Prac 2, MATUMDS

Let $F : \underbrace{[a, \infty)}_{\text{set of all possible values of } x \text{ for which function } F \text{ is defined.}} \to \mathbb{R}, \quad F(x) = 1 - \left(\frac{a}{x}\right)^b$

Q1

(a) $F(a) = ?$

$F(a) = 1 - \left(\frac{a}{a}\right)^b = 1 - 1^b = 1 - 1 = 0$

This means that at startin point $x = a$, the function value is 0.

(b) If $x \to \infty$, $\left(\frac{a}{x}\right)^b \xrightarrow[x \to \infty]{} 0$ $(a > 0, b > 0)$

(x gehts very large)

$$\Rightarrow \boxed{F(x) = 1 - \underbrace{\left(\frac{a}{x}\right)^b}_{\to 0} \xrightarrow[x \to \infty]{} 1}$$

This means it does not matter how large x is, the value of function f will be never greater than 1. So if we drow horizontal line $y = 1$, the graph of F will get close to this line, but it will never touch it or cross it (in mathematics we say: the line $y=1$ is an assymtote for the function F)

2 We continue to work with

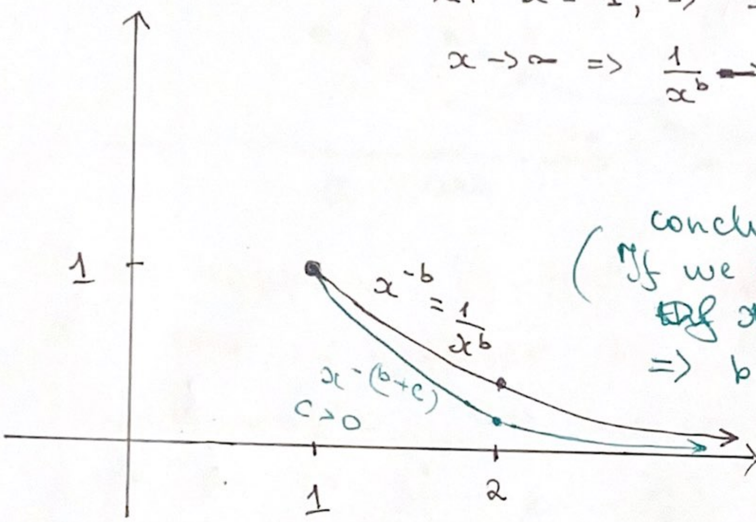
$F : [a, \infty) \to \mathbb{R}$, $F(x) = 1 - \left(\frac{a}{x}\right)^b$, $a, b > 0$

(a) Sketh $x^{-b} = \frac{1}{x^b}$, $x \geq 1$

Let $x = 1$, $\Rightarrow$ $x^{-b} = \frac{1}{x^b} = \frac{1}{1} = 1$

$x \to \infty \Rightarrow \frac{1}{x^b} \to 0$

$(b > 0)$



conclusion
(If we change b, shape ~~of~~ x^{-b} changes $\Rightarrow$ b is called shape parameter)

If we consider to different b, i.e. what will happen to F ?

Let us consider an exampe :

x^{-2} agains x^{-10}

$\frac{1}{x^2}$, $\frac{1}{x^{10}}$, both function will start at point 1, 1,
But for example at $x = 2$, $\frac{1}{2^2}$ will be much bigger than $\frac{1}{2^{10}}$
and similarly for any $x > 1$, i.e. $\frac{1}{x^{10}}$ will go to the line $y = 0$ (x-axis) much faster then $\frac{1}{x^2}$. In general if we consider $\frac{1}{x^b}$, $\frac{1}{x^{b+c}}$ $(c > 0)$, $\frac{1}{x^{b+c}}$ will go to 0 faster than $\frac{1}{x^b}$

