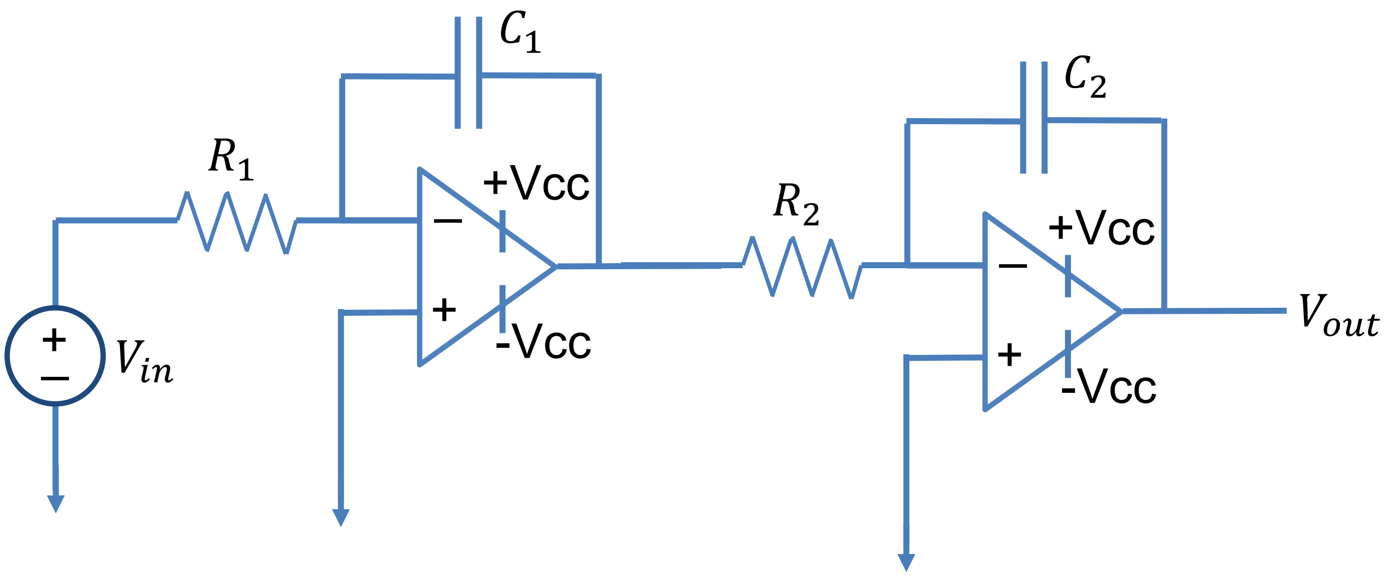
# Second Order Op Amp Circuits

We often build second order circuits with op amps rather than using inductors as inductors are difficult to integrate onto a chip and therefore do not lend themselves well to miniaturized circuits. The simplest way to do this would be to cascade two first order op amp circuits.




In the circuit above we have strung two integrators together. You should be able to convince yourself that the input and output of this circuit are related by:

$$\frac{d^2V_{out}}{dt^2} = \frac{V_{in}}{R_1C_1R_2C_2}$$

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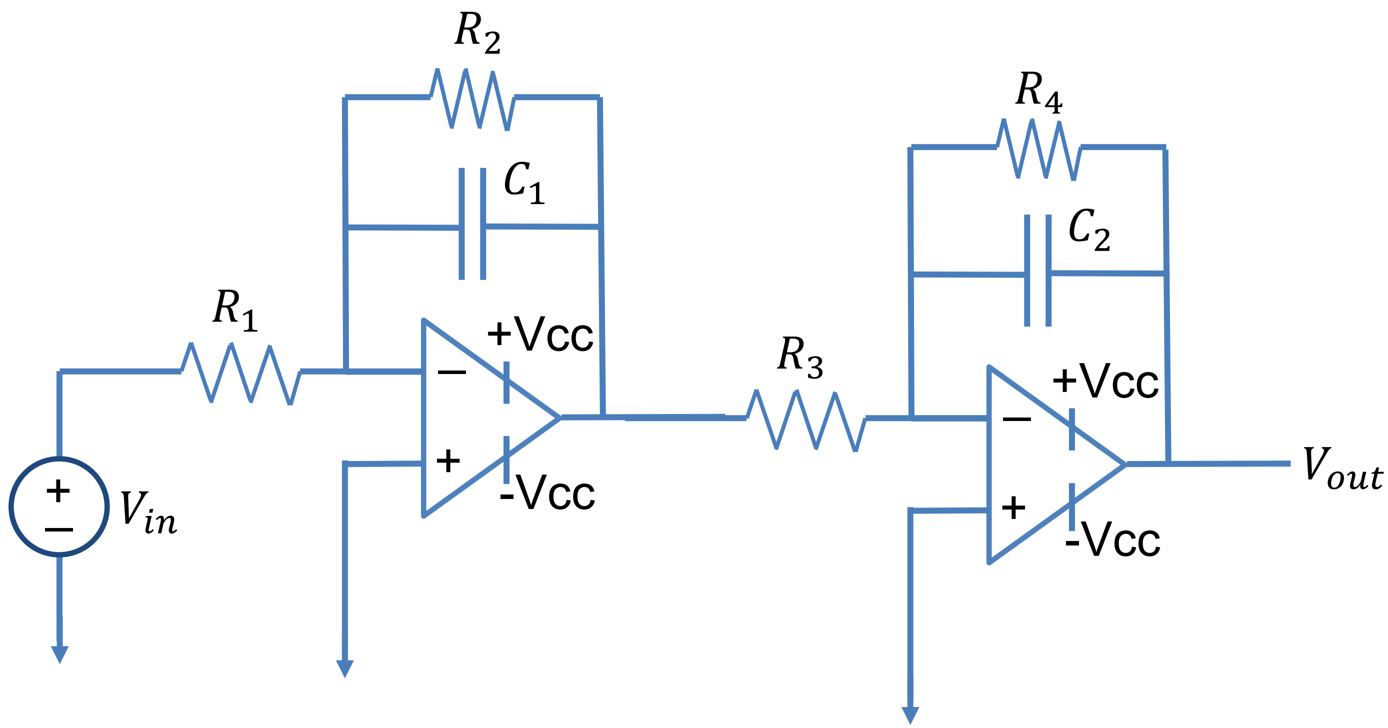
Now suppose we add an extra resistor in the feedback path of each stage.




Now what is the differential equation that describes the relationship between the input and output?

Can this produce an output that is underdamped? Overdamped? Critically damped?

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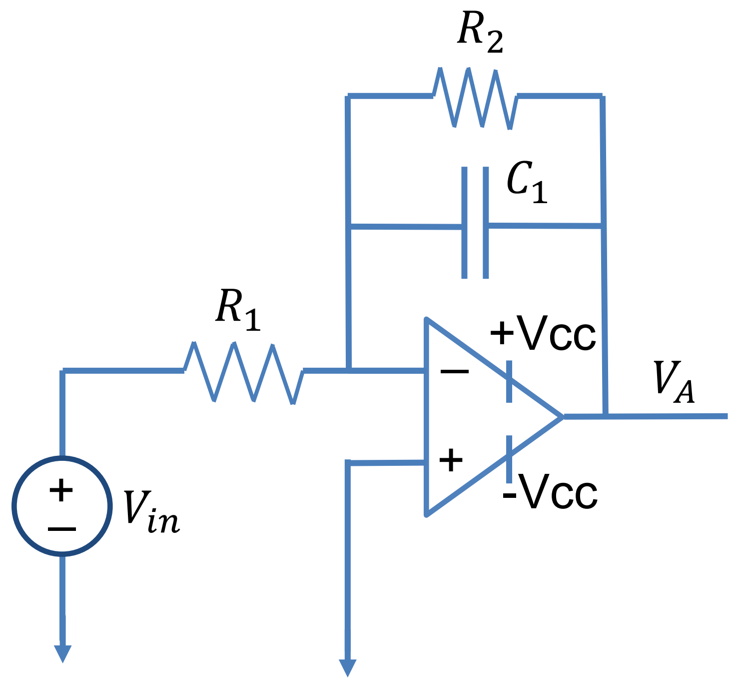
Looking at just one stage from the circuit on the previous slide we get:

$$\frac{V_{in}}{R_1} + C_1\frac{dV_A}{dt} + \frac{V_A}{R_2} = 0$$

or

$$\frac{dV_A}{dt} + \frac{V_A}{R_2C_1} = -\frac{V_{in}}{R_1C_1} \quad (1)$$




Similarly, for the second stage we would get:

$$\frac{dV_{out}}{dt} + \frac{V_{out}}{R_4C_2} = -\frac{V_A}{R_3C_2} \quad (2)$$

Substituting (2) into (1) we get:

$$-\frac{d}{dt}\left(R_3C_2\frac{dV_{out}}{dt} + \frac{R_3V_{out}}{R_4}\right) - \frac{R_3C_2}{R_2C_1}\left(\frac{dV_{out}}{dt} + \frac{V_{out}}{R_4C_2}\right) = -\frac{V_{in}}{R_1C_1}$$

