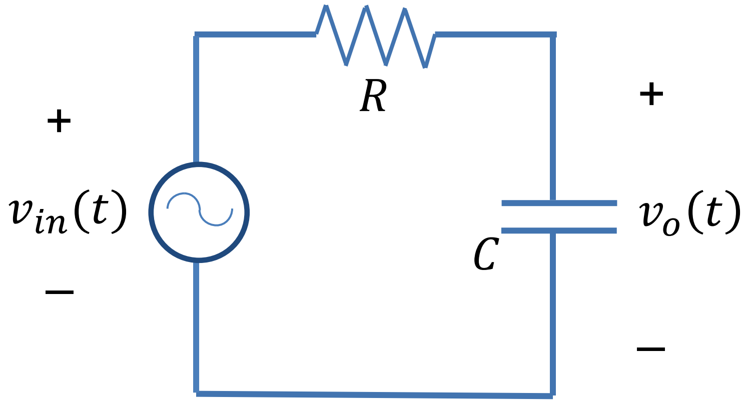
First Order Circuits with AC Sources

In this set of notes, we examine the behavior of first order circuits with AC (sinusoidal) sources.

Suppose in the RC circuit shown, the input voltage is a sinusoid:

$$v_{in}(t) = A\cos(\omega t + \theta),$$

where
- A is the amplitude,
- ω is the frequency (in radians per second) and
- θ is the phase of the sinusoid.



The differential equation that describes this circuit is:

$$RC\frac{dv_o}{dt} + v_o(t) = v_{in}(t)$$

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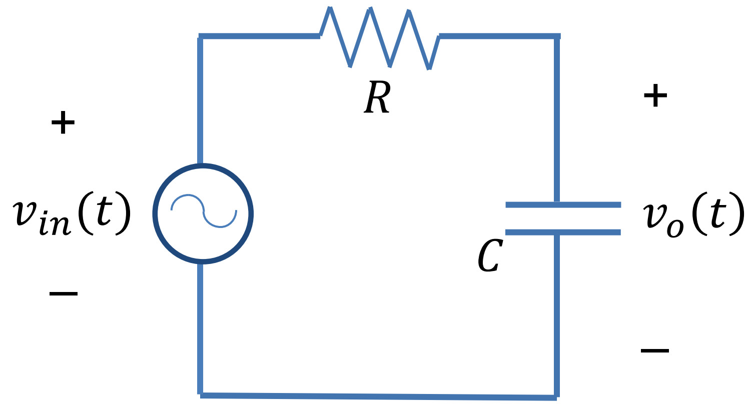
$$RC\frac{dv_o}{dt} + v_o(t) = Acos(\omega t + \theta),$$



The homogeneous solution to this equation is:

$$v_h(t) = Be^{-t/RC}.$$

To find the particular solution, use the method of undetermined coefficients.

Assume a particular solution of the form:

$$v_p(t) = Dcos(\omega t + \theta) + Esin(\omega t + \theta).$$

Then

$$\frac{dv_p}{dt} = \omega Ecos(\omega t + \theta) - \omega Dsin(\omega t + \theta).$$

Plugging these into the differential equation produces:

$$\omega RCEcos(\omega t + \theta) - \omega RCDsin(\omega t + \theta)$$
$$+Dcos(\omega t + \theta) + Esin(\omega t + \theta)$$
$$= Acos(\omega t + \theta).$$

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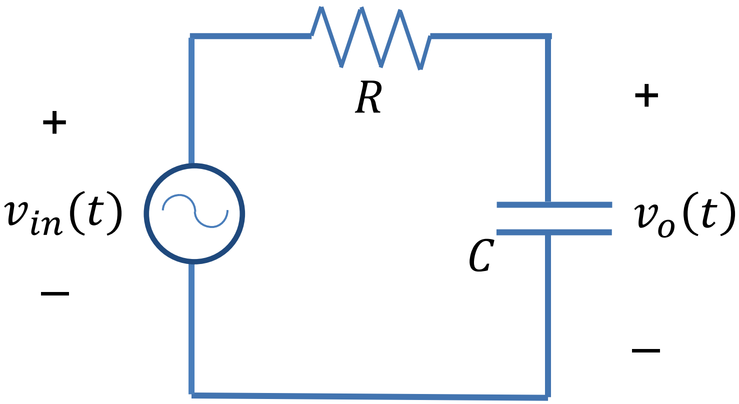
$$RC\frac{dv_o}{dt} + v_o(t) = A\cos(\omega t + \theta),$$

$$\omega RCE\cos(\omega t + \theta) - \omega RCD\sin(\omega t + \theta)$$
$$+D\cos(\omega t + \theta) + E\sin(\omega t + \theta)$$
$$= A\cos(\omega t + \theta).$$



Equating coefficients of the cos( ) terms and coefficients of the sin( ) terms produces the following set of linear equations to solve for the constants D and E.

$$\omega RCE + \mathrm{D} = \mathrm{A}$$
$$-\omega RCD + \mathrm{E} = 0$$

or in matrix form

$$\begin{bmatrix} 1 & \omega RC \\ -\omega RC & 1 \end{bmatrix}\begin{bmatrix} D \\ E \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix}$$

