Question 1

$$\underset{\sim}{u} = u\underset{\sim}{i} + v\underset{\sim}{j}, \text{ where } \begin{cases} u(x,y,t) = \dfrac{x}{2+t} \\ v(x,y,t) = \dfrac{-y}{1+t^2} \end{cases}$$

Assignment 3

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i) Streamlines in parametric form:

$$\frac{d\underset{\sim}{r}_{st}(s)}{ds} = \underset{\sim}{u}(\underset{\sim}{r}_{st}(s), t) \quad \ldots\ldots ①$$

Let $\underset{\sim}{r}_{st}(s) = (x(s), y(s))$.

So L.H.S. of ① $= \dfrac{d}{ds}(x(s), y(s))$

$=$ R.H.S. of ①

$= \underset{\sim}{u}((x(s), y(s)), t)$

$= \left(\dfrac{x}{2+t}, \dfrac{-y}{1+t^2}\right)$

From above, we have $\dfrac{d}{ds}(x,y) = \left(\dfrac{x}{2+t}, \dfrac{-y}{1+t^2}\right)$

$$\Rightarrow \begin{cases} \dfrac{dx}{ds} = \dfrac{x}{2+t} \quad \ldots\ldots ② \\ \dfrac{dy}{ds} = \dfrac{-y}{1+t^2} \quad \ldots\ldots ③ \end{cases}$$

We need to solve ② & ③.

② $\Rightarrow$: $\int \frac{1}{x}\, ds = \frac{1}{2+t}\int ds$

$\Rightarrow \log|x| = \dfrac{s+A}{2+t}$, $A \in \mathbb{R}$ is a constant

$\Rightarrow |x| = e^{\frac{s}{2+t}+\frac{A}{2+t}} = B(t)e^{\frac{s}{2+t}}$, where $B(t) = e^{\frac{A}{2+t}} \in \mathbb{R}$ is a constant

$\Rightarrow x = C(t)e^{\frac{s}{2+t}} \ldots\ldots ④$, where $C(t) = \pm B(t)$ is a constant.

Using the similary method, we can solve for y. and

$y = D(t)e^{-\frac{s}{1+t^2}} \ldots\ldots ⑤$, where $D(t)$ is a constant.

Combining eq. ④ & ⑤, we get the streamlines in parametric form:

$$\boxed{\mathcal{L}_{st}(s) = \left(C(t)e^{\frac{s}{2+t}},\ D(t)e^{\frac{-s}{1+t^2}}\right)} \quad \ldots\ldots\ ⑥$$

ii) Curves giving the shape of the streamlines in Cartesian form:

From ⑥, we get:

$$\begin{cases} x = C(t)e^{\frac{s}{2+t}} \\ y = D(t)e^{\frac{-s}{1+t^2}} \end{cases} \Rightarrow \begin{cases} \dfrac{x^{2+t}}{C^{2+t}} = e^{s} \\ \dfrac{y^{-(1+t^2)}}{D^{-(1+t^2)}} = e^{s} \end{cases} \Rightarrow \underbrace{\frac{x^{2+t}}{C^{2+t}} = \frac{y^{-(1+t^2)}}{D^{-(1+t^2)}}}_{\ldots\ ⑦}$$

Rearranging the equation ⑦ gives:

$$y = x^{-\frac{(2+t)}{1+t^2}} \cdot \frac{D}{C^{-\frac{(2+t)}{1+t^2}}}$$

that is, $\boxed{y = E(t)\,x^{-\frac{t+2}{1+t^2}}}$ ... ⑧, where $E(t) = \dfrac{D}{C^{-\frac{(2+t)}{1+t^2}}}$ is a constant.

iii) Streamlines passing through a point $(x,y)=(1,1)$:

From ⑧, we get

$$y = E(t)\,x^{-\frac{t+2}{1+t^2}}$$

If the streamlines passing through the point $(x,y)=(1,1)$:

$$1 = E(t)\,1^{-\frac{t+2}{1+t^2}}, \quad \forall t \geq 0$$

$$\Rightarrow E(t) = 1, \quad \forall t \geq 0$$

$\forall t \geq 0$, the streamlines passing through $(x,y)=(1,1)$ takes the form $\boxed{y = x^{-\frac{t+2}{1+t^2}}}$ ...... ⑨

- when $\boxed{t=0:}$

⑨ $\Rightarrow \boxed{y = x^{-2}}$

- when $t \to \infty$:

$$\lim_{t\to\infty} x^{\frac{-(t+2)}{1+t^2}} = x^{\lim_{t\to\infty}\frac{-(t+2)}{1+t^2}} = x^0 = 1$$

$\Rightarrow \boxed{y \to 1 \text{ when } t \to \infty.}$

iv). <u>the streaklines eminating from $(x,y)=(1,1)$, $\forall t \geq 0$,</u>

- In order to calculate the streaklines, we 1st calculate the particle path:

$$\frac{d\underline{r}_p(t)}{dt} = \underline{u}(\underline{r}_p(t), t), \text{ where } \underline{r}_p(t) = (x(t), y(t)). \quad \text{......⑩}$$

$$\text{L.H.S. of eq.⑩} = \frac{d}{dt}(x(t), y(t))$$
$$= \text{R.H.S. of eq.⑩}$$
$$= \left(\frac{x}{2+t}, \frac{-y}{1+t^2}\right) \quad \text{.......⑪}$$

From eq.⑪, we get $\frac{d}{dt}(x(t), y(t)) = \left(\frac{x}{2+t}, \frac{-y}{1+t^2}\right)$

$$\Rightarrow \begin{cases} \frac{dx}{dt} = \frac{x}{2+t} & \text{......⑫} \\ \frac{dy}{dt} = \frac{-y}{1+t^2} & \text{......⑬} \end{cases}$$

We need to solve ⑫ & ⑬:

$$⑫ \Rightarrow \int \frac{1}{x}\,dx = \int \frac{dt}{2+t}$$

$\log|x| = \log|2+t| + c$, where c is a constant of integration

$\Rightarrow \log|x| = \log(2+t) + c$, since $t \geq 0$, thus $2+t > 0$

$\Rightarrow |x| = A(2+t)$, where $A = e^c$ is a constant.

$\Rightarrow x = B(2+t)$, where $B = \pm A$ is a constant...... ⑭

Similarly,

⑬ $\Rightarrow -\int \frac{1}{y}\,dy = \int \frac{1}{1+t^2}\,dt$

$\Rightarrow -\log|y| = \arctan(t) + C'$, where c' is a $\overbrace{\text{constant of integration.}}$

$\Rightarrow |y|^{-1} = e^{\arctan(t)+c'}$, where $y \neq 0$

$\Rightarrow |y|^{-1} = A'e^{\arctan(t)}$, where $A' = e^{c'}$ is a constant

$\Rightarrow |y| = \frac{1}{A'} e^{-\arctan(t)}$

$\Rightarrow y = \pm \frac{1}{A'} e^{-\arctan(t)}$

$\Rightarrow y = De^{-\arctan(t)}$, where $D = \pm \frac{1}{A'}$ is a constant........ ⑮

Combining ⑭ & ⑮ gives:

$\mathcal{L}_p(t) = (B(2+t), De^{-\arctan(t)})$ .... ⑯

- Now, note that the governing equations to determine streaklines are the same for particle paths, except that the initial condition are $(x,y) = (1,1)$ at $t = \tau$, where $\tau > 0$.

Imposing this condition to ⑯ gives:

$$\left\{\begin{array}{l} B(2+\tau) = 1 \\ De^{-\arctan(\tau)} = 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} B = \frac{1}{2+\tau} & \cdots\cdots ⑰ \\ D = e^{\arctan(\tau)} & \cdots\cdots ⑱ \end{array}\right.$$

Substituting ⑰ & ⑱ into ⑯ gives:

$$\left\{\begin{array}{ll} x = \frac{1}{2+\tau}(2+t) & \cdots\cdots\cdots ⑲ \\ y = e^{\arctan(\tau)} e^{-\arctan(t)} & \cdots\cdots\cdots ⑳ \end{array}\right.$$

From ⑲, we can write

$\tau = \frac{2+t-2x}{x}$, where $x \neq 0$ ...... ㉑ ☺

From ⑳, we can get:

$$\tau = \tan(\arctan(t) + \log y), \ \underline{\text{where } y > 0} \quad ㉑$$

$$= \frac{\tan(\arctan(t)) + \tan(\log y)}{1 - \tan(\arctan(t))\tan(\log(y))}$$

$$= \frac{t + \tan(\log y)}{1 - t\tan(\log y)} \quad \cdots\cdots ㉒$$

Equate ㉑ & ㉒, we'll get:

$$\frac{2+t-2x}{x} = \frac{t+\tan(\log y)}{1-t\tan(\log y)}$$

$$\Rightarrow (1-t\tan(\log y))(2+t-2x) = xt + x\tan(\log y)$$

$$\Rightarrow 2+t-2x-2t\tan(\log y) - t^2\tan(\log y) + 2xt\tan(\log y)$$

$$= xt + x\tan(\log y)$$

$$\Rightarrow \tan(\log y) = \frac{xt+2x-2-t}{2xt-t^2-2t-x}$$

$$\Rightarrow \boxed{y = \exp\left(\arctan\left(\frac{xt+2x-2-t}{2xt-t^2-2t-x}\right)\right)} \quad \cdots\cdots ㉓$$

Therefore, eq. ㉓ is our streakline's equation emanating from the point $(x, y) = (1, 1)$, $\forall t \geq 0$.

