Circuits with Resistors, Capacitors and/or Inductors

When we studied resistive circuits, the equations involving voltages and currents were linear and we needed concepts from (linear) algebra to solve those equations.

Our next step will be to add inductors and/or capacitors to our circuits. The equations involving voltages and currents will now contain derivatives and hence we will need to use differential equations to solve the relevant circuit equations.

The approach to solve these problems will follow a general three step approach:
1. Use KVLs/KCLs to find the differential equation which describes the circuit.
2. Use the physical initial state of the circuit to develop a set of initial conditions necessary to solve the differential equation.
3. Solve the differential equation together with the initial conditions.

Finding the Differential Equation for an RL Circuit

This slide shows how to find the differential equation involving the current in a simple circuit consisting of a single resistor and a single inductor.
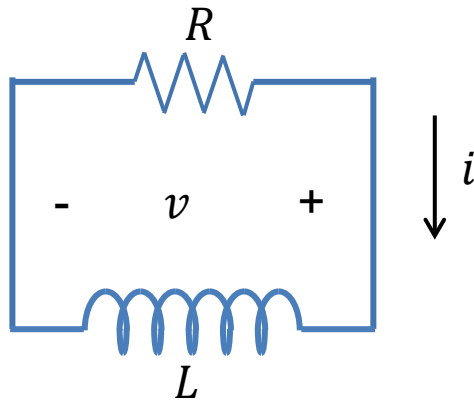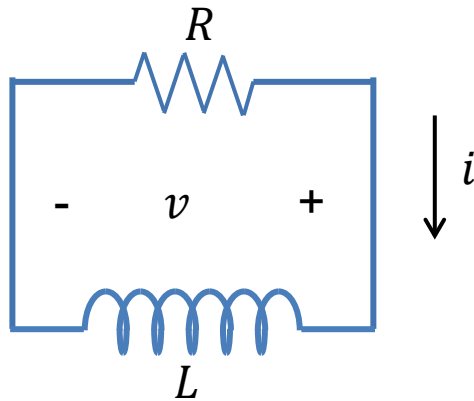
For the inductor: $v = L\dfrac{di}{dt}$

For the resistor: $v = -iR$
(note the sign convention)

Since the resistor and inductor are in parallel, the voltages across them are the same:

$$v = -iR = L\frac{di}{dt} \quad \Longrightarrow \quad \frac{di}{dt} + \frac{R}{L}i = 0.$$

Finding the Differential Equation for an RL Circuit

With a slight modification, we can find the differential equation that the voltage in the same circuit obeys

For the inductor: $v = L\dfrac{di}{dt}$

For the resistor: $v = -iR$ $\rightarrow$ $\dfrac{dv}{dt} = -R\dfrac{di}{dt}$
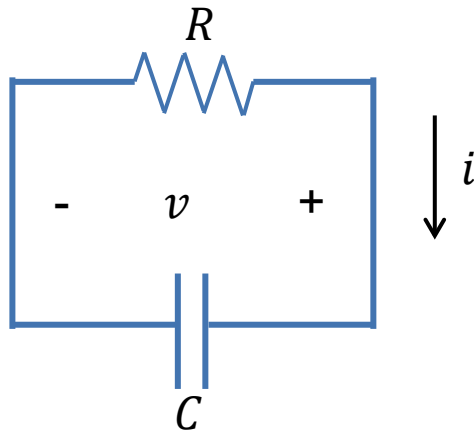(note the sign convention)

Since the current through the resistor and inductor are the same:

$$\frac{di}{dt} = \frac{v}{L} = -\frac{1}{R}\frac{dv}{dt} \implies \boxed{\frac{dv}{dt} + \frac{R}{L}v = 0.}$$

Note that this is the same equation that the current satisfied.

Finding the Differential Equation for an RC Circuit

We can follow an identical procedure to find the differential equation in a circuit with a single capacitor and a single resistor.

For the capacitor: $i = C \dfrac{dv}{dt}$

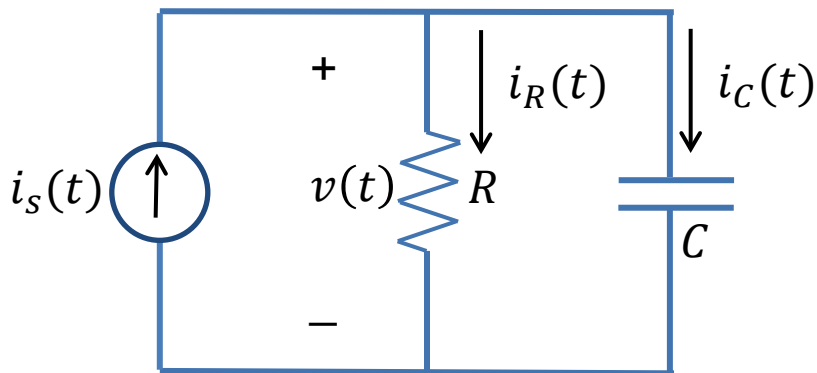For the resistor: $v = -iR$ ➡️ $\dfrac{dv}{dt} = -R\dfrac{di}{dt}$
(note the sign convention)

