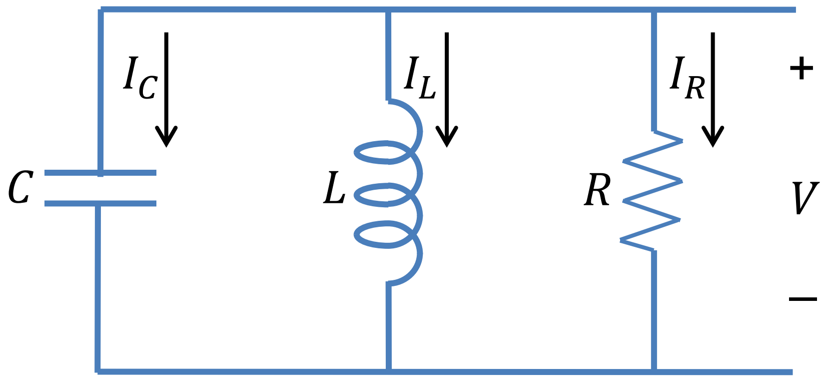
Second Order Circuits

Analysis of circuits involving resistors, capacitors and inductors will often lead to 2nd order differential equations. The basic approach to solving these equations is the same as with first order circuits (although some of the details are a little more complicated):

1. Use KVL or KCL to find a suitable differential equation to describe the circuit.
2. Use the physical initial state of the circuit to establish a suitable set of mathematical initial conditions needed for the differential equation found in step 1.
3. Solve the differential equation together with the initial conditions.
4. If necessary, translate the quantity you found in steps 1-3 to other related voltages, currents, energies, or powers.

Parallel RLC Circuit

This slide demonstrates how to find the differential equation which describes a parallel RLC circuit. Since the voltage, V, is common to all three elements, it is natural to find an equation for V.



Using KCL: $I_C + I_L + I_R = 0$ (1)

Also: $I_C = C\frac{dV}{dt}$, $V = L\frac{dI_L}{dt}$, $V = I_R R$

$I_L = \frac{1}{L}\int V(t)dt$, $I_R = \frac{V(t)}{R}$

Substituting the expressions for I_C, I_L, and I_R into (1) produces:

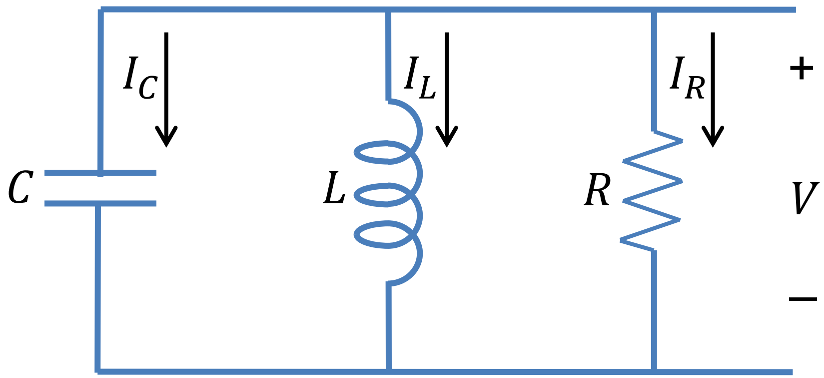
$$C\frac{dV}{dt} + \frac{1}{L}\int V(t)dt + \frac{V(t)}{R} = 0 \xrightarrow{d/dt} C\frac{d^2V}{dt^2} + \frac{V(t)}{L} + \frac{1}{R}\frac{dV}{dt} = 0.$$

Dividing by C and rearranging terms produces:

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{V(t)}{LC} = 0$$

Parallel RLC Circuit

If we were interested in finding some other quantity besides V, (e.g., one of the currents), then we could develop an equation for that quantity as well. Suppose for example, we wanted to solve for I_L.



Using KCL: $I_C + I_L + I_R = 0$ (1)

Also: $I_C = C\frac{dV}{dt}$, $V = L\frac{dI_L}{dt}$, $V = I_R R$

$$I_C = LC\frac{d^2 I_L}{dt^2}, \qquad I_R = \frac{L}{R}\frac{dI_L}{dt}$$

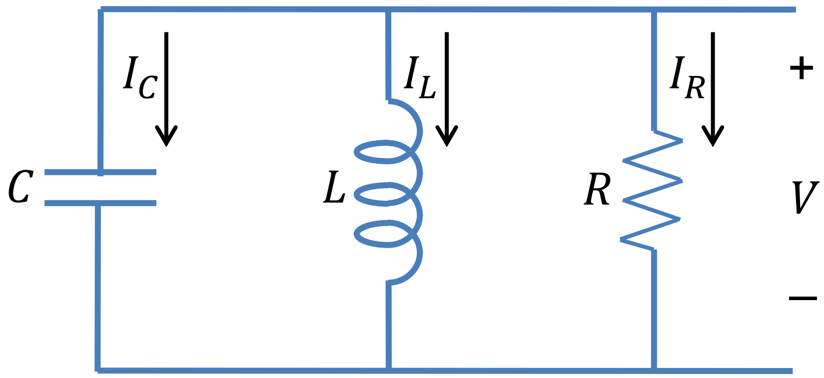
Substituting the expressions for I_C, I_L, and I_R into (1) produces:

$$LC\frac{d^2 I_L}{dt^2} + I_L + \frac{L}{R}\frac{dI_L}{dt} = 0.$$

Dividing by LC and rearranging terms produces:

$$\frac{d^2 I_L}{dt^2} + \frac{1}{RC}\frac{dI_L}{dt} + \frac{I_L(t)}{LC} = 0$$

Parallel RLC Circuit



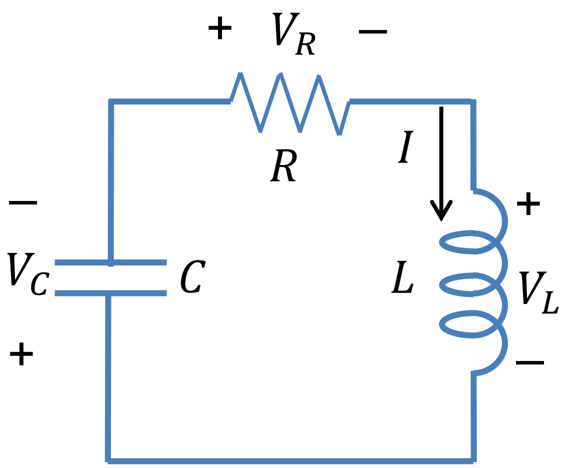
At home, you should convince yourself that the resistor current and the capacitor current also satisfy the same equation that the inductor current and the voltage were shown to satisfy on the previous slides.

In summary, we find that all of the currents and voltages in the parallel RLC circuit of this example satisfy the same second order differential equation:

$$\frac{d^2y}{dt^2} + \frac{1}{RC}\frac{dy}{dt} + \frac{y(t)}{LC} = 0.$$

Series RLC Circuit

Now suppose the RLC combination is in series rather than in parallel. Following the same procedure used for the parallel RLC case produces:



Using KVL: $V_C + V_L + V_R = 0$ (1)

Also: $\mathrm{I} = C\frac{dV_C}{dt}$, $V_L = L\frac{dI}{dt}$, $V_R = IR$

$$V_C = \frac{1}{C}\int I(t)dt$$

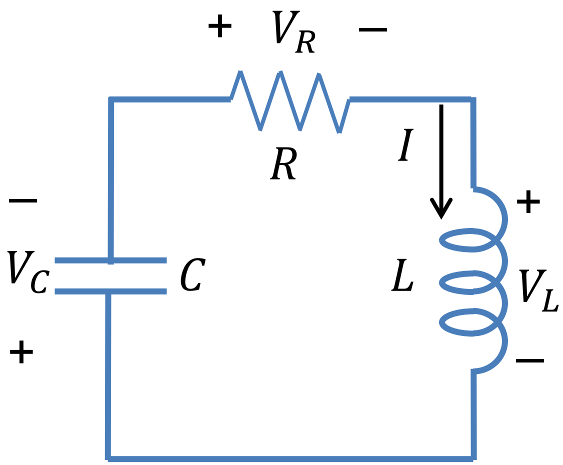
Substituting the expressions for V_C, V_L, and V_R into (1) produces:

$$\frac{1}{C}\int I(t)dt + L\frac{dI}{dt} + RI(t) = 0 \xrightarrow{d/dt} \frac{I(t)}{C} + L\frac{d^2I}{dt^2} + R\frac{dI}{dt} = 0$$

Dividing by L and rearranging terms produces:

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I\ (t)}{LC} = 0$$

Series RLC Circuit



- Note that this second order differential equation is slightly different than the one for the parallel RLC case. The coefficient of the middle term is different.

