Q2) We have, $y = 1_n\mu + X_2\tau + \varepsilon$, that is, $X_1 = 1_n$.

hat matrix

Thus, $X_1^TX_1 = n = kb$, $(X_1^TX_1)^c = 1/n$, $H_1 = 1/n\, J_{n}$ and $X_{2|1} = (I_n - \frac{1}{n}J_n)X_2$ $= X_2 - \frac{1}{k}J_{n,k}$, where $J_{n,k}$ is the $n\times k$ matrix of ones.

$(X_1^TX_1)^c$

The reduced normal equations are $X_{2|1}^T X_{2|1} t = X_{2|1}^T y$, that is,

$$b\left(I_k - \frac{1}{k}J_k\right)t = X_2^T\left(I_n - \frac{1}{n}J\right)y \qquad \boxed{\left((X^TX)b = (X^TX)y\right)}$$

Recall!!

Since $\left[b\left(I_k - \frac{1}{k}J_k\right)\right]^c = \frac{1}{b}I_k$, we have, (parameters) t, not b!!!

$$t = \frac{1}{b}X_2^T\overbrace{\left(I_n - \frac{1}{n}J_n\right)}^{H_1}y = \frac{1}{b}\left(X_2^T - \frac{1}{k}J_{k,n}\right)y = \begin{bmatrix} \bar{y}_1 - \bar{y}_. \\ \vdots \\ \bar{y}_k - \bar{y}_. \end{bmatrix}$$

$$t = \frac{1}{b}X_2^T\left(I_n - (X_1^TX_1)^c J_n\right)y = \frac{1}{b}\Big(X_2^T I_n - \underbrace{X_2^T(X_1^TX_1)^c J_n}_{X_2^T \frac{1}{kb}J_n}\Big)y$$

noting that,

$$X_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 \\ \hline & \ddots & & \\ \hline 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

, subject to a labelling the indices, this is exactly the same sol^n as for the ~~very~~ reduced normal equations for the CBD. (Complete Block Design)

see p. 45-50 (Thm 8.2) similar to that.

$y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij}$, Additive model!

standard error?!!

(Q3a) $\sum_j c_j \bar{y}_{\cdot j} \pm z_{\alpha/2}\, s \sqrt{\sum_j c_j^2 / b}$ where s (sample standard deviation)

SP.55 (Experimental design)

$t^T\tau$ in the lectures. degrees of freedom is large, $(n-p)$ is large!! → becomes standard normal (i.e. becomes close to a standard normal)

(Q3b) To have power $100(1-\alpha)$ against this alternative we need

$$\underbrace{|c^T\tau^*|}_{\text{critical value/point}} > z_{\alpha/2}\, \sigma \sqrt{\sum_j c_j^2 / b}, \quad \text{which gives}$$

$$b > z^2_{\alpha/2}\, \sigma^2 \sum_j c_j^2 \,/\, (c^T\tau^*)^2 = z^2_{\alpha/2}\, \sigma^2 c^T c \,/\, (c^T\tau^*)^2.$$

→ learn from prior MAST20005 Knowledge about "hypothesis testing"!!

(Q4a) ~~Eac~~ Each treatment occurs at most once in a block, and in exactly 3 blocks. Each pair of treatments occurs in exactly 1 block.
Thus we have a BIBD with $t=4$, $b=6$, $k=2$, $r=3$ and $\lambda=1$.

no# of design variables

~~no# of blocks~~

no# of blocks.

no# of treatments.

$\lambda = r\left(\frac{k-1}{t-1}\right)$ $\quad r = \frac{bk}{t}$

(Q5) From the lectures, we have that,

$$X_{2|1}^T X_{2|1} = \frac{\lambda t}{k}\left(I_t - \frac{1}{t} J_t\right)$$

SP. 93

$$(X_{2|1}^T X_{2|1})^c = \frac{k}{\lambda t} I_t, \qquad \boxed{r = \frac{bk}{t} \Rightarrow \frac{rt}{k} = b} \qquad \boxed{\lambda = \frac{bk}{t}\left(\frac{k-1}{t-1}\right),\ b = \frac{t}{k}\frac{(t-1)}{(k-1)}}$$

and therefore for any contrast $c^T\tau$,

$$c^T (X_{2|1}^T X_{2|1})^c X_{2|1}^T X_{2|1} = c^T\left(I_t - \frac{1}{t} J_t\right)$$

$$= c^T - \frac{1}{t} c^T J_t = c^T$$

normal equations get used for these notations!!

