MAST30022 Decision Making
2021
Tutorial 5

1. **(PS3-6(a)(b)(c))**

   (a) Find all pure strategy equilibria for the 2-person non-zero-sum game with bi-matrix

   $$\begin{bmatrix} (3,2) & (2,1) \\ (4,3) & (1,4) \end{bmatrix}$$

   (b) Find security level pure strategies (i.e. if the players just wanted to use pure strategies, what would give best security levels) and associated expected payoffs. What would be the actual payoff if both followed these strategies?

   (b) Find an equilibrium pair for this game by trying to set $\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T}$ = constant for all $\boldsymbol{x}$ and $\boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^T$ = constant for all $\boldsymbol{y}$, as explained in lectures. Find also the expected payoffs for this pair.

Solution

(a) To find all pure strategy equilibria, in the payoff bi-matrix, locate the entries that give the maximum in the column for Player 1 and the maximum in the row for Player 2. In this case there are no pure strategy equilibria.

(b) From the given payoff bi-matrix we have

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix},$$

and $L = \max\{2,1\} = 2 = U = \min\{4,2\}$. Therefore a_{12} is a saddle point and the optimal (pure) strategy for Player 1 is $\boldsymbol{x} = (1,0)$, and the optimal security level is $u^* = 2$.

For Player 2

$$\boldsymbol{B}^T = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix},$$

and $L = \max\{2,1\} = 2 = U = \min\{2,4\}$. Therefore b_{11} is a saddle point and the optimal (pure) strategy for Player 2 is $\boldsymbol{y} = (1,0)$, and the optimal security level is $v^* = 2$.

Using these strategies the expected payoff for Player 1 is

$$\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3.$$

For Player 2 the expected payoff is

$$\boldsymbol{x}\boldsymbol{B}\boldsymbol{y}^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2.$$

(c) $\boldsymbol{x}^*\boldsymbol{B} = [w, w]$ and $x_1 + x_2 = 1$ gives the following system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 - w &= 0 \\ x_1 + 4x_2 - w &= 0 \\ x_1 + x_2 &= 1. \end{aligned}$$

Solving the system of equations gives $\boldsymbol{x}^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the expected payoff to Player 2 is $w = \frac{5}{2}$. $\boldsymbol{A}\boldsymbol{y}^{*T} = [z, z]^T$ and $y_1 + y_2 = 1$ gives the following system of linear equations

$$\begin{aligned} 3y_1 + 2y_2 - z &= 0 \\ 4y_1 + y_2 - z &= 0 \\ y_1 + y_2 &= 1. \end{aligned}$$

Solving the system of equations gives $\boldsymbol{y}^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the expected payoff to Player 1 is $z = \frac{5}{2}$.

The equilibrium solution is better than the optimal security level solution, but not better than $(\boldsymbol{x}, \boldsymbol{y}) = ((1, 0), (1, 0))$, at least for Player 1.

2. **(PS3-7)**

(a) For any a and b, find an equilibrium pair $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ of mixed strategies for the 2-person non-zero-sum game with payoff matrix below, using the method of setting $\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} =$ constant for all $\boldsymbol{x}$ and $\boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^T =$ constant for all $\boldsymbol{y}$.

$$\begin{bmatrix} (2,0) & (0,2) & (4,4) \\ (0,4) & (1,2) & (2,0) \\ (1,a) & (1,b) & (0,0) \end{bmatrix}$$

(b) Show that this method does not work if the entry in the third row and first column of this matrix is changed to give:

$$\begin{bmatrix} (2,0) & (0,2) & (4,4) \\ (0,4) & (1,2) & (2,0) \\ (-1,a) & (1,b) & (0,0) \end{bmatrix}$$

Solution

(a) From the given payoff bi-matrix we have

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix},$$

and

$$\boldsymbol{B} = \begin{bmatrix} 0 & 2 & 4 \\ 4 & 2 & 0 \\ a & b & 0 \end{bmatrix}.$$

$\boldsymbol{x}^*\boldsymbol{B} = [w, w, w]$ and $x_1+x_2+x_3 = 1$ gives the following system of linear equations

$$\begin{aligned} 4x_2 + ax_3 - w &= 0 \\ 2x_1 + 2x_2 + bx_3 - w &= 0 \\ 4x_1 - w &= 0 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Solving the system of equations gives $\boldsymbol{x}^* = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and the expected payoff to Player 2 is $w = 2$.

