1. (a) $A_1(\mathbf{x}) = \mathbf{x}^T\mathbf{A}_1\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2.$

$A_2(\mathbf{x}) = \mathbf{x}^T\mathbf{A}_2\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 5x_2 \end{pmatrix} = 2x_1^2 - 3x_1x_2 + 5x_2^2.$

$A_3(\mathbf{x}) = \mathbf{x}^T\mathbf{A}_3\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} -2x_1 + 2x_2 \\ 2x_1 - 5x_2 \end{pmatrix} = -2x_1^2 + 4x_1x_2 - 5x_2^2$

$A_4(\mathbf{x}) = \mathbf{x}^T\mathbf{A}_4\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} -x_1 - x_2 \\ -x_1 + 3x_2 \end{pmatrix} = -x_1^2 - 2x_1x_2 + 3x_2^2$

(b) For $\mathbf{A}_1$, it can be seen directly from the definition: as $A_1(\mathbf{x}) = x_1^2 + x_2^2 > 0$ for all $(x_1, x_2) \neq (0,0)$, the matrix is positive definite.

For $\mathbf{A}_2$ and $\mathbf{A}_3$, we will apply Sylvester's criterion:

- The leading principal minors of $\mathbf{A}_2$ are $\Delta_1 = 2$ and $\Delta_2 = \det(\mathbf{A}_2) = 10 - 1 = 9$. They are both positive, so $\mathbf{A}_2$ is positive definite.
- The leading principal minors of $\mathbf{A}_3$ are $\Delta_1 = -2$ and $\Delta_2 = \det(\mathbf{A}_2) = 10 - 4 = 6$. As $-\Delta_1 = 2 > 0$ and $\Delta_2 > 0$, the matrix is negative definite.

For $\mathbf{A}_4$, observe that taking $\mathbf{x}^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$ gives $\mathbf{x}^T\mathbf{A}_4\mathbf{x} = -1$, and taking $\mathbf{x}^T = \begin{pmatrix} 0 & 1 \end{pmatrix}$ gives $\mathbf{x}^T\mathbf{A}_4\mathbf{x} = 3$. As the quadratic form $A_4(\mathbf{x})$ can take values of opposite signs, the matrix $\mathbf{A}_4$ is indefinite.

(c) On the subspace $\{\mathbf{x} : x_1 + x_2 = 0\}$ we have $x_2 = -x_1$. This gives

$$A_4(\mathbf{x}) = -x_1^2 + 2x_1^2 + 3x_1^2 = 4x_1^2,$$

which is always positive for $(x_1, x_2) \neq (0,0)$. Thus $\mathbf{A}_4$ is positive definite on $\{\mathbf{x} : x_1 + x_2 = 0\}$.

2. (a) The quadratic form corresponding to $\mathbf{A}$ is

$$A(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} = 2x_1x_2 + 2x_1x_3 + 2x_2x_3.$$

(b) If $\mathbf{x}^T = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$ then $A(\mathbf{x}) = 2$, and if $\mathbf{x} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$, then $A(\mathbf{x}) = -2$. As the quadratic form $A(\mathbf{x})$ can take values of opposite signs, the matrix $\mathbf{A}$ is indefinite.

(c) As $x_1 + x_2 + x_3 = 0$, we have $x_3 = -(x_1 + x_2)$. Then,

$$\begin{aligned} A(\mathbf{x}) = A(x_1, x_2, -(x_1+x_2)) &= 2x_1x_2 - 2x_1(x_1+x_2) - 2x_2(x_1+x_2) \\ &= -2(x_1^2 + x_2^2 + x_1x_2) \\ &= -2\left((x_1 + \tfrac{1}{2}x_2)^2 + \tfrac{3}{4}x_2^2\right). \end{aligned}$$

This is negative for all $(x_1, x_2) \neq 0$, which makes $\mathbf{A}$ negative definite on the set $\{\mathbf{x} : x_1 + x_2 + x_3 = 0\}$.

Alternatively, note that

$$A(x_1, x_2, -(x_1+x_2)) = -2(x_1^2 + x_2^2 + x_1x_2) = \mathbf{x}^T\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}\mathbf{x}.$$

The leading principal minors of $\mathbf{B} := \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$ are

$$\Delta_1 = \det(-2) = -2 \text{ and } \Delta_2 = \det(\mathbf{B}) = 4 - 1 = 3,$$

which satisfy $(-1)^k\Delta_k > 0$, so $\mathbf{B}$ is negative definite.