The solution to this set of equations is:

$$\begin{bmatrix} D \\ E \end{bmatrix} = \frac{A}{1 + (\omega RC)^2}\begin{bmatrix} 1 \\ \omega RC \end{bmatrix}$$

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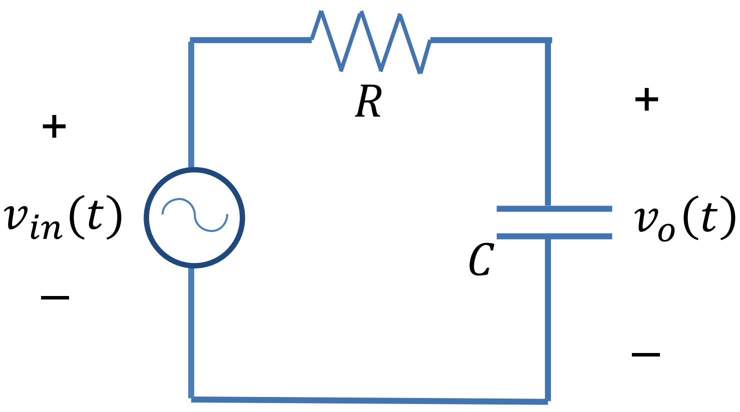

Summarizing our results. The solution to this differential equation is

$$v_o(t) = Be^{-\frac{t}{RC}} + \frac{A\cos(\omega t + \theta)}{1 + (\omega RC)^2} + \frac{A\omega RC\sin(\omega t + \theta)}{1 + (\omega RC)^2}$$

Or by using a few trig identities, we can rewrite this in the following equivalent form:

$$v_o(t) = Be^{-\frac{t}{RC}} + \frac{A\cos(\omega t + \theta - \tan^{-1}\omega RC)}{\sqrt{1 + (\omega RC)^2}}$$

Quite often, we are not concerned with the transient part of the solution. In this case, the part of the output that remains is the sinusoidal part.

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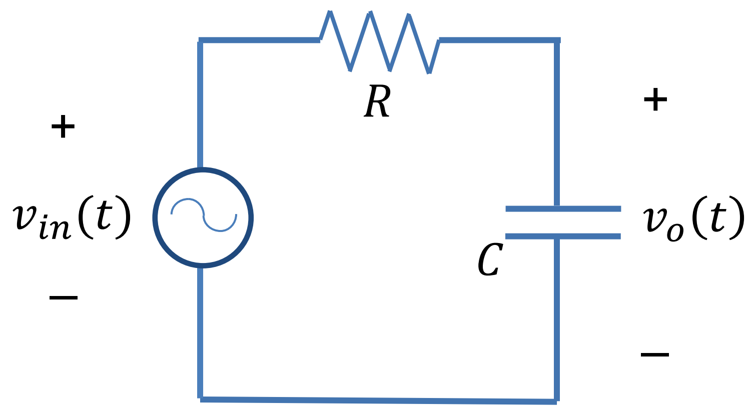

"Sinusoidal steady state" output:

$$v_o(t) = \frac{A\cos(\omega t + \theta - \tan^{-1}(\omega RC))}{\sqrt{1 + (\omega RC)^2}}$$

Notice that the output voltage is a sinusoid of the same frequency as the input sinusoid. Furthermore:

1) The amplitude of the output has been scaled by a factor of $\frac{1}{\sqrt{1+(\omega RC)^2}}$. This is known as the magnitude response of the circuit.

2) The phase of the output has been shifted by $-\tan^{-1}(\omega RC)$ radians. This is known as the phase response of the circuit.

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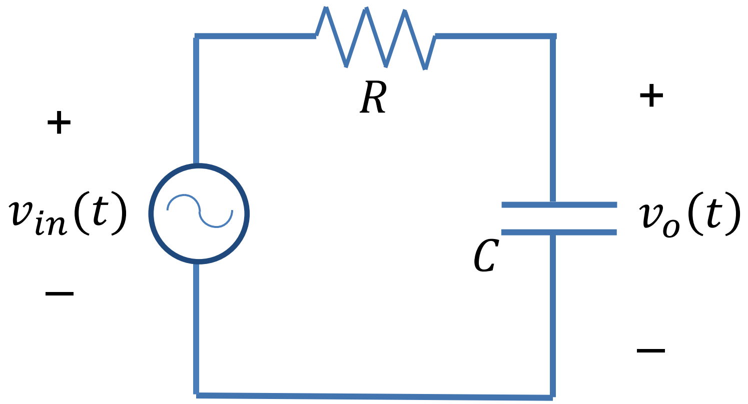
Magnitude Response: $\frac{|v_o|}{|v_{in}|} = \frac{1}{\sqrt{1+(\omega RC)^2}}$

Phase Response:

$$\angle v_o - \angle v_{in} = -\tan^{-1}(\omega RC)$$



For low frequency inputs, $\omega \ll 1/RC$, so that:

$$\frac{|v_o|}{|v_{in}|} \approx 1 \text{ and } \angle v_o - \angle v_{in} \approx 0.$$

For high frequency inputs, $\omega \gg 1/RC$, so that:

$$\frac{|v_o|}{|v_{in}|} \approx 0 \text{ and } \angle v_o - \angle v_{in} \approx -\frac{\pi}{2}.$$

This circuit is known as a *low-pass filter* since it allows low frequencies to pass but attenuates high frequencies.