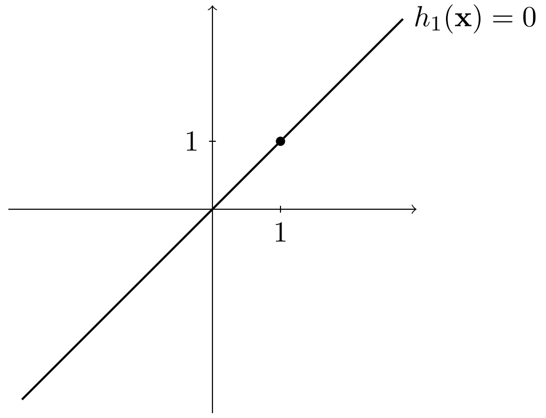
1. (a) As $h_1(1,1) = 0$, the level set is given by $x_1 - x_2 = 0 \iff x_2 = x_1$:

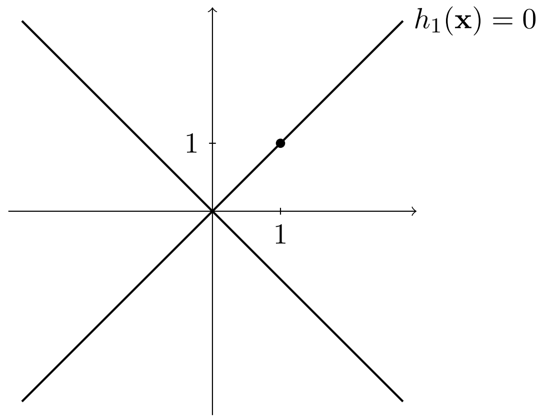


We have

$$\nabla h_1(x_1, x_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which is not the zero vector, so $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ is regular.

(b) As $h_2(1,1) = 0$, the level set is given by $x_1^2 - x_2^2 = 0 \iff |x_2| = |x_1|$:

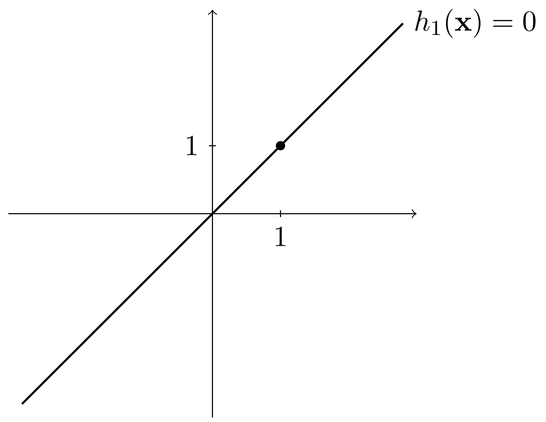


We have

$$\nabla h_2(x_1, x_1) = \begin{pmatrix} 2x_1 \\ -2x_2 \end{pmatrix},$$

and with $x_1 = x_2 = 1$, this is not the zero vector, so $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ is regular.

(c) As $h_3(1,1) = 0$, the level set is given by $(x_1 - x_2)^2 = 0 \iff x_2 = x_1$:



We have

$$\nabla h_2(x_1, x_1) = \begin{pmatrix} 2(x_1 - x_2) \\ -2(x_1 - x_2) \end{pmatrix},$$

and with $x_1 = x_2 = 1$, this is the zero vector, so $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ is not regular.

2. (a) We have

$$\begin{aligned}\mathcal{L}(\mathbf{x};\boldsymbol{\lambda}) &= f(\mathbf{x}) + \lambda_1(h_1(\mathbf{x}) - 3) + \lambda_2(h_2(\mathbf{x}) - 6) \\ &= x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3 + \lambda_1(x_1 + 2x_2 - 3) + \lambda_2(4x_1 + 5x_3 - 6)\end{aligned}$$

(b) We have

$$D^T\mathcal{L}(\mathbf{x};\boldsymbol{\lambda}) = \begin{pmatrix} 2x_1 + 2x_2 + 4 + \lambda_1 + 4\lambda_2 \\ 2x_1 + 6x_2 + 5 + 2\lambda_1 \\ 6 + 5\lambda_2 \\ x_1 + 2x_2 - 3 \\ 4x_1 + 5x_3 - 6 \end{pmatrix} = \begin{pmatrix} 2x_1 + 2x_2 + \lambda_1 + 4\lambda_2 + 4 \\ 2x_1 + 6x_2 + 2\lambda_1 + 5 \\ 5\lambda_2 + 6 \\ x_1 + 2x_2 - 3 \\ 4x_1 + 5x_3 - 6 \end{pmatrix}$$

