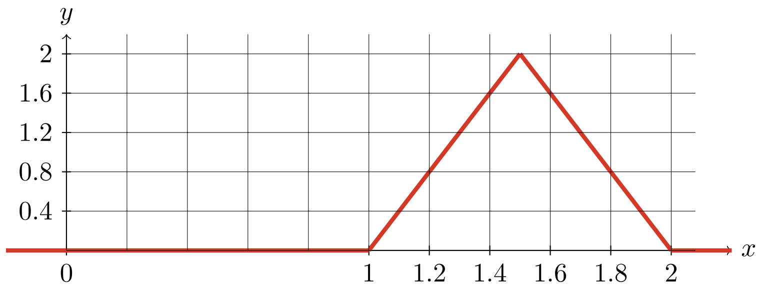
STM4PSD – Workshop 5

1. Consider the function $f(x)$ defined below, with its graph shown on the right.

$$f(x) = \begin{cases} 4x - 4 & \text{if } 1 \leqslant x \leqslant 1.5, \\ 8 - 4x & \text{if } 1.5 \leqslant x \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$



Suppose that X is a random variable whose probability density function is given by $f(x)$.

(a) Verify that $f(x)$ is a valid probability density function.

(b) Which of the following is more likely: $X \approx 1.2$ or $X \approx 1.6$? Explain.

(c) For each of the following, determine the appropriate region of the graph needed to calculate the probability. Then, use facts about triangles to calculate each probability.

i. $P(X \leqslant 1.4)$

ii. $P(X \geqslant 1.7)$

iii. $P(1.4 \leqslant X \leqslant 1.6)$

iv. $P(X \leqslant 1.6)$ **Hint**: there is a way to do this using just one triangle.

(d) Suppose $1 \leqslant x \leqslant 1.5$. Give a general formula for $P(X \leqslant x)$.

(e) Suppose $1.5 < x \leqslant 2$. Give a general formula for $P(X \leqslant x)$.

(f) Combine the previous two answers to give a general formula for the cumulative density function of X. Make sure the formula uses cases, and don't forget the "otherwise". Then, sketch its graph.

2. Suppose $Z \sim U(0, 2)$, and let $f_Z(x)$ denote the probability density function for Z.

(a) Sketch the graph of $f_Z(x)$ on an appropriate interval.

(b) Determine $P(Z \geqslant 1.5)$.

(c) Suppose six random numbers are independently generated according to a $U(0, 2)$ distribution.

i. What is the probability that at least one of the numbers is greater than 1?

ii. What is the probability that all six are greater than 1?

Hint: think of "greater than 1" as a "success".

3. Let $X \sim \mathrm{Exp}(2)$.

(a) Sketch the graph of the density function of X. Label the y-intercept.

(b) On the graph, shade in the region corresponding to $P(1 \leqslant X \leqslant 3)$.

(c) Using an appropriate formula from Readings 5.6, calculate the following, giving exact answers:

(i) $P(X \leqslant 1)$ (ii) $P(X \leqslant 3)$ (iii) $P(1 \leqslant X \leqslant 3)$

(d) Use the formulas from Readings 5.6 to determine the mean, median and mode of X. Then, on your graph, label the points corresponding to these values.

4. Let $X \sim \mathrm{Gamma}(4, 1)$.

(a) What are the values of the parameters k and θ?

(b) The formula for the density function of X requires the number $\Gamma(4)$. Determine $\Gamma(4)$.
Hint: the gamma function has an important property involving factorials.

(c) Use your answer to (a) to write down the formula for the density function of X.

(d) State the values of $E(X)$ and $\mathrm{Var}(X)$.

Poisson processes and queues

5. Suppose that cars arrive northbound to a roundabout in peak hour traffic at a rate of $\lambda = 3$ cars per minute. Assume that these cars arrive according to a Poisson process.
   (a) What is the probability that there are no cars in the first 2 minutes of peak hour?
   (b) What is the probability that at least two cars arrive in the first 3 minutes of peak hour?
   (c) Given that no car has arrived for the first 3 minutes of peak hour, what is the probability that no cars arrive in the next 2 minutes?
6. Continuing Question 5, accounting for other traffic, northbound cars must queue behind other cars before they can enter the roundabout. On average, a car will wait 10 seconds before they can enter the roundabout. Assume that this system can be modelled as an M/M/1 queue.
   (a) State the arrival rate λ, including appropriate units.
   (b) Based on the average waiting time, what is the departure rate μ? Ensure the units of measurement match the units in part (a).
   (c) Determine the traffic intensity ρ.
   (d) Determine the long term average number of cars in the queue.
   (e) Determine the expected amount of time a car spends in the queue before it enters the roundabout.
   (f) Determine the expected waiting time for a car before it gets to the front of the queue.
   (g) Suppose that the local council fears that nearby road works will increase the amount of traffic arriving northbound to the roundabout. This potential increase in traffic would result in an increased arrival rate λ. Assume that the departure rate μ is unchanged.
      i. What would happen if $\lambda \geqslant 6$?
      ii. Write down the total time spent in the queue in terms of λ.
      iii. The council will not take any additional steps to prevent congestion unless the expected time that a car spends in the queue exceeds one minute. What is the largest arrival rate that can be allowed before the council takes additional steps?
7. Cars arrive at a drive through restaurant according to a Poisson process, with a rate of 19 cars per hour. From the moment an individual begins placing their order, it takes an average time of 2.5 minutes before they pick up and pay for their order from the next window.

   Assume that this time taken follows an exponential distribution. There is just one service window; hence we can model this scenario as an M/M/1 queue. Assume that the queue has been running for long enough that its steady state properties apply.
   (a) Write down the service rate μ and the arrival rate λ for this queue (including units).
   (b) What is the long term average number of customers in the system?
   (c) On average, after arriving at the queue, how long will a customer need to wait before they reach the drive through speaker?
   (d) On average, after arriving at the queue, how long will a customer need to wait before they leave the restaurant?
   (e) Suppose that, once the drive through contains 3 cars or more, the queue will extend beyond the restaurant and start impeding traffic. The council has received several complaints from locals, and has informed the restaurant that fines will be imposed if traffic is impeded for more than 30% of the time the drive through is in operation.
   Let N denote the number of cars in the drive through at any one time.
      i. With the current value of ρ, should the restaurant be concerned about being fined?
      ii. Determine a suitable value of ρ that would be best for the restaurant.

   Hint: the probability distribution for the long-term number of individuals in an M/M/1 queue is given in Result 6.6.2.