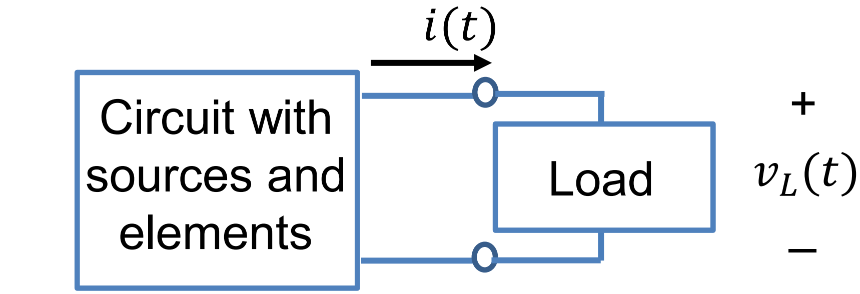
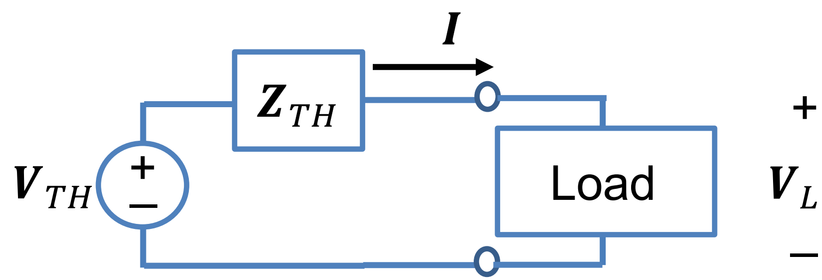
Power Transfer in AC Circuits

Suppose we have some circuitry which is trying to transfer power to a load.



We can model it in terms of a Thevenin equivalent with:



$$\boldsymbol{Z}_{TH} = R_{TH} + jX_{TH},$$
$$\boldsymbol{Z}_L = R_L + jX_L.$$

The current and load voltage (in phasor form) are:

$$\boldsymbol{I} = \frac{\boldsymbol{V}_{TH}}{\boldsymbol{Z}_{TH}+\boldsymbol{Z}_L}, \quad \boldsymbol{V}_L = \boldsymbol{I}\boldsymbol{Z}_L = \frac{\boldsymbol{V}_{TH}\boldsymbol{Z}_L}{\boldsymbol{Z}_{TH}+\boldsymbol{Z}_L}.$$

The power at the load is:

$$\boldsymbol{S}_L = \boldsymbol{V}_L\boldsymbol{I}^* = |\boldsymbol{I}|^2\boldsymbol{Z}_L = \frac{|\boldsymbol{V}_{TH}|^2\boldsymbol{Z}_L}{|\boldsymbol{Z}_{TH}+\boldsymbol{Z}_L|^2}$$

- Suppose we are interested in the following problem:

For a given (fixed) source circuitry, what value of $\boldsymbol{Z}_L$ will result in maximum real power transferred to the load?

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The real power in the load is calculated as:

$$P_L = Re[\boldsymbol{S}_L] = \frac{|\boldsymbol{V}_{TH}|^2 R_L}{|\boldsymbol{Z}_{TH} + \boldsymbol{Z}_L|^2} = \frac{|\boldsymbol{V}_{TH}|^2 R_L}{(R_{TH}+R_L)^2 + (X_{TH}+X_L)^2}.$$

To solve this optimization problem, we form

$$\frac{\partial P_L}{\partial R_L} = 0,$$
$$\frac{\partial P_L}{\partial X_L} = 0.$$

- Two equations to solve for the two unknowns (R_L, X_L).

$$\frac{\partial P_L}{\partial R_L} = \frac{|\boldsymbol{V}_{TH}|^2\left((R_{TH}+R_L)^2 + (X_{TH}+X_L)^2\right) - 2|\boldsymbol{V}_{TH}|^2 R_L(R_{TH}+R_L)}{\left((R_{TH}+R_L)^2 + (X_{TH}+X_L)^2\right)^2} = 0$$

$$\frac{\partial P_L}{\partial X_L} = \frac{-2|\boldsymbol{V}_{TH}|^2 R_L(X_{TH}+X_L)}{\left((R_{TH}+R_L)^2 + (X_{TH}+X_L)^2\right)^2} = 0$$

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The previous slide provides two equations in two unknowns to solve in order to find the optimal load impedance. For fractions to be equal to zero, it is sufficient for the numerators to be equal to zero, hence the equations simplify to:

$$\left((R_{TH}+R_L)^2 + (X_{TH}+X_L)^2\right) - 2R_L(R_{TH}+R_L) = 0$$

$$R_L(X_{TH}+X_L) = 0$$

Looking at the second equation, if we choose $R_L = 0$, this would result in $P_L = 0$. This is not what we want (it minimizes power rather than maximizing it). Therefore, we need to choose $X_L = -X_{TH}$. With that choice, the first equation simplifies to

$$(R_{TH}+R_L)^2 - 2R_L(R_{TH}+R_L) = 0$$
$$(R_{TH}+R_L)(R_{TH}-R_L) = 0$$

This leads to $R_L = \pm R_{TH}$. Since we can't use negative resistances, we end up with $R_L = R_{TH}$.

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In summary, power in the load will be maximized if:

$$\begin{matrix} R_L = R_{TH} \\ X_L = -X_{TH} \end{matrix} \Rightarrow \mathbf{Z}_L = R_{TH} - j\,X_{TH} = \mathbf{Z}_{TH}^{\ *}.$$

With this solution, the maximum power delivered to the load is:

$$P_L = \frac{|\mathbf{V}_{TH}|^2}{4R_{TH}}.$$

Note: The previous derivation was done assuming RMS phasors. If we had used peak amplitudes rather than RMS amplitudes, the result would be

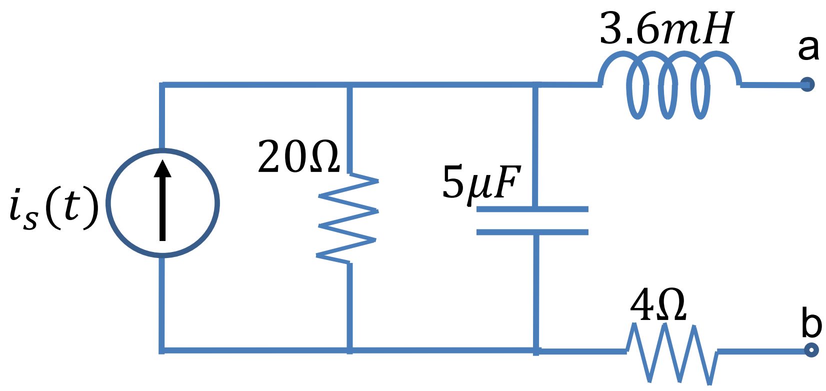
$$P_L = \frac{|\mathbf{V}_{TH}|^2}{8R_{TH}}.$$

Example

If $i_s(t) = 3\cos(5000t)$ Amps,

(a) What impedance should be connected across terminals a,b for maximum average power transfer?

(b) For the load impedance found in part (a), what average power is transferred to the load?



Example

If $i_s(t) = 3\cos(5000t)$ Amps, and the load is restricted to be purely resistive:

(a) What resistance should be connected across terminals a,b for maximum average power transfer?

(b) For the load resistance found in part (a), what average power is transferred to the load?