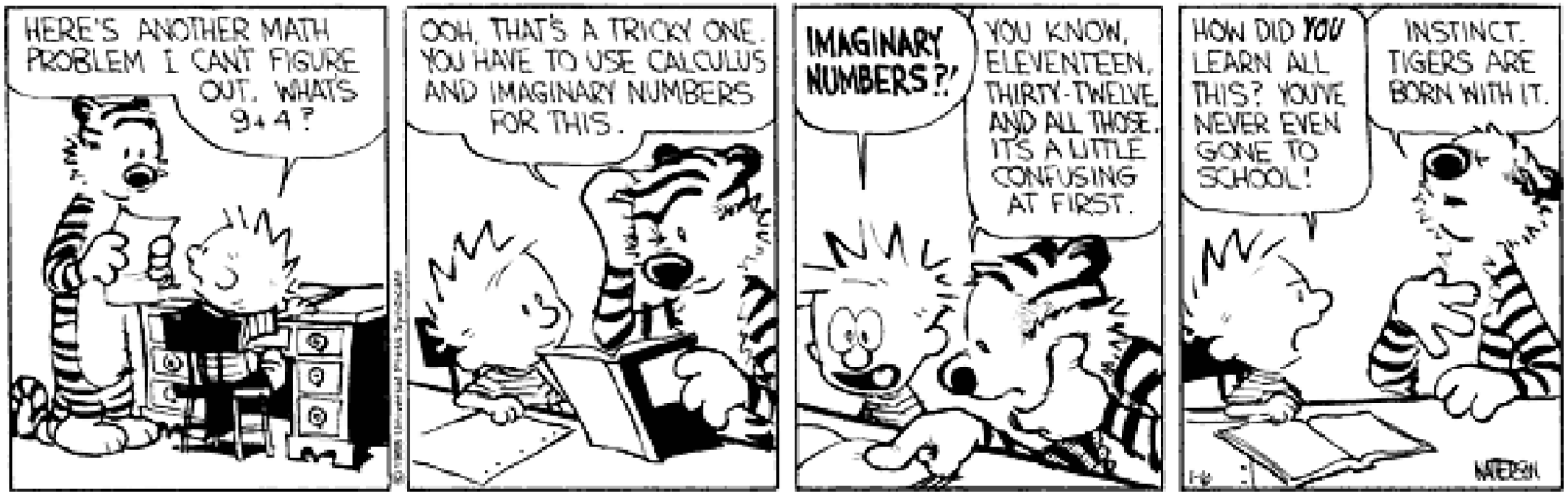
Complex Numbers



Complex Numbers - Representation

We form complex numbers around the basic building block

$$j = \sqrt{-1}.$$

Note: Most math books use the letter i to represent $\sqrt{-1}$. In electrical engineering, we prefer to use the letter j since we commonly use i to represent current.

Cartesian Representation of Complex Numbers

In its *Cartesian* (rectangular) *form*, a complex number is written in terms of a *real part* and an *imaginary part*.

Note: The j is NOT included in the imaginary part.

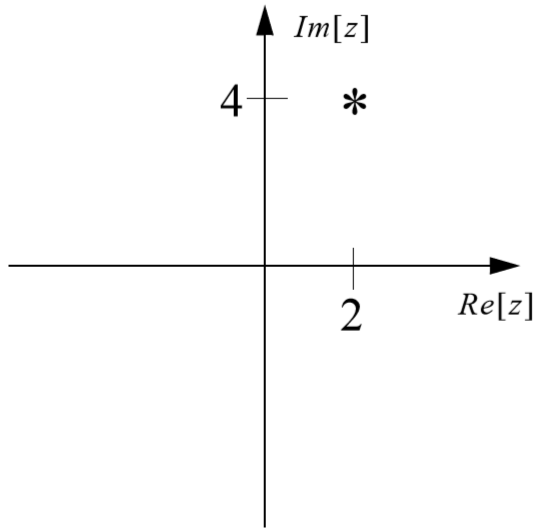
$$\mathrm{Im}[2 + 4j] \neq 4j$$

Example: $z = 2 + 4j$

$$Re[2 + 4j] = 2 \text{ (real part)}$$

$$\mathrm{Im}[2 + 4j] = 4 \text{ (imaginary part)}$$

We visualize complex numbers as points in a 2-D plane where the x-axis is the real part and the y-axis is the imaginary part.