This is a system of linear equations, which we express in matrix form:

$$\left(\begin{array}{ccccc|c} 2 & 2 & 0 & 1 & 4 & -4 \\ 2 & 6 & 0 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 5 & -6 \\ 1 & 2 & 0 & 0 & 0 & 3 \\ 4 & 0 & 5 & 0 & 0 & 6 \end{array}\right) \equiv \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 16/5 \\ 0 & 1 & 0 & 0 & 0 & -1/10 \\ 0 & 0 & 1 & 0 & 0 & -34/25 \\ 0 & 0 & 0 & 1 & 0 & -27/5 \\ 0 & 0 & 0 & 0 & 1 & -6/5 \end{array}\right),$$

so the solution is

$$\mathbf{x}^* = \begin{pmatrix} 16/5 \\ -1/10 \\ -34/25 \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} -27/5 \\ -6/5 \end{pmatrix}$$

(c) We have

$$\nabla h_1(\mathbf{x}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \nabla h_2(\mathbf{x}) = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix},$$

which are independent, so $\mathbf{x}^*$ is regular.

(d) As $T\mathcal{H}(\mathbf{x}^*) = \mathrm{Ker}(D\mathbf{h}(\mathbf{x}^*))$, we solve

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 4 & 0 & 5 & 0 \end{array}\right) \equiv \left(\begin{array}{ccc|c} 1 & 0 & 5/4 & 0 \\ 0 & 1 & -5/8 & 0 \end{array}\right)$$

to find $T\mathcal{H}(\mathbf{x}^*) = \mathrm{Sp}(-5/4, 5/8, 1) = \mathrm{Sp}(-10, 5, 8)$.

(e) Differentiating with respect to $\mathbf{x}$ only, we have

$$D^2\mathcal{L}(\mathbf{x};\boldsymbol{\lambda}) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $\mathbf{v} \in \mathrm{Sp}(-10, 5, 8)$ then $\mathbf{v} = a\begin{pmatrix} -10 & 5 & 8 \end{pmatrix}^T$, and

$$\mathbf{v}^T D^2\mathcal{L}(\mathbf{x};\boldsymbol{\lambda})\mathbf{v} = a\begin{pmatrix} -10 & 5 & 8 \end{pmatrix}\begin{pmatrix} 2 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} a\begin{pmatrix} -10 \\ 5 \\ 8 \end{pmatrix} = 150a^2,$$

which is positive when $\mathbf{v} \neq \mathbf{0}$. So $D^2\mathcal{L}(\mathbf{x};\boldsymbol{\lambda})$ is positive definite on $T\mathcal{H}(\mathbf{x}^*)$, implying $\mathbf{x}^*$ is a strict local minimiser.

3. The Lagrangian is

$$\mathcal{L}(\mathbf{x};\lambda) = 4x_1 + x_2^2 + \lambda(x_1^2 + x_2^2 - 9).$$

Then,

$$D\mathcal{L}(\mathbf{x};\lambda) = \begin{pmatrix} 4 + 2\lambda x_1 & 2x_2 + 2\lambda x_2 & x_1^2 + x_2^2 - 9 \end{pmatrix}.$$

Setting this equal to $\mathbf{0}$ results in the system of equations

$$2 + \lambda x_1 = 0 \tag{1}$$
$$(\lambda + 1)x_2 = 0 \tag{2}$$
$$x_1^2 + x_2^2 - 9 = 0. \tag{3}$$

From equation (2) we obtain $\lambda = -1$ or $x_2 = 0$, and from equation (1) we obtain $x_1 = -\frac{2}{\lambda}$.

Note that

$$Dh(\mathbf{x}) = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix},$$

which equals zero only when $x_1 = x_2 = 0$. So the only non-regular point is $\mathbf{x} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$.

Case 1: $\lambda = -1$. Then $x_1 = -\frac{2}{\lambda} = 2$, and substituting this into (3) gives

$$4 + x_2^2 - 9 = 0 \implies x_2 = \pm\sqrt{5}.$$

This gives two candidate solutions,

$$\mathbf{x}_1^* = \begin{pmatrix} 2 \\ -\sqrt{5} \end{pmatrix}, \quad \mathbf{x}_2^* = \begin{pmatrix} 2 \\ \sqrt{5} \end{pmatrix}.$$

These are both regular by the observation above. We will find the remaining candidates before classifying them.

Case 2: $\lambda \neq -1$. Then $x_2 = 0$ from equation (2). Substituting this into equation (3) gives $x_1^2 = 9 \implies x_1 = \pm 3$. This gives two candidate solutions,

$$\mathbf{x}_3^* = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \quad \mathbf{x}_4^* = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

These are both regular by the observation above. Using $x_1 = -\frac{2}{\lambda}$ we have $\lambda = \frac{2}{3}$ for $\mathbf{x}_3^*$ and $\lambda = -\frac{2}{3}$ for $\mathbf{x}_4^*$.