- Now, we need to specify the domain when eq. ㉓ holds:

From our strategy to work out the streakline, we need

$$0 < \tau < t \quad \cdots\cdots ㉔$$

But note that $\tau = \frac{2+t-2x}{x}$ from eq. ㉑

$\Rightarrow$ ㉔ becomes:

$$0 < \frac{2+t-2x}{x} < t \quad \cdots\cdots ㉕$$

*). When $x > 0$:

(25) $\Rightarrow$ $0 < 2+t-2x < xt$

$0 < 2+t-2x$ gives: $x < \frac{2+t}{2}$ ...... (26)

$2+t-2x < xt$ gives: $(2+t) < x(2+t)$ since $2+t > 0$

$\Rightarrow$ $x > 1$ ...... (27)

Combining (26) & (27) gives: $1 < x < \frac{2+t}{2}$ ...... (28)

*). When $x \le 0$:

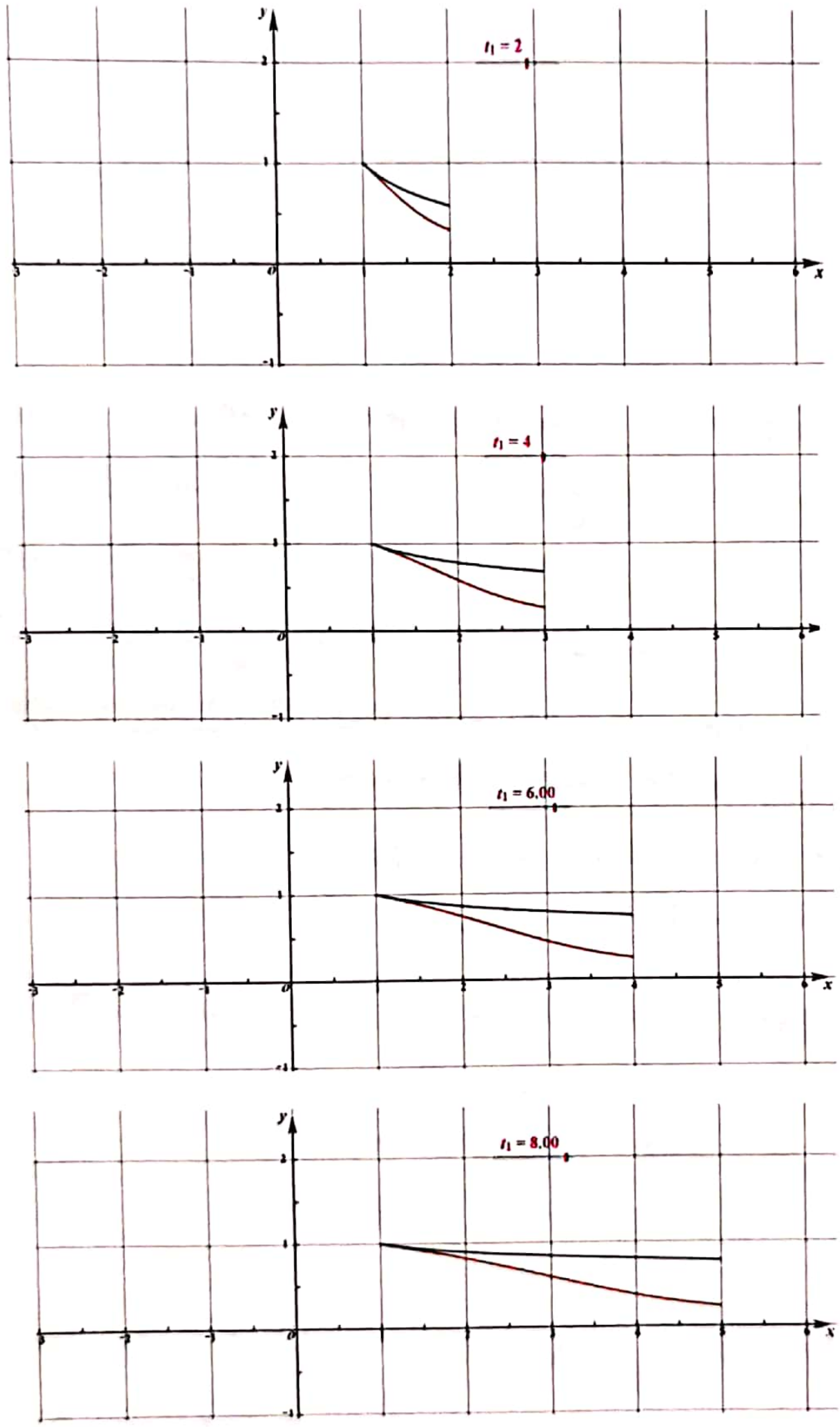
(25) $\Rightarrow$ $xt < \underbrace{2+t-x} < 0$

Since $t > 0$ and $x < 0$, $2+t-x > 2$, which contradicts with $2+t-x < 0$!

So we can't have $x \le 0$! .... (29)

Hence, combining (28), (29), ⊛ on page 5, ☺ on page 4, we conclude that the domain is: $\boxed{\left\{ \begin{array}{l} 1 < x < \frac{2+t}{2} \\ y > 0 \end{array} \right\} \forall t \geqslant 0.}$

b)



Red: streak line;
Green: streamline.

(To Be Continued......)

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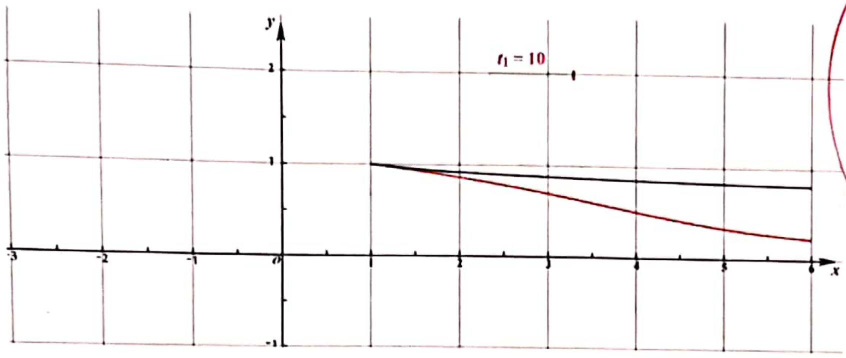

These 2 lines (streakline and streamline) are different because the flow isn't steady. i.e.: $u(x,y,t) = \frac{x}{2+t}$, $v(x,y,t) = \frac{-y}{1+t^2}$ the velocity has t dependance.

This actually makes sense since:

*). streaklines: are the loci of points of all the fluid particles that have passed continously through a particular spatial point, $(x,y)=(1,1)$ in our case, in the past.

*). Streamlines: are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction in which a massless fluid element will travel at any point in time.

Hence, if the flow isn't steady, the streaklines & streamlines will be different.

However, both the streaklines & streamlines are decreasing when time evolves. This is because of velocity fied is of the form: $\underset{\sim}{u} = \frac{x}{2+t}\underset{\sim}{i} - \frac{y}{1+t^2}\underset{\sim}{j}$

Question 2.

(i). Use Cartesian tensor method to prove:

① ……
$$\oint_S \underset{\sim}{n}\cdot(\underset{\sim}{u}'\cdot \underset{\sim}{T} - \underset{\sim}{u}\cdot \underset{\sim}{T}')\,dS = \int_V \underset{\sim}{u}\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{T}') - \underset{\sim}{u}'\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{T})\,dV$$

proof:

Step①: List the information we know:

For incompressible Newtonian fluid, its Navier-Stokes eq. is: $\rho(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}) = \underset{\sim}{\nabla}\cdot\underset{\sim}{T} + \underset{\sim}{f}$, …… ②

where $\underset{\sim}{\nabla}\cdot\underset{\sim}{T} = -\underset{\sim}{\nabla}p + \mu\nabla^2\underset{\sim}{u}$, and

$\underset{\sim}{f} = \rho\underset{\sim}{b}$ denotes the body force.

If we have Stokes flow, $\rho(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}) = \underset{\sim}{0}$, i.e.

① reduces to $\underset{\sim}{\nabla}\cdot\underset{\sim}{T} + \underset{\sim}{f} = \underset{\sim}{0}$ …… ③

However, we know that for incompressible viscous fluid, the constitutive eq. is: $\underset{\sim}{T} = -p\underset{\sim}{I} + \underset{\sim}{d}$ …… ④,

where $\underset{\sim}{d}$ is the deviatoric stress tensor, also

$\underset{\sim}{d} = \mu(\underset{\sim}{\nabla}\underset{\sim}{u} + (\underset{\sim}{\nabla}\underset{\sim}{u})^T)$ …… ⑤ where μ is the dynamic viscosity, assumed to be constant.

