MAST30001 Stochastic Modelling – 2020

$\boxed{\textbf{Exam Solutions}}$

---

1. (a) (2 + 2 marks)

$$p_{23}p_{33}^{(2)} = \frac{1}{2}\left(\frac{16}{25} + \frac{1}{10}\right) = \frac{37}{100} = 0.37.$$

$$\frac{1}{3}\left(p_{23}^{(2)}p_{33}^{(2)} + p_{33}^{(2)}p_{33}^{(2)}\right) = \frac{37}{150}\left(\frac{13}{20} + \frac{74}{100}\right) = \frac{37 \cdot 139}{15000} = 0.342867.$$

(b) (6 marks)

Inspection of the transition diagram gives $S_1 = \{1, 5\}$, $S_2 = \{2, 3\}$, $S_3 = \{4\}$.

All classes have loops, so have period 1

S_1 and S_2 are essential and S_3 is non-essential.

The essential communicating classes are positive recurrent since they are finite.

(c) (3 marks) The chain will eventually end up in the essential communicating classes S_1 or S_2, and since they are aperiodic and positive recurrent, these classes are ergodic. The long run probabilities are given by the stationary distributions solving $\pi P = \pi$ restricted to each class, and we easily find:

$$(\pi_2, \pi_3) = (2/7, 5/7),$$

and

$$(\pi_1, \pi_5) = (2/5, 3/5).$$

(d) (3 marks) We use first step analysis. Let A be the event "reach state 1 before state 2" and $f_i = P(A|X_0 = i)$. We then have

$$\begin{aligned} f_4 &= \frac{1}{4}f_3 + \frac{1}{4}f_4 + \frac{1}{4}f_5, \\ f_3 &= \frac{4}{5}f_3, \\ f_5 &= \frac{1}{3} + \frac{2}{3}f_5, \end{aligned}$$

and solving yields $f_4 = 1/3$. Alternatively, note from state 4, the only way to enter states 1 or 5 is through state 5, and given we leave state 4, the chance of jumping to state 5 is 1/3.

2. We model the system as a renewal process with inter-arrival distribution uniform between 10 and 20 minutes.

(a) (7 marks) If τ is a random variable uniformly distributed in the interval $(10, 20)$, then

$$\mu := \mathbb{E}[\tau] = 15, \quad \sigma^2 := \mathrm{Var}(\tau) = 25/3.$$

The renewal LLN says $N_t/t \to 1/[\tau] = 1/15$ as $t \to \infty$, and so we expect $N_{300} \approx 300/15 = 20$.
The renewal CLT says that

$$N_{300} \approx \mathrm{Normal}(300/\mu, 300\sigma^2/\mu^3) = \mathrm{Normal}(20, 2500/15^3) = \mathrm{Normal}(20, 20/27),$$

and so there is approximately a 95% chance that the number of trains that stop will fall in the interval

$$20 \pm (1.96)\sqrt{20/27} \approx 20 \pm 1.687.$$

(b) (4 marks) We know that for large t, $\big(T_{N_t+1} - t\big)$ has approximate the distribution of $U\tau^s$, where U is uniform on $(0, 1)$ and is independent of τ^s having the size-bias distribution of τ, here having density

$$\frac{x f_\tau(x)}{\mu} = \frac{x}{150}, \quad 10 < x < 20,$$

and thus

$$\mathbb{E}[U\tau^s] = \frac{1}{300}\int_{10}^{20} x^2 dx = \frac{7000}{900} = 7.7778.$$

Alternatively, $\big(T_{N_t+1} - t\big) = Y_t$ where Y_t is the residual lifetime, and has density on $x \geq 0$

$$\frac{1 - F_\tau(x)}{\mu} = \begin{cases} \frac{1}{15}, & 0 < x < 10, \\ \frac{1-(x-10)/10}{15}, & 10 < x < 20, \end{cases}$$

and thus mean

$$\frac{1}{15}\int_0^{10} x dx + \frac{1}{15}\int_{10}^{20} x(2 - x/10)dx = \frac{100}{30} + \frac{200}{45} = \frac{70}{9}.$$

(c) (3 marks) This is now a renewal process where the inter-arrivals are distributed as

$$\tilde{\tau} = \sum_{i=1}^{M} \tau_i,$$

where M is positive geometric with parameter $9/10$ and the τ_i are iid uniform on $(10, 20)$. Using conditional expectation, we have

$$\mathbb{E}[\tilde{\tau}] = \mathbb{E}[M]\mathbb{E}[\tau_1] = \frac{150}{9} = \frac{50}{3},$$

and so again using the renewal LLN, we expect around

$$\frac{300}{\mathbb{E}[\tilde{\tau}]} = 18$$

trains to stop.

