Q1.3

$$f_{X_1}(x) = \cancel{\int f_{X_1}}$$

$$\int f_{X_1 | X_0 = x_0}(x) \cdot f_{X_0}(x)\, dx_0$$

$$= \alpha_0 \alpha_1 \int \exp\left(-\frac{1}{2} \times \frac{(x_0-\mu_0)^2}{\sigma_0^2} + \frac{(x_1-x_0)^2}{\sigma^2}\right) dx_0$$

$$= \alpha_0 \alpha_1 \int \exp\left(-\frac{1}{2} \times \frac{\sigma^2(x_0-\mu_0)^2 + \sigma_0^2(x-x_0)^2}{\sigma_0^2 \cdot \sigma^2}\right) \cdot dx_0$$

$$= \alpha_0 \alpha_1 \int \exp\left(-\frac{1}{2} \times \frac{(\sigma^2+\sigma_0^2)x_0^2 - 2(\sigma^2\mu_0 + \sigma_0^2 x)x_0 + \sigma^2\mu_0^2 + \sigma_0^2 x^2}{\sigma_0^2 \sigma^2}\right)$$

$$= \alpha_0 \alpha_1 \int \exp\left(-\frac{1}{2} \times \frac{x_0^2 - 2\left(\frac{\sigma^2\mu_0 + \sigma_0^2 x_1}{\sigma^2+\sigma_0^2}\right)x_0 + \left(\frac{\sigma^2\mu_0 + \sigma_0^2 x}{\sigma^2+\sigma_0^2}\right)^2}{\sigma_0^2 \cdot \sigma^2 / (\sigma^2+\sigma_0^2)}\right) dx_0$$

$$\cdot \exp\left(-\frac{1}{2} \times \frac{\sigma^2\mu_0^2 + \sigma_0^2 x^2 - \frac{(\sigma^2\mu_0+\sigma_0^2 x_1)^2}{\sigma^2+\sigma_0^2}}{\sigma_0^2\sigma^2}\right) dx_0$$

$$= \alpha \cdot \exp\left(-\frac{1}{2} \times \frac{(\sigma^2\mu_0^2 + \sigma_0^2 x^2)(\sigma^2+\sigma_0^2) - (\sigma^2\mu_0 + x_0^2 x)^2}{\sigma_0^2 \sigma^2 (\sigma^2+\sigma_0^2)}\right)$$

$$= \alpha \cdot \exp\left(-\frac{1}{2} \frac{x^2 - 2\mu_0 x_1 + \mu_0^2}{\sigma^2+\sigma_0^2}\right)$$

$$= \alpha \cdot \exp\left(-\frac{1}{2} \frac{(x_0-\mu_0)^2}{\sigma^2+\sigma_0^2}\right)$$

Compared with $\alpha \cdot \exp\left(-\frac{1}{2} \frac{(x-\mu_1)^2}{\sigma_1^2}\right)$

$\Rightarrow$ ① $\mu_1 = \mu_0$

② $\sigma_1 = \sqrt{\sigma^2+\sigma_0^2}$

③ $\alpha = \dfrac{1}{\sqrt{2\pi(\sigma^2+\sigma_0^2)}}$