3. (a) $\nabla f(\mathbf{x}) = \begin{pmatrix} x_1^2 - 4 \\ x_2^2 - 16 \end{pmatrix}$, $D^2 f(\mathbf{x}) = \begin{pmatrix} 2x_1 & 0 \\ 0 & 2x_2 \end{pmatrix}$

(b) As the interior of $\mathbb{R}^2$ is $\mathbb{R}^2$, it follows from Corollary 3 of the subject notes.

(c) We require both $x_1^2 - 4 = 0$ and $x_2^2 - 16 = 0$, which has solutions $\mathbf{x}^T = \begin{pmatrix} \pm 2 & \pm 4 \end{pmatrix}$.

(d) For $\mathbf{p}^T = \begin{pmatrix} p_1 & p_2 \end{pmatrix}$, the Hessian is

$$D^2 f(\mathbf{x}) = \begin{pmatrix} 2p_1 & 0 \\ 0 & 2p_2 \end{pmatrix},$$

and the leading principal minors are $\Delta_1 = 2p_1$ and $\Delta_2 = 4p_1 p_2$. By Sylvester's criterion,

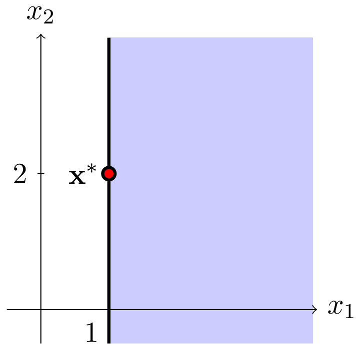
- If $\Delta_1, \Delta_2 > 0$, then $D^2 f(\mathbf{x})$ is positive definite, and the SOSC implies $\mathbf{x}$ is a local minimiser.
- If $\Delta_1 < 0$ and $\Delta_2 > 0$, then $D^2 f(\mathbf{x})$ is negative definite, and the SOSC implies $\mathbf{x}$ is a local maximiser.
- If $\Delta_1 > 0$ and $\Delta_2 < 0$, then $D^2 f(\mathbf{x})$ is indefinite, and the SO<u>N</u>C implies $\mathbf{x}$ is neither a local minimiser nor local maximiser.

So,

- For $\mathbf{p}^T = \begin{pmatrix} 2 & 4 \end{pmatrix}$, we have $\Delta_1 = 4$ and $\Delta_2 = 32$, so $\begin{pmatrix} 2 & 4 \end{pmatrix}^T$ is a local minimiser.
- For $\mathbf{p}^T = \begin{pmatrix} -2 & -4 \end{pmatrix}$, we have $\Delta_1 = -4$ and $\Delta_2 = 32$, so $\begin{pmatrix} -2 & -4 \end{pmatrix}^T$ is a local maximiser.
- For $\mathbf{p}^T = \begin{pmatrix} 2 & -4 \end{pmatrix}$, we have $\Delta_1 = 4$ and $\Delta_2 = -32$, so $\begin{pmatrix} 2 & -4 \end{pmatrix}^T$ is not an extremiser.
- For $\mathbf{p}^T = \begin{pmatrix} -2 & 4 \end{pmatrix}$, we have $\Delta_1 = -4$ and $\Delta_2 = -32$, so $\begin{pmatrix} -2 & 4 \end{pmatrix}^T$ is not an extremiser.

None of the points are global extremisers because f is neither bounded from above nor from below.

4. (a) The point $\mathbf{x}^*$ is on the boundary of the constraint set

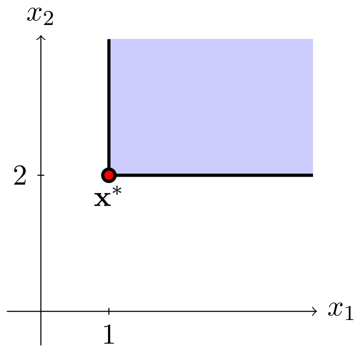


The feasible directions are $\mathbf{d} = \begin{pmatrix} d_1 & d_2 \end{pmatrix}^T \neq \begin{pmatrix} 0 & 0 \end{pmatrix}$ with $d_1 \geqslant 0$. We then have

$$\nabla^T f(\mathbf{x}^*)\mathbf{d} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = d_1 + d_2.$$

Taking, for example, $\mathbf{d}^T = \begin{pmatrix} 0 & -1 \end{pmatrix}$, gives $\nabla^T f(\mathbf{x}^*)\mathbf{d} = -1 < 0$, so by the FONC it is not a minimiser.