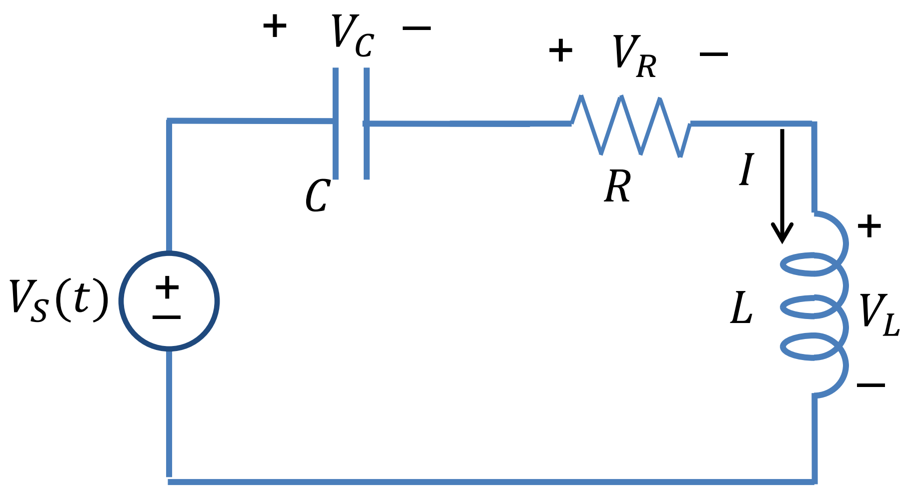
- At home, convince yourself that all three voltages satisfy the same differential equation as the current.

In summary, we find that all of the currents and voltages in the series RLC circuit of this example satisfy the same second order differential equation:

$$\frac{d^2y}{dt^2} + \frac{R}{L}\frac{dy}{dt} + \frac{y\,(t)}{LC} = 0.$$

RLC Circuits with Sources

When sources are added to the RLC circuits, sometimes the relevant differential equation will become non-homogeneous.



Using KCL: $V_C + V_L + V_R = V_S$ (1)

$$\mathrm{I} = C\frac{dV_C}{dt}, \quad V_L = L\frac{dI}{dt}, \quad V_R = IR$$

$$V_C = \frac{1}{C}\int I(t)dt$$

Substituting the expressions for V_C, V_L, and V_R into (1) produces:

$$\frac{1}{C}\int I(t)dt + L\frac{dI}{dt} + RI(t) = V_S$$

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I\ (t)}{LC} = \frac{1}{L}\frac{dV_S}{dt}$$

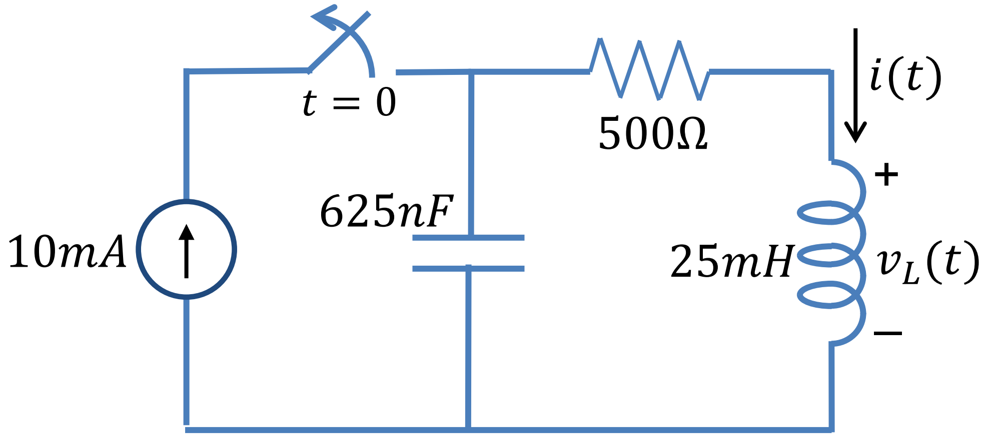
If $V_S(t)$ is constant (dc source), then $\frac{dV_S}{dt} = 0$ and the equation is homogeneous. Otherwise, the equation is non-homogeneous and will have a particular solution.

Example 1

In the circuit shown, the switch in the circuit has been closed for a long time before opening at time $t = 0$.

(a) Find the differential equation satisfied by $i(t)$ for $t > 0$.

(b) Find the differential equation satisfied by $v_L(t)$ for $t > 0$.

