# LAB 8: TRANSIENT RESPONSE OF A 2ND ORDER CIRCUIT

## Theory and Introduction

**GOALS FOR LAB 8 –** Students will learn about a relatively simple op-amp circuit that creates a 2nd order response. Students will measure the transient (step) response of the circuit and compare with theory and simulations results.

### THEORY

In class we often used circuits with resistors, capacitors and inductors to produce 2nd order responses (i.e., circuits whose voltages and currents are described by 2nd order differential equations). In practice, we tend to avoid using circuits with inductors as they are not easily integrated onto a chip and therefore result in circuits that cannot easily be miniaturized. In this lab we will use an op-amp circuit with resistors and capacitors that can synthesize any desired 2nd order response. The circuit is show in Figure 8.1 and is known as a Sallen-Key circuit.

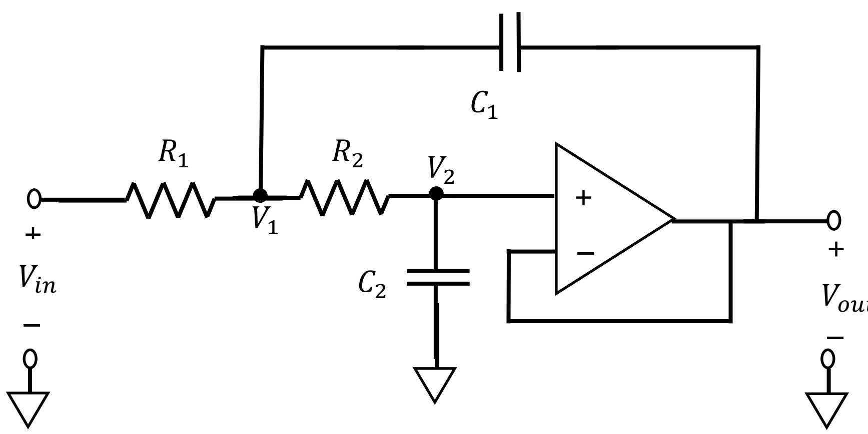


Figure 8.1 – A Sallen-Key 2nd Order Circuit

This circuit is analyzed as follows. First, note that $V_2 = V_{out}$ (why?). Then perform a KCL at node 1 resulting in

$$\frac{V_{in}-V_1}{R_1} + \frac{V_{out}-V_1}{R_2} + C_1\frac{d}{dt}(V_{out}-V_1) = 0.$$

This equation can be rearranged to produce:

$$V_{in} = V_{out} + \left(1+\frac{R_1}{R_2}\right)(V_1 - V_{out}) + R_1C_1\frac{d}{dt}(V_1 - V_{out}).$$

Next, perform a KCL at node 2 to produce the equation:

$$\frac{V_1 - V_{out}}{R_2} = C_2 \frac{dV_{out}}{dt}.$$

Plugging this equation into the one before it produces a 2nd order differential equation which describes the operation of this circuit,

$$R_1 C_1 R_2 C_2 \frac{d^2 V_{out}(t)}{dt^2} + (R_1 + R_2) C_2 \frac{dV_{out}(t)}{dt} + V_{out}(t) = V_{in}(t). \qquad (1)$$

The characteristic equation associated with this differential equation is

$$R_1 C_1 R_2 C_2 s^2 + (R_1 + R_2) C_2 s + 1 = 0,$$

whose roots are:

$$s = \frac{-(R_1 + R_2) C_2 \pm \sqrt{(R_1 + R_2) C_2{}^2 - 4 R_1 C_1 R_2 C_2}}{2 R_1 C_1 R_2 C_2}.$$

By carefully selecting component values, we can get real, complex, or repeated roots. Recall that these cases lead to overdamped, underdamped, and critically damped responses respectively.

It is common to express the roots of the characteristic equation of a 2nd order circuit in the form, $s = -\alpha \pm j\omega_d$, so that in the underdamped case (complex roots), the solution (homogeneous) to the differential equation will be of the general form:

$$V_{out}(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t).$$

This is the form of a damped oscillation where $\alpha$ is the *damping factor* (also known as the *neper frequency*) and $\omega_d$ is the *damped radian frequency*. Other common terms to describe a 2nd order circuit can be defined by expressing the roots of the characteristic equation in the form $s = -\alpha \pm j\sqrt{\omega_o{}^2 - \alpha^2}$, where $\omega_o$ is the *resonant radian frequency* of the circuit. This is the frequency that the circuit would oscillate at if it was completely undamped ($\alpha = 0$). In that case, the circuit would be a simple harmonic oscillator that you studied in your physics class. Finally it is common to refer to the *quality factor* or *Q-factor* of a 2nd order circuit which is defined as