$\boldsymbol{A}\boldsymbol{y}^{*T} = [z, z, z]^T$ and $y_1+y_2+y_3 = 1$ gives the following system of linear equations

$$\begin{aligned} 2y_1 + 4y_3 - z &= 0 \\ y_2 + 2y_3 - z &= 0 \\ y_1 + y_2 - z &= 0 \\ .y_1 + y_2 + y_3 &= 1. \end{aligned}$$

Solving the system of equations gives $\boldsymbol{y}^* = \left(\frac{2}{9}, \frac{2}{3}, \frac{1}{9}\right)$ and the expected payoff to Player 1 is $z = \frac{8}{9}$.

(b) Changing the third equation to $-y_1 + y_2 - z = 0$ gives $\boldsymbol{y}^* = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ and $z = 0$. $\boldsymbol{y}^*$ is not a mixed strategy, so the method does not work in this case.

3. **(PS3-8)**

In checking whether the pair $\boldsymbol{x}^* = \left(\frac{1}{3}, \frac{1}{4}, \frac{5}{12}\right)$, $\boldsymbol{y}^* = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ is in equilibrium for the 2-person game:

$$\begin{bmatrix} (5,3) & (2,6) & (1,7) \\ (6,2) & (3,5) & (7,1) \\ (4,4) & (2,6) & (4,4) \end{bmatrix},$$

it is worked out that

$$\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{y}^{*T} = \tfrac{175}{48}, \;\; \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} = (108x_1 + 276x_2 + 168x_3)/48.$$

(a) CLEARLY explain how we can get that this is not an equilibrium pair from these results.

(b) In going on to show that the pair $\boldsymbol{x}^* = (0,1,0), \boldsymbol{y}^* = (0,1,0)$ is an equilibrium pair for this game, it is worked out that

$$\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{y}^{*T} = 3, \;\; \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} = 2x_1 + 3x_2 + 2x_3$$

$$\boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^{*T} = 5, \;\; \boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^{T} = 2y_1 + 5y_2 + y_3.$$

CLEARLY explain how we can get that this is an equilibrium pair from these results.

Solution

For a strategy pair $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ to be in equilibrium it must satisfy

$$\begin{aligned} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} &\leq \boldsymbol{x}^*\boldsymbol{A}\boldsymbol{y}^{*T} \qquad \text{for all } \boldsymbol{x} \in X \\ \boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^{T} &\leq \boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^{*T} \qquad \text{for all } \boldsymbol{y} \in Y. \end{aligned}$$

(a)

$$\begin{aligned} 48\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} &= 108x_1 + 276x_2 + 168x_3 \\ &= 276 \\ &> 175 \end{aligned}$$

if $\boldsymbol{x} = (0,1,0)$. In this case $\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} > \boldsymbol{x}^*\boldsymbol{A}\boldsymbol{y}^{*T}$, so $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is not an equilibrium pair.

(b)

$$\begin{aligned} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^{*T} &= 2x_1 + 3x_2 + 2x_3 \\ &= 2x_1 + 3x_2 + 2 - 2x_1 - 2x_2 \\ &= x_2 + 2 \\ &\leq 3 \end{aligned}$$

for all $\boldsymbol{x} \in X$.

$$\begin{aligned} \boldsymbol{x}^*\boldsymbol{B}\boldsymbol{y}^{T} &= 2y_1 + 5y_2 + y_3 \\ &= 2y_1 + 5y_2 + 1 - y_1 - y_2 \\ &= y_1 + 4y_2 + 1 \\ &\leq 5 \end{aligned}$$

for all $\boldsymbol{y} \in Y$.