# Second Order Op Amp Circuits

After a little cleaning up, the differential equation that describes this circuit is:

$$\frac{d^2V_{out}}{dt^2} + \left(\frac{1}{R_2C_1} + \frac{1}{R_4C_2}\right)\frac{dV_{out}}{dt} + \frac{V_{out}}{R_2R_4C_1C_2} = \frac{V_{in}}{R_1R_3C_1C_2}$$

The characteristic equation associated with this differential equation is:

$$s^2 + \left(\frac{1}{R_2C_1} + \frac{1}{R_4C_2}\right)s + \frac{1}{R_2R_4C_1C_2} = 0$$

This equation factors as

$$\left(s + \frac{1}{R_2C_1}\right)\left(s + \frac{1}{R_4C_2}\right) = 0.$$

Clearly the roots of this equation are real which means that this circuit could produce an overdamped response or possibly a critically damped response.

The circuit on the next slide will provide us with a means to synthesize any second order response (i.e., underdamped, overdamped, or critically damped).

## Example

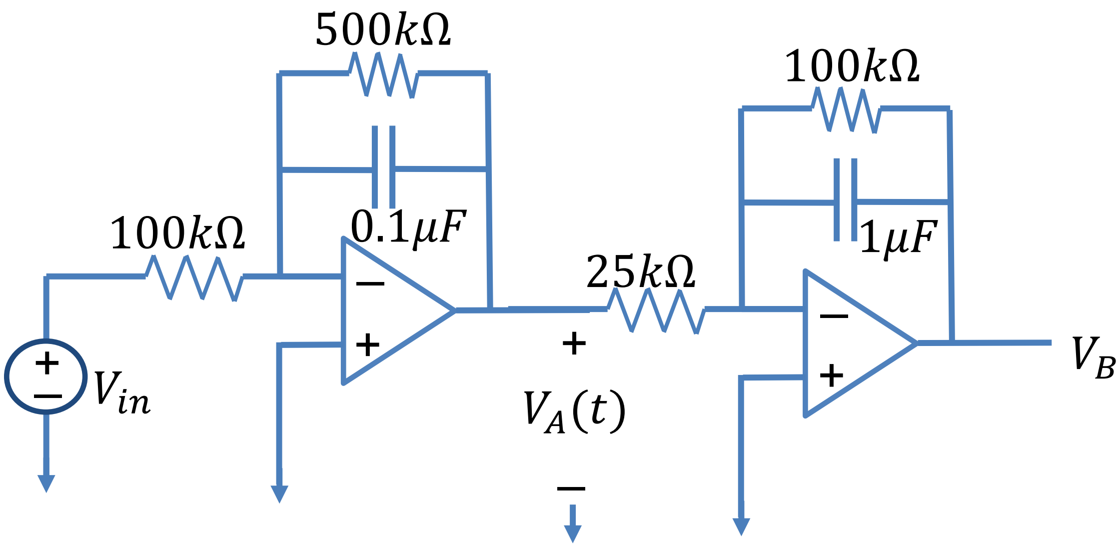
The input voltage to the circuit shown is

$$V_{in}(t) = \begin{cases} 250mV, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Assuming that there is no energy stored in the capacitors at time $t = 0$,

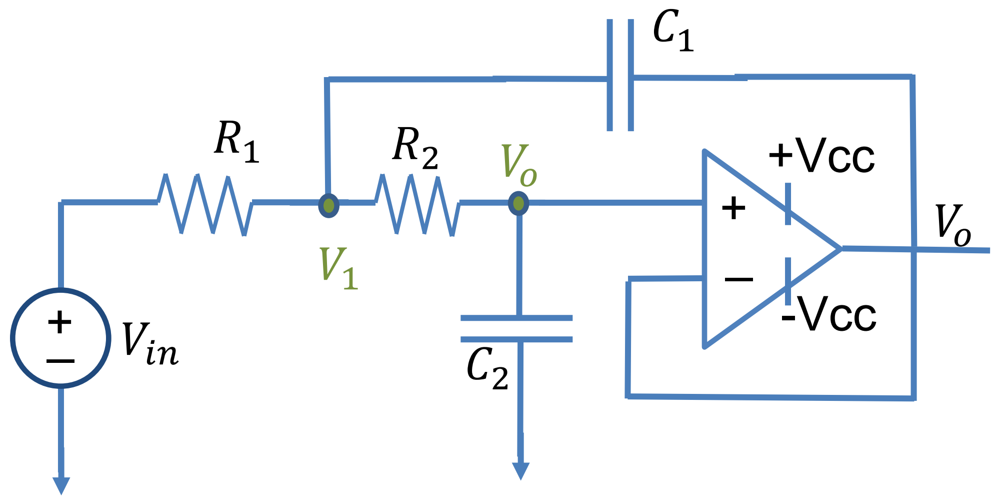
(a) Find $V_B(t)$ for $t > 0$.