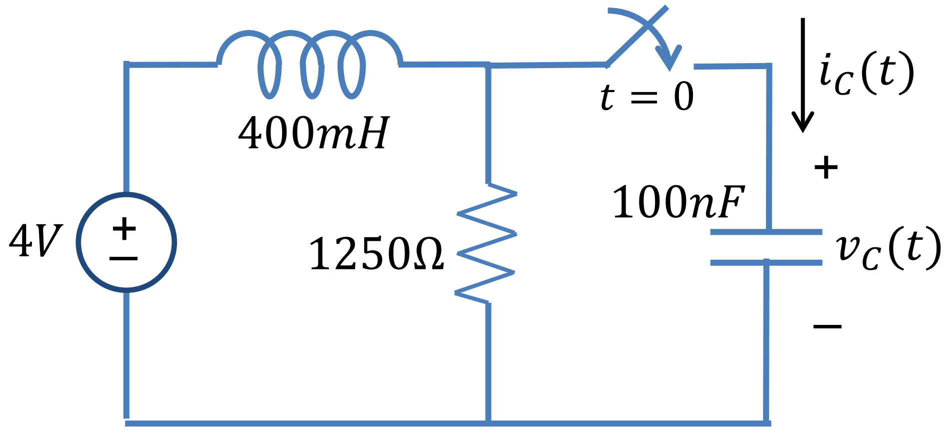


Example 2

In the circuit shown, the switch in the circuit has been open for a long time before closing at time $t = 0$. Assuming that there is no charge on the capacitor just prior to the switch moving,

(a) Find the differential equation satisfied by $v_C(t)$ for $t > 0$.

(b) Find the differential equation satisfied by $i_C(t)$ for $t > 0$.



Second Order Circuits

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1. Use KVL or KCL to find a suitable differential equation to describe the circuit.
2. Use the physical initial state of the circuit to establish a suitable set of mathematical initial conditions needed for the differential equation found in step 1.
3. Solve the differential equation together with the initial conditions.
4. If necessary, translate the quantity you found in steps 1-3 to other related voltages, currents, energies, or powers.

Initial Conditions

In order to find appropriate initial conditions, we start with the physical state of the circuit. This will usually involve initial currents in inductors and voltages in capacitors (since those are the things that do not change instantaneously when switches are flipped).

Using KVLs, KCLs, Ohm's Law, etc., we can translate the initial state of a circuit into appropriate mathematical initial conditions. These will usually take on the form of:

$$i(0) = a, \frac{di}{dt}(0) = b,$$

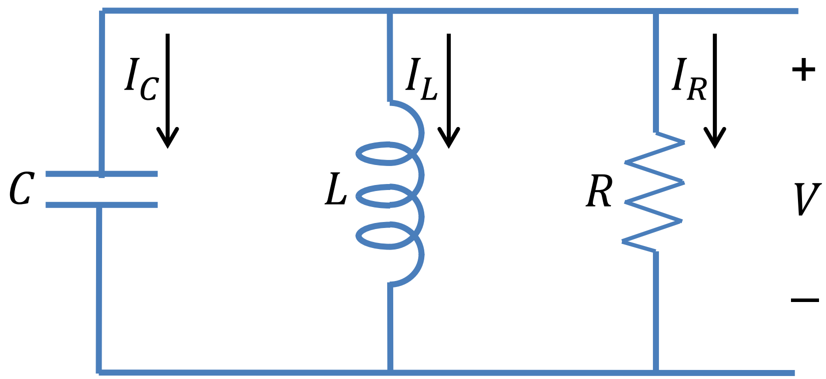
or

$$\mathrm{v}(0) = a, \frac{dv}{dt}(0) = b,$$

where a and b are constants. The differential equation we are trying to solve will specify the form of the initial conditions needed.

Parallel RLC Circuit

This slide demonstrates how to find the initial conditions needed for the voltage in a parallel RLC circuit.



Recall that the differential equation satisfied by the voltage in this circuit was:

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{V(t)}{LC} = 0$$

To completely solve this ODE we need two initial conditions, v(0) and $\frac{dV}{dt}(0)$.

Suppose we know the physical initial conditions of the circuit, namely $I_L(0)$ and $V_C(0)$. In this circuit, finding the initial voltage $\mathrm{V}(0)$ is trivial,

$$V(0) = V_C(0).$$

To find $\frac{dV}{dt}(0)$ we seek to find the capacitor current since $I_C(t) = C\,\frac{dV}{dt}$.

Using KCL:

$$I_C + I_L + I_R = 0, \Longrightarrow C\,\frac{dV}{dt} + I_L + \frac{V(t)}{R} = 0.$$

Finally:

$$\frac{dV}{dt}(0) = -\frac{I_L(0)}{C} - \frac{V_C(0)}{RC}$$

Parallel RLC Circuit

If we wanted to, we could also find initial conditions on any of the currents in the circuit. At home, try to show that the initial conditions on the various currents are given by:

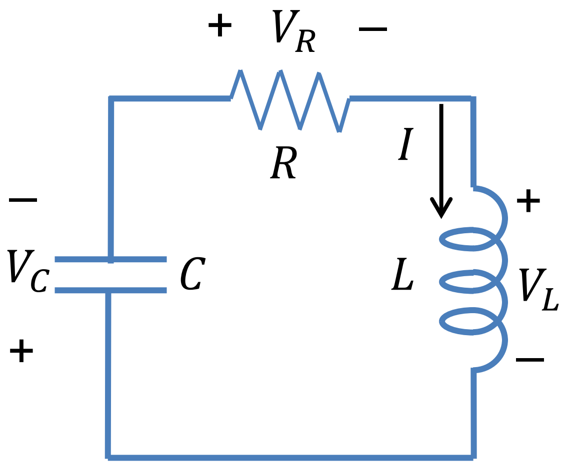
Inductor Current: $\frac{dI_L}{dt}(0) = \frac{V_C(0)}{L}$.

Resistor Current: $I_R(0) = \frac{V_C(0)}{R}$, $\quad \frac{dI_R}{dt}(0) = -\frac{1}{RC}\left(I_L(0) + \frac{V_C(0)}{R}\right)$.

Capacitor Current: $I_C(0) = -I_L(0) - \frac{V_C(0)}{R}$, $\quad \frac{dI_C}{dt}(0) = \frac{I_L(0)}{RC} + \frac{V_C(0)}{R}\left(\frac{1}{RC} - \frac{R}{L}\right)$.

Series RLC Circuit

This slide demonstrates how to find the initial conditions needed for the current in a series RLC circuit.



Recall that the differential equation satisfied by the current in this circuit was:

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I\,(t)}{LC} = 0$$

To completely solve this ODE we need two initial conditions, $\mathrm{I}(0)$ and $\frac{dI}{dt}(0)$.

Again starting with knowledge of $I_L(0)$ and $V_C(0)$, in this circuit, finding the initial current $\mathrm{I}(0)$ is trivial,

$$I(0) = I_L(0).$$

To find $\frac{dI}{dt}(0)$ we seek to find the inductor voltage since $V_L(t) = L\frac{dI}{dt}$. Using KVL:

$$V_C + V_L + V_R = 0, \quad \Longrightarrow \quad V_C + L\frac{dI}{dt} + RI(t) = 0.$$

Finally:

$$\frac{dI}{dt}(0) = -\frac{R}{L} I_L(0) - \frac{V_C(0)}{L}$$

Series RLC Circuit

If we wanted to we could also find initial conditions on any of the voltages in the circuit. At home, try to show that the initial conditions on the various voltages are given by:

Capacitor Voltage: $\frac{dV_C}{dt}(0) = \frac{I_L(0)}{C}$.

Resistor Voltage: $V_R(0) = RI_L(0)$,

$$\frac{dV_R}{dt}(0) = -\frac{R}{L}\left(RI_L(0) + V_C(0)\right).$$

Inductor Voltage: $V_L(0) = -RI_L(0) - V_C(0)$,

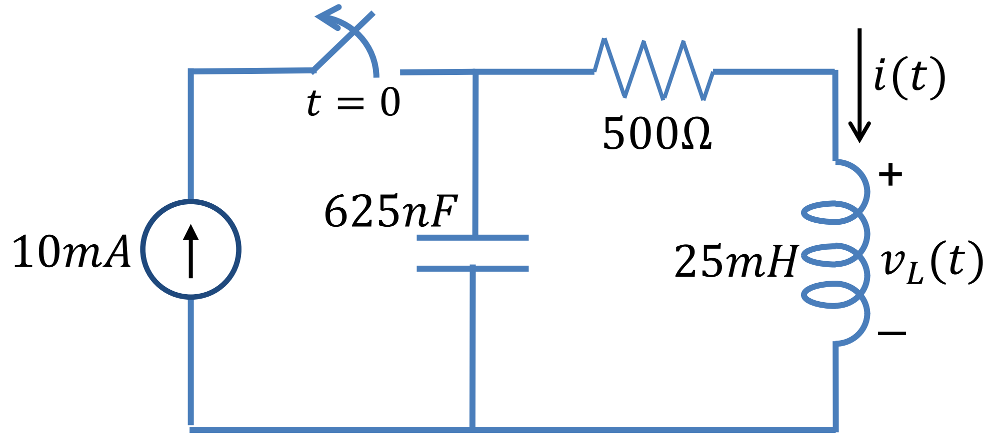
$$\frac{dV_L}{dt}(0) = \frac{R}{L}V_C(0) + R\left(\frac{R}{L} - \frac{1}{RC}\right)I_L(0).$$

Example 1

In the circuit shown, the switch in the circuit has been closed for a long time before opening at time $t = 0$.

