MAST30022 Decision Making
2021
Tutorial 9

1. **(PS7-1)** Suppose that Bob's utility function for asset position x is given by $u(x) = \log x$.

   (a) Is Bob risk-averse, risk-neutral, or risk-seeking?

   (b) Assume that Bob has \$20,000 and he is considering the following two lotteries:
   L_1: With probability 1, one will lose \$1,000
   L_2: With probability 0.9 one will gain \$0, and with probability 0.1 one will lose \$10,000.
   Determine which lottery Bob prefers and the risk premium of L_2.

Solution

(a) We have $u'(x) = 1/x$ and $u''(x) = -1/x^2$. Since $u''(x) < 0$, Bob is risk-averse.

(b) Let us look at the asset positions associated with the two lotteries. For L_1, with probability 1, Bob will have \$19,000. For L_2, with probability 0.9 he will have \$20,000, and with probability 0.1 he will have \$10,000. Since the utility function is $u(x) = \log x$,

$$\mathbb{E}(\text{U of } L_1) = \log 19000 = \log 10000 + \log 1.9$$

$$\mathbb{E}(\text{U of } L_2) = 0.9 \log 20000 + 0.1 \log 10000 = \log 10000 + 0.9 \log 2.$$

Since $\mathbb{E}(\text{U of } L_1) > \mathbb{E}(\text{U of } L_2)$, Bob prefers L_1 to L_2 (if he believes in the von Neumann-Morgenstern theory).

The expected value of L_2 is

$$EV(L_2) = 0.9 \times 20000 + 0.1 \times 10000 = 19000.$$

The certainty equivalent of L_2 is the number x^* such that Bob is indifferent between L_2 and receiving x^* with certainty. Solving

$$\log x^* = \log 10000 + 0.9 \log 2$$

yields $x^* \approx 18661$. Hence

$$RP(L_2) = 19000 - 18661 \approx 339.$$

2. **(PS7-2)** Answer Question 1 above for a utility function $u(x) = x^2$.

Solution

(a) $u'(x) = 2x$ and $u''(x) = 2$. Since $u''(x) > 0$, Bob is risk-seeking.

(b)

$$\mathbb{E}(\text{U of } L_1) = 19000^2 = 3.61 \cdot (10000)^2$$

$$\mathbb{E}(\text{U of } L_2) = 0.9 \cdot (20000)^2 + 0.1 \cdot (10000)^2 = 3.7 \cdot (10000)^2$$

Since $\mathbb{E}(\text{U of } L_2) > \mathbb{E}(\text{U of } L_1)$, Bob prefers L_2 to L_1 (if he believes in the von Neumann-Morgenstern theory).

Solving

$$x^{*2} = 3.7 \cdot (10000)^2$$

yields the certainty equivalent $x^* = \sqrt{3.7} \cdot 10000$ of L_2. Since $EV(L_2) = 19000$, the risk premium of L_2 is

$$RP(L_2) = 19000 - \sqrt{3.7} \cdot 10000 \approx \; -235.$$

3. **(PS7-4)** Alice is trying to determine which of two courses to take. If she takes the decision making course, she believes that she has a 10% chance of receiving an A, a 40% chance for a B, and a 50% for a C. If Alice takes a statistics course, she has a 70% chance for a B, a 25% chance for a C, and a 5% chance for a D. Alice is indifferent between $\langle 1, C\rangle$ and $\langle 0.25, A; 0.75, D\rangle$. She is also indifferent between $\langle 1, B\rangle$ and $\langle 0.7, A; 0.3, D\rangle$. If Alice wants to take the course that maximizes the expected utility of her final grade, which course should she take?

Solution

Alice's decision making problem can be expressed in terms of lotteries. That is,

$$\begin{aligned} L_{DM} &= \langle 0.1, A; 0.4, B; 0.5, C\rangle \\ L_{ST} &= \langle 0.7, B; 0.25, C; 0.05, D\rangle. \end{aligned}$$

Since

$$\begin{aligned} \langle 1, B\rangle &\sim \langle 0.7, A; 0.3, D\rangle \\ \langle 1, C\rangle &\sim \langle 0.25, A; 0.75, D\rangle, \end{aligned}$$

the utilities for the grades are $u(A) = 1$, $u(B) = 0.7$, $u(C) = 0.25$, and $u(D) = 0$. Therefore

$$\begin{aligned} \mathbb{E}[\text{U of } L_{DM}] &= 0.1 \times 1 + 0.4 \times 0.7 + 0.5 \times 0.25 \;=\; 0.505 \\ \mathbb{E}[\text{U of } L_{ST}] &= 0.7 \times 0.7 + 0.25 \times 0.25 + 0.05 \times 0 \;=\; 0.5525. \end{aligned}$$

Using the expected utility criterion, since $\mathbb{E}[\text{U of } L_{DM}] < \mathbb{E}[\text{U of } L_{ST}]$, she will choose the statistics course.