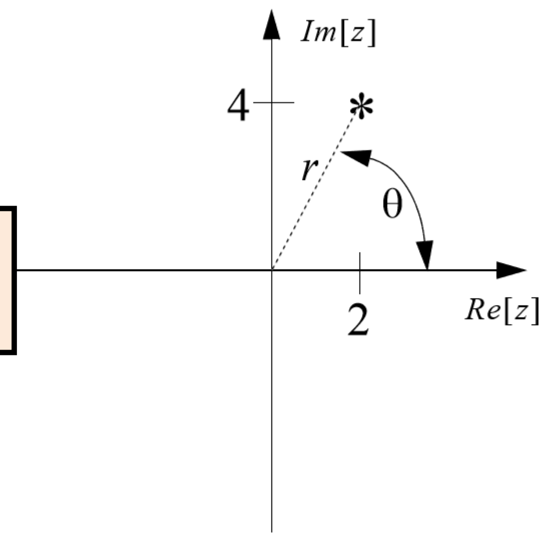
Complex Numbers - Representation

<u>Polar Representation of Complex Numbers</u>

In its *polar form*, a complex number is written in terms of a *magnitude*, r, and a *phase*, θ. The polar form of a complex number stems from *Euler's Identity*

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

Can you prove this?



Multiplying both sides by r produces

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta).$$

A complex number with magnitude, r, and a phase, θ, is written in its polar form as $z = re^{j\theta}$. The Cartesian representation of the same number would be $z = x + jy$ where

$$x = Re\left[re^{j\theta}\right] = r\cos(\theta),$$
$$y = Im\left[re^{j\theta}\right] = r\sin(\theta).$$

Note: You may also see the notation $r\angle\theta$ instead of $re^{j\theta}$

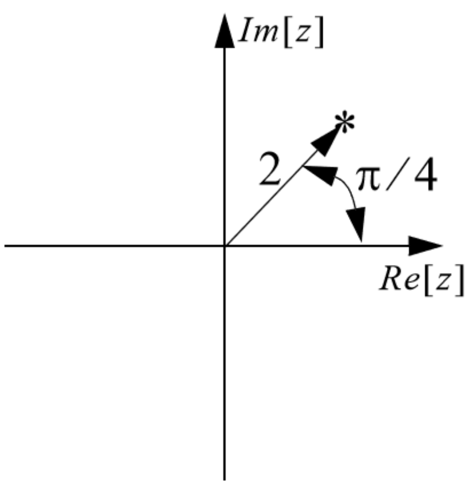
Note: This is the same conversion from polar to Cartesian co-ordinates that you saw in your multivariable Calculus course. The inverse transformation (Cartesian to polar) is then

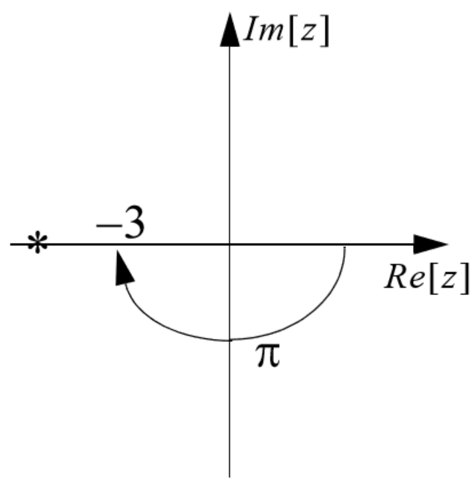
$$r = \sqrt{x^2 + y^2}$$
$$\theta = tan^{-1}(y/x)$$

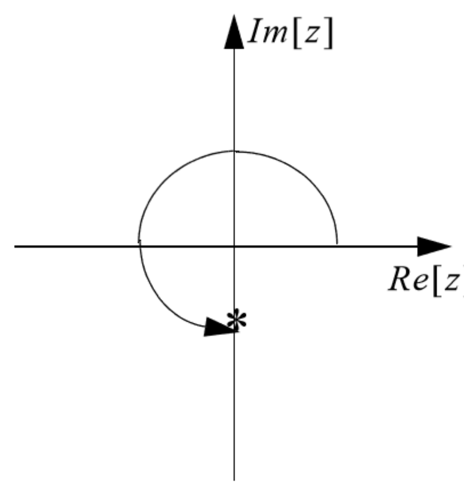
Complex Numbers - Representation

<u>Examples:</u> Convert each of the following complex numbers from polar to Cartesian form.

