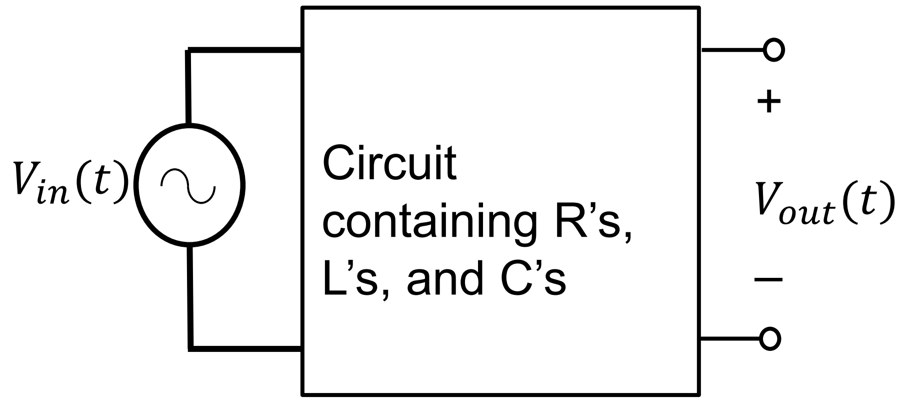
Transfer Functions



- If we apply an AC input voltage to a circuit the output voltage will be a sinusoid of the same frequency, however the amplitude and phase of the output voltage will generally be different that that of the input and will depend on the frequency of the input.

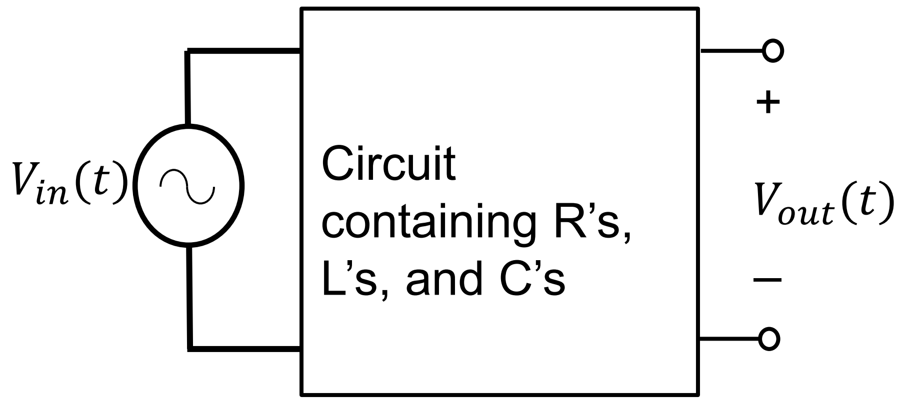
$$V_{in}(t) = A_{in}cos(\omega t + \theta_{in}) \leftrightarrow \boldsymbol{V}_{in} = A_{in}e^{j\theta_{in}}$$
$$V_{out}(t) = A_{\mathrm{out}}cos(\omega t + \theta_{out}) \leftrightarrow \boldsymbol{V}_{out} = A_{out}e^{j\theta_{out}}$$

- We are often interested in the relationship between the input/output amplitude and phase as they depend on frequency.

- The ratio of the output phasor to the input phasor is a function of frequency known as the *transfer function* and can be used to keep track of these relationships

$$H(\omega) = \frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}}$$

Transfer Functions



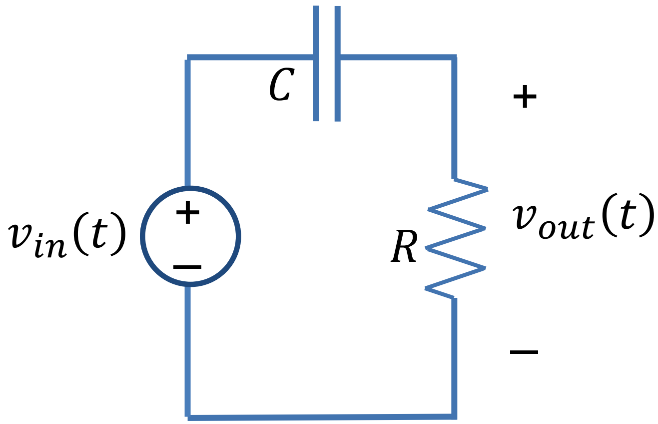
- The magnitude of the transfer function is known as the *magnitude response* and it keeps track of the ratio of the output amplitude to the input as it depends on frequency

$$|H(\omega)| = \left|\frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}}\right| = \frac{|\boldsymbol{V}_{out}|}{|\boldsymbol{V}_{in}|} = \frac{A_{out}}{A_{in}}$$

- The angle of the transfer function is known as the *phase response* and it keeps track of the phase difference between the input and the output as it depends on frequency

$$\angle H(\omega) = \angle\left(\frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}}\right) = \angle\boldsymbol{V}_{out} - \angle\boldsymbol{V}_{in} = \theta_{out} - \theta_{in}$$

Example



For this simple RC circuit, the output can be found using a voltage divider,

$$\boldsymbol{V}_{out} = V_{in}\frac{R}{R+\frac{1}{j\omega C}} = \boldsymbol{V}_{in}\frac{j\omega RC}{1+j\omega RC}$$