To classify them, we need the Hessian of $\mathcal{L}$ (with respect to $\mathbf{x}$ only):

$$D^2\mathcal{L}(\mathbf{x}; \lambda) = \begin{pmatrix} 2\lambda & 0 \\ 0 & 2 + 2\lambda \end{pmatrix},$$

and the tangent space of the level set $\mathcal{H} = \{\mathbf{x} : h(\mathbf{x}) = 9\}$,

$$T\mathcal{H}(\mathbf{x}) = \operatorname{Ker}(Dh(\mathbf{x})) = \operatorname{Ker}\begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix} = \operatorname{Sp}(-x_2, x_1).$$

Now we apply the SOSC:

- For $\mathbf{x}_1^* = \begin{pmatrix} 2 & -\sqrt{5} \end{pmatrix}^T$ and $\lambda = -1$, we have

  $$D^2\mathcal{L}(\mathbf{x}_1^*; -1) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix},$$

  which is indefinite, so we check on the tangent space $T\mathcal{H}(\mathbf{x}_1^*) = \operatorname{Sp}(\sqrt{5}, 2)$. For $\mathbf{v} \in T\mathcal{H}(\mathbf{x}_1^*)$, we have $\mathbf{v} = a\begin{pmatrix} \sqrt{5} & 2 \end{pmatrix}^T$, so

  $$\mathbf{v}^T D^2\mathcal{L}(\mathbf{x}_1^*; -1)\mathbf{v} = a^2 \begin{pmatrix} \sqrt{5} & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{5} \\ 2 \end{pmatrix} = -10a^2.$$

  Hence $D^2\mathcal{L}(\mathbf{x}_1^*; -1)$ is negative definite on $T\mathcal{H}(\mathbf{x}_1^*)$, implying $\mathbf{x}_1^*$ is a strict local maximiser.

- For $\mathbf{x}_2^* = \begin{pmatrix} 2 & \sqrt{5} \end{pmatrix}^T$ and $\lambda = -1$, we have

  $$D^2\mathcal{L}(\mathbf{x}_2^*; -1) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix},$$

  which is indefinite, so we check on the tangent space $T\mathcal{H}(\mathbf{x}_2^*) = \operatorname{Sp}(-\sqrt{5}, 2)$. For $\mathbf{v} \in T\mathcal{H}(\mathbf{x}_2^*)$, we have $\mathbf{v} = a\begin{pmatrix} -\sqrt{5} & 2 \end{pmatrix}^T$, so

  $$\mathbf{v}^T D^2\mathcal{L}(\mathbf{x}_2^*; -1)\mathbf{v} = a^2 \begin{pmatrix} -\sqrt{5} & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\sqrt{5} \\ 2 \end{pmatrix} = -10a^2.$$

  Hence $D^2\mathcal{L}(\mathbf{x}_2^*; -1)$ is negative definite on $T\mathcal{H}(\mathbf{x}_2^*)$, implying $\mathbf{x}_2^*$ is a strict local maximiser.

- For $\mathbf{x}_3^* = \begin{pmatrix} -3 & 0 \end{pmatrix}^T$ and $\lambda = 2/3$, we have

  $$D^2\mathcal{L}(\mathbf{x}_3^*; \tfrac{2}{3}) = \begin{pmatrix} 4/3 & 0 \\ 0 & 10/3 \end{pmatrix},$$

  which is positive definite as its leading principal minors are $\frac{4}{3}$ and $\frac{40}{9}$, implying $\mathbf{x}_2^*$ is a strict local minimiser.

- For $\mathbf{x}_4^* = \begin{pmatrix}3 & 0\end{pmatrix}^T$ and $\lambda = -2/3$, we have

$$D^2\mathcal{L}(\mathbf{x}_4^*; -\tfrac{2}{3}) = \begin{pmatrix}-4/3 & 0 \\ 0 & 2/3\end{pmatrix},$$

which is indefinite, so we check on the tangent space $T\mathcal{H}(\mathbf{x}_4^*) = \mathrm{Sp}(0, 3)$. For $\mathbf{v} \in T\mathcal{H}(\mathbf{x})$, we have $\mathbf{v} = a\begin{pmatrix}0 & 3\end{pmatrix}^T$, so

$$\mathbf{v}^T D^2\mathcal{L}(\mathbf{x}_4^*; -\tfrac{2}{3})\mathbf{v} = a^2\begin{pmatrix}0 & 3\end{pmatrix}\begin{pmatrix}-4/3 & 0 \\ 0 & 2/3\end{pmatrix}\begin{pmatrix}0 \\ 3\end{pmatrix} = 6a^2.$$

Hence $D^2\mathcal{L}(\mathbf{x}_4^*; \frac{2}{3})$ is positive definite on $T\mathcal{H}(\mathbf{x})$, implying $\mathbf{x}_4^*$ is a strict local minimiser.