(a) $z = 2e^{j(\pi/4)}$ (b) $z = 3e^{-j\pi}$ (c) $z = e^{j3\pi/2}$







$$2e^{j(\pi/4)} = 2\cos\left(\frac{\pi}{4}\right) + j2\sin\left(\frac{\pi}{4}\right)$$

$$= \sqrt{2} + j\sqrt{2}$$

$$3e^{-j\pi} = 3\cos(-\pi) + j3\sin(-\pi)$$

$$= -3 + j0 = -3$$

$$e^{j3\pi/2} = \cos\left(\frac{3\pi}{2}\right) + j\sin\left(\frac{3\pi}{2}\right)$$

$$= 0 + j(-1) = -j$$

Complex Numbers - Representation

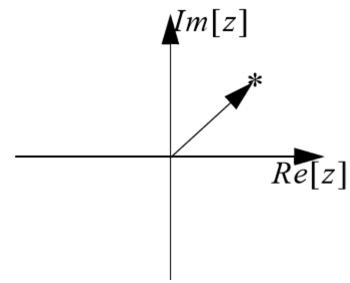
Examples: Convert each of the following complex numbers from Cartesian to polar form.

(a) $z = 1 + j$ (b) $z = 2j$ (c) $z = -1 - 2j$

(a) $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

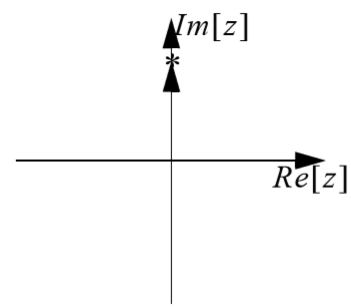
$z = 1 + j = \sqrt{2}e^{j(\pi/4)}$



(b) $r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 2^2} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{0}\right) = \frac{\pi}{2}$

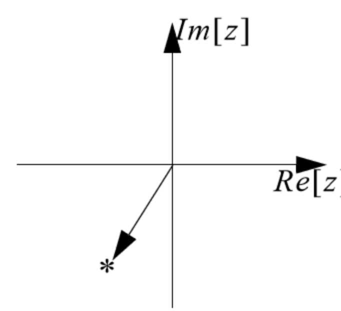
$z = 2j = 2e^{j(\pi/2)}$



(c) $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-2}{-1}\right) = 4.25\text{rad}$

$z = -1 - 2j = \sqrt{5}e^{j(4.25)}$



Note: For this example, your calculator will probably tell you the angle is 1.11rad. Why?

Complex Numbers - Conjugation

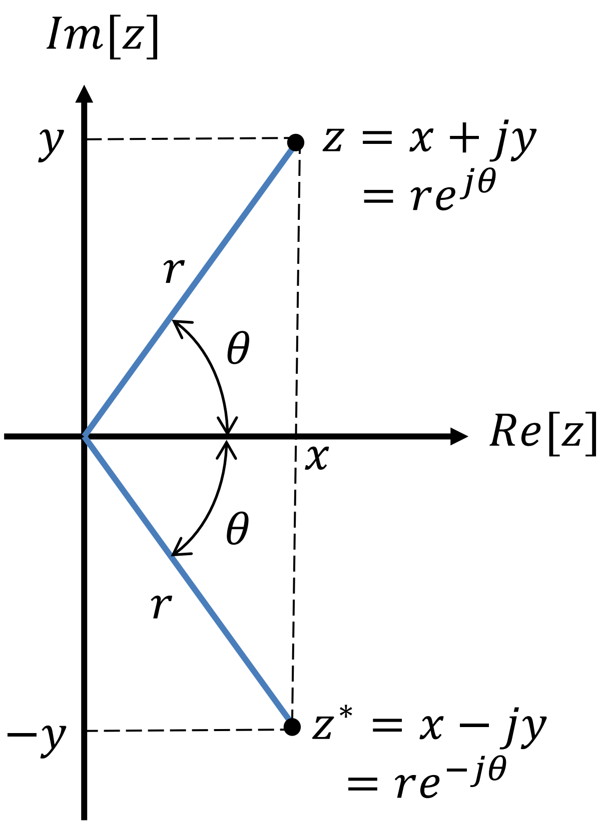
The conjugate z^* of a complex number z is formed by changing all of the j's to $-j$. You may also see the notation $\bar{z}$ used to represent conjugation.

