Sinusoids

A sinusoidal voltage signal has the general form

$$v(t) = V_o \cos(\omega t + \theta)$$

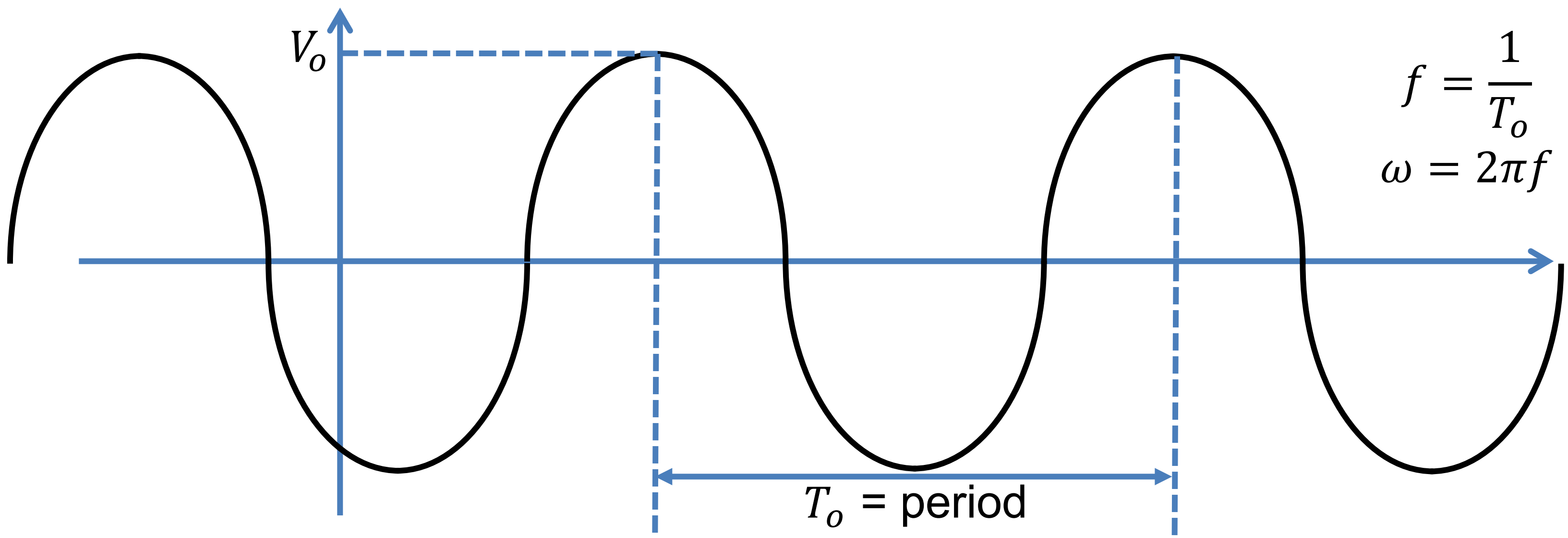
and is described by three parameters

V_o = amplitude (Volts)

θ = phase (radians or degrees)

ω = frequency (rad/sec)

or $f = \omega/2\pi$ = frequency in Hz (cycles per sec)



Sinusoids

The polar representation (in terms of amplitude and phase) of the sinusoid on the previous slide can also be written in an equivalent Cartesian (rectangular) form.

$$v(t) = V_o \cos(\omega t + \theta) \qquad \text{(polar form)}$$

Using the trig identity $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, the above expression becomes

$$v(t) = V_o \cos(\theta)\cos(\omega t) - V_o \sin(\theta)\sin(\omega t)$$

Let $A_o = V_o \cos(\theta)$ and $B_o = V_o \sin(\theta)$. Then

$$v(t) = A_o \cos(\omega t) - B_o \sin(\omega t) \qquad \text{(Cartesian form)}$$

If we have a sinusoid in its Cartesian form, we can convert to polar form using

$$V_o = \sqrt{{A_o}^2 + {B_o}^2} \text{ and } \theta = tan^{-1}(\tfrac{B_o}{A_o}).$$

Sinusoids

From earlier in the course:

- It is common to describe a periodic waveform in terms of its "root mean squared" (RMS) value.

$$V_{rms} = \sqrt{\langle v^2(t) \rangle}$$

In the above definition, the angular brackets represent a time average:

$$\langle x(t) \rangle = \frac{1}{T_o} \int_{t_o}^{t_o+T_o} x(t)dt$$

where T_o is the period and t_o is any convenient starting point.

- For a sinusoidal waveform,

$$V_{rms} = \frac{V_o}{\sqrt{2}}.$$

That is, the RMS amplitude is the peak value divided by 1.414. But this result does not hold for non-sinusoidal signals.

Phasors

In electrical engineering, it is common to represent sinusoidal waveforms in terms of complex numbers known as phasors. Doing so greatly simplifies the circuit analysis.