Also, we know that for incompressible flow: $\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$ ⑥

Now, we put ③.④.⑤.⑥ in the Cartesian Tensor form:

$$\begin{cases} \frac{\partial}{\partial x_i}T_{ik} + f_k = 0 & \text{……⑦} \\ T_{ik} = -p\delta_{ik} + d_{ik} & \text{……⑧} \\ d_{ik} = \mu(\frac{\partial}{\partial x_i}u_k + \frac{\partial}{\partial x_k}u_i) & \text{……⑨} \\ \frac{\partial}{\partial x_i}u_i = 0 & \text{……⑩} \end{cases}$$

Step ②: Use the information we gather to prove the identity:

Now, suppose there is a second Stokes flow in the same domain with T'_{ik}, f'_k, d'_{ik}, p', u'_i satisfying the same system of equations (as eq. ⑦–⑩).

Multiplying ⑦ by u'_k we get:

$$u'_k \frac{\partial}{\partial x_i} T_{ik} + u'_k f_k = 0 \quad \ldots\ldots ⑪$$

Similarly, interchanging primed & unprimed variables gives:

$$u_k \frac{\partial}{\partial x_i} T'_{ik} + u_k f'_k = 0 \quad \ldots\ldots ⑫$$

⑪-⑫ gives:

$$u'_k \frac{\partial}{\partial x_i} T_{ik} - u_k \frac{\partial}{\partial x_i} T'_{ik} + u'_k f_k - u_k f'_k = 0 \quad \ldots\ldots ⑬$$

However, $T_{ik} = d_{ik} - p\delta_{ik}$ and $T'_{ik} = d'_{ik} - p'\delta_{ik}$,

so ⑬ $\Rightarrow$ $u'_k \frac{\partial}{\partial x_i}(d_{ik} - p\delta_{ik}) - u_k \frac{\partial}{\partial x_i}(d'_{ik} - p'\delta_{ik}) + u'_k f_k - u_k f'_k = 0$

$$\Rightarrow u'_k \frac{\partial}{\partial x_i} d_{ik} - u_k \frac{\partial}{\partial x_i} d'_{ik} - u'_i \frac{\partial}{\partial x_i} p + u_i \frac{\partial}{\partial x_i} p' + u'_k f_k - u_k f'_k = 0 \quad ⑭$$

Now, we can use product rule: $\frac{\partial}{\partial x_i}(u'_k d_{ik}) = u'_k \frac{\partial}{\partial x_i} d_{ik} + d_{ik} \frac{\partial}{\partial x_i} u'_k$, and $\frac{\partial}{\partial x_i}(u'_i p) = u'_i \frac{\partial}{\partial x_i} p + p \frac{\partial}{\partial x_i} u'_i = u'_i \frac{\partial}{\partial x_i} p$ since the flow is incompressible, i.e. $\frac{\partial}{\partial x_i} u'_i = \nabla \cdot \underline{u}' = 0$.

Then ⑭ becomes:

$$\frac{\partial}{\partial x_i}(u'_k d_{ik}) - d_{ik} \frac{\partial}{\partial x_i} u'_k - \frac{\partial}{\partial x_i}(u_k d'_{ik}) + d'_{ik} \frac{\partial}{\partial x_i} u_k - \frac{\partial}{\partial x_i}(u'_i p) + \frac{\partial}{\partial x_i}(u_i p') + u'_k f_k - u_k f'_k = 0 \quad \ldots\ldots ⑮$$

However, note that $\frac{\partial}{\partial x_i}(u'_i p) = \frac{\partial}{\partial x_i}(u'_k p \delta_{ik})$.

hence, ⑮ will become:

$$\frac{\partial}{\partial x_i}(u'_k d_{ik} - u'_k p\delta_{ik}) - \frac{\partial}{\partial x_i}(u_k d'_{ik} - u_k p' \delta_{ik}) - d_{ik}\frac{\partial}{\partial x_i}u'_k + d'_{ik}\frac{\partial}{\partial x_i}u_k + u'_k f_k - u_k f'_k = 0 \quad \text{⑯}$$

$$\Rightarrow \frac{\partial}{\partial x_i}(u'_k T_{ik}) - \frac{\partial}{\partial x_i}(u_k T'_{ik}) - d_{ik}\frac{\partial}{\partial x_i}u'_k + d'_{ik}\frac{\partial}{\partial x_i}u_k + u'_k f_k - u_k f'_k = 0$$

However, $-d_{ik}\frac{\partial}{\partial x_i}u'_k + d'_{ik}\frac{\partial}{\partial x_i}u_k$

$$= -\mu\left(\frac{\partial}{\partial x_i}u_k + \frac{\partial}{\partial x_k}u_i\right)\frac{\partial}{\partial x_i}u'_k + \mu\left(\frac{\partial}{\partial x_i}u'_k + \frac{\partial}{\partial x_k}u'_i\right)\frac{\partial}{\partial x_i}u_k$$

$$= \mu\left(\cancel{-\frac{\partial}{\partial x_i}u_k\frac{\partial}{\partial x_i}u'_k} - \frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_i}u'_k + \cancel{\frac{\partial}{\partial x_i}u'_k\frac{\partial}{\partial x_i}u_k} + \frac{\partial}{\partial x_k}u'_i\frac{\partial}{\partial x_i}u_k\right)$$

$$= \mu\left(-\frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_i}u'_k + \frac{\partial}{\partial x_k}u'_i\frac{\partial}{\partial x_i}u_k\right)$$

$$= 0 \quad \ldots\ldots \text{⑰}$$

since the dummy summation indices can be exchanged so the 2nd term, $\frac{\partial}{\partial x_k}u'_i\frac{\partial}{\partial x_i}u_k = \frac{\partial}{\partial x_i}u'_k\frac{\partial}{\partial x_k}u_i$

Substituting ⑰ into ⑯ gives:

$$\frac{\partial}{\partial x_i}(u'_k T_{ik}) - \frac{\partial}{\partial x_i}(u_k T'_{ik}) + u'_k f_k - u_k f'_k = 0 \quad \ldots\ldots \text{⑱}$$

Comparing ⑬ & ⑱, we get:

$$\frac{\partial}{\partial x_i}(u'_k T_{ik}) - \frac{\partial}{\partial x_i}(u_k T'_{ik}) = u'_k\frac{\partial}{\partial x_i}T_{ik} - u_k\frac{\partial}{\partial x_i}T'_{ik} = -(u'_k f_k - u_k f'_k)$$

i.e. we have:

$$\frac{\partial}{\partial x_i}(u'_k \underbrace{T_{ki}}) - \frac{\partial}{\partial x_i}(u_k \underbrace{T'_{ki}}) = u'_k\frac{\partial}{\partial x_i}T_{ik} - u_k\frac{\partial}{\partial x_i}T'_{ik} \quad \ldots\ldots \text{⑲}$$

since $\underline{T}$ & $\underline{T}'$ are symmetric

In dyadic notation, ⑱ reads as follows: ⑳

$$\underline{\nabla}\cdot(\underline{u}'\cdot\underline{T}) - \underline{\nabla}\cdot(\underline{u}\cdot\underline{T}') = \underline{u}'\cdot(\underline{\nabla}\cdot\underline{T}) - \underline{u}\cdot(\underline{\nabla}\cdot\underline{T}')$$

Integrating both sides over the volume $\mathcal{V}$ of the domain. (10) becomes:

$$\int_V \underset{\sim}{\nabla}\cdot(\underset{\sim}{u}'\cdot\underset{\sim}{I}) - \underset{\sim}{\nabla}\cdot(\underset{\sim}{u}\cdot\underset{\sim}{I}')\, dV = \int_V \underset{\sim}{u}'\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{I}) - \underset{\sim}{u}\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{I}')\, dV$$

Using the divergence theorem, we obtain:

$$\oint_S (-\underset{\sim}{n})\cdot(\underset{\sim}{u}'\cdot\underset{\sim}{I}) - (-\underset{\sim}{n})\cdot(\underset{\sim}{u}\cdot\underset{\sim}{I}')\, dS = \int_V \underset{\sim}{u}'\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{I}) - \underset{\sim}{u}\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{I}')\, dV$$

NOTE: here $\underset{\sim}{n}$ is the unit vector INTO the fluid domain.

Rearranging the terms gives us the desired results:

$$\oint_S \underset{\sim}{n}\cdot(\underset{\sim}{u}'\cdot\underset{\sim}{I} - \underset{\sim}{u}\cdot\underset{\sim}{I}')\, dS = \int_V \underset{\sim}{u}\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{I}') - \underset{\sim}{u}'\cdot(\underset{\sim}{\nabla}\cdot\underset{\sim}{I})\, dV$$

$\square$

(ii)

(a) From the formula given in Lecture 22. we have:

$$\underset{\sim}{u}' = \underset{\sim}{\nabla}\times\left(\frac{\Psi'(r,\theta)}{r\sin\theta}\,\hat{\underset{\sim}{e}}_\phi\right), \text{ where } \Psi'(r,\theta) = \frac{U}{4}\left(3ar - \frac{a^3}{r}\right)\sin^2\theta$$

$$\Rightarrow \frac{\Psi'(r,\theta)}{r\sin\theta} = \frac{U}{4}\left(3a - \frac{a^3}{r^2}\right)\sin\theta$$

$$\Rightarrow \underset{\sim}{u}' = \underset{\sim}{\nabla}\times\left[\frac{U}{4}\left(3a - \frac{a^3}{r^2}\right)\sin\theta\,\hat{\underset{\sim}{e}}_\phi\right]$$

$$= \frac{\frac{\partial}{\partial\theta}\left(\frac{U}{4}\left(3a - \frac{a^3}{r^2}\right)\sin\theta\sin\theta\right)}{r\sin\theta}\,\hat{\underset{\sim}{r}} - \frac{\frac{\partial}{\partial r}\left(\frac{Ur}{4}\left(3a - \frac{a^3}{r^2}\right)\sin\theta\right)}{r}\,\hat{\underset{\sim}{\theta}}$$

$$= \frac{\frac{U}{4}\left(3a - \frac{a^3}{r^2}\right)\frac{\partial}{\partial\theta}(\sin^2\theta)}{r\sin\theta}\,\hat{\underset{\sim}{r}} - \frac{\sin\theta\,\frac{\partial}{\partial r}\left(\frac{3aUr}{4} - \frac{Ua^3}{4r}\right)}{r}\,\hat{\underset{\sim}{\theta}}$$