3. (a) (2 marks) We view the system as a CTMC with states $\{(0,0),(0,1),1,2,3,4\}$, where $(0,0)$ ($(0,1)$) means there are no cars in the shop and the mechanic is napping (not napping), and otherwise the states represent the number of cars in the shop (including those in service). When there are cars in the shop, the system behaves as a bounded birth-death process with birth and death rates equal to 3. When there are no cars in the shop, the process moves from $(0,0) \mapsto (0,1)$ at rate 10, and moves from $(0,1) \mapsto 1$ at rate 3 and $(0,1) \mapsto (0,0)$ at rate 1. Therefore the generator satisfies

$$A = \begin{pmatrix} -10 & 10 & 0 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 & 0 \\ 0 & 3 & -6 & 3 & 0 & 0 \\ 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix}.$$

The phrasing for this question was a bit ambiguous and an alternative interpretation is for a CTMC on $\{(0,0),(0,1),1,2,3\}$ with generator

$$A = \begin{pmatrix} -10 & 10 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 \\ 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 3 & -3 \end{pmatrix}.$$

(b) (3 marks) The system's stationary distribution satisfies $\pi A = 0$. Solving these equations gives

$$\pi = \frac{1}{51}(1, 10, 10, 10, 10, 10).$$

For alternative setup,

$$\pi = \frac{1}{41}(1, 10, 10, 10, 10).$$

(c) (2 marks) The average number of customers in the system is

$$\sum_{i=1}^{4} i\pi_i = \frac{100}{51} = 1.96078. \tag{1}$$

For alternative setup,

$$\sum_{i=1}^{3} i\pi_i = \frac{60}{41} = 1.46341.$$

(d) (1 mark) Using PASTA, an arriving customer finds the system in stationary, and is rejected if the system is in state $(0,0)$ or 4, which occurs with probability

$$\pi_{(0,0)} + \pi_4 = 11/51.$$

This is also the long run proportion of customers rejected.
Alternative setup is $\pi_{(0,0)} + \pi_3 = 11/41$.

(e) (3 marks) Using PASTA, an arriving customer finds the system in stationary. The chance they enter the system is $(1 - \pi_{(0,0)} - \pi_4) = 40/51$, and if they enter the system, the time they wait for service is the sum of the exponential service times of all customers already in the system, which is zero if there are no customers, and, if there are $i \geq 1$ customers, is distributed gamma with parameters i and rate 3. Thus the expected waiting time is

$$\frac{(\pi_1 + 2\pi_2 + 3\pi_3)(1/3)}{1 - \pi_{(0,0)} - \pi_4} = \frac{20/51}{40/51} = 1/2.$$

Alternatively, Little's law says that the total time in the system is equal to (1) divided by the rate of arrival, which, with rejections, is $3(1 - \pi_{(0,0} - \pi_4) = 120/51$; thus the delay is $5/6$, and then the time waiting for service is this minus the expected service time $1/3$, which gives $5/6 - 1/3 = 1/2$.

For alternative setup, the answer is

$$\frac{(\pi_1 + 2\pi_2)(1/3)}{1 - \pi_{(0,0)} - \pi_3} = \frac{10/41}{30/41} = 1/3.$$

4. Let $(N_t)_{t \geq 0}$ be the number of calls received to time t minutes.

(a) (2 marks) N_2 is Poisson distributed with mean 4, so

$$\mathbb{P}(N_2 \geq 2) = 1 - e^{-4} - 4e^{-4}.$$

(b) (3 marks) Let $(M_t)_{t \geq 0}$ be the process of calls lasting exactly one minute. By the thinning theorem, $(M_t)_{t \geq 0}$ is a Poisson process with rate $2/3$, and so

$$\mathbb{P}(M_2 \geq 2) = 1 - e^{-4/3} - (4/3)e^{-4/3}.$$

(c) (2 marks) Given the number of arrivals in an interval, their positions are i.i.d. uniformly distributed. Thus the number of the 10 that arrived in the first minute of the five is binomially distributed with parameters 10 and $1/5$, and so the desired probability is

$$\mathbb{P}(N_1 = 3 | N_5 = 10) = \binom{10}{3} (1/5)^3 (4/5)^7.$$

Alternatively, this can be done with direct calculation using the Poisson process.

(d) (3 marks) Following from the previous part, each of the 10 calls has chance $1/5$ of landing in the first minute, and then the chance they last exactly one minute is $1/3$ and so the number of the 10 that arrived in the first minute and last exactly one minute is binomially distributed with parameters 10 and $1/15$, and so the desired probability is

$$\mathbb{P}(N_1 = 3 | N_5 = 10) = \binom{10}{3} (1/15)^3 (14/15)^7.$$

Alternatively, this can be done with direct calculation using the Poisson process.