The transfer function is then

$$H(\omega) = \frac{\boldsymbol{V}_{out}}{\boldsymbol{V}_{in}} = \frac{j\omega RC}{1+j\omega RC}$$

Magnitude Response

$$|H(\omega)| = \left|\frac{j\omega RC}{1+j\omega RC}\right|$$

$$= \frac{|j\omega RC|}{|1+j\omega RC|} = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}}$$

Phase Response

$$\angle H(\omega) = \angle\left(\frac{j\omega RC}{1+j\omega RC}\right)$$

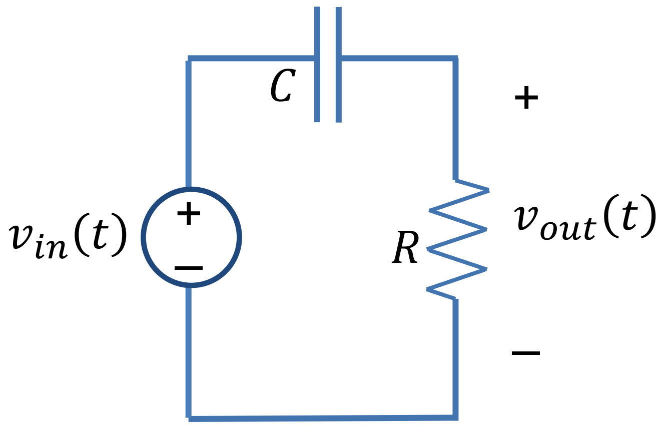
$$= \angle(j\omega RC) - \angle(1+j\omega RC)$$

$$= \frac{\pi}{2} - \tan^{-1}(\omega RC)$$

Example

Transfer Function

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$



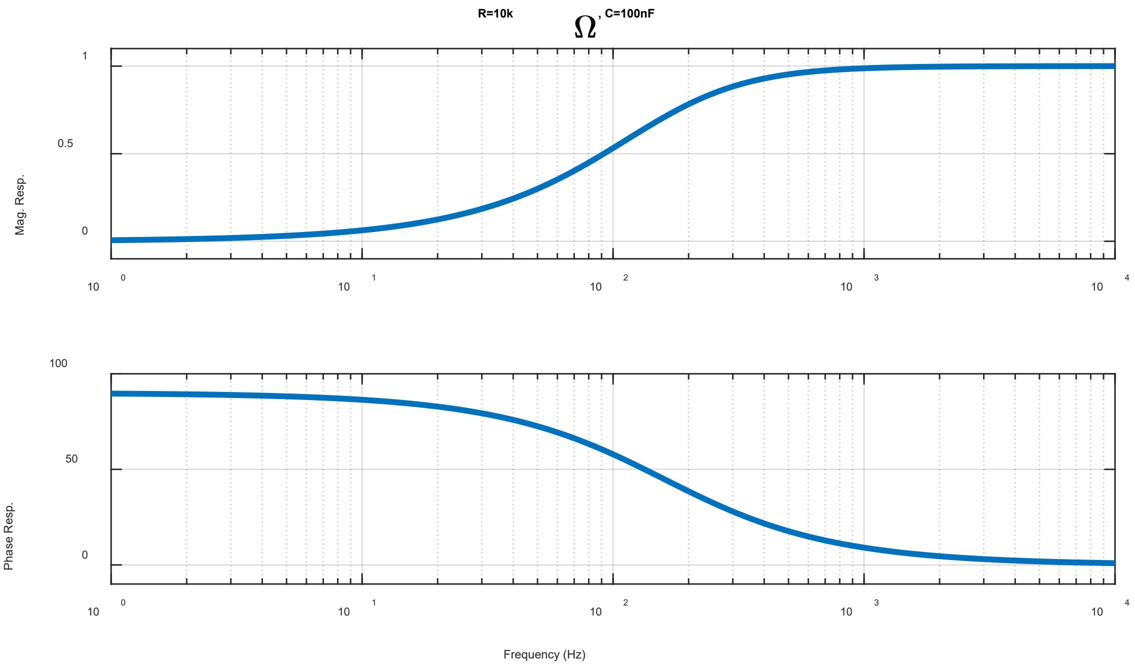
This circuit passes high frequencies and blocks low frequencies and is called a high-pass filter.

Magnitude Response

$$|H(\omega)| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

Phase Response

$$\angle H(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega RC)$$



Filters

Circuits can play different roles in terms of their frequency domain characteristics:

- Low-pass filters (LPF) – blocks high frequencies, but allows low frequencies to pass
- High-pass filters (HPF) – blocks low frequencies, but allows high frequencies to pass
- Band-pass filters (BPF) – block low and high frequencies but allows mid range frequencies to pass
- Band stop filters – block mid range filters but allows low and high frequencies to pass
- All pass filters (phase shifters) – allows all frequencies to pass but may alter the phase depending on frequency

The cutoff frequency of a filter, ω_o, is the frequency at which

$$|H(\omega_o)| = \frac{1}{\sqrt{2}} \max_{\omega} |H(\omega)|$$

The cutoff frequency serves as a quantitative dividing line between the pass band and the stop band of a filter.

A BPF or a band stop filter will have two cut-off frequencies.