(b) The point $\mathbf{x}^*$ is on the boundary of the constraint set



The feasible directions are $\mathbf{d} = \begin{pmatrix} d_1 & d_2 \end{pmatrix}^T \neq \begin{pmatrix} 0 & 0 \end{pmatrix}$ with $d_1, d_2 \geqslant 0$. We then have

$$\nabla^T f(\mathbf{x}^*)\mathbf{d} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = d_1,$$

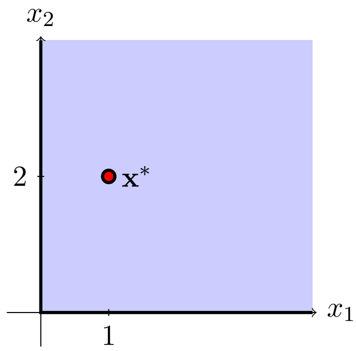
As $\nabla^T f(\mathbf{x}^*)\mathbf{d} = d_1 \geqslant 0$ for feasible $\mathbf{d}$, the FONC is satisfied, so $\mathbf{x}^*$ is possibly a minimiser.

(c) The sketch and feasible directions are identical to that in part (b). The only difference is that

$$\nabla^T f(\mathbf{x}^*)\mathbf{d} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = d_1 + d_2,$$

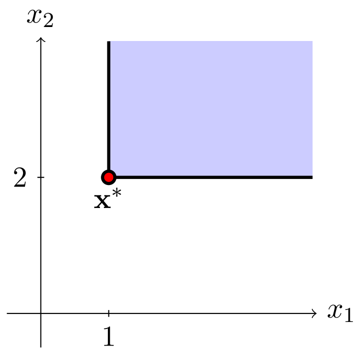
As $d_1, d_2 \geqslant 0$ and $\mathbf{d}^T \neq \begin{pmatrix} 0 & 0 \end{pmatrix}$, we have $\nabla^T f(\mathbf{x}^*)\mathbf{d} = d_1 + d_2 > 0$ for feasible $\mathbf{d}$. Hence the FOSC is satisfied, and $\mathbf{x}^*$ is a local minimiser.

(d) The point $\mathbf{x}^*$ is in the interior of the constraint set.



All directions are feasible. We have $\nabla f(\mathbf{x}^*) = \mathbf{0}$ so the FONC is satisfied. As $D^2 f(\mathbf{x}^*)$ is positive definite (it is the matrix $\mathbf{A}_1$ of Question 1), the SOSC is also satisfied and hence $\mathbf{x}^*$ is a minimizer.

(e) The point $\mathbf{x}^*$ is on the boundary of the constraint set.



The feasible directions are $\mathbf{d} = \begin{pmatrix} d_1 & d_2 \end{pmatrix}^T \neq \begin{pmatrix} 0 & 0 \end{pmatrix}$ with $d_1, d_2 \geqslant 0$. The FONC is satisfied, because

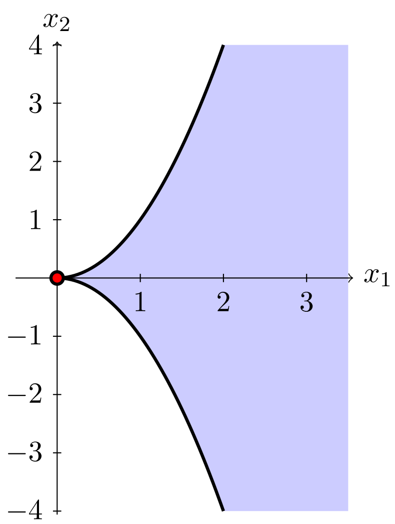
$$\nabla^T f(\mathbf{x}^*)\mathbf{d} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = d_1 \geqslant 0,$$

but for the feasible direction $\mathbf{d} = \begin{pmatrix} 0 & d_2 \end{pmatrix}^T$, we have

$$\mathbf{d}^T D^2 f(\mathbf{x}^*)\mathbf{d} = \begin{pmatrix} 0 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ d_2 \end{pmatrix} = -d_2^2 < 0,$$

so by the SONC, the point $\mathbf{x}^*$ is not a local minimiser.

