MAST30013 – Techniques in Operations Research

Semester 1, 2021

Tutorial 7 Solutions

1. The resource constraint can be written as:

$$10x + 20y = 1000.$$

The optimization problem is

$$\begin{aligned} \max \quad & R(x, y) = 5xy \\ \text{s.t.} \quad & 10x + 20y = 1000. \end{aligned}$$

(a) The Lagrangian for the above problem is

$$L(x, y, \eta) = 5xy + \eta(1000 - 10x - 20y).$$

The Lagrangian condition is

$$\nabla L(x, y, \eta) = 0 \quad \Rightarrow \quad \begin{bmatrix} 5y - 10\eta \\ 5x - 20\eta \\ 1000 - 10x - 20y \end{bmatrix} = 0. \tag{1}$$

The first two equations give $\eta = y/2 = x/4$, which implies $y = x/2$. Substituting this in the third equation, and solving for x gives $x = 50$. Then $y = 25$ and $\eta = 12.5$. The stationary point is $(50, 25)$.

Note since the constraint is an affine function, the constraint qualifications are satisfied at the stationary point. We use the second-order sufficiency condition to check the stationary point $(50, 25)$ is a point of local maxima. That is to check

$$d^T \nabla^2_{x,y} L(x^*, y^*, \eta^*) d < 0 \quad d \in \mathcal{C}(x^*, y^*),$$

where

$$\mathcal{C}(x^*, y^*) = \{d \in \mathbb{R}^2 : d \neq 0, \nabla h(x^*, y^*)^T d = 0\}.$$

The Jacobian is

$$\nabla h(x^*, y^*) = \nabla h \left(\begin{bmatrix} 50 \\ 25 \end{bmatrix} \right) = \begin{bmatrix} -10 \\ -20 \end{bmatrix}.$$

Now

$$\begin{aligned} & \nabla h(x^*, y^*)^T d = (-10, -20) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \\ & \Rightarrow -10d_1 - 20d_2 = 0 \\ & \Rightarrow d_2 = -d_1/2 \\ & \Rightarrow d = \begin{bmatrix} d_1 \\ -d_1/2 \end{bmatrix} \end{aligned}$$

So

$$d^T \nabla^2_{x,y} L(x^*, y^*, \eta^*) d = (d_1, -d_1/2) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ -d_1/2 \end{bmatrix} = -d_1^2 < 0.$$

The second order sufficient condition implies that the stationary point $(50, 25)$ is a local maxima.

(b) According to the economic interpretation of Lagrangian multipliers, the change in the objective function value due to a change ϵ in budge is approximately to $\eta^*\epsilon$. Thus, if the company increases its budget by \$100, the corresponding increase in the revenue is expected to be $\$12.5 \times 100 = \$1,250$.

2. The problem can be formulated to maximize (revenue - cost) or minimize (cost - revenue):

$$\begin{aligned} \max \quad & 3x^{\frac{1}{3}}y^{\frac{1}{3}} - wx - vy \\ \text{s.t.} \quad & x = 1000; \end{aligned}$$

$$\begin{aligned} \min \quad & wx + vy - 3x^{\frac{1}{3}}y^{\frac{1}{3}} \\ \text{s.t.} \quad & x = 1000. \end{aligned}$$

(a) (Now take the minimization problem as example.) The Lagrange function is

$$L(x, y, \eta) = wx + vy - 3x^{\frac{1}{3}}y^{\frac{1}{3}} + \eta(x - 1000)$$

$$\nabla_{x,y} L = 0 \quad \Rightarrow \quad \begin{bmatrix} w - x^{-\frac{2}{3}}y^{\frac{1}{3}} + \eta \\ v - x^{\frac{1}{3}}y^{-\frac{2}{3}} \end{bmatrix} = 0 \tag{2}$$

$$h(x) = 0 \quad \Rightarrow \quad x = 1000 \tag{3}$$

From the bottom line of (2),

$$y^{\frac{2}{3}} = \frac{x^{\frac{1}{3}}}{v} \quad \Rightarrow \quad y = \left(\frac{10}{v}\right)^{\frac{3}{2}}.$$

Now, from the top line of (2),

$$w - 1000^{-\frac{2}{3}}\left(\frac{10}{v}\right)^{\frac{1}{2}} + \eta = 0.$$

Then

$$\eta = \frac{1}{100}\left(\frac{10}{v}\right)^{\frac{1}{2}} - w.$$

Note since the constraint is an affine function, the constraint qualifications are satisfied at the stationary point $(1000, (10/v)^{\frac{3}{2}})$. We use the second-order sufficiency condition to check the stationary point is a point of local minima.
Given

$$\nabla h(x, y) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \nabla h(x^*, y^*)^T d = 0 \Rightarrow d = \begin{bmatrix} 0 \\ d_2 \end{bmatrix} \quad \forall d_2 \neq 0.$$

Then

$$\begin{aligned} d^T \nabla^2_{xy} L(x^*, y^*, \eta^*) d &= (0, d_2) \begin{bmatrix} \frac{\partial^2 L}{\partial x \partial x} & \frac{\partial^2 L}{\partial y \partial x} \\ \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial y \partial y} \end{bmatrix} \begin{bmatrix} 0 \\ d_2 \end{bmatrix} \\ &= \frac{\partial^2 L}{\partial y \partial y} d_2^2 \\ &= \frac{2}{3} x^{*\frac{1}{3}} y^{*-\frac{5}{3}} d_2^2 \\ &= \frac{20}{3}\left(\frac{10}{v}\right)^{-\frac{5}{2}} d_2^2 > 0. \end{aligned}$$

So the stationary point is a local minima.

(b) If η^* is positive, then the firm is willing to pay $\eta^*\epsilon$ to increase ϵ in the x quote.