Thus, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an equilibrium pair.

4. **(PS3-9)**

(a) The following problem is adapted from an example in "Game Theory and the Law" by D. G. Baird, R. H. Gertner and R. C. Picker, Harvard University Press, 1994.

Suppose that it is in the interest of a landowner to build and maintain a levee to prevent flood damage if and only if an adjacent landowner builds and maintains a levee too. Find all equilibria if the following payoff matrix applies. (Note that we assume here that each makes a decision without knowing what the other will do.)

		Landowner 2	
		Maintain	Don't Maintain
	Maintain	$(-4, -4)$	$(-10, -6)$
Landowner 1	Don't Maintain	$(-6, -10)$	$(-6, -6)$

(b) You should find that the above produces three equilibria. Suppose we now introduce a legal rule such that a landowner pays damages if they fail to maintain it when others do. This changes the payoffs as follows

		Landowner 2	
		Maintain	Don't Maintain
	Maintain	$(-4, -4)$	$(-4, -12)$
Landowner 1	Don't Maintain	$(-12, -4)$	$(-6, -6)$

Find all equilibria for the new payoffs.

We see that this gives us a way of analysing the changes brought by the legal rule.

Solution

(a) We first note that there are two equilibria of pure strategies, $(\boldsymbol{x}^*, \boldsymbol{y}^*) = ((1,0),(1,0))$ and $(\boldsymbol{x}^*, \boldsymbol{y}^*) = ((0,1),(0,1))$.

Now,

$$\boldsymbol{A} = \begin{bmatrix} -4 & -10 \\ -6 & -6 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -4 & -6 \\ -10 & -6 \end{bmatrix}.$$

$$\begin{aligned} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^T &= \begin{bmatrix} x_1 & 1-x_1 \end{bmatrix} \begin{bmatrix} -4 & -10 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ 1-y_1 \end{bmatrix} \\ &= (6y_1 - 4)x_1 - 6 \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{x}\boldsymbol{B}\boldsymbol{y}^T &= \begin{bmatrix} x_1 & 1-x_1 \end{bmatrix} \begin{bmatrix} -4 & -6 \\ -10 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ 1-y_1 \end{bmatrix} \\
&= (6x_1 - 4)y_1 - 6
\end{aligned}
$$

For fixed y_1, $\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^T$ is maximised when

$x_1 = 0$, for $6y_1 - 4 < 0$, i.e. $y_1 < 2/3$
$x_1 = 1$, for $6y_1 - 4 > 0$, i.e. $y_1 > 2/3$
$0 \leq x_1 \leq 1$, for $6y_1 - 4 = 0$, i.e. $y_1 = 2/3$.

For fixed x_1, $\boldsymbol{x}\boldsymbol{B}\boldsymbol{y}^T$ is maximised when

$y_1 = 0$, for $6x_1 - 4 < 0$, i.e. $x_1 < 2/3$
$y_1 = 1$, for $6x_1 - 4 > 0$, i.e. $x_1 > 2/3$
$0 \leq y_1 \leq 1$, for $6y_1 - 4 = 0$, i.e. $x_1 = 2/3$.

