School of Mathematics and Statistics
MAST30030
Applied Mathematical Modelling

Assignment 2. Due: 5pm Wednesday May 27th

This assignment counts for 5% of the marks for this subject.

Question 1

Due to the global outbreak of COVID-19, in order to limit the spread of the virus, the government has placed limits on the number of people that can be inside a building at any given time. Due to this initiative, a queue often forms at the entrance to the Woolworths on Lygon Street; assume this queue is one long, straight, line. The government regulations also mandate that there must be a minimum of 1.5m between each customer—this determines the maximum density of customers in the queue. While waiting for their morning coffee, a research scientist has observed that, if the queue is empty, a customer entering the queue will walk at 1.4m/s—this is the maximum velocity of customers in the queue. They have also determined, again by observation, that there is a linear relationship between the density of customers and their velocity, which is defined by the bounds given above.

The first 10 metres of the queue is sectioned off and so people are only able to enter if they are already in the queue. However, behind this point, people become confused about where to enter the queue and so they enter at a random position, at a rate of α customers per metre per second; model this random entry as a constant (per unit time, per unit length).

(a) Give the conservation equation for the queue in its dimensional form (without derivation - you may quote the lecture notes), relating the customer density, customer flux and source of customers entering the queue. Assume that the sectioned off portion of the queue is located between $x \in [0, 10)$, and that the remainder of the queue is unbounded, i.e. $x < 0$.

(b) Rescale the governing equation in (a) by using the maximum density and maximum velocity, and using a length scale of 10 metres. What is the time scale required to balance the two terms on the LHS of the conservation equation? You should find a unitless parameter that controls the source term, call this β.

(c) During the lunch time peak hour, the supermarket is at its maximum government mandated capacity, however it is functioning efficiently and so customers in the queue are moving at maximum flux. At time $t = 0$, a malfunction in the checkout machines results in customers unable to exit the supermarket, and so customers at the beginning of the queue cannot enter the building. Solve the governing equations using the method of characteristics and construct the space-time diagram. In so doing, identify the presence of any shocks and fans (if either exist).

Hint 1: You may find Lambert W functions (also known as product logarithm functions) to be useful in your solution. Specifically, the Lambert W function is a *multivalued* function defined as the solutions to $we^w = z$ where $w(z)$ is a Lambert W function. Lambert W functions also arise as solutions to the ODE,

$$\frac{dw}{dz} = \frac{w}{z(w+1)}. \tag{1}$$

Hint 2: You may find the substitution $z = ae^{bt}$ to be useful.

(d) The same research scientist has returned to the cafe at lunch time to get their second coffee of the day. While they waited for their coffee (which took 5 minutes to make), they counted that 7 customers entered the queue in the 5 metres that was visible to them—use these values to determine α. Plot the (dimensional) density for the queue at 10, 20 and 30 seconds after the checkouts malfunctioned. Discuss what the plots tell you, in physical terms, about the queue.

(e) Due to their excellent technical expertise, one of the checkout operators was able to identify the malfunction, and within 1 minute, they had fixed the issue. Customers were immediately able to start entering the supermarket again. Explain how you would continue to solve the problem (but do not actually solve it) and what you would expect to happen to the queue; think about the presence of any new shocks or fans that might form.