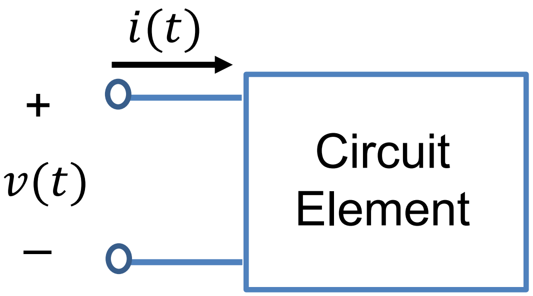
Power in AC Circuits

When voltages and currents are sinusoids, power will be a function of time.

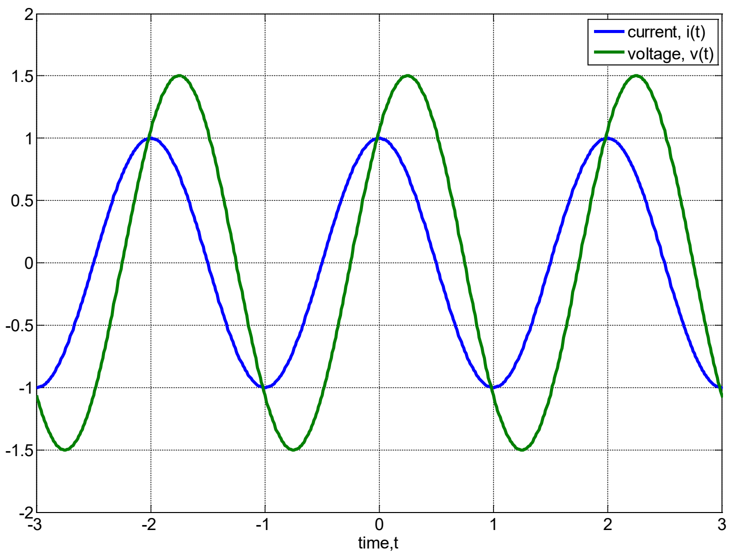
$$p(t) = i(t)v(t)$$

We refer to $p(t)$ as the *instantaneous power*.



Writing $i(t)$ and $v(t)$ explicitly as sinusoids:

$$i(t) = I_m \cos(\omega t + \theta_i)\,,$$
$$v(t) = V_m \cos(\omega t + \theta_v)\,.$$



It is common to define a time reference so that the current has a positive maximum at $t = 0$.

In that case, both $i(t)$ and $v(t)$ must be shifted by θ_i so that:

$$i(t) = I_m \cos(\omega t)\,,$$
$$v(t) = V_m \cos(\omega t + \theta_v - \theta_i)\,.$$

Power in AC Circuits

Using the time reference convention from the previous slide, the instantaneous power becomes:

$$p(t) = i(t)v(t)$$
$$= I_m V_m \cos(\omega t)\cos(\omega t + \theta_v - \theta_i)$$

Use the trig identity:

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$$

$$= \frac{I_m V_m}{2}\cos(\theta_v - \theta_i) + \frac{I_m V_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$

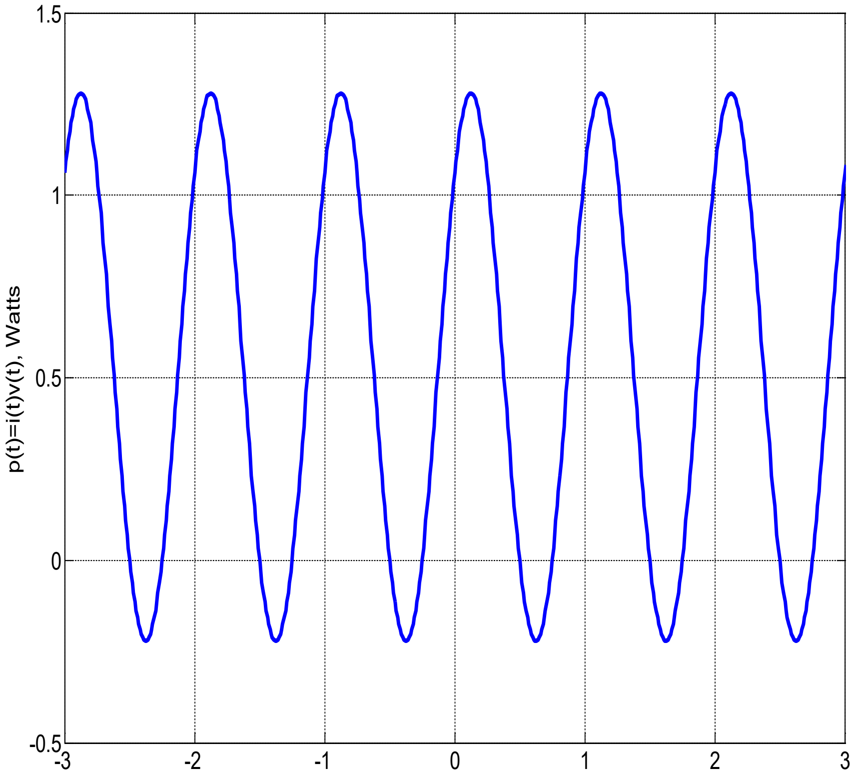
Use the trig identity:

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$= \frac{I_m V_m}{2}\cos(\theta_v - \theta_i) + \frac{I_m V_m}{2}\cos(\theta_v - \theta_i)\cos(2\omega t) - \frac{I_m V_m}{2}\sin(\theta_v - \theta_i)\sin(2\omega t)$$

Power in AC Circuits

$$p(t) = \frac{I_m V_m}{2}\cos(\theta_v - \theta_i) + \frac{I_m V_m}{2}\cos(\theta_v - \theta_i)\cos(2\omega t) - \frac{I_m V_m}{2}\sin(\theta_v - \theta_i)\sin(2\omega t)$$



Some observations:

- Instantaneous power oscillates at twice the frequency of current/voltage.
- Instantaneous power can be negative, even though we are using passive elements.
- The average value of the instantaneous power is

$$P = \frac{I_m V_m}{2}\cos(\theta_v - \theta_i)$$

- Average power will be positive and depends not only on the peak current and voltage but also on the phase relationship between them.

Average and Reactive Power

$$p(t) = \frac{I_m V_m}{2}\cos(\theta_v - \theta_i) + \frac{I_m V_m}{2}\cos(\theta_v - \theta_i)\cos(2\omega t) - \frac{I_m V_m}{2}\sin(\theta_v - \theta_i)\sin(2\omega t)$$

Some Definitions:

- *Average Power* or *Real Power*, measured in Watts

$$P = \langle p(t)\rangle \ = \frac{1}{T}\int_{t_o}^{t_o+T} p(t)dt = \frac{I_m V_m}{2}\cos(\theta_v - \theta_i)\,.$$

- *Reactive Power*, measured in VARs (volt-amp-reactive)

$$Q = \frac{I_m V_m}{2}\sin(\theta_v - \theta_i)\,.$$

With these definitions, instantaneous power is written as:

$$p(t) = P + P\cos(2\omega t) - Q\sin(2\omega t)\,.$$

Average and Reactive Power

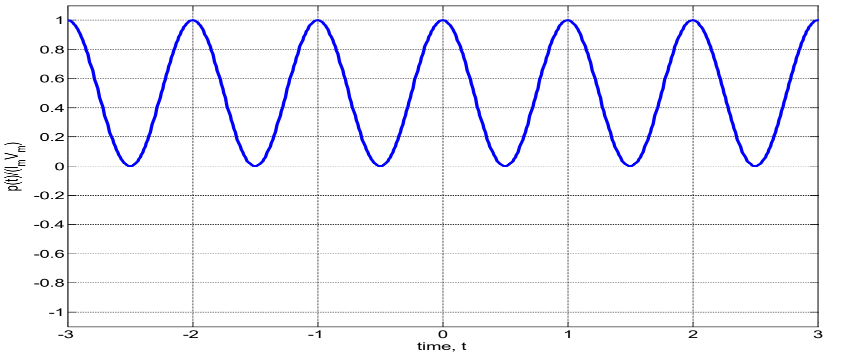
Resistors

$\theta_v = \theta_i,$

$\cos(\theta_v - \theta_i) = 1, \sin(\theta_v - \theta_i) = 0.$

$P = \frac{I_m V_m}{2}, \ \ Q = 0.$

$p(t) = P + P\cos(2\omega t).$



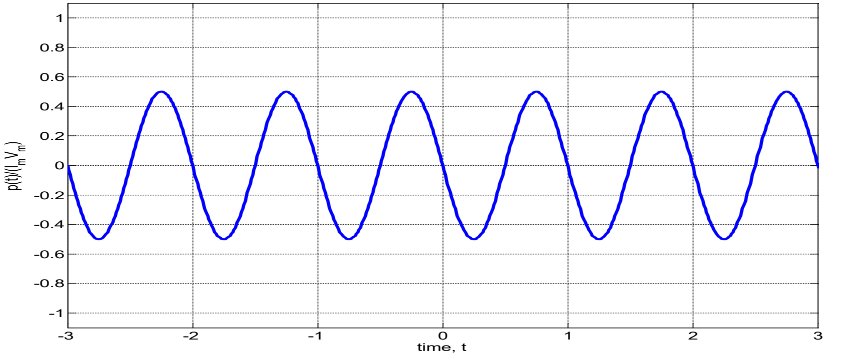
Inductors

$\theta_v = \theta_i + 90°,$

$\cos(\theta_v - \theta_i) = 0, \sin(\theta_v - \theta_i) = 1.$

$P = 0, \ \ Q = \frac{I_m V_m}{2}.$

$p(t) = -\frac{I_m V_m}{2}\sin(2\omega t).$



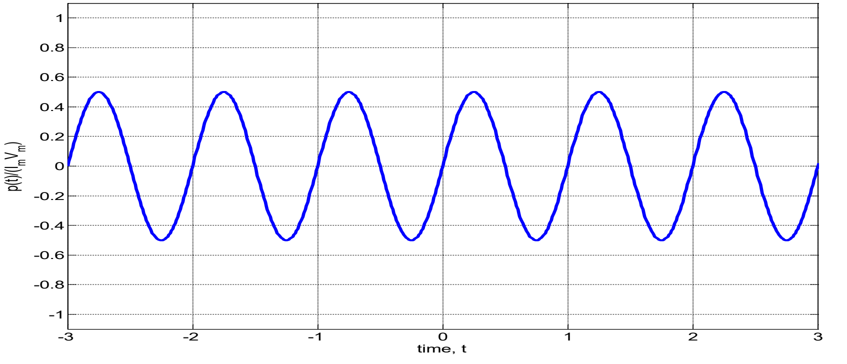
Capacitors

$\theta_v = \theta_i - 90°,$

$\cos(\theta_v - \theta_i) = 0, \sin(\theta_v - \theta_i) = -1.$

$P = 0, \ \ Q = -\frac{I_m V_m}{2}.$

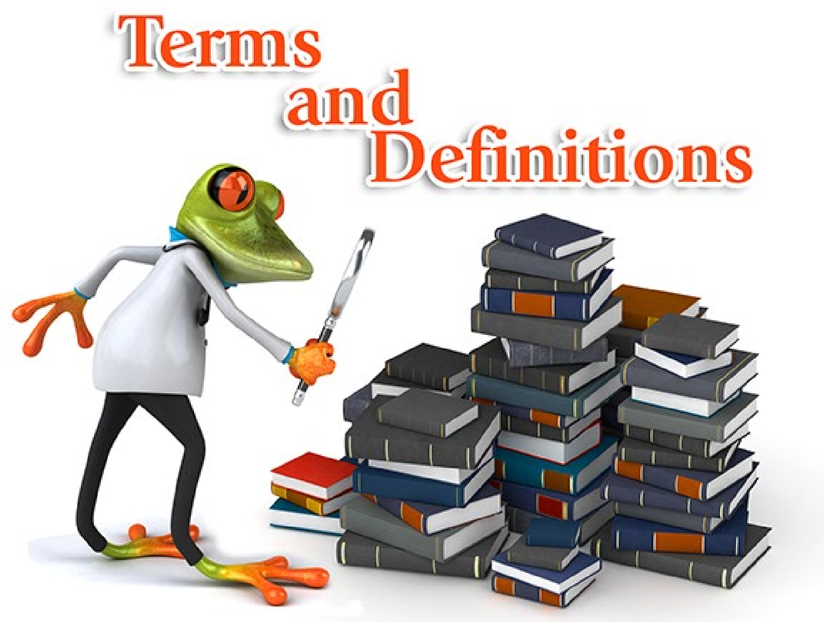
$p(t) = +\frac{I_m V_m}{2}\sin(2\omega t).$