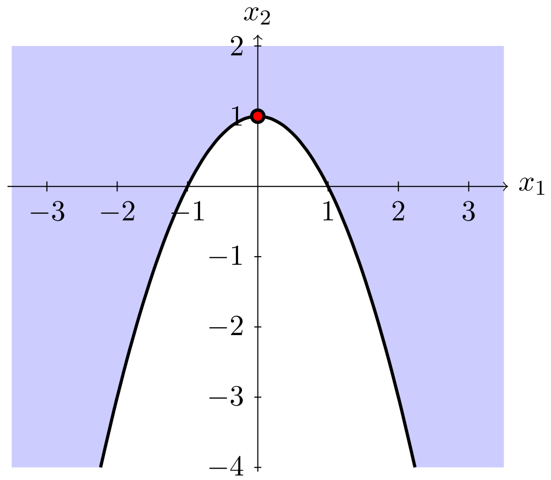
5. (a) The feasible region is:



(b) We have $\nabla^T f(\mathbf{x}) = \begin{pmatrix} 0 & -2x_2 \end{pmatrix}$, so $\nabla^T f(\mathbf{0}) = \mathbf{0}$, and hence the FONC is satisfied.

(c) No, it is a non-strict global maximiser. For all $\mathbf{x} \in \Omega$, we have $f(\mathbf{x}) \leqslant 0$, and $f(\mathbf{x}) = 0$ for all $\mathbf{x}$ with $x_2 = 0$.

6. (a) The feasible region is:

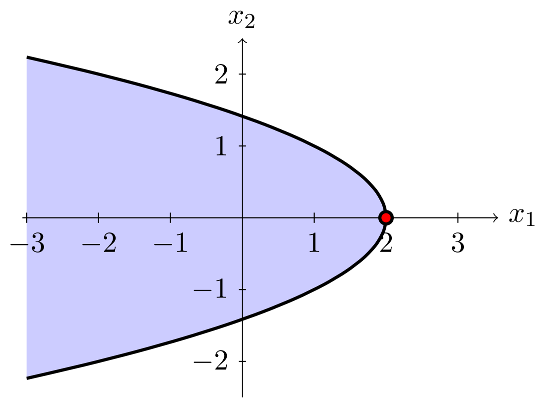


(b) We have $\nabla^T f(\mathbf{x}) = \begin{pmatrix} 0 & 5 \end{pmatrix}$, so $\nabla^T f(\mathbf{x}^*) = \begin{pmatrix} 0 & 5 \end{pmatrix}$.
The feasible directions are $\mathbf{d}$ with $d_2 \geqslant 0$, so $\nabla^T f(\mathbf{x}^*)\mathbf{d} = 5d_2 \geqslant 0$, and the FONC is satisfied.

(c) The Hessian is $D^2 f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so the SONC is satisfied.

(d) No, because the value at $\mathbf{x}^*$ is 5, but the value at $(\epsilon, 1 - \epsilon^2)$ is $5(1 - \epsilon^2) < 5$.

7. (a) The feasible region is:



(b) We have $\nabla^T f(\mathbf{x}) = \begin{pmatrix} -3 & 0 \end{pmatrix}$, so $\nabla^T f(\mathbf{x}^*) = \begin{pmatrix} -3 & 0 \end{pmatrix}$.
The feasible directions are $\mathbf{d}$ with $d_1 < 0$, so $\nabla^T f(\mathbf{x}^*)\mathbf{d} = -3d_1 \geqslant 0$, and the FONC is satisfied.

(c) The Hessian is $D^2 f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so the SONC is satisfied.

(d) The point $\mathbf{x}^* = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ is in fact a global minimiser; the value of f at $\mathbf{x}^*$ is -6 and for all $\mathbf{x} \in \Omega$, $\mathbf{x} \neq \mathbf{x}^*$, we have $x_1 < 2$ which gives $f(\mathbf{x}) > -6$.

8. (a) Let $\mathbf{g}(\mathbf{x}) = \mathbf{A}^T$ and let $\mathbf{h}(\mathbf{x}) = \mathbf{x}$. Then $f(\mathbf{x}) = \mathbf{g}(\mathbf{x})^T \mathbf{h}(\mathbf{x})$, so

$$\begin{aligned} Df(\mathbf{x}) &= \mathbf{g}(\mathbf{x})^T D\mathbf{h}(\mathbf{x}) + \mathbf{h}(\mathbf{x})^T D\mathbf{g}(\mathbf{x}) \\ &= (\mathbf{A}^T)^T I_n + \mathbf{x}^T \mathbf{0} = \mathbf{A}. \end{aligned}$$

(b) Let $\mathbf{g}(\mathbf{x}) = \mathbf{x}$ and let $\mathbf{h}(\mathbf{x}) = \mathbf{A}$. Then $f(\mathbf{x}) = \mathbf{g}(\mathbf{x})^T \mathbf{h}(\mathbf{x})$, so

$$\begin{aligned} Df(\mathbf{x}) &= \mathbf{g}(\mathbf{x})^T D\mathbf{h}(\mathbf{x}) + \mathbf{h}(\mathbf{x})^T D\mathbf{g}(\mathbf{x}) \\ &= (\mathbf{x}^T)^T \mathbf{0} + \mathbf{A}^T I_n = \mathbf{A}^T. \end{aligned}$$