Since the voltage across the resistor and capacitor are the same:

$$\frac{dv}{dt} = \frac{i}{C} = -R\frac{di}{dt} \implies \boxed{\frac{di}{dt} + \frac{1}{RC}i = 0.}$$

At home: Show that the voltage satisfies the same differential equation.

Finding the Differential Equation for 1st Order Circuits with Sources

If we add a voltage/current source to the circuit, the resulting O.D.E. may no longer be homogeneous.

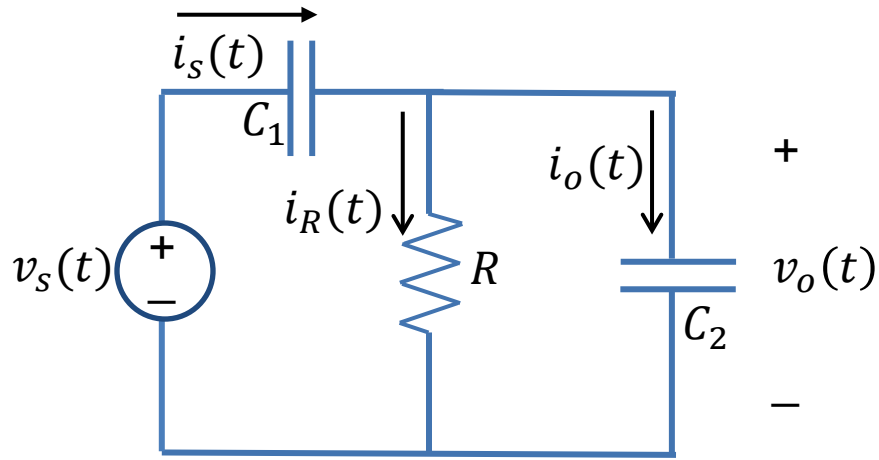For the capacitor: $i_C = C\dfrac{dv}{dt}$

For the resistor: $v = i_R R$

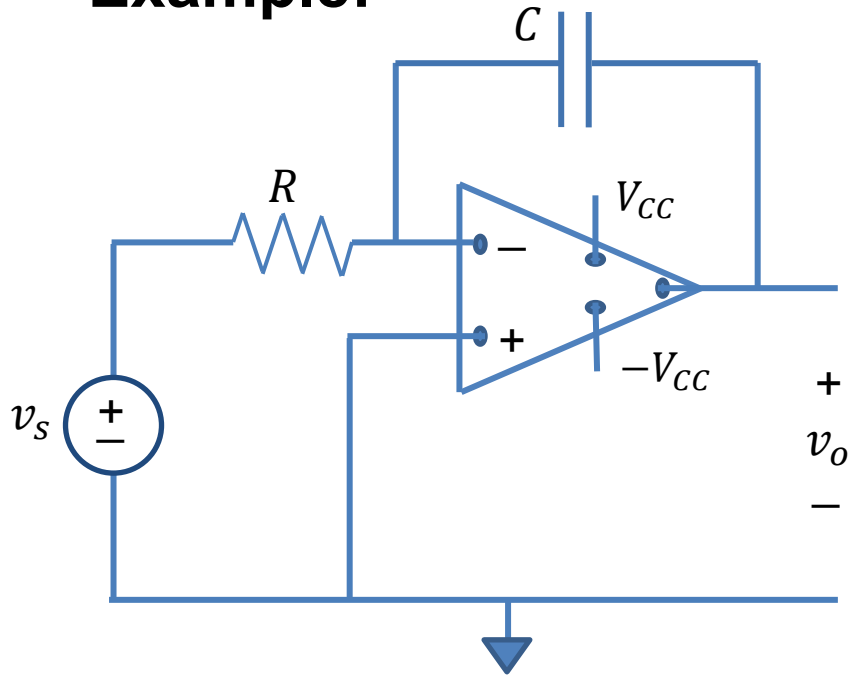Using KCL:

$$i_S = i_R + i_C = \frac{v}{R} + C\frac{dv}{dt}$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{i_S}{C}$$

Example:

(a) Find the differential equation satisfied by the output voltage, $v_o(t)$.

