THE UNIVERSITY OF MELBOURNE

Student Number

Semester 1 Assessment, 2017

School of Mathematics and Statistics

MAST10007 Linear Algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

Common content with: MAST10008 Accelerated Maths 1

This paper consists of 5 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Calculators, tablet devices or computers must not be used.
- No handwritten or print materials may be brought into the exam venue.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- The following notation is used throughout this paper:
  $\mathcal{P}_n$ is the real vector space of polynomials of degree at most n with real coefficients,
  $M_{m,n}$ is the real vector space of $m \times n$ matrices with real entries.
- You should attempt all questions. Marks for individual questions are shown. Show all of your work for all questions.
- The total number of marks available is 99.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

Question 1 (6 marks)

Consider the following linear system:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & -2 \\ -x_1 & + & x_2 & - & 2x_3 & = & 3 \\ 2x_1 & - & x_2 & + & 3x_3 & = & -7 \end{array}$$

(a) Write down the augmented matrix corresponding to the linear system.

(b) Reduce the matrix in (a) to reduced row-echelon form.

(c) Use the reduced row-echelon form to give all solutions in $\mathbb{R}^3$ to the linear system.

Question 2 (6 marks)

Use row-reduction to find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

or explain why it does not exist.

Question 3 (4 marks)

For the matrix A below compute both $\det(A)$ and $\det(2A)$.

$$A = \begin{bmatrix} 4 & -2 & 5 \\ -1 & -7 & 10 \\ 0 & 1 & -3 \end{bmatrix}$$

Question 4 (6 marks)

Let L be the line with vector equation

$$(x, y, z) = (1, 1, 1) + t(2, 4, 4), \qquad t \in \mathbb{R}$$

and let M be the line given by

$$(x, y, z) = (0, 1, 1) + s(4, -2, -2), \qquad s \in \mathbb{R}$$

(a) Find Cartesian equations for L and M.

(b) Determine whether the lines intersect. If they intersect, find the point of intersection.

Question 5 (6 marks)

(a) Find a Cartesian equation for the plane containing three points: $P = (1, 4, -7)$, $Q = (2, -1, 4)$, $R = (0, -9, 18)$.

(b) Find the area of the triangle with the vertices P, Q and R.

Question 6 (6 marks)

Let

$$B = \begin{bmatrix} 2 & -4 & -2 & 1 & 2 & -3 \\ -1 & 2 & 1 & 0 & 0 & -1 \\ -4 & 8 & 4 & -1 & -2 & 1 \\ 10 & -4 & -2 & -2 & 4 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix C is obtained from B by elementary row operations. Use this information to answer the following.

(a) Write down a basis for the row space of B.

(b) Write down a basis for the column space of B.

(c) Find a basis for the solution space (or null space) of B.

Question 7 (6 marks)

For each of the following decide if the following set S is a subspace of the given vector space V. Justify your answer by citing appropriate theorems or providing a counter-example.

(a) $V = \mathcal{P}_3$ and $S = \{p \in \mathcal{P}_3 \mid p(2) = 0\}$.

(b) $V = M_{3,3}$ and $S = \{A \in M_{3,3} \mid A^T + A = \mathbf{0}\}$ where $\mathbf{0}$ denotes the zero matrix in $M_{3,3}$.

(c) $V = \mathbb{R}^2$ and $S = \{w = (x, y) \in \mathbb{R}^2 \mid x \geq 0\}$.

Question 8 (6 marks)

Consider the subset of $\mathcal{P}_2$ given by

$$S = \{x + 3x^2, 1 + x^2, 1 + x + x^2, 2 + x + 5x^2\}$$

(a) Determine whether or not S is a linearly independent set.

(b) Determine whether S spans $\mathcal{P}_2$.

(c) Find a subset of S that is a basis for the span of S.

Question 9 (10 marks)

(a) Show that the following does *not* define an inner product on $\mathbb{R}^2$.

$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + x_2 y_2$$

(b) Let $V = \mathcal{P}_3$ be the real vector space of polynomials in x of degree ≤ 3 with the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

Let W be the subspace of V spanned by $\{x, x^2\}$.

(i) Use the Gram-Schmidt procedure to find an orthonormal basis for W.

(ii) Find the polynomial $p \in W$ that minimises the integral

$$\int_0^1 (p(x) - 1)^2\, dx.$$

Question 10 (6 marks) Use least squares to find the equation of the line $y = a + bx$ that will best approximate the points $(-3, 7)$, $(1, 2)$ $(-7, 11)$ and $(5, -3)$.

Question 11 (10 marks)

Consider two bases for $\mathcal{P}_2$ given by

$$\mathcal{B} = \{1, x, x^2\} \qquad \text{and} \qquad \mathcal{C} = \{5x^2 - 1, -4x, 2\}.$$

(a) Calculate the transition matrix $P_{\mathcal{C},\mathcal{B}}$ (which converts $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates).

(b) Calculate the transition matrix $P_{\mathcal{B},\mathcal{C}}$ (which converts $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates).

(c) Determine the polynomial $p \in \mathcal{P}_2$ that has the coordinate vector $[p]_{\mathcal{C}} = \begin{bmatrix} -4 \\ 3 \\ 11 \end{bmatrix}$.

(d) Find the coordinate vector $[q]_{\mathcal{C}}$ for $q = 1 + 2x - 3x^2$.

(e) Differentiation defines a linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ where $T(p(x)) = p'(x)$. Find the matrix of T with respect to

(i) the basis $\mathcal{B}$,

(ii) the basis $\mathcal{C}$.

Question 12 (9 marks)

Define a function $T : M_{2,2} \to M_{2,2}$ by $T(X) = AX - XA$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(a) Prove that T is a linear transformation.

(b) Find bases for the image and kernel of T.

(c) Verify the rank-nullity theorem for T.

Question 13 (6 marks)

For each of the following three matrices decide whether or not the matrix is diagonalizable over $\mathbb{R}$. You should justify your answers.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & -3 \\ 0 & 6 & 5 \end{bmatrix}$

Question 14 (8 marks) Consider the matrix

$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}.$$

(a) Find all eigenvalues and corresponding eigenvectors for the matrix A.

(b) Find and invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(c) Find a formula for A^n valid for each integer $n \geq 1$.

(d) Let $\boldsymbol{v}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ where $a + b = 1$. Describe the limiting behaviour of $A^n \boldsymbol{v}_0$ as $n \to \infty$.

Question 15 (5 marks)

Let $T : V \to W$ be an injective (one-to-one) linear transformation and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in V$. Assume that $\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_k)\}$ is a basis for W. Prove that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is a basis for V.

End of Exam—Total Available Marks = 100