So (12) $$\boxed{\underset{\sim}{u}' = \frac{1}{2}\left(\frac{3Ua}{r} - \frac{a^3U}{r^3}\right)\cos\theta\,\hat{\underset{\sim}{r}} - \left(\frac{3aU}{4r} + \frac{a^3U}{4r^3}\right)\sin\theta\,\hat{\underset{\sim}{\theta}}}$$

(b)

Step ①: Approximate $\underset{\sim}{e}$ by corresponding results for a continuum flow with $\underset{\sim}{u}'$:

From (a), we have $\underset{\sim}{u}' = \frac{1}{2}\left(\frac{3Ua}{r} - \frac{a^3U}{r^3}\right)\cos\theta\,\hat{\underset{\sim}{r}} - \left(\frac{3aU}{4r} + \frac{a^3U}{4r^3}\right)\sin\theta\,\hat{\underset{\sim}{\theta}}.$

$\underset{\sim}{e} \approx \underset{\sim}{e}'$

But what is $\underset{\sim}{e}'$? From formula sheet, we have:

- $e_{rr} = \frac{\partial}{\partial r}\left[\frac{1}{2}\left(\frac{3Ua}{r} - \frac{a^3U}{r^3}\right)\cos\theta\right] = -\frac{3}{2}U\cos\theta\left(\frac{a}{r^2} - \frac{a}{r^4}\right)$

- $e_{\theta\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[-\left(\frac{3aU}{4r} + \frac{a^3U}{4r^3}\right)\sin\theta\right] + \frac{1}{2}\left(\frac{3Ua}{r^2} - \frac{a^3U}{r^4}\right)\cos\theta$

  $= \frac{3U}{4}\cos\theta\left(\frac{a}{r^2} - \frac{a^3}{r^4}\right)$

- $e_{\varphi\varphi} = \frac{1}{2}\left(\frac{3Ua}{r^2} - \frac{a^3U}{r^4}\right)\cos\theta - \left(\frac{3aU}{4r^2} + \frac{a^3U}{4r^4}\right)\sin\theta\,\frac{\cos\theta}{\sin\theta}$

  $= \frac{3U}{4}\cos\theta\left(\frac{a}{r^2} - \frac{a^3}{r^4}\right)$

- $e_{r\theta} = e_{\theta r} = \frac{r}{2}\frac{\partial}{\partial r}\left[\left(-\frac{3aU}{4r^2} - \frac{a^3U}{4r^4}\right)\sin\theta\right] + \frac{1}{2r}\frac{1}{2}\left(\frac{a^3U}{r^3} - \frac{3Ua}{r}\right)\sin\theta$

  $= \frac{3Ua^3}{4r^4}\sin\theta$

- $e_{r\varphi} = e_{\varphi r} = 0$

- $e_{\theta\varphi} = e_{\varphi\theta} = 0$

Hence combining above, we have

$$\underset{\sim}{e} \approx \underset{\sim}{e}' = \begin{bmatrix} -\frac{3}{2}U\cos\theta\left(\frac{a}{r^2} - \frac{a}{r^4}\right) & \frac{3Ua^3}{4r^4}\sin\theta & 0 \\ \frac{3Ua^3}{4r^4}\sin\theta & \frac{3U}{4}\cos\theta\left(\frac{a}{r^2} - \frac{a^3}{r^4}\right) & 0 \\ 0 & 0 & \frac{3U}{4}\cos\theta\left(\frac{a}{r^2} - \frac{a^3}{r^4}\right) \end{bmatrix}$$

On the surface of the sphere (i.e. $r=a$), our $\underset{\sim}{e}$ reduces to

$$\underset{\sim}{e} = \begin{bmatrix} 0 & \frac{3U}{4a}\sin\theta & 0 \\ \frac{3U}{4a}\sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $\underset{\sim}{e}\cdot\underset{\sim}{n} = \underset{\sim}{e}\cdot\underset{\sim}{r} = \frac{3U}{4a}\sin\theta\,\hat{\underset{\sim}{\theta}}$ ...... (21)

However, we also know that $\underset{\sim}{u} = \underset{\sim}{U}_s + 2\lambda\,\underset{\sim}{n}\times([\underset{\sim}{e}\cdot\underset{\sim}{n}]\times\underset{\sim}{n})$ (22)

Substituting (21) into (22) gives when $r=a$:

$$\underset{\sim}{u} = U\hat{\underset{\sim}{k}} + 2\lambda\underset{\sim}{r}\times\left(\frac{3U}{4a}\sin\theta\,\hat{\underset{\sim}{\theta}}\times\hat{\underset{\sim}{r}}\right)$$

$$= U\hat{\underset{\sim}{k}} + \frac{3U\lambda}{2a}\sin\theta\;\underset{\sim}{r}\times(\hat{\underset{\sim}{\theta}}\times\hat{\underset{\sim}{r}})$$

$$= U\hat{\underset{\sim}{k}} + \frac{3U\lambda}{2a}\sin\theta\,\hat{\underset{\sim}{\theta}} \quad \text{...... (23)}$$

Also, we know that when $r=a$:

$$\underset{\sim}{u}' = U\hat{\underset{\sim}{k}} = U\cos\theta\,\hat{\underset{\sim}{r}} - U\sin\theta\,\hat{\underset{\sim}{\theta}} \quad \text{....... (24)}$$

<u>Step (2): Now we use the reciprocal theory to find out the drag force F_{drag}:</u>

In this part, there is no body force. From eq. (18) & (19) in part (a), we have: $\int_s \underset{\sim}{n}\cdot(\underset{\sim}{u}'\cdot\underset{\sim}{T} - \underset{\sim}{u}\cdot\underset{\sim}{T}')dS = 0$

$$\Rightarrow \int_s \underset{\sim}{n}\cdot(\underset{\sim}{u}'\cdot\underset{\sim}{T})ds = \int_s \underset{\sim}{n}\cdot(\underset{\sim}{u}\cdot\underset{\sim}{T}')\,ds \quad \text{...... (25)}$$

Since T is symmetry, (25) will become:

$$\int_s \underset{\sim}{u}'\cdot(\underset{\sim}{n}\cdot\underset{\sim}{T})ds = \int_s \underset{\sim}{u}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{T}')ds \quad \text{..... (26)}$$

Substituting (23) & (24) into (26) gives:

$$\int_s U\hat{\underset{\sim}{k}}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{T})ds = \int_s U\hat{\underset{\sim}{k}}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{T}')ds + \int_s \frac{3U\lambda}{2a}\sin\theta\,\hat{\underset{\sim}{\theta}}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{T}')ds$$

We can take $\hat{\mathbf{k}}$ outside the integral since $\hat{\mathbf{k}}$ is cartesian!

$$\Rightarrow U\hat{\mathbf{k}}\cdot\int_s(\mathbf{n}\cdot\mathbf{T})ds = U\hat{\mathbf{k}}\cdot\int_s(\mathbf{n}\cdot\mathbf{T}')ds + \frac{3U\lambda}{2a}\int_s \sin\theta\,\hat{\boldsymbol{\theta}}\cdot(\mathbf{n}\cdot\mathbf{T}')ds$$

However, we know $\int_s(\mathbf{n}\cdot\mathbf{T})ds = \mathbf{F}$ and $\int_s(\mathbf{n}\cdot\mathbf{T}')ds = \mathbf{F}'$

$$\Rightarrow U\hat{\mathbf{k}}\cdot\mathbf{F} = U\hat{\mathbf{k}}\cdot\mathbf{F}' + \frac{3U\lambda}{2a}\int_s \sin\theta\,\hat{\boldsymbol{\theta}}\cdot(\mathbf{n}\cdot\mathbf{T}')ds$$

Also, we know that $\mathbf{F}\cdot\hat{\mathbf{k}} = -F_{drag}$ and $\mathbf{F}'\cdot\hat{\mathbf{k}} = -F'_{drag}$

$$\Rightarrow -F_{drag} = -F'_{drag} + \frac{3\lambda}{2a}\int_s \sin\theta\,\hat{\boldsymbol{\theta}}\cdot(\mathbf{n}\cdot\mathbf{T}')\,ds \quad \ldots\ldots (27)$$

*) Now, let's investigate $\int_s \sin\theta\,\hat{\boldsymbol{\theta}}\cdot(\mathbf{n}\cdot\mathbf{T}')ds$!