(a) Find $i(0^+)$ and $\frac{di}{dt}(0^+)$.

(b) Find $v_L(0^+)$ and $\frac{dv_L}{dt}(0^+)$.

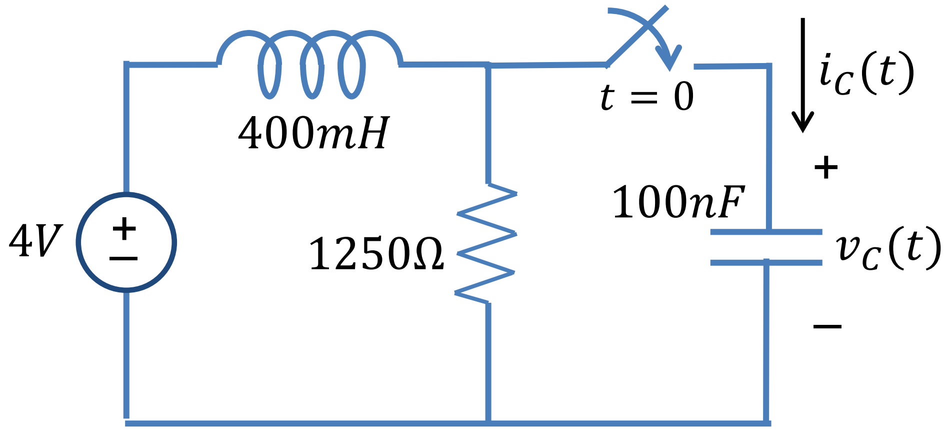


Example 2

In the circuit shown, the switch in the circuit has been open for a long time before closing at time $t = 0$. Assuming that there is no charge on the capacitor just prior to the switch moving,

(a) Find $v_C(0^+)$ and $\frac{dv_C}{dt}(0^+)$.

(b) Find $i_C(0^+)$ and $\frac{di_C}{dt}(0^+)$.



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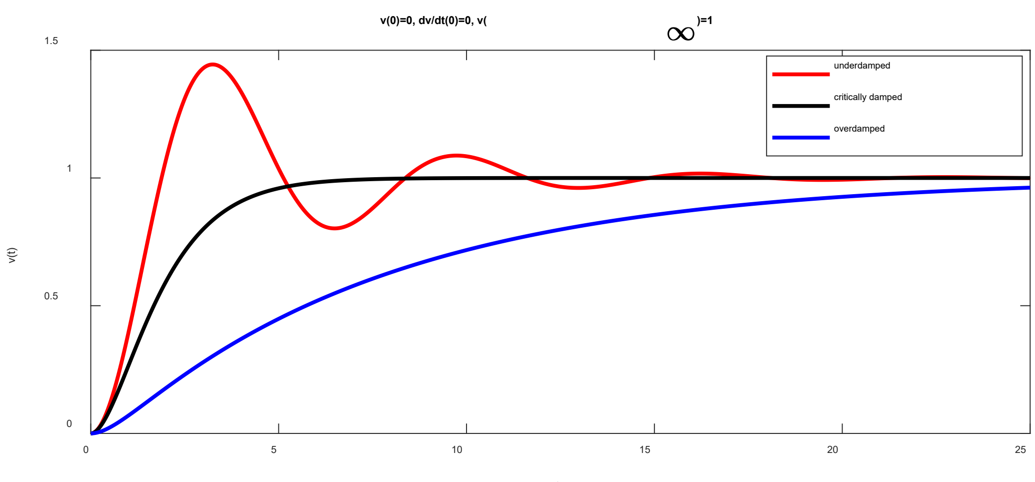
Damping in Second Order Circuits

When solving for the homogeneous solution of a second order equation, the characteristic equation will have one of three types of solutions:

Case	Roots	Homogeneous Solution	Response
I	Distinct real roots $s = s_1, s_2$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$	Overdamped
II	Complex roots $s = \sigma \pm j\omega$	$B_1 e^{\sigma t} \cos(\omega t) + B_2 e^{\sigma t} \sin(\omega t)$	Underdamped
III	Repeated real roots $s = s_0, s_0$	$C_1 e^{s_0 t} + C_2 t e^{s_0 t}$	Critically damped



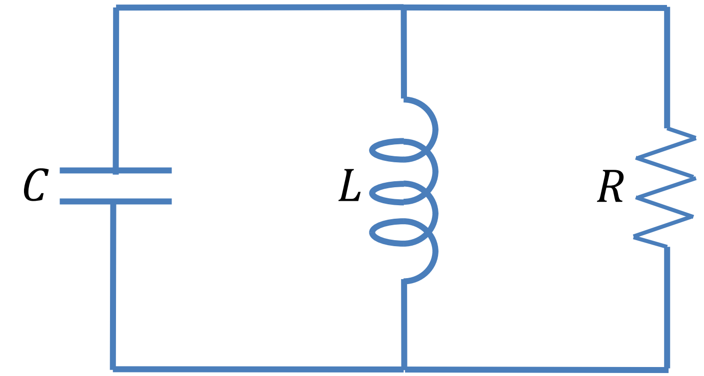
Overdamped – relatively slow to reach steady state value.