So the shape of x^{-b} will be diffenent from $x^{-(b+c)}$

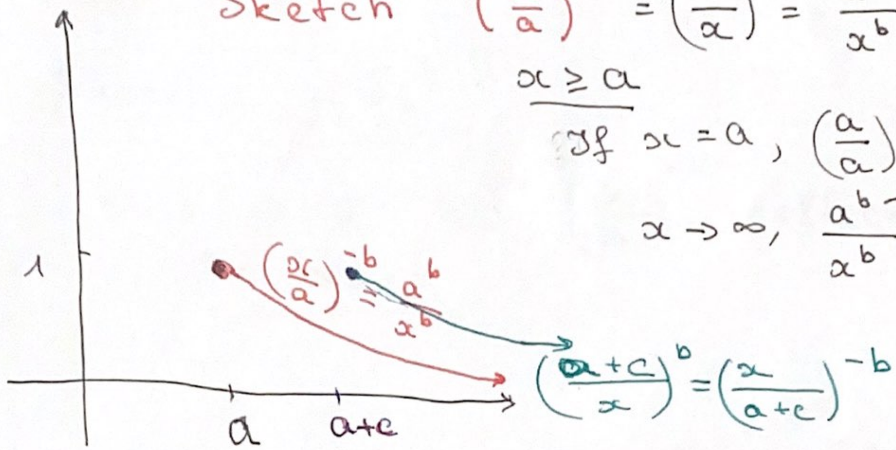
Q2 (continue)

(b) Sketch $\left(\frac{x}{a}\right)^{-b} = \left(\frac{a}{x}\right)^{b} = \frac{a^b}{x^b}$ (const)

$x \geq a$

If $x = a$, $\left(\frac{a}{a}\right)^{-b} = 1$

$x \to \infty$, $\frac{a^b}{x^b} \xrightarrow[x\to\infty]{} 0$ (a^b - constant)



What happens if we chang 2 different a, one is greater than other.

For example Let us consider

$\left(\frac{x}{2}\right)^{-b} = \left(\frac{2}{x}\right)^{+b} = \frac{2^b}{x^b}$ & $\left(\frac{x}{5}\right)^{-b} = \left(\frac{5}{x}\right)^{b} = \frac{5^b}{x^b}$

