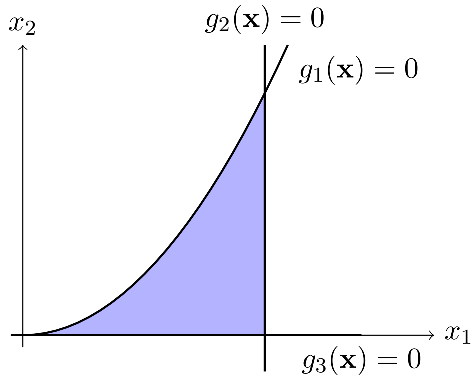
1. The set $\mathcal{G}$ looks like this:



We have

$$\nabla g_1(x_1, x_2) = \begin{pmatrix} -2x_1 \\ 1 \end{pmatrix}, \qquad \nabla g_2(x_1, x_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \nabla g_3(x_1, x_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

We must consider each possible combination of active constraints:

- If there is just one active constraint $\nabla g_i(x_1, x_2) = 0$, then the set $\{\nabla g_i(x_1, x_2)\}$ is linearly dependent provided that $\nabla g_i(x_1, x_2) \neq 0$. This is always true, so does not lead to any non-regular points.
- If there are two active constraints, then this corresponds to the intersection of two of the boundary curves in the diagram above.
  - The point $(x_1, x_2) = (1, 0)$ has active constraints g_2 and g_3.
    The set $\{\nabla g_2(1, 0), \nabla g_3(1, 0)\} = \{\begin{pmatrix}1 & 0\end{pmatrix}^T, \begin{pmatrix}0 & -1\end{pmatrix}^T\}$ is linearly independent, so $(1, 0)$ is regular.
  - The point $(x_1, x_2) = (1, 1)$ has active constraints g_1 and g_2.
    The set $\{\nabla g_1(1, 1), \nabla g_2(1, 1)\} = \{\begin{pmatrix}-2 & 1\end{pmatrix}^T, \begin{pmatrix}1 & 0\end{pmatrix}^T\}$ is linearly independent, so $(1, 1)$ is regular.
  - The point $(x_1, x_2) = (0, 0)$ has active constraints g_1 and g_3.
    The set $\{\nabla g_1(0, 0), \nabla g_3(0, 0)\} = \{\begin{pmatrix}0 & 1\end{pmatrix}^T, \begin{pmatrix}0 & -1\end{pmatrix}^T\}$ is linearly dependent, as $\nabla g_1(0, 0) = -\nabla g_2(0, 0)$, so $(0, 0)$ is non-regular.
- There are no points where all three constraints are active.

So the only non-regular point is $\mathbf{0}$. Observe in the diagram above that the two level-sets $g_1(\mathbf{x}) = 0$ and $g_3(\mathbf{x}) = 0$ do not intersect transversally.

2. We multiply the inequality constraint $g(\mathbf{x}) \leqslant \mathbf{0}$ by -1 to obtain $-\mathbf{g}(\mathbf{x}) \leqslant \mathbf{0}$. The problem is then to minimise $f(\mathbf{x})$ subject to $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ and $\mathbf{q}(\mathbf{x}) \leqslant \mathbf{0}$, where $\mathbf{q}(\mathbf{x}) = -\mathbf{g}(\mathbf{x})$.

   Using $\boldsymbol{\rho}$ in place of $\boldsymbol{\mu}$, the Lagrangian is then

$$\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda}, \boldsymbol{\rho}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x}) + \boldsymbol{\rho}^T \mathbf{q}(\mathbf{x}).$$

   A minimiser will have $\boldsymbol{\rho} \geqslant \mathbf{0}$ and

$$Df(\mathbf{x}) + \boldsymbol{\lambda}^T D\mathbf{h}(\mathbf{x}) + \boldsymbol{\rho}^T D\mathbf{q}(\mathbf{x}) = \mathbf{0} \iff Df(\mathbf{x}) + \boldsymbol{\lambda}^T D\mathbf{h}(\mathbf{x}) - \boldsymbol{\rho}^T D\mathbf{g}(\mathbf{x}) = \mathbf{0},$$

   and taking $\boldsymbol{\mu} = -\boldsymbol{\rho}$ will have $\boldsymbol{\mu} \leqslant \mathbf{0}$.