4. **(PS7-5)** Consider the following decision table under risk, in which the consequences are monetary payoffs.

Consequences	x_{ij}	States θ_1	θ_2	θ_3
Actions	a_1	\$100	\$110	\$120
	a_2	\$90	\$100	\$120
	a_3	\$100	\$110	\$100
		Probabilities 1/4	1/2	1/4

Convert this problem into one of choosing between lotteries. The decision maker holds the following indifferences with reference lotteries:

$$\$100 \sim \langle 0.5, \$120; 0.5, \$90 \rangle$$

$$\$110 \sim \langle 0.8, \$120; 0.2, \$90 \rangle$$

Which action should the decision maker choose?

Solution

The three actions a_1, a_2, and a_3 correspond to the three lotteries, L_1, L_2, and L_3, respectively, where

$$\begin{aligned} L_1 &= \langle 0.25, \$100, 0.5, \$110; 0.25, \$120 \rangle \\ L_2 &= \langle 0.25, \$90, 0.5, \$100; 0.25, \$120 \rangle \\ L_3 &= \langle 0.25, \$100, 0.5, \$110; 0.25, \$100 \rangle. \end{aligned}$$

Since

$$\begin{aligned} \langle 1, \$100 \rangle &\sim \langle 0.5, \$120; 0.5, \$90 \rangle \\ \langle 1, \$110 \rangle &\sim \langle 0.8, \$120; 0.2, \$90 \rangle, \end{aligned}$$

the utilities for the payoffs are $u(\$120) = 1$, $u(\$110) = 0.8$, $u(\$100) = 0.5$, and $u(\$90) = 0$. Therefore

$$\begin{aligned} \mathbb{E}[\text{U of } L_1] &= 0.25 \times 0.5 + 0.5 \times 0.8 + 0.25 \times 1 \;=\; 0.775 \\ \mathbb{E}[\text{U of } L_2] &= 0.25 \times 0 + 0.5 \times 0.5 + 0.25 \times 1 \;=\; 0.5 \\ \mathbb{E}[\text{U of } L_3] &= 0.25 \times 0.5 + 0.5 \times 0.8 + 0.25 \times 0.5 \;=\; 0.65. \end{aligned}$$

Using the expected utility criterion, since $\mathbb{E}[\text{U of } L_1] > \mathbb{E}[\text{U of } L_3] > \mathbb{E}[\text{U of } L_2]$, $a_1 \succ a_3 \succ a_2$. So the decision maker should choose a_1.

5. **(PS7-10)** Let $\mathcal{L}$ be a set of simple and finitely compounded lotteries and X the set of all possible rewards. Denote the maximum value of X by $x_{\max}$ and the minimum value of X by $x_{\min}$. Let A be the union of X, $\mathcal{L}$, and all lotteries of the type $\langle p, x_{\max}; 1-p, x_{\min}\rangle$. Assume that a decision maker's preferences over A satisfy all the seven axioms of von Neumann-Morgenstern.

   (a) By continuity (Axiom 7), for each $x_i \in X$, there exists u_i with $0 \leq u_i \leq 1$ such that
   $$x_i \sim \langle u_i, x_{\max}; 1-u_i, x_{\min}\rangle.$$
   Using Axiom 1 (weak order) and Axiom 6 (monotonicity), show that u_i is unique.

   (b) Using Axiom 1 (weak order), Axiom 3 (reduction of compound lotteries), Axiom 4 (substitutability), and Axiom 7 (continuity), show that for each $x_i \in X$
   $$x_i \sim \langle 0, x_1; \ldots; 1, x_i; \ldots; 0, x_r\rangle.$$

Solution:

