School of Mathematics and Statistics
MAST30030
Applied Mathematical Modelling

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Assignment 3. Due: 11:59pm Friday June 5th

This assignment counts for 15% of the marks for this subject.

Question 1 [20 marks]

At time $t \geq 0$, a velocity field $\mathbf{u} = u\hat{\mathbf{x}} + v\hat{\mathbf{y}}$ is given by

$$u(x, y, t) = -y; \; v(x, y, t) = x + t.$$

(a) Find the streamlines in parametric form.

(b) Are there any requirements for the streamfunction to exist? Calculate the streamfunction under these conditions (if any), and use this result to calculate the streamlines in Cartesian form.

(c) Using the results in (a) and (b), plot the streamline originating at $(x, y) = (1, 0)$ for $t \in \{0, 1, 2\}$.

(d) Find the path of the particle that was located at (X, Y) at $t = 0$.

(e) Using (d), plot the path of the particle originating at $(X, Y) = (1, 0)$. Discuss (briefly) any similarities or differences you observe in comparison to (c).

(f) Verify from the particle path that

$$\frac{\partial \mathbf{u}}{\partial t}|_{\mathbf{R}} = \frac{D\mathbf{u}}{Dt}.$$

Question 2 [20 marks]

(a) The generalised Lorentz reciprocal theorem for two incompressible, Stokes flows (which may or may not be subject to body forces) in the same domain, with velocity and stress tensor $(\mathbf{u}, \mathbf{T})$ and $(\mathbf{u}', \mathbf{T}')$, respectively, is

$$\int_S \mathbf{n} \cdot (\mathbf{u}' \cdot \mathbf{T} - \mathbf{u} \cdot \mathbf{T}')\, dS = \int_V \mathbf{u} \cdot (\nabla \cdot \mathbf{T}') - \mathbf{u}' \cdot (\nabla \cdot \mathbf{T})\, dV \tag{1}$$

where $\mathbf{n}$ is the unit vector into the fluid domain, V, and S is the surface of the domain. Use Cartesian tensor methods to prove this identity. Hint: You may need to use the constitutive equation for an incompressible, viscous fluid and the continuity equation.

(b) Consider two plates located at positions $z = -h/2$ and $z = h/2$ where h is a positive constant; the plates are infinitely long in both the x and y directions. The space between the plates is filled with a fluid of shear viscosity, μ. The top and bottom plates have velocity $\mathbf{u} = (U/2)\hat{\mathbf{x}}$ and $\mathbf{u} = -(U/2)\hat{\mathbf{x}}$ respectively where U

is a positive constant. Solve the Stokes equations (without body force and no applied pressure gradient) for the velocity field. Use your result for the velocity field to calculate the force per unit area exerted on each of the plates by the fluid.

(c) In addition to the problem specification in (b), a body force is acting on the fluid which decelerates fluid particles in close proximity to the walls, this can be modelled as $\mathbf{b} = -A\,(z/h)^2 \sin(2\pi z/h)\hat{\mathbf{x}}$ where A is a positive constant. Using your results from (b) and the Lorentz reciprocal theorem in (a), calculate the force per unit area exerted on the two plates, without explicitly solving for the velocity and stress fields.

Question 3 [10 marks]

A baby seal is swimming in the southern ocean when it spots an orca swimming below it. The baby seal, wanting to avoid becoming the orca's next meal, knows that it can swim fastest if it does not break the surface of the water. However, it also wants to swim as close to the surface as possible to avoid detection by the orca.

Calculate the maximum height the baby seal can swim at before it breaks the surface. You may find a diagram to be helpful (creativity encouraged but not required).

You may make the following assumptions: (i) The flow is steady (this approximation will hold well when the baby seal is gliding). (ii) The flow is inviscid. (iii) The baby seal is 1m long and 40cm thick at its widest point. It swims with its length parallel to the surface of the water. (iv) The gravitational acceleration is 9.8m/s^2. (v) The maximum speed the baby seal can reach while gliding is 5km/h. (vi) It is a calm day and so the surface of the sea, far from the seal, is flat.

Q1) $\underset{\sim}{u} = u\hat{x} + v\hat{y}$, where $\begin{cases} u(x,y,t) = -y \\ v(x,y,t) = x+t \end{cases}$

i). Streamlines in parametric form:

$$\frac{d\underset{\sim}{r}_{st}(s)}{ds} = \underset{\sim}{u}(\underset{\sim}{r}_{st}(s), t) \quad -① $$

let $\underset{\sim}{r}_{st}(s) = (x(s), y(s))$

so, L.H.S of ① $= \frac{d}{ds}(x(s), y(s))$

$= $ R.H.S of ①

$= \underset{\sim}{u}((x(s), y(s)), t)$

$= (-y, x+t)$

from above, we have $\frac{d}{ds}(x,y) = (-y, x+t)$

$$\begin{cases} \frac{dx}{ds} = -y \quad -② \\ \frac{dy}{ds} = x+t \quad -③ \end{cases}$$

From MAST20030: Different Equations.

$$\frac{d\underset{\sim}{r}_{st}(s)}{ds} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} r_{st}(s) = \underbrace{\begin{bmatrix} 0 \\ t \end{bmatrix}}_{b(s)} \quad -④$$

$$\det(\lambda I - A) = \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \right|$$

$= \lambda^2 + 1 = 0$

$\lambda = \pm i$

from ② & ④

$$\cancel{\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} A\cos(s) - B\sin(s) \\ A\cos(s) \cdot B\sin(s) \end{bmatrix}}$$

Refer to page 349 (A first course in DE w/ Modelling Applications)

$$X_c = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = c_1 \begin{pmatrix} \cos(s)+\sin(s) \\ \cos(s) \end{pmatrix} + c_2 \begin{pmatrix} \cos(s) - \sin(s) \\ -\sin(s) \end{pmatrix}$$

$X_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ ← particular sol^n vector.