3. (a) *Note carefully: this is a maximisation problem, so the KKT condition requires* $\boldsymbol{\mu} \leqslant \mathbf{0}$.
   Let $f(\mathbf{x}) = 2x_1^2 + 5x_1 - x_2$ and let $g(\mathbf{x}) = (x_1 + 1)^2 + (x_2 - 3)^2 - 4$. The Lagrangian is

$$\mathcal{L}(\mathbf{x}; \mu) = f(\mathbf{x}) + \mu g(\mathbf{x}) = 2x_1^2 + 5x_1 - x_2 + \mu((x_1 + 1)^2 + (x_2 - 3)^2 - 4).$$

   By setting $D\mathcal{L}(\mathbf{x}; \mu)$ to $\mathbf{0}$ and including the remaining conditions, the KKT condition is the existence of a multiplier μ such that

$$4x_1 + 5 + 2\mu(x_1 + 1) = 0, \tag{1}$$
$$-1 + 2\mu(x_2 - 3) = 0, \tag{2}$$
$$\mu g(\mathbf{x}) = 0, \tag{3}$$
$$g(\mathbf{x}) \leqslant 0, \tag{4}$$
$$\mu \leqslant 0. \tag{5}$$

(b) If $\mu = 0$, then equation (2) implies $-1 = 0$, which is certainly false. Thus, when $\mu = 0$, there is not a maximum.

(c) Equation (1) gives

$$\begin{aligned} 4x_1 + 5 + 2\mu(x_1 + 1) = 0 &\iff x_1(4 + 2\mu) + 5 + 2\mu = 0 \\ &\iff x_1 = -\frac{5 + 2\mu}{4 + 2\mu}, \end{aligned}$$

and equation (2) gives

$$\begin{aligned} -1 + 2\mu(x_2 - 3) = 0 &\iff 2\mu x_2 = 1 + 6\mu \\ &\iff x_2 = \frac{1 + 6\mu}{2\mu}. \end{aligned}$$

As we assume $\mu \neq 0$, equation (3) gives $g(\mathbf{x}) = 0$, and substituting x_1 and x_2 as above gives

$$\begin{aligned} (x_1 + 1)^2 + (x_2 - 3)^2 - 4 = 0 &\iff \left(-\frac{5 + 2\mu}{4 + 2\mu} + 1\right)^2 + \left(\frac{1 + 6\mu}{2\mu} - 3\right)^2 - 4 = 0 \\ &\iff \left(\frac{-5 - 2\mu + 4 + 2\mu}{4 + 2\mu}\right)^2 + \left(\frac{1 + 6\mu - 6\mu}{2\mu}\right)^2 - 4 = 0 \\ &\iff \frac{1}{(4 + 2\mu)^2} + \frac{1}{4\mu^2} - 4 = 0 \\ &\iff 4\mu^2 + (4 + 2\mu)^2 - 16\mu^2(4 + 2\mu)^2 = 0 \\ &\iff -64\mu^4 - 256\mu^3 - 248\mu^2 + 16\mu + 16 = 0 \\ &\iff -8\mu^4 - 32\mu^3 - 31\mu^2 + 2\mu + 2 = 0. \end{aligned}$$

The second-to-last line was obtained using the following MATLAB code:

```
syms mu
expand( 4*mu^2 + (4+2*mu)^2 - 16*mu^2*(4 + 2*mu)^2 )
```

We can now solve by running

```
syms mu
vpasolve(-8*mu^4 - 32*mu^3 - 31*mu^2 + 2*mu + 2)
```

Running `solve(-8*mu^4 - 32*mu^3 - 31*mu^2 + 2*mu + 2)` will also find them exactly.
Without performing the algebra manually, a suitable equation could also be arrived at using the following MATLAB code:

```
syms x1 x2 mu
f = 2*x1^2 + 5*x1 - x2;
g = (x1+1)^2 + (x2-3)^2 - 4;
L = f + mu*g;
DL = jacobian(L, [x1 x2]);
sols = solve(DL, [x1 x2]);
% x/y = 0 provided that x = 0, so extract the numerator only
eqn = numden(subs(g, sols));
disp(eqn);
```

And then solve by running `vpasolve(eqn)`. Running `solve(eqn)` will also find them exactly.
We find the following solutions for μ:

$$\begin{aligned} &-2.2515554759053468954115918536742 \\ &-1.7474014254276885219640631547406 \\ &-0.25259857457231147803593684525942 \\ &0.25155547590534689541159185367416 \end{aligned}$$

(d) Of the three given zeros, the μ_3 is positive so it does not correspond to a maximum. We verify the SOSC for the other two. The Hessian of $\mathcal{L}$ is

$$D^2\mathcal{L}(\mathbf{x};\mu) = \begin{pmatrix} 4+2\mu & 0 \\ 0 & 2\mu \end{pmatrix}.$$

As $\mu \neq 0$, the constraint $g(\mathbf{x}) = 0$ is active. The tangent space to the active constraint is

$$\mathrm{Ker}(Dg(\mathbf{x})) = \mathrm{Ker}\begin{pmatrix} 2(x_1+1) & 2(x_2-3) \end{pmatrix} = \mathrm{Sp}(x_2-3, -x_1-1).$$

