MAST30025 Assignment 1 S1 2021 (2/3)

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Question 4 Solution:

Part a:

Given information:

Let,

x_1,x_2,$x_3$$\sim$ $(N(\mu,\sigma^2))$ be a sequence of independent normal random variables,

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$\boldsymbol{x^T} = (x_1, x_2, x_3)^T$

Supposed to be x^T as noted!

$\boldsymbol{y}$ = (x$_1$ - ,x_2- ,x_3-)^T

To solve A from:

$$\boldsymbol{y} = A\boldsymbol{x}$$

$\begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

A $= \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Where A is symmetric and idempotent!

Part b: Finding the rank of A

```{r}
A = matrix(c(2,-1,-1,-1,2,-1,-1,-1,2)/3,3,3)

```

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          [,1]       [,2]       [,3]
[1,]  0.6666667 -0.3333333 -0.3333333
[2,] -0.3333333  0.6666667 -0.3333333
[3,] -0.3333333 -0.3333333  0.6666667
```

Finding rank of A

```{r}
rankMatrix(A)[1]
```

```
[1] 2
```

Part c: Computing E[$y^T y$]

Finding E[$\boldsymbol{y^T y}$]

$= \text{E}[(\frac{2x_1-x_2-x_3}{3}, \frac{-x_1+2x_2-x_3}{3}, \frac{-x_1-x_2+2x_3}{3}) \begin{bmatrix} \frac{2x_1-x_2-x_3}{3} \\ \frac{-x_1+2x_2-x_3}{3} \\ \frac{-x_1-x_2+2x_3}{3} \end{bmatrix}]$

$= \text{E}[(\text{x}_1 - \bar{x})^2 + (x_2 - \bar{x})^2(x_3 - \bar{x})^2)]$

$= \text{E}[(\text{x}_1 - \bar{x})^2 + (x_2 - \bar{x})^2(x_3 - \bar{x})^2)]$

$= \text{E}[\sum_{i=1}^{3}(x_i - \bar{x})^2]$

$= \text{E}[\sum_{i=1}^{3} x_i^2 - 2x_i\bar{x} + \bar{x}^2]$

$= \text{E}[\sum_{i=1}^{3} x_i^2 - n\bar{x}^2]$

Since we have 3 x's that are random variables!!

$= \text{E}[\sum_{i=1}^{3} x_i^2 - 3\bar{x}^2]$

$= \text{E}[(\sum_{i=1}^{3} x_i^2) - 3\bar{x}^2]$

since x_1, x_2 *and* x_3 *are identical independent distributions*!!

$= (3\text{-}1)\sigma^2 = 2\sigma^2$

Assuming that the sample variance is unbiased! and we can imply $\lambda = 0$! *Following similarly to Theorem*3.2. *for the Non−central distribution*!

Part d:

Using Theorem 3.5:

Proof:

Assuming that A is idempotent and has rank k. Because it is symmetric, it can be diagonalised. Let the (orthogonal) diagonalising matrix be P.

$\text{D} = P^T \text{ AP} = \begin{bmatrix} \lambda_1 & \ldots\ldots & 0 \\ \ldots\ldots\ldots & \lambda_2 & \ldots\ldots \\ 0 & \ldots\ldots\ldots & \lambda_k \end{bmatrix}$

since A is symmetric and idempotent, all eigenvalues are either 0 or 1. We know from definition:

$$tr(A) = r(A) = k$$

$\text{A} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$A^2 = \text{A} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

from Part 4b, we find out the rank and trace of matrix A we found in Part 4a. Is also is the same number of degrees of freedom for the chi squared distribution.

$$tr(A) = r(A) = 2$$

Therefore, A must have two eigenvalues of 1 and one eigenvalue of 0.
Using Theorem 3.5 and Corollary 3.7:
with our non central parameter λ!
$\lambda = \frac{1}{2}\mu^T A\mu$
$= \frac{1}{2}\begin{bmatrix}\mu\\\mu\\\mu\end{bmatrix}\frac{1}{3}\begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}\begin{bmatrix}\mu & \mu & \mu\end{bmatrix}$
$= 0$
$\Longleftrightarrow : if\ and\ only\ if$
E[y] = E[$\begin{bmatrix}x_1 - \mu\\x_2 - \mu\\x_3 - \mu\end{bmatrix}$]
Since x_1, x_2 *and* x_3 *is identically independently distributed*! *and taking* the expectation of the expectation is the expectation itself!
E[y] = E[$\begin{bmatrix}\mu - \mu\\\mu - \mu\\\mu - \mu\end{bmatrix}$] = 0
NOTE: $\mu = \bar{x}$
In which case,
$\frac{y^T y}{\sigma^2}$
is just the sum of two independent standard normal's. This is just an ordinary (central) chi squared distribution $\chi^2{}_2$.
with expectation of 2 and variance of 4.