Question 1. *Total: 7 marks*

Consider the matrix

$$M = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 64 & 0 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of M. *4 marks*

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 64 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ 64 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3-\lambda)\left[(4-\lambda)^2 - 64\right] = 0$$

$$\Rightarrow \quad (3-\lambda)(4-\lambda-8)(4-\lambda+8) = 0$$

$$\Rightarrow \quad (3-\lambda)(-\lambda-4)(12-\lambda) = 0$$

So eigenvalues are $3, -4$ and 12.

(b) *3 marks*

(i) Is M of full rank? Why or why not?

M has 3 non-zero eigenvalues, and M is 3x3,
so M <u>is</u> of full rank.

(ii) Is M invertible? Why or why not?

Yes M is invertible because $\det(m) \neq 0$

$(\det(m) = -4 \times 3 \times 12 \neq 0)$

Question 2. *Total: 10 marks*

Four portfolios of similar stocks were observed in 2001 and 2002. Their percentage returns in both years are given in the following table – profits are positive, losses are negative. It is plausible that there is an underlying linear relationship between the returns given by $y = \alpha x + \beta$ where x is the return in 2001, and y is the return in 2002.

Return 2001 (%)	-2	-1	0	3	x
Return 2002 (%)	1	-3	5	9	y

(a) Fill in the entries in the matrix-vector equation $AX = b$ which would correspond to a single line passing through all 4 points. *2 marks*

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -3 \\ 5 \\ 9 \end{bmatrix}$$

(b) Explain why the equation $AX = b$ does not have a solution. *1 mark*

The points do not lie exactly on a straight line.

(c) Write out the modified 2×2 matrix-vector equation which uniquely determines the coefficients of the linear least squares line of best fit, $y = \alpha x + \beta$. *2 marks*

$$A^T A X = A^T b$$

$$\begin{bmatrix} -2 & -1 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 3 & 1 \end{bmatrix} X = \begin{bmatrix} -2 & -1 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \\ 9 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 14 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2+3+27 \\ 1-3+5+9 \end{bmatrix} = \begin{bmatrix} 28 \\ 12 \end{bmatrix}$$

(d) Solve the system you found in the previous part for α and β, and hence write down the linear least squares line of best fit. *1 mark*

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{56} \begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} 28 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4 \times 28}{56} \\ \frac{14 \times 12}{56} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Thus $y = 2x + 3$

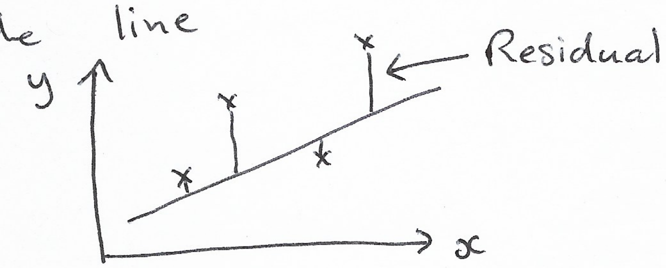
(e) What is the best estimate of the return in 2002 of a similar portfolio of stocks which increased by 2% in 2001. *1 mark*

$y = 2 \times 2 + 3 = 7$

7% is estimate given by LLSR.

(f) Briefly and carefully explain what is meant by linear least squares line of best fit: illustrate your answer with a sketch. *2 marks*

The line of best fit minimises the sum of squares of the "residuals" – the difference between the data points and the line



(g) Suppose there is a portfolio of stocks which increased by 1% in 2002. Explain briefly why the linear least squares regression line you obtained above is not appropriate to use to estimate the return in 2001. *1 mark*

The y values can't be used to estimate x values, because the LLSR line minimises vertical residuals. A different line would minimise horizontal residuals.

