MAT4MDS — Practice 3 Worked Solutions

Model Answers to Practice 3

Question 1.

(a) $C+D=\begin{bmatrix}7 & 5\\ 3 & 5\end{bmatrix}$

(b) C is 2×2 and E is 2×3. Matrices of different orders can't be added.

(c) $3F=\begin{bmatrix}3 & 0 & 0\\ 21 & 15 & -9\\ 12 & -6 & 3\end{bmatrix}$

(d) $G-4D=\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}-\begin{bmatrix}4 & 8\\ 4 & 16\end{bmatrix}=\begin{bmatrix}-2 & -8\\ -5 & -15\end{bmatrix}$

(e) $F+F^T=\begin{bmatrix}1 & 0 & 0\\ 7 & 5 & -3\\ 4 & -2 & 1\end{bmatrix}+\begin{bmatrix}1 & 7 & 4\\ 0 & 5 & -2\\ 0 & -3 & 1\end{bmatrix}=\begin{bmatrix}2 & 7 & 4\\ 7 & 10 & -5\\ 4 & -5 & 2\end{bmatrix}$

(f) E is 2×3,so E^T is 3×2. They can't be added because their orders are different.

Question 2. Since M is of size $p\times q$, the matrix M^TM is $(q\times p)(p\times q)\rightsquigarrow(q\times q)$ i.e. square.

Question 3.

(a) $CE=\begin{bmatrix}6 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & -1\end{bmatrix}=\begin{bmatrix}6+6 & 0+3 & 6-3\\ 2+2 & 0+1 & 2-1\end{bmatrix}=\begin{bmatrix}12 & 3 & 3\\ 4 & 1 & 1\end{bmatrix}$.

(b) F is 3×3 and E is 2×3 so it is not possible to calculate the product FE. The number of columns of F is not the same as the number of rows of E.

(c) EC cannot be calculated, as E is 2×3 and C is 2×2. The number of columns of E is not the same as the number of rows of C.

(d) $EF=\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 7 & 5 & -3\\ 4 & -2 & 1\end{bmatrix}=\begin{bmatrix}5 & -2 & 1\\ 5 & 7 & -4\end{bmatrix}$

(e) $CD=\begin{bmatrix}6 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}9 & 24\\ 3 & 8\end{bmatrix}$

(f) $DC=\begin{bmatrix}1 & 2\\ 1 & 4\end{bmatrix}\begin{bmatrix}6 & 3\\ 2 & 1\end{bmatrix}=\begin{bmatrix}10 & 5\\ 14 & 7\end{bmatrix}$

(g) $DG=\begin{bmatrix}1 & 2\\ 1 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}=\begin{bmatrix}0 & 2\\ -2 & 4\end{bmatrix}$.

We notice that the order of matrix multiplication matters. We have an example ($CD\neq DC$) that shows that it cannot be true in general that there is a commutative property $AB=BA$. In fact, one product may be defined and the other not.

Question 4. Using answers from Question 3,

$$C(DG) = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & 24 \\ -2 & 8 \end{bmatrix}$$

$$(CD)G = \begin{bmatrix} 9 & 24 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 24 \\ -2 & 8 \end{bmatrix}.$$

In this example, grouping does not matter. As we know, one example in which matrix multiplication is associative is not enough to prove that it is true in general!

(However, it *is* a true property, but a general proof, using the notation of Section 2.3 of the unit text, is messy.)

Question 5.

(a) $(E^T)^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} = E$

Since the rows of $(E^T)^T$ are the columns of E^T, which are the rows of E, we can see this is true for any matrix.

(b) $F + F^T$ is symmetric about the diagonal, so that it is its own transpose. This cannot be true for all matrices, because Question 1 (f) shows that it may not be defined. But for square matrices, we would suspect that it is true.

(c) Possibly one might suspect that $C^T D^T$ is $(CD)^T$, but it is not:

$$C^T D^T = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 5 & 7 \end{bmatrix} \qquad D^T C^T = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 24 & 8 \end{bmatrix} = (CD)^T$$

Question 6.

(a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

(b) $\det \begin{pmatrix} 2 & -3 \\ 5 & 7 \end{pmatrix} = 14 - (-15) = 29$

(c) $\begin{vmatrix} 1 & -2 \\ -3 & -4 \end{vmatrix} = -4 - 6 = -10$

(d) $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb$

Question 7.

(a) $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 3 & 4 & 1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & 4 \\ 3 & 4 \end{vmatrix} = 0 + 0 + 0 = 0.$

(b) $\begin{vmatrix} 0 & -2 & 1 \\ 3 & 1 & 1 \\ 3 & 1 & 0 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 \\ 3 & 0 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = 2(-3) + 0 = -6.$

(c) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 3 & 0 \\ 0 & 0 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = -27.$

(d) We can expand using any row or column, so we use the third column:

$$\begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 10 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix} = -10 \begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= -10\left\{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} -2 & 5 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 5 \\ 2 & 3 \end{vmatrix}\right\}$$

$$= -10\{(4-3) - (-4-5) + 3(-6-10)\} = 380.$$

Question 8.