$x \geq 2$ $\qquad$ $x \geq 5$

$\left(\frac{x}{2}\right)^{-b}$ will start at $x = 2$,

and $\left(\frac{x}{5}\right)^{-b}$ will start first at $x = 5$,

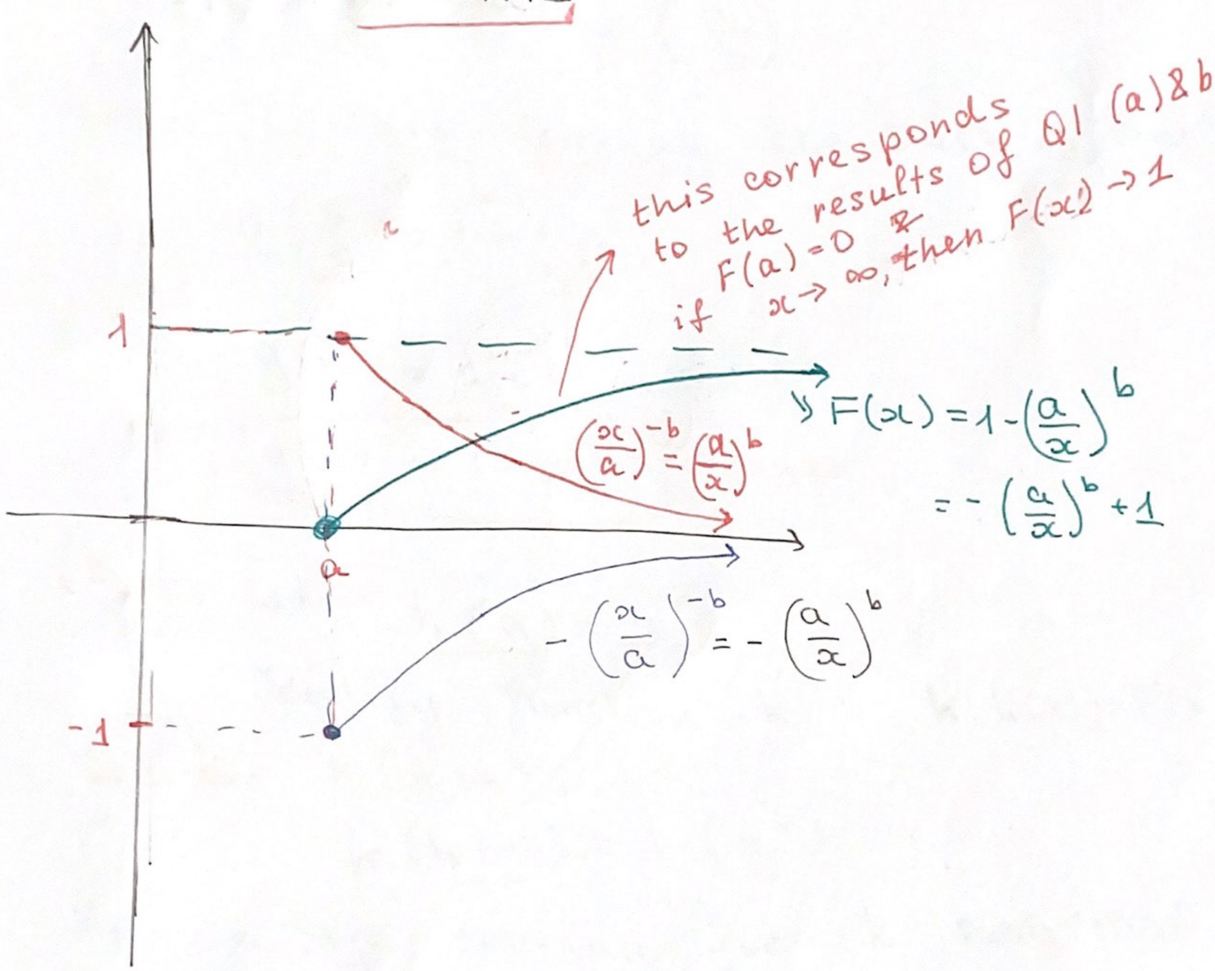
that means $\left(\frac{x}{5}\right)^{-b}$ will be skaled horisontaly with respecto $\left(\frac{x}{2}\right)^{-b}$.

Simmilary it will be in general case.

Q2 (continue)

(c) Sketch $-\left(\frac{x}{a}\right)^{-b}$

this will be $\left(\frac{x}{a}\right)^{-b}$ reflected in x-axis



(d) To obtain graph of $F(x) = 1-\left(\frac{a}{x}\right)^{b}$

$$= -\left(\frac{a}{x}\right)^{b} + \underline{\underline{1}}$$

we have to take graph $-\left(\frac{a}{x}\right)^{b}$ (blue curve) and move it by 1 unit up.

3 (a) Let $\underline{f(x)} = \underline{\frac{a}{x}}$, $F(x) = 1 - \underbrace{\left(\frac{a}{x}\right)^b}_{(*)}$, $x \geq a$

Identify such g that

$$g(f(x)) = F(x).$$

Consider $g(f(x))$:

$$g(\underline{f(x)}) = g\left(\underline{\frac{a}{x}}\right)$$

Therefore we are looking for such g, that

$$g\Big(\underbrace{\frac{a}{x}}_{u}\Big) = \overbrace{1 - \Big(\underbrace{\frac{a}{x}}_{u}\Big)^b}^{(*) = F(x)}$$

$$\Rightarrow \quad g(u) = 1 - u^b$$

$$\Rightarrow \quad \boxed{g(x) = 1 - x^b} \text{ - Answer}$$

(b)