NOTE: depends on the eigenvectors you use!! see bottom ??? Assumed knowledge from calc 2 & DE!!!

$$\begin{aligned} 0 &= -b_1 + 0 \qquad b_1 = 0 \\ 0 &= a_1 + t \qquad a_1 = -t. \end{aligned} \Rightarrow X_p = \begin{pmatrix} -t \\ 0 \end{pmatrix}$$

$$-A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then $X = X_c + X_p$ is,

i.e. $$\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = c_1 \begin{pmatrix} \cos(s)+\sin(s) \\ \cos(s) \end{pmatrix} + c_2 \begin{pmatrix} \cos(s) - \sin(s) \\ -\sin(s) \end{pmatrix} + \begin{pmatrix} -t \\ 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} (c_1+c_2)\cos(s) + (-c_2+c_1)\sin(s) - t \\ c_1\cos(s) - c_2\sin(s) \end{pmatrix}}$$

where c_1 & c_2 are constants.

FINAL answER

Case 1: $\lambda = i$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \xrightarrow{①\times i} \begin{bmatrix} -1 & -i \\ 1 & i \end{bmatrix} \xrightarrow{①+②} \begin{bmatrix} 0 & 0 \\ 1 & i \end{bmatrix} \xrightarrow{①\leftrightarrow②} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

let $y = r$, $x = -ir$ $\quad \underset{\sim}{v}_1 = r\begin{bmatrix} -i \\ 1 \end{bmatrix}$ $\boxed{r=-1}$ $\Rightarrow \underset{\sim}{v}_1 = \begin{bmatrix} i \\ -1 \end{bmatrix}$

Case 2: $\lambda = -i$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow{①\times i} \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \xrightarrow{②+①} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \qquad \underset{\sim}{v}_2 = u\begin{bmatrix} i \\ 1 \end{bmatrix}$$

let $y = u \Rightarrow x + u(-i) = 0$, $x = ui$ $\qquad u = 1, \Rightarrow \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\nabla \cdot \vec{V} = 0$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Then, $\varphi(x, y, t)$ is the streamline

$u = \frac{\partial \psi}{\partial y}$, $v = \frac{\partial \varphi}{\partial x}$ $\Rightarrow$ Stream function in Cartesian coordinates.

$\varphi(x, y, t) = -$

$u = \frac{\partial \psi}{\partial y} = -y$, ~~$\psi = -\frac{y^2}{2}$~~ — ①

$v = \frac{\partial \psi}{\partial x} = x + t$ ~~$\psi = \frac{x^2}{2} + tx$~~ — ②

$\Rightarrow \psi = -\frac{y^2}{2} + c$ from ①

$\Rightarrow \psi = \frac{x^2}{2} + tx + c$ from ②

Then $\therefore \varphi(x, y, t) = \frac{x^2}{2} - \frac{y^2}{2} + tx + c$

let $c = 0$

$\therefore \varphi(x, y, t) = \frac{x^2}{2} - \frac{y^2}{2} + tx + C$

Q1(d) $\cancel{\ }$ + $\cancel{\ }$

$$\boxed{\frac{d\underline{r}_p}{dt} = \underline{u}^L} \quad (1)$$

$$\underline{u}^L(\underline{R}, t) = \underline{u}^E(r_p(t), t) = \underline{u}^E(\underline{r}, t)$$

$$\frac{dr_p}{dt} = \underline{u}(r_p(t), t)$$

$$(x'(t), y'(t)) = (-y, x+t)$$

$$\begin{cases} \dfrac{dx}{dt} = -y & (2) \\[2ex] \dfrac{dy}{dt} = x+t & (3) \end{cases}$$

Decoupled Equation.

~~from Eq 2 $x(t) = -yt + c$, $c \in \mathbb{R}$ constant.~~ (4)

~~sub into (3) $\frac{dy}{dt} = -yt + c + t$~~ → problem. wrong

$\cancel{\text{from (3), } y(t) = xt + \frac{t^2}{2} + c}$ — (4)

~~sub (4) into (2),~~

$$\cancel{\boxed{\frac{dx}{dt} = -xt + \frac{t^2}{2} + c}} \quad (5)$$

$$\cancel{\frac{dx}{dt} + xt = \frac{t^2}{2} + c}$$

~~IF: $e^{\int t\,dt}$~~

2

x,

from (2), $\frac{dx}{dt} = -y$

$$\frac{d^2x}{dt^2} = -\frac{dy}{dt}$$

sub into (3)

$$-\frac{d^2x}{dt^2} = x+t$$

$$\frac{d^2x}{dt^2} + x = -t$$

$$x'' + x = -t$$

let, $x_c(t) = e^{\lambda t}$

$x_c'(t) = \lambda e^{\lambda t}$

$x_c''(t) = \lambda^2 e^{\lambda t}$

$$\lambda^2 e^{\lambda t} + e^{\lambda t} = 0$$

$$e^{\lambda t}(\lambda^2+1) = 0$$

$$\lambda^2+1=0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$

$x_c(t) = A\cos t + B\sin t$, $A, B \in \mathbb{R}$.