Power Factors and Reactive Factors

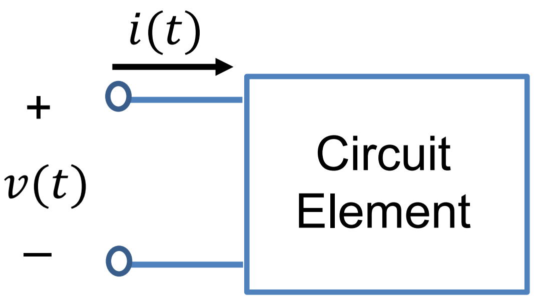
More Definitions:

- $\theta_v - \theta_i =$ power factor angle
- $\cos(\theta_v - \theta_i) =$ power factor (pf)
- $\sin(\theta_v - \theta_i) =$ reactive factor (rf)
- $\theta_v > \theta_i \rightarrow$ lagging power factor
  current lags voltage
  $rf > 0$
  inductive element
- $\theta_v < \theta_i \rightarrow$ leading power factor
  current leads voltage
  $rf < 0$
  capacitive element



Example

Find:
(a) Average power, P.
(b) Reactive power, Q.
(c) Power factor, pf.
(d) Reactive factor, rf.



$v(t) = 100\cos(\omega t + 15°)$ V
$i(t) = 4\sin(\omega t - 15°)$ A

RMS Power

It is common to express power calculations in terms of RMS values. For example, when we say that a wall outlet produces 120V AC, this is actually an RMS value.

Recall that for sinusoidal voltages and currents $V_{rms} = V_m/\sqrt{2}$ and $I_{rms} = I_m/\sqrt{2}$. Then:

$$P = I_{rms}V_{rms}\cos(\theta_v - \theta_i)\,,$$
$$Q = I_{rms}V_{rms}\sin(\theta_v - \theta_i)\,.$$

In a resistor,

$$P = I_{rms}V_{rms},$$
$$V_{rms} = I_{rms}\mathrm{R},$$

so that formulas for average power correspond to what we are used to for DC circuits:

$$P = {I_{rms}}^2 R = \frac{{V_{rms}}^2}{R}.$$

Power in AC Circuits Using Phasors

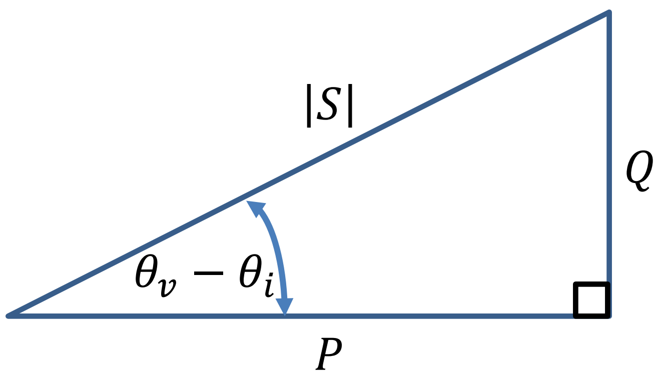
It is sometimes convenient to use phasors to compute various power quantities. To aid in this, define a "complex power" which is a complex number formed from average and reactive power.

Complex Power, measured in VAs (volt-amps)

$$S = P + jQ$$

Apparent Power, measured in VAs (volt-amps)

$$|S| = \sqrt{P^2 + Q^2}.$$



You can visualize all of these quantities with a right triangle as shown.

Note: If any two of the quantities in the figure above are known, the others can be found using the appropriate trig/geometry.