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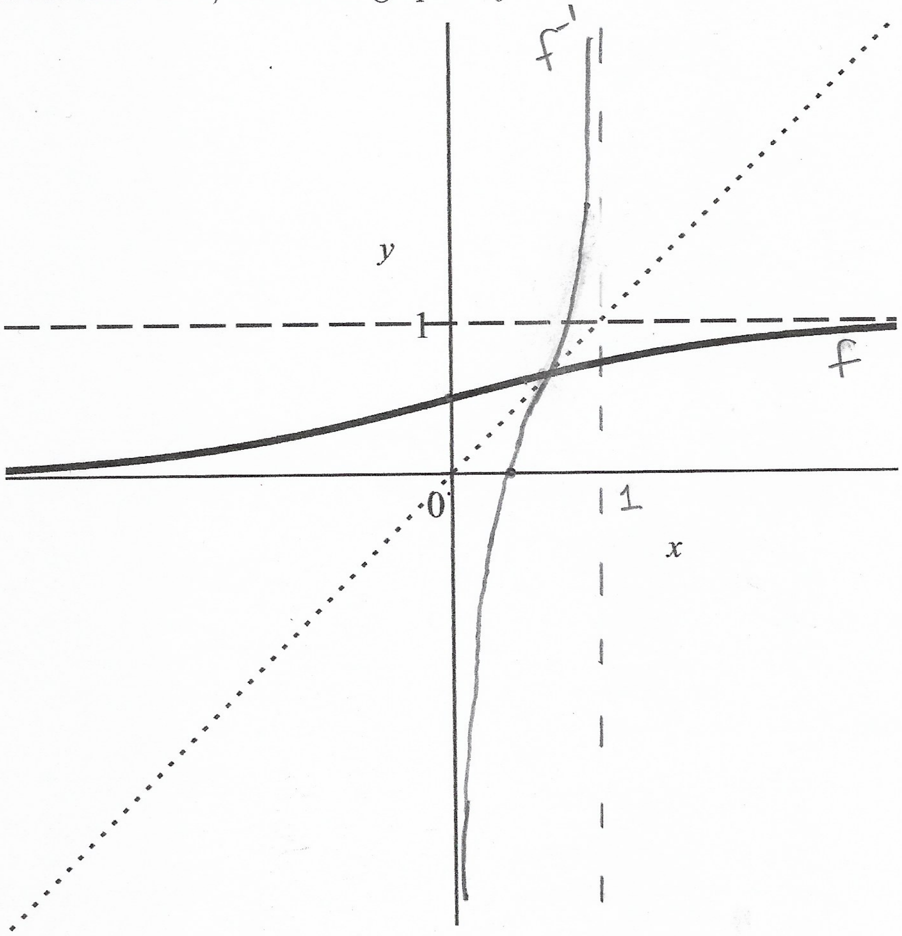
Question 3. *Total: 7 marks*

Let

$$f(x) = \frac{e^x}{1 + e^x}$$

(a) The graph of f is shown on the axes below (solid line). Also indicated is the line $y = x$ (dotted) and an asymptote of f (dashed).

On the same axes, sketch the graph of f^{-1}. *3 marks*



Reflection of graph in line $y = x$.

(b) Find the rule for the inverse function f^{-1}, where *3 marks*

$$f(x) = \frac{e^x}{1+e^x}$$

Let $y = f^{-1}(x)$. Then $f(f^{-1}(x)) = x$

$\Rightarrow \quad x = \dfrac{e^y}{1+e^y}$

$\Rightarrow \quad x + xe^y = \cancel{e}^y$

$\Rightarrow \quad e^y(x-1) = -x$

$\Rightarrow \quad e^y = \dfrac{x}{1-x}$

$\Rightarrow \quad y = \log_e\left(\dfrac{x}{1-x}\right)$

$\Rightarrow \quad f^{-1}(x) = \log_e\left(\dfrac{x}{1-x}\right)$

(c) State the domain (allowed input values) of f^{-1}. *1 mark*

$\operatorname{dom}(f^{-1}) = \operatorname{ran}(f) = (0,1)$

(Either from graph <u>or</u> from (b).)

Question 4. *Total: 5 marks*

(a) Using an appropriate substitution, find an antiderivative of $\frac{x}{x^2+1}$. *2 marks*

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\frac{du}{dx}\,dx$$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\log_e(u) + C$$

(b) By an appropriate method, find an antiderivative of $\frac{x^2}{x+1}$. *3 marks*

$$\int \frac{x^2}{x+1}\,dx = \int \frac{x^2+x-x}{x+1}\,dx$$

$\frac{x^2+x}{x+1}$

$= \frac{x\cancel{(x+1)}}{\cancel{(x+1)}}$

$$= \int x - \frac{x}{x+1}\,dx$$

$$= \frac{x^2}{2} - \int \left(\frac{x+1}{x+1}\right) - \frac{1}{x+1}\,dx$$

1

$$= \frac{x^2}{2} - x + \log_e|x+1| + C$$

Question 5. *Total: 15 marks*

Consider the function f (which is a probability density function) with rule

$$f(y) = \begin{cases} \dfrac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

in which α and β are both positive parameters.