The estimate follows from the solution to the reduced normal equations. (i.e. it is estimable Thm. 6.10)

$$\boxed{\lambda = \frac{12}{4}\left(\frac{1}{3}\right) = 1}$$

Q7b) ~~Differentiating~~ solution and setting the derivative to 0, we obtain the generalised normal equations,

$$X_2^T V^{-1} X_2 t = X_2^T V^{-1} y.$$ (from ~~in~~ your understanding from the rank model) full/or less than

Thus a solution is $t = (X_2^T V^{-1} X_2)^c X_2^T V^{-1} y$. Similar to $b = (X^T X)^c X^T y$ considering $(X^T X)^c$ (regardless what c holds this true!

Q7c) We have (by substitution) $y_1 = U^T(X_2\tau + \varepsilon') = y_1 U^T X_2 \tau + \varepsilon_1$

$y_1 = U^T X_2 \tau + \varepsilon_1$, $E\varepsilon_1 = 0$, $Var\,\varepsilon_1 = \sigma^2 U^T U$

$y_2 = k\mu 1 + X_1^T X_2 \tau + \varepsilon_2$, $E\varepsilon_2 = 0$, $Var\,\varepsilon_2 = (k^2\sigma_\beta^2 + k\sigma^2)I$.

$= X_1^T(X_2\tau + \varepsilon') =$

Q7d) We have,

$Cov(y_1, y_2) = E\,U^T(y - Ey)(y - Ey)^T X_1 = U^T E[y - E[y]][y - E[y]]^T X_1$

$= U^T\, Var\, y\, X_1 = U^T(\sigma^2 I + \sigma_\beta^2 X_1 X_1^T) X_1$

$= \sigma^2 U^T X_1 + \sigma_\beta^2 U^T X_1 X_1^T X_1$

$= 0$. ?!! How

in detail.

Q7e) $y_1 = U^T y = U^T(X_2\tau + \varepsilon + X_1\beta)$

$= U^T X_2 \tau + \underbrace{U^T \varepsilon}_{\approx \varepsilon_1} + \underbrace{U^T X_1 \beta}$

Assume it is symmetric

$X_1 U^T \beta = 0$.

$= U^T X_2 \tau + \varepsilon_1$

$U^T E[\varepsilon_1] = 0$, $Var[\varepsilon_1] = \sigma^2 U^T U$

$Var[U^T \varepsilon'] = U^T Var(\varepsilon') U$

$= U^T U \sigma^2$

$y_2 = X_1^T(X_2\tau + \varepsilon') = X_1^T(X_2\tau + \varepsilon + X_1\beta) = X_1^T X_2 \tau + \underbrace{X_1^T\varepsilon}_{= \varepsilon_2} + \underbrace{X_1^T X_1 \beta}_{= k\mu 1}$

$E[\varepsilon_2] = 0$, $Var(\varepsilon_2) = (k^2\sigma_\beta^2 + k\sigma^2)I$.

Q7A) Tutorial 12+13

$$E[y] = E[X_1\beta + X_2\tau + \varepsilon]$$

$$= X_1 E[\beta] + E[X_2\tau] + \underbrace{E[\varepsilon]}_{=0}$$

$$= \cancel{X_1 \mu 1}$$

$$= \mu X_1 1 + X_2 E[\tau] + \cancel{0}$$

Assume $E[\tau] = \tau$.

$$= \mu X_1 1 + X_2 \tau = \mu 1 + X_2 \tau$$

$$Var(X_1\beta + X_2\tau + \varepsilon) = X_1 Var(\beta) X_1^T + X_2 Var(\tau) X_2^T + Var(\varepsilon)$$

$$= X_1 \sigma_\beta^2 I X_1^T + \underbrace{X_2 \quad X_2^T}_{=0} + \sigma^2 I$$

$$= \cancel{X_1} \sigma_\beta^2 X_1 X_1^T + \sigma^2 I$$

constant / scalar ~~the~~ for the random block effects!!

Question 5 b) Show that b is biased if $\lambda \neq 0$!!!! Assignment 2

$b^* = (X^T X + \lambda I)^{-1} X^T y.$

$(X^T X + \lambda I) b = X^T y.$

Theorem 4.4 (Gauss-Markov Theorem).

$E[b^*] = [(X^T X + \lambda I)^{-1}] X^T E[y]$

$= [(X^T X + \lambda I)^{-1}] X^T X \beta$

$= (X^T X + \lambda I)^{-1} X^T X \beta \neq \beta$ is biased $\lambda \neq 0$!!

So b^* is an unbiased estimator for β, we know that $E[b^*] = \beta$
therefore $[X^T X + \lambda I]^{-1} X^T X \beta$ which means $\lambda = 0$ then
$(X^T X)^{-1} X^T X \beta = I\beta = \beta$

$Var(b)$
$b = (X^T X)^{-1} X^T y$

$\Rightarrow Var((X^T X)^{-1} X^T y)$