Try $x_p(t) = Ct^2 + Dt + E$

for Inhomogeneous part.

$x_p'(t) = 2Ct + D$

$x_p''(t) = 2C$

$$2C + Ct^2 + Dt + E = -t$$

$D = -1$

$C = E = 0,$

$$x_p(t) = -t$$

$$x(t) = x_c(t) + x_p(t) = A\cos t + B\sin t - t.$$

Now to find $y(t)$,

from $\frac{dx(t)}{dt} = -y(t)$

$$\Rightarrow y(t) = -\frac{dx(t)}{dt}$$

$$= -(-A\sin t + B\cos t - 1)$$

$$= A\sin t - B\cos t + 1$$

$\therefore \underline{r}_p(t) = (A\cos t + B\sin t - t,\ A\sin t - B\cos t + 1)$

$r_p(0) = (X, Y)$

$= (A, -B+1)$

$Y = -B+1$

$B = 1-Y$

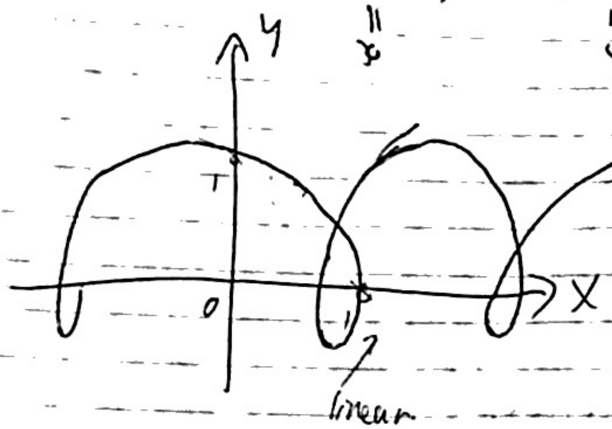
$= (X\cos t + (1-Y)\sin t - t,$
$X\sin t - (1-Y)\cos t + 1)$

Q2(e) $(1,0) = (x, y)$

$= (\cos t + \sin t - 1, \ \sin t - \cos t + 1)$

$\quad\quad\quad\quad \| \quad\quad\quad\quad\quad\quad \|$

$\quad\quad\quad\quad x \quad\quad\quad\quad\quad\quad y.$



$y = -x + 1$

These 2 lines (streak line & streamline from w) are different because the flow isn't steady, i.e. $u(x,y,t) = -y$, $v(x,y,t) = x + t$. the velocity has t dependence.

This actually makes sense since:

*). Streaklines: are the loci of points of all the fluid particles that passed continuously through a particular spatial point, $(x,y) = (1,0)$. In our case, in the past

*) Streamlines: are a family of curves that are instantaneously tangent to the velocity vector of the flow. these show the direction in which a massless fluid element will travel at any point in time.

$\Rightarrow$ streaklines & streamlines are different. (not steady),

$\Rightarrow$ Streaklines & streamlines are ~~different~~ decreasing as time increases.

This is because of velocity field is of the form: $u(x,y,t) = -y$

$v(x,y,t) = x + t.$

Q1(f)

$$\left.\frac{\partial u}{\partial t}\right|_R = \frac{Du}{Dt}.$$

$$\frac{\partial u^L(R,t)}{\partial t} = \frac{Du}{Dt}$$

LHS: $\underline{u}^L(R,t) : \underline{u}(\underbrace{r_p(t)}_{R}, t)$ velocity field.

$$\frac{\partial}{\partial t}\underline{u}(r_p(t),t) = \frac{\partial}{\partial t}u\left[\underbrace{(\overbrace{X\cos t+(1-Y)\sin t - t}^{x(t)}, \overbrace{X\sin t-(1-Y)\cos t+1}^{y(t)})}_{r_p(t)}, t\right]$$

$$= \frac{\partial}{\partial t}\left\{-y, x+t\right\} = \frac{\partial}{\partial t}\left\{\underbrace{-X\sin t+(1-Y)\cos t - 1}_{-y}, \underbrace{X\cos t+(1-Y)\sin t - t}_{x} + t\right\}$$

$$= \left\{X\cos t - (1-Y)\sin t, -X\sin t + (1-Y)\cos t\right\}$$

velocity vector.