From the constitutive equation: $\mathbf{T}' = -p\mathbf{I} + 2\mu\mathbf{e}$.

$$\text{so } \mathbf{n}\cdot\mathbf{T}' = \mathbf{n}\cdot(-p\mathbf{I} + 2\mu\mathbf{e})$$

$$= -p\,\mathbf{n}\cdot\mathbf{I} + 2\mu\,\mathbf{n}\cdot\mathbf{e} \quad \ldots\ldots (28)$$

However, from eq. (21), we have $\mathbf{n}\cdot\mathbf{e} = \frac{3U}{4a}\sin\theta\,\hat{\boldsymbol{\theta}}$ ……(29)

Substituting (29) into (28) and substituting (28) into (27):

$$-F_{drag} = -F'_{drag} + \frac{3\lambda}{2a}\int_s \sin\theta\,\hat{\boldsymbol{\theta}}\cdot\left(-p\,\mathbf{n}\cdot\mathbf{I} + 2\mu\frac{3U}{4a}\sin\theta\,\hat{\boldsymbol{\theta}}\right)ds \quad \ldots (30)$$

However, $\hat{\boldsymbol{\theta}}\cdot(-p\,\mathbf{n}\cdot\mathbf{I}) = 0$. Also $F'_{drag} = 6\pi a\mu U$

so (30) becomes:

$$-F_{drag} = -6\pi a\mu U + \frac{9U\lambda\mu}{4a^2}\int_s \sin^2\theta\, d\theta \quad \ldots\ldots (31)$$

After integrating and rearranging terms, we will get:

$$\boxed{F_{drag} = 6\pi a\mu U - 6\pi\mu K_n U} \quad \ldots (32)$$

λ

$$\int_s \sin^2\theta\, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}(\sin^2\theta)\,a\sin\theta\, d\theta\, d\phi$$
$$= \int_0^{2\pi}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi} d\phi$$
$$= 4/3 \times 2\pi = \frac{8}{3}\pi$$

(C). from (32), we get: they don't have the same units.

$$F_{drag} = 6\pi a\mu U - 6\pi K_n U\mu.$$

$$\Rightarrow \boxed{F_{drag} = F'_{drag} - 6\pi K_n U\mu.} \quad \cdots\cdots (33)$$

the non-continuum effects add a term '$-6\pi K_n U\mu$' to the drag force, i.e. the drag force becomes smaller in the non-continuum system.

When $K_n \to 0$:
(33) tells us that $F_{drag} \to F'_{drag}$.

This actually makes sense since when $K_n \to 0$, $\frac{\lambda}{a} \to 0$ i.e. the sphere's radius isn't tiny compared to the mean free path, i.e. we will have the continuum hypothesis, i.e. $\lim_{K_n \to 0} F_{drag} = F'_{drag} = 6\pi a\mu U$.

outer surface.

-1

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Question 3.

(i). general idea:

superposition

uniform flow — rotating flow

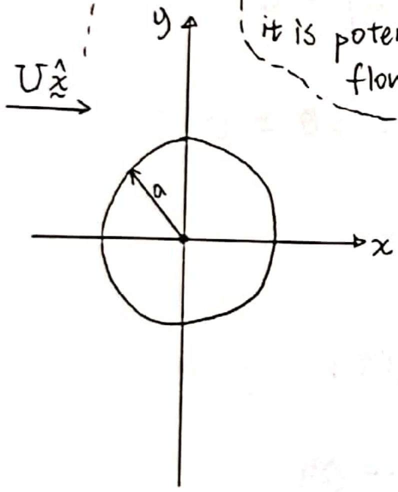
Uniform flow:

Known:

- $\nabla^2 \phi = 0$ ...... ①   since $\underset{\sim}{\omega} = \underset{\sim}{\nabla} \times \underset{\sim}{u} = \underset{\sim}{0}$
- $\nabla^2 \psi = 0$ ...... ②

I mean, it is potential flow.

NOTE: Here my $\hat{\underset{\sim}{\theta}}$ & $\hat{\underset{\sim}{r}}$ are polar coordinates



- $\underset{\sim}{u} = \underset{\sim}{\nabla}\phi = \frac{\partial \phi}{\partial r}\hat{\underset{\sim}{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\underset{\sim}{\theta}}$ ...... ③
- $\underset{\sim}{u} = \underset{\sim}{\nabla} \times (\psi(r,\theta)\,\hat{\underset{\sim}{z}}) = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\underset{\sim}{r}} - \frac{\partial \psi}{\partial r}\hat{\underset{\sim}{\theta}}$ ...... ④
- boundary conditions:

– $\underset{\sim}{u}_\infty = U\hat{\underset{\sim}{x}} = U\cos\theta\,\hat{\underset{\sim}{r}} - U\sin\theta\,\hat{\underset{\sim}{\theta}}$ ...... ⑤

Combining ③.④.⑤ gives:

$$\begin{cases} \dfrac{\partial \phi_\infty}{\partial r} = U\cos\theta & \text{...... ⑥} \\ \dfrac{1}{r}\dfrac{\partial \phi_\infty}{\partial \theta} = -U\sin\theta & \text{...... ⑦} \end{cases}$$

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial \psi_\infty}{\partial \theta} = U\cos\theta & \text{...... ⑧} \\ \dfrac{\partial \psi_\infty}{\partial r} = U\sin\theta & \text{...... ⑨} \end{cases}$$

– $\underset{\sim}{n} \cdot \underset{\sim}{u}_{r=a} = \underset{\sim}{n} \cdot \underset{\sim}{U}_s = 0$

$\Rightarrow \underset{\sim}{r} \cdot \underset{\sim}{u}_{r=a} = 0$ ...... ⑩

Combining ③.④.⑩ gives:

$\frac{\partial \phi_{r=a}}{\partial r} = 0$ ...... ⑪ and $\frac{1}{r}\frac{\partial \psi_{r=a}}{\partial \theta} = 0$ ...... ⑫

*). Velocity potential ϕ for uniform flow:

From ⑥, ⑦, we know that there must be $\cos\theta$ dependence for ϕ. Also, since the solution to Laplace eq. is separable and unique, so let's try the simplest form of ϕ:

$$\phi = f(r)\cos\theta \quad \text{.......} ⑬$$

Now, let's investigate $f(r)$ more closely!:

we know that for uniform flow, $\nabla^2\phi = 0$

$$\Rightarrow \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(f(r)\cos\theta)}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2(f(r)\cos\theta)}{\partial\theta^2}\right) = 0$$

$$\Rightarrow \frac{1}{r}\left(\cos\theta\frac{\partial f(r)}{\partial r} + r\cos\theta\frac{\partial^2(f(r))}{\partial r^2}\right) - \frac{f(r)}{r^2}\cos\theta = 0$$

$$\Rightarrow \frac{1}{r}\left(\frac{d(f(r))}{\partial r} + r\frac{d^2(f(r))}{dr^2}\right) - \frac{f(r)}{r^2} = 0$$

$\Rightarrow$ multiplying r^2 on both sides, we get:

$$\left(r\frac{d}{dr}-1\right)\underbrace{\left[\left(r\frac{d}{dr}+1\right)f\right]}_{P} = 0 \quad \text{.......} ☺$$

$$\Rightarrow \left(r\frac{d}{dr}-1\right)P = 0 \quad \text{.......} ⑭$$

$r\frac{dP}{dr} = P$

$\int\frac{1}{P}dP = \int\frac{1}{r}dr$

However, the eq. in ⑭ is separable! So we have:

$$\frac{\frac{dP}{dr}}{P} = \frac{1}{r} \Rightarrow \int\frac{1}{P}dP = \int\frac{1}{r}dr$$

$\Rightarrow P = ar$ ......⑮, where a is a constant of integration.

However, from ☺, we have: $P = \left(r\frac{d}{dr}+1\right)f$

$$\Rightarrow r\frac{df}{dr} + f = ar$$

$$\Rightarrow \frac{d}{dr}(rf) = ar$$

integrating both sides gives:

$rf = \frac{c_1 r^2}{2} + c_2$, where c_2 is another constant of integra-tion.

i.e. $f = c_3 r + \frac{c_2}{r}$, where $c_3 = \frac{c_1}{2}$. .... (16)

Substituting (16) into (13) gives:

$$\Phi = c_3 r \cos\theta + \frac{c_2}{r}\cos\theta \quad .... (17)$$

Now, we use B.C.'s to solve for constants c_2, c_3:

$$\frac{\partial\Phi}{\partial r} = c_3\cos\theta - \frac{c_2}{r^2}\cos\theta$$

However, from (11), we know $\frac{\partial\Phi_{r=a}}{\partial r} = 0$.

$$\Rightarrow c_3\cos\theta - \frac{c_2}{a^2}\cos\theta = 0$$

$$\Rightarrow c_3 = \frac{c_2}{a^2} \quad .... (18)$$

Moreover, from (6), we know $\frac{\partial\Phi_\infty}{\partial r} = U\cos\theta$.

$$\Rightarrow c_3\cos\theta = U\cos\theta$$

$$\Rightarrow c_3 = U \quad .... (19)$$

Combining (18) & (19) we have: $\begin{cases} c_2 = Ua^2 \\ c_3 = U \end{cases}$

Hence, for uniform flow, we have

$$\boxed{\Phi_u = Ur\cos\theta\left(1 + \frac{a^2}{r^2}\right) \quad .... (20)}$$

*) Streamfunction ψ for uniform flow:

From (8), (9), we know that there must be $\sin\theta$ dependence for ψ. Also, since the solution to Laplace e.q. is separable &

unique. So using the similar method as before, let's try the simplest form of ψ:

$$\psi = \sin\theta\, f(r) \quad \ldots\ldots (21)$$

Similarly as before, $\nabla^2\psi = 0$ gives:

$$\frac{1}{r}\left(\sin\theta \frac{df}{dr} + r\sin\theta \frac{d^2 f}{dr^2}\right) - \frac{f}{r^2}\sin\theta = 0$$, which will reduce to the same form as before:

$$\left(r\frac{d}{dr} - 1\right)\left[\left(r\frac{d}{dr} + 1\right)f\right] = 0 \quad \ldots\ldots ☺$$

So we can write down the solution for ☺:

$f = D_1 r + \frac{D_2}{r}$, where D_1, D_2 are two constants of integration ..... (22)

Substituting (22) into (21) gives:

$$\psi = D_1 r\sin\theta + \frac{D_2}{r}\sin\theta \quad \ldots\ldots (23)$$

Now, we use boundary conditions to solve for D_1, D_2:

$$\frac{\partial\psi}{\partial r} = D_1\sin\theta - \frac{D_2}{r^2}\sin\theta$$

However from (9), we know $\frac{\partial\psi_\infty}{\partial r} = U\sin\theta$

$$\Rightarrow D_1 = U.$$

Also, $\frac{\partial\psi}{\partial\theta} = Ur\cos\theta + \frac{D_2}{r}\cos\theta$

However from (12), we have $\frac{1}{r}\frac{\partial\psi_{r=a}}{\partial\theta} = 0$

$$\Rightarrow \frac{Ua\cos\theta}{a} + \frac{D_2}{a^2}\cos\theta = 0$$

$$\Rightarrow D_2 = -Ua^2.$$

Hence, for uniform flow, we have

$$\boxed{\psi_u = Ur\sin\theta\left(1 - \frac{a^2}{r^2}\right) \quad \ldots\ldots (24)}$$

Rotating Flow:

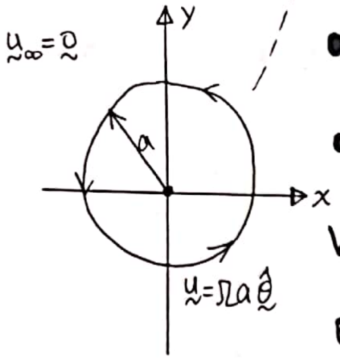
Known:

- $\nabla^2\phi = 0$ ...... (25)

I mean, since it is a potential flow, ϕ satisfies the Laplace eq.

so does ψ

- $\nabla^2\psi = 0$ ...... (26)



- The system looks the same when rotating, i.e. we have rotation symmetry $\Rightarrow$ ψ doesn't depend on θ, i.e. we have $\psi(r)$ ...... (27)
- $\underset{\sim}{u} = \underset{\sim}{\nabla} \times (\psi(r)\hat{\underset{\sim}{z}}) = -\frac{\partial\psi}{\partial r}\hat{\underset{\sim}{\theta}}$ ...... (28)
- However, since $\underset{\sim}{u} = \underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial r}\hat{\underset{\sim}{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\underset{\sim}{\theta}}$, from (28), we have $\frac{\partial\phi}{\partial r} = 0$, i.e. ϕ doesn't depend on r $\Rightarrow$ we have $\phi(r)$ and $\underset{\sim}{u} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\underset{\sim}{\theta}}$ ..... (29)
- Boundary conditions:
  - $\underset{\sim}{u} = \Omega a \hat{\underset{\sim}{\theta}}$ when $r = a$:

    so (28) $\Rightarrow$ $-\frac{\partial\psi_{r=a}}{\partial r} = \Omega a$ ...... (30)

    (29) $\Rightarrow$ $\frac{1}{a}\frac{\partial\phi_{r=a}}{\partial\theta} = \Omega a$ ...... (31)
  - $\underset{\sim}{u} = \underset{\sim}{0}$ when $r \to \infty$:

    so (28) $\Rightarrow$ $-\frac{\partial\psi_\infty}{\partial r} = 0$ ...... (32)

    $\frac{1}{r}\frac{\partial\phi_\infty}{\partial\theta} = 0$ ...... (33)

*) Streamfunction $\psi(r)$ for rotating flow:

From (26), we have $\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = 0$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 0$$

$$\Rightarrow \psi'' + \frac{1}{r}\psi' = 0 \quad \text{......} \quad (34)$$

Now, we need to solve eq. 34 :

Let $p = \psi'$

$\Rightarrow$ 34 will become: $p' + \frac{1}{r}p = 0$

$\Rightarrow rp' + p = 0$ we know $r \geq a$ because the fluid cannot penetrate through the solid cylinder

$\Rightarrow \frac{d}{dr}(rp) = 0$ 35

Solving 35 gives: $p = \psi' = \frac{c_1}{r}$, where c_1 is a constant of integration.

$\Rightarrow \psi = c_1 \log r + c_2$, where c_2 is another constant of integration.

However, from B.C.'s, we have 30 : $-\frac{\partial \psi_{r=a}}{\partial r} = \Omega a$

$\Rightarrow -\frac{\partial \psi_{r=a}}{\partial r} = \frac{-c_1}{a} = \Omega a \quad \Rightarrow c_1 = -\Omega a^2.$

For c_2, it doesn't matter what value it takes, so we simply choose $c_2 = 0$

Hence, for rotating flow, we have:

$$\psi_r = -\Omega a^2 \log r \quad \text{.........} \quad (36)$$

*). Velocity Potential for $\phi(\theta)$ for rotating flow:

From 25, we have $\nabla^2 \phi = 0$

$\Rightarrow \nabla^2 \phi = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$

$\Rightarrow \frac{1}{r^2}\frac{d^2 \phi}{d\theta^2} = 0$

$\Rightarrow \frac{1}{r}\phi'' = 0$

$\Rightarrow \phi = A\theta + B$, where A, B are two constants of integration.

How

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However, from B.C's, we have (31): $\frac{1}{a}\frac{\partial \phi_{r=a}}{\partial \theta} = \Omega a$

$$\Rightarrow \frac{1}{a}\frac{\partial \phi_{r=a}}{\partial \theta} = \frac{1}{a}A = \Omega a$$

$$\Rightarrow A = \Omega a^2.$$

For B, it doesn't matter what value it takes, so we simply choose $B=0$

Hence, for rotating flow, we have:

$$\boxed{\phi_r = \Omega a^2 \theta} \quad \ldots\ldots (37).$$

What we have so far:

Combining what we discussed before, we have

$$\begin{cases} \phi_u = Ur\cos\theta\left(1+\frac{a^2}{r^2}\right) \\ \psi_u = Ur\sin\theta\left(1-\frac{a^2}{r^2}\right) \\ \phi_r = \Omega a^2\theta \\ \psi_r = -\Omega a^2 \log r \end{cases}$$

Using Superposition principle, we have

$$\phi = \phi_u + \phi_r = Ur\cos\theta\left(1+\frac{a^2}{r^2}\right) + \Omega a^2\theta \quad \ldots\ldots (38)$$

$$\psi = \psi_u + \psi_r = Ur\sin\theta\left(1-\frac{a^2}{r^2}\right) - \Omega a^2 \log r \quad \ldots\ldots (39)$$

So $\underset{\sim}{u} = \underset{\sim}{\nabla}\phi = \hat{\underset{\sim}{r}}\frac{\partial \phi}{\partial r} + \hat{\underset{\sim}{\theta}}\frac{1}{r}\frac{\partial \phi}{\partial \theta}$

$$\Rightarrow \underset{\sim}{u} = \underbrace{U\left(1-\frac{a^2}{r^2}\right)\cos\theta}_{U_r}\,\hat{\underset{\sim}{r}} + \underbrace{\left(\frac{\Omega a^2}{r} - U\left(1+\frac{a^2}{r^2}\right)\sin\theta\right)}_{U_\theta}\hat{\underset{\sim}{\theta}} \quad (40)$$

ii).

pressure:

For inviscid flow, the constitutive eq. is:

$$\underset{\sim}{T} = -p\underset{\sim}{I} \quad \ldots\ldots \text{(41)}$$, where p- the pressure is what we need to solve for.

Since from i). we know $\underset{\sim}{w} = \underset{\sim}{0}$ for this complete flow, Bernoulli's eq is valid everywhere!

i.e. $\frac{1}{2}q^2 + \frac{p}{\rho} + \chi$ = constant everywhere. $q = |\underset{\sim}{u}||\underset{\sim}{u}|$

Since there is no body force in this question, we have

$$\frac{1}{2}q^2 + \frac{p}{\rho} = \text{constant everywhere} \quad \ldots\ldots \text{(42)}$$

*). (Infinitely) far from the cylinder:

$\Rightarrow \frac{1}{2}U^2 + \frac{P_\infty}{\rho} = \text{constant} \quad \ldots\ldots \text{(43)}$, where P_∞ denotes the pressure (infinitely) far from the cylinder.

*). At r = a:

$\frac{1}{2}U_\theta^2 + \frac{P_{r=a}}{\rho} = \text{constant} \quad \ldots\ldots \text{(44)}$. where $P_{r=a}$ denotes the pressure at r=a and $U_\theta = \frac{\Omega a^2}{r} - U(1+\frac{a^2}{r^2})\sin\theta$.