Let $\underline{h(x)} = \underline{x^b}$

Identify function k : $k(h(x)) = F(x)$

Consider $k(h(x))$:

$$k(\underline{h(x)}) = k(\underline{x^b})$$

So we are looking for k, such that

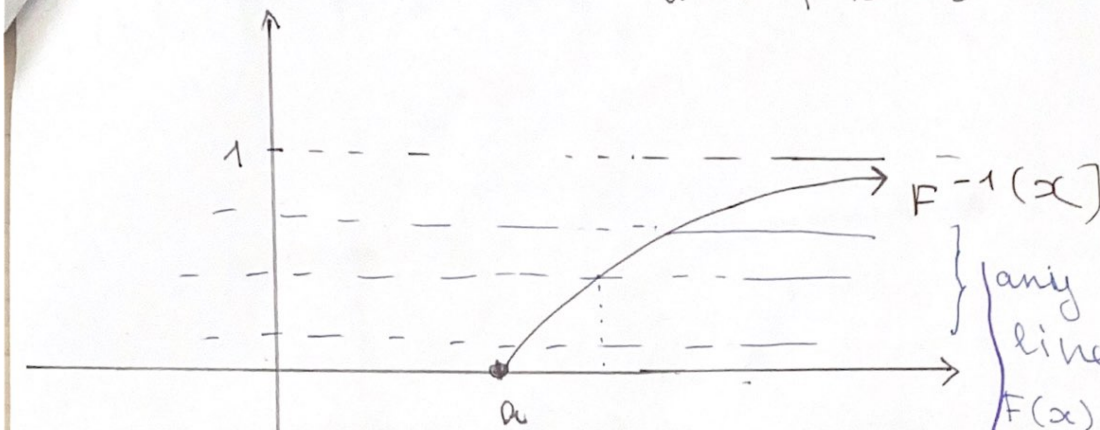
$$k(\underbrace{x^b}_{u}) = \underbrace{1 - \left(\frac{a}{x}\right)^b}_{(*)\ F(x)} = 1 - \frac{a^b}{\underbrace{x^b}_{u}}$$

$$\Rightarrow \quad k(u) = 1 - \frac{a^b}{u}$$

Or $\boxed{k(x) = 1 - \frac{a^b}{x}}$ - Answer

Q4 $F : [a, \infty) \to \mathbb{R}$, $F = 1 - \left(\frac{a}{x}\right)^b$

$a > 0$, $b > 0$



any horisontal line crosses $F(x)$ at most once, so F^{-1} (inverse) exist. (every value on vert. axis we can trace back to only one input)

Input values of F^{-1} are from $[0, 1)$ (so they are output values of F)

Output values of F^{-1} are from $[a, \infty)$, so they are input values of F.

In general:

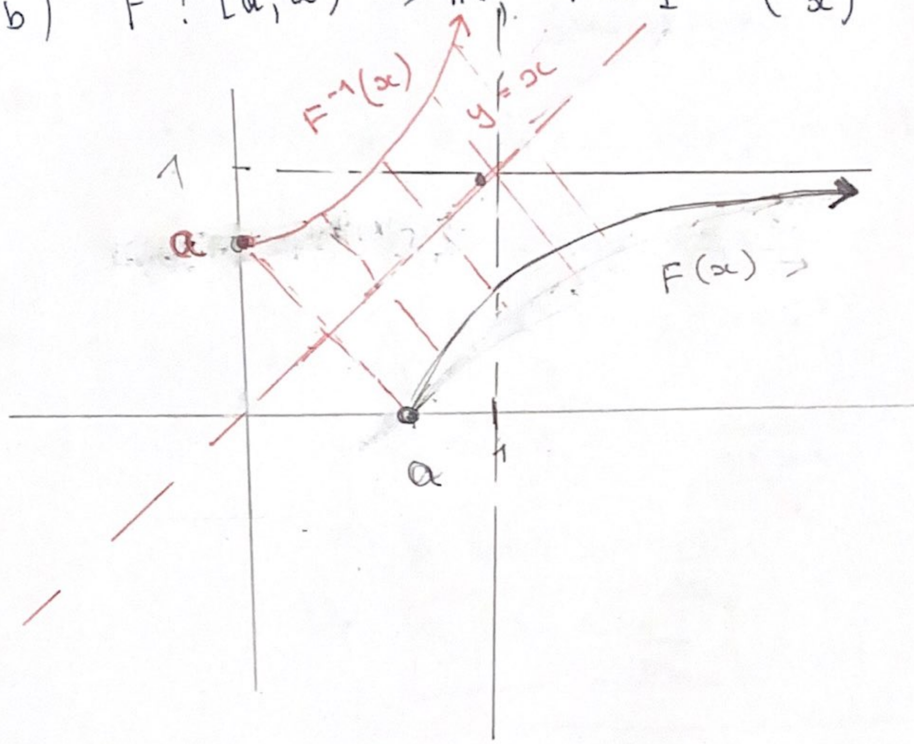
F^{-1} (invers function) exists is F is one-to-one function i.e. is every value on vertical axis can be traced back to only one input value.