(a) (i) $\det(J_{2\times2}) = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

(ii) $(J_{2\times2})^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1^2+1^2 & 1^2+1^2 \\ 1^2+1^2 & 1^2+1^2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2J_2$

(iii) This is an inner product:

$$J_{m\times1}^T J_{m\times1} = [1 \quad 1 \quad \dots \quad 1]\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = [1^2 + 1^2 + \dots + 1^2] = [m]$$

(b) (i) $\det(J_n) = 0$, because all determinants are expanded in terms of smaller determinants, and $|J_2| = 0$

(ii) $(J_n)^2 = nJ_n$, because each entry is the sum of n entries, each of which is 1^2.

Question 9.

(a) Each entry in the matrix product $J_{13}D$ is the sum of the entries in the corresponding column of D.

(b) Each entry in the matrix $\frac{1}{13}J_{13}D$ is the average (algebraic mean) of the entries in the corresponding column of D. For example, in column one each entry is the average sepal length of the 13 flower specimens.

(c) The result of forming the product $C_{13}D$, where the matrix $C_{13} := I_{13} - \frac{1}{13}J_{13}$, is to form a matrix in which each entry is the difference of each original measurement from the overall mean measurement of the corresponding property.

Question 10.

(a) The mean is a measure of centre for a data set. The matrix C_n is used to form the difference of data arranged in a column from the mean of the data in that column, finding how far it is from the centre. (See answer to Question 9 (c).)

(b)

$$C_n^2 = \left(I_n - \frac{1}{n}J_n\right)\left(I_n - \frac{1}{n}J_n\right)$$

$$= I_n^2 - \frac{1}{n}I_nJ_n - \frac{1}{n}J_nI_n + \frac{1}{n^2}J_n^2$$

$$= I_n - \frac{2}{n}J_n + \frac{n}{n^2}J_n \quad \text{using properties of } I_n \text{ and } J_n$$

$$= I_n - \frac{1}{n}J_n = C_n$$

After C_n has been applied once, we have a column of entries which express difference from the mean of the data. The mean of this new column of information is zero. Applying C_n again to this new data finds the difference of each of these entries from zero hence they will be the values obtained by centering them the first time.

Question 11.

(a)

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) The statement applied to **square** matrices. In this case A is (2×3) and B is (3×2). We also see that BA is not an identity matrix.

(c) A is not square, and hence has no inverse.

Question 12.

(a) Only C is not invertible, because: $|C| = \begin{vmatrix} 6 & 3 \\ 2 & 1 \end{vmatrix} = 0$ $|D| = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2$

$$|F| = \begin{vmatrix} 1 & 0 & 0 \\ 7 & 5 & -3 \\ 4 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 5 & -3 \\ -2 & 1 \end{vmatrix} = -1, \quad |G| = \begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix} = 1.$$

(b) $D^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}$ and $G^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$.

Question 13.

(a) $X = A^{-1}B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

Check: $AX = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = B$. ✓

(b) $A^{-1}B = -\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -6 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 10 \\ -4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$.

Check: $AX = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix} = B$. ✓

Question 14.

(a) Assume that A^{-1} exists. Then

$$\begin{aligned} & AA^{-1} = I_n \\ \Rightarrow \quad & \det(AA^{-1}) = \det(I_n) \\ \Rightarrow \quad & \det(A)\det(A^{-1}) = 1 \qquad \text{(by the Multiplicative Property)} \\ \Rightarrow \quad & \det(A^{-1}) = \frac{1}{\det(A)}. \end{aligned}$$

(b) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and hence

$$\det(A^{-1}) = \det\left(\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right) = \det\begin{pmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{pmatrix} = \frac{da}{|A|^2} - \frac{bc}{|A|^2} = \frac{da - bc}{|A|^2} = \frac{|A|}{|A|^2} = \frac{1}{|A|}.$$

(c) (i) $\det(D^{-1}) = \begin{vmatrix} 2 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} = \frac{1}{\det(D)}$; $\det(G^{-1}) = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1 = \frac{1}{\det(G)}$.

(ii)

$$DG = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

$$GD = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

So $|DG| = 2$, and $|GD| = 2$, both of which equal $|D||G|$ as expected.

(iii) The socks-and-shoes formula introduced in the readings tells us that $(DG)^{-1} = G^{-1}D^{-1}$. This can be calculated using G^{-1} and D^{-1}:

$$(DG)^{-1} = G^{-1}D^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}.$$

Then checking this answer:

$$(DG)^{-1}(DG) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I_2. \ \checkmark$$

(If you try the other way around, you will see that $D^{-1}G^{-1}(DG) \neq I_2$.)