(c) Let $\mathbf{g}(\mathbf{x}) = \mathbf{x}$ and let $\mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Then $f(\mathbf{x}) = \mathbf{g}(\mathbf{x})^T \mathbf{h}(\mathbf{x})$, so

$$\begin{aligned} Df(\mathbf{x}) &= \mathbf{g}(\mathbf{x})^T D\mathbf{h}(\mathbf{x}) + \mathbf{h}(\mathbf{x})^T D\mathbf{g}(\mathbf{x}) \\ &= \mathbf{x}^T \mathbf{A} + (\mathbf{A}\mathbf{x})^T I_n \\ &= \mathbf{x}^T \mathbf{A} + \mathbf{x}^T \mathbf{A}^T = \mathbf{x}^T(\mathbf{A} + \mathbf{A}^T). \end{aligned}$$

For $n = 1$, (a) and (b) correspond to the fact that $\frac{d}{dx}(ax) = a$ and (c) corresponds to the fact that $\frac{d}{dx}(ax^2) = 2ax$.

9. A boldface letter indicates that it is a vector-valued function while an italics letter indicates that it is a real-valued function.

10. (a) We have $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{B} + 6$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

so

$$\begin{aligned} Df(\mathbf{x}) &= \mathbf{x}^T(\mathbf{A} + \mathbf{A}^T) + \mathbf{B}^T = \mathbf{x}^T\begin{pmatrix} 2 & 6 \\ 6 & 14 \end{pmatrix} + \begin{pmatrix} 3 & 5 \end{pmatrix} \\ \implies \nabla f(\mathbf{x}) &= \begin{pmatrix} 2 & 6 \\ 6 & 14 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} \\ \implies D^2 f(\mathbf{x}) &= \begin{pmatrix} 2 & 6 \\ 6 & 14 \end{pmatrix} \end{aligned}$$

(b) As the function is unconstrained, its feasible set is all of $\mathbb{R}^2$; the interior of $\mathbb{R}^2$ is $\mathbb{R}^2$, so it follows from Corollary 3 of the subject notes that if $\mathbf{x}^*$ is a minimiser then $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

(c) From (a), we have $\nabla f(\mathbf{x}^*) = \mathbf{0}$ if and only if

$$\begin{pmatrix} 2 & 6 \\ 6 & 14 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we require

$$\begin{aligned} 2x_1 + 6x_2 + 3 &= 0 \\ 6x_1 + 14x_2 + 5 &= 0, \end{aligned}$$

which has solution $(x_1, x_2) = (\frac{3}{2}, -1)$. So there is only one point satisfying the FONC: $\mathbf{p}^T = \begin{pmatrix} \frac{3}{2} & -1 \end{pmatrix}$.

(d) The leading principal minors of $D^2 f(\mathbf{x})$ are $\Delta_1 = 2$ and $\Delta_2 = 28 - 36 = -8$. By Sylvester's criterion, $D^2 f(\mathbf{x})$ is indefinite, so by the SONC, $\mathbf{p}$ is neither a minimiser nor a maximiser.

(e) It is a saddle point.

11. As the function is unconstrained, its feasible set is all of $\mathbb{R}^2$; the interior of $\mathbb{R}^2$ is $\mathbb{R}^2$, so any local extremiser $\mathbf{x}^*$ must satisfy $\nabla f(\mathbf{x}^*) = \mathbf{0}$. We have

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

so $\nabla f(\mathbf{x}^*) = \mathbf{0}$ if and only if

$$\begin{aligned} 4x_1 - 2x_2 + 3 &= 0 \\ -2x_1 + 2x_2 + 2 &= 0, \end{aligned}$$

which has solution $(x_1, x_2) = (-\frac{5}{2}, -\frac{7}{2})$. So this is the only possible local extremiser.

We characterise it using the SOSC. The Hessian is

$$D^2 f(\mathbf{x}) = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix},$$

which has leading principal minors $\Delta_1 = 4$ and $\Delta_2 = 8 - 4 = 4$. As both are positive, $D^2 f(\mathbf{x})$ is positive definite, and hence the point $\mathbf{x}^* = \begin{pmatrix} -\frac{5}{2} & -\frac{7}{2} \end{pmatrix}$ is a local minimiser.