Power in AC Circuits Using Phasors

Recalling the definitions of P, Q, S, complex power can be written as

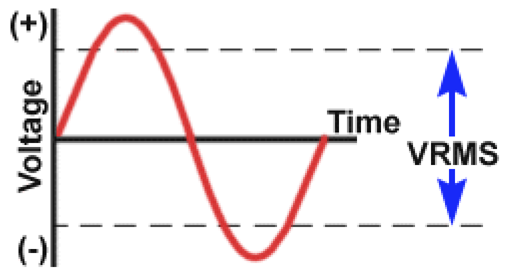
$$S = P + jQ$$
$$= \frac{I_m V_m}{2}\cos(\theta_v - \theta_i) + j\,\frac{I_m V_m}{2}\sin(\theta_v - \theta_i)$$

$$= \frac{I_m V_m}{2}\exp(\mathrm{j}(\theta_v - \theta_i))$$

$$= \frac{1}{2}\left(V_m e^{j\theta_v}\right)\left(I_m e^{-j\theta_i}\right)$$

$$S = \frac{1}{2}\boldsymbol{V}\boldsymbol{I}^*$$

We can compute S directly from the voltage and current phasors and the P and Q can be extracted from $Re[S]$ and $Im[S]$.



Alternatively, we can use RMS voltages and currents (and phasors):

$$\boldsymbol{V}_{rms} = V_{rms}e^{j\theta_v} = \frac{V_m}{\sqrt{2}}e^{j\theta_v}$$

$$\boldsymbol{I}_{rms} = I_{rms}e^{j\theta_v} = \frac{I_m}{\sqrt{2}}e^{j\theta_i}$$

$$S = \boldsymbol{V}_{rms}\boldsymbol{I}^*{}_{rms}$$

Power in AC Circuits Using Phasors

For an element with impedance Z, the voltage and current phasors are related by

$$\boldsymbol{V}_{rms} = \boldsymbol{I}_{rms} Z,$$

so that various forms of the power expressions can be developed.

$$S = \boldsymbol{V}_{rms}\boldsymbol{I}^*{}_{rms}$$

$$= (\boldsymbol{I}_{rms} Z)\boldsymbol{I}^*{}_{rms}$$

$$= |\boldsymbol{I}_{rms}|^2 Z$$

$$= |\boldsymbol{I}_{rms}|^2 (X + jY)$$

Therefore,

$$P = |\boldsymbol{I}_{rms}|^2 X,$$

$$Q = |\boldsymbol{I}_{rms}|^2 Y.$$

$$S = \boldsymbol{V}_{rms}\boldsymbol{I}^*{}_{rms}$$

$$= \boldsymbol{V}_{rms}\left(\frac{\boldsymbol{V}_{rms}}{Z}\right)^* = \frac{|\boldsymbol{V}_{rms}|^2}{Z^*}$$

$$= \frac{|\boldsymbol{V}_{rms}|^2}{X - jY} = \frac{|\boldsymbol{V}_{rms}|^2 (X + jY)}{X^2 + Y^2}$$

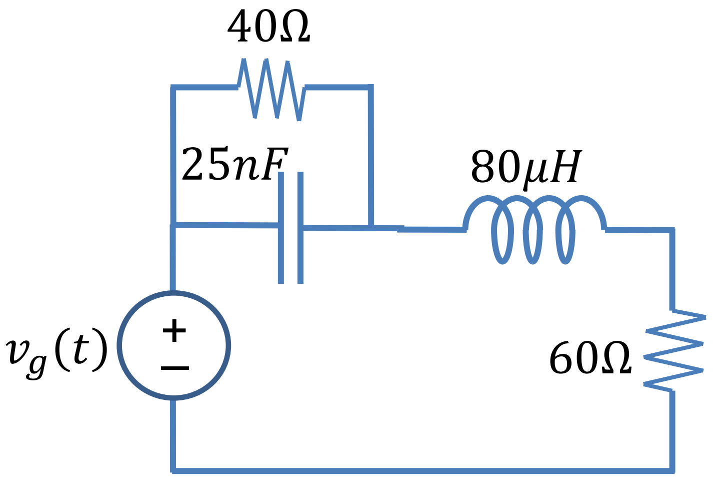
Therefore,

$$P = \frac{|\boldsymbol{V}_{rms}|^2 X}{|Z|^2},$$

$$Q = \frac{|\boldsymbol{V}_{rms}|^2 Y}{|Z|^2}.$$

Example

Find:
(a) Average power,
(b) Reactive power,
(c) Apparent power,
Supplied by the voltage source.



$v_g(t) = 40\cos(10^6 t)$ V