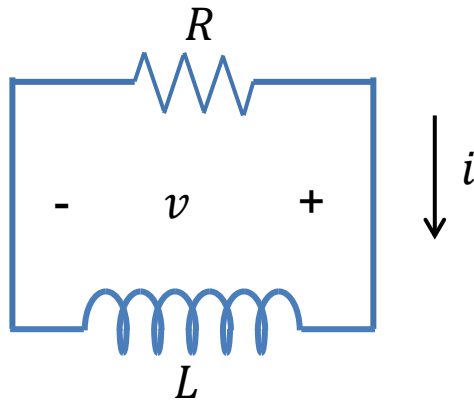(b) Find the differential equation satisfied by the output current, $i_o(t)$.

Example:

(a) Find the relationship between the output voltage, $v_o(t)$, and the input voltage, $v_s(t)$.

(b) What mathematical operation does this circuit perform?

(c) How would the operation of this circuit change if we exchanged the positions of the resistor and the capacitor?

Solving the Differential Equation for a 1st Order Circuit

In order to completely solve the differential equation that describes a circuit, we need to know something about the "initial" state of the circuit.

Recall that for this circuit, the current satisfies the differential equation:
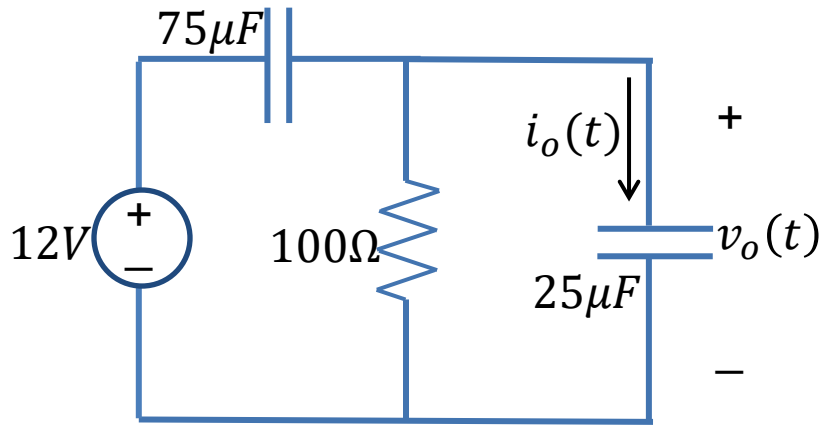
$$\frac{di}{dt} + \frac{R}{L}i = 0.$$

The general solution to this first order ODE is

$$i(t) = Ae^{-\frac{R}{L}t}.$$

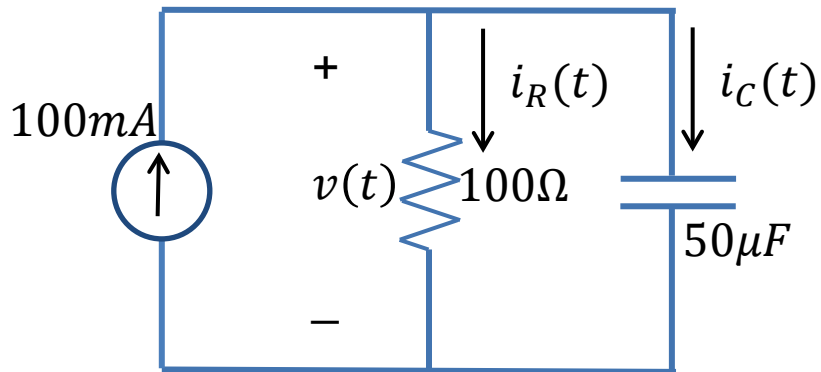In the solution above, A is an arbitrary constant. It is determined from the initial conditions.

e..g, suppose we know that at time $t = 0$, $i(t) = 0.25$Amps. Then, from the general solution, $i(0) = Ae^{-\frac{R}{L}0} = A \Rightarrow A = 0.25 \Rightarrow i(t) = 0.25e^{-\frac{R}{L}t}$.

Example:

(a) Suppose it is known that at time $t = 0$, $v_o(0) = 6$ Volts, Find $v_o(t)$ for $t > 0$.

(b) Suppose it is known that at time $t = 0$, $i_o(0) = 50$ mA, Find $i_o(t)$ for $t > 0$.

Example:

Suppose it is known that at time $t = 0$, $i_C(0) = 75\text{mA}$, Find $v(t)$ for $t > 0$.

Note: There are two different ways one could apply the initial conditions here:

1) Use $i_C(0)$ to find $i_R(0)$ (via KCL) and then find $v(0)$ (via Ohm's Law).

2) Use $i_C(0)$ to find $\frac{dv}{dt}(0)$ (via $i_C(t) = C\frac{dv}{dt}$) and then use $\frac{dv}{dt}(0)$ as the initial condition.

Try it both ways.