For $\mathbf{v} \in \mathrm{Sp}(x_2-3, -x_1-1)$, we have $\mathbf{v} = a\begin{pmatrix} x_2-3 & -x_1-1 \end{pmatrix}^T$, so

$$\mathbf{v}^T D^2\mathcal{L}(\mathbf{x};\mu)\mathbf{v} = a^2 \begin{pmatrix} x_2-3 & -x_1-1 \end{pmatrix} \begin{pmatrix} 4+2\mu & 0 \\ 0 & 2\mu \end{pmatrix} \begin{pmatrix} x_2-3 \\ -x_1-1 \end{pmatrix}.$$

For $\mu_1 = -1.747$ we use MATLAB:

```
mu = -1.747;
x1 = -(5+2*mu)/(4+2*mu);
x2 = (1+6*mu)/(2*mu);
v = [x2 - 3; -x1-1];
hess = [4+2*mu 0; 0 2*mu];
v'*hess*v
```

This shows that

$$\mathbf{v}^T D^2\mathcal{L}(\mathbf{x};\mu_1)\mathbf{v} \approx -13.605a^2,$$

and so $\mu_1 = -1.747$ corresponds to a strict local maximiser.
Similarly, for $\mu_2 = -0.253$, we have

$$\mathbf{v}^T D^2\mathcal{L}(\mathbf{x};\mu_2)\mathbf{v} \approx 13.605a^2,$$

and so $\mu_2 = -0.253$ does not correspond to an optimiser (as $\mu_2 < 0$ would make it a maximiser but $D^2\mathcal{L}(\mathbf{x};\mu_2) > 0$ would make it a minimiser).

4. The KKT condition is

$$\begin{aligned} 2x_1 + \lambda(2x_1+2x_2) + 2\mu x_1 &= 0, & (1)\\ 2x_2 + \lambda(2x_1+2x_2) - \mu &= 0, & (2)\\ \mu(x_1^2 - x_2) &= 0, & (3)\\ x_1^2 + 2x_1x_2 + x_2^2 &= 1, & (4)\\ x_1^2 - x_2 &\leqslant 0, & (5)\\ \mu &\geqslant 0. & (6) \end{aligned}$$

We distinguish two cases.

Case 1: $\mu > 0$. Then (3) gives $x_1^2 - x_2 = 0 \implies x_2 = x_1^2$. Substituting this into (4) gives

$$\begin{aligned} x_1^2 + 2x_1x_2 + x_2^2 = 1 &\iff x_1^2 + 2x_1^3 + x_1^4 = 1 \\ &\iff x_1^2(x_1+1)^2 = 1 \\ &\iff (x_1(x_1+1))^2 - 1 = 0 \\ &\iff (x_1^2+x_1-1)(x_1^2+x_1+1) = 0. \end{aligned}$$

This has two real solutions,

$$x_1 = \frac{-1\pm\sqrt{5}}{2}, \quad x_2 = \frac{3\mp\sqrt{5}}{2}.$$

Subtracting (2) from (1) gives

$$2x_1 - 2x_2 + \mu(2x_1+1) = 0 \iff \mu = \frac{2(x_1-x_2)}{2x_1+1} = -2 \pm \tfrac{4}{5}\sqrt{5} < 0.$$

This violates (6), so this case does not lead to a minimiser.

Case 2: $\mu = 0$. Then subtracting (2) from (1) gives

$$2x_1 - 2x_2 = 0 \iff x_1 = x_2.$$

Substituting into (4) gives

$$x_1^2 + 2x_1^2 + x_1^2 = 1 \iff 4x_1^2 = 1 \iff x_1 = \pm\tfrac{1}{2}.$$

So we have $x_1 = x_2 = \pm\frac{1}{2}$. Inequality (5) is satisfied only when $x_1 = x_2 = \frac{1}{2}$, and so from equation (1) we have

$$2 + 4\lambda = 0 \implies \lambda = -\frac{1}{2}.$$

Thus, our only candidate for a minimiser is $\mathbf{x}^* = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\end{pmatrix}^T$ with $\lambda = -\frac{1}{2}$ and $\mu = 0$.

To see if it is a local minimiser, we use the SOSC.

We have

$$D^2\mathcal{L}(\mathbf{x};\lambda,\mu) = \begin{pmatrix} 2+2\lambda+2\mu & 2\lambda \\ 2\lambda & 2+2\lambda \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

for $\lambda = \frac{1}{2}$ and $\mu = 0$.

The tangent space to $x_1^2 + 2x_1x_2 + x_2^2 = 1$ is

$$\operatorname{Ker}\begin{pmatrix} 2(x_1+x_2) & 2(x_1+x_2) \end{pmatrix} = \operatorname{Sp}(1,-1).$$

For all $a \in \mathbb{R}\backslash\{0\}$ we have

$$\begin{pmatrix} a & -a \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ -a \end{pmatrix} = 4a^2 > 0.$$

Therefore, by the SOSC, $(\frac{1}{2}, \frac{1}{2})$ is a strict local minimizer.