RHS: $\frac{Du}{Dt} = \underbrace{\frac{\partial u}{\partial t}}_{(1)} + \underbrace{(u.\nabla)\underline{u}}_{(2)}$

$$= \frac{\partial}{\partial t}\left[(-y, x+t)\right] = (0,1) \quad -(1)$$

(2), $\underbrace{(u.\nabla)\underline{u}}_{\text{vector}} = \left\{(u,v)\cdot(\partial/\partial x, \partial/\partial y)\right\}(u,v)$

$$= \left(u\underbrace{\frac{\partial u}{\partial x}}_{=0} + v\underbrace{\frac{\partial u}{\partial y}}_{=-1}, u\underbrace{\frac{\partial v}{\partial x}}_{=1} + v\underbrace{\frac{\partial v}{\partial y}}_{=0}\right)$$

$$= (-v, u)$$
$$= (-x-t, -y)$$

$$\Rightarrow \frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u.\nabla)\underline{u} = (0,1) + (-x-t, -y)$$

$$= (-x-t, -y+1)$$

$$= (-(X\cos t+(1-Y)\sin t - t) - t, -(X\sin t-(1-Y)\cos t+1)+1)$$

$$= (X\cos t - (1-Y)\sin t, -X\sin t + (1-Y)\cos t)$$

Hence. $\left.\frac{\partial u}{\partial t}\right|_R = \frac{Du}{Dt}$

Q2 a) use Cartesian tensor method to prove:

$$① \quad \oint_S \underset{\sim}{n} \cdot (\underset{\sim}{u}' \cdot \underset{\sim}{T} - \underset{\sim}{u} \cdot \underset{\sim}{T}')\,dS$$

$$= \int_V \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}') - \underset{\sim}{u}' \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T})\,dV$$

Proof: step ①: List the information we know:

For incompressible Newtonian fluid, it's Navier-Stokes eq. is: $\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}\right) = \underset{\sim}{\nabla} \cdot \underset{\sim}{T} + \underset{\sim}{f}$, ——②

Where $\underset{\sim}{\nabla} \cdot \underset{\sim}{T} = -\underset{\sim}{\nabla} p + \mu \nabla^2 \underset{\sim}{u}$; and $\underset{\sim}{f} = \rho \underset{\sim}{G}$ denotes the body force.

If we have Stokes flow,

$\rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}\right) = 0$, i.e. ① reduces to

$$\boxed{\underset{\sim}{\nabla} \cdot \underset{\sim}{T} + \underset{\sim}{f} = \underset{\sim}{0}} \text{ —— ③}$$

However, we know that for incompressible viscous fluid, the constitutive eq. is: $\underset{\sim}{T} = -p\underset{\sim}{I} + \underset{\sim}{d}$ ——④

where $\underset{\sim}{d}$ is the deviatoric stress tensor, also $\underset{\sim}{d} = \mu(\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T)$ ——⑤, where μ is the dynamic viscosity, assumed to be constant.

Also, we know that for incompressible flow: $\nabla \cdot u = 0$ ——⑥. Now, we put ③, ④, ⑤, ⑥ in the Cartesian tensor form:

$$\begin{cases} \frac{\partial}{\partial x_i} T_{ik} + f_k = 0 & \text{——⑦} \\ T_{ik} = -p\,\delta_{ik} + d_{ik} & \text{——⑧} \\ d_{ik} = \mu\left(\frac{\partial}{\partial x_i} u_k + \frac{\partial}{\partial x_k} u_i\right) & \text{——⑨} \\ \frac{\partial}{\partial x_i} u_i = 0 & \text{——⑩} \end{cases}$$

i.e. $\frac{\partial}{\partial x_i} u_i' = \underset{\sim}{\nabla}\cdot \underset{\sim}{u}' = 0.$

Then (14) becomes:

$$\boxed{\frac{\partial}{\partial x_i}(u_k' d_{ik}) - d_{ik}\frac{\partial}{\partial x_i}u_k' - \frac{\partial}{\partial x_i}(u_k d_{ik}') + d_{ik}'\frac{\partial}{\partial x_i}u_k - \frac{\partial}{\partial x_i}(u_i' p) + \frac{\partial}{\partial x_i}(u_i p') + u_k' f_k - u_k f_k' = 0} \quad (15)$$

However, note that $\frac{\partial}{\partial x_i}(u_i' p) = \frac{\partial}{\partial x_i}(u_k' p\, \delta_{ik})$,

hence, (15) will become:

$$\frac{\partial}{\partial x_i}(u_k' d_{ik} - u_k' p\,\delta_{ik}) - \frac{\partial}{\partial x_i}(u_k d_{ik}' - u_k p' d_{ik}) - d_{ik}\frac{\partial}{\partial x_i}u_k' +$$

$$d_{ik}'\frac{\partial}{\partial x_i}u_k + u_k' f_k - u_k f_k' = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x_i}(u_k' T_{ik}) - \frac{\partial}{\partial x_i}(u_k T_{ik}') - d_{ik}\frac{\partial}{\partial x_i}u_k' + d_{ik}'\frac{\partial}{\partial x_i}u_k + u_k' f_k - u_k f_k' = 0} \quad (16)$$

However, $d_{ik}\frac{\partial}{\partial x_i}u_k' + d_{ik}'\frac{\partial}{\partial x_i}u_k$

$$= -\mu\left(\frac{\partial}{\partial x_i}u_k + \frac{\partial}{\partial x_k}u_i\right)\frac{\partial}{\partial x_i}u_k'$$

$$+ \mu\left(\frac{\partial}{\partial x_i}u_k' + \frac{\partial}{\partial x_k}u_i'\right)\frac{\partial}{\partial x_i}u_k$$

$$= \mu\left(-\cancel{\frac{\partial}{\partial x_i}u_k\frac{\partial}{\partial x_i}u_k'} - \frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_i}u_k' + \cancel{\frac{\partial}{\partial x_i}u_k'\frac{\partial}{\partial x_i}u_k}\right.$$

$$\left.+ \frac{\partial}{\partial x_k}u_i'\frac{\partial}{\partial x_i}u_k\right)$$