$$Q = \frac{\omega_o}{2\alpha}.$$

The Q-factor is a quantitative measure of how damped a 2nd order circuit is.  Note that the case of $\omega_o = \alpha$ will lead to repeated roots of the characteristic equation and a value of $Q = 1/2$.  Thus, $Q = 1/2$ is the dividing line between underdamped and overdamped.  For $Q > 1/2$, the roots are complex, and the circuit is underdamped, while for $Q < 1/2$, the roots are real, and the circuit is overdamped.  Plots of typical output voltages for the underdamped and overdamped cases are shown in Figure 8.2.  For an overdamped system, the real roots will cause the voltage response to have an exponential form, while in an underdamped system, the complex roots will produce a decaying sinusoidal response.

Over damped

Under damped

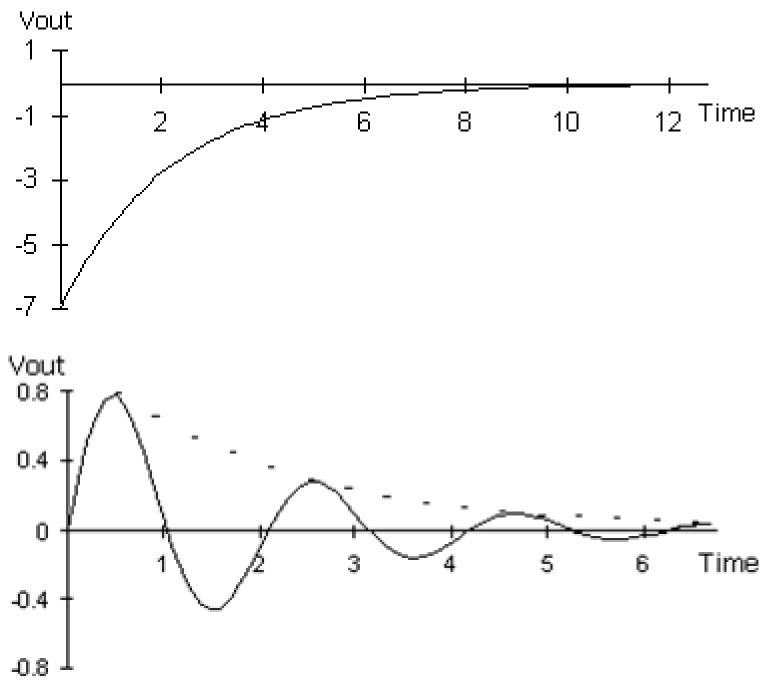


Figure 8.2 – Responses of 2nd order systems.

The textbook provides values for $\alpha, \omega_o, \omega_d$ for the case of simple series and parallel RLC circuits, but the formulation above allows us to compute these quantities for more complicated 2nd order circuits like our Sallen-Key circuit.

Table 8.1 summarizes these results.

| | $\alpha$ | $\omega_o$ | $\omega_d$ | $Q$ |
|---|---|---|---|---|
| Series RLC | $\frac{R}{2L}$ | $\sqrt{\frac{1}{LC}}$ | $\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$ | $\sqrt{\frac{L}{R^2C}}$ |
| Parallel RLC | $\frac{1}{2RC}$ | $\sqrt{\frac{1}{LC}}$ | $\sqrt{\frac{1}{LC}-\left(\frac{1}{2RC}\right)^2}$ | $\sqrt{\frac{R^2C}{L}}$ |
| Sallen-Key | $\frac{1}{2R_{eq}C_1}$, $R_{eq} = \frac{R_1R_2}{R_1+R_2}$ | $\sqrt{\frac{1}{R_1C_1R_2C_2}}$ | $\sqrt{\frac{1}{R_1C_1R_2C_2}-\left(\frac{1}{2R_{eq}C_1}\right)^2}$ | $\sqrt{\frac{C_1}{C_2}\frac{R_{eq}}{R_1+R_2}}$ |

Table 8.1 – Parameters for Various 2nd Order Circuits.

If we needed to completely solve the second order differential equation from equation (1), it will be necessary to establish appropriate initial conditions. Since the output voltage appears across the capacitor $C_2$ in Figure 8.1, we can determine initial conditions from the state of the capacitor, $C_2$.