Recal Euler's identity:

$$e^{j\theta} = \cos(\theta) + j \cdot \sin(\theta).$$

Also,

$$e^{-j\theta} = \cos(\theta) - j \cdot \sin(\theta).$$

Using these expressions we can write sin and cos in terms of complex exponentials:

$$\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2} \quad \text{or} \quad \cos(\theta)=\text{Re}[e^{j\theta}]$$

$$\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j} \quad \text{or} \quad \sin(\theta)=\text{Im}[e^{j\theta}]$$

Phasors

A general sinusoidal waveform can be written as

$$v(t) = V_o \cos(\omega t + \emptyset) = Re\left[V_o e^{j(\omega t+\emptyset)}\right] = Re\left[V_o e^{j\emptyset} e^{j\omega t}\right]$$

Let $\boldsymbol{V}$ be the complex number $V_o e^{j\emptyset}$. Then:

$$v(t) = Re[\boldsymbol{V} e^{j\omega t}]$$

sinusoidal signal

Phasor

So in this context, a phasor is not a futuristic handgun, but rather is a complex number used as a shorthand representation of a sine wave.

Note that the magnitude and angle of the phasor represent the amplitude and phase of the sine wave.

Phasors

Examples

$\boldsymbol{V}_1 = 2$ (real and positive) $\quad v_1(t) = Re\left[2e^{j\omega t}\right] = 2\cos(\omega t)$

$\boldsymbol{V}_2 = -3$ (real and negative) $\quad v_2(t) = Re\left[-3e^{j\omega t}\right] = -3\cos(\omega t)$

or $\quad = Re\left[3e^{j\pi}e^{j\omega t}\right] = 3\cos(\omega t + \pi)$

$\boldsymbol{V}_3 = 2j$ (imaginary and positive)

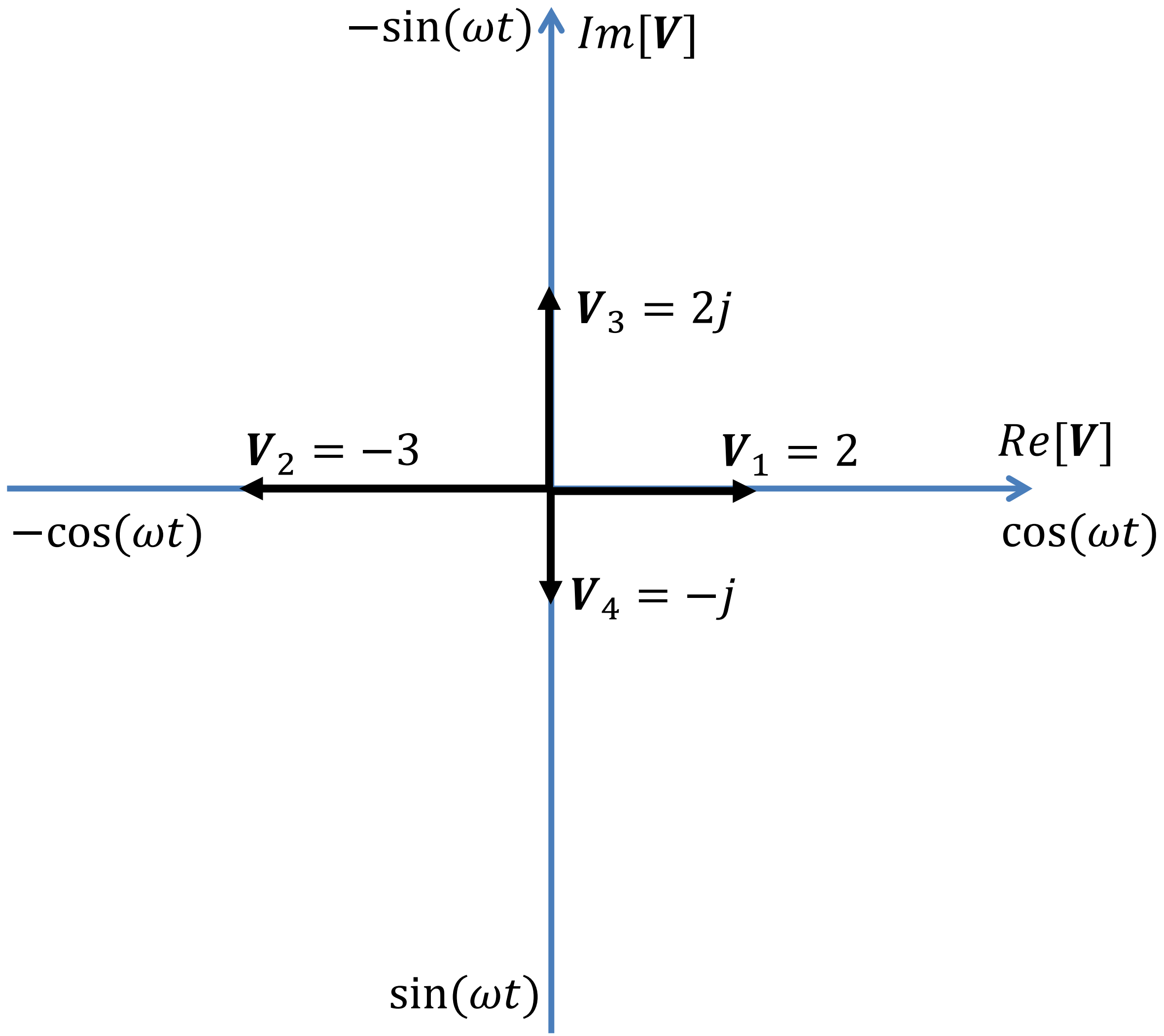
$$
\begin{aligned}
v_3(t) &= Re\left[2je^{j\omega t}\right] \\
&= Re[2j(\cos(\omega t) + j\sin(\omega t))] \\
&= Re[-2\sin(\omega t) + 2j\cos(\omega t)] \\
&= -2\sin(\omega t)
\end{aligned}
$$