12. We have to minimize $f(\bar{x}) = \sum_{i=1}^{n} (\bar{x} - x_i)^2$. By the FONC, we require $\frac{df}{d\bar{x}} = 0$, resulting in

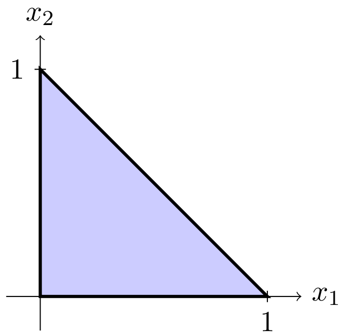
$$\begin{aligned} \sum_{i=1}^{n} 2(\bar{x} - x_i) = 0 &\iff 2n\bar{x} - 2\sum_{i=1}^{n} x_i = 0 \\ &\iff \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 0, \end{aligned}$$

i.e., $\bar{x}$ is the average of the numbers. To confirm this is indeed a minimiser, we apply the SOSC:

$$\frac{d^2 f}{d\bar{x}^2} = \frac{d}{d\bar{x}}\left(\sum_{i=1}^{n} 2(\bar{x} - x_i)\right) = \sum_{i=1}^{n} 2 = 2n.$$

This is positive, so by the SOSC, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a local minimiser. In fact, it is a global minimiser, as $f(\bar{x})$ is a quadratic function, for which any local minimiser is global.

13. The feasible region is



We will minimize $f(\mathbf{x}) = -c_1x_1 - c_2x_2$.

(a) The minimizer $\mathbf{x}^*$ does not lie in the interior of the constraint set as $\nabla f(\mathbf{x}) = -\mathbf{c} \neq \mathbf{0}$.

(b) If $\mathbf{x}^* \in L_1$, then a feasible direction is $\mathbf{d} = \begin{pmatrix}0 & 1\end{pmatrix}^T$ and $\mathbf{d}^T\nabla f(\mathbf{x}^*) = -c_2 < 0$.
If $\mathbf{x}^* \in L_2$, then a feasible direction is $\mathbf{d} = \begin{pmatrix}1 & 0\end{pmatrix}^T$ and $\mathbf{d}^T\nabla f(\mathbf{x}^*) = -c_1 < 0$.
If $\mathbf{x}^* \in L_3$, then a feasible direction is $\mathbf{d} = \begin{pmatrix}1 & -1\end{pmatrix}^T$ and $\mathbf{d}^T\nabla f(\mathbf{x}^*) = -c_1 + c_2 < 0$.

(c) At $\begin{pmatrix}1 & 0\end{pmatrix}^T$, the set of feasible directions is $\{\mathbf{d} : d_1 < 0, \quad 0 \leqslant d_2 < -d_1\}$.
For feasible $\mathbf{d}$ we have $\mathbf{d}^T\nabla f = -c_1d_1 - c_2d_2 > (-c_1 + c_2)d_1 > 0$, so the FONC is satisfied.

Since the set Ω is compact, according to Weierstrass f achieves its minumum on Ω. Since $\begin{pmatrix}1 & 0\end{pmatrix}^T$ is the only point that can be the minimum, it is the minimum. Alternatively, the FOSC can be employed here.

14. (a) Define the function file `rosenbrock.m`:

```
function z = rosenbrock(x1, x2)
% The Rosenbrock function
z = 100*(x2-x1.^2).^2 + (1-x1).^2;
end
```

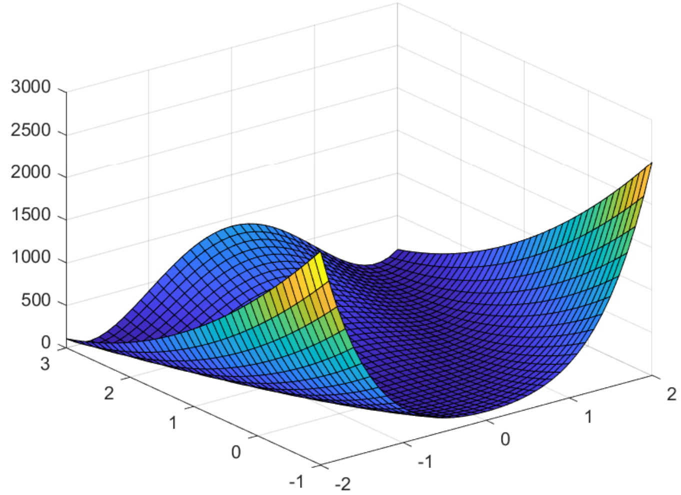
(b) We have $\nabla f(\mathbf{x}) = \begin{pmatrix}-400x_1(x_2 - x_1^2) - 2(1 - x_1) & 200(x_2 - x_1^2)\end{pmatrix}^T$, so define the function file `grosenbrock.m`:

```
function [g1, g2] = grosenbrock(x1,x2)
% Gradient of the Rosenbrock function
g1 = -400*x1.*(x2-x1.^2) - 2*(1-x1);
g2 = 200*(x2 - x1.^2);
end
```

