Complementary mathematical topics

These topics will not be tested in STM5001 assignments. They are given for students interested in the mathematical background and justifications of models considered in this week lectures.

Covariance functions and positive definiteness

Recall that by the definition, $B(t,s) \in \mathcal{P}_T$ if and only if

$$\boxed{\sum_{k=1}^{n} \sum_{l=1}^{n} B(t_k, t_l) c_k \overline{c}_l \geq 0}$$

for all integer n, elements $t_k, k = 1, ..., n$, from the set T, and complex numbers $c_k, k = 1, ..., n$.

Let X_t be a random field on T. Suppose that

$$E|X_t|^2 < +\infty \quad \text{and} \quad E(X_t) = 0.$$

Let $B(t, s) = EX_t\overline{X_s}$ denote the covariance function of X_t.

For any $t_1, \ldots, t_n \in T; c_1, \ldots, c_n \in \mathbb{C}$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{l=1}^{n} B(t_k, t_l)c_k\overline{c}_l &= \sum_{k=1}^{n}\sum_{l=1}^{n} EX_{t_k}\overline{X_{t_l}}c_k\overline{c}_l \\
&= E\left(\sum_{k=1}^{n} c_k X_{t_k} \sum_{l=1}^{n} \overline{c_l X_{t_l}}\right) \\
&= E\left|\sum_{k=1}^{n} c_k X_{t_k}\right|^2 \geq 0.
\end{aligned}$$

Thus, the **covariance function $B(t, s)$ is indeed a positive definite function.**

Properties of elements in $\mathcal{P}_T$

We start with some properties of positive definite functions. If a function does not satisfy these properties, then this function is not a covariance functions.

Let $B(t,s) \in \mathcal{P}_T$. Then:

(1) $B(t,t) \geq 0$ **for all** $t \in T$.
Proof. Let us choose $n = 1,\ t_1 = t$. Then from
$\sum_{k=1}^{n} \sum_{l=1}^{n} B(t_k, t_l) c_k \overline{c}_l \geq 0 \Rightarrow$ for all $c_1,\ t$:
$B(t,t)|c_1|^2 \geq 0 \Rightarrow B(t,t) \geq 0.$

(2) $B(t,s) = \overline{B(s,t)}$.
Proof. Let $n = 2, t_1 = t, t_2 = s$. Then for any $a_1, a_2 \in \mathbb{C}$

$$
\begin{aligned}
\sum_{k=1}^{2}\sum_{i=1}^{2} a_k B(t_k, t_i)\bar{a}_i &= B(t,t)|a_1|^2 + B(t,s)a_1\bar{a}_2 \\
&\quad + B(s,t)a_2\bar{a}_1 + B(s,s)|a_2|^2 \geq 0.
\end{aligned}
$$

$$
\left.\begin{aligned}
B(t,t)|a_1|^2 &\geq 0 \\
\\
B(s,s)|a_2|^2 &\geq 0
\end{aligned}\right\} \Rightarrow \; Im\left\{B(t,s)a_1\bar{a}_2 + B(s,t)a_2\bar{a}_1\right\} = 0
$$

Let $B(t,s) = \alpha_1 + i\beta_1$, $B(s,t) = \alpha_2 + i\beta_2$. If $a_1 = a_2 = 1$ then $(*)$ becomes

$$Im(\alpha_1 + \alpha_2 + i(\beta_1 + \beta_2)) = 0 \Rightarrow \beta_1 = -\beta_2.$$

If $a_1 = i, a_2 = 1$ then $(*)$ becomes

$$Im(\alpha_1 i - \beta_1 - \alpha_2 i + \beta_2) = 0 \Rightarrow \alpha_1 = \alpha_2.$$

Thus $B(t,s) = \overline{B(s,t)}$.

(3) $|B(t,s)|^2 \leq B(t,t)B(s,s)$.

Proof. Let $n = 2, t_1 = t, t_2 = s$. Then from $\sum_{k=1}^{n}\sum_{l=1}^{n} B(t_k, t_l)c_k\overline{c}_l \geq 0$, it follows that

$$\sum_{k=1}^{2}\sum_{l=1}^{2} B(t_k, t_l)c_k\overline{c}_l \geq 0 \Rightarrow \left(c_1, c_2\right)\begin{pmatrix} B(t,t) & B(t,s) \\ B(s,t) & B(s,s) \end{pmatrix}\begin{pmatrix} \overline{c}_1 \\ \overline{c}_2 \end{pmatrix} \geq 0$$

$$\Rightarrow \left|\begin{pmatrix} B(t,t) & B(t,s) \\ B(s,t) & B(s,s) \end{pmatrix}\right| \geq 0 \Rightarrow B(t,t)B(s,s) - B(t,s)B(s,t) \geq 0,$$

$$\Rightarrow (\text{by } \ (2)) \Rightarrow B(t,t)B(s,s) \geq |B(t,s)|^2.$$

Methods to obtain basic covariance functions.

1. **If** $f(t) : T \to \mathbb{C}$ **then** $B(t,s) = f(t)\overline{f(s)} \in \mathcal{P}_T$.

To show it, note that for any $t_1, \ldots, t_n \in T; c_1, \ldots, c_n \in \mathbb{C}$, we have

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{l=1}^{n} B(t_k, t_l) c_k \overline{c}_l &= \sum_{k=1}^{n}\sum_{l=1}^{n} f(t_k)\overline{f(t_l)} c_k \overline{c}_l \\
&= \sum_{k=1}^{n} f(t_k) c_k \times \sum_{l=1}^{n} \overline{f(t_l) c_l} \\
&= \left| \sum_{k=1}^{n} c_k f(t_k) \right|^2 \geq 0.
\end{aligned}$$

Hence $f(t)\overline{f(s)} \in \mathcal{P}_T$.

2. **If** $f(t,\lambda): T \times \Lambda \to \mathbb{C}$, $\mu(\lambda) \geq 0$ **and** $\int_\Lambda |f(t,\lambda)|^2 \mu(\lambda) d\lambda < +\infty$, **then** $B(t,s) = \int_\Lambda f(t,\lambda)\overline{f(s,\lambda)}\mu(\lambda)d\lambda \in \mathcal{P}_T$.

To show it, note that for any $t_1, \dots, t_n \in T$; $c_1, \dots, c_n \in \mathbb{C}$ we have

$$\begin{aligned}\sum_{k=1}^{n}\sum_{l=1}^{n} B(t_k, t_l)c_k\overline{c}_l &= \int_\Lambda \sum_{k=1}^{n}\sum_{l=1}^{n} f(t_k,\lambda)\overline{f(t_l,\lambda)}c_k\overline{c}_l\mu(\lambda)d\lambda \\ &= \int_\Lambda \left|\sum_{k=1}^{n} c_k f(t_k,\lambda)\right|^2 \mu(\lambda)d\lambda \geq 0.\end{aligned}$$