Suppose that prior to time $t = 0$, the circuit has reached a steady state. Then at time $t = 0^-$, both capacitors are acting like open circuits. No current flows anywhere in the circuit and as a result there are no voltage drops across the two resistors. As a result, $V_2(0^-) = V_1(0^-) = V_{in}(0^-)$. At $t = 0^+$, the input voltage may change instantaneously, but the capacitors will hold the other voltages at the same values. As a result: $V_2(0^+) = V_1(0^+) = V_{in}(0^-)$. Now, resistor $R_1$ has a voltage drop across it of $V_{in}(0^+) - V_1(0^+) = V_{in}(0^+) - V_{in}(0^-)$ and a resulting current flowing through it (left to right) of $I_1(0^+) = (V_{in}(0^+) - V_{in}(0^-))/R_1$. The other resistor, $R_2$, has a voltage drop across it of $V_1(0^+) - V_2(0^+) = V_{in}(0^-) - V_{in}(0^-) = 0$ and will have no current through it.

As a result, $I_1(0^+)$ will flow through capacitor $C_1$ and no current will flow through $C_2$. The resulting initial conditions needed for the differential equation will then be:

$$V_{out}(0^+) = V_{in}(0^-) \quad \text{and} \quad \frac{dV_{out}}{dt}(0^+) = 0.$$

## PRELAB

A. Design a Sallen-Key circuit as shown in Figure 8.1. Choose component values so that the circuit produces a critically damped response ($Q = 1/2$) and a resonant radian frequency of $\omega_o = 2000\pi$ rad/sec ($f_o = 1$kHz). Be sure to choose component values that are available to you in your lab kit. You will not be able to exactly achieve the design goals with the restrictions of the component values, but you should try to get as close as possible with what you have. Repeat your design for each of the following cases:

   a. $Q = 0.25$ (Slightly over damped)
   b. $Q = 0.1$ (Over damped)
   c. $Q = 1$ (Slightly under damped)
   d. $Q = 2.5$ (Under damped).

**Provide a table with your component values for each of the five cases.** Save a copy for yourself as you will need it in the lab this week. (Do not forget to write units.)

| CASE | $R_1$ | $R_2$ | $C_1$ | $C_2$ | Q | $\omega_o$ |
|---|---|---|---|---|---|---|
| **1. Critically Damped** | | | | | | |
| **2. Slightly Over Damped** | | | | | | |
| **3. Over Damped** | | | | | | |
| **4. Slightly Under Damped** | | | | | | |
| **5. Under Damped** | | | | | | |

Table 8.2 Component value for different case

B. Using a 100Hz square wave with 2 Volts (peak-to-peak) as your input source, run SPICE simulations for each case calculated in part A (total 5 cases). **Print one copy of the schematic and print a graph of the transient response for each case in part A to submit with your prelab.** Be sure to label your graphs.

## PROCEDURE

Parts and Equipment needed:

Several resistors

Several (non-polarized) capacitors

1—741 OPAMP

Power Supply (+/- 5V)

Function Generator

Oscilloscope

**Task 1** – Transient Response of $2^{nd}$ Order Circuit

A. Build the Sallen-Key circuit in Figure 8.1. Use the component values from your prelab that produced a critically damped response ($Q = 1/2$). Use a 100Hz square wave with 2 Volts (peak-to-peak) as your input. You can use the function generator on your AD2 to provide the square wave voltage source or you can use the signal generator in the lab. Display the input on CH1 and the output on CH2 of the oscilloscope. **Save the waveforms for your lab report.** Be sure to set the triggering to "edge" on your scope and adjust the scales on the vertical and horizontal axes so that the waveforms are plainly visible on the scope display.

   **Note:** Your AD2 can simultaneously play the role of the power supply (for the op amp), function generator (to provide the input voltage), and the scope (to view the input and output voltages).

   Repeat part A for the other cases you worked out in the prelab, namely, $Q = 0.25$ (slightly over damped), $Q = 0.1$ (over damped), $Q = 1$ (slightly under damped), and $Q = 2.5$ (under damped). **Be sure to save the waveforms for your lab report.**

**Before you leave the lab…**

Bring your circuit into the lab to show to your TA. If you did all your measurements with your AD2, your TA will also show you how to use the bench signal generator and the bench oscilloscope.

## Lab Report Requirements

1. Title Page
2. Procedure – Summarize in your own words what you have done.
3. Data and Results – For each of the five cases, compare the transient responses found in each of the following manners:
    a. Theoretical – solve the appropriate differential equation with the appropriate initial conditions (show your calculations),
    b. Simulations – Provide the results from your SPICE simulations in the prelab,
    c. Measured – Provide measured results from the lab.

    Make sure to point out any significant differences you observe.

4. Discussion –
    a. Write comments on each case (5 cases), comparing differences of theoretical, simulation and measured plot. Comments can be made regarding what differences you have observed and why those differences are there. Try your best to explain the cause of those differences.
    b. Any changes you could make to the procedure to make your results come out better if you had to do it all over again.
5. Conclusion
6. Appendix