(c) The script

```
[X,Y] = meshgrid(-2:1/10:2,-1:1/10:3);
Z = rosenbrock(X,Y);
surf(X,Y,Z)
```

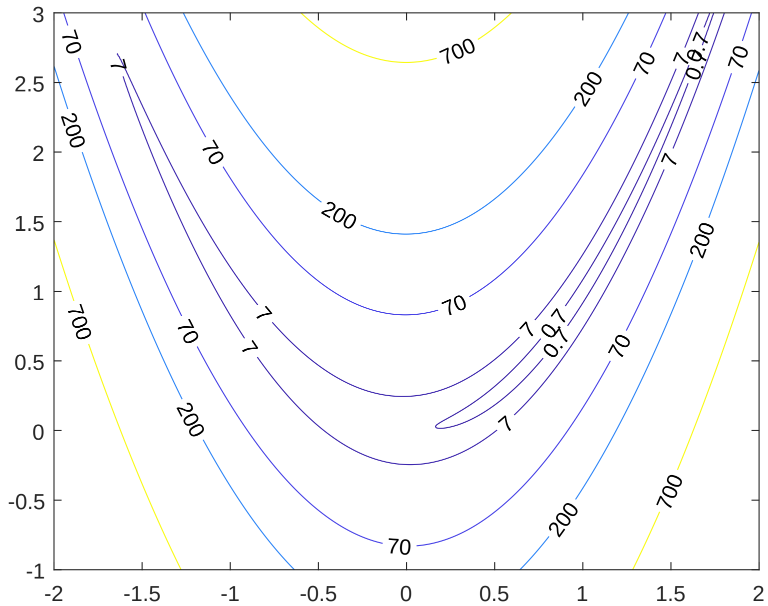
produces



(d) The script

```
[X,Y] = meshgrid(-2:1/100:2,-1:1/100:3);
Z = rosenbrock(X,Y);
contour(X,Y,Z,[.7,7,70,200,700],'ShowText','on')
```

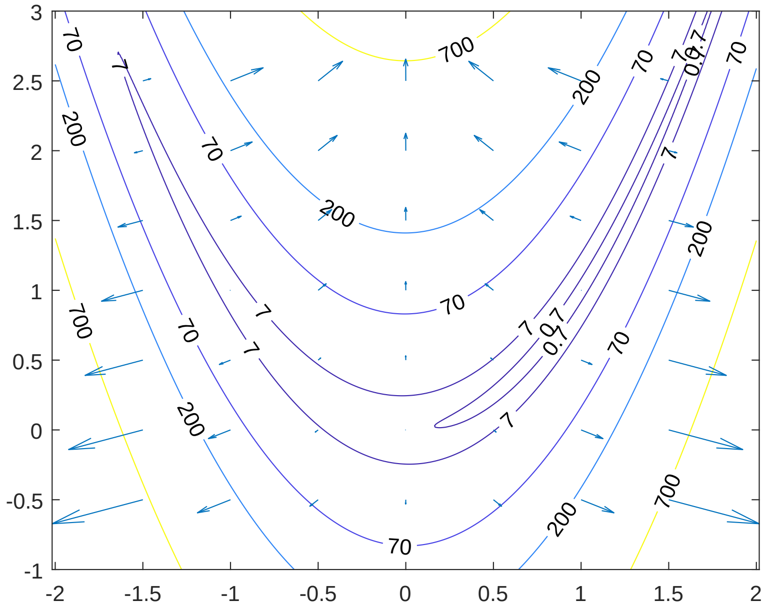
produces



(e) The script

```
[X,Y] = meshgrid(-2:1/100:2,-1:1/100:3);
Z = rosenbrock(X,Y);
contour(X,Y,Z,[.7,7,70,200,700],'ShowText','on')
hold on;
[X,Y] = meshgrid(-1.5:1/2:1.5,-.5:1/2:2.5);
[A,B] = grosenbrock(X,Y);
quiver(X,Y,A,B)
hold off;
```