In Cartesian form: $z = x + jy \rightarrow z^* = x - jy$.

In polar form: $z = re^{j\theta} \rightarrow z^* = re^{-j\theta}$.

Some useful properties involving complex conjugates:

- $zz^* = |z|^2 = r^2 = x^2 + y^2$.
- $(z^*)^* = z$.
- If $f(z)$ is a polynomial function of z with real coefficients, $f(z) = 0 \Leftrightarrow f(z^*) = 0$. That is, roots of real polynomials occur in complex conjugate pairs.

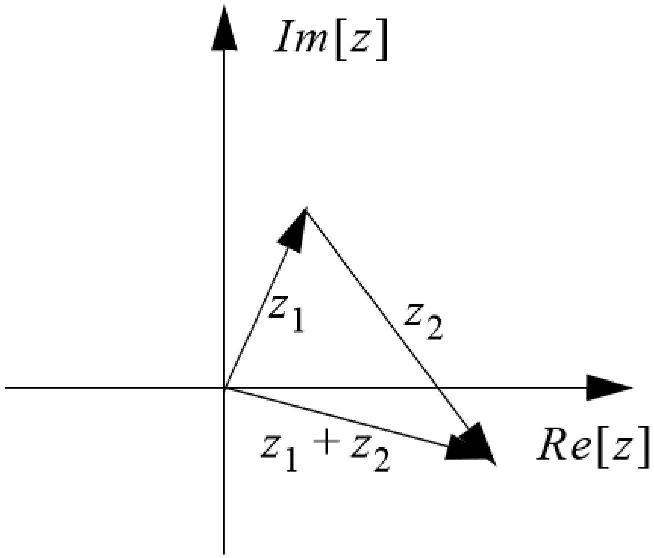


Complex Numbers - Arithmetic

It is easiest to add and subtract complex numbers in their Cartesian form.

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$
$$= (x_1 + x_2) + j(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 + jy_1) - (x_2 + jy_2)$$
$$= (x_1 - x_2) + j(y_1 - y_2)$$



Visually, this works just like vector addition and subtraction.

If you are given complex numbers in their polar form and you need to add or subtract them, you will need to:

1) Convert both complex numbers from polar to Cartesian form.
2) Add or subtract them as shown above.
3) Convert the answer back to polar form (if desired).

Advanced Note: If we view $Re[\,]$ and $Im[\,]$ as operators we see that:

$$Re[z_1 \pm z_2] = Re[z_1] \pm Re[z_2]$$
$$Im[z_1 \pm z_2] = Im[z_1] \pm Im[z_2]$$

➔ We can exchange the order of operations involving Re/Im and +/-.

We will see the same is not true for multiplication/division.

Complex Numbers - Arithmetic

It is easiest to multiply and divide numbers in polar form:

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$$

From this we can infer a few more useful properties of complex numbers:

- $|z_1 z_2| = |z_1||z_2|$ (magnitude of product is product of magnitudes)
- $\angle(z_1 z_2) = \angle z_1 + \angle z_2$ (angle of product is the sum of the angles)
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (magnitude of quotient is quotient of magnitudes)
- $\angle\left(\frac{z_1}{z_2}\right) = \angle z_1 - \angle z_2$ (angle of quotient is the difference of the angles)

Complex Numbers - Arithmetic

It is possible to multiply and divide complex numbers in their Cartesian form, but it is a little more complicated than using the polar form.