(a) *3 marks*

(i) Is α a shape parameter or a scale parameter? Explain.

α appears as $y^{\alpha-1}$ so it determines the shape of $f(y)$

(ii) Is β a shape parameter or a scale parameter? Explain.

β appears as $(\beta y)^\alpha e^{-(\beta y)}$ ie (βy) together. β determines the scale.

(b) (i) Calculate *2 marks*

$$\int_{-\infty}^{\infty} y f(y) dy$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^\alpha e^{-\beta y} dy$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty u^\alpha e^{-u} \frac{1}{\beta} du \qquad u = \beta y \qquad \frac{dy}{du} = \frac{1}{\beta}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta}$$

$$= \frac{\alpha}{\beta} \quad \text{because } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

From fact sheet:

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

(ii) Calculate *2 marks*

$$\int_{-\infty}^{\infty} y^2 f(y) dy$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-\beta y} dy$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{u}{\beta}^{\alpha+1} e^{-u} \frac{1}{\beta} du$$

$$= \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\beta^2}$$

$$= \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\beta^2\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\beta^2}$$

(iii) Using your answers to (i) and (ii), show that the variance associated with this probability distribution is $\frac{\alpha}{\beta^2}$. *3 marks*

$$E(y^2) - [E(y)]^2 = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2$$

$$= \frac{\alpha}{\beta^2}$$

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(c) It can be shown that

$$f'(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-2} e^{-\beta y} [\alpha - 1 - \beta y]$$

so that there is a stationary point at $y = \frac{\alpha - 1}{\beta}$.

(Note: This information is being given to you, and you do not have to show it.)

Using the extended product rule, and the second derivative test, show that this point is a maximum (for $\alpha > 1$). *5 marks*

$$f''(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \Big[(\alpha-2) y^{\alpha-3} e^{-\beta y} [\alpha-1-\beta y]$$
$$- \beta y^{\alpha-2} e^{-\beta y} [\alpha-1-\beta y]$$
$$+ y^{\alpha-2} e^{-\beta y} [-\beta] \Big]$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-3} e^{-\beta y} \Big[(\alpha-2)(\alpha-1-\beta y)$$
$$- \beta(\alpha-1-\beta y)y - \beta y \Big]$$

For $y = \frac{\alpha-1}{\beta}$

$$f''(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\alpha-1}{\beta}\right)^{\alpha-3} e^{-\cancel{\beta}\frac{(\alpha-1)}{\cancel{\beta}}} [0 - 0 - (\alpha-1)]$$

< 0 for $\alpha - 1 > 0$

By 2nd deriv. test, since $f''\left(\frac{\alpha-1}{\beta}\right) < 0$, the stationary point is a maximum.

Question 6. *Total: 11 marks*

Consider the function of two variables

$$f(x,y) = xe^{-y^2} + x^2y.$$

(a) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ *2 marks*

$$\frac{\partial f}{\partial x} = e^{-y^2} + 2xy$$

$$\frac{\partial f}{\partial y} = -2yxe^{-y^2} + x^2$$

(b) Find $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ *3 marks*

$$\frac{\partial^2 f}{\partial x^2} = 2y$$

$$\frac{\partial^2 f}{\partial y^2} = -2xe^{-y^2} + 4y^2xe^{-y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = -2ye^{-y^2} + 2x$$

(c) Hence find the second order Taylor polynomial for $f(x,y)$ near $(1,0)$. *4 marks*

$f(1,0) = 1 \qquad \dfrac{\partial f}{\partial x}(1,0) = 1 \qquad \dfrac{\partial f}{\partial y} = 1$

$\dfrac{\partial^2 f}{\partial x^2} = 0 \qquad \dfrac{\partial^2 f}{\partial y^2} = -2 \qquad \dfrac{\partial^2 f}{\partial x \partial y} = 2$

$$T_2 f(x,y) \approx 1 + (x-1) + y + \frac{1}{2}\left[-2y^2 + 4y(x-1)\right]$$

(d) Find *2 marks*

$$\int_0^6 f(x,y)\,dx$$

$$= \int_0^6 x e^{-y^2} + x^2 y \, dx$$

$$= \left[\frac{x^2}{2} e^{-y^2} + \frac{x^3}{3} y \right]_0^6$$

$$= 18 e^{-y^2} + \frac{6 \cdot 36}{3} y$$

$$= 18 e^{-y^2} + 72y$$

*** * * * End of Questions * * * ***