From Figure 1 we can see that there are three equilibrium pairs, given by

$$
\begin{aligned}
&\boldsymbol{x}^* = (0, 1), \quad \boldsymbol{y}^* = (0, 1) \\
&\boldsymbol{x}^* = (1, 0), \quad \boldsymbol{y}^* = (1, 0) \\
&\boldsymbol{x}^* = (2/3, 1/3), \quad \boldsymbol{y}^* = (2/3, 1/3)
\end{aligned}
$$

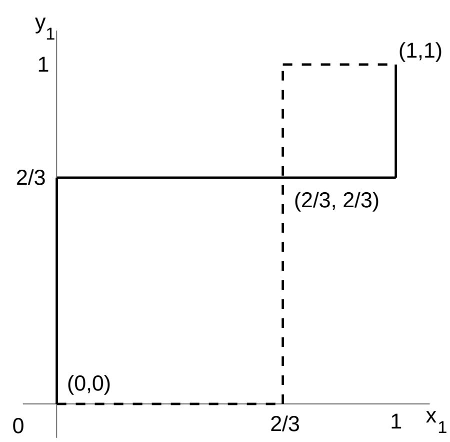

Figure 1: PS3-9

(b) We now note that there is only one equilibrium of pure strategies, $(\boldsymbol{x}^*, \boldsymbol{y}^*) = ((1, 0), (1, 0))$.
Now,

$$
\boldsymbol{A} = \begin{bmatrix} -4 & -4 \\ -12 & -6 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -4 & -12 \\ -4 & -6 \end{bmatrix}.
$$

$$
\begin{aligned}
\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^T &= \begin{bmatrix} x_1 & 1-x_1 \end{bmatrix} \begin{bmatrix} -4 & -4 \\ -12 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ 1-y_1 \end{bmatrix} \\
&= (6y_1+2)x_1 - 6y_1 - 6 \\
\boldsymbol{x}\boldsymbol{B}\boldsymbol{y}^T &= \begin{bmatrix} x_1 & 1-x_1 \end{bmatrix} \begin{bmatrix} -4 & -6 \\ -10 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ 1-y_1 \end{bmatrix} \\
&= (6x_1+2)y_1 - 6x_1 - 6
\end{aligned}
$$

For fixed y_1, $\boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^T$ is maximised when $x_1 = 1$. For fixed x_1, $\boldsymbol{x}\boldsymbol{B}\boldsymbol{y}^T$ is maximised when $y_1 = 1$. From Figure 2 we can see that there is one equilibrium pair

$$\boldsymbol{x}^* = (1,0), \quad \boldsymbol{y}^* = (1,0)$$

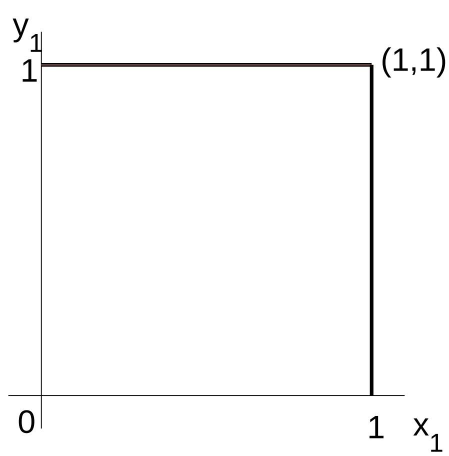

Figure 2: PS3-9

Here, imposing a payment for damages if one landowner does not maintain the levee and the other does, means that the equilibrium solution is that both landowners maintain the levee - if either one deviates unilaterally, they are worse off.

5. **(PS3-12)** Find the solution generated by Nash's axioms, for the game with payoff bi-matrix below, using

   (a) the point $(0, 0)$ as the status quo point;

   (b) the point $(1, 2)$ as the status quo point.

$$\begin{bmatrix} (1,4) & (-1,-4) \\ (-4,-1) & (4,1) \end{bmatrix}$$

Solution

The cooperative payoff set C is shown in Figure 3.

