First Order Differential Equations

- Now that we will be putting inductors and capacitors in our circuits, when we perform KVLs and KCLs, the resulting equations will involve derivatives and so we will need to solve differential equations.
- To start with, we will only need to solve first order constant coefficient DEQs of the form

$$a\frac{dy}{dt} + by(t) = x(t),$$

along with an initial condition

$$y(0) = y_o.$$

First Order Differential Equations

$$a\frac{dy}{dt} + by(t) = x(t), \quad y(0) = y_o.$$

- The general solution to this kind of DEQ will consist of:
  - A homogeneous (common) solution –
    - ✓ We get by finding the solution when $x(t) = 0$.
    - ✓ Will contain an arbitrary constant
  - A particular solution – depends on the form of $x(t)$.

- The approach to solving these equations will be as follows:
  1. Find the homogeneous solution
  2. Find the particular solution
  3. Combine the two solutions to find the general solution
  4. Apply initial conditions to resolve arbitrary constant.

Finding the Homogeneous Solution

$$a\frac{dy}{dt} + by(t) = 0$$

- Assume a solution of the form $y(t) = Ae^{st}$.
- Plug this solution into the original equation

$$a\frac{d}{dt}(Ae^{st}) + b(Ae^{st}) = 0$$

$$Ae^{st}(\mathrm{as} + \mathrm{b}) = 0$$

$$\mathrm{as} + \mathrm{b} = 0$$

This is known as the "Characteristic equation"

$$s = -b/a.$$

- The homogeneous solution is then found to be

$$y_H(t) = Ae^{-bt/a}.$$

- Note that this solution works for any arbitrary constant A.

Finding the Particular Solution

$$a\frac{dy}{dt} + by(t) = x(t)$$

- In this class, we will use the method of undetermined coefficients to find the particular solution.
- Based on the form of $x(t)$ we will guess at the form of $y_p(t)$ as given in the table below

$x(t)$	$y_p(t)$
Constant - b_0	c_1
Linear - $b_0 + b_1 t$	$c_0 + c_1 t$
Quadratic - $b_0 + b_1 t + b_2 t^2$	$c_0 + c_1 t + c_2 t^2$
Exponential - e^{rt}	$c_1 e^{rt}$
Sine - $\sin(\omega t)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$
Cosine - $\cos(\omega t)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$

Finding the Particular Solution

- Once we have a general guess at the form of the particular solution, we will plug that guess into the DEQ to determine the coefficients.

Example

$$4\frac{dy}{dt} + 3y(t) = -2$$

Since the RHS is a constant, we guess that the particular solution is also a constant:

$$y_p(t) = c \rightarrow \frac{dy_p}{dt} = 0.$$

$$4(0) + 3c = -2 \rightarrow c = -2/3.$$

Therefore

$$y_p(t) = -\frac{2}{3}$$

Putting it all Together

Example

$$2\frac{dy}{dt} + 5y(t) = 1, \quad y(0) = 3.$$

Step 1 – Find the Homogeneous solution

$$2\frac{dy}{dt} + 5y(t) = 0$$

$$y = e^{st} \rightarrow 2s + 5 = 0 \rightarrow s = -5/2.$$

$$y_H(t) = Ae^{-5t/2}$$

Step 2 – Find the Particular solution

Guess $y(t) = c \rightarrow 2(0) + 5c = 1 \rightarrow c = 1/5.$

$$y_P(t) = 1/5.$$

Putting it all Together

Step 3 – Combine homogeneous and particular solution to form complete solution.

$$y(t) = y_H(t) + y_P(t) = Ae^{-5t/2} + 1/5.$$

Step 4 – Apply ICs to the complete solution to find the unknown constant.

$$y(0) = 3 \rightarrow A + \frac{1}{5} = 3 \rightarrow A = 14/5.$$

$$y(t) = \frac{14}{5}e^{-5t/2} - \frac{1}{5}.$$

Warning: Don’t apply the initial condition until after you have formed the complete solution. This is a common mistake.

Let's try one together

$$\frac{dy}{dt} + 3y(t) = 2t + 5, \quad y(0) = -4.$$

Homogeneous solution:

char. eqn.: $s + 3 = 0 \rightarrow s = -3.$

$y_H(t) = Ae^{-3t}.$

Particular solution:

assume $y_P(t) = at + b \ \rightarrow \ \frac{dy_p}{dt} = a$

plug into diff. eq.: $a + 3(at + b) = 2t + 5$

rearrange: $3at + (a + 3b) = 2t + 5$

equate coeff of t: $3a = 2 \rightarrow a = \frac{2}{3}.$

equate constant term: $a + 3b = 5 \rightarrow b = \frac{5-a}{3} = 13/9$

$y_P(t) = \frac{2}{3}t + \frac{13}{9}.$

Let's try one together

$$\frac{dy}{dt} + 3y(t) = 2t + 5, \quad y(0) = -4.$$

Form complete solution:

$$\mathrm{y}(t) = Ae^{-3t} + \frac{2}{3}t + \frac{13}{9}.$$

Apply initial condition:

$$y(0) = A + \frac{13}{9} = -4 \rightarrow \mathrm{A} = -\frac{49}{9}.$$

Final answer:

$$y(t) = -\frac{49}{9}e^{-3t} + \frac{2}{3}t + \frac{13}{9}$$

If you need more practice, try some of these

Find the solution to each of the following differential equations together with the given initial conditions.

1. $2\frac{dy}{dt} + 4y(t) = -3$, $y(0) = 5$.

2. $3\frac{dy}{dt} + 3y(t) = 1$, $\left.\frac{dy}{dt}\right|_{t=0} = 2$.

3. $2\frac{dy}{dt} + y(t) = \cos(6\pi t + 30°)$, $y(0) = 0$.

4. $\frac{dy}{dt} + y(t) = \cos(4\pi t) + \sin(4\pi t)$, $y(0) = -1$.

5. $\frac{dy}{dt} + 3y(t) = 2t + 3$, $y(0) = 3$.

6. $4\frac{dy}{dt} + 2y(t) = t^2$, $\left.\frac{dy}{dt}\right|_{t=0} = 0$.

7. $3\frac{dy}{dt} + y(t) = t\exp(-2t)$, $y(0) = 1$.

8. $\frac{dy}{dt} + 2y(t) = t^2\exp(-3t)$, $y(0) = 0$.

9. $\frac{dy}{dt} + y(t) = \exp(-t)\cos(2\pi t)$, $y(0) = 1$.

10. $\frac{dy}{dt} + 3y(t) = \exp(-3t)$, $y(0) = -1$.