Step ②: Use the information we gather to prove the identity:

Now suppose there is a second stokes flow in the same domain with T_{ik}', f_k', d_{ik}', p', u' satisfying the same system of equations (as eq ⑦—⑩)

Multiplying ⑦ by u_k', we get:

$$\boxed{u_k' \frac{\partial}{\partial x_i} T_{ik} + u_k' f_k = 0} \quad (11)$$

Similarly, interchanging primed and unprimed variables gives

$$\boxed{u_k \frac{\partial}{\partial x_i} T_{ik}' + u_k f_k' = 0} \quad (12)$$

(11) - (12) gives:

$$\boxed{u_k' \frac{\partial}{\partial x_i} T_{ik} - u_k \frac{\partial}{\partial x_i} T_{ik}' + u_k' f_k - u_k f_k' = 0} \quad (13)$$

However, $T_{ik} = d_{ik} - p\,\delta_{ik}$ and $T_{ik}' = d_{ik}' - p'\delta_{ik}$

so

↳ (13) $\Rightarrow u_k' \frac{\partial}{\partial x_i}(d_{ik} - p\,\delta_{ik}) - u_k \frac{\partial}{\partial x_i}(d_{ik}' - p'\delta_{ik}) + u_k' f_k - u_k f_k' = 0$

$$\Rightarrow \boxed{u_k' \frac{\partial}{\partial x_i} d_{ik} - u_k \frac{\partial}{\partial x_i} d_{ik}' - u_i' \frac{\partial}{\partial x_i} p + u_i \frac{\partial}{\partial x_i} p' + u_k' f_k - u_k f_k' = 0} \quad (14)$$

Now, we can use product rule: $\frac{\partial}{\partial x_i}(u_k' d_{ik}) = u_k' \frac{\partial}{\partial x_i} d_{ik} + d_{ik} \frac{\partial}{\partial x_i} u_k'$, and $\frac{\partial}{\partial x_i}(u_i' p) = u_i' \frac{\partial}{\partial x_i} p + p \frac{\partial}{\partial x_i} u_i'$

$= u_i' \frac{\partial}{\partial x_i} p$, since the flow is incompressible,

$$= \mu \left(-\frac{\partial}{\partial x_k} u_i \frac{\partial}{\partial x_i} u'_k + \frac{\partial}{\partial x_k} u'_i \frac{\partial}{\partial x_i} u_k \right)$$

since the dummy summation indices can be exchanged so the 2nd term: $\frac{\partial}{\partial x_k} u'_i \frac{\partial}{\partial x_i} u_k = \frac{\partial}{\partial x_i} u'_k \frac{\partial}{\partial x_k} u_i$

$$= 0 \quad \text{— (17)}$$

Substituting (17) into (16) gives:

$$\boxed{\frac{\partial}{\partial x_i}(u'_k T_{ik}) - \frac{\partial}{\partial x_i}(u_k T'_{ik}) + u'_k f_k - u_k f'_k = 0} \quad \text{— (18)}$$

Comparing (13) & (18), we get:

$$\boxed{\frac{\partial}{\partial x_i}(u_k T_{ki}) - \frac{\partial}{\partial x_i}(\underbrace{u_k T'_{ki}}_{\text{since } \underset{\sim}{T} \,\&\, \underset{\sim}{T'} \text{ are symmetric.}}) = u'_k \frac{\partial}{\partial x_i} T_{ik} - u_k \frac{\partial}{\partial x_i} T'_{ik}} \quad \text{— (19)}$$

In dyadic notation - (18) reads as follows:

$$\boxed{\underset{\sim}{\nabla} \cdot (\underset{\sim}{u}' \cdot \underset{\sim}{T}) - \underset{\sim}{\nabla} \cdot (\underset{\sim}{u} \cdot \underset{\sim}{T}') = \underset{\sim}{u}' \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}) - \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}')} \quad \text{- (20)}$$

Integrating both sides over the volume V of the domain, (20) becomes:

$$\int_V \underset{\sim}{\nabla} \cdot (\underset{\sim}{u}' \cdot \underset{\sim}{T}) - \underset{\sim}{\nabla} \cdot (\underset{\sim}{u} \cdot \underset{\sim}{T}') \, dV = \int_V \underset{\sim}{u}' \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}) - \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}') \, dV$$

Using the divergence theorem, we obtain:

$$\oint_S (-\underset{\sim}{n}) \cdot (\underset{\sim}{u}' \cdot \underset{\sim}{T}) - (-\underset{\sim}{n}) \cdot (\underset{\sim}{u} \cdot \underset{\sim}{T}') \, dS = \int_V \underset{\sim}{u}' \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}) - \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}') \, dV$$

NOTE: here $\underset{\sim}{n}$ is the unit vector INTO the fluid domain. Rearranging the terms gives us the desired results:

$$\boxed{\oint_S \underset{\sim}{n} \cdot (\underset{\sim}{u}' \cdot \underset{\sim}{T} - \underset{\sim}{u} \cdot \underset{\sim}{T}') \, dS = \int_V \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}') - \underset{\sim}{u}' \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{T}) \, dV}$$

Q2b)

U positive constant $\leftarrow$ $u = (U/2)\hat{x}$, $u = -(U/2)\hat{x}$ (a fluid of shear viscosity μ)

$z = -\frac{h}{2}$, $z = \frac{h}{2}$

from the lectures (lecture 11A Fluid Mechanics)

Exact Solutions to Navier-Stokes Equation.