Easy way to test it: if any horisontal line crosses a function at most once, then inverse exists.

If there exist a horisontal line, which crosses the graph more than once, then inverse does not exist.

Q4 (continue)

(b) $F : [a, \infty) \to \mathbb{R}$, $F = 1 - \left(\frac{a}{x}\right)^b$



Q4 (c)

By def. of inverse $F(\underbrace{F^{-1}(x)}_{y}) = x$

Let $y = F^{-1}(x)$, so $F(y) = x$

$$\iff 1 - \left(\frac{a}{y}\right)^b = x$$

$$\iff -\left(\frac{a}{y}\right)^b = x - 1$$

$$\iff \left(\frac{a}{y}\right)^b = -x + 1$$

$$\iff \frac{a}{y} = (-x+1)^{\frac{1}{b}}$$

$$\boxed{F^{-1}(x) = \frac{a}{(1-x)^{1/b}}} \Leftarrow y = \frac{a}{(1-x)^{1/b}} \Leftarrow \frac{y}{a} = \frac{1}{(1-x)^{1/b}}$$

Q5. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$

(Cauchy distribution function)

(a) find g & h: $f(x) = g(h(x))$

For example: $\boxed{h(x) = 1 + x^2}$

$\Rightarrow g(h(x)) = g(1+x^2)$

So we are looking for g:

$$g(\underbrace{1+x^2}_{u}) = \frac{1}{\underbrace{1+x^2}_{u}}$$

$$\Rightarrow g(u) = \frac{1}{u}$$

Or $\boxed{g(x) = \frac{1}{x}}$

Answ: (One exampe)
$$\boxed{\begin{array}{l} h(x) = 1 + x^2 \\ g(x) = \frac{1}{x}, \quad \text{then } f(x) = g(h(x)) \end{array}}$$

This answer is not unique

(b) Let $k(x) = 1 + x^2$

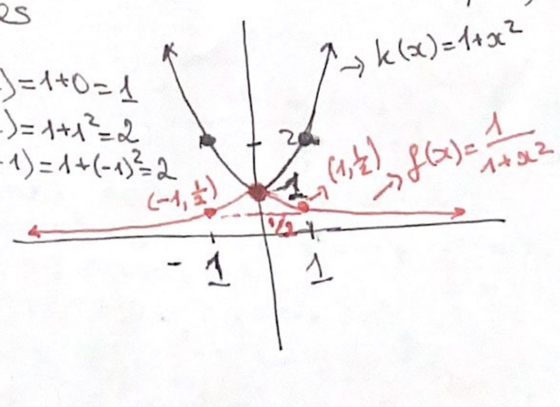
The graph is quadr. function wit T.P at $(0, 1)$

Som values of $k(x)$:

$x = 0 \Rightarrow k(0) = 1 + 0 = 1$

$x = 1 \Rightarrow k(1) = 1 + 1^2 = 2$

$x = -1 \Rightarrow k(-1) = 1 + (-1)^2 = 2$



Some values of $f(x)$:

