MAST20005/MAST90058: Week 5 Solutions

1. $\left[\sqrt{\frac{12}{23.34}} \times 37.751, \sqrt{\frac{12}{4.404}} \times 37.751\right] = [4.41, 10.1]$

2. (a) The pooled estimate of the standard deviation is

$$s_p = \sqrt{\frac{9 \times 0.323^2 + 9 \times 0.210^2}{18}} = 0.2724.$$

   Hence a 95% confidence interval is

$$2.548 - 1.564 \pm 2.101 \times 0.2724\sqrt{\frac{1}{10} + \frac{1}{10}} = [0.728, 1.240].$$

   (b) Yes. The confidence interval is quite far from zero so we have good evidence that the mean force required to remove the seal when the wedge is in place is larger than when it is not.

   (c) Inspect a box plot or compute a CI for $\operatorname{var}(X)/\operatorname{var}(Y)$.

3. A 90% confidence intervals for σ_x/σ_y is

$$\left[\sqrt{0.3821} \times \frac{0.197}{0.318}, \sqrt{2.475} \times \frac{0.197}{0.318}\right] = [0.383, 0.975].$$

   This interval lies completely below 1, so we have good evidence that the standard deviations *differ* between the two shifts.

4. (a) $\hat{p} = 24/642 = 0.0374$

   (b) $\hat{p} \pm 1.96\sqrt{\hat{p}(1-\hat{p})/n} = [0.0227, 0.0521]$

   (c) Upper bound: $\hat{p} + 1.645\sqrt{\hat{p}(1-\hat{p})/n} = 0.0497$

5. Need $1.96 \times \sqrt{4.84}/\sqrt{n} = 0.4 \Rightarrow n = (1.96^2 \times 4.84/0.4^2) = 116.2$, so we take $n = 117$.

6. The error is $\epsilon = c\sqrt{p(1-p)/n}$, which gives the formula:

$$n = \frac{c^2 p(1-p)}{\epsilon^2}.$$

   We don't know the value of p, but the maximum error occurs when $p = 0.5$ so we use that value for calculating the sample size. For 95% CIs we use $c = \Phi^{-1}(1 - 0.025) = 1.960$ and for 90% CIs we use $c = \Phi^{-1}(1 - 0.05) = 1.645$. In each case, we always round up to the nearest integer to get a valid sample size. This gives:

   (a) 1068

   (b) 2401

   (c) 752

7. From the sample we have $n = 10$, $\bar{x} = 30.84$ and $s = 2.908$.

   (a) $\hat{\mu} = 30.8$; 90% CI for μ: $30.84 \pm 1.833 \times 2.908 \times \sqrt{\frac{1}{10}} = [29.2, 32.5]$.

   (b) $\hat{\sigma} = 2.91$; 95% CI for σ: $\left[\sqrt{\frac{9}{19.02}} \times 2.908, \sqrt{\frac{9}{2.700}} \times 2.908\right] = [2.00, 5.31]$.

(c) 90% PI for X: $30.84 \pm 1.833 \times 2.908 \times \sqrt{\frac{11}{10}} = [25.2, 36.4]$.

8. (a) Using the CLT approximation for $\hat{p}$,

$$\Pr\left(p - c\sqrt{\frac{p(1-p)}{n}} < \hat{p} < p + c\sqrt{\frac{p(1-p)}{n}}\right) \approx 1 - \alpha$$

where $c = \Phi^{-1}(1 - \alpha/2)$.

(b) Rearranging the above inequality gives,

$$-c\sqrt{\frac{p(1-p)}{n}} < \hat{p} - p < c\sqrt{\frac{p(1-p)}{n}}.$$

Since the middle term is smaller in absolute value than both endpoints, which are equal in absolute value, we can square all of the terms and write:

$$(\hat{p} - p)^2 < c^2\,\frac{p(1-p)}{n}.$$

(c) Expanding both sides gives,

$$p^2 - 2p\hat{p} + \hat{p}^2 < -\frac{c^2}{n}p^2 + \frac{c^2}{n}p.$$

Moving all terms to one side and collecting them together based on p,

$$\left(1 + \frac{c^2}{n}\right)p^2 - 2\left(\hat{p} + \frac{c^2}{2n}\right)p + \hat{p}^2 < 0.$$

This is a quadratic in p. You can verify that it has two roots by calculating the discriminant and showing it is positive. Let the roots be a and b. Since the leading coefficient is positive (i.e. it is an ‘upward’ parabola), the inequality will be satisfied when $a < p < b$. We can calculate the roots either by completing the square and factorising, or by applying the quadratic formula. Either method will give you the required endpoints.