MAT4MDS — Practice 7 Worked Solutions

Model Answers to Practice 7

Question 1.

(a) $f(x) = x^2e^x \Rightarrow f'(x) = 2xe^x + x^2e^x$ and $f'' = 2e^x + 2xe^x + 2xe^x + x^2e^x = 2e^x + 4xe^x + x^2e^x$.

This gives $f(0) = 0,\ f'(0) = 0$ and $f''(0) = 2$ so that $(T_2f)(x) = 0 + 0x + \frac{2x^2}{2!} = x^2$.

(b) Let $y = (x+1)\ln(x+1) = uv$ where $u = (x+1)$ and $v = \ln(x+1)$. Then

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v = (x+1)\frac{1}{x+1} + \ln(x+1) = 1 + \ln(x+1)$$

It follows that $\frac{d^2y}{dx^2} = \frac{1}{x+1}$. Now, when $x = 0$ we have

$$y = 1\ln(1) = 0, \quad \frac{dy}{dx} = 1 + \ln(1) = 1 \quad \textit{and} \quad \frac{d^2y}{dx^2} = 1$$

so that $(T_2f)(x) = 0 + 1x + \frac{1}{2!}x^2 = x + \frac{1}{2}x^2$.

(c) Using the product rule first, $f'(x) = e^{x^2} + x \times 2xe^{x^2} = e^{x^2} + 2x^2e^{x^2}$.

We now get, using the sum, chain and product rules,

$$f''(x) = 2xe^{x^2} + 4xe^{x^2} + 2x^2 \times 2xe^{x^2} = 6xe^{x^2} + 4x^3e^{x^2}.$$

Since $f(0) = 0,\ f'(0) = 1$ and $f''(0) = 0$, we have $(T_2f)(x) = x$.

Question 2.

(a) $f(x) = f'(x) = f''(x) = f'''(x) = e^x$, so $f(0) = 1 = f'(0) = f''(0) = f'''(0)$.

This gives $(T_3f)(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

(b) $g(x) = xe^x \Rightarrow g'(x) = e^x + xe^x = (1+x)e^x, g''(x) = e^x + (1+x)e^x = (2+x)e^x$ and $g'''(x) = (3+x)e^x$, giving $g(0) = 0,\ g'(0) = 1,\ g''(0) = 2$ and $g'''(0) = 3$ so that $(T_3g)(x) = x + x^2 + \frac{x^3}{2}[= x(T_2f)(x)]$.

Question 3. (a), (d) and (f) are correct statements.

Question 4. $(T_2f)'(x) = f'(0) + f''(0)\,x$ and $(T_2f)''(x) = f''(0)$ so $(T_2f)(0) = f(0)$, $(T_2f)'(0) = f'(0)$ and $(T_2f)''(0) = f''(0)$ so the graphs are wrong because

(a) $(T_2f)(0) \neq f(0)$ (wrong value at 0),

(b) $(T_2f)'(0) \neq f'(0)$ (wrong slope at 0) and

(c) $(T_2f)''(0) \neq f''(0)$ (wrong curvature at 0).

Question 5. From $f(x) = e^{-x^2}$, we obtain

$$f'(x) = -2xe^{-x^2} \qquad \Rightarrow \quad f'(0) = 0$$

$$f''(x) = [4x^2 - 2]e^{-x^2} \qquad \Rightarrow \quad f''(0) = -2$$

$$f'''(x) = [(4x^2 - 2)(-2x) + 8x]e^{-x^2} \qquad \Rightarrow \quad f'''(0) = 0$$

$$f^{(iv)}(x) = [12 - 24x^2 + (-2x)(12x - 8x^3)]e^{-x^2} \qquad \Rightarrow \quad f^{(iv)}(0) = 12$$

Thus $(T_4 f)(x) = 1 - \frac{2x^2}{2!} + \frac{12x^4}{4!} = 1 - x^2 + \frac{x^4}{2}$. It appears that we could obtain the Taylor polynomial for the Gaussian by replacing x by $-x^2$ in the Taylor polynomial for e^x. (This gives a polynomial of order $2n$, from the polynomial of order n.

Question 6.

(a) Using a truncation of the geometric series, with x replaced by $-x$, we obtain

$$(T_5 g)(x) = 1 - x + x^2 - x^3 + x^4 - x^5$$

(b) We replace x by x^2 in (b) and note that we do not need any terms of order higher than x^5:

$$(T_5 h)(x) = 1 - x^2 + x^4$$

Question 7. Let $g = f'$. Then

$$(T_n f)(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$(T_n g)(x) = (T_n f')(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n+1)}(a)}{n!}(x-a)^n$$

Now differentiating $(T_n f)(x)$ we obtain

$$(T_n f)'(x) = 0 + f'(a) + \frac{2f''(a)}{2!}(x-a) + \cdots + \frac{nf^{(n)}(a)}{n!}(x-a)^{n-1}$$

$$= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$$

We can conclude that $(T_n f)'(x) = (T_{n-1} f')(x)$.

Question 8. Using $a = 1$, we obtain $f(1) = e^{-1}$ and $f'(1) = -2e^{-1}$. This gives a linear approximation

$$e^{-x^2} \approx e^{-1}(1 - 2(x-1))$$