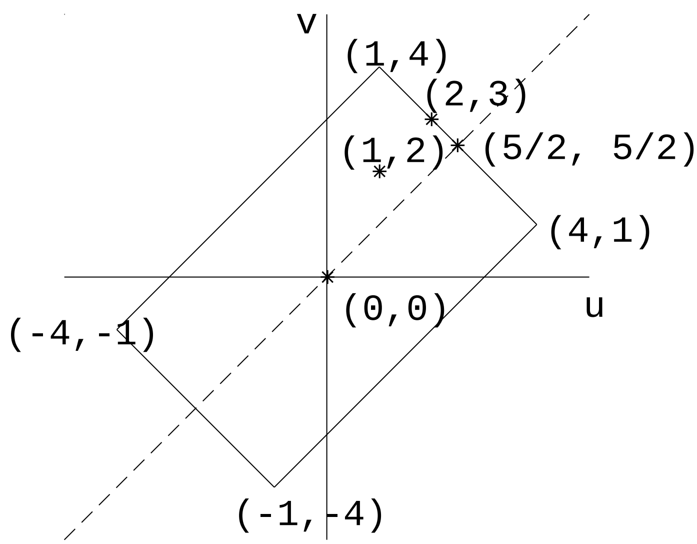

Figure 3: PS3-12

(a) We note that both C and the status quo point are symmetric, so the Nash solution is also symmetric. Here $PB(C) = NS(C) = \{(u, v)|v = -u + 5, 1 \le u \le 4\}$. Therefore, $(\underline{u}, \underline{v}) = \left(\frac{5}{2}, \frac{5}{2}\right)$. For the players to achieve this payoff vector they need to play strategy pairs (a_1, A_1) and (a_2, A_2) both with probability $\frac{1}{2}$.

(b) $NS(C) = \{(u, v)|v = -u + 5, 1 \le u \le 3\}$. To find the Nash solution we need to maximise

$$\begin{aligned} f(u) &= (u-1)(v-2) \\ &= (u-1)(-u+3) \\ &= -u^2 + 4u - 3, \end{aligned}$$

over $NS(C)$. Now, $f'(u) - 2u + 4 = 0 \Longrightarrow u = 2 \in [1, 3]$. Therefore $(\underline{u}, \underline{v}) = (2, 3)$. Since

$$\begin{aligned} (2,3) &= t(1,4) + (1-t)(4,1) \\ \Longrightarrow -3t &= -2 \\ \Longrightarrow t = \tfrac{2}{3}. \end{aligned}$$

For the players to achieve the Nash solution they need to play strategy pairs (a_1, A_1) with probability $\frac{2}{3}$ and (a_2, A_2) with probability $\frac{1}{3}$.

6. **(PS3-15)** Show that Nash's solution obeys Nash's (a) symmetry axiom and (b) "invariant under linear transformation" axiom.

Solution

We only consider the case when there is at least one point$(u_1, v_1) \in C$ such that $u_1 > u_0$ and $v_1 > v_0$. The other cases are similar.

(a) Let the status quo point be $\boldsymbol{q} = (u_0, u_0) \in C$. Suppose $\underline{u} \neq \underline{v}$. Since C is symmetric, $(\underline{u}, \underline{v}) = (\underline{v}, \underline{u})$. Then

$$\begin{aligned} g(\underline{u}, \underline{v}) &= (\underline{u} - u_0)(\underline{v} - u_0) \\ &= g(\underline{v}, \underline{u}). \end{aligned}$$

But since g attains its maximum at only one point, we must have $\underline{u} = \underline{v}$.

(b) Let $L(u, v) = (au + b, cv + d)$ where $a, c > 0$. If (u_0, v_0) is the status quo point in C, then $(au_0 + b, cv_0 + d)$ is the status quo point in $L(C)$. Let $u' = au + b$ and $v' = cv + d$. Then

$$\begin{aligned} h(u', v') = &\ (u' - (au_0 + b))(v' - (cv_0 + d)) \\ = &\ ac(u - u_0)(v - v_0). \end{aligned}$$

Since $a, c > 0$, if the maximum of the right hand side is attained at $(u, v) = (\underline{u}, \underline{v})$, the maximum of the left hand side is attained at $(u', v') = (a\underline{u} + b, c\underline{v} + d)$.