Multiplication

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2)$$

$$= x_1 x_2 + jy_1 x_2 + jy_2 x_1 + j^2 y_1 y_2 \quad \text{(using FOIL)}$$

$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \quad \text{(using } j^2 = -1\text{)}$$

Division

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2}$$

$$= \frac{x_1 + jy_1}{x_2 + jy_2} * \frac{x_2 - jy_2}{x_2 - jy_2}$$

(multiply numerator and denominator by conjugate of denominator)

$$= \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} \quad \text{(using } zz^* = |z|^2 = x^2 + y^2 \text{ in denominator)}$$

$$= \frac{(x_1 x_2 + y_1 y_2) + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \quad \text{(using FOIL and } j^2 = -1 \text{ in numerator)}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \quad \text{(just cleaning up a little)}$$

Complex Numbers – Exponentials and Logs

Exponentials: Evaluating exponentials of complex numbers is easiest to do using the Cartesian form. Euler's identity along with properties of exponents can be used to help us evaluate exponentials of complex numbers.

$$\begin{aligned} e^z &= e^{x+jy} = e^x e^{jy} \\ &= e^x \cos(y) + j e^x \sin(y). \end{aligned}$$

Logs (Natural): Evaluating logs of complex numbers is easiest using the polar form. Here we just use log properties to help us.

$$\begin{aligned} \log(z) &= \log\left(re^{j\theta}\right) \\ &= \log(r) + \log\left(e^{j\theta}\right) \\ &= \log(r) + j\theta. \end{aligned}$$

Advanced Note: When dealing with complex numbers, $f(z) = \log(z)$ is what is called a multi-valued function. Suppose for example we wanted to find $\log(-5)$. The number -5 can be written as

$$-5 = 5e^{j\pi}$$

so that

$$\begin{aligned} \log(-5) &= \log\left(5e^{j\pi}\right) \\ &= \log(5) + j\pi. \end{aligned}$$

Complex Numbers – More on Exponentials

Euler's identity allows us to relate complex exponentials and trig functions.

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$ ← Euler's identity (1)

$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$ ← Conjugate (2)

➔ $\cos(\theta) = \mathrm{Re}[e^{j\theta}]$

➔ $\sin(\theta) = \mathrm{Im}[e^{j\theta}]$

This relationship plays a fundamental role in the analysis of AC circuits

Adding (1) and (2) and dividing by 2 gives

$$\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}.$$

Subtracting (1) and (2) and dividing by $2j$ gives

$$\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j}.$$

Note: These relationships between exponentials and trig functions make it quite easy to develop all sorts of trig identities. For example:

$$\cos^2(\theta) = \frac{e^{j2\theta} + 2 + e^{-2j\theta}}{4}$$

$$\sin^2(\theta) = \frac{-e^{j2\theta} + 2 - e^{-2j\theta}}{4}$$

Adding these two expressions proves that $\cos^2(\theta) + \sin^2(\theta) = 1$.

Complex Numbers – Practice Problems

1. Convert the following complex numbers from polar to Cartesian form:

   (a) $z_1 = 2\exp(j\pi/3)$, (b) $z_2 = 5\exp(j7\pi/4)$, (c) $z_3 = \exp(-j3\pi/2)$.

2. Convert the following complex numbers from Cartesian to polar form:

   (a) $z_4 = -1 + 2j$, (b) $z_5 = -1$, (c) $z_6 = -1 - j$.

3. Simplify the following quantities:

   (a) $\exp\left(j\frac{\pi}{4}\right) + 2\exp(-j\pi)$, (b) $j(2-3j)$, (c) $\frac{1+3j}{2-2j}$,

   (d) $\frac{2\exp\left(j\frac{\pi}{2}\right)}{3\exp\left(-j\frac{\pi}{3}\right)}$, (e) $(1-3j)^2$, (f) $(2-2j)^{-1}$,

   (g) $\exp(-2+4j)$, (h) $\exp\left(-5\exp\left(j\frac{\pi}{3}\right)\right)$, (i) $(1+j)^{12}$.