Equate ② & ③ then gives:

$$P_{r=a} = \left(\frac{1}{2}U^2 - \frac{1}{2}U_\theta^2 + \frac{P_\infty}{\rho}\right)\rho \quad \ldots\ldots \text{(45)}$$

How to calculate $P_{r=a}$? To avoid somewhat horrible calculation, Let $-2\pi a^2\Omega = \Gamma$.

Then $U_\theta = -U(1+\frac{a^2}{r^2})\sin\theta - \frac{\Gamma}{2\pi a} \quad \ldots\ldots \text{(46)}$

So on r=a: $U_\theta = -2U\sin\theta - \frac{\Gamma}{2\pi a} \quad \ldots\ldots \text{(47)}$

Substituting (47) into (45) gives:

$$P_{r=a} = \left(\tfrac{1}{2}U^2 - \tfrac{1}{2}\left(-2U\sin\theta - \frac{\Gamma}{2\pi a}\right)^2 + \frac{P_\infty}{\rho}\right)\rho.$$

i.e. $\boxed{P_{r=a} = P_\infty + \tfrac{1}{2}\rho U^2 - 2U^3\sin^2\theta\,\rho - \frac{\rho\Gamma^2}{8\pi^2 a^2} - \frac{U\Gamma\rho\sin\theta}{\pi a}, \text{ where } \Gamma = -2\pi a^2\Omega.}$

<u>Net Force/unit length</u>:

Since we don't have body force in this question:

$$\underset{\sim}{F}_{net} = \oint \underset{\sim}{T}\cdot\underset{\sim}{n}\,dl$$

$$= -\oint (P_{r=a})\,\underset{\sim}{n}\,dl \quad \ldots\ldots (48)$$

from the constitutive eq.: $\underset{\sim}{T} = -P\underset{\sim}{I}$.

However, $\underset{\sim}{n} = \underset{\sim}{r}$. So we now need to convert $\underset{\sim}{n}$ into Cartesi-an coordinate s.t. we can bring $\hat{\underset{\sim}{i}}$ & $\hat{\underset{\sim}{j}}$ outside the integral: $\Rightarrow \underset{\sim}{r} = \cos\theta\,\hat{\underset{\sim}{i}} + \sin\theta\,\hat{\underset{\sim}{j}}$.

So eq. (48) will become:

$$\underset{\sim}{F}_{net} = -\int_0^{2\pi} P_{r=a}(\cos\theta\,\hat{\underset{\sim}{i}} + \sin\theta\,\hat{\underset{\sim}{j}})\,a\,d\theta$$

$$= \underbrace{\left(-a\int_0^{2\pi} P_{r=a}\cos\theta\,d\theta\right)\hat{\underset{\sim}{i}}}_{(A)} + \underbrace{\left(-a\int_0^{2\pi} P_{r=a}\sin\theta\,d\theta\right)\hat{\underset{\sim}{j}}}_{(B)} \quad \ldots\ldots (49)$$

<u>Solve for (A) first</u>:

$$(A) \Rightarrow -a\hat{\underset{\sim}{i}}\int_0^{2\pi} P_\infty\cos\theta + \tfrac{1}{2}\rho U^2\cos\theta - 2U^3\sin^2\theta\cos\theta\,\rho - \frac{\rho\Gamma^2}{8\pi^2 a^2}\cos\theta - \frac{U\Gamma\rho}{\pi a}\sin\theta\cos\theta\,d\theta$$

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$$\Rightarrow -a\hat{i}\left\{\left[P_\infty \sin\theta + \frac{1}{2}\rho U \sin\theta - \frac{\rho\Gamma^2}{8\pi^2 a^2}\sin\theta + \frac{U\Gamma\rho}{2\pi a}\cos 2\theta\right]\Big|_0^{2\pi} - 2\int_0^{2\pi} U^2 \sin^2\theta \cos\theta\, \rho\, d\theta\right\} \cdots\cdots (50)$$

0

*). What is $-2U^2\rho\int_0^{2\pi}\sin^2\theta\cos\theta\, d\theta$ then?

Let $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$$\Rightarrow -2U^2\rho\int_0^0 u^2\, du = 0 \quad \cdots\cdots (51)$$

Substituting (51) into (50), we end up getting:

$$(A) = -a\hat{i}\int_0^{2\pi} P_{r=a}\cos\theta\, d\theta = 0\,\hat{i} \quad \cdots\cdots (52)$$

Next, solve for (B):

$$(B) \Rightarrow -a\hat{j}\int_0^{2\pi} P_\infty \sin\theta + \frac{1}{2}\rho U^2 \sin\theta - 2U^2\sin^3\theta\,\rho - \frac{\rho\Gamma^2}{8\pi^2 a^2}\sin\theta - \frac{U\Gamma\rho}{\pi a}\sin^2\theta \; d\theta$$

$$\Rightarrow -a\hat{j}\left[\left(-P_\infty\cos\theta - \frac{1}{2}\rho U^2\cos\theta + \frac{\rho\Gamma^2}{8\pi^2 a^2}\cos\theta\right)\Big|_0^{2\pi}\right] + 2aU^2\rho\hat{j}\int_0^{2\pi}\sin^3\theta\, d\theta + \frac{U\Gamma\rho}{\pi}\hat{j}\int_0^{2\pi}\sin^2 d\theta \quad \cdots\cdots (53)$$

*). What is $\int_0^{2\pi}\sin^3\theta\, d\theta$ then?

$$\Rightarrow \int_0^{2\pi}\sin\theta\,(1-\cos^2\theta)\, d\theta$$

$$= \left[-\cos\theta\right]_0^{2\pi} + \int_1^1 u^2\, du$$

$$= 0 \quad \cdots\cdots (54)$$

*). What is $\int_0^{2\pi}\sin^2 d\theta$ then?

$$\Rightarrow \frac{1}{2}\int_0^{2\pi} 1-\cos 2\theta\, d\theta = \pi \quad \cdots\cdots (55)$$

Substituting 54, 55 into 53 gives:

$$\textcircled{B} = -a\hat{j}\int_0^{2\pi} p_{r=a}\sin\theta\, d\theta$$

$$= \frac{U\Gamma\rho}{\pi}\pi\,\hat{\underset{\sim}{j}}$$

$$= U\Gamma\rho\,\hat{\underset{\sim}{j}} \quad \cdots\cdots (56)$$

Finally, combining 52 & 56, we'll get:

$$\underset{\sim}{F}_{net} = U\Gamma\rho\,\hat{\underset{\sim}{j}} + 0\,\hat{\underset{\sim}{i}} = U\Gamma\rho\,\hat{\underset{\sim}{j}} \quad \cdots\cdots (57)$$

per unit length

iii).

<u>drag & lift on the cylinder:</u>

From e.g. 57, we have $\underset{\sim}{F}_{net} = \underbrace{U\Gamma\rho\,\hat{\underset{\sim}{j}}}_{\underset{\sim}{F}_{lift}} + \underbrace{0\,\hat{\underset{\sim}{i}}}_{\underset{\sim}{F}_{drag}}$

So we have

$$\underset{\sim}{F}_{lift} = U\Gamma\rho\,\hat{\underset{\sim}{j}} = -2\pi a^2\Omega U\rho\,\hat{\underset{\sim}{j}} \quad \cdots\cdots (58)$$

$$\underset{\sim}{F}_{drag} = \underset{\sim}{0} \quad \cdots\cdots (59)$$

per unit length

<u>Relation between $\underset{\sim}{F}_{lift}$ & $\underset{\sim}{F}_{drag}$ & the cylinder's direction of rotation?</u>

*) <u>$\underset{\sim}{F}_{drag}$:</u>

Since $\underset{\sim}{F}_{drag} = \underset{\sim}{0}$, it has no dependence on the angular velocity of the spinning cylinder, i.e. no matter in which direction the cylinder rotates, $\underset{\sim}{F}_{drag}$ will always be zero. (D'Alembert's Paradox).

*). $\underset{\sim}{F}_{lift}$:

From (58), we know that $\underset{\sim}{F}_{lift} = -2\pi a^2 \Omega U \rho \hat{\underset{\sim}{j}}$.

There is an Ω dependence!

When solving for the velocity potential in part (i), we use the B.C. : $\underset{\sim}{u}_{r=a} = \Omega a \hat{\underset{\sim}{\theta}}$ to solve for the constant of integration. However, if we change the cylinder's direction of rotation, i.e. Ω is in $-\hat{\underset{\sim}{z}}$ direction, the B.C. will become $\underset{\sim}{u}'_{r=a} = -\Omega a \hat{\underset{\sim}{\theta}}$! Tracing each step of calculating the lift force, we can see that if we change the cylinder's rotation direction from $\Omega \hat{\underset{\sim}{z}}$ to $-\Omega \hat{\underset{\sim}{z}}$, $\underset{\sim}{F}'_{lift} = 2\pi a^2 \Omega U \rho \hat{\underset{\sim}{j}} = -\underset{\sim}{F}_{lift}$!

This actually intuitively makes sense since

- if cylinder is rotating in $\hat{\underset{\sim}{z}}$ direction with angular velocity Ω:

the fluid's velocity near the bottom boundary of the cylinder is bigger than that near the top boundary of the cylinder. So P_{bottom} is smaller than P_{top}, i.e. the cylinder will experience a $-\hat{\underset{\sim}{j}}$ direction lift force, and vice versa.

