Second Order Differential Equations

- A general second order constant coefficient DEQs is of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy(t) = x(t),$$

along with a pair of initial conditions

$$y(0) = y_o, \frac{dy}{dt}(0) = y_1.$$

- We will use the same techniques to solve 2nd order differential equations as we did to solve 1st order equations.

- As with the first order equations, the general solution consists of a homogeneous solution and a particular solution.

Finding the Homogeneous Solution

$$a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+cy(t)=0$$

Assume a solution of the form $y(t)=Ae^{st}$, plug this solution into the original equation, producing the characteristic equation:

$$as^2+bs+c=0.$$

The solution to the quadratic characteristic equation will have one of three different forms:

Case	Roots	Homogeneous Solution
I	Distinct real roots $s=s_1,s_2$	$A_1e^{s_1t}+A_2e^{s_2t}$
II	Complex roots $s=\sigma\pm j\omega$	$B_1e^{\sigma t}\cos(\omega t)+B_2e^{\sigma t}\sin(\omega t)$
III	Repeated real roots $s=s_0,s_0$	$C_1e^{s_0t}+C_2te^{s_0t}$

Finding the Particular Solution

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy(t) = x(t),$$

- As before, we will use the method of undetermined coefficients to find the particular solution.
- Based on the form of $x(t)$ we will guess at the form of $y_p(t)$ as given in the table below

$x(t)$	$y_p(t)$
Constant - b_0	c_1
Linear - $b_0 + b_1 t$	$c_0 + c_1 t$
Quadratic - $b_0 + b_1 t + b_2 t^2$	$c_0 + c_1 t + c_2 t^2$
Exponential - e^{rt}	$c_1 e^{rt}$
Sine - $\sin(\omega t)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$
Cosine - $\cos(\omega t)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$

Example

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 4, \quad y(0) = 1, \frac{dy}{dt}(0) = 0.$$

Step 1 – Find the Homogeneous solution

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 0$$

$$y = e^{st} \rightarrow s^2 + 3s + 2 = 0$$

$$\rightarrow (s+1)(s+2) = 0$$

$$\rightarrow s = -1, -2$$

$$y_H(t) = A_1e^{-t} + A_2e^{-2t}$$

Example

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 4, \quad y(0) = 1, \frac{dy}{dt}(0) = 0.$$

Step 2 – Find the Particular solution

Assume $y_p(t) = c$

$$\rightarrow \frac{dy_p}{dt} = 0,$$

$$\rightarrow \frac{d^2y_p}{dt^2} = 0$$

Plug into equation: $2c = 4 \rightarrow c = 2.$

$$y_p(t) = 2.$$

Example

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 4, \quad y(0) = 1, \frac{dy}{dt}(0) = 0.$$

Step 3 – Combine homogeneous and particular solution to form complete solution.

$$y(t) = y_H(t) + y_P(t) = A_1e^{-t} + A_2e^{-2t} + 2.$$

Step 4 – Apply ICs to the complete solution to find the unknown constants.

$$y(0) = 1 \rightarrow A_1 + A_2 + 2 = 1$$

$$\frac{dy}{dt}(0) = 0 \rightarrow -A_1 - 2A_2 = 0$$

$$\rightarrow A_1 = -2, A_2 = 1 \rightarrow y(t) = -2e^{-t} + e^{-2t} + 2$$

Practice problems

Find the solution to each of the following differential equations together with the given initial conditions. Plot your results and visually inspect that your solution matches the initial conditions given.

1. $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = 0$, $y(0) = 2$, $\left.\frac{d}{dt}y(t)\right|_{t=0} = -1$.

2. $\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) + y(t) = 0$, $y(0) = 0$, $\left.\frac{d}{dt}y(t)\right|_{t=0} = 1$.

3. $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}(y(t)) + 4y(t) = 0$, $y(0) = -1$, $\left.\frac{d}{dt}y(t)\right|_{t=0} = 1$.

4. $\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = -6$, $y(0) = 3$, $\left.\frac{d}{dt}y(t)\right|_{t=0} = -2$.

5. $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}(y(t)) + 2y(t) = \exp(-t)$, $y(0) = -1$, $\left.\frac{d}{dt}y(t)\right|_{t=0} = 1$.

6. $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}(y(t)) + y(t) = \cos(t)$, $y(0) = 2$, $\left.\frac{d}{dt}y(t)\right|_{t=0} = -2$.