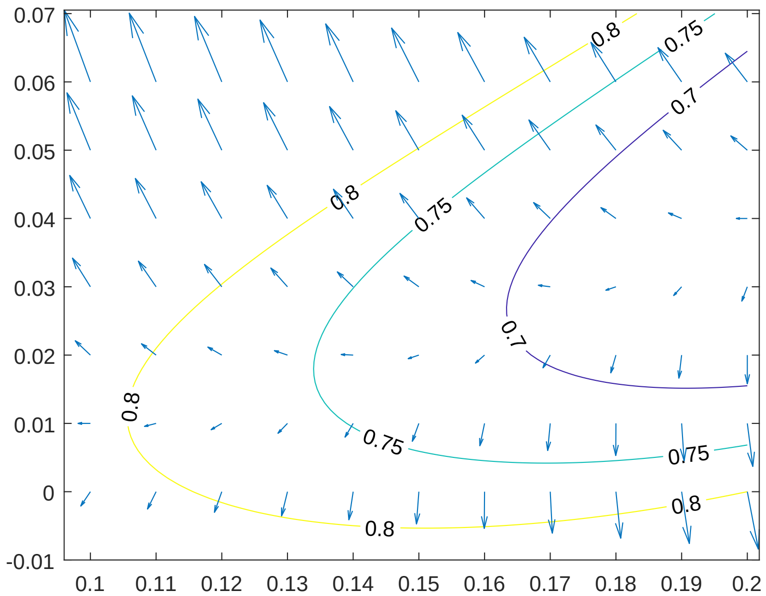
produces



(f) We plot the .7-, .75- and the .8-level sets using

```
[X,Y] = meshgrid(0.1:1/1000:0.2,-0.01:1/1000:0.07);
Z = rosenbrock(X,Y);
contour(X,Y,Z,[0.7, 0.75, 0.8],'ShowText','on')
hold on;
[X,Y] = meshgrid(0.1:0.01:0.2,0:0.01:0.06);
[A,B] = grosenbrock(X,Y);
quiver(X,Y,A,B)
hold off;
```

This gives

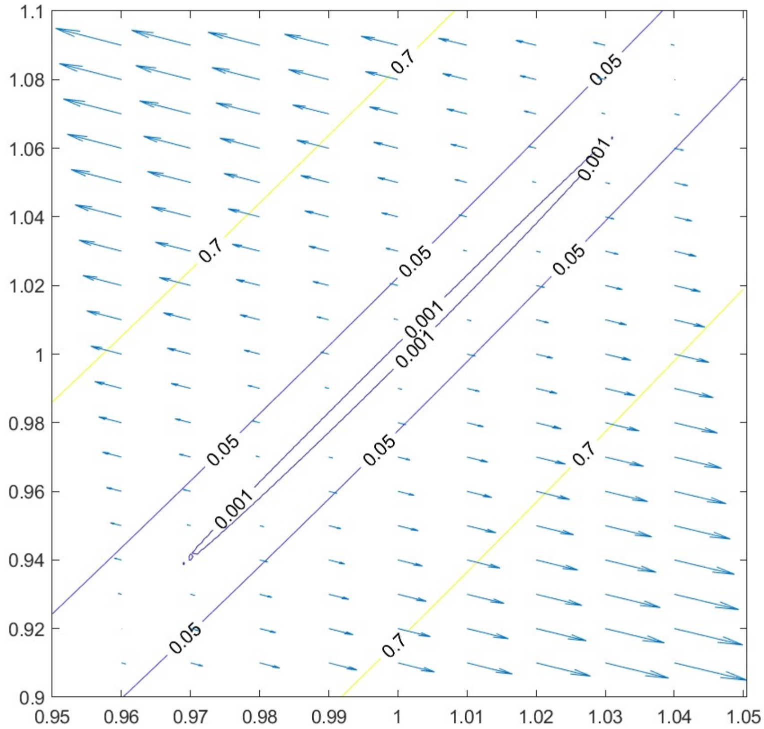


One can observe that the arrows have greater magnitude when the curves are closer together.

(g) The MATLAB code

```
[X , Y ] = meshgrid (0.95:1/1000:1.05 , 0.9:1/1000:1.1);
Z = rosenbrock (X , Y );
contour (X ,Y ,Z ,[0.001 0.05 0.7] , ’ShowText’,’on’)
hold on;
[X , Y ] = meshgrid (0.96:0.01:1.04 ,0.91:0.01:1.09);
[A , B ] = grosenbrock (X, Y);
quiver (X ,Y ,A , B )
hold off;
```

produces



which shows that there seems to be a minimum of 0 at (1,1). Looking at the function this is clearly the case.