To conclude in this question:

rotational direction (angular velocity)
- $\hat{\underset{\sim}{z}}$ direction : $-\hat{\underset{\sim}{j}}$-direction lift
- $-\hat{\underset{\sim}{z}}$ direction : $+\hat{\underset{\sim}{j}}$- direction lift

(v).

Streamfunction:

$$\psi = Ur\left(1-\frac{a^2}{r^2}\right)\sin\theta - \Omega a^2 \log r \quad ③$$ from before.

From the geometry of this problem:

*). Appropriate length scale is:

$$l_c = a$$

So the dimensionless length: $\bar{r} = \frac{r}{a}$ ...... ①

*). Velocity scale is:

$$u_c = U$$

So the dimensionless velocity: $\bar{u} = \frac{u}{U}$ ...... ②

Substituting ① & ② into ③ gives:

$$\psi = Ua\bar{r}\left(1-\frac{\cancel{a^2}}{\cancel{a^2}\bar{r}^2}\right)\sin\theta - \Omega a^2\log(a\bar{r})$$

$$= Ua\left(\bar{r}-\frac{1}{\bar{r}}\right)\sin\theta - \Omega a^2\log(a\bar{r})$$

since the constant 'a' doesn't matter

$$\Rightarrow \bar{\Psi} = \frac{\psi}{Ua} = \left(\bar{r}-\frac{1}{\bar{r}}\right)\sin\theta - \frac{\Omega a}{U}\log(\bar{r})$$

Since $\bar{\Omega} = \frac{\Omega a}{U}$, we have

$$\boxed{\bar{\Psi} = \left(\bar{r}-\frac{1}{\bar{r}}\right)\sin\theta - \bar{\Omega}\log(\bar{r})}$$

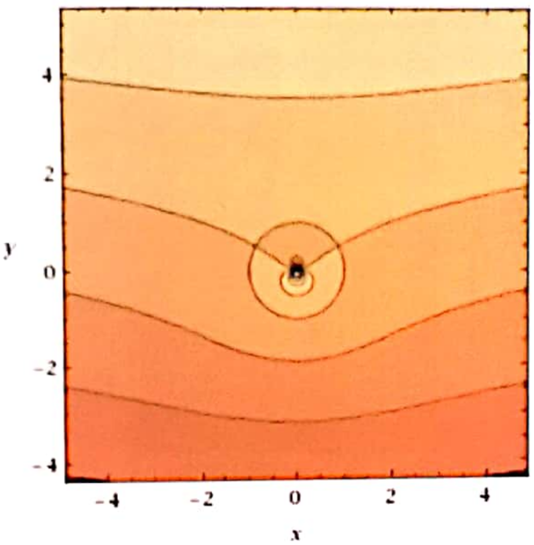
What's the Physical significance of $\bar{\Omega}$?

$\bar{\Omega} = \frac{\Omega a}{U}$.

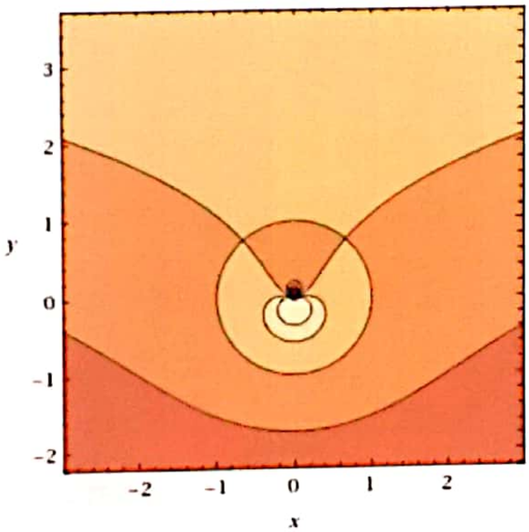
$\bar{\Omega}$ tells us the relative rotating velocity of the cylinder w.r.t. to the flow velocity. i.e. $\bar{\Omega}$ tells us the infomation about how quickly the cylinder's surface is rotating compared to how quickly the flow is moving.

the streamlines are the level sets of the streamfunction. From Wolfram Alpha, we got:

$\bar{\Omega} = 1$:



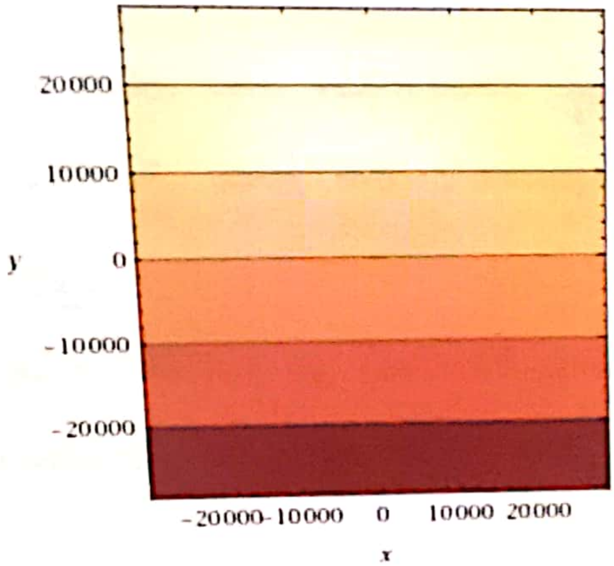
$\bar{\Omega} = 1.5$:



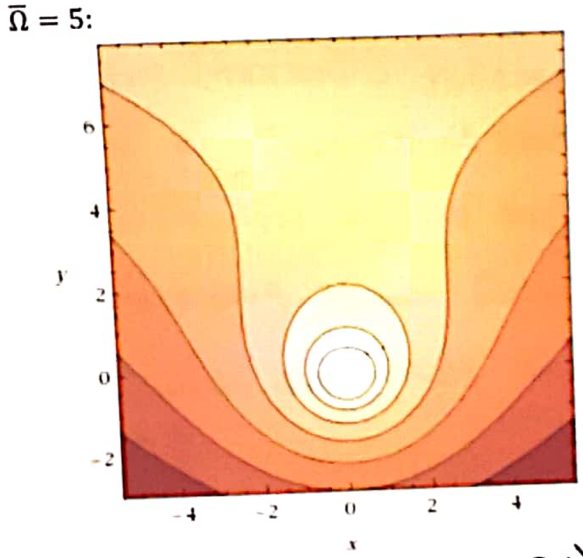
$\bar{\Omega} = 2$:

< to be continued >

$\bar{\Omega} = 2.$



$\bar{\Omega} = 5$:



From our graph, we can see that the bigger the angular velocity of the rotating cylinder, the greater lift force the cylinder will experience, which actually makes sense physically. This is indicated by the density of streamlines. From Lecture we know that higher density of streamlines means lower pressure, so the lift force is created (in $-\hat{i}$ direction in this problem).

think about the Mill-Race effect.

However, $\bar{\Omega} = 2$ seems to be a critical point while the cylinder will receive no lift force since the density of the streamlines/the pressure is constant from my above diagram.

V).

Common Occurrence.

This flow can represent football, tennis, table tennis rotating and translating through air.

Why?

Think about a football, say, rotating & translating in $-\hat{\underset{\sim}{x}}$ direction through air, as follows:

In the football's reference frame, i.e. in the frame where the football is stationary, the air's motion can be approximately modelled by the superposition of two flows —

1). rotating steady flow which is stationary far from the ball and whose angular velocity at the ball's surface is $\Omega\hat{\underset{\sim}{z}}$.

2). uniform flow with velocity $\Omega\hat{\underset{\sim}{x}}$.

which is exactly an analogy of our Q3.

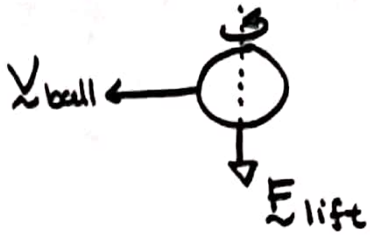
Physical Insight:

If the lift force due to the ball's rotation isn't ignorable compared to the ball's weight, the trajectory of the ball is closely related to its rotation direction.

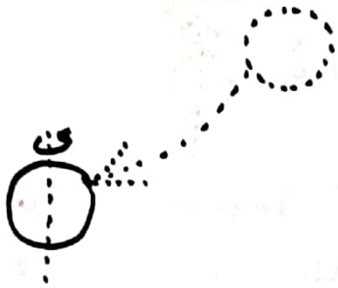
i.e:

< TO BE CONTINUED >

at $t=0$:

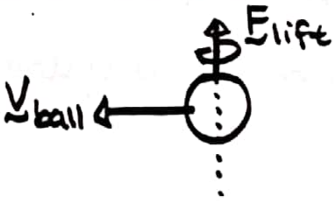


at $t=\tau>0$:

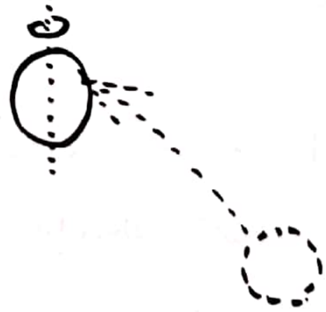


while

at $t=0$:



at $t=\tau>0$:



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