$x = 0 \Rightarrow f(0) = \frac{1}{1+0^2} = 1$

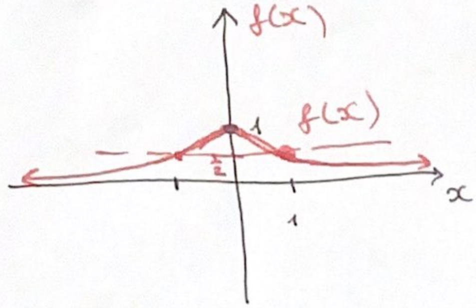
$x = 1 \Rightarrow f(1) = \frac{1}{1+1^2} = \frac{1}{2}$

$x = -1 \Rightarrow f(-1) = \frac{1}{1+(-1)^2} = \frac{1}{2}$

$x \to \infty \Rightarrow f(x) \to 0$

$x \to -\infty \Rightarrow f(x) \to 0$

15 (c) Does $f(x)$ have inverse



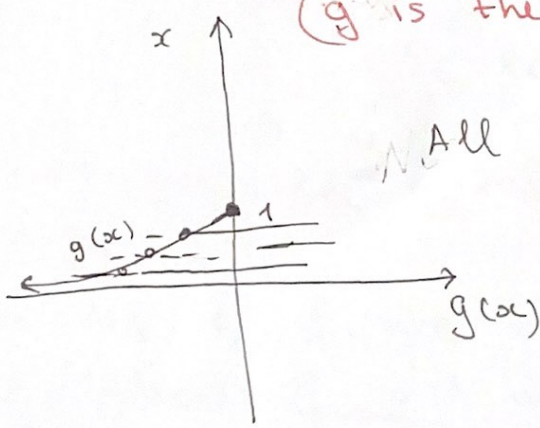
$f(x)$ does not have inverse since for $y = \frac{1}{2}$, there are two input values $x = -1$ and $x = 1$

(A horisont. line $y = \frac{1}{2}$ crosses $f(x)$ twice).

set of inputs of g.

Consider $g: (-\infty, 0) \to \mathbb{R}$, $g(x) = \frac{1}{1+x^2}$

(g is the lef handside of f)



All horis. line cross $g(x)$ at most once, so g^{-1} exists. (inverse)

(d) By definition $g(g^{-1}(x)) = x$

Let $g^{-1}(x) = y \Rightarrow g(y) = x$

$\Rightarrow \frac{1}{1+y^2} = x$ def. of g

$\Rightarrow 1 + y^2 = \frac{1}{x}$

$\Rightarrow y^2 = \frac{1}{x} - 1$

$\Rightarrow y = \pm\sqrt{\frac{1}{x} - 1}$

No possible to have 2 options for g^{-1}.

How to decide which option to take for g^{-1}: $+\sqrt{\frac{1}{x} - 1}$ or $-\sqrt{\frac{1}{x} - 1}$?

We know that outputs set of invers must be the same as input of original. So input set of $g = (-\infty, 0)$, $\Rightarrow$ outputs of g^{-1} must be from $(-\infty, 0) \Rightarrow \boxed{g^{-1}(x) = -\sqrt{\frac{1}{x} - 1}}$

MAT4MDS Prac 2 (10)

Q6 $f: [0, \infty) \to \mathbb{R}$, $f(x) = e^{-x^2}$ (1)

Find f^{-1}:

By def. $f(f^{-1}(x)) = x$

Let $y = f^{-1}(x) \Rightarrow f(y) = x$

$$\underset{(1)}{\Rightarrow} e^{-y^2} = x$$

$$\log_e e^{-y^2} = \log_e x$$

$$-y^2 = \log_e x$$

$$y^2 = -1\log_e x$$

or $y^2 = \log_e x^{-1}$

$$y^2 = \log_e \frac{1}{x}$$

$$\Rightarrow y = \pm\sqrt{\log_e \frac{1}{x}}$$

It is not possible to have two options for f^{-1}!

So from def. of f, we know that outputs values of f^{-1} must be from $[0, \infty)$ (set of inputs of f)

$$\Rightarrow f^{-1}(x) = +\sqrt{\log_e \frac{1}{x}}$$