Question 4 Solution:

Part a:

Given information:

Let,

x_1,x_2,$x_3$$\sim$ $(N(\mu$,$\sigma^2))$ be a sequence of independent normal random variables,

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}$$

$\boldsymbol{x^T} = (x_1, x_2, x_3)^T$

Supposed to be x^T as noted!

$\boldsymbol{y}$ = (x$_1$ - ,x_2- ,x_3- $)^T$

To solve A from:

$$\boldsymbol{y} = A\boldsymbol{x}$$

$$\begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A $= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

Where A is symmetric and idempotent!

Part b: Finding the rank of A

Proof: that there is a linear combination for any columns?

```{r}
A = matrix(c(2,-1,-1,-1,2,-1,-1,-1,2)/3,3,3)
```

```
           [,1]       [,2]       [,3]
[1,]  0.6666667 -0.3333333 -0.3333333
[2,] -0.3333333  0.6666667 -0.3333333
[3,] -0.3333333 -0.3333333  0.6666667
```

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# Finding rank of A
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```{r}
rankMatrix(A)[1]
```

```
[1] 2
```

Each column all added up together gives us 0!.
$x_1 + x_2 + x_3 = 0$
Can be written as,
$x_1 = -x_2 - x_3$
That are linearly dependant and similar for $x_2 and x_3$
Hence r(A) = 2

<u>Part c: Computing E[$y^T y$]</u>

Finding E[$\boldsymbol{y^T y}$]

$$= E[(\frac{2x_1-x_2-x_3}{3}, \frac{-x_1+2x_2-x_3}{3}, \frac{-x_1-x_2+2x_3}{3}) \begin{bmatrix} \frac{2x_1-x_2-x_3}{3} \\ \frac{-x_1+2x_2-x_3}{3} \\ \frac{-x_1-x_2+2x_3}{3} \end{bmatrix}]$$

$= E[(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 (x_3 - \bar{x})^2)]$
$= E[(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 (x_3 - \bar{x})^2)]$

$= \mathrm{E}[\sum_{i=1}^{3}(x_i - \bar{x})^2]$
$= \mathrm{E}[\sum_{i=1}^{3} x_i^2 - 2x_i\bar{x} + \bar{x}^2]$
$= \mathrm{E}[\sum_{i=1}^{3} x_i^2 - n\bar{x}^2]$
Since we have 3 x's that are random variables!!
$= \mathrm{E}[\sum_{i=1}^{3} x_i^2 - 3\bar{x}^2]$
$= \mathrm{E}[(\sum_{i=1}^{3} x_i^2) - 3\bar{x}^2]$
since x_1, x_2 *and* x_3 *are identical independent distributions*!!
$= (3\text{-}1)\sigma^2 = 2\sigma^2$

Assuming that the sample variance is unbiased! and we can imply $\lambda = 0$! *Following similarly to Theorem* 3.2. *for the Non−central distribution*!

Alternative method:
Theorem 3.5:
$\mathrm{E}[\mathrm{y}^T Ay] = tr(AV) + \mu^T \mathrm{A}\mu$
since A = I,
$= \mathrm{tr(V)} + \mu^T \ \mu$
V = vary = varAx = AvarxA^T
since A is symmetric and idempotent!!

$$\mathrm{var(x}_i) = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$

$$\mathrm{V} = \frac{1}{3}\begin{bmatrix} 2\sigma^2 & -1 & -1 \\ -1 & 2\sigma^2 & -1 \\ -1 & -1 & 2\sigma^2 \end{bmatrix}$$

$\mu = E[y] = E[Ax] = AE[x]$

$$=\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} = 0$$

$$\mathrm{E}[\mathrm{y}^T y] = tr(\frac{1}{3}\begin{bmatrix} 2\sigma^2 & -1 & -1 \\ -1 & 2\sigma^2 & -1 \\ -1 & -1 & 2\sigma^2 \end{bmatrix}) + 0$$

$= 2\sigma^2$

Part d:
Using Theorem 3.5:
Proof:

Assuming that A is idempotent and has rank k. Because it is symmetric, it can be diagonalised. Let the (orthogonal) diagonalising matrix be P.

D = P^T AP = $\begin{bmatrix} \lambda_1 & ...... & 0 \\ .... & \lambda_2 & ...... \\ 0 & .... & \lambda_k \end{bmatrix}$

since A is symmetric and idempotent, all eigenvalues are either 0 or 1. We know from definition:

$$tr(A) = r(A) = k$$

A $=\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

A^2 = A = $\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

from Part 4b, we find out the rank and trace of matrix A we found in Part 4a. Is also is the same number of degrees of freedom for the chi squared distribution.

$$tr(A) = r(A) = 2$$

Therefore, A must have two eigenvalues of 1 and one eigenvalue of 0.

<u>Using Theorem 3.5 and Corollary 3.7:</u>

with our non central parameter λ!

$\lambda = \frac{1}{2}\mu^T A\mu$

$= \frac{1}{2}\begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \mu & \mu & \mu \end{bmatrix}$

$= 0$

$\iff : if\ and\ only\ if$

E[y] = E[$\begin{bmatrix} x_1 - \mu \\ x_2 - \mu \\ x_3 - \mu \end{bmatrix}$]

Since x_1, x_2 *and* x_3 *is identically independently distributed*! *and taking* the expectation of the expectation is the expectation itself!

E[y] = E[$\begin{bmatrix} \mu - \mu \\ \mu - \mu \\ \mu - \mu \end{bmatrix}$] = 0

NOTE: $\mu = \bar{x}$

In which case,

$\frac{y^T y}{\sigma^2}$

is just the sum of two independent standard normal's. This is just an ordinary (central) chi squared distribution $\chi^2{}_2$.

with expectation of 2 and variance of 4.