Unidirectional flows (consider the velocity field)

$$\boxed{\underline{u} = w(x,y,z,t)\,\underline{k}} \quad (1)$$

i.e., straight ~~and~~ and parallel streamlines.

From the continuity equation we have,

$\nabla \cdot u = 0$

$\Rightarrow \frac{\partial w}{dz} = 0$

$\therefore \underline{u} \cdot \nabla \underline{u} = w \frac{\partial w}{\partial z} \underline{k} = 0$

The NS equation is linear:

$$\boxed{\rho \frac{\partial \underline{u}}{\partial t} = -\nabla p + \mu \nabla^2 \underline{u}} \quad (2)$$

Substituting (2) into (1), and extracting components:

$$\boxed{\hat{x} : 0 = -\frac{\partial p}{\partial x}} \quad (3a) \qquad \boxed{\hat{y} : 0 = -\frac{\partial p}{\partial y}} \quad (3b)$$

$$\boxed{\rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)} \quad (3c)$$

$$\boxed{(4) \Rightarrow p = p(z,t)}$$

(from (3a) & (3b) we have)

Rearranging (3c) gives,

$$\frac{\partial p}{\partial z} = -\rho\frac{\partial w}{\partial t} + \mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \quad -(5)$$

Since w is independent of z, from (4) & (5) we obtain,

$$\frac{\partial p}{\partial z} = -G(t) \quad (6)$$

* where $G(t)$ is some function of t only!!

Solving (6) gives,

$$\boxed{p = -G(t)z + p_0(t)} \quad (7)$$

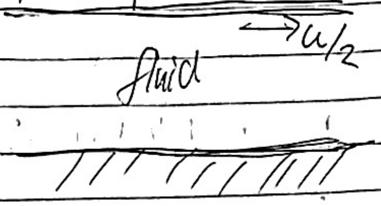
substituting (7) into (3c) then gives.

$$\boxed{\rho\frac{\partial w}{\partial t} = G(t) + \mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)}$$

Governing equation for unidirectional flow.

Couette Flow

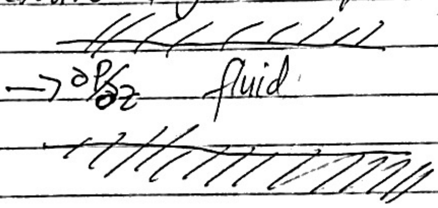
2 plane parallel surface, one is moving.



$u=\left(\frac{U}{2}\right)\hat{x}$, $u=\left(-\frac{U}{2}\right)\hat{x}$

Poiseuille flow

2 plane parallel surfaces, both stationary with pressure gradient.



Combined Couette/Poiseville Flow:

2 plane parallel surfaces are moving and pressure gradient

Conditions

(1) constant pressure gradient $G = 0$

(2) Top plate has constant velocity $\underset{\sim}{u} = \frac{U}{2}\hat{\underset{\sim}{x}}$

& Bottom plate has constant velocity $\underset{\sim}{u} = -\frac{U}{2}\hat{\underset{\sim}{x}}$

(3) Plates infinite in x-y directions.

$\Rightarrow$ fluid velocity field. $\underset{\sim}{u} = W(z)\hat{\underset{\sim}{x}}$

Governing eqn is: $0 = \underbrace{G}_{\text{pressure force}} + \underbrace{\mu \frac{d^2 w}{dy^2}}_{\text{viscous force}}$ — (1)

Boundary conditions

$\underset{\sim}{u} = \underset{\sim}{0}$ at $z = 0$, $\underset{\sim}{u} = \frac{U}{2}\hat{x}$ at $z = \frac{h}{2}$ $\Rightarrow$

$$\begin{cases} w(0) = 0 & \text{— (2a)} \\ w\left(\frac{h}{2}\right) = \frac{U}{2} & \text{— (2b)} \end{cases}$$

$L_c = d$ (gap width)

$U_c = U = \frac{U}{2}\hat{x}$ (top plate) if $U = 0$, then pressure drives flow & U isn't a characteristic velocity! From (1) we obtain:

$$(1) \quad = G + \mu \frac{d^2 w}{dz^2}$$

Solving this equation gives

$$\sim G + \mu \frac{w}{4}$$

$\Rightarrow$

$$w \sim G + \mu \frac{w}{4}$$

$$w \sim \frac{4G}{\mu}$$

velocity scale is

$u_c = \frac{4G}{\mu}$ (top plate) , $u_c = -\frac{4G}{\mu}$ (bottom plate)

SUMMARY 2 possible velocity scales:

(1). $u_c = U$, if $U \neq 0$

(2). $u_c = \frac{G \times 4}{\mu_{ax}}$, if $G \neq 0$, $U = 0$

consider scaling of $u_c = U$, $l_c = d \simeq U$

dimensionless velocity & position as,

$$\hat{w} = \frac{w}{u_c} = \frac{w}{U} = \frac{w}{U\frac{X}{2}} = \frac{2w}{UX} \quad \text{— (3a)}$$

$$\hat{z} = \frac{z}{l_c} = \frac{z}{d} \quad \text{— (3b)}$$

$$= \frac{2z}{h}$$

$$\hat{w} = \frac{w}{u_c} = \frac{w}{U} = \frac{w}{\frac{u}{2}\hat{x}} \quad \text{— 3a}$$

$$\hat{z} = \frac{\cancel{z}}{l_c} = \frac{z}{d} = \frac{2\cancel{z}}{\cancel{h}} \quad \text{— 3b}$$

sub ③ into ① gives.