Underdamped – reacts to change in input quickly, but response rings.

Critically damped – quickest approach to steady state values without overshoot.

Damping in Second Order Circuits

Suppose we had a circuit with just an inductor and a capacitor (RLC circuit with no R). This is an example of what you called a simple harmonic oscillator in your Physics class.



Diff. Eq. → $\frac{d^2y}{dt^2} + \frac{y\,(t)}{LC} = 0$

Char. Eq. → $s^2 + \frac{1}{LC} = 0$

Roots: $s = \pm j\sqrt{\frac{1}{LC}}$

Behavior: $B_1 \cos(\omega_o t) + B_2 \sin(\omega_o t)$
→ purely sinusoidal response (no damping)

$\omega_o = \sqrt{\frac{1}{LC}}$ = natural resonant frequency

As we add some resistance to the circuit (underdamped → parallel resistance finite but still large), the response becomes a damped oscillation.

Diff. Eq. → $\frac{d^2y}{dt^2} + \frac{1}{RC}\frac{dy}{dt} + \frac{y\,(t)}{LC} = 0$

Char. Eq. → $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$

Roots: $s = -\frac{1}{2RC} \pm j\sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$

$$\omega_d = \sqrt{\omega_o^2 - \left(\frac{1}{2RC}\right)^2}$$

= damped resonant frequency

Damping in Second Order Circuits

There is a quantity known as the Q-factor that measures how underdamped or overdamped a second order system is.

For a series RLC circuit:

Diff. Eq. → $\frac{d^2y}{dt^2} + \frac{R}{L}\frac{dy}{dt} + \frac{y(t)}{LC} = 0$

Char. Eq. → $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$

Roots: $s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

Q-factor: $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$

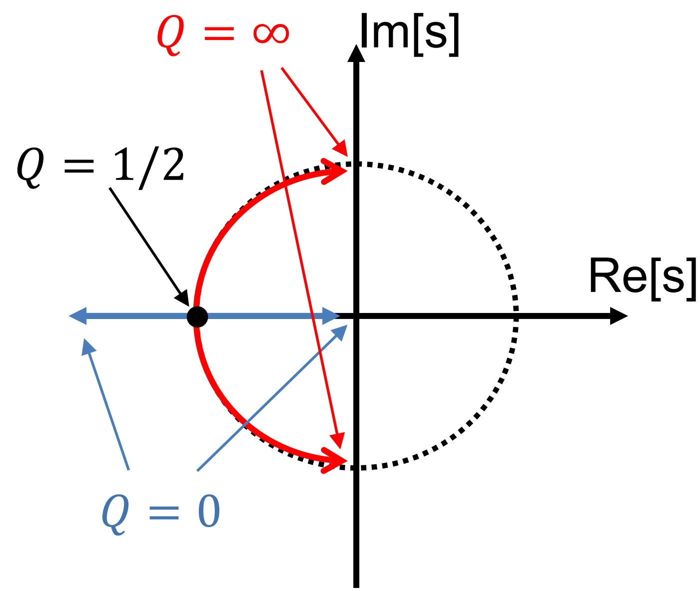
→ roots, $s = \frac{\omega_o}{2Q}\left(-1 \pm \sqrt{1-(2Q)^2}\right)$

