Complementary mathematical topics

These topics will not be tested in STM5001 assignments. They are given for students interested in the mathematical background and justifications of models considered in this and previous weeks lectures.

Spatio-temporal modelling

In previous lectures we mainly focused on spatial modelling and prediction problems. However, usage of spatio-temporal data is rapidly increasing due to technological advances. These data vary in space and time, for example, measurements of weather characteristics, air quality, fire spread, crop yields, etc.

Analysis of not only spatial but also spatio-temporal correlations is important to predict values not only from neighbouring observations, but previous time periods. Spatio-temporal interpolation provides more accurate predictions than spatial interpolation.

We consider Gaussian spatio-temporal random field Z defined over a spatial domain S and temporal domain T.

A sample

$$z = (z(s_1, t_1), ..., z(s_n, t_n))$$

is observed at a set of distinct spatio-temporal locations $(s_1, t_1), ..., (s_n, t_n) \in S \times T$.

In many applications it is important to model Z from the sample z

- to predict at unobserved locations in space and time;
- to simulate the random field in $S \times T$.

We assume that the field is stationary and spatially isotropic. Hence, it can be characterised through a mean m and a covariance function $C(s, t)$, where the spatio-temporal covariance only depends on the separating distances across space $s \in S$ and time $t \in T$.

Spatio-temporal covariance models

Now we give several ways to construct spatio-temporal covariance models that are often used in applications.

1. The separable covariance model

This spatio-temporal covariance function can be represented as the product of spatial and temporal covariances:

$$\boxed{C_{sep}(s,t) = C_1(s)C_2(t).}$$

The value that the variogram approaches to at the range (the range can be equal $+\infty$) is called the sill.

If the covariance function approaches 0 at $+\infty$, then by $\gamma(s) = C(0) - C(s)$ it follows that the sill equals $C(0)$.

Let us denote the normalised (equal to 1 at the origin) covariance and variogram functions by $\tilde{C}_1(s)$, $\tilde{C}_2(t)$ and $\tilde{\gamma}_{sep}(s,t)$ respectively. Then,

$$\gamma_{sep}(s,t) = sill \cdot \tilde{\gamma}_{sep}(s,t)$$

and

$$C_{sep}(s,t) = sill \cdot \tilde{C}_1(s)\tilde{C}_2(t)$$

$$\begin{aligned}\gamma_{sep}(s,t) &= C_{sep}(0,0) - C_{sep}(s,t) = C_{sep}(0,0) - C_1(s)C_2(t) \\ &= sill \cdot (1 - \tilde{C}_1(s)\tilde{C}_2(t)) = sill \cdot (1 - (1-\tilde{\gamma}_1(s))(1-\tilde{\gamma}_2(t)) \\ &= sill \cdot (\tilde{\gamma}_1(s) + \tilde{\gamma}_2(t) - \tilde{\gamma}_1(s)\tilde{\gamma}_2(t)).\end{aligned}$$

Thus the separable variogram is given as:

$$\boxed{\gamma_{sep}(s,t) = sill \cdot (\tilde{\gamma}_1(s) + \tilde{\gamma}_2(t) - \tilde{\gamma}_1(s)\tilde{\gamma}_2(t)).}$$

2. Product-sum covariance model

This covariance model given by

$$\boxed{C_{ps}(s,t) = kC_1(s)C_2(t) + C_1(s) + C_2(t),}$$

where $k > 0$.

The corresponding variogram can be computed as

$$\boxed{\gamma_{ps}(s,t) = (k \cdot sill_2 + 1)\gamma_1(s) + (k \cdot sill_1 + 1)\gamma_2(t) - k\gamma_1(s)\gamma_2(t),}$$

where $\gamma_1(s)$ and $\gamma_2(t)$ are spatial and temporal variograms and $sill_1$ and $sill_2$ are models spatial and temporal sills.

The following identity defines the overall sill of the model

$$\boxed{sill_{ps} = k \cdot sill_1 \cdot sill_2 + sill_1 + sill_2.}$$

3. Metric covariance model

This model assumes identical spatial and temporal covariance functions. It introduces spatio-temporal anisotropy by an anisotropy correction κ, which let treat spatial and temporal distances on equal scales.

The metric covariance functions have the following form:

$$\boxed{C_m(s,t) = C_0\left(\sqrt{||s||^2 + (\kappa \cdot t)^2}\right),}$$

where $C_0(x)$ is a covariance function on $\mathbb{R}$.

Then, the corresponding variogram is

$$\boxed{\gamma_m(s,t) = \gamma_0\left(\sqrt{||s||^2 + (\kappa \cdot t)^2}\right).}$$

4. Sum-metric covariance model

A combination of the spatial, temporal and metric models is called a sum-metric covariance. It is given by the formula

$$C_{sm}(s,t) = C_1(s) + C_2(t) + C_0\left(\sqrt{||s||^2 + (\kappa \cdot t)^2}\right).$$

The corresponding variogram is given by

$$\gamma_{sm}(s,t) = \gamma_1(s) + \gamma_2(t) + \gamma_0\left(\sqrt{||s||^2 + (\kappa \cdot t)^2}\right).$$

Spatio-temporal predictions

The estimated estimated spatio-temporal variogram is used to predict at spatio-temporal locations (s_0, t_0) analogously to the spatial kriging:

$$\boxed{\hat{Z}(s_0, t_0) = \hat{m}(s_0, t_0) + \hat{\eta}(s_0, t_0),}$$

where $\hat{m}(s_0, t_0)$ is the predicted trend component and $\hat{\eta}(s_0, t_0)$ is the space-time residual. The term $\hat{\eta}(s_0, t_0)$ is obtained as

$$\hat{\eta}(s_0, t_0) = \lambda_O \eta,$$

where η is a vector of space-time residuals at the observed locations and λ_O is a vector of kriging weights estimated by

$$\lambda_O = \Gamma_O^{-1} \gamma_O.$$

Now the matrix Γ_O contains the variogram values between all possible combinations of space-time locations and moments.