$$0 = G + \mu \frac{d^2(U\hat{w})}{d(d\hat{z})^2}$$

$$\rightarrow \boxed{\frac{d^2w^2}{d\hat{z}^2} = -\frac{Gd^2}{\mu U}} \quad \text{— ④}$$

B.C.'s becomes

$\hat{w} = 1$ at $\hat{z} = \frac{h}{2}$ (since $w = U$, $z = d$)

$U = \frac{u}{2}\hat{x}$, $d = \frac{h}{2}$

$$\alpha \equiv \frac{Gd^2}{\mu U} = \frac{G \times h^2/4}{\mu \frac{u}{2}\hat{x}} = \frac{\frac{Gh^2}{4}}{\mu \frac{u}{2}\hat{x}} = \frac{Gh^2}{4} \times \frac{2}{\mu U\hat{x}}$$

$$= \frac{Gh^2}{4} \times \frac{2}{\mu U\hat{x}} = \frac{Gh^2}{2\mu U\hat{x}}$$

$\alpha = \pm \frac{Gh^2}{2\mu U\hat{x}}$ (top & bottom plate)

$\hookrightarrow$ single dimensionless

(Q3) $u = 5km/hr \Rightarrow 1.39 m/s$.
$g = -9.8 m/s^2$ (or keep g as it is)
Pre-knowledge (SCROLL DOWN TO DYNAMIC SIMILARITY)
$\Rightarrow$ In dimensional variables (from the lectures).

$$\underbrace{\rho\left(\frac{\partial u}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}\right)}_{\text{inertial force}} = -\underbrace{\underset{\sim}{\nabla} p}_{\text{pressure force}} + \underbrace{\mu \nabla^2 \underset{\sim}{u}}_{\text{viscous force.}} \quad -①$$

$\Rightarrow$ define the dimensionless variables:

$$\begin{cases} \hat{r} = \underset{\sim}{r}/L_c \\ \hat{u} = \underset{\sim}{u}/u_c \\ \hat{t} = \dfrac{t}{t_c} \end{cases} \quad -②$$

sub ② into ①,

NOTE

$\underset{\sim}{\nabla} = \frac{1}{L_c}\hat{\underset{\sim}{\nabla}}$

$$\frac{\rho u_c^2}{L_c}\left(\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{\underset{\sim}{u}}\cdot\hat{\underset{\sim}{\nabla}}\hat{\underset{\sim}{u}}\right) = -\frac{1}{L_c}\hat{\underset{\sim}{\nabla}} p + \frac{\mu u_c}{L_c^2}\hat{\nabla}^2\hat{\underset{\sim}{u}} \quad -③$$

all order 1

from ③,
we see that

$$\left.\begin{aligned} &\text{Inertial forces} \sim \frac{\rho u_c^2}{L_c}, \\ &\text{Viscous forces} \sim \frac{\mu u_c}{L_c^2} \end{aligned}\right] -④$$

Therefore, if we define the Reynolds Number:

$$\boxed{Re = \frac{\text{Inertial forces}}{\text{Viscous forces}}} \rightarrow Re = \frac{\rho u_c L_c}{\mu} \quad -⑤$$

a single parameter:

Substitituting (5) into (3) gives:

$$Re\left(\frac{\partial \hat{\underset{\sim}{u}}}{\partial \hat{t}} + \hat{\underset{\sim}{u}}\cdot\hat{\underset{\sim}{\nabla}}\underset{\sim}{u}\right) = -\frac{L_c}{\mu u_c}\hat{\underset{\sim}{\nabla}}p + \hat{\nabla}^2\hat{\underset{\sim}{u}} \quad (6)$$

Implication:

(1) If viscous forces >> inertial forces:
$Re \ll 1$

when $Re \to 0$: $\underset{\sim}{0} = \underbrace{-\frac{L_c}{\mu u_c}\hat{\underset{\sim}{\nabla}}p + \hat{\nabla}^2\hat{\underset{\sim}{u}}}_{\text{force equillibrium}}$

Therefore, an appropriate pressure scaling in this case is:

$$\boxed{P_c = \frac{\mu u_c}{L_c}}$$, when $Re \ll 1$.

In this cause, the (Navier Stokes eqn) reduces to:

$$\nabla P = \mu \nabla^2 \underset{\sim}{u}$$

(2) If inertial forces >> viscous forces:
$Re \gg 1$:

(6) then becomes,

$$\boxed{Re\underbrace{\left(\frac{\partial \hat{\underset{\sim}{u}}}{\partial t} + u\cdot\hat{\nabla}\underset{\sim}{u}\right)}_{\text{Mass acceleration}} = \underbrace{-\frac{L_c}{\mu u_c}\hat{\underset{\sim}{\nabla}}P}_{\text{net force}}}$$

Dynamic Similarity

*

↳ dimensionless form:

$$Re\left(\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u}\cdot\nabla\hat{u}\right) = -\nabla \hat{p} + \nabla^2 \hat{u}$$

* Seal travelling at 5 km/hr *
with measured 1m long and 40cm.