→ $Q = \frac{1}{2}$ ←critically damped

→ $Q > \frac{1}{2}$ ←underdamped

$$\omega_d = \omega_o\sqrt{1-\left(\frac{1}{2Q}\right)^2}$$

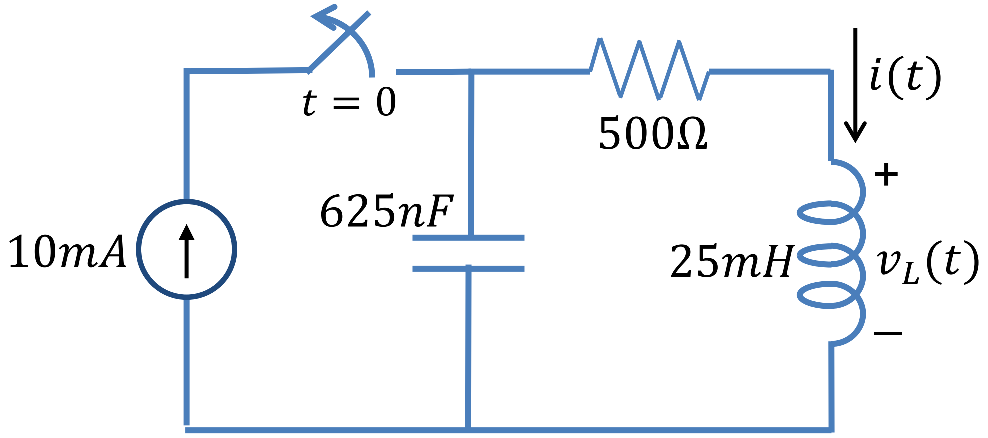
→ $Q < \frac{1}{2}$ ←overdamped



- For a parallel RLC circuit: $Q = R\sqrt{\frac{C}{L}}$
- The concept of Q-factor is not all that important for DC circuits
- It will be more meaningful in the context of AC circuits.

Example 1

In the circuit shown, the switch in the circuit has been closed for a long time before opening at time $t = 0$.
(a) Find $i(t)$ for $t > 0$.
(b) Find $v_L(t)$ for $t > 0$.



We found previously that:

(a) $\frac{d^2 i}{dt^2} + 20{,}000\frac{di}{dt} + 64{,}000{,}000\frac{i\,(t)}{LC} = 0$

$i(0^+) = 10mA,$

$\frac{di}{dt}(0^+) = 0A/s.$

(b) $\frac{d^2 v_L}{dt^2} + 20{,}000\frac{dv_L}{dt} + 64{,}000{,}000\frac{v_L(t)}{LC} = 0$

$v_L(0^+) = 0V,$

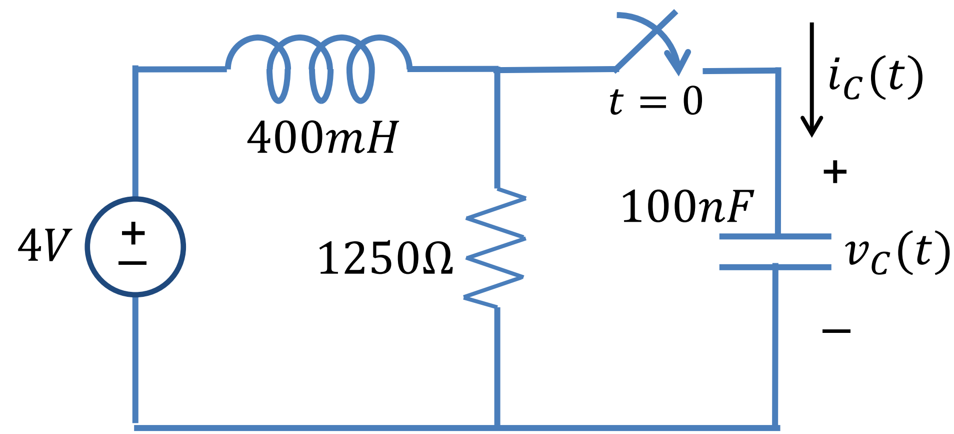
$\frac{dv_L}{dt}(0^+) = -16{,}000V/s.$

Example 2

In the circuit shown, the switch in the circuit has been open for a long time before closing at time $t = 0$. Assuming that there is no charge on the capacitor just prior to the switch moving,

(a) Find $v_C(t)$ for $t > 0$.

(b) Find $i_C(t)$ for $t > 0$.



We found previously that:

(a) $\frac{d^2 v_C}{dt^2} + 8{,}000\frac{dv_C}{dt} + 25{,}000{,}000 v_C(t) = 100{,}000{,}000$

$v_C(0^+) = 0V,$

$\frac{dv_C}{dt}(0^+) = 32{,}000 V/s.$

(b) $\frac{d^2 i_C}{dt^2} + 8{,}000\frac{di_C}{dt} + 25{,}000{,}000 i_C(t) = 0$

$i_C(0^+) = 3.2mA,$

$\frac{di_C}{dt}(0^+) = -15.6 A/s.$

Second Order Circuits

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3. Solve the differential equation together with the initial conditions.

4. If necessary, translate the quantity you found in steps 1-3 to other related voltages, currents, energies, or powers.

Example 2 (revised)

In the circuit shown, the switch in the circuit has been open for a long time before closing at time $t = 0$. Assuming that there is no charge on the capacitor just prior to the switch moving, find $i_R(t)$ for $t > 0$.