(b) Find $V_A(t)$ for $t > 0$.



# Second Order Op Amp Circuits

This circuit is known as a Sallen-Key circuit and can synthesize any second order circuit with the appropriate selection of component values.




To analyze this circuit we will do a couple KCLs at the nodes marked with the ●

$$\frac{V_{in}-V_1}{R_1}+\frac{V_o-V_1}{R_2}+C_1\frac{d(V_o-V_1)}{dt}=0 \qquad (1)$$

$$\frac{V_o-V_1}{R_2}+C_2\frac{dV_o}{dt}=0 \qquad (2)$$

# Second Order Op Amp Circuits

Using equation (2), we can solve for $V_1$ as

$$V_1 = V_o + R_2 C_2 \frac{dV_o}{dt}.$$

Then, substitute into (1) resulting in a single equation involving $V_o$ and $V_{in}$.

$$\frac{V_{in} - (V_o + R_2 C_2 \frac{dV_o}{dt})}{R_1} + \frac{V_o - (V_o + R_2 C_2 \frac{dV_o}{dt})}{R_2} + C_1 \frac{d(V_o - (V_o + R_2 C_2 \frac{dV_o}{dt}))}{dt} = 0$$

After a little cleaning up, this simplifies to

$$\frac{d^2 V_o}{dt^2} + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right)\frac{dV_o}{dt} + \frac{V_o}{R_1 C_1 R_2 C_2} = \frac{V_{in}}{R_1 C_1 R_2 C_2}.$$

# Second Order Op Amp Circuits

The corresponding characteristic equation for the Sallen-Key circuit is

$$s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)s + \frac{1}{R_1C_1R_2C_2} = 0.$$

The roots of the characteristic equation are

$$s = -\left(\frac{1}{2R_1C_1} + \frac{1}{2R_2C_1}\right) \pm \sqrt{\left(\frac{1}{2R_1C_1} + \frac{1}{2R_2C_1}\right)^2 - \frac{1}{R_1C_1R_2C_2}}$$

By appropriate selection of component values, we can make the roots:

Real → overdamped
Complex → underdamped
Repeated → critically damped

You will use the Sallen-Key circuit in Labs 7 & 8 to study the behavior of second order circuits.

# Second Order Op Amp Circuits

Using some of the terminology we introduced in the context of RLC 2nd order circuits, for the Sallen-Key circuit:

- Natural resonant frequency: $\omega_o = \sqrt{\frac{1}{R_1 C_1 R_2 C_2}}$,
- Damped resonant frequency: $\omega_d = \sqrt{\omega_o^2 - \left(\frac{1}{2R_1C_1} + \frac{1}{2R_2C_1}\right)^2}$,
- Q-factor: $Q = \sqrt{\frac{R_1R_2C_1}{(R_1+R_2)^2C_2}}$,
- roots, $s = \frac{\omega_o}{2Q}\left(-1 \pm \sqrt{1-(2Q)^2}\right)$
  - $Q = \frac{1}{2}$ ←critically damped
  - $Q > \frac{1}{2}$ ←underdamped

    $$\omega_d = \omega_o\sqrt{1-\left(\frac{1}{2Q}\right)^2}$$

  - $Q < \frac{1}{2}$ ←overdamped

If we define

$R_s = R_1 + R_2$ (series combination),

$R_p = \frac{R_1R_2}{R_1+R_2}$ (parallel combination),

then the Q-factor can be written in a slightly more convenient form

$$Q = \sqrt{\frac{R_pC_1}{R_sC_2}}.$$

## Example

Design (that is, find the component values for) a Sallen-Key circuit with an underdamped response whose characteristic equation roots are

$$s = -10{,}000 \pm 24{,}000j.$$