$\boldsymbol{V}_4 = -j$ (imaginary and negative)

$$
\begin{aligned}
v_4(t) &= Re\left[-je^{j\omega t}\right] \\
&= Re[-j(\cos(\omega t) + j\sin(\omega t))] \\
&= Re[\sin(\omega t) - j\cos(\omega t)] \\
&= \sin(\omega t)
\end{aligned}
$$

Phasors

It is common to view these phasors as vectors in a complex plane.



Phasors

Example: Write each phasor as a sinusoid and draw the phasor as a vector in the complex plane.

(a) $\boldsymbol{V}_a = 3 + 4j$

(b) $\boldsymbol{V}_b = -1 + j$

Phasors

Example: Express each sinusoid in terms of their phasor representations

(a) $v_a(t) = -3\cos(\omega t + 30°)$

(b) $v_b(t) = 2\sin(\omega t)$

Phasors

Phasors can be particularly useful when combining sinusoidal waveforms. Suppose

$$v(t) = V_1 \cos(\omega t + \theta_1) + V_2 \cos(\omega t + \theta_2).$$

The resulting signal $v(t)$ will be a sinusoid of the same frequency,

$$v(t) = V_0 \cos(\omega t + \theta_0).$$

We can write all three signals in terms of their phasor representations

$$v(t) = V_1 \cos(\omega t + \theta_1) + V_2 \cos(\omega t + \theta_2)$$
$$Re\left[V_0 e^{j\theta_0} e^{j\omega t}\right] = Re\left[V_1 e^{j\theta_1} e^{j\omega t}\right] + Re\left[V_2 e^{j\theta_2} e^{j\omega t}\right]$$
$$= Re\left[\left(V_1 e^{j\theta_1} + V_2 e^{j\theta_2}\right) e^{j\omega t}\right]$$

Therefore

$$V_0 e^{j\theta_0} = V_1 e^{j\theta_1} + V_2 e^{j\theta_2}$$

or

$$\boldsymbol{V}_0 = \boldsymbol{V}_1 + \boldsymbol{V}_2$$

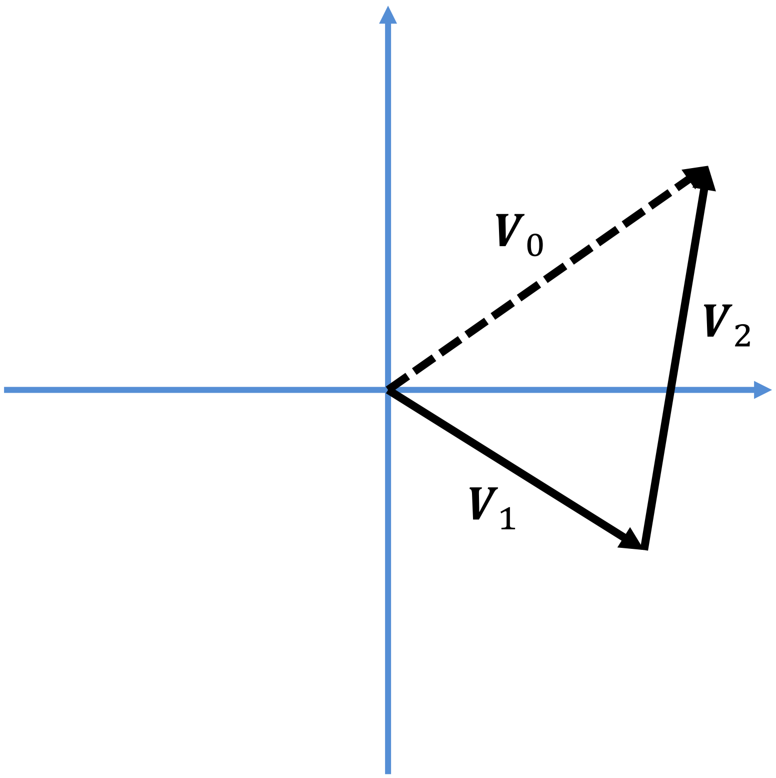
Phasors

We can pictorially add phasors to see the phasor of the resultant sinusoid.

Note: It is common to write the phasor $re^{j\theta}$ using the notation $r\angle\theta$ where θ is specified in degrees.

Example:

$$1 + j = \sqrt{2}e^{\frac{j\pi}{4}} = \sqrt{2}\angle 45°$$



Phasors

Example: Use phasors to find $y(t) = 20\cos\left(\omega t - \frac{\pi}{6}\right) + 40\cos(\omega t + \frac{\pi}{3})$