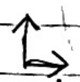
* ~~Assumption~~ Free surface (air/liquid) remains flat.

⇒ Unidirectional flow $\boxed{u\cdot\nabla u = 0}$

Velocity field is: $u = w(y)\,e$ — $\boxed{I}$

Since gravity - is driving the flow we must include the body force term in the eq. of motion:

$$\boxed{0 = -\nabla p + \mu\nabla^2 u + \rho\, b}$$ body force/mass $= F/M$



$$b = \frac{F}{M} = a = -g\cos\alpha\,\hat{y} + g\sin\alpha\,\hat{z} \quad \text{— (3)}$$

Substituting (3) into (2) & extracting components gives:

$\hat{u}$: ~~...~~ → $\boxed{0 = -\frac{\partial P}{\partial y} + 0 - \rho g\cos\alpha}$ — (4a)

$\hat{z}$: $\boxed{0 = -\frac{\partial P}{\partial z} + \mu\frac{d^2 w}{dy^2} + \rho g\sin\alpha}$ — 4b

Solving (4a) gives: $\boxed{P = -y\rho g\cos\alpha + f(z)}$ — (5)

since $\mu \frac{d^2 w}{dy^2} + \rho g \sin\alpha$ depends only on y!!

Therefore, from (5) we obtain

$$\boxed{\frac{\partial p}{\partial z} = -G \text{ (constant)}}$$

Substituting (6) into (4b) gives

$$\boxed{0 = G + \mu \frac{d^2 w}{dy^2} + \rho g \sin\alpha} \quad (7)$$

with BC: $w = 0$ at $y = 0$ (no-slip) — (8)

$\Rightarrow$ Free surface BC ($y = d$)? stress vector $\underline{t}$ must be continuous, $\boxed{\underline{t} = \underline{n} \cdot \underline{T}}$ — (9)

Where $\underline{n}$ is normal to surface

We know that

$$\boxed{\underline{T} = -p\underline{I} + 2\mu\underline{e}} \quad \ldots\ldots (10)$$

BUT, $\mu_{gas} << \mu_{liq}$! Therefore, if magnitude of $\underline{e}$ is comparable gas and liquid $\Rightarrow$

$$\underline{T}_{gas} = -p_{gas}\underline{I} \quad (11a)$$

$$\underline{T}_{liq} = -p_{liq}\underline{I} + 2\mu_{liq}\underline{e}_{liq} \quad (11b)$$

Therefore, at $y = d$ we require

$$\boxed{\underline{n} \cdot \underline{T}_{liq} = \underline{n} \cdot \underline{T}_{gas}} \quad (12)$$

Since $\underline{n} = \underline{j}$, (12) becomes

$$\boxed{-p_{liq}\underline{j} + 2\mu_{liq}\underline{j} \cdot \underline{e}_{liq} = -p_{gas}\underline{j}} \quad (13)$$

NOTE: Solution depends only on a single dimensionless navrion...

Extracting j and $\underline{k}$ components of (13):

$$j: \; -p_{liq}\, j \cdot j + 2\mu_{liq}\, j \cdot e_{liq} \cdot j = -p_{gas}\, j \cdot j$$

$$\Rightarrow -p_{liq} + 2\underbrace{\mu_{liq}\, e_{yy}}_{\frac{\partial v}{\partial y}=0} = -p_{gas}$$

$$\Rightarrow \boxed{p_{liq} = p_{gas}} \;\Big|_{y=d} \quad \text{(14)}$$

$$\underline{k}: \; -p_{liq}\, j \cdot \underline{k} + 2\mu_{liq}\, j \cdot \underline{e}_{liq} \cdot \underline{k} = -p_{gas}\, j \cdot \underline{k}$$

$$\Rightarrow 0 = 2\mu_{liq}\, e_{yz}$$

$$= 2\mu_{liq} \times \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$= \mu_{liq} \frac{dw}{dy}$$

$$\Rightarrow \boxed{\frac{dw}{dy} = 0} \;\Big|_{y=d} \quad \text{(15)}$$

Solving (7), (8), (15) gives,

$$\boxed{w(y) = \frac{-\rho g \sin\alpha}{\mu}\left(\frac{1}{2}y^2 - dy\right)} \quad \text{(16)}$$

and, $\boxed{p = p_{air} + \rho g\,(d-y)\cos\alpha}$ (17)

$\boxed{~~W(y)~\&~P~\text{from (17) doesn't matter}~~}$ ~~same pressure at both~~

One may argue using the Bernoulli's

$$\boxed{\cancel{P_1} + \frac{1}{2}v_1^2\rho_1 + \cancel{\rho_1 g h_1} = \cancel{P_2} + \cancel{\frac{1}{2}v_2^2}\rho_2 + \rho_2 g h_2} \quad (18)$$

$\boxed{P_1 = P_2}$ at the ground,

Imply that $w(y) = P_1$
(from (17)) $P = P_2$

$\boxed{\rho_1 = \rho_2}$

$$\frac{1}{2}v_1^2\cancel{\rho_1} = \cancel{\rho_2} g h_2$$

$$h_2 = \frac{v_1^2}{2g} = \text{maximum height} = \frac{(1.39)^2}{2 \times 9.8} \approx 10\,\text{cm}$$