MAT4MDS — Practice 10

INTEGRATION II

Substitution rule

For suitable functions f and g we have

$$\int_a^b f(u)\frac{du}{dx}\ dx = \int_{g(a)}^{g(b)} f(u)\ du$$

where $u = g(x)$.

Question 1. Use integration by substitution to do the following integrals:

(a) $\int_0^1 (1+x^4)^5x^3\ dx$.

(b) $\int_1^2 \frac{3x}{1+x^2}\ dx$.

(c) $\int_0^1 \frac{e^x}{1+e^x}\ dx$.

(d) $\int_0^\infty x^2e^{-x^3}\ dx$. (Remember, you will have to take a limit here!)

The following example shows how the substitution rule can be used in the reverse direction.

Example: Find $\int_0^1 x(x+1)^8\ dx$ with the substitution $x = t-1$.

Let $x = g(t) = t-1$ giving $\frac{dx}{dt} = 1$. Note that $g(1) = 0$ and $g(2) = 1$, so

$$\begin{aligned}
\int_0^1 x(x+1)^8\ dx &= \int_1^2 (t-1)(t)^8\frac{dx}{dt}\ dt \quad \text{by substitution} \\
&= \int_1^2 (t-1)(t)^8\ dt \quad as\ \frac{dx}{dt} = 1 \\
&= \int_1^2 t^9 - t^8\ dt \\
&= \left[\frac{t^{10}}{10} - \frac{t^9}{9}\right]_1^2 = \frac{512}{5} - \frac{512}{9} - \frac{1}{10} + \frac{1}{9} = \frac{49}{20}.
\end{aligned}$$

Question 2. Following the example above, calculate the integral $\int_1^5 x(x-1)^{\frac{1}{2}}\ dx$ using the substitution $x = t+1$.

Integration by Parts

$$\int_a^b u\frac{dv}{dx}\ dx = uv|_a^b - \int_a^b v\frac{du}{dx}\ dx$$

Choosing $\mathbf{u}$ and $\frac{\mathbf{dv}}{\mathbf{dx}}$: There is no hard and fast way of doing this but the following may help.

(1) If the integral contains a function you don't know the integral of, choose this as u.

(2) If the integral contains a function whose integral is more complicated, choose this as u.

(3) Polynomials are often good candidates for u

(4) Functions like e^x are often good candidates for $\frac{du}{dx}$.

(5) If one choice doesn't get you anywhere, try choosing the other way around.

Question 3. Use integration by parts to find the following:

(a) $\int_2^4 x^3 \ln(x)\ dx$.

(b) $\int_0^1 xe^{-x}\ dx$

(c) $\int_0^1 x^2e^{-x}\ dx$ (Hint: after applying integration by parts once, use your answer to (b).)

Question 4. The trapezoidal rule for numerical integration was introduced in Reading 8.3. In this question, we apply the trapezoidal rule to the function $f(x) = x^2 + 1$ on the interval $[0,1]$

(a) In each of (i), (ii), and (iii): Sketch the graph of f for $0 \le x \le 1$, shade the regions that give the trapezoidal rule approximation on the given number of sub-intervals to $\int_0^1 f(x)\ dx$ and **mark the relevant heights on your diagram**:

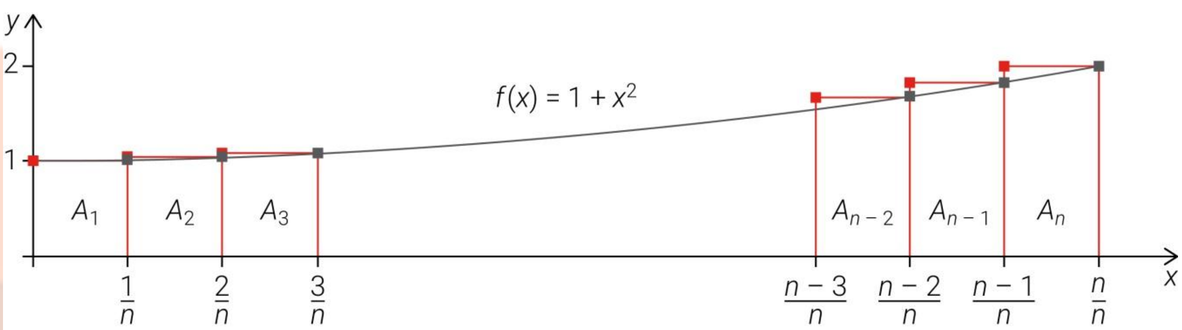
(i) 1 sub-interval,

(ii) 3 sub-intervals,

(iii) 5 sub-intervals.

(b) Use each of your diagrams from (a) to calculate an approximation to $\int_0^1 f(x)\ dx$. Give answers to 4 decimal places.

Question 5. Let $f(x) = x^2 + 1$. The diagram below shows what happens when we find an approximation to $\int_0^1 f(x)\ dx$ by using the right-hand end point approximation with n sub-intervals.



(a) Use the diagram to get a formula for the Riemann sum approximation to $\int_0^1 f(x)\ dx$ on n subintervals by writing down the first 3 terms of the sum, then $\cdots$, and then the last 3 terms of the sum, writing $f(1)$ as $f(\frac{n}{n})$, with no simplifications.

(b) By rearranging the terms, taking out common factors etc., show that your formula simplifies to

$$1+\frac{1}{n^3}(1^2+2^2+\cdots+(n-2)^2+(n-1)^2+n^2).$$

(c) Use the formula $1^2+2^2+\cdots+(n-2)^2+(n-1)^2+n^2=\frac{1}{6}n(n+1)(2n+1)$ to show that the formula in (b) simplifies to $1+\frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n})$.

(d) Use the formula $\int_0^1 x^2+1\,dx \approx 1+\frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n})$ to find the approximation to the definite integral $\int_0^1 x^2+1\,dx$ when $n=10$ and when $n=100$.

(e) We define $\int_0^1 x^2+1\,dx := \lim_{n\to\infty}\left(1+\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right)$. Take this limit to evaluate $\int_0^1 x^2+1\,dx$.

(f) Find $\int_0^1 x^2+1\ dx$. Is it the same as (e)?

The cumulative distribution function F is an anti-derivative of the probability density function f for continuous data. That is:

$$P(X\leqslant x)=F(x)=\int_{-\infty}^{x} f(t)dt$$

The **mean** value is given by

$$\int_{-\infty}^{\infty} xf(x)dx$$

Question 6.

(a) Find the mean value of the normal distribution, where the probability density function is $f(x)=ce^{-x^2}$. (Note that c is a constant such that f is correctly normalised to be a probability density.)

(b) For the Cauchy distribution, what happens when you attempt to calculate the mean?

(c) The exponential distribution has

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & x\geqslant 0\\ 0 & x<0\end{cases}.$$

Calculate the mean of this distribution.