Recap: density versus distribution

A continuous random variable X has two functions associated to it:

- a probability **density** function, and
- a probability **distribution** function.

A probability density function describes the relative likelihood of events.

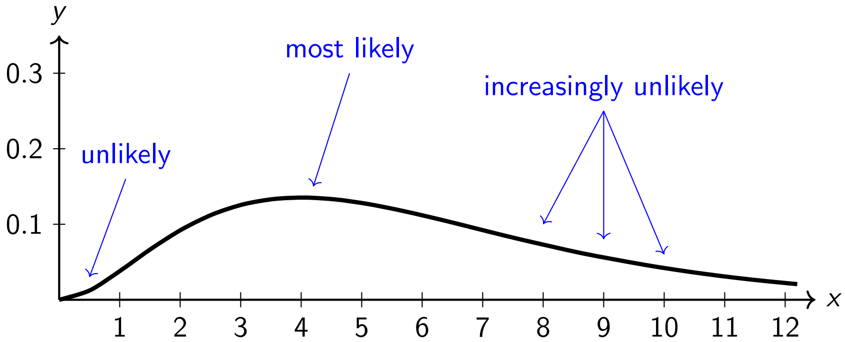
A probability distribution function determines the probability $P(X \leqslant x)$.

In formal terms:

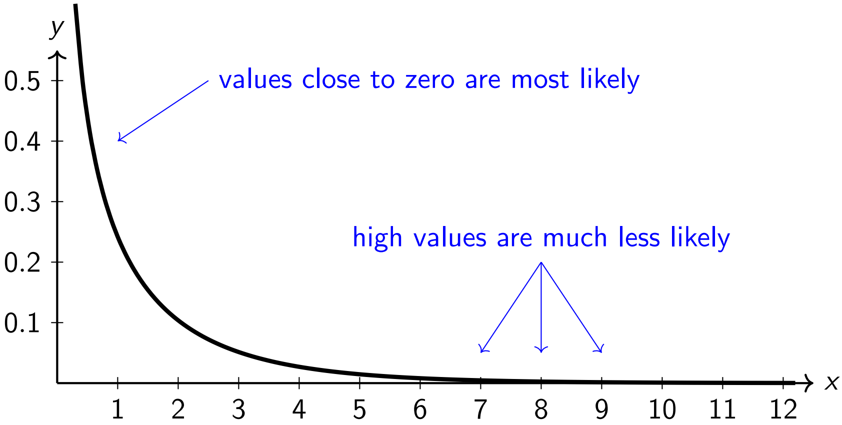
- the density function is the derivative of the distribution function,
- the distribution function is an integral of the distribution function.

Recap: density functions

Probability density functions measure relative likelihood:



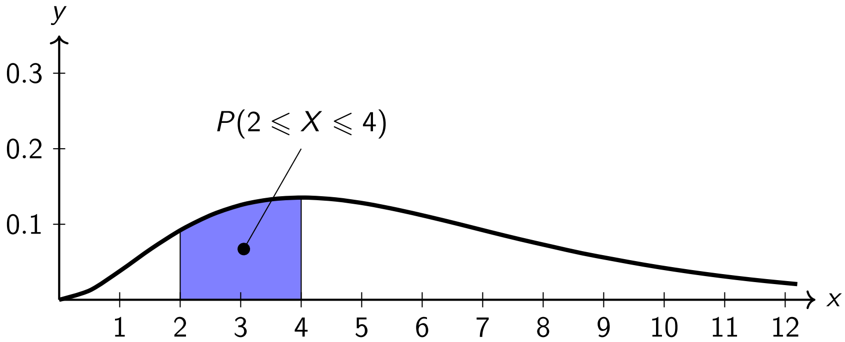
Recap: density functions



But probabilities can't be directly read from the graph.

Recap: probability is area

The area under the density function is used to determine probabilities.



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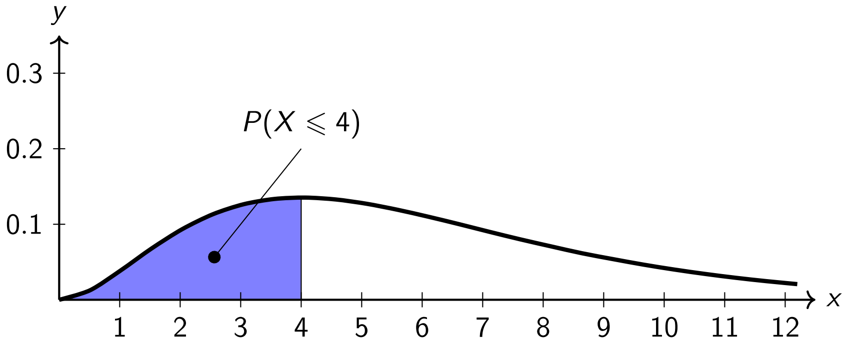

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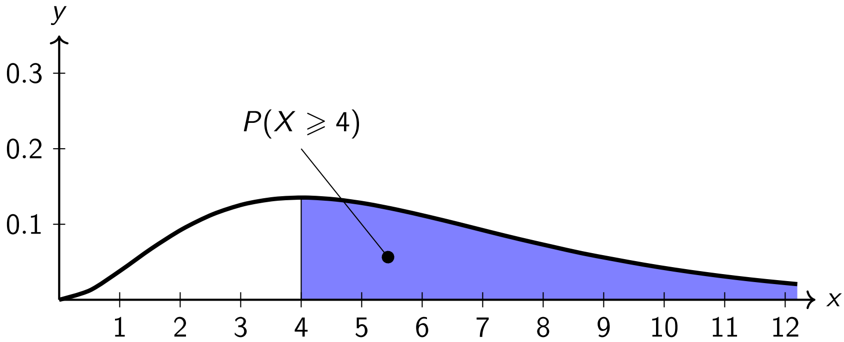

