{e}_1\frac{du_1}{dt} + h_2\boldsymbol{e}_2\frac{du_2}{dt} + h_3\boldsymbol{e}_3\frac{du_3}{dt}.$$

The grad, div, curl and Laplacian in orthogonal curvilinear coordinates is given by the formulae:

1. $\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3$

2. $\nabla \cdot \boldsymbol{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial u_1} + \frac{\partial(h_1h_3F_2)}{\partial u_2} + \frac{\partial(h_1h_2F_3)}{\partial u_3}\right]$

3. $\nabla \times \boldsymbol{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$

4. $\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$

The volume element in orthogonal curvilinear coordinates is given by:

$$\boxed{\begin{aligned} |\text{Jacobian}| &= h_1 h_2 h_3 \\ dV = \ dx\,dy\,dz &= h_1 h_2 h_3 \, du_1 \, du_2 \, du_3 \end{aligned}}$$

The surface area element in orthogonal curvilinear coordinates is given by:

$$\boxed{\begin{aligned} |\boldsymbol{T_u} \times \boldsymbol{T_v}| &= h_u h_v \\ dS &= h_u h_v \, du \, dv \end{aligned}}$$