(a) Let $x_i \in X$. Suppose $x_i \sim \langle u_{i1}, x_{\max}; 1-u_{i1}, x_{\min}\rangle$ and $x_i \sim \langle u_{i2}, x_{\max}; 1-u_{i2}, x_{\min}\rangle$. We show that $u_{i1} = u_{i2}$, which proves uniqueness.
By transitivity of "$\sim$" (Axiom 1) we have that
$$\langle u_{i1}, x_{\max}; 1-u_{i1}, x_{\min}\rangle \sim \langle u_{i2}, x_{\max}; 1-u_{i2}, x_{\min}\rangle.$$
In particular this means that
$$\begin{aligned}\langle u_{i1}, x_{\max}; 1-u_{i1}, x_{\min}\rangle &\succeq \langle u_{i2}, x_{\max}; 1-u_{i2}, x_{\min}\rangle\\ \langle u_{i2}, x_{\max}; 1-u_{i2}, x_{\min}\rangle &\succeq \langle u_{i1}, x_{\max}; 1-u_{i1}, x_{\min}\rangle.\end{aligned}$$
Applying Axiom 6 (monotonicity) twice, we get $u_{i1} \geq u_{i2}$ and $u_{i2} \geq u_{i1}$, hence $u_{i1} = u_{i2}$ and the value u_i is unique.

(b) Let $x_i \in X$. Without loss of generality we assume that $x_1 = x_{\max}$ and $x_r = x_{\min}$. By Axiom 7 (continuity) there exists a u_i, with $0 \leq u_i \leq 1$ such that
$$x_i \sim \langle u_i, x_{\max}; 1-u_i, x_{\min}\rangle.$$
Using Axiom 4 (substitutability) we have
$$\langle 0, x_1; \ldots; 1, x_i; \ldots; 0, x_r\rangle \sim \langle 0, x_1; \ldots; 1, \langle u_i, x_{\max}; 1-u_i, x_{\min}\rangle; \ldots; 0, x_r\rangle.$$
By Axiom 3 (reduction of compound lotteries) it follows that
$$\langle 0, x_1; \ldots; 1, \langle u_i, x_{\max}; 1-u_i, x_{\min}\rangle; \ldots; 0, x_r\rangle \sim \langle u_i, x_1; \ldots; 1-u_i, x_r\rangle.$$
Finally, by Axiom 7
$$\langle u_i, x_1; \ldots; 1-u_i, x_r\rangle \sim x_i.$$
It now follows by transitivity of "$\sim$" (Axiom 1) that
$$x_i \sim \langle 0, x_1; \ldots; 1, x_i; \ldots; 0, x_r\rangle.$$

6. **(PS8-5)** Oilco must determine whether or not to drill for oil in the South China Sea. It costs \$100,000, and if oil is found the value is estimated to be \$600,000. At present, Oilco believes that there is a 45% chance that the field contains oil. Before drilling, Oilco can hire (for \$10,000) a geologist to obtain more information about the likelihood that the field will contain oil. There is a 50% chance that the geologist will issue a favorable report, and a 50% chance of an unfavorable report. Given a favorable report, there is an 80% chance that the field contains oil. Given an unfavorable report, there is a 10% chance that the field contains oil. Determine Oilco's optimal course of action. (Adapted from "Operations Research: Appl. & Alg.", W. L. Winston, 4th ed., 2004.)

   Solution

   See the decision tree below. (Unit: \$000s)

   Optimal strategy: Hire a geologist to obtain more information. If the report is favorable, drill for oil; otherwise, do not drill.

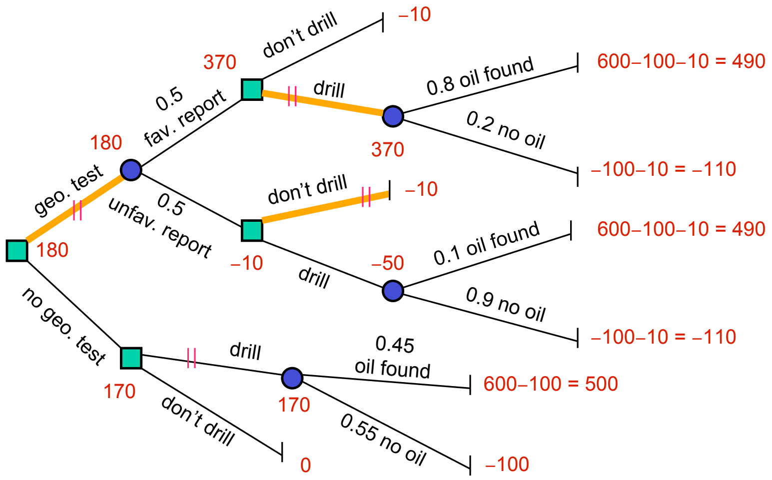

Figure 1: PS8-5