Recap: area uses integrals

Calculating area is generally done using integrals. For a probability density function $f_X(x)$:

$$P(a \leqslant X \leqslant b) = \text{area from } a \text{ to } b = \int_a^b f_X(x)\,dx$$

$$P(X \geqslant a) = \text{entire area right of } a = \int_a^\infty f_X(x)\,dx$$

$$P(X \leqslant a) = \text{entire area left of } a = \int_{-\infty}^a f_X(x)\,dx$$

In this subject, we use integrals as a notational aid. Calculating integrals for a lot of probability density functions is sometimes mathematically impossible. For this reason it is more suitable to numerically approximate them using a computer.

Recap: probability distribution functions

Probability is represented by a probability distribution function. If X is a random variable, then

$$F_X(x) := P(X \leqslant x)$$

is the distribution function. It holds the information about probabilities, and only sometimes has an actual formula. More often, it is calculated with a computer.

E.g., for a continuous variable X:

- $P(X < 3)$ means to calculate $F_X(3)$
- $P(-5 \leqslant X \leqslant 10) = P(X \leqslant 10) - P(X < -5)$, so we need to calculate $F_X(10)$ and $F_X(-5)$.
- $P(X \geqslant 9) = 1 - P(X < 9)$, so we need to calculate $F_X(9)$.

Note use of the fact that $P(X \leqslant a) = P(X < a)$ for continuous variables.

Expected value and variance

For a continuous random variable X with density function $f_X(x)$, the expected value and variance are also defined in terms of integrals:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f_X(x)\, dx$$

In STM4PSD, we will either perform calculations numerically using R, or use known formulas for specific distributions.

Some continuous distributions

- The continuous uniform distribution.
  - Notation: $X \sim U(a, b)$.
- The normal distribution.
  - Notation: $X \sim N(\mu, \sigma^2)$.
- The exponential distribution.
  - Notation: $X \sim \mathrm{Exp}(\lambda)$.
- The gamma distribution.
  - Notation: $X \sim \mathrm{Gamma}(k, \theta)$.

Continuous uniform distribution

If $X \sim U(a, b)$ then:

- The probability density function is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leqslant x \leqslant b, \\ 0 & \text{otherwise.} \end{cases}$$

- The probability distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if } x < a, \\ \frac{x-a}{b-a} & \text{if } a \leqslant x \leqslant b, \\ 1 & \text{otherwise.} \end{cases}$$

- $E(X) = \frac{1}{2}(a + b)$
- $\text{Var}(X) = \frac{1}{12}(b - a)^2$

For some examples involving the continuous uniform distribution, refer to the previous videos.

Normal distribution

If $X \sim N(\mu, \sigma^2)$ then:

- The probability density function is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

- The probability distribution function cannot be expressed in terms of elementary functions.
- $E(X) = \mu$
- $\mathrm{Var}(X) = \sigma^2$

For some examples involving the normal distribution, refer to the previous videos.

Exponential distribution

If $X \sim \mathrm{Exp}(\lambda)$ then:

- The probability density function is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geqslant 0, \\ 0 & \text{otherwise} \end{cases}$$

  The parameter λ is called the *rate*.

- The probability distribution function is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geqslant 0, \\ 0 & \text{otherwise} \end{cases}$$

- $E(X) = \frac{1}{\lambda}$
- $\mathrm{Var}(X) = \frac{1}{\lambda^2}$

The exponential distribution is important in the study of Poisson processes and M/M/1 queues, modelling the time between events.

Example

Let $X \sim \text{Exp}(3.5)$.

1. Write down the formulas for the probability density function and the probability distribution function of X.
2. Determine $P(X > 1)$, $P(X \geqslant 2)$ and $P(X \geqslant 2 \mid X > 1)$.
3. Determine $E(X)$ and $\text{Var}(X)$.

Gamma distribution

If $X \sim \text{Gamma}(k, \theta)$ then:

- The probability density function is

$$f_X(x) = \begin{cases} \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases}$$

  The parameter k is called the *shape* and θ is called the *scale*.
- A general formula for the probability distribution function cannot be expressed in terms of elementary functions. But for some particular values of k it can be determined.
- $E(X) = k\theta$
- $\text{Var}(X) = k\theta^2$

The gamma distribution is important in the study of Poisson processes and M/M/1 queues, modelling the time of event k.

The gamma function

Recall that the density function for a gamma distribution is:

$$\begin{cases} \dfrac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases}$$

The function $\Gamma(x)$ is called the gamma function.

Most of the time it is hard to calculate exactly, but occasionally it is easy. For example, if n is a positive integer, then

$$\Gamma(n) = (n-1)!$$

Example

Let $X \sim \text{Gamma}(5, 2.5)$.

1. Write down the formula for the probability density function of X.
2. Determine $E(X)$ and $\text{Var}(X)$.
3. Use R to determine $P(X > 1)$, $P(X \geqslant 2)$ and $P(X \geqslant 2 \mid X > 1)$.