Circuit Analysis Using Laplace Transforms

It is common to use Laplace Transforms to aid in the analysis of circuits. Recall for a signal $x(t)$ defined on $0 \leq t < \infty$, its Laplace Transform is defined as

$$\mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}\,dt.$$

Examples:

- $x(t) = 1$ (constant) $\leftrightarrow X(s) = \int_0^\infty e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$
- $x(t) = e^{-at} \leftrightarrow X(s) = \int_0^\infty e^{-at}e^{-st}\,dt = \int_0^\infty e^{-(s+a)t}\,dt = \left.\frac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty = \frac{1}{(s+a)}$
- $x(t) = te^{-at} \leftrightarrow X(s) = \int_0^\infty te^{-at}e^{-st}\,dt = -\left.\frac{1+(s+a)t}{(s+a)^2}e^{-(s+a)t}\right|_0^\infty = \frac{1}{(s+a)^2}$
- $x(t) = \cos(\omega t) \leftrightarrow X(s) = \int_0^\infty \cos(\omega t)\,e^{-st}\,dt = \left.\frac{e^{-st}}{s^2+\omega^2}(\omega\sin(\omega t) - s\cos(\omega t))\right|_0^\infty = \frac{s}{s^2+\omega^2}$
- $x(t) = \sin(\omega t) \leftrightarrow X(s) = \int_0^\infty \sin(\omega t)\,e^{-st}\,dt = \left.\frac{e^{-st}}{s^2+\omega^2}(-s\sin(\omega t) - \omega\cos(\omega t))\right|_0^\infty = \frac{\omega}{s^2+\omega^2}$

These and a few other relevant Laplace Transforms are summarized on the table on the next slide.

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Signal, $x(t)$	Transform, $X(s)$
1	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$

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For our applications, an important property of Laplace Transforms is that it transforms derivatives into multiplication by s as seen below:

$$\begin{aligned}\mathcal{L}\left\{\frac{dx}{dt}\right\} &= \int_0^\infty \frac{dx}{dt} e^{-st}\,dt \\ &= x(t)e^{-st}\Big|_0^\infty + s\int_0^\infty x(t)e^{-st}\,dt \qquad \text{(using integration by parts once).} \\ &= -x(0) + sX(s).\end{aligned}$$

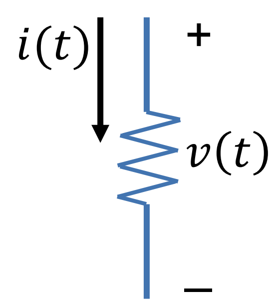
In a similar way we can also show that integration corresponds to dividing by s in the Laplace domain.

In the following, we will use this property of Laplace transforms to observe the behavior of typical circuit elements in the Laplace domain. The first, resistors, is trivial:

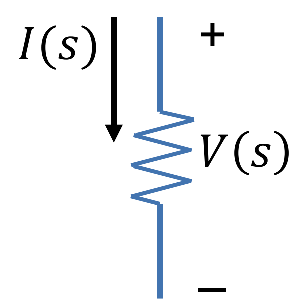
Resistors:

Time domain: $v(t) = Ri(t)$

Laplace domain: $V(s) = RI(s)$



Time Domain



Laplace Domain

Circuit Analysis Using Laplace Transforms

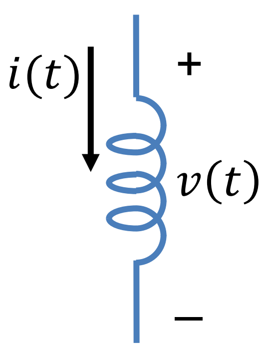
Inductors:

Time domain: $v(t) = L\frac{di}{dt}$

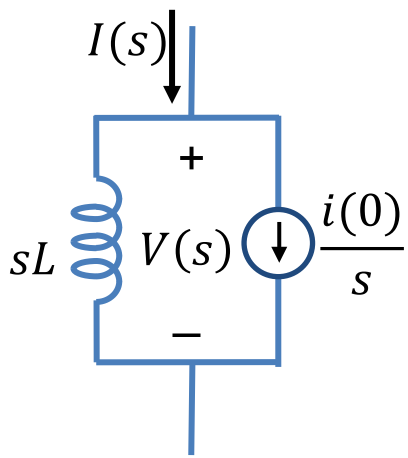
Laplace domain:

$$V(s) = sLI(s) - Li(0)$$

$$I(s) = \frac{V(s)}{sL} + \frac{i(0)}{s}$$



Time Domain



Laplace Domain

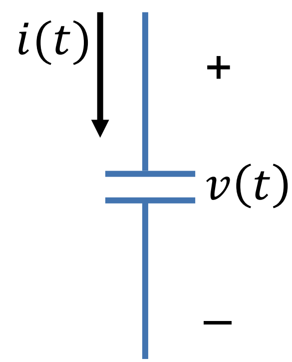
Capacitors:

Time domain: i$(t) = C\frac{dv}{dt}$

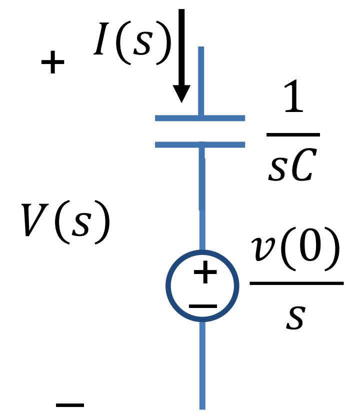
Laplace domain:

$$I(s) = sCV(s) - Cv(0)$$

$$V(s) = \frac{I(s)}{sC} + \frac{v(0)}{s}$$



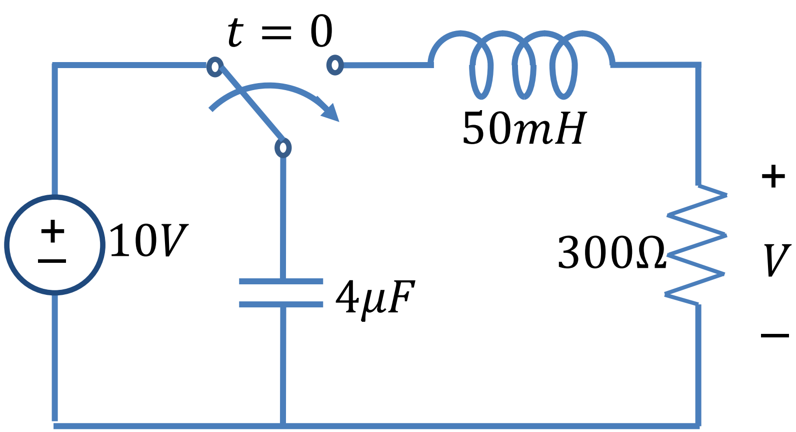
Time Domain



Laplace Domain

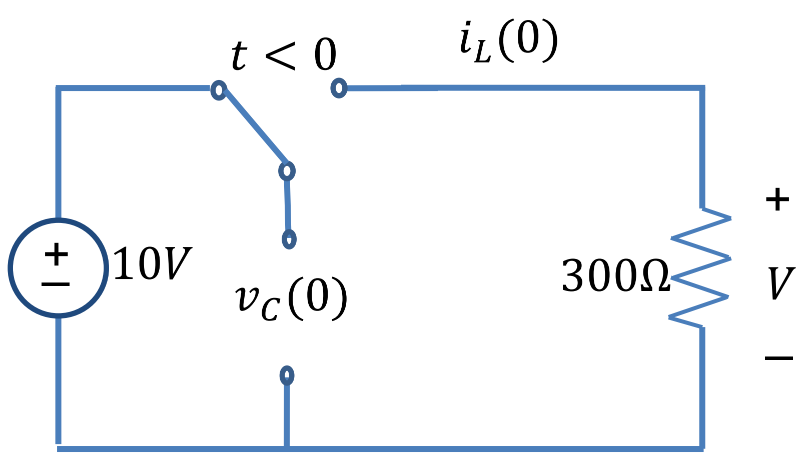
Circuit Analysis Using Laplace Transforms

Here is an example we worked previously when studying 2nd order circuits. We'll look at it now using Laplace domain techniques.



In the circuit shown, steady state has been reached before the switch moves at time $t = 0$. Find the voltage on the resistor for $t > 0$

Step 1: Find the initial inductor current and capacitor voltage using steady state techniques



$i_L(0) = 0$

$v_C(0) = 10$

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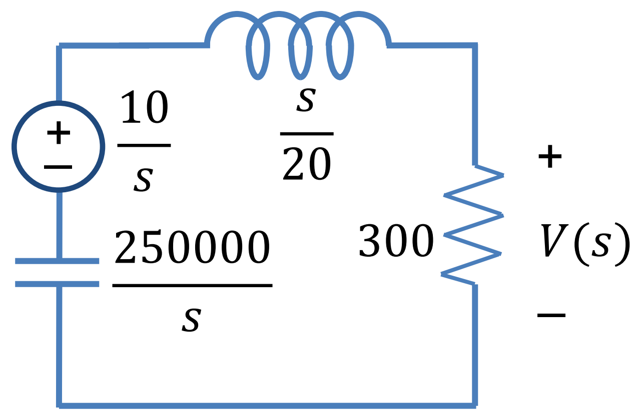
Step 2: Re-draw the circuit in the Laplace domain for $t > 0$.

Note: In this case there is no current source in parallel with the inductor since we found $i_L(0) = 0$.



Step 3: Solve for the voltage/current of interest using resistive circuit techniques.

$$V(s) = \frac{10}{s}\frac{300}{\frac{s}{20} + 300 + \frac{250000}{s}}$$ (using a voltage divider)

$$= \frac{60000}{s^2 + 6000s + 5000000}$$ (simplifying)

$$= \frac{60000}{(s + 5000)(s + 1000)}$$ (factoring)

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Step 4: Bring voltage/current of interest back to the time domain by finding inverse Laplace transform.

$$V(s) = \frac{60000}{(s+5000)(s+1000)}$$

$$= \frac{15}{s+1000} - \frac{15}{s+5000}$$

(the previous step was accomplished using partial fractions)

$$v(t) = 15e^{-1000t} - 15e^{-5000t}$$

(for $t \geq 0$).

- Since we found the roots of the denominator (poles) to be real, we are expecting exponential solutions in the time domain.

- As such, we should manipulate the form of $V(s)$ to match the form of an exponential in our table of Laplace Transforms.

- The relevant entry in the table is: $e^{-at} \leftrightarrow \frac{1}{s+a}$.

➢ The advantage of using this technique is that we never had to "translate" the initial conditions. All we needed to find in terms of initial conditions were the inductor current and the capacitor voltage.

Example

In the circuit shown, the current source produces:

$$I_S(t) = \begin{cases} 1/2, & t < 0, \\ 1/4, & t > 0. \end{cases}$$

Find $v_C(t)$ for $t > 0$.