MAST20005/MAST90058: Week 7 Solutions

1. $\alpha = \Pr(X \in \{2,3\} \mid p = 1/3) = 0.22222 + 0.03703 = 0.25926$
   $\beta = \Pr(X \in \{0,1\} \mid p = 2/3) = 0.03703 + 0.22222 = 0.25926$

2. (a) Assuming H_0 gives $\mathbb{E}(Y) = 8$ and $\text{var}(Y) = 7.36 = 2.713^2$. Using a normal approximation,

$$\alpha = \Pr(Y \leqslant 6 \mid p = 0.08) \approx \Pr(Z < \frac{6-8}{2.713}) = \Phi(-0.737) = 0.23.$$

   In this case we should ideally be using continuity correction because it makes a noticeable difference,

$$\alpha = \Pr(Y \leqslant 6 \mid p = 0.08) \approx \Pr(Z < \frac{6.5-8}{2.713}) = \Phi(-0.553) = 0.29.$$

   (b) When $p = 0.04$, we have $\mathbb{E}(Y) = 4$ and $\text{var}(Y) = 3.84 = 1.96^2$. Using a normal approximation,

$$\alpha = \Pr(Y \geqslant 7 \mid p = 0.04) \approx \Pr(Z > \frac{7-4}{1.96}) = \Pr(Z > 1.531) = 1 - \Phi(1.531) = 0.063.$$

   If we use continuity correction we get,

$$\alpha = \Pr(Y \geqslant 7 \mid p = 0.04) \approx \Pr(Z > \frac{6.5-4}{1.96}) = \Pr(Z > 1.276) = 1 - \Phi(1.276) = 0.10.$$

3. Under H_0, $\mathbb{E}(Y) = 14.63$ and $\text{var}(Y) = 13.606 = 3.689^2$. Hence,

$$z = \frac{23 - 14.63}{3.689} = 2.269$$

   (a) $z > 1.645$ so reject H_0 at 5% level of significance.
   (b) $z < 2.326$ so don't reject H_0 at the 1% level of significance.
   (c) The p-value is $\Pr(Z \geq 2.269) = 1 - \Phi(2.269) = 1 - 0.9883 = 0.0117$

4. $\hat{p}_m = 124/894 = 0.1387$ and $\hat{p}_f = 70/700 = 0.1$, which gives,

$$\hat{p}_m - \hat{p}_f \pm 1.96\sqrt{\frac{\hat{p}_m(1-\hat{p}_m)}{n_m} + \frac{\hat{p}_f(1-\hat{p}_f)}{n_f}} = (0.007, 0.07).$$

   Under H_0, $\hat{p} = 194/1594 = 0.1217$. Reject H_0 if $|z| > 1.96$.

$$|z| = \frac{|\hat{p}_m - \hat{p}_f|}{\sqrt{\hat{p}(1-\hat{p})\,(1/n_m + 1/n_f)}} = 2.345 > 1.96$$

   so we reject H_0.

5. (a) The critical region is:

$$T = \frac{\bar{X} - 47}{S/\sqrt{20}} < -1.729 \quad \text{(0.05 quantile of } t_{19}\text{)}$$

   (b) $t = (46.94 - 47)/(0.15/\sqrt{20}) = -1.789$. This is less than -1.729 so we reject H_0.

(c) Comparing the test statistic to the quantiles of t_{19} provided, we can deduce that the p-value is between 0.025 and 0.05.

6. (a) $H_0 \colon \mu = 1.9$
   (b) $H_1 \colon \mu \neq 1.9$
   (c) $|T| = |\bar{X} - 1.9|/(S/3) > 2.306$
   (d) $|t| = |2.05 - 1.9|/(0.17/3) = 2.647$
   (e) $|t| > 2.306$ so we reject H_0.
   (f) $2.306 < 2.647 < 2.896$ so the area of one tail (i.e. extreme values in one direction) is between 0.01 and 0.025. Since we have a two-sided alternative, the p-value will be double this, so we have: $0.02 <$ p-value < 0.05.

7. (a) The critical region is

$$\chi^2 = \frac{19S^2}{(0.095)^2} < 10.117$$

   and the observed value is

$$\chi^2 = \frac{19 \times (0.065)^2}{(0.095)^2} = 8.895$$

   so we reject H_0 and conclude there is evidence that the company was successful.
   (b) Since the 0.025 quantile of χ^2_{19} is $8.906 \approx 8.895$, the p-value is approximately 0.025.

8. (a) The critical region is given by:

$$|T| = \frac{|\bar{X} - \bar{Y}|}{\sqrt{\frac{12S_X^2 + 15S_Y^2}{27}\left(\frac{1}{13} + \frac{1}{16}\right)}} > 2.052 \quad (0.975 \text{ quantile of } t_{27})$$

   (b) The observed value is:

$$|t| = \frac{|72.9 - 81.7|}{\sqrt{\frac{12 \times 25.6^2 + 15 \times 28.3^2}{27}\left(\frac{1}{13} + \frac{1}{16}\right)}} = 0.869 < 2.052$$

   so we cannot reject H_0.
   (c) $0.684 < 0.869 < 1.314$ so the area of one tail (i.e. extreme values in one direction) is between 0.1 and 0.25. Since we have a two-sided alternative, the p-value will be double this, so we have: $0.2 <$ p-value < 0.5.
   (d) The test of interest is $H_0 \colon \sigma_X^2 = \sigma_Y^2$ against $H_1 \colon \sigma_X^2 \neq \sigma_Y^2$.

$$\frac{s_X^2}{s_Y^2} = \frac{25.6^2}{28.3^2} = 0.818 \in (0.314, 2.96) \quad (0.025 \text{ and } 0.975 \text{ quantiles of } \mathrm{F}_{12,15})$$

   so there is insufficient evidence that the variances differ (cannot reject H_0).