(e) (3 marks) In order for a call to still be in the system at time n, it must arrive in an interval $(k-1,k)$, for some $k=1,\dots,n$, and take longer than $n-k$ minutes. Let Y_k be the number of such calls. By the thinning theorem, the Y_k are Poisson with mean

$$2\mathbb{P}(X>n-k)=2(2/3)^{n-k},$$

and because of independent increments, they are independent. Thus their sum, which is the number of calls in the system at time n, is Poisson with mean

$$2\sum_{k=1}^{n}(2/3)^{n-k}=6(1-(2/3)^n).$$

(f) (3 marks) Let $(L_t)_{t\geq 0}$ be the process of calls lasting exactly one minute from the additional source. By the thinning theorem, $(L_t)_{t\geq 0}$ is a Poisson process with rate 5/3. Letting $(M_t)_{t\geq 0}$ as above, the superposition theorem implies that $(L_t+M_t)_{t\geq 0}$ is a Poisson process with rate 7/3, and so

$$\mathbb{P}(L_2+M_2\geq 2)=1-e^{-14/3}-(14/3)e^{-14/3}.$$

5. (a) (3 marks) By independent increments,

$$\mathbb{P}(B_{10}\geq -2|B_4=-1)=\mathbb{P}(B_{10}-B_4\geq -1)=1-\Phi(-1/\sqrt{6}).$$

(b) (1 mark) By the Markov property this is the same as the previous part.

(c) (4 marks) (B_4,B_{10}) is bivariate normal with covariance 4, and so we can represent

$$\frac{B_4}{2}=(1/5)B_{10}+\sqrt{3/5}Z,$$

where Z is standard normal and independent of B_{10}. Thus

$$\mathbb{P}(B_4\geq -2|B_{10}=-1)=\mathbb{P}(\sqrt{3/5}Z-1/5\geq -1)=1-\Phi\big(-4/\sqrt{15}\big).$$

(d) (3 marks) It is enough to show that $(X_t)_{0\leq t\leq 1}$ has continuous paths and that the finite dimensional distributions are the same as those of a Brownian motion. The continuity is because X_t a sum of continuous functions. The finite dimensional distributions are multivariate normal because they are a linear function of multivariate normals and the means are zero. So we only need to check the covariances match those of Brownian motion. For $0\leq s<t\leq 1$, we have

$$\begin{aligned}\mathbb{E}[X_sX_t]&=\mathbb{E}[B_s(B_t-tB_1+tZ)]-s\mathbb{E}[B_1(B_t-tB_1+tZ)]+s\mathbb{E}[Z(B_t-tB_1+tZ)]\\&=s-ts-2st+st=s,\end{aligned}$$

where we have used the independence of Z and $(B_t)_{t\geq 0}$ and the covariance structure of Brownian motion.

6. The first step analysis equations are, for $2\leq i\leq K-2$,

$$e_i=1+(1/2)e_{i+1}+(1/2)e_{i-1}, \tag{2}$$

and

$$\begin{aligned}
e_0 &= 1 + \alpha e_1, \\
e_1 &= 1 + (1/2)e_2 \\
e_K &= 1 + \alpha e_{K-1} \\
e_{K-1} &= 1 + (1/2)e_{K-2}.
\end{aligned}$$

Using the second equation to write e_2 in terms of e_1 and then using (2) starting from $i = 2$ and building up leads us to guess the formula for $1 \le i \le K - 1$

$$e_i = i(e_1 - i + 1),$$

which can be checked to satisfy the recursions above. Now using that symmetry implies that $e_{K-1} = e_1$, we have

$$e_1 = (K - 1)(e_1 - K + 2),$$

and solving implies that $e_1 = K - 1$, so the general formula is for $1 \le i \le K - 1$

$$e_i = i(K - i),$$

and the boundary conditions imply

$$e_K = e_0 = 1 + \alpha(K - 1)$$

7. The embedded discrete time Markov chain is a biased random walk drifting to infinity, and so it is transient and only visits each state finitely many times. Because the exponential holding times don't affect the number of times a state is visited, the continuous time chain is also transient. Thus it is enough to show that $\pi A = 0$ has a probability solution. But since the chain is a birth-death chain, we know a solution is

$$\pi_i = \pi_0 \prod_{j=1}^{i} \frac{a_{j-1,j}}{a_{j,j-1}} = \pi_0 2^{-i/2},$$

and this is summable so a probability solution exists.